Guidelines for effective technology facilitation of Realistic Mathematics Education to enhance teaching practice

DJ Laubscher

10218343

Thesis submitted for the degree *Doctor Philosophiae* in Mathematics Education at the Potchefstroom Campus of the North-West University

Promoter: Prof AS Blignaut

Co-Promoter Prof HD Nieuwoudt

May 2017

It all starts here ™

NORTH-WEST UNIVERSITY
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Declaration

I the undersigned, hereby declare that the work contained in this dissertation / thesis is my own original work and that I have not previously in its entirety or in part submitted it at any university for a degree.



Signature

November 2016

Date

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This thesis is dedicated to my late parents Louis and Joy Holmes, who instilled in me the desire to work hard, strive for the best and never give up. They taught me to live by the motto “If something is worth doing, do it well.”

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Abstract

Different teaching styles and approaches that are employed in a teaching environment have a huge impact on the achievement of learners. The value of using the Realistic Mathematics Education (RME) teaching approach to teaching Mathematics is evident in a number of studies. This study aimed to determine how the RME approach, facilitated by technology, could be used to enhance teaching practice. It also aimed to establish what support needs in-service Mathematics teacher-students in the South African context had with regard to using the RME approach, and it intended to yield guidelines for the effective use of technology in implementing the RME approach, thereby generating theoretically relevant knowledge to the body of scholarship. A purposeful stratified sample was used to understand what needs in-service Mathematics teachers had in terms of making Mathematics more realistic for their learners. Participants were selected from a group of in-service teachers enrolled for the BEd Honours post-graduate degree in Mathematics education at the NWU. Qualitative design-based research was an appropriate methodology for this study. I produced, evaluated and adjusted content based on the RME approach, presented by means of a mobile application to assist teacher-students in presenting learning content to their learners in a more realistic and relevant way. The first stage of the study consisted of a systematic literature review (SLR) to determine how the RME approach could be facilitated by technology to enhance teaching practice. Individual interviews with participants followed, to determine their needs in terms of effectively implementing the RME approach in their teaching practice. Thereafter iterative cycles of design and intervention took place. A mobile application (app) was used to present content to the participants, based on the needs identified in the individual interviews and the literature review. The re-design of the intervention was guided by testing different versions of the prototype of the app with various users, as well as by literature. Focus group interviews were used to establish the teacher-students' perceptions and evaluation of the app. The final stage of the study produced guidelines based on the findings of the study. Data were coded, sorted and summarised into themes using a computer assisted qualitative data analysis software (CAQDAS), ATLAS.ti™. The data analysis and interpretation were integrated with the literature review to answer the research questions. Based on these findings, guidelines for the effective use of technology in implementing the RME approach in teaching practice were designed. These guidelines were designed around the five dominant themes in the study, namely Mathematics, RME, ICT, role-players and aspects relating to the app. The guidelines will be of value to teachers, their learners, lecturers, curriculum specialists and instructional designers in the design, implementation and adaptation of mathematical content and course material.

Keywords:

Mathematics Education; Realistic Mathematics Education (RME); mLearning; Mobile App; Systematic Literature Review (SLR); qualitative data analysis; Interpretivist; ATLAS.ti™; teacher-students; open distance learning (ODL).

Opsomming

Verskillende onderrigstyle en benaderings tot onderrig het ’n geweldige impak op die prestasie van leerders. Die waarde van die gebruik van die Realistiese Wiskunde-onderrig (RWO)-benadering word genoegsaam gesteun deur navorsing. Hierdie studie het gepoog om te bepaal hoe die RWO-benadering, gefasiliteer deur tegnologie, gebruik kan word om die praktyk van onderrig en leer te verbeter. Dit het ook gepoog om te bepaal watter ondersteuningsbehoeftes praktiserende Wiskunde-onderwysers wat deeltyds studeer, spesifiek in die Suid-Afrikaanse konteks, het t.o.v. die gebruik van die RWO-benadering. Voorts het dit gepoog om ook spesifieke riglyne daar te stel vir die effektiewe gebruik van tegnologie tydens die implementering van die RWO-benadering, wat ’n bydrae sal maak tot die relevante teoretiese kennis in hierdie vakgebied. ’n Doelgerigte gestratifiseerde steekproef is gebruik om te bepaal watter ondersteuningsbehoeftes praktiserende Wiskunde-onderwysers het, om Wiskunde meer realisties te maak vir hulle leerders. Deelnemers is geselekteer vanuit ’n groep praktiserende Wiskunde onderwysers wat deeltyds studeer, en ingeskrewe BEd Honneurs-studente (Wiskunde-onderrig) by die Noordwes-Universiteit is. ’n Kwalitatiewe ontwerp-gebaseerde navorsingsmetodologie is gebruik vir hierdie studie. Ek het die inhoud geproduseer, beoordeel en aangepas n.a.v. die RWO-benadering. Dit is gedoen m.b.v. ’n mobiele toepassing wat praktiserende Wiskunde onderwysers wat deeltyds studeer, kan gebruik in ’n poging om die leerinhoud meer realisties en relevant aan te bied. Die eerste fase van die studie bestaan uit ’n sistematiese literatuurstudie om te bepaal hoe die RWO-benadering gefasiliteer kan word d.m.v. tegnologie in ’n poging om onderrigpraktyk te verbeter. Individuele onderhoude met die deelnemers is daarna gedoen met die doel om die behoeftes t.o.v. die effektiewe implementering van die RWO-benadering in die onderrigpraktyk te bepaal. Ontwerpsiklusse wat beide herhalend en ingrypend van aard was, is toegepas. ’n Mobiele toepassing wat ontwerp is na gelang van die bepaling van behoeftes soos geïdentifiseer deur die individuele onderhoude en die literatuurstudie, is gebruik om inhoud aan te bied aan die deelnemers. Die herontwerp van die ingryping is bepaal deur die toetsing en evaluasie van verskillende weergawes van die prototipe, en is voordurend aangepas na gelang van toetsing en terugvoer van verskeie gebruikers, ondersteun deur die literatuur. Fokusgroep-onderhoude is gebruik om die deelnemers se persepsies en evaluasie van die toepassing te bepaal. Die finale fase van die studie het riglyne opgelewer gebaseer op die bevindings van die studie. Data is gekodeer, sorteer en opgesom en vervat in verskillende temas deur gebruik te maak van ’n kwalitatiewe rekenaardata-analise-sagteware, ATLAS.ti™. Die data-analise en interpretasie is geïntegreer met die literatuuroorsig met die doel om die navorsingsvraag te beantwoord. Riglyne vir die effektiewe gebruik van tegnologie in die implementering van die RWO-benadering in onderrigpraktyk is ontwerp, gebaseer op die bevindings uitgewys deur die data. Hierdie riglyne is ontwerp en het gefokus op die vyf dominante temas in die studie, naamlik Wiskunde, RWO, IKT, die verskillende rolspelers en aspekte m.b.t. die toepassing. Die riglyne sal van waarde wees vir onderwysers, die leerders, lektore, kurrikulumspesialiste en onderrigontwerpers tydens die ontwerp, implementering en aanpassing van wiskundige inhoud en Wiskundige kursusmateriaal.

Sleutelwoorde

Wiskunde-onderrig, Realistiese Wiskundeonderrig, mobiele leer, Mobiele Toepassing, Sistematiese literatuuroorsig, kwalitatiewe data-analise, interpretivisties, ATLAS.ti™; praktiserende onderwysstudente, oop afstandsleer (OAL).

List of Acronyms and Abbreviations

ACE	Advanced Certificate in Education
ANA	Annual National Assessment
App	Application
BEd	Bachelor of Education
CAQDAS	Computer Assisted Qualitative Data Analysis Software
DBE	Department of Basic Education
DE	Distance Education
ECAR	Educause Centre for Analysis and Research
FET	Further Education and Training
HEI	Higher Education Institution
HHT	Hand-held Technology
HLT	Hypothetical Learning Trajectory
HOD	Head of Department
HU	Hermeneutic Unit
ICT	Information and Education Technology
iOS	iPhone Operating System
IT	Information Technology
IWB	Interactive Whiteboard
MKT	Mathematical Knowledge for Teaching
NCTM	The National Council of Teachers of Mathematics
NSLA	National Strategy for Learner Attainment Framework
NWU	North-West University
ODL	Open and Distance Learning
PCK	Pedagogical Content Knowledge
RME	Realistic Mathematics Education
SDL	Self-directed Learning
TCK	Technological Content Knowledge
TPACK	Technological Pedagogical Content Knowledge
TPK	Technological Pedagogical Knowledge
UNISA	University of South Africa
UODL	Unit for Open Distance Learning
VLE	Virtual Learning Environment

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Addendum 9	ATLAS.ti™ Hermeneutic Unit

Addenda are available on the CD at the back of the dissertation.

The ATLAS.ti™ PDF file is available at https://goo.gl/kkVzX6

Chapter One

Orientation to the Research Journey

1.1 Introduction

The effective teaching and learning of Mathematics remains a priority in South Africa (Fleisch & Schöer, 2014, p. 1). Different approaches to the teaching of Mathematics influence learner achievement. The significance of using the Realistic Mathematics Education (RME) teaching approach to assist learners previously taught by traditional approaches is evident in literature. This study aims to determine the needs of in-service Mathematics teacher-students in the South African context with regard to teaching using the RME approach. It also intends to yield guidelines for the effective use of technology in implementing the RME approach; thereby generating theoretically relevant knowledge to the body of scholarship (Gravemeijer, 1999, p. 113).

The sections that follow in this chapter give an overview of the study in terms of the problem statement and motivation to do the research; a brief literature review which also highlights the gaps in the current literature; the research design and methodology; ethical issues; the way in which data were generated; and also some important terminology referred to in this study. Figure 1.1 represents an overview of what follows in Chapter One.

1.2 Motivation and Problem Statement

A primary concern of the South African Government, on which the Department of Basic Education has to deliver, is to improve the quality of basic education (DBE, 2012, p. 2). The Annual National Assessment (ANA) and the National Senior Certificate (NSC) examination play crucial roles in the Government's action plan to improve the quality of basic education (DBE, 2013, p. 6). Since the 2015 ANA tests were postponed, the NSC examinations remain the main essential measure for monitoring progress in achieving the targets that have been set in terms of learner achievement for Grade 12s.

The Annual National Assessment (ANA) which monitored learner achievement progress was implemented from 2011 to 2014. It was a diagnostic testing programme that required all schools in the country to conduct the same Language and Mathematics tests, which was grade-specific, for grades 1 to 6 and 9 (DBE, 2012, p. 2). The 2012 ANA, a huge undertaking, assessed the literacy and numeracy of more than seven million learners. The national average percentage marks for Mathematics in 2014 were: grade1: 68%, grade 2: 62%, grade 3: 56%, grade 4: 37%, grade 5: 37%, grade 6: 43%, and grade 9:11% (DBE, 2014, p. 9).

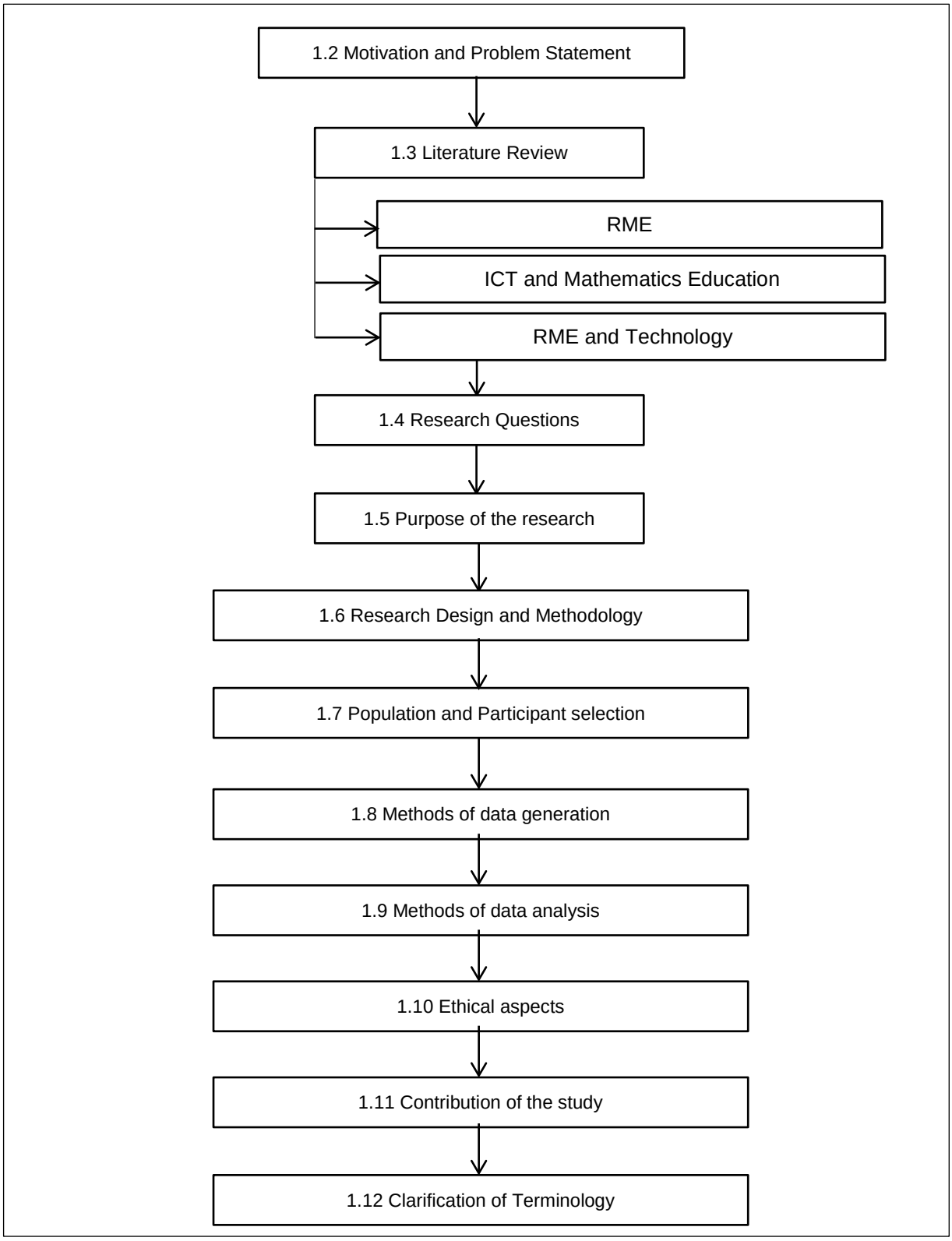


Figure 1.1: Orientation of the study

In most cases, except in grade 6 and grade 9, learners performed slightly better in 2014 than in 2013, however there remains a huge concern about the declining trend from lower to higher grades, culminating in the extremely poor results of the grade 9s (DBE, 2014, p. 9).

Not only are there concerns with the Foundation, Intermediate and Senior Phase Mathematics, but also with the grade 12s in the Further Education and Training (FET) phase. One of the targets of the government's Action Plan to 2014 was to increase the number of grade 12 learners who pass Mathematics (DBE, 2011, p. 8). The percentage pass rates for Grade 12 Mathematics for the past eight years are as follows: 2008 was 45,4%, 2009 was 46,0%, 2010 was 47,4%, 2011 was 46,3%, 2012 was 54%, 2013 was 59,1%, 2014 was 53,5% and in 2015 it was 49,1% (DBE, 2015, p. 58). Despite occasional improvement, there is still great concern about the Mathematics results.

One of the challenges identified by the government during the first five years of the implementation of the National Senior Certificate (NSC) is that there is a concern about the large numbers of candidates enrolling for Mathematical Literacy rather than Mathematics. Since the ratio at present is 2:3 of Mathematics to Mathematical Literacy for grade 10 to 12 learners, the Department of Basic Education (DBE) would like to see more learners taking Mathematics (DBE, 2015, p. 19).

Specific intervention strategies for 2013 have been devised as part of a turnaround strategy within the National Strategy for Learner Attainment (NSLA) framework. One such strategy expects teachers to, amongst other important aspects, not only ensure full curriculum coverage; provide opportunities for more written work by learners, but also improve the quality of teaching and assessment tasks given to learners (DBE, 2013, p. 14). The DBE has prioritised Mathematics, Science and Technology skills in line with the national human resource development priorities. One example of such support is the Dinaledi[1] schools' programme which provides focused support and intervention in Mathematics, Science and Technology in the form of funding for Mathematics and Science equipment, developing learning and teaching support material for Mathematics and Science and training for Mathematics and Science teachers on subject content knowledge (DBE, 2013, p. 15). Furthermore, the Telkom Masters Agreement was signed on 27 March 2012 for the first phase of the Connectivity Plan, which will provide internet connectivity to 1650 schools for a period of three years (DBE, 2013, p. 15) and aims at assisting with the learners' performance in Mathematics Science and Technology.

Different teaching styles influence learner achievement, and the selection of teaching approaches that are employed in a teaching environment, have a huge impact on the achievement of learners (Samuelsson, 2010, p. 61). Particularly, the traditional approach, where *talk-and-chalk* is the preferred teaching style, is linked to the poor quality of Mathematics education (Bishop, Hart, Lerman, & Nunes, 1993, p. 18). Often learners do not like learning Mathematics because they do not learn the Mathematics that they need in life (Fauzan, Plomp, & Gravemeijer, 2013, p. 162). Various studies have indicated that the RME approach offers great value to learners who have failed to benefit adequately from traditional teaching approaches (Barnes, 2004; Cobb, Zhao, & Visnovska, 2008; Fauzan et al., 2013; Gravemeijer & Doorman, 1999; Webb, 2011).

[1] In 2001 the Department of Education initiated the Dinaledi School Project to try to improve the number of university-entrance passes in Mathematics and Science for Grade 12 learners. The project involved selecting certain secondary schools that demonstrated potential for improving learner performance in Mathematics and Science. These schools were provided with resources and support to improve the teaching and learning of these subjects (DoE, 2009, p. 6).

The development of what is now known as Realistic Mathematics Education (RME) started in the late 1960s and is still under active development. In essence, RME is concerned with the idea of Mathematics as a human activity of making sense of reality so as for it to be useful (Freudenthal, 1973). The relevance for Mathematics education is that the subject is not a closed system, but an activity. The focus should be on the process of mathematizing reality (Freudenthal, 1968, p. 7). This process of mathematization involves providing learners the opportunity to reinvent Mathematics by reorganising or mathematizing real world situations or mathematical processes (Cobb et al., 2008, p. 105). By using technology, teacher-students can learn the RME approach as and when they wish to do so. This can happen during a guided programme or even after the completion of such a programme, since technology makes the availability of information very easy to access (Zulkardi, 2002, p. 157).

The National Council of Teachers of Mathematics (NCTM) claims in their position statement that "it is essential that teachers and learners have regular access to technologies that support and advance mathematical sense making, reasoning, problem solving, and communication" (National Council of Teachers of Mathematics, 2011, p. 1). The importance of technology in teacher education is also emphasised by the NCTM. They assert that teacher education programmes and professional development must continually update specialists' knowledge of technology and the application thereof to support learning (National Council of Teachers of Mathematics, 2011, p. 1).

Drijvers (2012, p. 1) valiantly raises the question of whether digital technology in Mathematics education works or not. Upon investigation of six cases, which are considered leading studies in the field, he reveals that both success and failure occur at levels of teaching, learning and research (Drijvers, 2012, p. 12). An important observation that is made in this study is that the integration of technology in Mathematics education does not reduce the importance of the teacher. The teacher plays an active role and needs to devise learning activities that relate experiences within the technological environment to mathematical activities (Drijvers, 2012, p. 12). Another important observation is that digital technology should be embedded in an educational context that is coherent and in which the technology is integrated in a natural way (Drijvers, 2012, p. 13). The importance of the teacher in the process of teaching with technology should not be underestimated. The role of the teacher has been acknowledged as a critical factor in the integration of technology into teaching and learning Mathematics. It is considered critical because the way in which teachers approach the use of technology has vast consequences for the effects of its use in the classroom (Drijvers, Doorman, Boon, Reed, & Gravemeijer, 2010a, p. 213). Ottenbreit-Leftwich et al. (2012, p. 400) assert that technology is an essential part of professional competency and teachers' appropriate use of technology can have positive academic benefits.

1.2.1 Gaps in the Existing Literature

A study concerning the design, development and evaluation of a learning environment (an RME course created with web support) for student teachers in Indonesia revealed important results (Zulkardi, 2002, p. 46). This learning environment could assist the student teachers in learning the mathematical, didactic and practice part of the RME course, thus promoting their understanding about RME and supporting student teachers in learning how to redesign lesson material for the classroom (Zulkardi, 2002, p. 168). The learning environment also had a positive impact on developing student teachers' performance, and on increasing the positive attitude of pupils in the secondary school towards Mathematics (Zulkardi, 2002, p. 168).

A research project (Widjaja & Heck, 2003) conducted at an Indonesian Junior School which investigated the effects of teaching and learning in an RME-based and Information and Communication Technology (ICT)-supported learning process revealed that teachers in general considered the experience as positive. The teachers believed that the chosen approach proved to be advantageous for both teachers and learners. Learners obtained results from their own efforts, rather than receiving descriptive material from the teacher, they were acquainted with new technology and their abilities and skills were explored and encouraged (Widjaja & Heck, 2003, p. 41). Advantages for the teacher included, amongst others, that the teachers did not have to spend much time and energy on explanations (Widjaja & Heck, 2003, p. 41). Some recommendations made in the study suggest that this idea should be developed within teacher-training institutes, more material should be provided for teachers and stronger networks among teachers and teacher-students should be established (Widjaja & Heck, 2003, p. 47).

A local study performed at UNISA also reports the successful adoption of the RME theory to teach introductory Calculus concepts within the context of Distance Education (Kizito, 2012b, p. 3). Some important recommendations in this study include that the concepts and mathematical structures that are to be represented should be carefully selected, and the learning activities should be researched, tested and developed by a team of experts which should include mathematicians and Mathematics subject didacticians (Kizito, 2012b, p. 3).

Some limitations highlighted by the studies listed above include the further need to investigate the extent to which the learning environment, which comprises a RME course with Web support, will improve the perceptions and Mathematics learning outcomes of pupils after they have been taught by prospective teachers using the RME approach with Web support (Zulkardi, 2002, p. 172). Additionally, Kizito (2012b, p. 284) suggests that intervention which involves mobile learning (mLearning) requires an institutional approach in order to be effective. Widjaja and Heck (2003, p. 45) also propose that the changing roles of teachers and learners need to be further researched.

Despite a few similar studies in the field, as summarised in Table 1, the need arises to investigate the use of the RME approach facilitated by a variety of technologies, for in-service Mathematics teacher-students in the South African context, with the aim to enhance teaching practice.

Table 1.1: Comparison of similar studies relating to Realistic Mathematics Education, Design-based research and technology

Key Factors	UNISA study	Indonesian study	Erkki Sutinen	Andreasen	Current Study
RME	Yes	Yes	Yes	Yes	Yes
Technology	Mobile	Web-based	Mobile	No technology	Mobile
Target	Undergraduate Mathematics	Undergraduate Education	Non-students	Undergraduate Mathematics education	In-service
Mode of delivery	Distance Education	Full-time	Incidental	Full-time	Open and Distance Learning (ODL) and face to face
Design	Design-based research	Design-based research	Action Research	Design-based research	Design-based research
Reference	(Kizito, 2012b)	(Widjaja & Heck, 2003)	In progress	(Andreasen, 2006)	

1.3 Overview of the Literature

1.3.1 Realistic Mathematics Education

Mathematics is of such a general nature that it applies to a richer variety of situations than any other teaching subject (Freudenthal, 1968, p. 5). Many learners are not able to apply their Mathematical classroom experiences in the most trivial situations in daily life (Freudenthal, 1968, p. 5). Freudenthal (1968, p. 5) distinguishes between two attitudes to teaching Mathematics, namely teaching Mathematics without any relation to its use other than hoping that learners will apply it when necessary; and the opposing attitude which is to teach useful Mathematics. Freudenthal (1973) believes that learners should be allowed to reinvent Mathematics by mathematizing real world situations. RME is rooted in Freudenthal's interpretation of Mathematics as a human activity, which implies an activity of looking for and solving problems, and also an activity of organising subject matter (Freudenthal, 1973).

Mathematizing is distinguished as having a horizontal and vertical component (Treffers, 1993, p. 94). In general terms, these components can be described as follows. In horizontal mathematization, learners devise mathematical tools which can help to organize and solve a problem in a real life situation. Vertical mathematization is the process of reorganization within the mathematical system itself, e.g. finding shortcuts and discovering connections between concepts and strategies and applying these discoveries (van den Heuvel-Panhuizen, 1998). Horizontal mathematization is a process in which learners transform problem situations that they perceive as realistic into a mathematical system (Widjaja & Heck, 2003, p. 9). In RME mathematization takes place in both

directions by means of a process of reinvention which is guided by the teacher and the instructional materials (Widjaja & Heck, 2003, p. 10).

In RME, Mathematics is considered to be a human activity connected with reality, where contextual problems are used as a starting point in learning (Widjaja & Heck, 2003, p. 3). Three basic RME heuristics have been identified by Gravemeijer, Bowers, and Stephan (2003, p. 52). These provide designers with information to assist in supporting learners within the cycles of design-based research. They include the following:

1. sequences must be experientially real for learners (learners must be able to engage in personally meaningful activity, the sequences must be realistic in learner terms);
2. guided reinvention (after the designer has engaged in planning, a sequence of instructional activities are developed);
3. emergence of learner-developed models (learners' modelling activity is developed to support the reinvention process) (Gravemeijer et al., 2003, p. 52).

These three heuristics work together to assist learners to participate in activities that will develop sophisticated mathematical practices (Gravemeijer et al., 2003, p. 52).

Besides the three heuristics, RME is also characterised by five types of activities (Gravemeijer, 1994, p. 451; Kizito, 2012b, p. 103):

1. *Phenomenological exploration*: Teaching learners ways to manipulate the means of organising phenomena that need to be organised.
2. *Using models and symbols for progressive mathematization*: Attention is given to models, model situations and schemata that arise from problem-solving activities.
3. *Learner contributions*: Learners are encouraged to create their own constructions and productions.
4. *Interactivity:* A learning process is constructed where learners' informal methods are used to negotiate, intervene, discuss, co-operate and evaluate.
5. *Intertwining:* Problem-solving consists of an interlinking of learning strands.

As far as possible, these activities should be included in a RME approach.

The notion of guided reinvention is closely associated with RME. As Freudenthal (1973) points out, Mathematics is an activity where the most important activity is mathematization, which implies a form of organization from a mathematical perspective. This organisation or mathematization is how the learners can reinvent Mathematics, but it is important to note that learners are not meant to reinvent on their own. Freudenthal (1973) refers to guided reinvention, where the emphasis is not on the invention, but rather on the process that takes place (Gravemeijer & Doorman, 1999, p. 116). The process starts where real life situations are mathematized, and continues where reinvention takes place in which learners also mathematize their own mathematical activity (Gravemeijer & Doorman, 1999, p. 116). In practice this would mean that specific contextual problems are given to learners where they are given the opportunity to create solutions within the given context. The reinvention

takes place when learners use their everyday language to structure the given problems into both informal and formal mathematical forms (Barnes, 2004, p. 56).

In order to apply the principles of RME, the development of a Hypothetical Learning Trajectory (HLT) is required. Simon (1995) refers to HLT which represents a framework for teachers to adapt instructional sequences that suit their needs in the classroom. An HLT comprises three components: the teacher's goal or aim for learner learning that determines the direction, the actual mathematical tasks that will be used to promote learner learning, and a predicted view or hypothesis regarding the process of learner thinking, learning and understanding that will shape in the context of the learning activities (Baroody, Cibulskis, Lai, & Li, 2004, p. 231; Simon, 1995, p. 133). HLT is the course along which learning might proceed, as predicted by the teacher. The creation and on-going modification and refinement of these learning activities form a cyclical process. As teachers evaluate learners' thinking and learning, new ideas for learning activities can be planned. These learning activities depend on the teacher's hypotheses about the development of learners' thinking and learning (Simon, 1995, p. 136).

A number of key aspects play a role in the teaching process. These aspects include: the teachers' knowledge of Mathematics, their views about learners' cognitions, their theories about Mathematics teaching and learning, their knowledge of learning with respect to specific mathematical content, and their knowledge of mathematical representations, materials and activities (Simon, 1995, p. 133). These ideas are reinforced by research done on Mathematical Knowledge for Teaching (MKT).

Mathematical Knowledge for Teaching (MKT) is defined as the "mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to learners" (Ball, Thames, & Phelps, 2008, p. 399). As an extension to the work of Shulman (1986), Ball et al. (2008, p. 403) distinguish between different domains for mathematical knowledge for teaching, namely subject matter knowledge and pedagogical content knowledge, which in turn are subdivided into the following categories: common content knowledge, horizon content knowledge, specialised content knowledge, knowledge of content and learners, knowledge of content and teaching and knowledge of content and curriculum. When referring to the mathematical knowledge that teachers need to teach, Ball et al. (2008, p. 395) consider "teaching" to entail the following activities: planning the lessons, evaluating learners' work, designing and assessment, explaining classwork to parents, designing and managing homework and dealing with equity. Mishra and Koehler (2006) also expand on the work of Shulman (1986) by introducing the knowledge of technology as an essential knowledge domain. The Technological Pedagogical Content Knowledge (TPACK) framework presents the three main components of teachers' knowledge: content, pedagogy and technology. Not only are these aspects important, but the interactions between them which are represented as PCK (pedagogical content knowledge), TCK (technological content knowledge), TPK (technological pedagogical knowledge) and TPACK (technological pedagogical content knowledge) are essential (Mishra & Koehler, 2006, p. 62). The role of a teacher's MKT is therefore vital in his or her construction of an HLT.

1.3.2 Information and Communication Technology and Mathematics Education

The positive effects of using technology in teaching and learning situations are becoming more apparent (Bennison & Goos, 2010, p. 31; Chinnappan, 2003, p. 35; Li, 2003, p. 72; Mistretta, 2005, p. 18). Training teachers to integrate technology in the classroom and continuing to investigate the effects of technology on teaching and learning are two ways to empower technology-based learning environments (Mistretta, 2005, p. 23). The need to assist both pre-service and in-service Mathematics educators to develop the ability to effectively make use of technology in their classrooms is becoming more pronounced and important. It is suggested that technology is best learned in context and should thus be integrated into coursework and field experience (Li, 2003, p. 62). Li (2003, p. 62) believes that teacher-students should see their professors modelling or demonstrating the use of technology.

In a study that explored the use of the Internet (discussion forums, online material, e-journaling, computer games) in a Mathematics education course, teacher-students felt that instructional technology could be an effective tool for their own learning. They also indicated that the use of the Internet in the course assisted them with aspects like improving communication with one another, which in turn could enhance their understanding of educational theories. Other advantages of using instructional technology that they mentioned, include that it gives time for synthesis, it enhances learning by providing visual and interactive experiences, and it saves time (Li, 2003, p. 72).

Although teachers use technology to display or present content, many are not aware of the potential that technology has to promote concept development in the Mathematics classroom (Serow & Callingham, 2011, p. 161). The role of the teacher in integrating technology in Mathematics education is extremely important. The teacher has to coordinate learning by creating technology-rich activities, use appropriate tool techniques and relate the experiences in the technological environment to mathematical activities. To achieve this, a process of professional development is needed (Drijvers, 2012, p. 12). Serow and Callingham (2011, p. 171) who conducted research on the use of the Interactive Whiteboard (IWB) in teaching primary school Mathematics, suggest that Mathematics teachers require time to explore and develop teaching materials and should be allowed to work with a mentor from an early stage when it comes to effective professional development.

Facebook was originally designed as a social networking website, but has proceeded to be used in educational settings as well. Grossecka, Branb, and Tiruc (2011, p. 1426) review characteristics from literature concerning learners and teachers. The learners recommend Facebook as a tool that has the potential to contribute significantly to the educational arena. For the learners, some advantages include the following: to be involved in achieving learning tasks; to develop communication, cognitive and social competencies; to create their own learning path by establishing links and connections; to consolidate self-confidence and self-esteem and to communicate with the teacher outside the class.

For the teacher, the following benefits are highlighted: to practise a differential pedagogy in the best interest of the learners; to perform mentoring; to interconnect learning experiences; to expand the communicative experience with the learners concerning didactic issues and to give up on old behavioural patterns (Grossecka et al., 2011, p. 1427).

GeoGebra is freely-available open-source dynamic Mathematics software which incorporates Geometry, Algebra, Calculus and a spread sheet into a single software package. To merely provide teachers with technology is not enough for them to successfully integrate the technology into their teaching (Hohenwarter & Lavicsa, 2007, p. 49). Hohenwarter and Jones (2007, p. 130) assert that, apart from online collaboration, teachers should also be offered professional development in terms of the use of technology in their teaching and also research activities should be coordinated, especially in relation to GeoGebra.

1.3.3 Realistic Mathematics Education and Technology

As the TPACK framework of Mishra and Koehler (2006) suggests, ICT is a suitable means to promote the realization of alternative approaches to Mathematics, such as the realistic approach (Widjaja & Heck, 2003, p. 3). In their study which focused on the applicability of ICT-supported lessons, based on a RME approach, Widjaja and Heck (2003) reported that learners were both positive towards the use of ICT in lessons before and after implementing different ICT related tasks. Not only did learners' performance improve in this study, but 77% of the group agreed or strongly agreed that doing activities using a computer was interesting and exciting and looked forward to making use of ICT in their next Mathematics lessons.

A study which explores teachers' views on Mathematics education and the role of technology therein, and was guided by RME principles, reveals that teachers see technology as a means to stimulate interaction in the class, a key principle in RME (Drijvers et al., 2010a, p. 222). The study also revealed that teachers see technology as an effective means to achieve their teaching goal in Mathematics; they believe that ICT could help learners to develop understanding; and that technology is a suitable and useful means to provide scaffolding for learners with mid-ability who benefit from clear demonstrations and explanations in a structured manner (Drijvers et al., 2010a, p. 223).

A literature study to explore the existing body of knowledge with regard to the adoption of a RME approach, facilitated by technology to enhance teaching practice, was conducted. A systematic literature review was performed to not only limit bias (Petticrew & Roberts, 2006, p. 9), but also to summarise the existing evidence relating to the topic, to identify any gaps in current research to suggest possible areas for further investigation and to provide an appropriate background to position new research (Kitchenham, 2004, p. 2). Although not a common feature, systematic literature reviews have been used in the social sciences for many decades and are increasingly used to not only direct new research, but to support practice and policy (Petticrew & Roberts, 2006, p. 23).

The systematic literature review was done with the aid of the databases Scopus, EBSCOHost, Web of Science, Science Direct, MathSciNet, and Google Scholar. Various combinations of the following keywords were used to perform the searches: realistic* or RME, teach* or learn* or educat*, math*, tech* or ict or educat* tech*.

1.4 Research Questions

The study aimed to address the following main research question: *What are the guidelines for the effective use of technology in implementing the RME approach in teaching practice?* In order to effectively answer this research question, two sub-questions were also posed to gain insight on the main question. These questions were:

1. How can the use of the RME approach, facilitated by technology, enhance teaching practice?
2. What support needs do teachers have in order to effectively implement the RME approach in their teaching practice?

1.5 Purpose of the Study

The purpose of the study was to:

- Develop an understanding of how the RME approach, facilitated by technology, can enhance teaching practice;
- Establish support needs of teachers in order to effectively implement the RME approach in their teaching practice;
- Develop guidelines for the effective use of technology in implementing the RME approach in teaching practice.

1.6 Research Design and Methodology

Each researcher brings with him or her certain beliefs and philosophical assumptions to their research, which are instilled in them through different influences (Creswell, 2013, p. 15). Philosophical ideas are often hidden in research, yet they still influence research practice. A worldview or paradigm is a researcher's orientation about the world and the nature of research (Creswell, 2009, p. 5). The significance of a paradigm is that it gives meaning to the world as we see it. The selected paradigm for this research was the interpretive paradigm. From an interpretive perspective, the focus is on describing, understanding or interpreting an experience (Merriam, 2009, p. 11).

Creswell (2013, p. 15) suggests that a qualitative researcher highlights the importance of understanding these beliefs and assumptions and also actively writes about them in a study, and this I have duly done in the study. The interpretive paradigm is "characterised by a concern for the individual" and aims to understand the "subjective world of human experience" (Cohen, Manion, & Morrison, 2011, p. 17). Interpretive approaches start with individuals and aim to understand those individuals' interpretation of the world around them (Cohen et al., 2011, p. 18). Interpretive research is the most common type of qualitative research and works from the premise that reality is socially constructed; and that there is no single observable reality, but rather multiple realities (Merriam & Tisdell, 2016, p. 9).

Researchers with interpretivist goals describe and attempt to explain the meaning or implications of phenomena related to educational factors such as teaching, learning, performance, assessment and social interaction (McKenney & Reeves, 2012, p. 29). Design experiments aim to support the composition of an empirically grounded local instruction theory. They also aim to place classroom events in a broader context and serve as the context for the development of theoretical frameworks that entail new scientific categories, which can be useful in generating, selecting and assessing design alternatives. The interpretive framework serves as an innovation for interpreting classroom discourse and communication, and also shows what norms to aim for to make the design experiment successful (Gravemeijer & Cobb, 2013, p. 80). The interpretive framework has a dual purpose: it acts as a lens for making sense of what is happening in a real world setting, and acts as a guideline for instructional design. It also offers guidelines on the characteristics of the classroom culture(Gravemeijer & Cobb, 2013, p. 89).

Qualitative research is a "situated activity that locates the observer in the world" and consists of interpretive practices that make the world visible (Denzin & Lincoln, 2011, p. 6). The qualitative researcher studies "things in their natural settings, attempting to make sense of or interpret phenomena in terms of the meanings people bring to them" (Denzin & Lincoln, 2011, p. 6). Merriam and Tisdell (2016, p. 15) point out that qualitative researchers are concerned with "understanding the meaning people have constructed; that is, how people make sense of their world and the experiences they have in the world." Qualitative researchers are concerned with "the meanings people attach to things in their lives" and identify with the people they study so as to understand how they see things (Taylor & Bogdan, 1998, p. 7).

As was the case in this study, various other qualitative studies including those of Baumann et al. (2013, p. 25); Juuti and Lavonen (2013, p. 49); Nathans and Revelle (2013, p. 164), have made use of a design-based research approach. Educational design-based research is an innovative and promising approach in which the iterative progression of solutions to complex problems provides the backdrop for scientific investigation (McKenney & Reeves, 2014, p. 131). Researchers who work with design-based research not only attempt to solve significant real world problems, but seek to unfold new knowledge (McKenney & Reeves, 2014, p. 131). The common features of design-based

research include the following: theories on learning and teaching are produced; it is interventionist; it is iterative and takes place in real life settings (Barab & Squire, 2004, p. 2). Design-based research is concerned with using design to develop models of how people think, know, act and learn; as well as to uncover, explore and confirm theoretical relationships (Barab & Squire, 2004, p. 5).

1.7 Population and Participant Selection

The teacher-students that participated in this research were enrolled at the North-West University (NWU) Potchefstroom Campus in South Africa. The NWU has three campuses across South Africa: Mafikeng Campus, Potchefstroom Campus and Vaal Triangle Campus. The Unit for Open Distance Learning (UODL) is situated on the Potchefstroom Campus and services in the region of 30 000 students who are registered to study various courses through distance education. The majority of these students are in the Faculty of Education Sciences, and are teachers who are under- or unqualified and who want to upgrade their qualifications (Pienaar, 2016, p. 1). The UODL has 65 study centres in South Africa, Namibia and other countries. Lectures are broadcast to these centres where students can attend contact classes and have access to a mini-library equipped with computers with Internet access (Spamer, 2016, p. 1). Figure 1.2 illustrates the location of the various study centres in South Africa and Namibia.

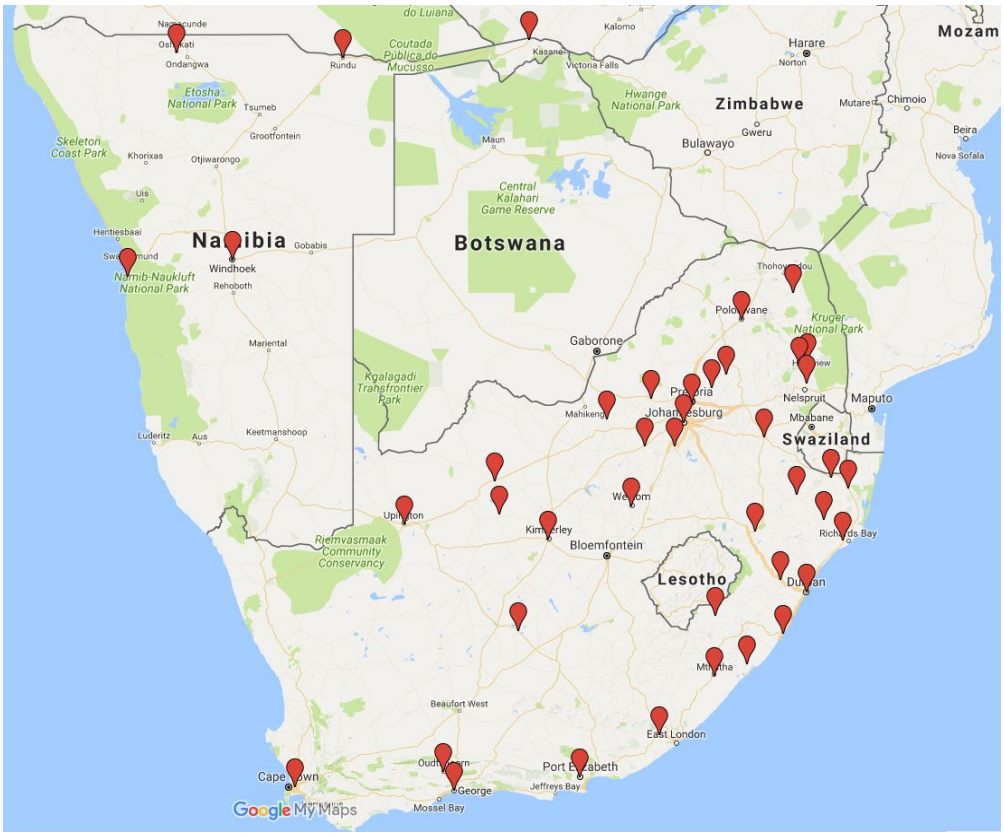


Figure 1.2: Distribution of NWU Study Centres across South Africa and Namibia

Many of the teacher-students are from disadvantaged schools and are women (Kok, 2009). They live in areas that range from deep rural areas where there is often no electricity or running water to fully urbanized communities such as Johannesburg and Cape Town. Many teacher-students work in classrooms that are often not well-equipped and teach large groups of learners (more than forty in a classroom). Despite these circumstances, the teacher-students still have the desire to improve their qualifications.

Stratified purposive sampling entails that participants are selected according to preselected criteria relevant to the research question (Nieuwenhuis, 2010c, p. 79). I selected this sampling method because I wanted to understand what needs in-service Mathematics teachers had in terms of making Mathematics more realistic for their learners. In this study, I selected participants from a group of in-service teachers enrolled for the BEd Honours post-graduate degree in Mathematics education at the NWU, which is delivered by both the dual face to face and ODL mode. The criteria for selection were as follows:

(i) teacher-students enrolled at the NWU for the BEd Honours degree through either face to face or ODL mode
(ii) teacher-students who attended contact classes for the degree in question
(iii) teacher-students who are currently teaching Mathematics in any Phase
(iv) male and female teachers at various schools.

1.8 Methods of Data Generation

After the systematic literature review had been completed, I designed an interview schedule for a semi-structured interview with teacher-students in order to establish what needs they had in relation to making the Mathematics content more realistic for their learners. Mathematics teaching that is related to real life has proved to enhance learners' understanding (Fauzan et al., 2013, p. 174) and can stimulate and address alternative conceptions of learners (Barnes, 2004, p. 63). I facilitated the semi-structured interviews with four teacher-students. When I coded and categorised the data, using ATLAS.ti™, I was able to determine two important aspects in this study: what areas of the curriculum were problematic to the teacher-students and also to their learners; and what areas they found difficult to present in a real life context. With the help of a programmer, I designed an application (app) to guide teacher-students in how to present mathematical content using the principles of RME. Teacher-students were given the opportunity to work with the app, and gave feedback at a follow-up focus group interview. These interviews were also coded with the aid of ATLAS.ti™.

1.9 Methods of Data Analysis

The data obtained through the interviews were transcribed and the documents were assigned to the Computer Assisted Qualitative Data Analysis Software (CAQDAS), ATLAS.ti™, where they were coded. ATLAS.ti™ offers a variety of tools to explore the phenomena concealed in the data. ATLAS.ti™ provided me with an environment in which to manage, compare, explore and reassemble meaning from the data in a systematic manner (Friese, 2014a, p. 10).

1.10 Ethical Aspects of this Research

The following ethical principles were applied throughout the project, namely that the participants were protected from any harm and the research data would at all times remain confidential (Fraenkel & Wallen, 2003, p. 57). The necessary ethics application form was completed and submitted to the North-West University's ethics committee, and permission was attained to commence the research. The ethical clearance number for this study is: NWU-HS-2014-0267. Participants were requested to complete a letter of consent, providing me with permission to continue with the project. The letter ensured participants that their participation was voluntary, their identity would remain anonymous, all information would be treated as confidential and that their decision whether to participate in the project or not, would not jeopardise their studies in any way.

1.11 Contribution of this Study

This contribution of this study would be to develop, implement and evaluate the use of the RME approach supported by technology, as presented to in-service teacher-students enrolled for a post-graduate degree in Mathematics education. The aim was to provide guidelines for the effective use of the RME approach, facilitated by technology, to enhance the usefulness of Mathematics in the classroom, and improve teaching and learning practice. The study contributed towards the subject area and discipline of Mathematics by providing insight into the needs of Mathematics teachers in terms of making Mathematics more realistic for their learners. The findings of this study could assist curriculum experts and lecturers in the designing and adaptation of study material that is more relevant for teaching practice. It could also assist teachers and learners by making the mathematical content more relevant and realistic for them. The study also produced guidelines as to how the RME approach could be successfully facilitated by technology to improve teaching and learning practice.

1.12 Clarification of Important Terminology

Table 1.2 presents the clarification of some important terminology used in the study.

Table 1.2: Table for Clarification of Terminology

Term	Clarification of term
Mathematics	Freudenthal (1968, p. 4) argues that Mathematics has proved indispensable for the understanding and the technological control of both the physical world and the social structure. Mathematics has been described by some as a static discipline developed abstractly, while others see it as a dynamic discipline constantly changing due to new discoveries (Dossey, 1992, p. 39). Other authors describe Mathematics as the science of pattern and order, which challenges the view that Mathematics is merely a discipline dominated by computation (Van de Walle, Karp, & Bay-Williams, 2013, p. 13). Mathematics is also described as a human endeavour in which ordinary people construct concepts, discover relationships, invent methods, execute algorithms, communicate and solve problems posed by the real world (Cangelosi, 2003, p. 7).
Realistic Mathematics Education (RME)	Realistic Mathematics Education (RME) is rooted in the work of Freudenthal (1973). Freudenthal views Mathematics as a human activity of sense-making where learners should be given the opportunity to reinvent Mathematics by organising or mathematizing real world situations. The basic principles of RME view the learning of Mathematics as a progressive reorganisation of real world situations or mathematical processes (Cobb et al., 2008, p. 107).
Learning technology	Learning technology is defined as the application of technology for the enhancement of teaching, learning and assessment (Rist & Hewer, 1996, p. 3). Learning technology is viewed as not only a technical skill or as a means of improving learning effectiveness, but also a way of shifting goals and processes of education (Law et al., 2008, p. 14).
Technology	Technology has the potential to promote generalisation and justification by enabling fast and accurate computation, collection and analysis of data (Goos, Galbraith, Renshaw, & Geiger, 2003, p. 74). It is also able to provide learners with rich learning environments, which allows them to adopt a wide range of perspectives on complex issues and cater for individual differences (Sang, Valcke, van Braak, & Tondeur, 2010, p. 103).
Student teacher	Student teacher refers to pre-service students who are studying in the field of education.
Teacher-student	Teacher-student refers to teachers, who are currently in the profession, but who are furthering their qualifications on a part-time basis, the mode of delivery could be either face-to-face or open-distance learning (ODL).

Chapter Two

Research Design and Methodology

2.1 Introduction

The literature offers various definitions for research. One such definition is that research is a step by step process of collecting and analysing information to increase our understanding of a topic or issue (Creswell, 2014, p. 17). Merriam and Tisdell (2016, p. 3) state that what most of these definitions have in common is the idea of "inquiring into, or investigating something in a systematic manner." Research is important because its adds to our knowledge, it improves practice and it informs policy debates (Creswell, 2014, p. 18). Researchers bring their own beliefs and philosophical assumptions to their research projects. There is a close connection between the philosophy that researchers hold and how they advance to use a framework to encompass the investigation (Creswell, 2013, p. 15).

Educational research is important because it helps us in understanding educational processes, guides our decision making and can enhance classroom, school and system accountability (McMillan & Schumacher, 2014, p. 11). This research aims to produce guidelines for the effective facilitation of technology to facilitate Realistic Mathematics Education to enhance teaching practice.

This chapter in detail describes the choices I made during my research journey and it informs all subsequent chapters in terms of the world view of the researcher, the design and methodology for this study, choices regarding research strategies, intervention strategies and the consequent guidelines. Figure 2.1 illustrates the research design and methodology that has been used in this study.

The study is driven by the research question as discussed in § 1.4, namely: *What are the guidelines for the effective use of technology in implementing the RME approach in teaching practice?* The two sub-questions that were posed to gain insight on the main question were:

1. How can the use of the RME approach, facilitated by technology, enhance teaching practice?
2. What support needs do teachers have in order to effectively implement the RME approach in their teaching practice?

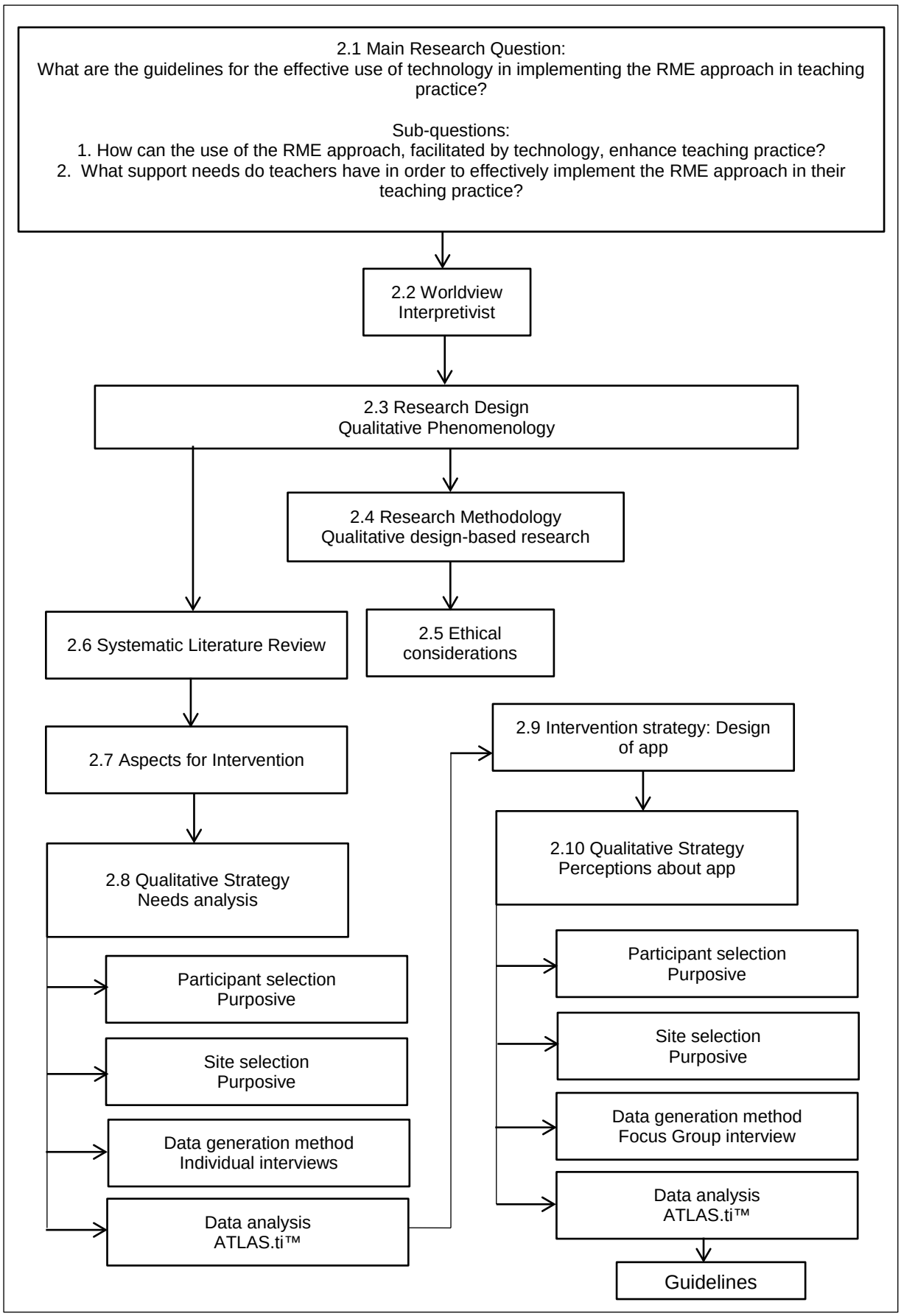


Figure 2.1: Research Design and Methodology for this Study

2.2 Worldview of the Research

Social scientists have particular assumptions about the nature of the social world, and they approach their subject with this in mind. These assumptions are of an ontological, epistemological, human, and methodological nature (Burrell & Morgan, 1994, p. 1). Burrell and Morgan (1994, p. 2) refer to assumptions of an ontological nature as the very essence of the phenomena being investigated; assumptions of an epistemological nature as the "assumptions about the grounds of knowledge," how one can understand the world and communicate this knowledge to others; and assumptions about human nature as the relationship between human beings and their environment. The three sets of assumptions directly impact the methodological nature. Different models of ontology, epistemology and human nature direct social scientists to different methodologies (Burrell & Morgan, 1994, p. 2).

As Merriam (2009, p. 8) and Creswell (2013, p. 19) aptly explain, there is little consistency regarding the philosophical foundations of qualitative research. As is typical of qualitative research, different authors make sense of the philosophical influences each in his or her own way. Various terms and descriptions are used such as: paradigms; philosophical assumptions; traditions and theoretical underpinnings; theoretical traditions and orientations; theoretical paradigms; worldviews and epistemology; and theoretical perspectives (Creswell, 2013, p. 19; Merriam, 2009, p. 8). In essence they are beliefs about ontology (the nature of reality), epistemology (what counts as knowledge and how knowledge claims are justified), axiology (the role of values in research) and methodology (the process of research) (Creswell, 2013, p. 20). These beliefs shape how the qualitative researcher views the world and acts in it (Denzin & Lincoln, 2013, p. 26). Berman and Smyth (2013, p. 6) suggest a model to illustrate the alignment of ontology, epistemology and methodology within a doctoral study.

Table 2.1 Pedagogical Model for the use of Conceptual Frameworks in a Doctoral Study (Berman & Smyth, 2013)

What is your conceptual framework?		
WHAT ontology	HOW methodology	WHY epistemology
Context for the research problem	Research theme(s)	New knowledge
Key concepts	Explicit research questions	New theorising
Relationships	Research design	Extent (limits) of generalising
Establishment of terminology and common language	Methodology (data gathering, organising, analysing, interpreting)	Implications for practice
Boundaries for conclusions and theorising	Validity (conceptual, practical ...)	

They refer to these aspects as the *what, why* and *how* of research and represent the alignment of ontology, epistemology and methodology. The ontology, or *what* is being researched, refers to the reality of the context of the research problem; it must be clear and confined so that a solid foundation for shared meaning can be established (Berman & Smyth, 2013, p. 6). These ontological concerns are linked to the meaning of the research, or the epistemology (the *why* of the research). As new knowledge is generated from comprehensive abstract thinking, epistemological outcomes can be

produced which could have both theoretical and practical implications. The higher order thinking and conceptualisation of reality and knowledge and the specific epistemology guides the nature of the methodology (the *how* of the research) (Berman & Smyth, 2013, p. 6).

2.2.1 Ontology

In classical philosophy, ontology referred to the science of being; however, the meaning has evolved during the contemporary era (Given, 2008, p. 577). Ontology is in essence a form of questioning (Given, 2008, p. 580) and refers to the nature of reality and its characteristics (Creswell, 2013, p. 20; Ormston, Spencer, Barnard, & Snape, 2014, p. 2; Willis, Jost, & Nilakanta, 2007, p. 9). Different ontological perspectives reflect different prescriptions of what can be real and what cannot be real (Willis et al., 2007, p. 9). One's beliefs about the nature of reality or ontology is an indication of one's philosophical positioning (Merriam, 2009, p. 8), which means that different researchers, the individuals being studied as well as the readers of the qualitative study, embrace different realities (Creswell, 2013, p. 20). The aim of qualitative research is to report these multiple realities using varied evidence in themes where actual words of different individuals are used to present different perspectives (Creswell, 2013, p. 20).

2.2.2 Epistemology

Epistemology is an essential part of a philosophical study that "includes the sources and limits, rationality and justification of knowledge" (Given, 2008, p. 264). It refers to the nature of knowledge (Merriam, 2009, p. 8) and how it can be acquired (Ormston et al., 2014, p. 2); theories of knowledge and perceptions in science (Flick, 2014, p. 536); the relationship between the inquirer and the known (Denzin & Lincoln, 2013, p. 26) or otherwise phrased, as the nature of the relationship between the researcher and what can be known (Terre Blanche & Durrheim, 2006, p. 6). Epistemology is concerned with what we know about reality and how we get to know it (Willis et al., 2007, p. 10). When doing a qualitative study, the researcher tries to get as close as possible to the participants being studied in order to accumulate subjective evidence based on individual views (Creswell, 2013, p. 20). The subjective experiences of people create knowledge, which means that studies need to be conducted in the field, where participants live and work (Creswell, 2013, p. 20).

The interpretive paradigm is informed by a concern to understand the world as it is, and strives to explain the world within the participant's frame of reference, rather than that of the observer (Burrell & Morgan, 1994, p. 28). Interpretive research starts out with the assumption that reality can only be accessed through social constructions like language, consciousness and shared meanings. Interpretive research is guided by a set of beliefs and emotions about the world and how it should be understood and studied (Denzin & Lincoln, 2011, p. 12). Interpretive studies attempt to understand phenomena through the meanings that people assign to them, and aim to offer a perspective of a

situation and analyse a situation to provide insight into the way in which a particular group of people make sense of their situation (Nieuwenhuis, 2010b, p. 60).

Burrell and Morgan (1994) have done extensive work relating to different approaches to social theory. They identify four paradigms for the analysis of social theory (Figure 2.2).

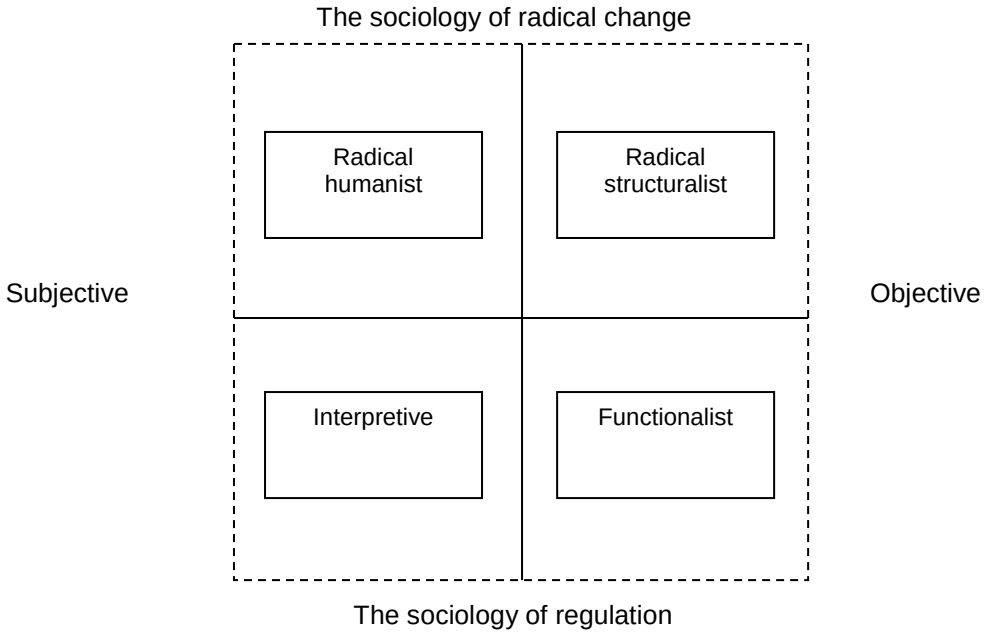


Figure 2.2: Four Paradigms for the Analysis of Social Theory (Burrell & Morgan, 1994, p. 22)

The four paradigms describe overlapping, but fundamentally different perspectives for analysing social phenomena (Burrell & Morgan, 1994, p. 23). These four paradigms represent four sets of basic assumptions in social theory, and they define four views of the social world based on different assumptions with regard to science and society (Burrell & Morgan, 1994, p. 24). When located in each different paradigm, the world is viewed in a particular way. The functionalist paradigm represents a perspective which approaches its subject matter from an objectivist point of view. It is rooted in the sociology of regulation. This paradigm is concerned with explaining the following issues: the status quo, social integration, solidarity, need satisfaction and actuality. These concerns are approached from an outlook that is realist, positivist, determinist and nomothetic. In general this paradigm provides rational explanations of social affairs; it is pragmatic in nature and is concerned with understanding society in a way which produces knowledge (Burrell & Morgan, 1994, pp. 25-26).

Since this study is rooted in the Interpretivist paradigm, this paradigm will be discussed in detail in § 2.2.3. The radical humanist paradigm is concerned with developing a sociology of radical change from a subjectivist point of view. It sees the social world from a nominalist, anti-positivist, voluntarist and ideographic perspective. In this paradigm social theorising is designed to critique the status quo.

In this paradigm, the greatest emphasis is based on radical change, modes of domination, emancipation, deprivation and potentiality (Burrell & Morgan, 1994, p. 32).

The radical structuralist paradigm promotes a sociology of fundamental change. It is dedicated not only to radical change, but also to emancipation and potentiality, and emphasises structural conflict, modes of domination, contradiction and deprivation. This paradigm approaches these issues from a realist, positivist, determinist and nomothetic perspective. The rational structuralist focuses on structural relationships within a realist social world (Burrell & Morgan, 1994, p. 34).

2.2.3 Interpretivist Paradigm

The interpretivist paradigm is driven by a concern to understand the world as it is, to understand the fundamental nature of the social world. It strives to explore the realm of individual consciousness and subjectivity within the participant's frame of reference. The interpretivist paradigm's approach to social science is nominalist, anti-positivist, voluntarist and ideographic. It views the social world as a developing social process, created by the concerned individuals (Burrell & Morgan, 1994, p. 28). Interpretivist theorists strive to understand the basis and source of social reality. Their stance is backed by the assumption that the world of human affairs is cohesive, ordered and integrated. Issues like conflict, domination, contradiction, potentiality and change do not feature in their framework. Their concern is with understanding the social world "as it is" and is involved with issues relating to the nature of social order (Burrell & Morgan, 1994, p. 31).

Interpretivist research is guided by specific beliefs and feelings about the world and how it should be understood and studied (Denzin & Lincoln, 2013). For interpretivists, the meaning of the world for the person or group being studied is critically important to good research in social sciences (Willis et al., 2007, p. 6). Denzin and Lincoln (2013, p. 27) categorise the interpretivist paradigm as positivist and postpositivist, constructivist-interpretivist, critical and feminist-poststructural. A core belief of this paradigm is that the reality we know is socially constructed (Merriam & Tisdell, 2016, p. 9; Willis et al., 2007, p. 95), which implies that researchers only have access to a socially constructed reality (Willis et al., 2007, p. 97). This serves as a justification as to why social constructivism is also known as interpretivism (Creswell, 2013, p. 24), and that there is no single observable reality (Merriam & Tisdell, 2016, p. 9). Understanding the context in which the research takes place is crucial to the interpretation of the data that are gathered (Willis et al., 2007, p. 98).

The purpose of interpretivist research is to understand a particular situation, and not to discover universal laws (Willis et al., 2007, p. 111). There is not a so-called "correct way" of viewing a particular situation, it is rather an understanding of multiple perspectives on the topic (Willis et al., 2007, p. 113). Individuals develop subjective meanings, which are directed towards certain objects, of their experiences. The meanings are many and diverse, which means that the researcher needs to seek the complexity of the views, rather than limit the meaning into a few selected categories (Creswell,

2013, p. 24). This kind of research addresses the process of interaction between individuals. The contexts in which people live and work need to be considered when attempting to understand the historical and cultural setting of the participant. The researchers' own backgrounds, such as their own personal, cultural and historical experiences, shape the way in which they interpret the research (Creswell, 2013, p. 25). Interpretivists do not consider any form of research to be completely objective and tend to prefer data sources that are close to the point of application because context is so important in the interpretation of data (Willis et al., 2007, p. 111). Qualitative research is most often located within an interpretive paradigm, which assumes that reality is socially constructed and that there are numerous realities or interpretations of a single event (Merriam, 2009, p. 8).

2.3 Research Design: Qualitative Phenomenology

Research design refers to the decisions about how research is conceptualised, the conducting of a research project and the type of contribution the research is envisioned to make to the body of knowledge (Given, 2008, p. 761). The process in which a research design is developed entails three related components, namely the theoretical, methodological and ethical considerations, which all shape the design and what the research is aiming to achieve (Given, 2008, p. 761). A research design describes a flexible course of action that links theoretical paradigms to two aspects, namely strategies of inquiry and methods of collecting empirical data (Denzin & Lincoln, 2008, p. 33). It also positions the reader in the empirical world and connects him or her to specific locations, persons, groups, organisations and forms of relevant interpretive material such as documents and archives (Denzin & Lincoln, 2008, p. 34).

2.3.1 Research Approach: Qualitative Design

One of the most generic definitions for qualitative research is offered by Denzin and Lincoln (2008, p. 4), namely that it is "a situated activity that locates the observer in the world." Qualitative research can be used to explore a problem or issue, to empower individuals to share their experiences, and/or to develop theories when partial or inadequate theories need to be further explored (Creswell, 2013, p. 48). It aims to thoroughly describe "life-worlds" from the participants' points of view and strives to create a better understanding of social realities (Flick, von Kardoff, & Steinke, 2010, p. 3). It involves the collection of different kinds of empirical material that describes moments and meanings in individuals' lives (Denzin & Lincoln, 2013, p. 7).

Qualitative researchers are concerned with understanding how people interpret their experiences, how they create their worlds and what meaning they ascribe to their experiences (Merriam & Tisdell, 2016, p. 6). They accentuate the importance of both understanding and writing about the beliefs and theories that inform their research (Creswell, 2013, p. 15); they collect data in a natural setting and are sensitive to the people and places that are being studied. The final report presents "the voices of the

participants, the reflexivity of the researcher, a complex description and interpretation of the problem, and its contribution to the literature" (Creswell, 2013, p. 44). Creswell (2013, p. 44) emphasises the process of research, namely the philosophical assumptions, interpretive lens, and the procedure for studying human or social problems.

Creswell (2013, p. 45) describes an extensive list of characteristics of qualitative research, which include the following aspects: the research is conducted in a natural setting; the researcher acts as a key instrument in data collection; it involves the use of multiple methods; complex reasoning through inductive and deductive logic is used; it focuses on participants' perspectives and meanings; it is situated within the context of the participants; it involves an emergent and evolving design; it is reflective and interpretive and sensitive to the researcher's background; and it presents a holistic, complex picture of the issue being studied. Flick (2014, p. 14) narrows this list down to the following essential features, namely: the correct choice of appropriate methods and theories; recognising the perspectives of the participants and their diversity; the researchers' reflections on the research (reflexivity); and the variety of approaches and methods in qualitative research. Qualitative research should be used when an issue needs to be explored and we need a detailed understanding of the issue, and to develop theories (Creswell, 2013, p. 47).

Qualitative research is concerned with understanding the meaning people have constructed, which means it has to do with how people make sense of their world and the experiences they have in the world (Merriam, 2009, p. 13). Although various characteristics have been explored to understand qualitative research, Merriam (2009, p. 14) highlights four characteristics that stand out as most important in understanding qualitative research. These are: the focus is on the process, understanding and meaning; the researcher is the main instrument in collecting and analysing the data; the process is inductive, and the product is richly descriptive.

Nieuwenhuis (2010b, p. 50) refers to qualitative research as research that attempts to collect rich descriptive data about a particular phenomenon or context in order to develop an understanding of what is being observed. The focus is on how individuals and groups view and understand the world and construct meaning from their experiences. Qualitative research is concerned with understanding the processes and the social and cultural contexts which underlie behavioural patterns. It studies people or systems by interacting with and observing the participants in their natural environment, the focus being on their interpretations and meanings. The emphasis of qualitative research is on the quality and depth of information, rather than on the scope or the breadth of the information. (Nieuwenhuis, 2010b, p. 51).

The most important concern is therefore to understand the phenomena being researched from the perspective of the participant rather than the researcher. In order to effectively understand the phenomena, it makes most sense that the researcher acts as the primary instrument to collect and analyse the data, since he or she could expand his or her understanding through verbal and non-

verbal communication, by processing data immediately, summarizing material, making certain that interpretation is accurate by checking with participants and exploring responses that were unusual or unexpected (Merriam, 2009, p. 15). Qualitative research is concerned with gathering data to build concepts, hypotheses or theories instead of deductively testing hypotheses. The product of qualitative research is abundantly descriptive because words and pictures rather than numbers are used to express what has been discovered about the phenomenon (Merriam, 2009, p. 16).

2.3.2 Qualitative Approach to Inquiry: Phenomenology

Phenomenology emphasises the subjectivity and relativity of reality and highlights the need to understand how humans view themselves and the world around them (Willis et al., 2007, p. 53). It is the study of the way in which people perceive the world, rather than trying to learn what the world is. The focus is on understanding from the perspective of the person(s) being studied (Willis et al., 2007, p. 107). Phenomenology focuses on the lived experience and how being subjected to an experience is converted into consciousness (Merriam & Tisdell, 2016, p. 26). The task of the phenomenologist is to portray the quintessence of the experience, which can often be an intense human experience (Merriam & Tisdell, 2016, p. 26).

A phenomenological study describes the mutual meaning for a number of individuals, of their lived experiences of a concept or a phenomenon, the focus being on the description of what participants have in common as a particular phenomenon is experienced (Creswell, 2013, p. 76). A study such as this produces a "composite description that presents the 'essence' of the phenomenon" (Creswell, 2013, p. 82). It refers to the study of the perceptions of the person being studied (Willis et al., 2007, p. 108). Phenomenology serves to reduce an individual experience with a phenomenon to an account of the universal principle. It is in many ways the study of consciousness (Willis et al., 2007, p. 172) and is ideally used in the social sciences (Creswell, 2013, p. 77). Two types of phenomenology exist, namely hermeneutic phenomenology; and empirical, transcendental, or psychological phenomenology. Hermeneutical phenomenology is research that is directed towards lived experiences where the transcripts of life are interpreted (Creswell, 2013, p. 79).

2.4 Research Methodology: Qualitative Design-based Research

A researcher's philosophical stance and paradigm informs the methodology used, including the research design and techniques used to gather, analyse and interpret the data (Creswell & Clark, 2011, p. 38). Methodology refers to the way in which we approach problems and look for answers (Taylor & Bogdan, 1998, p. 3); how we know the world or gain knowledge of it (Denzin & Lincoln, 2011, p. 12); the process of how we look for new knowledge (Lincoln, Lynham, & Guba, 2013, p. 213). Methodology is interwoven with and develops from the nature of a particular discipline and perspective (Lincoln et al., 2013, p. 200). The methodology of qualitative research is considered to be inductive,

emerging and guided by the researcher's experience in collecting and analysing the data (Creswell, 2013, p. 22).

Qualitative research served as a suitable approach within design-based research methodology in a number of studies (Guler & Altun, 2010, p. 118; Gunn & Lefoe, 2013, p. 45; Nathans & Revelle, 2013, p. 164). Design-based research is a suitable approach for a short-term project such as a PhD provided the relevant phases are followed and the required outcomes are achieved (Pool & Laubscher, 2016, p. 10). It necessitates a significant literature review and theory generation and makes use of different data analysis and collection methods which are used in quantitative or qualitative research, and is considered to be an effective and efficient approach to investigate processes in educational settings (Wang & Hannafin, 2005, p. 6). Design-based research is considered to be a series of approaches set on producing new theories, artefacts and practices that can have an impact on teaching and learning (Barab & Squire, 2004, p. 2).

The design-based research approach is rooted in two approaches in the teaching of Mathematics, namely the socio-constructivist approach and the RME approach (Gravemeijer & Cobb, 2013, p. 74). The theory of RME offers design heuristics and also functions as an interpretivist framework to interpret student activity in the learning of Mathematics (Gravemeijer & Cobb, 2013, p. 89). Plomp (2007, p. 13) describes educational design-based research as "the systematic study of designing, developing and evaluating educational interventions (such as programmes, teaching-learning strategies and materials, products and systems) as solutions for complex problems in educational practice, which also aims at advancing our knowledge about the characteristics of these interventions and the processes of designing and developing them." Design-based research in education is described as research in which the design of materials such as computer tools or learning tasks forms an essential part of the research. The design of learning environments is interwoven with the developing of theory (Bakker & van Eerde, 2012, p. 2). Design-based research is defined by the goals of those who pursue it, rather than by its methods (McKenney & Reeves, 2012, p. 28). Plomp (2007, p. 11) distinguishes between different research functions which characterise design-based research, namely to describe, to compare, to evaluate, to explain or to predict, to design and develop. The focus of design-based research is on understanding (Gravemeijer & Cobb, 2013, p. 74).

Wang and Hannafin (2005, p. 6) define design-based research as a systematic and flexible methodology aimed at improving educational practices through iterative analysis, design, development and implementation. This collaboration takes place between researchers and practitioners and real world settings. Design-based research is a series of approaches where the aim is to produce new theories, artefacts and practices that can impact teaching and learning (Barab & Squire, 2004, p. 2). It is further described as being a means to capture social interaction, and is considered to involve flexible design revision (Barab & Squire, 2004, p. 3). Amiel and Reeves (2008, p. 35) claim that, due to the intentions and lifecycle proposed by a design-based framework, it is in a position to address the complexities that are characterised by educational technology research. Researchers not only design

research processes, but also design and implement interventions systematically to improve designs and advance pragmatic and theoretic aims (Wang & Hannafin, 2005, p. 6).

The research process in design-based research is cyclical in nature. Activities involving analysis, design, evaluation and revision are iterated (Plomp, 2007, p. 13). Authors represent this process of design-based research in different ways, one such representation by Amiel and Reeves (2008, p. 34) (Figure 2.3), is as follows:

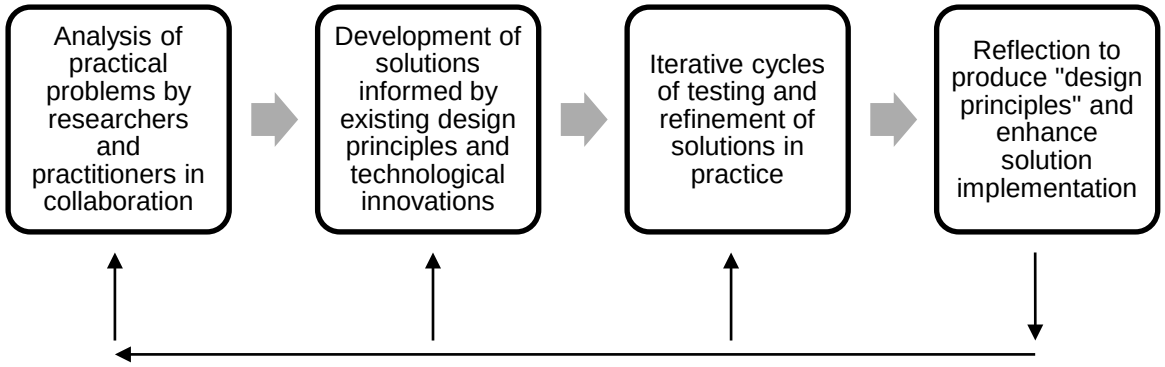


Figure 2.3 The Process of Design-based Research (Amiel & Reeves, 2008, p. 34)

Another illustration is presented by McKenney (2001, p. 12) and is presented in Figure 2.4.

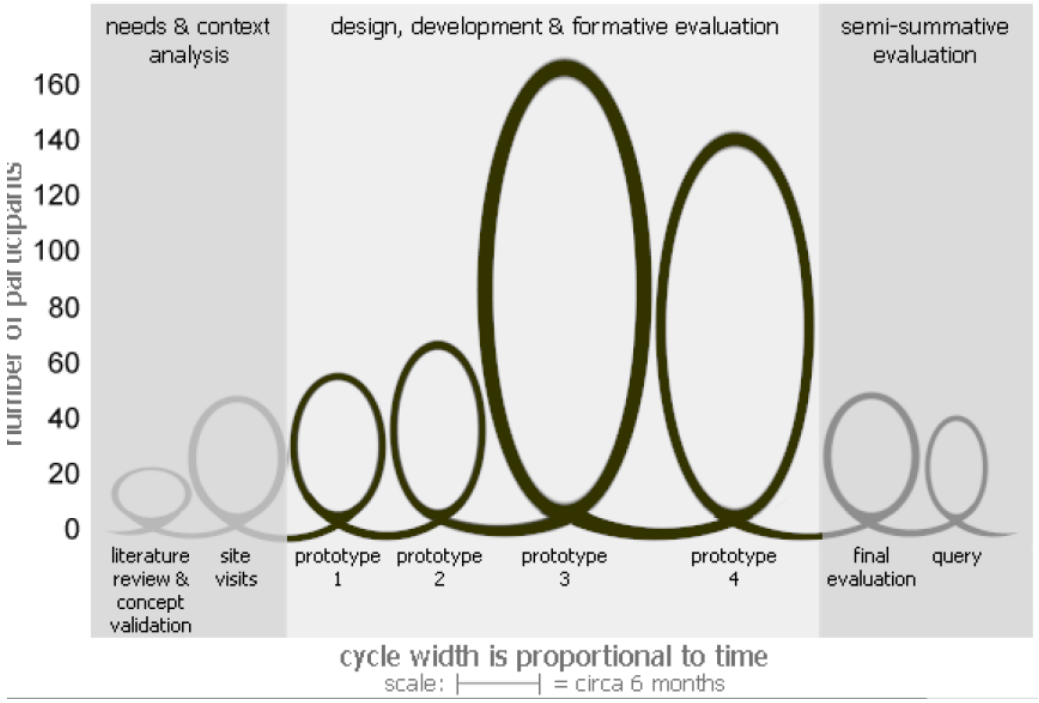


Figure 2.4 Illustration of the CASCADE-SEA Study by McKenney (2001, p. 12)

As illustrated in Figure 2.4, McKenney (2001, p. 12) indicates that design-based research features three main phases, namely needs or contextual analysis, design or formative evaluation of the prototype tools and a summative assessment of the final product. These phases are represented on

the horizontal axis. Each phase consists of a number of cycles of activities which involve different groups of participants, which could include experts or users. The vertical axis represents the number of participants in relation to the phases of the research. Although authors represent the design-research process in different ways, there are essentially three main stages in the process: preliminary research, prototyping and assessment. Throughout these stages researchers will perform systematic reflection and documentation to produce the required scientific harvest in the form of theories or design principles (Plomp, 2007, p. 15). An adapted version of the illustration presented by McKenney (2001, p. 12), specific to this study, will be discussed later in this section.

McKenney and Reeves (2012, p. 76) build on models such as these, and represent a visual model (Figure 2.5) that depicts the overall process from the point of view of the researcher.

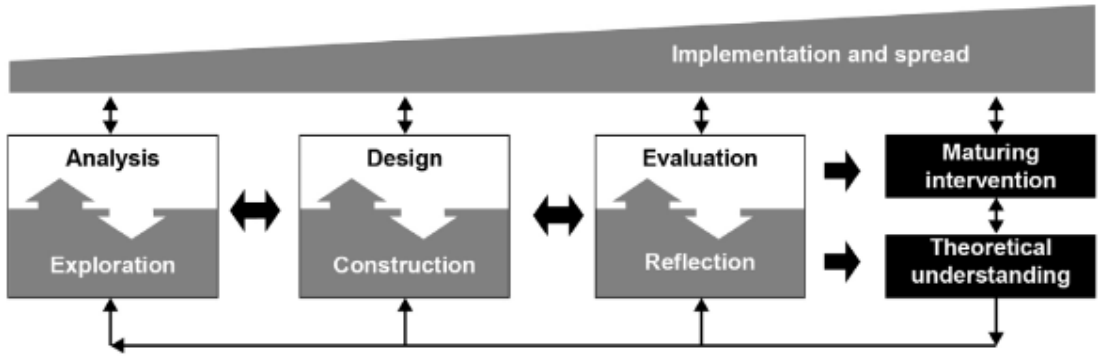


Figure 2.5 Generic Model for Conducting Design Research in Education (McKenney & Reeves, 2012, p. 77)

The three core phases of their model are: analysis and exploration; design and construction and evaluation and reflection, followed closely by a fourth output phase which includes maturing intervention and theoretical understanding. The iterative nature of the process is emphasised by the arrows between the different components. As is emphasised by various specialists on design-based research, McKenney and Reeves (2012, p. 77) also accentuate both the iterative and flexible nature of design-based research. The process is iterative because results from certain elements lead into others on a continual basis, and flexible because many different routes can be followed. The output that is produced in such a process is both theoretical and practical. The implementation and spread referred to in the figure indicate the increasing interaction with practice (McKenney & Reeves, 2012, p. 77).

The process of design-based research starts with an analysis of the problem that exists in educational practice. In this step expertise is pursued in the form of a literature review to establish theoretical contributions that help to understand the problem and context. The literature also helps to develop a scientifically related approach for the study (McKenney & Reeves, 2012, p. 78). Once a solid understanding of the problem and context has been established, an unrestricted exploration can take place in which similar problems and their solutions are explored (McKenney & Reeves, 2012, p. 79).

In this study, the poor Mathematics results in South Africa in particular, and the influence of a teaching approach on learners' achievement in Mathematics, created the need to investigate the effect of using technology to implement the principles of the RME approach in teaching practice. In this study, the systematic literature review (§ 2.6) which focused on the literature relating to RME, Mathematics education and ICT, explored current work on existing design principles and technological innovations. This first step was further explored by doing a needs analysis of the teacher-students (§ 2.8) to establish which areas of the curriculum they found problematic to teach and also in which sections they had difficulty applying the principles of RME.

The second step in the process of design-based research is to develop solutions informed by existing design principles and innovations in technology (Amiel & Reeves, 2008, p. 34). In this stage, an articulate process is followed and recorded in which a tentative solution to the problem is sought (McKenney & Reeves, 2012, p. 79). Here a conceptual model is created—during design, solutions to the problem are "generated, explored and considered" and eventually charted (McKenney & Reeves, 2012, p. 79). It is important here as well that the guidelines for the actual building of solutions are defined. Construction is where the ideas of the design stage are applied to manufacture a potential solution (McKenney & Reeves, 2012, p. 79). In this study, the design phase was realised by the development of a mobile application tool that aimed to address the needs expressed by the teacher-students. A detailed description of the design process is presented in Chapter Five.

The third step in the process of design-based research is that of evaluation and reflection where iterative cycles of refinement of solutions in practice are performed (Amiel & Reeves, 2008, p. 34). Evaluation refers to the empirical testing of the design or constructed intervention. Reflection comprises "active and thoughtful consideration of what has come together in both research and development" in order to produce theoretical understanding (McKenney & Reeves, 2012, p. 80). As one reassesses and reflects upon the data, new designs are created and employed to produce an on-going cycle of design-reflection-design. In this study the evaluation and reflection took place throughout the design process. Various prototypes of the app were created and tested by novices, experts in the field of Mathematics education, an expert in the field of RME as well as the design team, which consisted of an expert programmer who is experienced in the design of mobile applications; my promoter who is an expert in the field of using technology in teaching and learning; my co-promoter, an expert in Mathematics and Mathematics education; and myself, who took on the role of content specialist.

In the output phase, design principles are produced and the implementation of solutions are enhanced (Amiel & Reeves, 2008, p. 34). The ultimate outcome of design-based research is to produce a set of design principles which are empirically derived and richly described (Amiel & Reeves, 2008, p. 35). In Figure 2.5, McKenney and Reeves (2012, p. 80) depict two main outputs from design-based research, namely maturing interventions, which contribute to practice; and theoretical understanding, which is

produced through (usually several) micro- and meso-cycles of design-based research. In this study the design-based research produced guidelines for the effective use of technology in implementing the RME approach. The guidelines are presented in detail in Chapter Seven. These guidelines contribute to practice by presenting ideas and suggestions for using technology to reinforce the principles of RME in teaching practice.

For the purposes of this study, it was necessary to adapt the illustration by McKenney (2001, p. 12) to suit the design-based process that was followed. Figure 2.6 presents this adapted illustration.

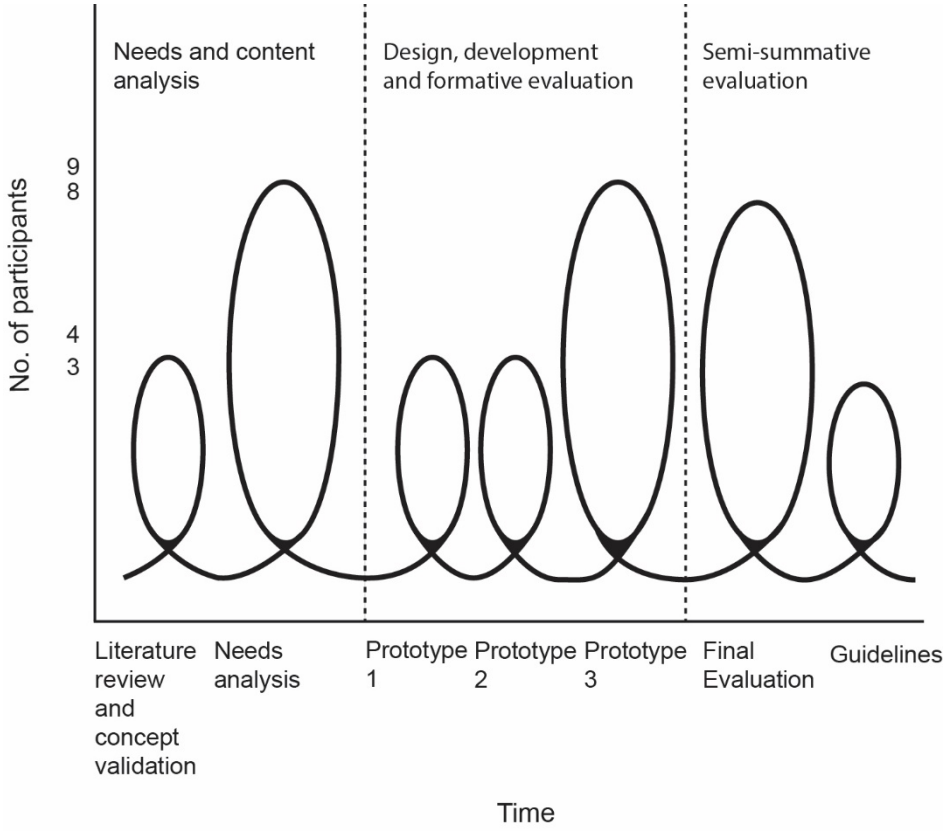


Figure 2.6: The Design-based Research Process Specific to this Study, adapted from McKenney (2001, p. 12)

In this study, the three phases as suggested by McKenney (2001, p. 12) were also relevant: needs and content analysis; design, development and formative evaluation; and semi-summative evaluation. Time is represented on the horizontal axis and the phases are also represented on the horizontal axis. The phases entailed the following activities: a systematic literature review in which concepts were validated; a needs analysis of the teacher-students' needs relating to their teaching practice; the development of three different prototypes of a mobile app; a final evaluation of the mobile app; and finally guidelines for the effective use of technology in implementing the RME approach. On the vertical axis, the number of participants, which include users and experts, are represented. The cycles had varying numbers of participants. This is illustrated in Table 2.2.

Table 2.2 The Participants that Played a Role in each Cycle of the Study

Cycle	Participants	Number of Participants
Literature review and concept validation	Myself, Promoter, Assistant promoter, Expert in Mathematics education and Systematic Literature Reviews	4
Needs analysis	Myself, Promoter, Assistant Promoter, four Participants, Two Experts in Mathematics education	9
Prototype 1	Myself, Promoter, Assistant Promoter, Expert Programmer	4
Prototype 2	Myself, Promoter, Assistant Promoter, Expert Programmer	4
Prototype 3	Myself, Promoter, Assistant Promoter, Expert Programmer, Two Experts in Mathematics education, three Novices	9
Final evaluation	Myself, Promoter, Assistant Promoter, Expert Programmer, four Participants	8
Guidelines	Myself, Promoter, Assistant Promoter	3

The emphasis on design-based research should be on the premise that the researcher should generate evidence-based claims about learning that addresses theoretical issues and generating theoretical knowledge in the field, rather than on showing that a particular design works (Barab & Squire, 2004, p. 6). This study aimed to determine what support needs in-service teacher-students had with regard to teaching using the RME approach, and it intended to yield guidelines for the effective use of technology in implementing the RME approach, thereby generating theoretically relevant knowledge to the body of scholarship.

One of the characteristics of design-based research is that it is interventionist, which means that the goal thereof is to design interventions in a real world setting, thus building a stronger connection between educational research and real world problems (Amiel & Reeves, 2008, p. 34; Joseph, 2004, p. 235; Plomp, 2007, p. 15). Other characteristics include: it is iterative—cycles of analysis, design and development, evaluation and revision take place; it is process oriented—the emphasis is on understanding and improving interventions. Additionally it is utility oriented—the value of a design is partly measured by how practical it is for users in real contexts; it is theory oriented—the design is partially based on a conceptual framework and theoretical proposals, and the evaluation of the prototypes in the intervention contribute to building theory. Added to that the involvement of practitioners is important—the active collaboration with and participation of practitioners increase the chances that the intervention is relevant and practical (Barab & Squire, 2004, p. 2; Plomp, 2013, p. 20).

This study was interventionist in that the interventions were not only designed in the real world setting of the Mathematics classroom, but the content that was designed was focused specifically on the real life application of Mathematics. The process was iterative in that a number of prototypes of the app were created and tested by novices and experts. The design of the intervention was grounded in the theory of RME. The five characteristics of RME were adhered to and the three principles of RME formed the foundation of the design. Each prototype was grounded in guided reinvention, focused on the use of models and adhered to the principle of didactical phenomenology. The experts who tested the app as well as the participants in the research played the role of practitioners. The experts were

all actively working in the field of Mathematics education, and IT education at a higher education institution (HEI), and dealt with Mathematics and technology in education on a daily basis. The participants were actively involved in Mathematics education at secondary school level. Following their needs analysis and opinions of the use of the RME approach in teaching Mathematics aided by technology, a suitable and appropriate intervention, in the form of the app, was designed to assist them in their teaching practice. The relevance of this intervention is discussed in detail in Chapter Six.

Design-based research is said to have a dual purpose (Plomp, 2013, p. 22), namely to develop prototypical products (curriculum documents and materials) along with experimental evidence of their quality; and to generate methodological directions for the design and evaluation of such products (Fauzan et al., 2013, p. 163). In developmental studies, not only should the research based intervention be functional and effective in addressing the solutions to the complex problem, but it should also contribute to the body of knowledge in the field as well, typically in the form of re-usable guidelines for addressing educational problems (Barab & Squire, 2004, p. 5; Plomp, 2013, p. 22). This links closely to the characteristic of theory orientation as discussed above.

Plomp (2013, p. 22) distinguishes between three types of design-based studies, namely developmental studies, validation studies and implementation studies. In developmental studies, the foundation is the identification of educational problems in which there are few validated principles available to structure and support the design and development activities (Plomp, 2013, p. 22). A characteristic of a developmental study is that the study is informed by prior research through a review of the literature. Researchers, together with practitioners, design and develop feasible and effective interventions by studying consecutive versions of the interventions in the relevant contexts. The final stage of a developmental study in design-based research is when a period of reflection follows where the researcher and practitioner reflect on the research process in order to produce design principles for developing innovative interventions which are significant for educational practice (Plomp, 2013, p. 22).

In validation studies the focus is on designing learning trajectories or environments in order to develop and validate theories about learning. The research goal is to develop or validate theory and it also has a dual purpose, namely to design learning environments with the purpose to validate design principles and to develop and validate theories about learning and learning environments (Plomp, 2013, p. 23). In such studies, researchers work in natural settings such as the classroom, rather than controlled settings (Plomp, 2013, p. 23).

Gravemeijer and Cobb (2013) distinguish between three phases in validation studies: preparing for the experiment; designing the experiment; and retrospective analysis. These phases are similar to those in developmental studies (Plomp, 2013, p. 26). In practice, it is possible that the design-based research may combine a developmental and validation study. Plomp (2013, p. 26) uses the work of Fauzan et al. (2013) as such an example. This research involved the development of a geometry

course based on the RME principles (developmental study) and also investigated the validity of design principles in a different context (validation study).

The findings in design-based research cannot be generalised to a larger universe. The researcher should try to generalise the design principles to form part of a broader theory. In order to do this, the design principles must be tested through a number of repetitions in various contexts (Plomp, 2013, p. 34).

2.5 Ethical Considerations

Given (2008, p. 273) describes ethics as "the part of human philosophy concerned with appropriate conduct and virtuous living." Because qualitative research revolves around human endeavours, it is viewed by scholars as moral ethical endeavours. The ethical principles as suggested by Christians (2008, p. 192), namely informed consent; free of deception; privacy and confidentiality; and accuracy, were applied throughout the project. The necessary ethical application was completed and submitted to the North-West University's ethical committee, and permission was obtained to commence the research. The ethical clearance number is: NWU-HS-2014-0267 (Addendum 2.1). The participants were provided with a letter describing their potential role in the research, which spelled out exactly what would be expected of them in the project. They were requested to complete a letter of consent, providing me with consent to continue with the project. This letter (Addendum 2.2) ensured the participants that participation was voluntary, that they were free to withdraw from the project at any time, that withdrawal would not jeopardise their studies in any way, and that all information would be treated as confidential. Names were not required at any point during the data generation which ensured that the participants remained anonymous.

2.6 Qualitative Systematic Literature Review

Kitchenham (2004, p. 1) defines a systematic literature review as "a means of identifying, evaluating and interpreting all available research relevant to a particular research question, or topic area, or phenomenon of interest." Systematic literature reviews also serve as a "method of making sense of large bodies of information, and a means of contributing to the answers to questions about what works and what does not" (Petticrew & Roberts, 2006, p. 2). They are also a method of mapping out areas of uncertainty, and can indicate where new studies are necessary (Petticrew & Roberts, 2006, p. 2). Systematic literature reviews are used in a variety of fields including medicine, health care, social policy and education (Briner & Denyer, 2012, p. 1).

The most common reasons for performing a systematic literature review is: to summarise the existing evidence concerning a treatment or technology; to identify any gaps in the existing research; to

provide a framework in which to position new research; to examine the extent to which hypotheses are supported or contradicted by empirical evidence; or to assist in the generation of new hypotheses (Kitchenham, 2004, p. 1).

One of the advantages of doing a systematic literature review is that it provides information about the effects of a phenomenon across a wide range of settings (Kitchenham, 2004, p. 2). Systematic literature reviews are increasingly replacing traditional narrative reviews because they attempt to bring the same level of rigour to reviewing evidence as should be present in producing the evidence (Hemingway & Brereton, 2009, p. 1). Systematic literature reviews are suitable ways of examining quantitative or qualitative evidence (Hemingway & Brereton, 2009, p. 1).

Systematic literature reviews should adhere to a number of core principles. They should be: systematic and organised, transparent, replicable and updatable. They should also be able to synthesise and summarise the evidence relating to the research question (Briner & Denyer, 2012, p. 3). Important features that differentiate between a systematic literature review and a traditional literature review are: systematic reviews start by defining a review protocol that specifies the research question being addressed as well as the methods that will be used to do the review; systematic reviews are based on a defined search strategy with the purpose of detecting as much relevant literature as possible; systematic reviews document the search strategy to verify its rigour and completeness; systematic reviews require explicit inclusion and exclusion criteria when assessing potential material; systematic reviews specify the information to be obtained from each primary study as well as quality criteria by which the material will be evaluated (Kitchenham, 2004, p. 3).

2.6.1 Process and Documentation for a Systematic Literature Review

Although quite similar, different authors have varied opinions about the key steps to follow when doing a systematic literature review. Hemingway and Brereton (2009, p. 1) classify the following steps: identifying the relevant evidence; selecting studies or reports for inclusion; assessing the quality of these reports or studies; synthesising the findings in an unbiased way; interpreting the findings, and presenting a balanced, impartial summary of the findings. Kitchenham (2004, p. 3) provides a more expanded version and identifies three main phases, each of which encompasses a number of stages. The phases are: planning, conducting and reporting the review. The planning phase includes the stages of identifying the need for a review and developing a review protocol. The phase of conducting the review includes the stages: identifying the research, selecting the primary studies, quality assessment of studies, the extraction and monitoring of data, and data synthesis. The final phase of reporting the review consists of a single stage (Kitchenham, 2004, p. 3). Briner and Denyer (2012, p. 3) identify the following five key steps: (i) planning the review, (ii) locating studies, (iii) appraising contributions, (iv) analysing and synthesising information, and (v) reporting evidence. Khan, Kunz, Kleijnen, and Antes (2003, p. 118) also identify five steps. These are (i) framing the question, (ii)

identifying relevant publications, (iii) assessing study quality, (iv) summarising the evidence, and (v) interpreting the findings.

For the purposes of this study, I have made use of the following steps to perform the systematic literature review:

i) Identify the need for a review (Kitchenham, 2004)
ii) Identify the relevant evidence (Hemingway & Brereton, 2009)
iii) Select studies and reports for inclusion (Briner & Denyer, 2012; Hemingway & Brereton, 2009; Khan et al., 2003)
iv) Assess quality of contributions (Briner & Denyer, 2012; Hemingway & Brereton, 2009; Khan et al., 2003; Kitchenham, 2004)
v) Analyse and synthesise information (Briner & Denyer, 2012; Hemingway & Brereton, 2009)
vi) Interpret findings (Hemingway & Brereton, 2009; Khan et al., 2003)
vii) Present a balanced, impartial report (Briner & Denyer, 2012; Hemingway & Brereton, 2009; Kitchenham, 2004).

2.6.2 Search Process Documentation

The systematic literature review was documented in detail (Briner & Denyer, 2012, p. 4; Kitchenham, 2004, p. 3). All searches were recorded in detail, including the date, keywords used, the database in which the search took place, and the number of hits. With reference to the inclusion and exclusion criteria (Kitchenham, 2004, p. 3), documents were identified. In total, thirteen searches were performed independently and in collaboration with expert librarians.

The systematic literature search was done using the following keywords: realistic* or RME, teach* or learn* or educat*, math*, tech* or ict or educat* tech*. Various combinations of these keywords were used to perform the searches. The databases that were used were: Scopus, EBSCOHost, Web of Science, Science Direct, MathSciNet, and Google Scholar.

The initial searches on the various databases were done using different combinations of the keywords. A record of this process was tabulated (Addendum 2.3). Thereafter, author maps for six authors who were dominant in the search results were drawn on each of the databases. An additional 36 searches were done. The six dominant authors were LM (Michiel) Doorman, Paul Drijvers, Koeno PE Gravemeijer, AJP (André) Heck, PBJ (Peter) Boon and Nicholas Zaranis. Table 2.3 summarises each of these authors' field of expertise.

Table 2.3 The Dominant Authors and their Fields of Expertise

The Dominant Authors	Field of Expertise
Prof Dr Michiel Doorman	Interests are context-based Mathematics education; Modelling as a lever for learning Mathematics; Inquiry based learning and Coherency between

The Dominant Authors	Field of Expertise
	Mathematics and science learning.
Prof Dr Paul Drijvers	ICT in Mathematics education; Algebra education; Teachers' professional development; Curriculum development; Mathematical thinking
Prof Dr Koeno Gravemeijer	Mathematics; Computer Science; Public and Law Administration; Educational theory
Dr Andre Heck	Research on ICT aspects of Mathematics and Science education; Development of ICT tools; Development of curriculum material
Prof Dr Peter Boon	Educational technology; Teaching methods; Mathematics education; Online learning; Teaching and learning; Didactics
Prof Dr Nicholas Zaranis	Computers in Education; Educational software; Internet in Education; Teaching of Mathematics by using new technologies

2.6.3 Selection Process Criteria Documentation

After the searches had been performed, documents were screened for their aptness for this study. In total 318 documents were perused and evaluated for their suitability. The process was recorded in detail in nine different Excel sheets (Addendum 2.4). The research revolved around four main concepts: i) Realistic Mathematics Education, ii) ICT, iii) Mathematics/teacher/education, and iv) Mathematics education, and documents were screened for content relating to these themes. Initially the title and abstract were perused for suitability. Thereafter the full text was read to determine its appropriateness for this study. Only documents that related to all four themes were selected. As discussed above, the author search was done and ten additional sources were included for analysis. Six articles were hand-picked and also included.

Once the coding and analysing of the documents took place, and I began to document the literature, I realized that the key authors in the field were regularly referred to by the identified sources. It was therefore necessary to include six core articles that relate specifically to the study. These sources are authored by the key role players of the RME movement in the Netherlands, namely Hans Freudenthal, Adrian Treffers and Martin Simon. Table 2.4 indicates each of these authors' field of expertise.

Table 2.4 The Key Role players of the Realistic Mathematics Education Movement

Key Role players of RME movement	Field of expertise
Prof Dr Hans Freudenthal	Known as the father of RME
Prof Dr Adrian Treffers	Mathematics curriculum researcher at Freudenthal Institute
Prof Dr Martin Simon	Mathematics taught with a constructivist mind set; how teachers and learners understand their mathematical knowledge

The additional sources were coded and included in the systematic review for consideration.

2.6.4 Quality Assessment of Primary Documents

In order to select suitable documents for inclusion in the review, I studied all documents that were identified in the various searches and applied the quality assessment. The assessment of the

documents was done in collaboration with peers in the field of Mathematics education who acted as additional independent reviewers. The criteria for the selection of documents related to documents that:

- addressed at least four of the key concepts encompassed in this research
- related to qualitative, quantitative or mixed-method research
- published in books, accredited journals or conference proceedings
- were published between 2000 and 2015.

Figure 2.7 presents the process that was followed to assess and select the final documents for inclusion.

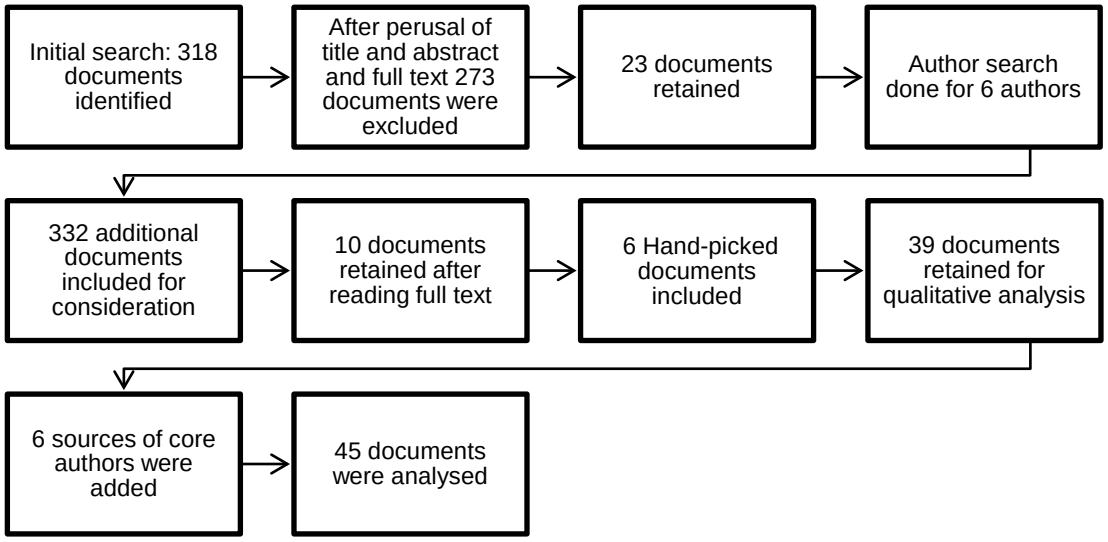


Figure 2.7: The Process of Primary Document Selection

2.6.5 Data Analysis for the Systematic Literature Review

When doing the systematic literature review, I used ATLAS.ti™ to code, sort and analyse the data. ATLAS.ti™ is part of the genre of Computer Assisted Qualitative Data Analysis Software (CAQDAS) (Friese, 2014b, p. 1). Software such as ATLAS.ti™ offers many advantages such as: it allows for easier systematic data analysis; large volumes of data and different media types can be structured and integrated quickly; it increases the validity of research results; codes and concepts can be modified as the analysis progresses (Friese, 2014b, p. 1). The three basic aspects of computer-assisted analysis are preparing data and creating a project file, coding the data, and using the software to sort and structure them with the aim of discovering patterns and relations (Friese, 2014b, p. 15). ATLAS.ti™ has very domain specific terminology that describes different aspects and processes of the software; this terminology is presented in Table 2.5.

Table 2.5 Domain Specific Terminology in ATLAS.ti™

Terminology	Meaning
Hermeneutic unit (HU)	A data file that stores everything that gets done to the data (Friese, 2014b, p. 37). The container that keeps track of all your data (Friese, 2015, p. 7). It helps to structure the data for each project (Friese, 2014a, p. 14).
Primary document	As documents are added to the project, ATLAS.ti™ creates a primary document (Friese, 2014b, p. 37). The primary document represents the data that you are analysing in your project (Friese, 2014a, p. 14).
Codes	Often a researcher-generated word or phrase that assigns a particular attribute to a portion of data (Saldaña, 2013, p. 262). A word or string of words used to name a category in data analysis (Boeije, 2010, p. 95). Keywords that are normally linked to quotations (Friese, 2014b, p. 36)
Categories	A group or cluster used to sort parts of the data during analysis (Boeije, 2010, p. 95) The aim of having a system to code data is to organise the data into categories and subcategories. Main category codes are developed by the process of conceptualisation, comparing and contrasting data segments (Friese, 2014b, p. 158).
Coding	Coding is the basic activity you engage in when working with ATLAS.ti™. It is the basis of everything else that you do further in ATLAS.ti™. It refers to the procedure of allocating categories, concepts or codes to sections of information that is of interest to your research goals (Friese, 2015, p. 7).
Quotation	Quotations are marked data segments that have a defined start and end point (Friese, 2014b, p. 38). A quotation refers to coded segments (Friese, 2015, p. 18).
Theme	That which the data are mainly concerned with (Boeije, 2010, p. 95).
Memos	Memos are places to write down all sorts of thoughts and ideas (Friese, 2014b, p. 37). Memos are independent objects that can stand alone or be connected to other objects (Friese, 2015, p. 36).
Code families	Code families are used to group codes together that belong together (Friese, 2014a, p. 234).
Network views	Network views allow one to visualise different aspects within the data. A few examples of what can be visualised are: your codings; relationships between codes; relationships between quotations; relationships between memos and quotes and many more (Friese, 2014b, p. 37).

There are a number of basic steps that need to be followed when working with ATLAS.ti™. As is illustrated in Figure 2.8, the process for this study began by creating a hermeneutic unit (HU), which contains the primary documents, quotes, code words, notes, memos, links, code families and network views (Friese, 2014b, p. 24).

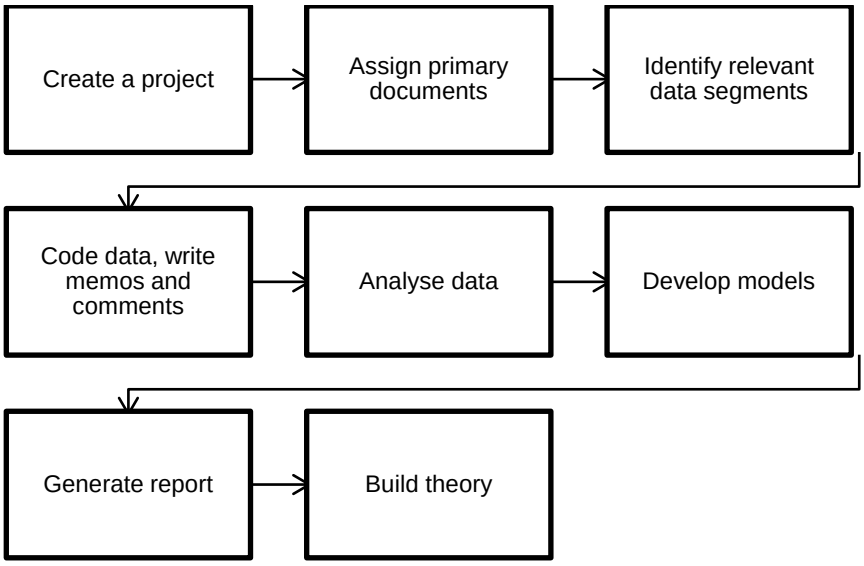


Figure 2.8 The ATLAS.ti™ Workflow Process adapted from Friese (2014a, p. 27)

It is not important that all the steps follow sequentially, logic dictates which can be shifted out of order (Friese, 2014a, p. 27). I did not follow all these steps sequentially. The six additional documents that were not assigned to the project initially were added to the project at a later stage as the need arose. I created the HU with the title "Systematic literature review relating to the use of ICT to facilitate the principles of Realistic Mathematics Education to enhance teaching practice." The 45 primary documents that were assigned to the HU included 29 journal articles, seven conference proceedings, six conference papers, one book section and two books. Relevant segments of the data were identified and coded. I did not use a pre-determined list of codes, and created codes as the different themes emerged in the data. I coded the data and created memos where necessary. Quotations, codes, categories and network views were created. I made use of inductive data analysis, since I did not have a pre-determined set of coding categories. This requires of the researcher to be immersed in the text until themes and concepts emerge from the data (Hesse-Biber & Leavy, 2006, p. 311).

As described in Table 2.5, a code is a word or phrase that assigns an attribute to a portion of data (Saldaña, 2009, p. 3). Codes are the building blocks for the construction of theory and models and form the basis of the analyst's argument; they exemplify the assumptions on which the analysis is built (MacQueen, McLellan, Kay, & Milstein, 1998, p. 31). Codes can be developed from existing theory or concepts; from raw data and from a project's research goals and questions (DeCuir-Gunby, Marshall, & McCulloch, 2011, p. 138). I created 58 codes in the process of analysing the documents.

A codebook is the compilation of the codes, the description of the content and a short data example for orientation (DeCuir-Gunby et al., 2011, p. 138; Saldaña, 2009, p. 21). Codebooks are especially critical when multiple team members work together on a project (Saldaña, 2009, p. 21). MacQueen et al. (1998, p. 32), who work with team-based qualitative analysis, describe the structure of the codebook as consisting of the following six components: "the code, a brief definition, a full definition, guidelines for when to use the code, guidelines for when not to use the code, and examples." DeCuir-Gunby et al. (2011, p. 138) on the other hand use only three components, namely "code name/label, full definition (an extensive definition that collapses inclusion and exclusion criteria), and an example."

Coding enables researchers to engage in reducing and simplifying the data and converting these into meaningful units. This allows researchers to make connections between ideas and concepts (DeCuir-Gunby et al., 2011, p. 138). DeCuir-Gunby et al. (2011, p. 141) describe the steps for developing theory-driven data as follows: generating the code; reviewing and revising the code and establishing the reliability of the code and coders. I followed this process when doing the systematic literature review. I generated codes (Table 2.6); reviewed and revised the codes within the context of the data; and established the reliability of the codes and coder by inviting a peer who is an expert in the field of Mathematics education and Systematic Literature Reviews to peer-code two articles of the systematic literature review. This will be discussed further on in this section.

Table 2.6 Codebook Table

Code	Description	Example
HLT	A hypothetical learning trajectory refers to what the teacher predicts about the path by which learning might progress(Simon, 1995, p. 135).	"The development of an HLT and the design of instructional activities are closely related: the HLT guides the design of the instructional activities, but choices made in the design process may lead to reconsideration of the HLT" (Bakker, Doorman, & Drijvers, 2003).
Design of activities	Activities or tasks that are designed based on the principles and characteristics or RME.	"Task design is widely recognized as an important, but complex and subtle activity" (Drijvers, Boon, Doorman, Bokhove, & Tacoma, 2013, p. 1).
Didactical phenomenology	One of the three key principles or heuristics of the design of teaching. It confronts students with phenomena that need to be organised by mathematical structures (Bakker et al., 2003, p. 5).	*"Didactical phenomenology* concerns the relation between the thought object−the 'nooumenon'−and the phenomenon–the 'phainomenon'−from the perspective of teaching and learning. In particular, it addresses the question how mathematical 'thought objects' can help in organizing and structuring phenomena in reality" (Drijvers et al., 2013, p. 1).
Emergent modelling	One of the principles of RME. Problem situation are used that lead to models that initially represent the problem situation, but that have the potential to develop into general models (Bakker et al., 2003, p. 5).	"According to the *emergent modeling* perspective, a model may play different roles during different phases of activity. Initially, a model is context-specific: it refers to a meaningful problem situation that is experientially real for the student, and is a model of that situation" (Drijvers et al., 2013, p. 1).
Guided reinvention	A principle of RME where students are given the opportunity to experience a process similar to how a topic was invented (Drijvers et al., 2013, p. 1).	"The principle of guided reinvention seems to apply well to digital design. ICT offers opportunities for exploration and investigation, and in this way for reinvention" (Drijvers et al., 2013, p. 7).
Advantages of using RME	Any positive aspect related to the implementation or use of RME characteristics and principles.	"The teacher also mentioned the fact that in the school's final assessment, the pupils involved in this research outscored those in the other class who he had taught the same topic in the traditional way and who he considered to be stronger in mathematics" (Widjaja & Heck, 2003, p. 20).
Characteristics of RME	The attributes or tenets of RME.	"The five characteristics of RME are: the use of contextual problems; the use of models; the use of pupils' contributions; interactivity; and intertwining of learning strands" (Widjaja & Heck, 2003, pp. 4-5).
Content easy to relate to real life	Teacher-students expressed their views on which content in the curriculum is easiest to relate to real life situations.	"Uhm, in Financial Mathematics as well. Through and through it every day life situation where you talk of borrowing, investments and other areas of activities" (P61:12).
Content where need help to relate to real life	Teacher-students discussed which areas of the curriculum they felt they needed help in relating the content to real life situations.	"I think exponents is problematic and algebra, functions" (P59:60).
Disadvantages of using RME	Any negative aspect related to the implementation or use of RME characteristics and principles.	"It should be noted that starting with real situations and real data is not at all easy" (Widjaja & Heck, 2003, p. 22).
Human activity	Freudenthal viewed Mathematics as a human activity, where reality is used as a foundation of mathematization.	"According to Freudenthal (1991), mathematics as human activity and mathematics must be connected to reality. Therefore, students should be able to understand the concepts of mathematics and its application in the real world" (Cahyono, 2012, p. 71).
Mathematization	Mathematization is the process of developing Mathematics concepts that originated in the real world (Cahyono, 2012, p. 72).	"It is based on the view of Freudenthal (1991) that mathematics is a human activity and that reality can be used as a source for mathematization." (Widjaja & Heck, 2003, p. 4).

Code	Description	Example
Principles of RME	The theory of RME produced three principles according to which activities are designed, namely: guided reinvention, didactical phenomenology and emergent modelling. (Drijvers et al., 2013, p. 1).	"this brief exploration of the issue suggests that the three RME principles are valuable for digital design, even if some appropriation is needed compared to the design of paper-and-pencil tasks, such as taking into account the constraints of the digital tools and the fact that the technological environment forms an additional 'world' to the student" (Drijvers et al., 2013, p. 7).
Real life	Situations that are linked to reality; experientially real, everyday life phenomena.	"In RME pupils learn mathematics based on activities they experience in their daily life; pupils have a big opportunity to construct their knowledge by themselves, etc." (Fauzan et al., 2013, p. 172).
Recommendations for RME based lessons	Any suggestions and recommendations presented in the literature for lessons based on the RME approach.	"The pupils could use the student book without any difficulties and they could learn the topic Area and Perimeter as intended according to the RME point of view. The teacher guide was useful for the teachers in implementing the RME-based geometry course" (Fauzan et al., 2013, p. 172).
RME tasks	Tasks and activities specifically designed using the principles and characteristics of RME.	"Gravemeijer and Doorman (1999) stressed that research on the design of Realistic Mathematics Education (RME) based activities has shown that the use of personalized contexts improved word problem solving by increasing the meaningfulness of contexts and enhancing student motivation" (Mousoulides, Sriraman, & Christou, 2007, p. 29).
Advantages, disadvantages and recommendations for RME	Positive and negative critique about RME as well as suggestions and commendations regarding RME.	"RME is a guide for student learning that focuses on the development of abstract thinking" (Fitzallen & Watson, 2014, p. 263).
Aspects of RME	These include features of RME other than the three principles or five characteristics, such as real life; human activity etc.	"According to Freudenthal (1983), mathematics is a human activity and therefore it must constitute a human value, must be close to reality of fact, be close to children and have a relationship with society" (Zaranis, Kalogiannakis, & Papadakis, 2013, p. 5).
RME	RME is the acronym for Realistic Mathematics Education which is a "domain specific instruction theory for the teaching and learning of Mathematics" (Drijvers et al., 2013, p. 1)	"In the concept of RME, mathematics is a human activity connected with reality" (Widjaja & Heck, 2003).
RME theory	The theoretical framework of Realistic Mathematics Education	"The Realistic Mathematics Education (RME) approach was the key theoretical framework for the design of the software and of the instructional material, and for the flanking educational research" (Heck, Boon, Bokhove, & Koolstra, 2007, p. 2).
Approaches to teaching Mathematics	Various teaching approaches such as the traditional approach, constructivism, problem-solving approach and RME.	"The traditional way of teaching had a negative influence on the pupils' attitudes towards mathematics which means that most pupils did not like to learn mathematics, and that some of them were even afraid of mathematics" (Fauzan et al., 2013, p. 161).
Modelling	This refers to mathematical modelling which can take on various forms such as physical models, equations, dynamic systems etc.	"The idea is to look for models that can be generalized and formalized to develop into entities of their own, which then can become models for mathematical reasoning" (Doorman et al., 2008, p. 2).
Learner reasoning	The process that learners go through where they ponder on and evaluate ideas.	"The RME philosophy also suggests instruction should build students' reasoning gradually from the concrete to the abstract, through the use of manipulatives, diagrams, and other imagery to reinforce students' reasoning" (Fitzallen & Watson, 2014, p. 264).
Learner understanding	When learners are given the opportunity to learn with deep	"The results of research in the Netherlands showed that the RME has shown satisfactory results. RME

Code	Description	Example
	understanding.	has the potential to improve students' understanding of mathematics (Streefland, 1991)" (Cahyono, 2012, p. 72).
Mathematics education	All aspects that relate to the teaching and learning of Mathematics.	"For over two decades, many stakeholders have highlighted the potential of digital technologies for mathematics education" (Drijvers, 2013, p. 1).
Maths strategies	Learning and teaching strategies used in Mathematics education.	"Context problems are selected, offering the students the opportunity to develop situation-specific representations and strategies" (Doorman et al., 2008, p. 2).
Meaningful learning of Maths	When that which is learned is fully understood in whole and related parts.	"active participation is promoted when students make artefacts that are meaningful and useful to them" (Fitzallen & Watson, 2014, p. 264).
Problem areas to teach/learn	Teacher-students discussed the areas of the content that they found problematic to teach and the areas that their learners found difficult to learn.	"Learners do have a problem with algebra and integers in fact. If they do have a problem with integers, it is going to give them a problem when dealing with algebra" (P59:12).
Teaching methods and strategies	General teaching methods and strategies	"This nation-wide reform now has an established record (Vos, 2010) that speaks directly to the building of mathematics competence, but also addresses issues of ways of working, cooperation, problem solving and metacognition"(Gordon, Rey, Siewiorek, Vivitsou, & von Reis Saari, 2012, p. 21).
Factors that influence teaching and learning	Any factor or aspect that plays a role in teaching and learning	"Another factor that is not so much elaborated in the case descriptions but is important to mention here, is assessment, which should be in line with the students' activities with technology; not doing so would suggest that in the end the use of digital technology is not important" (Drijvers, 2013, p. 15).
Learner aspects	Any aspect that influences the learner specifically in relation to the learning of Mathematics, with particular attention to learner understanding and learner reasoning	"The effectiveness of the RME-based geometry course had also to be investigated in order to show its effect on the pupils' learning" (Fauzan et al., 2013, p. 171).
Mathematics	Freudenthal (1968, p. 4)argues that Mathematics has proved indispensable for the understanding and the technological control of both the physical world and the social structure. Mathematics has been described by some as a static discipline developed abstractly, while others see it as a dynamic discipline constantly changing due to new discoveries (Dossey, 1992, p. 39). Other authors describe Mathematics as the science of pattern and order, which challenges the view that Mathematics is merely a discipline dominated by computation (Van de Walle et al., 2013, p. 13). Mathematics is also described as a human endeavour in which ordinary people construct concepts, discover relationships, invent methods, execute algorithms, communicate and solve problems posed by the real world (Cangelosi, 2003, p. 7).	"This is also in line with the objective of realistic mathematics education (RME), where instructional design is aimed at creating optimal opportunities for the emergence of formal mathematical knowledge from situation specific reasoning" (Doorman et al., 2008, p. 1).

Code	Description	Example
Mathematics content	Content areas at any level of the Mathematics school curriculum. Both formal and informal Mathematics is included in this category.	"Within this perspective, we can distinguish between formal and informal by denoting formal mathematical reasoning as a form of reasoning that builds on arguments that are located in the newly formed mathematical reality. Seen this way, the distinction between informal and formal is a relative distinction-a distinction that is relative to a certain topic and that can be made especially from an observer perspective" (Gravemeijer, 1999, p. 160).
Community	One of the role players that can influence the teaching and learning of Mathematics as expressed by the teacher-students.	"But, maybe only, let me say ten per cent of our children have got a good background, they have got a satellite dish. So mostly they are from shacks, where they do not have electricity, you know..." (P60:211).
Context	Another factor that the teacher-students identified that serves as a role player in the teaching and learning of Mathematics.	"Ja, last time when we were using the overhead projectors, but now the state of our school we cannot, we have so many projectors here, but they are not functioning" (P49:76).
Learner	Learners also are role players in the teaching and learning of Mathematics. Teacher-students voiced their opinions about the role that learners play.	"You see, most of our learners they are not, how can I say it, they are not familiar with the situations" (P49:38).
Role players	Any person or aspect that plays a role in the teaching and learning of Mathematics.	"Involving different stakeholders and especially teachers in the design process is a key factor for an innovative learning activity to find its way to teachers' everyday practice" (Perez, 2012, p. 21).
Teacher	Teacher-students expressed their views about teachers as role players in the teaching and learning of Mathematics.	"Then, recently we have Euclidean geometry which has been introduced. Uhm, it's also a major problem, but I think the problem is not the learners but maybe the majority of teachers who are teaching that, they did not even do it during their times at school. It's something that was brought in as a crash programme for teachers to be taught as well. So, those are the main three areas: Euclidean geometry, trigonometry and functions" (P61:6).
Integration across strands	One of the characteristics of RME. This deals with the integration of content across the curriculum.	"So this is how now I'm learning, right from grade twelve. Because if you're teaching it at grade twelve, there are some of the concepts that you don't know at the grade eleven and grade ten level" (P49:18).
Learning environment	The atmosphere and space that is created in which learning can take place.	"Yes, the learners present and they present it so well, they present it so well. In the meantime they are also learning, you understand? They are also learning. So, I also tell some other learners they must also do it, because you gain more knowledge by just standing there and going home to prepare the topic to do it. They come prepared and then they do it so nicely" (P60:235).
Student support	Support that is rendered to the students in terms of their learning.	"While pre-enrolment testing establishes learner readiness for participating in a course, the pre-course diagnostic testing is conducted to determine the amount and level of support a learner will require to advance successfully within the course" (Kizito, 2012a, p. 39).
Distance education	Distance learning that offers students flexible learning opportunities at their convenience.	"In the South African distance education environment, where the majority of students simply cannot afford access to web-based learning, the use of a mLearning strategy seems to be a viable and rational alternative for providing instructional support" (Kizito, 2012a, p. 39).
Motivation	Factors that encourage and inspire teachers or learners in the process of teaching and learning.	"In our opinion, digital assignments and diagnostic testing add to the motivation and performance of students in mathematics education" (Heck et al., 2007, p. 18).

Code	Description	Example
Pedagogy	An aspect that has to do with the theory and practice of education.	"even if the digital technology´s affordances and constraints are important design factors, the main guidelines and design heuristics should come from pedagogical and didactical considerations rather than being guided by the technology´s limitations or properties" (Drijvers, 2013, p. 15).
Social aspects	Any social aspect in the teaching and learning of Mathematics. This can include social norms and social interaction.	"The third level of the teaching procedure included activities using appropriate social interaction to refine the models of the problem" (Zaranis, 2014a, p. 62).
TPACK	The three main components of teachers' knowledge: content, pedagogy and technology. These aspects, as well as the interaction between them are important.	"To be able to do so, a process of professional development is required, which includes the teacher's own instrumental genesis, or, in terms of the TPACK model, the development of his technological and pedagogical content knowledge" (Drijvers, 2013, p. 15).
Challenges when working with ICT	Any aspect relating to ICT that is problematic.	"There are some disadvantages, too. When a school already uses a VLE, with all kinds of communication facilities such as email and bulletin board available, it could be considered cumbersome that yet another environment is added to the already long list" (Heck et al., 2007, p. 8).
ICT systems: virtual learning environment (VLE)	Various ICT systems and virtual learning environments are represented here.	"As technology for teaching mathematics the Freudenthal Institute's *Digital Mathematics Environment* (DME) is used. The DME integrates a content management system, a learning management system and an authoring environment. The content consists of online modules in the form of Java applets. The learning management system offers means to distribute content among students and to monitor student progress" (Drijvers et al., 2013, p. 2).
Teacher-student ICT factors	Factors relating to ICT that play a role in the teacher-students' teaching environment.	"Ja, we could prepare in advance. And then presently we've got only one projector in the school, so it is difficult to use the laptop when teaching. And some of the classes do not have electricity. That's another problem" (P49:80).
Technology as educational tool	An aspect where technology is used as an educational tool or has potential to be used as an educational tool.	"Yes, they have the tablet. We have the interactive whiteboard, but we the teachers are still struggling to use the interactive whiteboard. Most especially for Maths" (P60:22).
Value of using ICT	The advantages or benefits of using ICT in the educational arena.	"In our opinion, digital assignments and diagnostic testing add to the motivation and performance of students in mathematics education" (Heck et al., 2007, p. 18).
Devices	Any technological device that can be used in educational practice, e.g. mobile phone, laptop, tablet.	"Observation of the students' behavior during the experimental lessons supports the premise that the graphics calculator can stimulate the use of realistic contexts, the exploratory and dynamic approach to mathematics, a more integrated view of mathematics, and a more flexible behavior in problem solving" (Drijvers & Doorman, 1996, p. 425).
ICT	This refers to the broad, encompassing term that includes all information and communication technology.	"In order to help teachers to benefit from technology in everyday mathematics teaching, therefore, it is important to have more knowledge about the new teaching techniques that emerge in the technology-rich classroom and how these relate to teachers' views on mathematics education and the role of technology therein" (Drijvers et al., 2010a, p. 214).
Tools	Technology tools that aim to help teachers teach learners more effectively, such as software programmes,	"Two kinds of applets can be distinguished: model applets, which help to develop mathematical understanding (concepts), and exercise applets, which support the development of mathematical

Code	Description	Example
	applications, websites etc.	skills" (Heck et al., 2007, p. 2).
Methodology	Different methodological approaches used in the literature, e.g. design-based research; action research.	"The three studies that are discussed in this paper combine the methodology of design research with a prominent role for the Hypothetical Learning Trajectory as a research instrument in all phases of design research (design, teaching experiment, retrospective analysis)" (Bakker et al., 2003, p. 1).

Interrater reliability was calculated using Cohen's kappa. This is an index of interrater reliability which is used to measure the level of agreement between two sets of ratings (Cohen, 1960, p. 37). Kappa is the portion of agreement between raters (Fleiss & Cohen, 1973, p. 613). Kappa ranges between -1.00 and +1.00, where a negative value indicates poor agreement, and a positive value indicates better than chance agreement (Fleiss & Cohen, 1973, p. 613). A kappa of zero indicates a random level of agreement or disagreement (Wood, 2007, p. 7). Wood (2007, p. 6) states that for research purposes, the kappa should be at least 0.60 or 0.70. For making applied decisions about a particular individual like doing intelligence tests, the kappa should be at least 0.80 or 0.90. The Cohen's Kappa for the systematic literature review was calculated to be 0.98 (Addendum 2.5).

2.6.6 Validity and Reliability in the Systematic Literature Review

The validity and reliability of a study to a large degree depend on the ethics of the researcher (Merriam, 2009, p. 228). If the ethical standards of the researcher are not acceptable, it will negate the reliability and validity of the study. I have at all stages of the research adhered to the strict ethical standards of the NWU (§ 2.5), as well as conducted all aspects of the research in an ethical manner.

The importance of producing valid and reliable results and knowledge in research is invaluable (Merriam, 2009, p. 210). Validity is an important key to effective research (Cohen et al., 2011, p. 133). The following terms are extensively embraced in qualitative research: credibility, transferability, dependability, conformability (Merriam, 2009, p. 211), trustworthiness, authenticity and plausibility (Given, 2008, p. 909). Reliability refers to the replicability of research findings which in essence in social sciences is problematic, since human behaviour constantly varies (Merriam, 2009, p. 220).

Reliability is often referred to as the "dependability, consistency and/or repeatability of a project's data collection, interpretation, and/or analysis" (Given, 2008, p. 753). To put these into practice, authors propose techniques such as spending extensive time in the field, triangulation and the use of thick description (Creswell, 2013, pp. 203-204).

Given (2008, p. 753) lists three of the commonly cited indicators of credibility and dependability that qualitative researchers use to demonstrate systematic attention to reliability issues as: i) methodological coherence, which refers to the appropriate and thorough collection, analysis and interpretation of data; ii) researcher responsiveness, which refers to the early and on-going verification of findings and analyses with participants; and iii) audit trails, which refer to the transparent

description of all procedures and issues relating to the research project. Creswell (2007, p. 206) suggests, amongst others, the following strategies to ensure validation in qualitative research: Prolonged engagement and persistent observation in the field; and rich, thick description, which helps readers make decisions about transferability. The manner in which I have complied with the indicators as suggested by Given (2008, p. 753) and Creswell (2007, p. 206) is illustrated in Table 2.7.

Table 2.7 Indicators of Credibility and Dependability in the Systematic Literature Review

Indicator	Description
Methodological coherence	I followed a scientific procedure to select relevant documents. The selection process was discussed in detail in § 2.6.1 – 2.6.3. Data analysis was systematically done with the aid of ATLAS.ti™.
Researcher responsiveness	The interpretation of the data was verified by a peer in the field of Mathematics education, who is also an expert in performing systematic literature reviews. Cohen's Kappa was calculated to verify the interrater reliability (§ 2.6.5)
Audit trails	All procedures that were followed in the selection, analysis and interpretation of the documents were described in detail in § 2.6.
Prolonged engagement	Over a period of a year, I selected, coded, analysed and documented the systematic literature review.
Rich, thick descriptions	I made use of an extensive scientific process that was documented and described in detail as to how documents were selected, coded and analysed. A detailed analysis of the literature follows in Chapter Three.

2.6.7 Limitations of the Systematic Literature Review

Although the systematic literature review has many benefits (§ 2.6), there are limitations involved in its use as well. Most importantly, systematic literature reviews can be performed badly (Hemingway & Brereton, 2009, p. 6). To overcome this problem, researchers should rigorously follow the steps documented for successful systematic literature reviews by various authors (§ 2.6.1). The most obvious limitation of a systematic literature review is the limited key words used in the search, the limited data-bases that were selected, as well as the limited time period of publication. The systematic literature review is a complex, intensive, rigorous and time-consuming method of doing a literature review. Despite these aspects, it still proved to be a suitable method for deriving a theoretical framework for the study.

2.7 Aspects for Intervention

The research questions and the exposition of the systematic literature review provided guidance and insight with regard to various aspects that required intervention. This overview was an ideal starting point which offered valuable insight into strategies that could be used to intervene. From this point the qualitative strategy, which took the form of a needs analysis, could be further used to refine the intervention process and assist the teacher-students in addressing their specific needs.

2.8 Qualitative strategy: Needs Analysis

One of the sub-questions posed to gain insight on the main research question for this study was: *What support needs do teachers have in order to successfully implement the RME approach in their teaching practice?* To answer this question, it was necessary to schedule a meeting with the teacher-students to establish what needs they had in terms of making the Mathematics content more realistic for their learners. I also used this opportunity to determine their views on what type of technology would be most suitable to address these needs.

2.8.1 Participant Selection

Two basic types of sampling can be used, namely probability sampling and nonprobability sampling. Nonprobability sampling is the most suitable choice for qualitative research, and the most common form of nonprobability sampling is purposive or purposeful sampling (Merriam & Tisdell, 2016, p. 96). Nonprobability sampling implies that the researcher targets a particular group knowing full well that it does not represent the wider population, it merely represents itself (Cohen et al., 2011, p. 155). Small-scale research often makes use of nonprobability sampling because the samples are less complicated to achieve, are considerable less costly and can prove perfectly adequate when researchers do not intend to generalise their findings, despite the disadvantages that can arise from the non-representativeness (Cohen et al., 2011, p. 155). Purposive sampling (participant selection) allows the researcher to select individuals and sites because they can provide an understanding of the research problem and principal phenomenon in the study (Creswell, 2013, p. 156). The number of participants in the sample varies, depending on the research. In phenomenology, the range of the number of participants can be from one to hundreds (Creswell, 2013, p. 157).

As mentioned in Chapter One, in this study, participants were selected from a group of in-service teachers enrolled for the BEd Honours post-graduate degree in Mathematics education at the NWU, which is delivered by means of both the face to face and ODL modes. The criteria for selection were:

(i) teacher-students enrolled at the NWU for the BEd Honours degree by means of either face to face or ODL mode,
(ii) teacher-students who attend contact classes for the degree in question,
(iii) teacher-students who are currently teaching Mathematics in any phase and
(iv) male and female teachers at various schools.

At the contact sessions in Potchefstroom, I made contact with the students who attended the presentations for the Mathematics modules in the course. I enquired of these students who would be willing to take part in this research, and collected their contact information. I approached these students for the initial interviews.

2.8.2 Site Selection

The selected site for this research was the Potchefstroom Campus of the North-West University. Part-time students enrolled at the Faculty of Education Sciences at the above-mentioned campus, for the BEd Honours post-graduate degree, specialising in Mathematics education, were invited to participate in the research.

There are a number of types of purposive sampling; the most appropriate for this study was convenience sampling. In convenience sampling, otherwise known as accidental or opportunity sampling (Cohen et al., 2011, p. 155), a sample is selected based on time, money, location and availability of sites or respondents (Merriam & Tisdell, 2016, p. 98). This sampling involves choosing the nearest individuals to serve as respondents until the sample size has been obtained (Cohen et al., 2011, p. 155). The advantage of using convenience sampling is that it saves time, money and effort, but this can impact on the information received and the credibility (Creswell, 2013, p. 158). Captive audiences like students or student teachers often serve as participants in convenience sampling (Cohen et al., 2011, p. 156). The researcher in this type of sampling needs to be particular about reporting that the parameters of being able to generalise are negligible (Cohen et al., 2011, p. 156). I made use of purposive sampling to select the site. Two of the interviews took place at the Potchefstroom Campus of the North-West University, which is a familiar terrain for the student-teachers as they regularly attend contact classes at this site. The third interview took place in the HOD's office at one of the participant's school in Potchefstroom. The fourth meeting took place at a coffee shop in Ventersdorp.

2.8.3 Methods of Data Generation or Collection

Interviews allow participants to discuss their interpretation of the world in which they live and to express their opinions on situations from their perspective (Cohen et al., 2011, p. 349). As Cohen et al. (2011, p. 349) describe it, the interview is "not simply concerned with collecting data about life: it is part of life itself, its human embeddedness is inescapable." I made use of semi-structured individual interviews with the teacher-students to establish their needs in terms of making Mathematics more realistic for their learners. The RME approach has proved to enhance and improve learners' understanding in Mathematics (Fauzan et al., 2013, p. 174). I also made use of observations of the participants' classroom practice to establish the teacher-students' needs with regard to making the curriculum content relevant for their learners. Individual as well as focus group interviews were used to further generate data. Focus group interviews should be used when interviewees are similar and the interaction between the interviewees will yield the best information (Creswell, 2007, p. 133). The interviews allowed me to gain in-depth knowledge of how teacher-students experience the intervention that is presented to them by means of different technologies, based on the RME approach. In order to ensure validity and reliability, the following measures were used: prolonged engagement and

persistent observation in the field, the use of thick, rich description and feedback from others (Creswell, 2007, p. 207).

2.8.3.1 Initial Testing of the Informal Discussion and Needs Analysis

An initial informal discussion was scheduled with a teacher with whom the researcher is familiar, to test the effectiveness and accuracy of the questions in the informal discussion. The teacher is currently teaching Mathematics at a rural secondary school in the Potchefstroom area. Some of the lecturers who are involved with distance education at the Faculty of Education Sciences at the NWU have since 2011 been involved in a school support programme with a particular school in the area. The support involves assisting grade 11 and 12 teachers and learners with Mathematics, via an Interactive White Board (IWB). The researcher is familiar with this teacher because of this support programme, since he is one of the Mathematics teachers at the school, and she is one of the lecturers involved in the support programme.

The initial informal discussion was recorded and transcribed. I together with my promoter coded the informal discussion with the aid of ATLAS.ti™. The existing codes that were created in the systematic literature review were used as well as a few additional codes that were necessary to create in the coding process. To ensure that the coding was done in sufficient detail, and to determine whether the questions were unambiguous and clear, the researcher and two peers perused the transcription again, and added additional codes where necessary. The two peers that assisted with this further coding are full-time Mathematics lecturers at the Faculty of Education Sciences at the NWU.

2.8.3.2 Individual Semi-structured Interviews

One of the most common ways of collecting qualitative data is the interview, which is used by researchers when they “cannot observe behaviour, feelings or how people interpret the world around them” (Merriam & Tisdell, 2016, p. 108). In the SAGE Encyclopaedia of Qualitative Research Methods, Given (2008, p. 422) distinguishes between different types of interviews. Merriam and Tisdell (2016, p. 109) use the same classification, namely structured, semi-structured and unstructured interviews. In-depth or semi-structured interviews are interviews in which participants are encouraged to talk in depth about a topic. The interview can be considered to be semi-structured, since the researcher maintains some control over the content and direction of the discussion, however participants are free to expand on their ideas and move the discussion into new but related directions (Given, 2008, p. 422). The conversation fluctuates between the researcher’s introduction of the topic, the participant’s description of his or her experiences and the researcher’s examining of these experiences for additional information (Given, 2008, p. 422). In a structured interview, the participant responds to a list of pre-determined topics or questions, and in an unstructured interview there are no preconceived topics or questions (Given, 2008, p. 422).

Four participants were approached to take part in the study. Individual semi-structured interviews were scheduled with each participant and an appropriate location was decided upon to do the interview. An interview schedule was loosely followed (Addendum 2.6). The questions in semi-structured interviews are flexibly worded and the interview is a mix of more or less structured questions (Merriam, 2009, p. 90). The main part of the interview is guided by a list of questions or issues that need to be explored. This allows the researcher to react to the situation as it arises (Merriam, 2009, p. 90). The first interview took place at the North-West University (NWU) Potchefstroom Campus.

The interviews were coded using ATLAS.ti™. Open-coding was used to code the interviews as this allowed me to explore ideas and meaning that were contained in the raw data (DeCuir-Gunby et al., 2011, p. 139). Codes that are driven by data involve five steps in order to create the codebook , namely: raw data are reduced; subsample themes are identified; themes are compared across subsamples; codes are created; and the reliability of codes is determined (DeCuir-Gunby et al., 2011, p. 141). These five steps were used in the study; firstly I perused paragraphs of the data to identify appropriate codes. The existing codes which were created in the systematic literature review were used, and where necessary, additional codes were created which were suited to the context and content of the data. This constituted step two, three and four. The final step to establish reliability was done by inviting two peers who are experts in the field of Mathematics education to peer code an interview. The Cohen's Kappa for the individual interview was calculated to be 0.99 (Addendum 2.7).

Data saturation occurs when "continued data collection produces no new information or insights into the phenomenon you are studying" (Merriam & Tisdell, 2016, p. 199). By the third and fourth interview, no new information emerged in this study. Another indication of data saturation is concerned with the analysis of the data—if the categories, themes or findings are robust enough to cover what emerges in later data, then saturation occurs (Merriam & Tisdell, 2016, p. 199). When coding in ATLAS.ti™, no new codes or themes were added in the third or fourth interview, so I deemed four a sufficient number of participants for the study.

2.8.3.3 Data Analysis

Data were coded, sorted and summarised into themes or patterns which would aid the understanding and interpretation of what is emerging (Nieuwenhuis, 2010a, p. 99). This analysis was done with a Computer Assisted Qualitative Data Analysis Software (CAQDAS), ATLAS.ti™. Based on these findings, I could establish what sections of the curriculum the teachers and learners found most problematic, and also with which sections they needed help to relate the content to real life situations. As a result of what was revealed in the interviews, I could also decide on the appropriate use of technology to present the intervention, which was based on the principles and characteristics of RME.

2.9 Intervention Strategy: Design of the App

The needs analysis revealed two important aspects that directed the rest of the study (a detailed discussion is presented in Chapter Four). Firstly, participants needed assistance with ideas on how to make the Mathematics more realistic for their learners; and secondly, they were keen to use technology, specifically mobile technology, as a tool to learn how to apply the principles of RME in their teaching practice. It was therefore necessary to devise a strategy to intervene, that could address the issues raised. This intervention strategy took the form of an app that was designed for Android. With the help of a programmer and specialist in the design of applications, we designed an app for the teacher-students that could help them to make the content more realistic for their learners. The design process is documented in detail in Chapter Five.

2.10 Qualitative strategy: Participants' Perceptions about the App

When the intervention tool, the Financial Mathematics app, had undergone various stages of refinement in the form of improved prototypes and it was completed, the programmer made it available on Google Play Store as a free app. It was necessary to make this app available to the teacher-students so as to establish their perceptions and opinions about the app.

2.10.1 Participant Selection

The same students that were selected for the initial interviews were approached for the follow-up focus group interview. One of the students that took part in the initial interview was not able to attend the focus group interview and an additional student who complied with the criteria in § 2.8.1 was approached to attend the focus group interview. This student was informed of the research process up to that point and agreed to take part in the focus group interview.

2.10.2 Site Selection

The same site was used as discussed in § 2.8.2. I met with teacher-students at an appropriate venue at the North-West University Potchefstroom Campus and made sure that all participants were comfortable.

2.10.3 Methods of Data Generation or Collection

A focus group interview was selected as the method of data generation. A focus group interview is an interview with a group of people who have a common interest or knowledge in a topic (Merriam, 2009, p. 93). It is beneficial when the interaction between the interviewees will produce the best information,

when the participants are similar and when they are willing to cooperate with each other (Creswell, 2007, p. 133). I decided to make use of the focus group interview for two reasons. Firstly, because I hoped that the interaction between participants would yield the most fruitful information. Secondly, I was also aware that the teacher-students were familiar with one another because they were studying the same course and attended contact classes together, so I hoped that that fact would encourage active cooperation between them. I designed an interview protocol (Creswell, 2007, p. 133) with ten questions. These questions were based on the literature review. The systematic literature review produced five main themes (Chapter Three), namely Mathematics, RME, ICT, role players and methodology. I designed questions which incorporated these themes in the process of evaluating and commenting on the app (Addendum 2.8).

2.10.3.1 Focus Group Interviews

Initially I planned to do one focus group interview with the four selected teacher-students. When contacting the students for a viable date and time to meet, I noted that one of the teacher-students was not able to attend the focus group interview at the same time as the others for logistical reasons. I decided to divide the group in two and proceeded with two focus group interviews, each attended by two students.

The purpose of the focus group interview was to introduce the teacher-students to the app on Financial Mathematics and to determine their views and opinions about it. The app was available for teacher-students to download free of charge on Google Play Store. I allocated time for them to familiarise themselves with the app before the scheduled questions (Addendum 2.8) were discussed. In this time, I was available to assist in any way they needed assistance. We then progressed to a discussion about the app and addressed the scheduled questions in the discussion. A detailed analysis of this discussion was presented in Chapter Six.

2.10.3.2 Data Analysis

I transcribed the two focus group interviews and analysed the data with the aid of ATLAS.ti™. The data analysis and interpretation were integrated with the literature review to answer the research questions. Based on these findings, guidelines for the effective use of technology in implementing the RME approach in teaching practice were designed. The detailed guidelines were presented in Chapter Seven.

All the data in the study were analysed in the same Hermeneutic Unit (HU) in ATLAS.ti™. The final list of codes as well as the code density, or the number of times each code was noted, is indicated in Table 2.8.

Table 2.8 Code Density Table

Code	Frequency
HLT	62
Design of activities	45
Didactical phenomenology	14
Emergent modelling	42
Guided reinvention	93
Advantages of using RME	26
Characteristics of RME	53
Content where need help to relate to real life	11
Disadvantages of using RME	6
Human activity	29
Mathematization	49
Principles of RME	22
Real life	252
Recommendations for RME based lessons	42
RME tasks	32
Aspects of RME	1
RME	362
RME theory	16
Approaches to teaching Mathematics	118
Modelling	72
Content related to real life	27
Learner reasoning	4
Learner understanding	8
Mathematics education	44
Maths strategies	4
Meaningful learning of Maths	66
Problem areas to teach/learn	17
Teaching methods and strategies	105
Factors that influence teaching and learning	249
Learner aspects	36
Mathematics	130
Mathematics content	125
Community	11
Context	35
Learner	19
Role players	205
Teacher	99
Integration across strands	6
Learning environment	7
Student support	5
Distance education	13
Motivation	14
Pedagogy	71
Social aspects	13
TPCK	6
Challenges when working with ICT	56
ICT systems: virtual learning environment (VLE)	85
Teacher-student ICT factors	79
Technology as educational tool	97
Value of using ICT	241
Devices	327
ICT	38
Tools	64
Methodology	97
Positive aspects of app	26
Suggestions to improve app	17
Usability of the app	5

2.10.4 Validity and Reliability in the Qualitative Inquiry

As discussed in § 2.6.6, unless a researcher is able to show the audience that the procedures that were followed to ensure that his/her methods were reliable and his/her conclusions were valid, there is no point in trying to conclude a research dissertation (Silverman, 2007, p. 209). I used the same criteria to ensure validity and reliability in the qualitative inquiry as I used in the systematic literature review. A detailed description of the theoretical underpinnings is given in § 2.6.6.

The manner in which I have complied with the indicators as suggested by Given (2008, p. 753) and Creswell (2007, p. 206) in the qualitative inquiry is illustrated in Table 2.9.

Table 2.9 Indicators of Credibility and Dependability in the Qualitative Inquiry

Indicator	Description
Methodological coherence	The initial test interview was peer coded to ensure validity and reliability of the coding (§2.8.3.1) All data analysis was systematically done with the aid of ATLAS.ti™. The design of the app followed nine systematic and coherent steps in the design and development phase (§ 5.5.2)
Researcher responsiveness	The interpretation of the interview data was verified by two expert peers in the field of Mathematics education. Cohen's Kappa was calculated to verify the interrater reliability (§ 2.8.3.2)
Audit trails	All procedures that were followed in the collection, analysis and description of the qualitative data were described in detail in § 2.8-2.10. The detailed analysis of the individual interviews was done in Chapter Four and the detailed analysis of the focus group interviews was done in Chapter Six.
Prolonged engagement	There was a period of nine months between doing the initial needs analysis of the teacher-students and the follow-up focus group interviews. In this time, myself and various other role players spent extensive time in conceptualising, developing and testing the mobile app, before meeting with the teacher-students for the second time.
Rich, thick descriptions	Chapters Four and Six present detailed analyses of the qualitative data. The actual words of participants were used to reinforce the analysis and findings. Reference is constantly made in these chapters to the conceptual framework from the literature review (Chapter Three) as well as the process that was followed in the research methodology of qualitative design-based research.

2.11 The Role of the Researcher

Denzin and Lincoln (2013, p. 8) describe the qualitative researcher as a "bricoleur," a person who engages in the construction of something from a wide range of available things, an amateur who is able to do practical repairs of any kind (Given, 2008, p. 65; Gravemeijer & Cobb, 2013, p. 78). Denzin and Lincoln (2013, p. 11) are of the opinion that qualitative researchers should be proficient in performing a large number of assorted tasks which include interviewing and rigorous self-reflection, and should read widely about the many interpretive paradigms. Gravemeijer and Cobb (2013, p. 78) state that the way a design-based researcher goes about doing things resembles the way a "bricoleur" works. The qualitative researcher should understand that the research is an interactive process which is shaped by not only his/her own personal history, biography, gender, social class, race and ethnicity, but also those of the people in the setting (Denzin & Lincoln, 2013, p. 11). The researcher is the

primary instrument for collecting and analysing data (Creswell, 2013, p. 45; Merriam & Tisdell, 2016, p. 16).

When discussing the process of design-based research, Amiel and Reeves (2008, p. 35) discuss the liaison between researchers and practitioners. They suggest that researchers rarely engage directly with practitioners as far as the design process is concerned. The design-based research should start with negotiations between researcher and practitioner regarding the research goals. The practitioner's role is to partner the researcher in establishing research questions and identify problems that need to be investigated (Amiel & Reeves, 2008, p. 35).

One of the requirements of a researcher in design-based research is that he or she should be "humble in approaching research by recognising the complexity of interactions that occur in real world environments and the contextual limitations of proposed designs" (Amiel & Reeves, 2008, p. 35). According to Plomp (2013, p. 43) the researcher needs to be adaptable, and the researcher needs to be adaptable in the following ways: in addition to being researcher, the researcher should be willing to take on the role of designer, advisor and facilitator; be tolerant with regard to role distinctions; and be willing to allow the study to be partially influenced by the role players (McKenney, Nieeven, & van den Akker, 2006, p. 84).

I am a lecturer in the Faculty of Education Sciences at the NWU, and work specifically with programmes that are delivered to in-service teacher-students, and specialise in the field of Mathematics education. I fulfilled a multi-faceted role in this research. Firstly I acted as researcher throughout the entire process. After negotiating with teacher-students and establishing what problem areas teacher-students (who acted as practitioners in this research) had in teaching Mathematics content to their learners, my second role, as a content specialist, was to design appropriate intervention in the form of a mobile app. The latter was based on the RME approach to assist the teacher-students to make the content which they teach to their learners more relevant and related to real life. The third role that I fulfilled in the study was to act as interviewer and facilitator of the teacher-students to establish what support-needs the teacher-students had in order to effectively implement the RME approach in their teaching, as well as to determine their perceptions about the mobile app as intervention. My fourth role was to design and refine guidelines appropriate for the application of the RME approach to be implemented with the aid of technology.

2.12 Summary of the Chapter

This chapter presented the research design and methodology for this study. The research questions formed the foundation of the method that was selected. I described my stance as an interpretivist and expressed the suitability of selecting to do a qualitative phenomenological study. I explicated the methodology of qualitative design-based research as well as the suitability thereof in this study. The

ethical considerations were taken into account. I detailed the description of the systematic literature review process as well as the qualitative inquiry. Concerning both aspects I discussed the rationale behind participant selection, site selection, data generation methods and the process of analysis. I also explored my role as researcher and discussed the trustworthiness of both the systematic literature review as well as the qualitative inquiry. Chapter Three presents the systematic literature review.

Chapter Three

Systematic Literature Review

3.1 Introduction

This chapter presents the review of the current literature with regard to the implementation of RME principles, aided by the use of ICT in Mathematics education. ATLAS.ti™ was used to analyse the literature in order to create themes for discussion and interpretation. Five themes emerged in this review which related to: (i) Mathematics, (ii) RME, (iii) ICT, (iv) Methodology, and (v) Role players. These themes guide the discussion in the chapter that follows. Figure 3.1 is an illustration from ATLAS.ti™ of the five main themes that emerged from the systematic literature review.

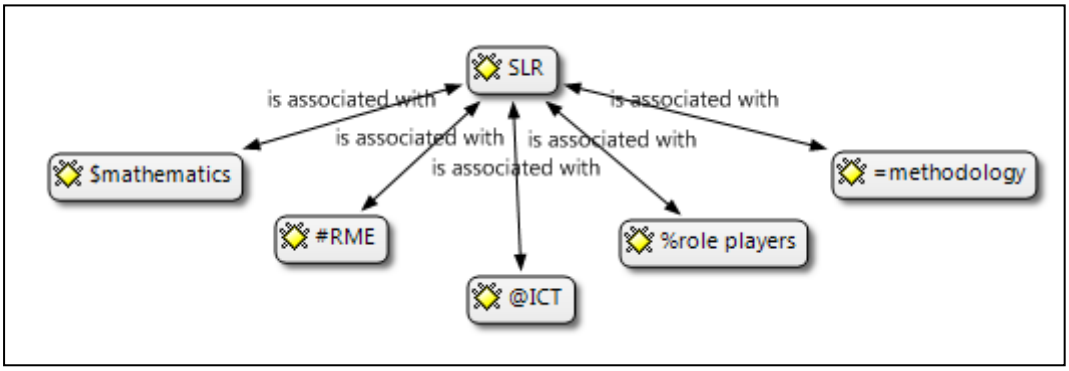


Figure 3.1: Emerging Themes from the Systematic Literature Review

Figure 3.2 presents the conceptual framework for the systematic literature review and serves as a guide for the rest of Chapter Three.

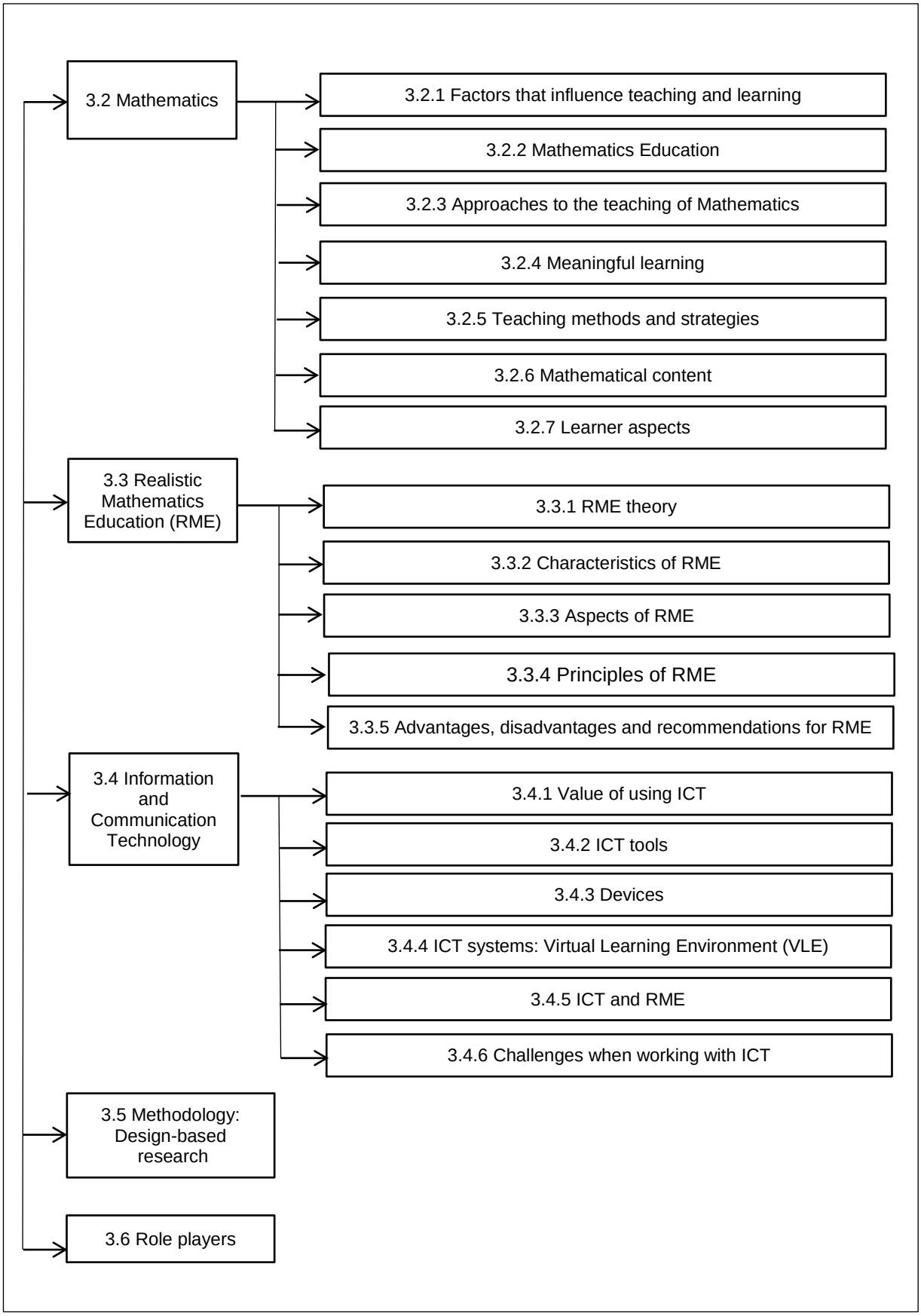


Figure 3.2: Conceptual Framework for the Systematic Literature Review

3.2 Mathematics

Mathematics is one of the main themes that arose from the systematic literature review as is illustrated in Figure 3.3.

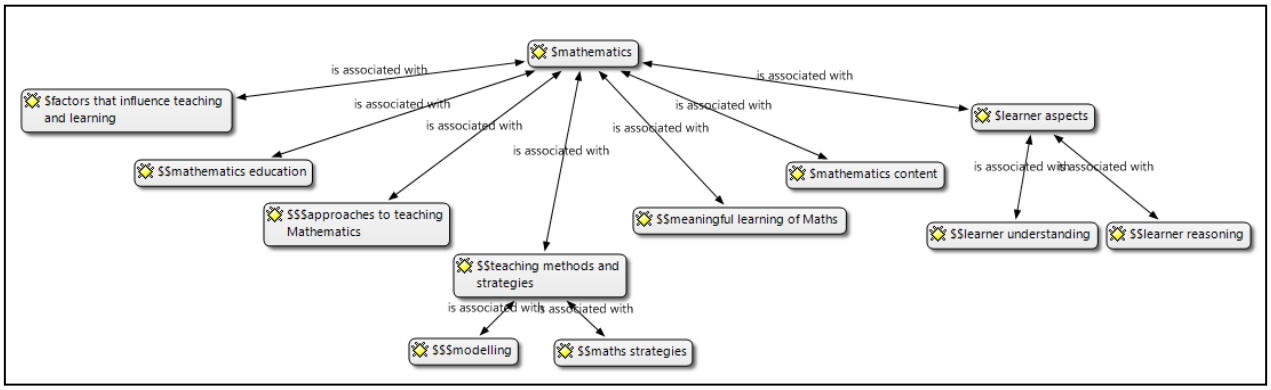


Figure 3.3: Mathematics as Theme from the Systematic Literature Review

Mathematics is not just an abstract body of knowledge, but a subject that contains skills and a way of thinking that can be applied in numerous real life situations (Domazet, Baranović, & Matić, 2013). The teaching and learning of Mathematics is a complex undertaking, but should be connected to reality in order to make sense to learners (Cahyono, 2012). One of the main expectations of Mathematics is that it will develop rational thinking, orderliness and accuracy of expression, all of which can be applied in everyday life (Domazet et al., 2013).

3.2.1 Factors that Influence Teaching and Learning

The rapidly changing information society influences the role of learners and teachers (Gravemeijer, 2012). Future jobs require what Gravemeijer (2012) refers to as 21st century skills, which include "flexibility, critical thinking, problem solving, collaboration, and communication." Gravemeijer (2012) emphasises the importance of addressing these demands by investigating educational practices that may nurture these goals. Heck et al. (2007) are of the opinion that digital assignments and diagnostic testing can increase motivation as well as performance in learners. Also, the use of real life contexts stimulate motivation because the relevance of Mathematics can then be seen in the real world (Risnawati, Kurniati, & Regita, 2014).

Mobile Learning (mLearning) is a field that is rapidly expanding and that affords new possibilities to improve learning, especially in a distance education (DE) environment (Kizito, 2012a). South Africa in particular is a suitable location to use mLearning in distance education due to its viability and affordability as opposed to web-based technology (Kizito, 2012a). Further research is required to determine how the mobile phone can effectively be used in testing in DE (Kizito, 2012a). The portability of mobile technology allows learners to work creatively and collaboratively (Zaranis et al.,

2013). When used correctly, technology could promote critical thinking, improved problem solving skills and collaboration (Stols, 2012).

Various reasons are offered for poor performance in Mathematics. These include: when topics taught seem to be removed from learners' daily lives (Fauzan et al., 2013); inaccurate learning material; inadequate and out-dated teaching methods; poor forms of assessment and learners' anxiety about Mathematics (Widjaja & Heck, 2003); and learners' negative attitudes towards Mathematics (Widjaja & Heck, 2003). The difficulties that learners experience in Mathematics in later years are associated with the inadequate development of mathematical thinking in the initial years (Zaranis, 2014c).

3.2.2 Mathematics Education

An essential goal of Mathematics education is the adoption of fundamental mathematical knowledge, the development of basic mathematical literacy and the ability to solve problems (Domazet et al., 2013). Mathematics education is not about transferring a mound of mathematical knowledge to young people, but rather about fulfilling broader social goals (Domazet et al., 2013). It is important to remember that pedagogy should guide the process of designing digital technology, rather than technology itself (Drijvers, 2013). The technology needs to be rooted in an educational context (Drijvers, 2013).

Mathematics teachers are challenged to develop Mathematics education that is in line with the dynamic conceptions of symbolising and the development of meaning (Bakker et al., 2003). The role of the teacher is essential in understanding how to teach effectively with technology (Drijvers, 2012). A teacher needs to plan for productive whole-class discussions, which demands that the teacher needs to account for differences in students. Students are at different levels of mathematical understanding, and the teacher needs to highlight problem areas in such discussions that stimulate a higher level of understanding (Gravemeijer, 2012). Tasks should be designed in such a way that students are encouraged to develop their own Mathematics. The teacher needs to guide the process and should know when to further explore a topic or concept and when to cease to investigate a topic (Drijvers et al., 2013). Digital assessment and diagnostic testing can increase the motivation and performance of students in Mathematics education (Heck, Boon, & van Velthoven, 2008).

3.2.3 Approaches to the Teaching of Mathematics

The traditional teaching approach tends to dominate in many classrooms still today (Fauzan et al., 2013). This approach is characterised by teachers' who actively explain material and provide examples and exercises, while learners listen, write and perform the tasks requested by the teacher (Gravemeijer, 2012; Widjaja & Heck, 2003). There is very little room for discussion, interaction and communication (Widjaja & Heck, 2003), and learners normally do not get involved in problem solving and reasoning (Gravemeijer, 2012). The social norms of the traditional classroom dictate that the

teachers' answers are always correct and that students should follow given procedures to reach correct answers, which are more important than reasoning (Gravemeijer, 2012). Learners are seldom given the opportunity to understand the rationale behind algorithms that are taught to them (Risnawati et al., 2014).

The traditional approach in teaching Mathematics has been labelled as a possible reason for the poor quality of Mathematics education in Indonesia (Fauzan et al., 2013; Widjaja & Heck, 2003). The reason for this is that most learners feel that they do not have the opportunity to learn significant Mathematics (Fauzan et al., 2013). Context problems in the traditional approach are used only as applications at the end of a section, rather than during the course of the teaching and learning process (Gravemeijer & Doorman, 1999).

Traditional teaching approaches tend to have a negative influence on learners, especially on their attitude towards Mathematics (Fauzan et al., 2013). Often in the traditional approach, learners work with problems that are not directly linked to school Mathematics, and they need to deal with unfamiliar situations and contexts (Mousoulides et al., 2007). Another consequence of traditional teaching is when students solve traditional word problems, high level cognitive and metacognitive processes are often absent (Mousoulides et al., 2007). The continuous change in Mathematics education is evident in the shift from traditional curricula to problem-oriented curricula (Widjaja & Heck, 2003).

A common thread of all alternative approaches is the belief that learning Mathematics should not be a process of assembling pieces of knowledge, but rather of cognitive growth (Gravemeijer & Doorman, 1999) and the development of core skills in learners (Heck, 2009). Reform from the traditional approach demands that curricula, teaching materials and assessment need to be adapted (Zulkardi, 2000). In essence, Freudenthal (1968), who is seen as the father of RME, opposes the traditional idea that the end result of the work of Mathematicians is the starting point for Mathematics education (Gravemeijer & Doorman, 1999).

Problem solving through modelling leads to the design of activities that allows for students to deal with non-routine problem situations that demand the development of important mathematical ideas that can be extended, explored and refined in other problem situations (Mousoulides et al., 2007). Technological tools create new prospects for problem solving in Mathematics (Doorman et al., 2007).

RME is a domain-specific instruction theory for the teaching and learning of Mathematics (Drijvers et al., 2013). The RME-based teaching and learning process promotes learner-centred learning (Fauzan et al., 2013). One of the cornerstones of RME is that learners are encouraged to not only receive information, but also question and process information (Widjaja & Heck, 2003), actively participate in the educational process and develop mathematical tools and insights (Drijvers et al., 2013). The RME approach can be used to address the shortcomings that the traditional approach presents, such as the remembering of facts and concepts that must be reproduced verbally, studying

computational aspects and applying formulas (Fauzan et al., 2013). RME does not aim to use abstract principles or rules in concrete situations (Widjaja & Heck, 2003).

Learners' achievements tend to improve in experimental situations, which is common to the RME approach as opposed to traditional teaching (Fauzan et al., 2013; Zaranis, 2014a) and the learners become more independent and actively involved in the teaching and learning process (Fauzan et al., 2013; Fitzallen & Watson, 2014). Learners learn by experience, and not because of what they are told by the teacher (Gravemeijer, 2012). The structure of RME is such that learners gradually reorganise their thinking toward more abstract ideas and relate those ideas to the practical context of the situation (Fitzallen & Watson, 2014). Learners reconstruct solution procedures by using problems that have meaning in their reality (Drijvers & Doorman, 1996).

Teachers are key role players in bringing about change in education. Teacher training both pre-service and in-service is therefore essential (Zulkardi, 2000). Fauzan et al. (2013) found that teachers like to use the RME approach and that they are able to implement the approach as it is intended. They do however sometimes revert back to the traditional teaching approach, perhaps because they are not yet fully familiar with the RME approach (Fauzan et al., 2013).

In the RME approach, context problems have a central role in Mathematics teaching. They are used to support a reinvention process which allows learners to come to terms with formal Mathematics (Gravemeijer & Doorman, 1999). There is an increased focus on the utility of Mathematics in daily practice. It should not be seen as a ready-made product, the emphasis should rather be on the process of doing Mathematics (Zulkardi, 2000). Mathematics that is built on everyday perceptions is in line with the objectives of RME where formal mathematical knowledge can be created from situation specific reasoning (Doorman et al., 2008).

Development in technology encourages innovation in the teaching and learning of Mathematics (Cahyono, 2012). If technology is used correctly it can enhance teaching and learning, specifically in Mathematics (Stols, 2012; Zaranis, 2014c). More specifically, a technology enriched environment helps to improve the conceptual geometric growth of learners (Stols, 2012). ICT is a good catalyst to realise activity-based approaches to teaching Mathematics such as the RME approach (Widjaja & Heck, 2003). Computer-assisted learning can contribute in developing mathematical skills and can cultivate deeper conceptual thinking as compared to teaching in the traditional approach (Zaranis, 2014a, 2014c). Digital activities are particularly effective when they are designed to examine a specific problem or to teach a specific skill (Zaranis et al., 2013). New technological tools create new possibilities for problem-solving in Mathematics since new horizons can be explored (Doorman et al., 2007).

RME instruction theory links closely to the work of Van Hiele (Gravemeijer, 1999; Treffers, 1978). The Van Hiele theory identifies five discrete levels of Geometric development that are ordered and

progressive, that are not dependent on age but rather on a learners' experience (Stols, 2012; Zaranis, 2014a). The Van Hiele theory has a significant influence on geometry education and can be used to measure the cognitive growth of students (Stols, 2012).

3.2.4 Meaningful Learning

The theory of RME can help to design activities that promote meaningful learning (Wijers, Jonker, & Drijvers, 2010). One way in which meaningful learning can be promoted in both formal and informal settings, is through playing games (Zaranis et al., 2013). The interaction between teacher and student and students with one another can provide them with opportunities to enhance their learning process through social interaction (Cahyono, 2012). This interaction also enhances reflection on work, engaging in explanations, justifying, agreeing and disagreeing, and questioning alternatives, which is an essential part of RME (Widjaja & Heck, 2003). Learners that are given the opportunity to discuss their own informal strategies in solving problems with the teacher and fellow pupils, have a greater opportunity to stimulate understanding than those who do not (Widjaja & Heck, 2003). Learning outcomes and learners' understanding can be improved as a result of collaboration in learning (Cahyono, 2012). A crucial aspect in the learning process is when students are given the opportunity in the classroom to discuss their solutions and pose new problems (Bakker et al., 2003).

Tasks form an important aspect of meaningful learning. The execution of tasks should guide learners to acquire the concepts that they are required to master. When too few tasks are given, this might hinder the process of acquiring knowledge, while too many tasks can result in learners that are bored or exhausted (Widjaja & Heck, 2003). Tasks should be designed in such a way that they start by investigating problem situations that can be systematised and solved by the Mathematics that needs to be learned (Boon, 2006). The purpose of investigation tasks is to give students opportunities to improve their general competencies, deepen existing mathematical knowledge and become more adept in applying their knowledge and skills in practice (Heck, 2009).

Assessment should be in line with learners' activities with technology, and should not be removed from the use of digital technology (Drijvers, 2013). Assessment can take on different forms such as writing an essay, doing an experiment, collecting data and drawing conclusions, designing exercises that can be used in a test, or designing a test for other learners in the classroom (Zulkardi, 2000).

3.2.5 Teaching Methods and Strategies

Learners' attitudes have been noted to improve as a result of RME-based courses (Fauzan et al., 2013). The use of real world situations in school Mathematics is necessary to nurture a positive attitude towards the subject (Mousoulides et al., 2007).

Modelling activities that take place within real contexts can promote motivation and self-regulation (Mousoulides et al., 2007; Widjaja & Heck, 2003). Practical work can be done in groups and creates the opportunity for interaction between learners with each other and the teacher, where discussion and reflection about Mathematics and mathematical thinking is stimulated (Widjaja & Heck, 2003).

Dolk, den Hertog, and Gravemeijer (2002) describe reflection as a prerequisite for learning and constructing meaning and knowledge, specifically when dealing with learning from one's own activities. This is important to remember in teacher education: beginner teachers need to be encouraged to reflect on their actions and experiences in the classroom (Dolk et al., 2002). Reflection and discussion can serve as encouragement to learners to do observation. When discussing what they have observed, they reflect on the strategies that they used to report and observe—this encourages better and deeper observations (Dolk et al., 2002).

3.2.5.1 Models

The term model refers to situation models and mathematical models that are generated by learners themselves (Zulkardi, 2000). Models are more than a visual representation of a piece of abstract Mathematics. They can be used to bridge the gap between abstract, informal strategies and real, formal Mathematics (Boon, 2006; Doorman, Drijvers, Gravemeijer, Boon, & Reed, 2013; Gravemeijer, 1999; Widjaja & Heck, 2003). Although modelling also necessitates computation and deduction processes, it is predominantly about focused description, explanation or conceptualisation (Mousoulides et al., 2007).

Gravemeijer (1999) explores a discussion about two types of models: a "model of" where reality is diminished into the form of a model (created after a concept is deliberated) and a "model for" where the same model can be utilised in other situations and in new realities. This implies that the "model of" informal mathematical thinking becomes a "model for" more formal mathematical reasoning (Doorman et al., 2013; Gravemeijer, 1999). The shift in the role of the model takes place alongside a shift in the learners' thinking.

Activities that involve modelling are intrinsically social experiences since students often work together in small groups to develop a product that can be shared with others in the classroom (Mousoulides et al., 2007). Activities that require the production of models, can lead to significant forms of learning because they involve mathematizing which includes activities such as quantifying, coordinating, categorizing and systematizing (Mousoulides et al., 2007). Activities such as these are intended to disclose the way in which students think about real life situations that can be modelled through Mathematics (Mousoulides et al., 2007). The creation of a model is often a cyclical process (Mousoulides et al., 2007).

The roles of context problems and modelling are closely intertwined (Gravemeijer & Doorman, 1999). When modelling activities are set within realistic contexts, this allows for learners to make use of multiple interpretations and approaches, thereby promoting intrinsic motivation and self-regulation (Mousoulides et al., 2007). It is important to work with and study modelling of intricate systems that occur in real life situations, even from the early grades, because such models are essential aspects of knowledge that need to be emphasised in teaching and learning. This is especially important for learners preparing to work in fields such as Mathematics, Science and Technology (Mousoulides et al., 2007). By using modelling, learners are able to reason with a model of the problem, but also think about how they are to act in the real life situation (Andresen, 2007). Eventually, learners will not need the support of a model for more formal mathematical reasoning (Andresen, 2007).

Models emerge from learners' activities. As learners reason, models are generated while discussions and activities take place in the classroom (Doorman et al., 2008). Learner sense-making can be enhanced when using realistic modelling problems in Mathematics. This also helps to nurture a positive attitude towards Mathematics (Mousoulides et al., 2007). The descriptions, explanations and justifications that learners make use of form an integral part of the models that they produce (Mousoulides et al., 2007).

Models clearly play a different role in RME than in the more conventional teaching approaches (Gravemeijer, 1999). In the traditional teaching approaches, models are most often used as didactic embodiments of the formal Mathematics that is taught. This means that the formal Mathematics is concretized in order to be made more tangible for the students (Gravemeijer, 1999). The RME philosophy advises that instruction should be designed to use models and diagrams to build learners' reasoning gradually from the concrete to the abstract, thereby reinforcing learners' reasoning (Fitzallen & Watson, 2014). For RME to work, learners need to know how to model new situations (Boon, 2006). Models are able to support mathematization of the intended concepts (Doorman et al., 2008). The models should be generalised to develop into entities of their own, which then can be used as models for mathematical reasoning (Doorman et al., 2008; Widjaja & Heck, 2003).

Virtual manipulations which are facilitated by computer software are similar to physical manipulations which means that ICT can effectively support the learning process, especially in Mathematics (Zaranis et al., 2013). Technology can successfully enhance the meaning of symbolic information by making it more visual and more easily manipulated, which makes the content more accessible (Gravemeijer, 2012).

3.2.5.2 Strategies

RME focuses on the development of abstract thinking (Fitzallen & Watson, 2014) by using context problems to encourage students to develop situation-specific representations and strategies (Doorman et al., 2008). The RME approach can assist in creating a positive classroom climate which

allows for interaction and collaboration (Fauzan et al., 2013; Wijers et al., 2010). Learners should adopt the necessary classroom social norms to suit an inquiry-oriented classroom culture (Gravemeijer, 2012). Learners enjoy performing experiments during lessons rather than just listening to the teacher and doing written exercises (Widjaja & Heck, 2003). This tends to create a livelier, noisier and messier classroom, but does provide a more enjoyable atmosphere for teachers and learners to work in, and learners tend to be more enthusiastic about the work (Widjaja & Heck, 2003; Wijers et al., 2010).

An inquiry-oriented classroom creates the opportunity for learners to explain and justify their solutions, to try and understand other learners' reasoning and to ask questions if they do not understand (Gravemeijer, 2012). Apart from these inquiry-based norms, learners should also be prepared to solve mathematical problems, discuss solutions and deliberate about the core concepts (Gravemeijer, 2012). An interactive classroom allows for learners to discover things for themselves, talk actively with one another and with the teacher, and to be more aware of their own mathematical thinking (Widjaja & Heck, 2003).

It is important that teachers are willing to adapt their role in the classroom by moving towards more learner-centred learning (Widjaja & Heck, 2003). Teachers often are required to improve their skills in conducting class discussions, addressing questions that challenge learners and encourage them to think critically (Widjaja & Heck, 2003). The shift from the more traditional teaching approach to a more inquiry-driven approach means that cooperation, problem solving and metacognition are promoted (Gordon et al., 2012; Mousoulides et al., 2007).

3.2.6 Mathematical Content

According to Gravemeijer (1999), one of the reasons that learning Mathematics has been branded as difficult is that the gap between everyday life and formal Mathematics is vast. Formal Mathematics should grow out of learners' activities and should not be seen as something abstract and removed from reality (Gravemeijer, 1999). Within the RME approach, learners are expected to develop formal Mathematics by mathematizing their informal mathematical activities (Gravemeijer, 1999).

The RME approach distinguishes itself from other approaches in that it surpasses the contrast between informal and formal knowledge. The design of a hypothetical learning trajectory (HLT) can assist students in reinventing formal Mathematics (Gravemeijer & Doorman, 1999). Ideally the learning trajectory develops in such a way that the formal Mathematics surfaces in the activities that the students do (Gravemeijer & Doorman, 1999; Heck et al., 2008). In the RME philosophy, different Mathematics topics should be integrated in one curriculum. This promotes the idea of an integrated view of Mathematics, and sanctions flexibility to connect sub-divisions of the curriculum with each other and also to connect abstract Mathematics to meaningful problems in the real world (Widjaja & Heck, 2003).

The intention of the RME approach is that students should experience formal Mathematics and informal Mathematics in the same way (Gravemeijer, 1999). What students ultimately understand of formal Mathematics should remain associated with their understanding of the everyday phenomena that they experience (Gravemeijer & Doorman, 1999).

In the RME approach, it is essential that formal Mathematics can be developed as an instrument for solving real problems for the students (Boon, 2006). Real contexts that are meaningful to learners should be used as a starting point for their learning. Situations in which concepts appear in reality should be used as a starting point for concept formation and should lead to more formal Mathematics (Doorman et al., 2013; Widjaja & Heck, 2003). Technology can also assist in bridging the gap between formal and informal Mathematics. Mathematical applets are able to visualize mathematical concepts and explore context situations (Heck et al., 2008).

It is important in the RME approach that Mathematics teaching is a process in which students develop and apply mathematical concepts in real life problem situations (Domazet et al., 2013). The development of algebraic understanding and skills is an important topic in Mathematics education; however it is an obstacle for many students (Heck et al., 2007). One of the reasons for this is that algebraic manipulations are often carried out in a superficial way where rules are used but are not meaningfully connected to the underlying concept (Heck et al., 2007). Various programmes have been designed and developed to help learners practice algebraic skills in ways that make sense (Heck et al., 2007).

Geometry is often the most difficult topic in school Mathematics for learners and teachers (Fauzan et al., 2013; Stols, 2012). Learners' poor performance in Geometry, as well as their negative attitude to Geometry is a challenge in many classrooms (Fauzan et al., 2013). Making use of real contexts and data that students can relate to is essential in statistics education (Fitzallen & Watson, 2014). Technology can provide many options for representing graphical data which can help improve student learning in statistics (Fitzallen & Watson, 2014). The appropriate use of technology can help students to understand mathematical concepts more effectively (Zaranis, 2014b). According to the principles of RME, learners should mathematize everyday subject matter as well as their own mathematical activity (Gravemeijer & Doorman, 1999).

3.2.7 Learner Aspects

The theory of RME is predominantly about knowledge construction (Gravemeijer, 1999). Activities based on the RME approach allow for learners to build their own mathematical knowledge (Gravemeijer, 1999). When designing activities, teachers should be sensitive about where their learners are in terms of their anticipated learning so that activities can be based on ideas that are less sophisticated than the Mathematics being taught (Kizito, 2012a). Through interaction with each

other, students can enhance the understanding of mathematical concepts and their application in real life, which is an important characteristic of RME (Cahyono, 2012).

Social interaction, one of the five tenets or characteristics of RME, is an essential part of RME (Cahyono, 2012; Widjaja & Heck, 2003; Zaranis, 2014a, 2014c; Zulkardi, 2000). The aim of RME is to allow learners to consider the knowledge that they acquire as knowledge for which they are responsible. For this to happen, certain social norms should be in place, such as "you should not guess the answers," and "you need to work things out yourself" (Gravemeijer & Doorman, 1999). The learning process is enhanced by social interaction in the form of discussions, between learners and learners and learners and teachers (Cahyono, 2012; Widjaja & Heck, 2003). Interaction can promote the understanding of mathematical concepts and the way in which those are applied in real life (Cahyono, 2012). Modelling activities are essentially social experiences because learners work together in groups to produce something that can be shared (Mousoulides et al., 2007; Zaranis, 2014a).

Digital learning activities can inspire learners to work collaboratively (Zaranis et al., 2013). A learning environment that includes a mobile device has the potential to encourage collaborative work in learners (Kizito, 2012a; Zaranis et al., 2013). A virtual learning environment (VLE) provides an opportunity for learners to interact with one another and with the teacher (Cahyono, 2012)

3.2.7.1 Learner Understanding

By allowing learners to use informal strategies to solve problems, and allowing for discussion with fellow pupils and the teacher, the process of understanding is stimulated (Widjaja & Heck, 2003). When learners apply their mathematical knowledge in a tangible context and in a meaningful way, it leads to a deepening of their knowledge and consolidation of the concepts involved (Heck, 2009).

In RME, the holistic view of learning is emphasised where students' needs are addressed and the realisation of educational goals and purposes are facilitated (Fitzallen & Watson, 2014). RME focuses on the development of abstract thinking which assists in relating ideas in the practical context of a situation (Fitzallen & Watson, 2014). The use of the RME approach has the potential to improve learners' understanding of Mathematics (Cahyono, 2012). A benefit of using the RME approach is that learners can use the new knowledge and skills that they acquired from one lesson in the lessons that follow (Fauzan et al., 2013).

Educational technologies provide alternatives for presenting data through different representations, thereby improving student learning (Fitzallen & Watson, 2014). Computer-assisted learning can augment the development of mathematical skills and can encourage a deeper perceptual aptitude for learners (Zaranis et al., 2013). Tools influence the way in which students make sense in Mathematics (Doorman et al., 2008)

3.2.7.2 Learner Reasoning

Poor performance of learners, especially in international testing, can possibly be attributed to the instructional setting (Widjaja & Heck, 2003). Learners' own constructions are an important part of the learning process because they form the basis of classroom discussions and interactive reflection (Wijers et al., 2010). Learner achievement improves as a result of the intervention of an RME course (Fauzan et al., 2013). Learners' reasoning can be reinforced by using models, diagrams and other imagery (Fitzallen & Watson, 2014).

The interactive classroom encourages learners to explore things for themselves, talk more actively and become more conscious of their own mathematical thinking (Widjaja & Heck, 2003). When working with symbols, it is important for teachers to remember that students do not always have the mathematical background to interpret those symbols correctly. The teacher should explain to students what can be seen, how it can be interpreted and how they need to reason with those symbolic representations (Bakker et al., 2003).

3.3 Realistic Mathematics Education (RME)

Figure 3.4 represents the theme of RME with its sub-themes as categorised in the systematic literature review.

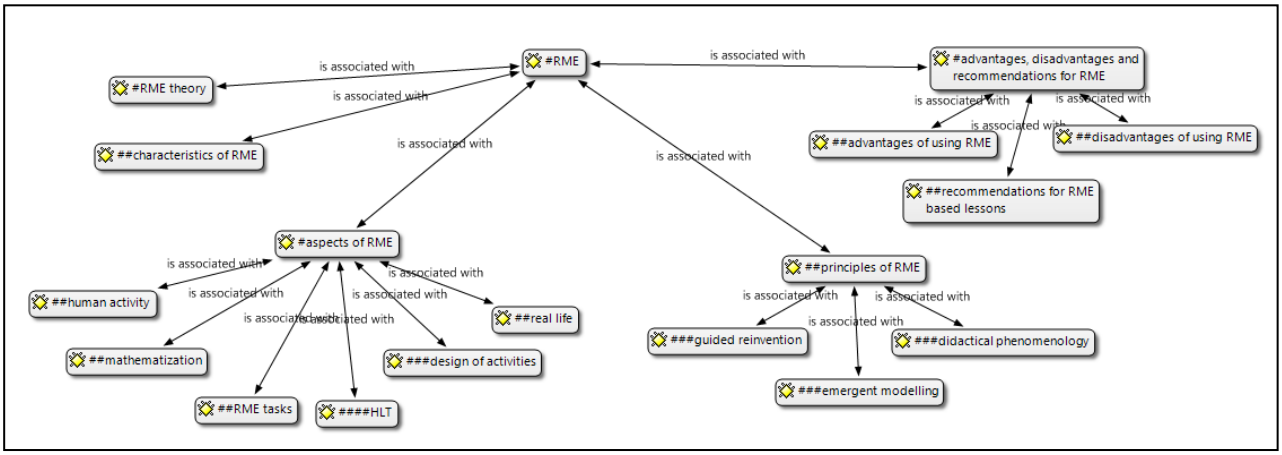


Figure 3.4: Realistic Mathematics Education (RME) as Theme from the Systematic Literature Review

According to Freudenthal (1991), students should be given the opportunity to reinvent Mathematics through a process of mathematization (Cobb et al., 2008; Doorman et al., 2008; Doorman et al., 2013; Gravemeijer, 1999), where they formalize their informal understandings and intuitions (Cobb et al., 2008) and are supported in re-creating or reinventing Mathematics (Doorman et al., 2013). The

central idea of Realistic Mathematics Education (RME) is that instruction is designed in which everyday situations and mathematical activities are mathematized. This allows for learners to reinvent Mathematics and thereby realise a better understanding of the subject (Kizito, 2012a). One of the aims of RME is to provide learners with the opportunity to consider the knowledge that they acquire as their own (Boon, 2006).

3.3.1 Realistic Mathematics Education Theory

RME, a teaching and learning theory in Mathematics education (Zulkardi, 2000), was based and developed on Freudenthal's view of Mathematics as a human activity (Cahyono, 2012; Cobb et al., 2008; Dolk et al., 2002; Domazet et al., 2013; Doorman et al., 2008; Doorman et al., 2013; Freudenthal, 1991; Kizito, 2012a; Perez, 2012; Widjaja & Heck, 2003; Zaranis, 2014c; Zulkardi, 2000). Various studies in Mathematics education have made use of the theoretical framework based on the RME approach (Andresen, 2007; Heck et al., 2007; Kizito, 2012a; Zaranis, 2014a).

RME is a foremost view on the learning of Mathematics in The Netherlands (Doorman et al., 2007). The main aim of the Freudenthal Institute is to do research and curriculum development so as to promote innovation and improve Mathematics education (Boon, 2006). The RME principles of discovering learning in a real context, where Mathematics is seen as a human activity, and where students make sense through participation in mathematical representation, is greatly treasured (Heck, 2009). The RME theory is a vision or philosophy of Mathematics that is being developed in an on-going process (Gravemeijer, 1999; Zaranis, 2014a). It is not fixed, but involves a process of experimentation, analysing and reflecting and is constantly under construction (Gravemeijer, 1999). RME develops in an iterative process of expansion, adjustment and refinement (Gravemeijer, 1999) and relates to the work of Van Hiele (Gravemeijer, 1999).

RME is more than simply situating Mathematics in a real world situation; instead the instructional tasks should draw on realistic situations as a foundation for students' mathematization. The activities should be designed in such a way that students can systematise their activity within the realistic context to reinvent significant Mathematics (Stephan & Akyuz, 2014).

3.3.2 Characteristics of Realistic Mathematics Education

Various researchers highlight the five tenets or characteristics of RME (Cahyono, 2012; Dolk et al., 2002; Drijvers & Doorman, 1996; Widjaja & Heck, 2003; Zulkardi, 2000), namely:

i) the use of contextual problems
ii) the use of models
iii) the use of student contribution
iv) interactivity (among pupils and between pupils and teacher)
v) intertwining of learning strands.

Real contexts that are meaningful and natural to pupils are important to use as a starting point for learners' learning (Boon, 2006; Dolk et al., 2002; Widjaja & Heck, 2003; Zulkardi, 2000). These contexts can be concrete or abstract and should be explored to develop intuitive notions that form the foundation of concept formation (Stols, 2012). This allows them to engage with the situation immediately. It is imperative that the instruction should not start with the formal Mathematics and end with the application in a real context, but rather that the contexts should be used as anchors for the formation of concepts (Widjaja & Heck, 2003). When using personalised contexts, where the meaningfulness of the context increases, students' solution of word problems improve and student motivation increases (Gravemeijer & Doorman, 1999).

Using models or bridging by vertical instrumentation is an important aspect in RME. Learners should develop and use models when solving problems as a bridge between what is abstract and what is real. Initially the model is a representation of what is familiar to learners, later by generalization and formalization, it becomes an object on its own that is used for mathematical reasoning and can assist in developing sophisticated mathematical concepts (Dolk et al., 2002; Widjaja & Heck, 2003). Learners should be given the opportunity to create concrete objects which assists in informal problem solving strategies. Teachers should design activities and instructional material in such a way that learners are guided through a bottom-up reinvention process (Widjaja & Heck, 2003).

Interaction in the classroom is an important aspect in RME since it encourages discussion and collaboration, both important aspects when reflecting on the work. The kind of interaction that takes place is explaining, justifying, agreeing and disagreeing, questioning alternatives, and reflecting (Widjaja & Heck, 2003). The integration of different topics in Mathematics into one curriculum is imperative in the RME philosophy. Learners should have a cohesive view of Mathematics as well as the flexibility to associate content with other disciplines or domains and also to problems in the real world (Widjaja & Heck, 2003).

With the introduction of RME principles in the classroom, interactivity can be promoted as learners take more responsibility for their learning (Widjaja & Heck, 2003). The classroom is inclined to become more energetic as learners do experimental work and are involved in discussions and debates (Widjaja & Heck, 2003). The open computer learning environment can assist with interactivity as learners explore on their own, and tend to be more aware of their own mathematical thinking (Widjaja & Heck, 2003).

A high quality RME-based course requires three quality criteria, namely: validity (content validity where the intervention includes new knowledge, and construct validity where interventions are linked to each other); practicality which means that teachers and learners consider the interventions to be engaging in normal conditions; and effectiveness which refers to the experiences and outcomes that are consistent with the proposed aims (Cahyono, 2012; Fauzan et al., 2013).

In a study that investigated the teaching of triangles and circles, with and without technology, teaching intervention was based on the characteristics of the theory of RME and the van Hiele model. These characteristics include: the introduction of a problem using a realistic context; the main objects of the problem are identified; social interaction and teacher intervention to refine the models of the problem; encouraging the process of reinvention; and focusing on the connections and aspects of Mathematics in general (Zaranis, 2014a).

3.3.3 Aspects of Realistic Mathematics Education

3.3.3.1 Mathematics as a Human Activity

Various researchers refer to Freudenthal's work (Freudenthal, 1971) when discussing his view of Mathematics as a human activity (Cahyono, 2012; Cobb et al., 2008; Dolk et al., 2002; Domazet et al., 2013; Doorman et al., 2008; Freudenthal, 1971; Kizito, 2012a; Perez, 2012; Widjaja & Heck, 2003; Zaranis, 2014c; Zulkardi, 2000), rather than a man-made system (Kizito, 2012a). Mathematics must be connected to reality (Cahyono, 2012; Widjaja & Heck, 2003; Zulkardi, 2000), where learning is viewed as a process within a social context (Perez, 2012) that can be used as a source for mathematization (Widjaja & Heck, 2003). Students should be supported (Doorman et al., 2013) and encouraged to reinvent Mathematics (Cobb et al., 2008; Dolk et al., 2002; Doorman et al., 2008).

According to Freudenthal (1968), in order for Mathematics to be of human value, it should be taught so as to be useful. This means that it should be closely related to reality, close to children and relevant to society (Zaranis, 2014a). Students should be able to understand the concepts of Mathematics and its application in the real world (Cahyono, 2012),

3.3.3.2 Mathematization

Freudenthal's view of Mathematics as a human activity (Freudenthal, 1968) suggests that Mathematics can be used as a source for mathematization (Widjaja & Heck, 2003), which means organising matter from a mathematical perspective (Gravemeijer & Doorman, 1999). The process of developing mathematical concepts and ideas that originate in a concrete situation in the real world is known as mathematization (Cahyono, 2012; Freudenthal, 1968; Zulkardi, 2000). The process forces students to explore the situation, find and identify relevant Mathematics, schematize and visualize, in order to develop a model which leads to the Mathematics concept. They can then apply mathematical concepts to new areas of the real world, and so reinforce the concept (Zulkardi, 2000).

Students should learn Mathematics by mathematizing subject matter from reality and their own mathematical activity (Freudenthal, 1968; Gravemeijer, 1999). Freudenthal highlights the importance of mathematical activities that are taken from reality, that consist largely of organizing, mathematizing

subject matter (Cahyono, 2012; Freudenthal, 1968). He believes that mathematizing can comprise the mathematizing of both subject matter from real life situations as well as mathematical subject matter (Freudenthal, 1968; Gravemeijer & Doorman, 1999).

Treffers (1978) differentiates between two kinds of mathematization, namely horizontal and vertical mathematization (Freudenthal, 1991; Treffers, 1978; Widjaja & Heck, 2003; Zulkardi, 2000). Horizontal mathematization refers to the modelling of experientially real situations into Mathematics and vice versa (Freudenthal, 1991; Treffers, 1978; Widjaja & Heck, 2003). This is the process in which learners convert problem situations that they perceive as real into some form of Mathematics. They use and create mathematical tools to organise and solve mathematical problems located in real life situations (Freudenthal, 1991; Treffers, 1978; Widjaja & Heck, 2003; Zulkardi, 2000). Vertical mathematization refers to the process of accomplishing a higher level of abstraction within Mathematics (Widjaja & Heck, 2003). This refers to reorganization within the mathematical system, such as finding short cuts, generalizing methods, and making connections between concepts (Freudenthal, 1991; Treffers, 1978; Widjaja & Heck, 2003). Freudenthal (1991) describes the two as follows: horizontal mathematization moves from the world of life to the world of symbols, while vertical mathematization means moving within the world of symbols (Widjaja & Heck, 2003). Progressive mathematization, also known as vertical and horizontal mathematization, allows learners to rediscover the knowledge, insight and mathematical procedures (Cahyono, 2012).

With regard to mathematization, Treffers (1978) divides Mathematics education into four types, namely the mechanistic, empiristic, structuralist and realistic approach. Firstly, in the mechanistic or traditional approach, horizontal and vertical mathematization are seldom or never used. Secondly in the empiristic approach, students are faced with situations in which they have to do horizontal mathematization, but do not need to produce a model or formula. Thirdly, in the structuralist approach, which is based on set theory, flowchart and games, the approach relates to horizontal mathematization in a created world, and is not linked to the real world. Finally, in the realistic approach, a real world situation is taken as the starting point for learning Mathematics (it is explored using horizontal mathematization), students then organize the problem, identify regularities and relations and by using vertical mathematization, they develop concepts (Freudenthal, 1991; Treffers, 1978; Zulkardi, 2000).

Zulkardi (2000) distinguishes between conceptual mathematization and applied mathematization. Conceptual mathematization is the process of extracting the applicable concept from a real situation. The process compels students to explore the situation, identify the relevant Mathematics, systematise and visualise to develop a model resulting in a mathematical concept. Applied mathematization occurs when the concept is applied to new areas of real life in which the concept is supported and reinforced (Zulkardi, 2000).

In the RME approach, learners are given the opportunity to create concrete objects that can be used as a point of departure for mathematization and the subsequent formalization of the informal strategies (Gravemeijer, 1999; Widjaja & Heck, 2003). Activities that stimulate the generation of a model can lead to imperative forms of learning because they involve mathematization (Mousoulides et al., 2007). The mathematization occurs by counting, dimensioning, classifying, algebraising, and organising different objects, associations, actions, arrangements and regularities (Mousoulides et al., 2007).

The process of solving contextual problems starts with the real world problem which the learner then will mathematize, reflect on the problem, do abstraction and formalisation and then return to the real world at the end of the process (Cahyono, 2012). The real world should be truly authentic for mathematization to be most effective (Doorman et al., 2007). Educational designs should be based on carefully chosen real world situations that can easily be mathematized (Boon, 2006). Software is an ideal means to stimulate such situations to recall familiarities and extend them with new ones (Boon, 2006).

3.3.3.3 The Use of Tasks in Realistic Mathematics Education

Tasks are defined as problematic situations that are experientially real for students. This means that they can experience the problem that they encounter, as real (Stephan & Akyuz, 2014). They need to be appropriate for mathematization (Stephan & Akyuz, 2014) and should be designed in such a way that students are encouraged to reinvent key mathematical concepts (Stephan & Akyuz, 2014).

Teachers involved in the design of instructional material participate in various ways: anticipating supportive mathematical imagery; creating challenging formative assessment; making use of their mathematical knowledge to change and improve the instructional sequence; and working and revising tasks that were already created (Stephan & Akyuz, 2014). Apart from the instructional sequence itself, which is considered the most important aspect by teachers in guiding their actions, lesson imaging is also an important aspect. Lesson imaging allows teachers to envision how students will solve problems, and how this can be exploited to result in powerful mathematical discussions, where objectives can be met (Stephan & Akyuz, 2014). Collaboration plays a crucial role in teachers' practices in their own success in the classroom (Stephan & Akyuz, 2014).

3.3.3.4 Hypothetical Learning Trajectory (HLT)

The hypothetical learning trajectory (HLT) refers to the consideration of the learning goal, the tools they will use, if technology is involved (Perez, 2012; Simon, 1995), the learning activities, and the thinking and learning in which students might engage (Perez, 2012; Simon, 1995). Originally, HLTs were designed for short teaching cycles of one or two lessons, but later were extended to sequences

of lessons varying from eight to twenty lessons (Bakker et al., 2003). The concept of HLT is a highly flexible instrument suitable when discussing different aspects of teaching and learning (Perez, 2012).

The development of an HLT involves the design of instructional activities that relate to the assessment of the initial level of understanding, the formulation of the end result and the envisaged mental activities of the students (Bakker et al., 2003). HLTs allow for students to reinvent formal Mathematics (Gravemeijer, 2012). They are designed in such a way that the formal Mathematics emerges from the mathematical activity that the students are involved in (Gravemeijer, 2012). The HLT should not have a rigid structure and should not expect students all to follow the same learning trajectory at the same speed; the HLT should represent a broader learning route (Bakker et al., 2003). The development of the HLT and the design of instructional activities are closely related.

The HLT guides the design of the instructional activities, however within the design process, the HLT may be reconsidered and adapted (Bakker et al., 2003). The HLT will normally be improved after a teaching experiment, based on classroom experiences. Thereafter, a new teaching cycle is started, and within design-based research, a new research cycle is also started (Bakker et al., 2003). An aspect of HLT that challenges designers is that each tool, although seemingly similar, can have varying effects on how students interact in the activity (Perez, 2012). What is important is that students' investigations, reasoning and solutions are shared, discussed and generalised to other problem situations (Boon, 2006).

The HLT deals with a small number of instructional activities and is tailored to the teacher's own classroom at any given moment (Gravemeijer, 1999). The HLT is important to teachers at the start of a project to help in orientating them (Stephan & Akyuz, 2014). The hypothetical nature of these learning trajectories implies that teachers need to examine students' reactions to determine whether the actual learning trajectory corresponds with what was intended (Bakker et al., 2003; Gravemeijer, 2012). As a result, teachers should interpret and create new instructional activities (Gravemeijer, 2012), depending on whether the conjectures can be verified or should be rejected (Bakker et al., 2003).

The HLT is a suitable research instrument in observing the development of instructional design activities and the associated hypotheses within design-based research (Bakker et al., 2003). As students' mental activities are emphasised, the expected results are motivated (Bakker et al., 2003). If an HLT is designed where technology is integrated, the designer ought to be knowledgeable about the tool and also how teachers and students perceive the use of technology in a learning activity (Perez, 2012).

3.3.3.5 Design of Activities

When designing learning activities, it is important to remember that the activity should encourage the students' work with Mathematics (Andresen, 2007). Designers should make use of their domain specific knowledge, teaching experience and their view on the teaching and learning of the topic when designing activities for the students (Bakker et al., 2003). The design of instructional activities is an essential part of executing a teaching experiment (Bakker et al., 2003). Instructional tasks should be designed so that students are given the opportunity to reinvent important mathematical concepts (Stephan & Akyuz, 2014). The development of an HLT is related to designing instructional activities (Bakker et al., 2003). Unlike with HLT's which are hypothetical, when instructional activities are performed, the teacher should look for evidence of whether the conjectures can be verified or not (Bakker et al., 2003).

3.3.3.6 Real Life Contexts

Context problems are defined as problems in which the problem situation is experientially real to the student (Gravemeijer & Doorman, 1999; Widjaja & Heck, 2003). Solving problems in realistic contexts, which students recognise as relevant and real, and which offer opportunities to develop situation-specific reasoning and tentative representations for organizing repeated calculations, plays a crucial role in the RME approach from the start (Boon, 2006; Doorman et al., 2008; Gravemeijer & Doorman, 1999; Stephan & Akyuz, 2014). In RME learners have the opportunity to learn Mathematics based on activities they experience in their everyday lives (Fauzan et al., 2013).

The students' understanding of the formal Mathematics should be entrenched in their understanding of the everyday-life occurrences (Gravemeijer & Doorman, 1999). Authentic contexts and real data that students can relate to are especially important in statistics (Fitzallen & Watson, 2014). This allows for optimal learning opportunities (Fitzallen & Watson, 2014). Students' solutions to word problems improve when personalized contexts are used. This enhances the meaningfulness of the context and can increase motivation (Gravemeijer & Doorman, 1999). Didactical phenomenology presents students with phenomena that need to be organised by mathematical structures. These phenomena can best be generated by meaningful contexts from real life which should occur naturally and with meaning (Bakker et al., 2003).

Even a pure Mathematics problem can be considered a context problem, if the Mathematics involved is experientially real for the student (Gravemeijer & Doorman, 1999). Context problems can act as anchoring points for students to learn and reinvent Mathematics (Gravemeijer & Doorman, 1999; Stephan & Akyuz, 2014; Widjaja & Heck, 2003), and should be selected so that they provide students with the opportunity to develop situation specific strategies (Bakker et al., 2003; Doorman et al., 2008). Well selected context problems provide opportunities for students to develop informal, context-specific solution strategies (Gravemeijer & Doorman, 1999; Widjaja & Heck, 2003).

Context problems form the foundation for progressive mathematization (Gravemeijer & Doorman, 1999) and should be observed as natural and meaningful in order to offer a foundation for the mathematization (Bakker et al., 2003). Solving contextual problems starts from real life problems that students mathematize and reflect on, form abstractions and formalise, then return to the real world at the end of the process (Cahyono, 2012).

Context problems are rooted in students' reality, yet by solving these context problems students are able to expand their own reality (Gravemeijer & Doorman, 1999). They allow for students to develop informal, context-specific models and strategies for solving the problems, which can later be formalized and generalized to establish a process of progressive mathematization (Doorman et al., 2008) and give students the opportunity to develop representations and strategies that relate to the situation (Doorman et al., 2008).

In the RME approach, models should be grounded in the contextual problems that students should solve (Gravemeijer, 1999). Context problems that offer students opportunities to develop situation-specific strategies should be chosen. The models that develop from these contexts emerge from the students' activities (Gravemeijer & Doorman, 1999). Modelling activities should take place in a situation that is experientially real for the students (Andresen, 2007; Doorman et al., 2008). Mathematical ideas should be fixed in meaningful real world contexts and are developed by students as they work on problems (Mousoulides et al., 2007). These ideas should be close to children's reality and should be relevant to everyday situations in life (Zulkardi, 2000).

Context problems are designed to support the reinvention process that gives students an opportunity to come to terms with formal Mathematics (Gravemeijer & Doorman, 1999). Activity sheets are one way in which students can be guided to advance from informal to more formal activities (Heck et al., 2008). RME not only uses contexts in the teaching and learning of Mathematics, but attaches substantial significance to Mathematics teaching as a process in which students develop and apply mathematical concepts and tools in realistic contexts (Domazet et al., 2013). The RME approach uses real world contexts in problem-solving to encourage motivation in students because the Mathematics is linked to the real world (Fauzan et al., 2013; Risnawati et al., 2014).

Realistic contexts are vital in not only developing sophisticated symbols, but also in developing meaning within the symbol (Bakker et al., 2003). Lessons should be designed to allow learners to use their knowledge of the context to support their thinking (Fitzallen & Watson, 2014). Instruction that is designed where everyday situations and mathematical activities are mathematized, assists students in reinventing Mathematics, and therefore achieving a better understanding of the subject (Kizito, 2012a). Many learners dislike Mathematics because they do not learn something useful and significant to them (Fauzan et al., 2013).

Digital technology should be entrenched in a context that is comprehensible and in which the technology is integrated in a natural way (Drijvers, 2013). It should be simple, attractive, easily accessible and close to the student's life (Risnawati et al., 2014). Real world situations should be chosen in such a way that software can simulate the situations which allow for mathematization to take place (Boon, 2006). ICT tools allow learners to utilise new learning platforms and also assist them in realising new knowledge through activities related to real life scenarios (Zaranis et al., 2013). ICT can assist in realising realistic analysis by allowing for the following: collecting good quality real-time data; analysing and visualising the data; doing otherwise complex calculations; building and using computer models; comparing results and producing reports (Heck, 2009). The graphics calculator can stimulate the use of realistic contexts (Drijvers & Doorman, 1996) and the mobile phone has the potential to allow students to engage in real, authentic tasks (Kizito, 2012a). Over and above the use of contexts in Mathematics teaching and learning, it is imperative in the RME approach that Mathematics is taught as a process in which students develop and apply Mathematical concepts and tools in problem situations with a realistic context (Domazet et al., 2013).

3.3.4 Principles of Realistic Mathematics Education

The three principles or key heuristics of RME that inform the design of instructional sequences are: guided reinvention or progressive mathematization, didactical phenomenology and emergent modelling (Andreasen, 2006; Bakker et al., 2003; Cahyono, 2012; Dolk et al., 2002; Drijvers et al., 2013; Stephan & Akyuz, 2014; Widjaja & Heck, 2003). These three principles should guide the design process to ensure that meaningful problems are used to foster students' cognitive development (Bakker et al., 2003). The instructional designer requires knowledge of the suitable design principles as well as their application in different context-specific situations (Doorman et al., 2013).

3.3.4.1 Guided Reinvention

The principle of guided reinvention (or progressive mathematization (Andreasen, 2006)), the first instructional design principle in RME (Gravemeijer, 1999), implies that learners should be given the opportunity to develop their own Mathematics (Bakker et al., 2003; Drijvers et al., 2013; Gravemeijer, 1999; Gravemeijer & Doorman, 1999). They should be allowed to experience a process which is similar to that by which a given mathematical topic was invented (Drijvers et al., 2013). In guided reinvention, learners will devise informal strategies to solve problems that may be formalized and generalized to establish a process of further abstraction (Doorman et al., 2008). The process requires guidance from the teacher who can help to direct the progression in a sensible manner (Drijvers et al., 2013).

In guided reinvention, Freudenthal (1991) believes that the emphasis should be on the character of the learning process, rather than on the invention itself (Gravemeijer & Doorman, 1999). Learners

should consider the knowledge that they acquire as their own, and should take responsibility for that knowledge (Gravemeijer & Doorman, 1999).

Guided reinvention encompasses reconstructing the expected way of developing a mathematical concept from a given problem situation (Bakker et al., 2003). The process begins with the mathematizing of real life subject matter, and leads to reinvention where students mathematize their own mathematical activities (Gravemeijer, 1999; Gravemeijer & Doorman, 1999). Formal Mathematics develops from students' activities (Gravemeijer, 1999). Guided reinvention can involve the development of new problem solving strategies which are necessary to solve tasks which cannot be solved with the existing strategies (Drijvers et al., 2013). Guided reinvention provides a manner in which to bridge the gap between formal and informal Mathematics (Gravemeijer & Doorman, 1999). This process, which provides students with the opportunity to reinvent Mathematics, surpasses the contrast between Mathematics as an activity and Mathematics as a body of knowledge (Gravemeijer, 1999).

Context problems play a key role in the reinvention process. Well-selected context problems provide students with opportunities to develop informal solution strategies that are relevant for that specific context (Gravemeijer & Doorman, 1999). Students are not expected to reinvent everything themselves (Gravemeijer & Doorman, 1999). The teacher's role is to facilitate learners through the process of guided reinvention, and provide students with the opportunity to reflect on the strategies they invented (Widjaja & Heck, 2003). It is important in teacher training, when learning about the reinvention process, that student teachers experience mathematization and reinvention first hand (Dolk et al., 2002). This can be extended to include activities such as anticipating learners' solutions and analysing their work (Dolk et al., 2002).

Guided reinvention is also applicable in digital design. ICT provides opportunities for exploration and investigation which is well suited to reinvention (Drijvers et al., 2013); however in some cases, technology may constrain students' opportunity for exploration as a result of pre-designed tools that provide too much guidance (Drijvers et al., 2013). There can also be tension between the reinvention process, which is considered to be bottom-up, and the nature of ICT, which tends to be top-down (Bakker et al., 2003).

3.3.4.2 Emergent Modelling

The term emergent modelling refers to the notion that models emerge from the activity of the students, and furthermore the required formal Mathematics emerges in the process (Gravemeijer, 1999; Gravemeijer & Doorman, 1999). The principle of emergent modelling focuses on the use of models during different phases of activities. A model is initially context-specific (Doorman et al., 2008; Doorman et al., 2013; Drijvers et al., 2013). This refers to a model which represents a meaningful, experientially real problem situation (Doorman et al., 2013; Drijvers et al., 2013; Kizito, 2012a). The

second phase continues as the learner works with the model where it gradually acquires a more generic character and progresses into a model for mathematical reasoning. This is possible because the model starts to refer to new mathematical objects in a more abstract framework of mathematical relations (Drijvers et al., 2013).

In the instructional design theory for RME, models are used to promote a process in which formal Mathematics is reinvented by the students themselves (Gravemeijer, 1999). It can be challenging for the designer of the tasks to find appropriate problem situations that are suited to the development of such models (Bakker et al., 2003; Drijvers et al., 2013; Gravemeijer, 1999). Instructional activities should be designed in such a way that students are able to shift from reasoning with models of informal activities to actual modelling of formal mathematical objects, relations and activities (Bakker et al., 2003; Doorman et al., 2013; Gravemeijer, 1999; Heck et al., 2007; Stephan & Akyuz, 2014). Thereby the formal Mathematics is made more accessible for the students (Gravemeijer, 1999). The task designer should support students in this transition from informal to formal by using tools that students have created, to explain their mathematical reasoning (Stephan & Akyuz, 2014).

Modelling and symbolising is an integral part of activities that aim to deal with problem situations (Gravemeijer & Doorman, 1999). The development of the model and the symbolization thereof complement the development of mathematical conceptualization of the problem situation (Gravemeijer & Doorman, 1999). Models are not derived from the intended Mathematics, but rather are grounded in the contextual problems that the students need to solve (Gravemeijer, 1999). The principle that underpins this notion is that students who work with the models will be motivated to reinvent the formal Mathematics (Gravemeijer, 1999). The purpose of using models is to bridge the phenomenological appearances of Mathematics in reality with formal Mathematics (Gravemeijer, 1999).

When analysing the design principle of emergent modelling, Gravemeijer (1999) refers to three interrelated processes. Firstly the process of the overall transition whereby the model emerges from informal mathematical activity, but gradually progresses into a model for more formal mathematical reasoning. The second process comprises the establishment of a new mathematical reality. Finally, the third process involves the emergence of a sequence of signification which serves as a concrete demonstration of the model in a sequence of signs, where each new sign can signify an activity with a previous sign (Gravemeijer, 1999). Emergent modelling can be a useful principle in digital design (Drijvers et al., 2013). When dealing with paper and pencil design, the models need to be suitable for further development in moving towards complexity and mathematical abstraction. In digital design, the emerging models must be supported by the digital tools that are available (Drijvers et al., 2013).

3.3.4.3 Didactical Phenomenology

The principle of didactic phenomenology is concerned with the relation between the thought object and the phenomenon itself (Drijvers et al., 2013). It deals with the concept of how mathematical structures can assist in organising and structuring phenomena in real life (Bakker et al., 2003; Drijvers et al., 2013; Zulkardi, 2000). Students are invited to develop mathematical concepts in this process (Bakker et al., 2003). The designer of the tasks is required to find meaningful phenomena that need to be structured and organised by the Mathematics (Drijvers et al., 2013).

The three RME principles can be well-utilised in digital design, although some adjustments may be necessary as compared to paper and pencil design (Drijvers et al., 2013). These principles however, are essential for digital design, despite the constraints of the digital tool (Drijvers et al., 2013). When dealing with digital design, the principle of didactical phenomenology is also valuable, however the phenomena that are dominant in the tasks do not necessarily come from real life (Drijvers et al., 2013). The student must contend with not only the world of Mathematics, but also ICT as a separate entity. Drijvers et al. (2013) are of the opinion that making use of a real life context together with these two domains can cause a cognitive overload.

3.3.5 Advantages, Disadvantages and Recommendations for Realistic Mathematics Education

3.3.5.1 Advantages

RME-based courses can help to improve learners' motivation, their understanding of Mathematics (Cahyono, 2012; Fauzan et al., 2013), their attitude towards Mathematics, their reasoning, activity and creativity and can assist in improving the classroom climate (Fauzan et al., 2013). RME is able to promote interactivity and collaboration in the classroom (Widjaja & Heck, 2003).

By following the RME approach, learners are able to make noteworthy progress in their performance in Mathematics, and both teachers' and learners' opinions about the teaching and learning activities are inclined to be positive (Widjaja & Heck, 2003). Learners tend to be more involved in the lessons and are more willing to try new and different things in classwork, and are less easily bored (Widjaja & Heck, 2003). Where technology is involved, learners become accustomed to new technologies (Widjaja & Heck, 2003). RME allows learners the opportunity to be actively involved in the learning process and promotes learning experiences that start with rich contexts (Fitzallen & Watson, 2014). Different classroom arrangements promote social interaction, stimulate the use of tools and improve reflection and generalisation (Doorman et al., 2013).

RME can be considered to be versatile, since an RME course designed for primary school Mathematics, which uses context problems and modelling, has proved to be suitable for an advanced

topic such as Calculus (Gravemeijer & Doorman, 1999). The use of the RME approach to teaching Mathematics allows for students to actively participate in the learning process. It sanctions the development of learning experiences that start with a rich and meaningful context that illustrates Mathematics in real life (Fitzallen & Watson, 2014).

3.3.5.2 Disadvantages

By following the RME approach, the classroom can become noisy and messy, but the positive effect of the approach outweighs this (Widjaja & Heck, 2003). Fauzan et al. (2013) report on some issues that emerged in classroom experiments when developing an RME-based course. They report that learners could not finish the contextual problems for reasons such as a negative attitude, dependence on the teacher, and they were not used to group work. They also report that with certain context problems, learners do not use the context to solve the problem because the statement did not guide them to do so (Fauzan et al., 2013).

3.3.5.3 Recommendations for Realistic Mathematics Education Based Lessons

The research of Widjaja and Heck (2003) has revealed recommendations for teachers who might want to introduce RME-based lessons that are supported by technology. Firstly, teachers should be willing to adapt their role in the classroom to more learner-centred and should refine their skills with regard to classroom discussions and dealing with challenging questions. Secondly, activities and tasks need to be carefully selected—there should not be too few activities, and also not too many, since either way, this could hamper the process of learning. They also point out that the logistical matters need to be attended to in advance for example preparing laboratories and making arrangement for group work (Widjaja & Heck, 2003).

3.4 Information and Communication Technology

Figure 3.5 illustrates ICT as one of the major themes that emerged from the systematic literature review.

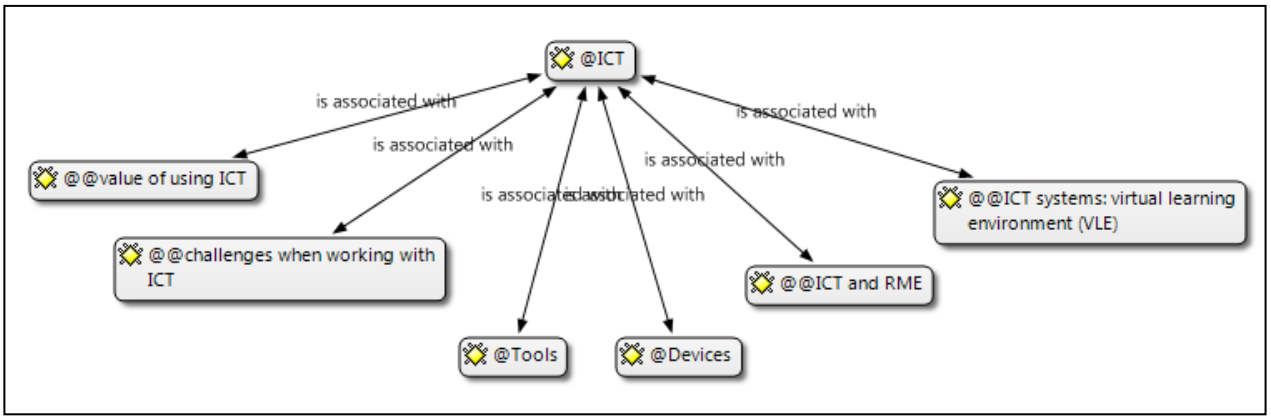


Figure 3.5: Information and Communication Technology as Theme from the Systematic Literature Review

3.4.1 The Value of Using Information and Communication Technology

There are numerous advantages discussed in the literature that highlight the value of using ICT in education. The use of ICT can promote student learning (Andresen, 2007) and promote mathematical reasoning (Widjaja & Heck, 2003; Zaranis, 2013, 2014b). Student learning can improve because educational technologies have various options to display data with different representations such as tables, graphs and formulas, which is particularly helpful when dealing with statistical concepts (Doorman et al., 2013; Fitzallen & Watson, 2014; Perez, 2012) and also to overcome the difficulty of integrating the operational and structural aspects of the function concept (Doorman et al., 2013).

The role of the teacher is described as both critical and problematic when it comes to integrating technology in teaching (Drijvers et al., 2010a). It is critical because the way in which teachers approach the use of technology has far-reaching consequences on the effects thereof in the classroom, and problematic because teachers find it difficult to adapt their teaching techniques when technology is incorporated, and can even avoid using technology when they do not perceive it to be useful (Drijvers et al., 2010a).

ICT has an imperative role to play in gathering information, acquiring data, processing and analysing data, solving problems, reporting and communication (Heck, 2009). It provides affordances for simulations, manipulation of data and is able to convert representations (Perez, 2012). Where possible, existing technologies should be extended and converted into concepts that work for school Mathematics (Heck et al., 2007) because technological developments encourage innovation in the teaching and learning of Mathematics by adjusting to conditions (Cahyono, 2012).

Not only have various research projects confirmed that the implementation of ICT in education can have positive effects on numerous subjects, but ideally, information technologies should be treated as

tools for teaching and learning (Zaranis, 2014b). ICT is a good facilitator to realise alternative approaches such as an activity-based approach or a realistic approach to Mathematics and Science instruction (Widjaja & Heck, 2003). Teaching and learning through ICT has a positive effect on the teaching of rectangles, circles, triangles and squares, particularly when using the RME approach (Zaranis, 2013, 2014c).

Computer tools provide opportunities for thinking and learning as they facilitate the learning activities that students engage with (Doorman et al., 2013). Computer aided learning is often a more enjoyable way of practising subjects that are otherwise seen as boring (Heck et al., 2008) and have shown to have a positive effect on the teaching of geometric shapes (Zaranis, 2014b). Educational software is found to be valuable in education, specifically Mathematics (Zaranis, 2013). Computer-assisted learning can significantly help in developing mathematical skills, and can promote deeper conceptual thinking than the traditional Mathematics teaching approach (Zaranis, 2014b, 2014c). When ICTs are treated as tools for teaching and learning, students can use these tools to become more familiar with new technology and can integrate investigation, communication and understanding across the curriculum (Zaranis, 2014c).

Technology also proves to be valuable for teachers (Dolk et al., 2002; Zulkardi, 2000). They can use web technology to store information like lessons or problems, can access information from anywhere in the world and discuss and share experiences after using the material in the classroom with colleagues or experts (Zulkardi, 2000). When using ICT in education, teachers that have less experience can be supported by experienced teachers, where shared experiences can be used to improve practice (Heck et al., 2007).

Learners' appropriate use of computers can be related to their ability to more efficiently understand the different mathematical concepts as well as develop mathematical thinking in school (Zaranis, 2014b). Heck et al. (2007) have found that learners who practice Mathematics using mathematical tools on computers are better prepared for writing exams than for doing written assignments. Technology provides social interaction and additional opportunities for a rich learning environment(Zaranis, 2014b), and has a positive effect on children's cognitive and sensory development (Zaranis, 2014c). The use of ICT can enhance students' comprehensiveness in learning Mathematics and engage students in the learning process (Domazet et al., 2013). According to Kizito (2012a), the partnership that needs to exist between a student and teacher in a distance learning environment cannot take place without some form of technological intervention or mediation.

The Internet can provide students with access to the same resources that professionals make use of. It allows for students to gather information and make contact with experts in their field of study. It also enables students to not only gather subject information, but also to collect good quality real-time data, do computations that are otherwise impossible, build models using dynamic software and report results (Heck, 2009).

With the aid of technology, a learning environment can be designed in which students investigate and acquire practical knowledge which can be generalised to the broader teaching context (Dolk et al., 2002). An integrated learning environment can provide students with the following benefits: students can practice Mathematics anywhere and anytime; assignments can be generated randomly which means that exercise material can almost be considered to be unlimited; student work (activities and assessment) can be stored for future reference; and quality feedback can be given to students automatically (Heck et al., 2007; Risnawati et al., 2014). Direct feedback that gives students hints and suggestions, and allows for a trial and error approach, makes it possible for students to explore different strategies for solving problems (Heck et al., 2007).

Learners who work in a computer-learning environment are more actively involved in discussions and debates relating to their experimental work. They tend to talk more actively and are more aware of their own mathematical thinking (Widjaja & Heck, 2003). Virtual learning environments are able to integrate a variety of tools that function in the following ways: provision of information, communication tool, means for collaboration, learning and management (Cahyono, 2012). Discussion in a technological environment can be a great advantage because as the discussions reveal issues, different representations and techniques can be tested with fast and dynamic feedback (Drijvers et al., 2010a).

Digital activities can contribute in the field of education in that they allow children to participate in a world in which they learn to speak, think and act in new and different ways (Zaranis et al., 2013). These activities can benefit learners in the following fields: learning (well-designed activities provide motivation and encourage learning), cognitive skills (through repetition, spatial ability, visual attention) and social interaction (cooperative interaction) (Zaranis et al., 2013). Digital media can introduce learners to abstract concepts that were previously considered too advanced for their age, and encourage learners to work together (Zaranis et al., 2013). Digital assessment adds to the motivation and performance of learners in Mathematics education (Heck et al., 2007).

When learning activities are designed using ICT, it is important that they are done in such a way that learners actively engage with the Mathematics (Andresen, 2007). Digital technology should be embedded in an educational context in such a way that the technology is integrated in a natural way (Drijvers, 2013). It is therefore not only important that teachers need to be knowledgeable about Mathematics, but they also need to be prepared in basic technological literacy (Widjaja & Heck, 2003).

For teachers to benefit from using technology in everyday Mathematics teaching, it is important that they are adequately equipped with not only basic knowledge, but knowledge about new teaching techniques that are necessary in a technology-rich classroom (Drijvers et al., 2010a). The teachers' knowledge of Mathematics as well as their beliefs and views about Mathematics education and the

role of technology in that, is an important aspect to consider when introducing technology into the classroom (Drijvers et al., 2010a). This will include aspects such as the teachers' knowledge and skills on how to incorporate technology in the classroom and their concerns about time constraints and behavioural control (Drijvers et al., 2010a).

3.4.2 Information and Communication Technology Tools

Information technology offers invaluable possibilities for creating educational tools that can help students come to grips with various ideas (Gravemeijer, 2012). Different technology tools makes it possible for teachers to track students' progress and diagnose problem areas (Heck et al., 2007), and create new possibilities for problem solving in Mathematics by opening new horizons that lend themselves for exploration and problem solving activities (Doorman et al., 2007). By providing direct feedback and giving students hints and tips, teachers create a trial and error approach which makes it more possible for students to explore different strategies in order to solve the problem (Heck et al., 2008). Technology tools also influence the process of students' sense making in Mathematics (Doorman et al., 2008), allow for students to review their mistakes and can store feedback which enhances the already positive effects of direct, instantaneous feedback (Heck et al., 2007).

Heck et al. (2007) distinguish between two types of applets, namely model applets, which aid in developing the mathematical understanding of concepts, and exercise applets, which can support the development of mathematical skills. Applets offer many benefits for learners: they are fun and motivate learners; they allow students to work at their own level, therefore addressing individual differences between learners; applets help to make the Mathematics easier to understand because of the visual, interactive and dynamic features they have; and their ability to do complex calculations allows learners to focus on the mathematical concepts and models. Students are more creative when using applets, have a more positive self-esteem and the practice and feedback features provide more benefits than pencil-and-paper exercises (Heck et al., 2007). Exercises on the applets help learners develop mathematical understanding and skills more thoroughly than with pencil and paper (Heck et al., 2008).

An important aspect to remember is that applets should be integrated in the daily classroom routine to gain their full benefits (Heck et al., 2007). Applets are able to provide feedback in cases of error as well as success. The feedback is both visual and audible, which makes it easy for learners to understand (Zaranis et al., 2013). For teachers, the benefits of using applets include being able to review what students actually did after a computer aided lesson: what progress they made, which problems arose and what areas need attention in future lessons (Heck et al., 2008).

GeoGebra is an open-source dynamic software for Mathematics that combines arithmetic, geometry, algebra, statistics and calculus (Stols, 2012). GeoGebra can be useful to provide competence development for teachers in terms of mathematical representations (Perez, 2012). Technology can

help create an active learning environment where students scan discover, explore and visualise (Stols, 2012). According to Stols (2012), in a study that investigated the geometric cognitive growth of pre-service Mathematics teachers in terms of the Van Hiele levels, it was found that dynamic geometry software could enhance students teachers' geometric visualisation, analysis and deduction, but did not extend to the next Van Hiele levels of reasoning, informal justification and formal deduction.

When students are exposed to new technology, their skills and abilities are expanded in the computer-learning environment and they have the opportunity to improve their presentation skills (Widjaja & Heck, 2003). Interactive computer tools are able to present symbolic information such as graphs, models or numerical measures, which allow these tools to be employed successfully in enhancing the meanings of the symbolic information by making them more visible, accessible and more easily manipulated (Gravemeijer, 2012). A technology-enriched environment can improve the conceptual geometric growth of learners (Stols, 2012).

3.4.3 Devices

mLearning is defined as learning that takes place in learning environments and areas which make use of the mobility of technology, learners and learning (Zaranis et al., 2013). The term mobile learning (mLearning) is used in different contexts and involves a number of mobile devices such as mobile phones, personal digital assistants, handheld devices and notebooks (Kizito, 2012a). mLearning is an emerging and rapidly expanding field that provides new opportunities for education (Kizito, 2012a). In the past, mLearning was hampered by slow networks, limited services and hesitancy to invest in devices with a relatively short shelf life. Device attributes such as the screen size, battery life, security, and limited resources also delayed the adoption of mLearning (Kizito, 2012a); however mLearning is becoming a popular component of education. An added advantage of mLearning is in the distance education environment; where the majority of students cannot afford access to web-based learning, mLearning is a more viable alternative to provide instructional support (Kizito, 2012a).

Drijvers (2013) differentiates between three didactical functionalities for digital technology: the tool function for doing Mathematics; the function of providing the learning environment for practising skills; and the function of providing a learning environment for promoting the development of conceptual understanding. When designing digital technology, the guidelines and design heuristic should be guided by pedagogy and didactical considerations rather than the limitations or properties of technology (Drijvers, 2013). Digital activities are considered to be most effective when they are aimed at examining a specific problem or teaching a specific skill such as Mathematics, where detailed objectives can be determined (Zaranis et al., 2013). Digital educational activities are appealing to students' interests as well as a pleasurable pastime and an attractive learning environment for them to work in (Zaranis et al., 2013).

Mobile devices offer many advantages in the learning process including stimulation, motivation, ease of use and availability. Mobile devices are becoming more valuable in education because they are attractive, portable, and more affordable in relation to desktop computers. They offer the opportunity for learning with wireless technology which allows for easy access to information, promotes digital literacy, and provides opportunities for independent learning (Zaranis et al., 2013).

Mobile devices fit better into the lifestyle of young children than desktop computers because they do not need to sit at a desk or in an office to use the device. The touchscreen is also irresistible for children (Zaranis et al., 2013). By introducing mobile technology in education, there is a positive correlation between the learners' everyday life and their school life. Mobile devices can be used for various activities such as comparing, sorting, matching and counting (Zaranis et al., 2013). Educators should be sure to design the technological integration in such a way that learners are equipped with the necessary knowledge and skills they need in life (Zaranis et al., 2013).

The use of handheld technology in Mathematics education traditionally focused on content such as algebra and graphing, and took place in regular teaching sessions. It is now more common that these devices offer options for other topics such as geometry, and can be used outside the classroom due to the wireless options available (Wijers et al., 2010). The wireless option allows for opportunities for situated learning outside of school and encourages characteristics such as situated authentic learning, peer collaboration and motivational power (Wijers et al., 2010).

One of the many devices that can be used in the Mathematics classroom is the graphics calculator. The graphics calculator has the potential to stimulate learners to engage in informative exploratory activities, and can kindle the use of realistic contexts (Drijvers & Doorman, 1996). The realistic quality of a problem can be enhanced when using a graphics calculator, since the device takes care of the time-consuming technical work, allowing learners freedom to concentrate on the process of mathematization (Drijvers & Doorman, 1996).

Mobile phones have the potential to open opportunities for learners due to their mobility, accessibility and flexibility. They also enable students to engage in real, authentic tasks (Kizito, 2012a). Their benefits range from a means of communication and social networking to being handy as a learning medium (Risnawati et al., 2014). The mobile phone offers the feature of working collaboratively (sharing graphs and solving problems collectively), which assists in establishing a community of Mathematics learners (Kizito, 2012a). In South Africa, a mobile tutoring system known as *Doctor Maths*, is a very popular instant-messaging service that caters to over three million school-aged subscribers to support the teaching and learning of Mathematics (Kizito, 2012a).

The term smartphone describes mobile phones which have similar features and characteristics of tablets on a smaller screen, as well as the usefulness of voice data over cellular networks (Zaranis et al., 2013). Smartphones with multimedia facilities, which allow for data collection, the recording of

video images or audio clips and short message services, can potentially be useful for teaching and learning in many subjects. These features can assist students in building new knowledge (Kizito, 2012a). Smart mobile devices can be used to present digital activities such as exercises relating to comparing, sorting, matching, counting and the general knowledge of numbers (Zaranis et al., 2013).

An advantage of using tablets in education is that most tablets do not have phone features, which means there is less opportunity for disruption by incoming calls or text messages than with mobile phones. Tablets offer the benefit of mobile apps in a broader context, at all levels of education, and provide a rich tool which can be used inside and outside the classroom (Zaranis et al., 2013). Students can store educational materials in a digital portfolio (Zaranis et al., 2013). In a study that compared the outcomes that students were able to achieve through the traditional teaching approach as opposed to tablet aided learning, it was revealed that the tablet aided approach produced better learning outcomes for the students (Zaranis et al., 2013). The portability of tablets allows teachers and learners to use various locations in the classroom which allows for collaboration and creativity (Zaranis et al., 2013).

Various governments, worldwide, of both developing and developed countries, have in the last few years paid special attention to the development and implementation of mLearning in teaching and learning (Zaranis et al., 2013). However, affordability and ease of use of mobile devices are issues which can hinder effective delivery and acceptance by users (Kizito, 2012a).

Multimedia studies can act as a mediating tool in surpassing the contrast between theory and practice. It can also be instrumental in helping teachers develop situated practical knowledge about teaching (Dolk et al., 2002).

3.4.4 Information Communication Technology Systems: Virtual Learning Environment

A Virtual Learning Environment (VLE) is a collection of integrated tools that facilitates the management of online learning, provides a means of delivery, allows for student tracking, assessment, and provides access to resources (Heck et al., 2007). A VLE offers various functions, such as providing information, sanctioning communication, collaboration, learning and management (Heck et al., 2007).

An advantage of using an existing VLE that students and teachers are familiar with, is that users know how to log on to the system, how to send e-mails, how to work with the built-in communication tools, and how to export results to the administrative system (Heck et al., 2007). VLEs can promote anytime communication and interaction with the teacher and with fellow students which can take the form of text-based, audio or video communication. The VLE can also stimulate the students to do homework (Heck et al., 2007). Students tend to like the fact that teachers have insight into their work

through the VLE. Teachers are able to assist with common mistakes and therefore can address problems more effectively (Heck et al., 2007).

Since virtual manipulations are similar to physical manipulations, using ICT could effectively support the learning process especially in Mathematics (Zaranis et al., 2013). Virtual worlds provide a promising structure for learners since they are allowed to play many different roles (Zaranis et al., 2013).

3.4.5 Information Communication Technology and Realistic Mathematics Education

Although there are challenges in the integration of ICT with the principles of RME (Drijvers et al., 2013), various studies have shown that technology can act as a catalyst in the RME approach to teaching Mathematics (Widjaja & Heck, 2003). Teaching by the principles of RME with the aid of educational software for tablets produces better outcomes than traditional teaching (Zaranis et al., 2013). Instruction involving technology should equip learners with the necessary skills and knowledge for later life (Zaranis et al., 2013).

When dealing with RME theory, where real world situations form the foundation of mathematical activities, software should be carefully chosen in order to stimulate these situations, recall old experiences and extend these to new ones by introducing purposeful interactions (Boon, 2006). The mobile phone is useful in quickly determining a student's starting point before designing instruction in the RME perspective (Kizito, 2012a). An important aspect when using the RME approach in designing applets is that the problems that are presented should be real and meaningful to the learners and worth solving (Heck et al., 2008).

3.4.6 Challenges when Working with Information Communication Technology

Teachers and learners alike can benefit from ICT in the form of different computer programs, lesson material and communication available on the Internet, and much more. However, not all stakeholders share the optimism about the effective use of ICT in teaching and learning (Drijvers et al., 2010a; Heck et al., 2007).

The use of technology in Mathematics education should not be seen as an instant remedy that reduces the importance of the teacher (Drijvers, 2013). When computers and technological tools are used continuously in the classroom, students can develop a dependency on these tools (Zaranis, 2014b). An added challenge for teachers is that students can be at different levels of computer literacy, which the teacher will have to deal with in order to progress (Zaranis, 2014b), and where online activities are prescribed for homework, a teacher needs to keep in mind that not all students have an Internet connection (Risnawati et al., 2014). The use of technology challenges the stability of teaching practices, and the way teachers coordinate student learning (Drijvers et al., 2010a).

Techniques that have always been used are often not suitable when technology is available. Teachers need to develop new teaching techniques that are applicable with the available tools (Drijvers et al., 2010a).

With regard to mobile technology, although there have been studies about the use of mobile phones in Mathematics education, Kizito (2012a) suggests that interviews should be used in future studies to explore the in-depth perceptions of students' experiences of mobile phone technology. One of the limitations of the mobile phone in teaching Mathematics is the difficulty in manipulating symbols and equations; most mobile phones and smart phones have limited features to deal with this effectively (Kizito, 2012a). With the rapid advancement of technology, it might be possible in the near future to have affordable systems which allow easy manipulation of symbols (Kizito, 2012a). Kizito (2012a) is of the opinion that more focused research is required to investigate the use of the mobile phone in distance education.

3.5 Methodology: Design-based Research

Design-based research is cyclical in nature where thought experiments and teaching experiments alternate, following the process of analysis, design, development, implementation, evaluation and reflection (Bakker et al., 2003; Dolk et al., 2002; Doorman et al., 2013; Fauzan et al., 2013). Bakker et al. (2003) distinguish between micro cycles (at the level of the lessons) and macro cycles (at the global level of the teaching experiment). Design-based research is a means for the researcher to understand teaching, learning and educational systems (Bakker et al., 2003). Design-based research comprises articulating, testing and refining conjectures, designing instructional activities, and teaching experiments (Doorman et al., 2008).

Design-based research consists of three phases, namely the preliminary design phase, the teaching experiment phase, and the phase of retrospective analysis (Bakker et al., 2003; Perez, 2012). In the preliminary design phase, an HLT is developed and instructional activities are designed, and usually adapted as a result of the teaching experiment (Bakker et al., 2003). Fauzan et al. (2013, p. 166) use the following terms to refer to the three main phases of design-based research, namely: front-end analysis, the prototype phase, and the assessment phase. The purpose of front-end analysis is to get "a picture of the starting point and the potential end points of the course". This normally involves needs analysis, literature reviews and the analysis of examples that are available (Fauzan et al., 2013). In the second phase, the instructional activities as well as the prior expectations rooted in the HLT are confronted with the reality of the classroom (Bakker et al., 2003). The third phase of design-based research involves data analysis and reflecting on the findings (Bakker et al., 2003).

Design based research has the following two purposes: to develop a prototypical product, with evidence of its quality, and to produce guidelines to design and evaluate that product (Fauzan et al.,

2013). The development methodology needs to be systematically reflected on in order to produce design principles (Fauzan et al., 2013). Fauzan et al. (2013) summarise two different types of design-based research as presented by various other authors, namely validation studies and development studies. Validation studies focus on the design of learning environments to validate theories about the learning process; they aim to promote learning and instruction theories like RME. Development studies focus on design principles for developing innovative interventions that are significant for educational practice, which is done through a process of systematic reflection on the development methodology (Fauzan et al., 2013). It can happen that a study can be both a validation study and a developmental study (Fauzan et al., 2013).

Some reasons offered as to why design-based research is used include the following: design-based research affords a perspective to develop theory, the results are useful in the sense that educators can use the relevant research products and it gives researchers the opportunity to improve education by creating innovative designs (Dolk et al., 2002). Design based research within RME would reflect the three key principles of RME, namely guided reinvention, didactical phenomenology and emergent modelling (Fauzan et al., 2013). A detailed description of the methodology of design-based research was done in Chapter Two.

3.6 Role players

Traditionally, the role of learners was to listen, write and perform tasks provided by the teacher (Widjaja & Heck, 2003); however understanding is stimulated when learners are given an opportunity to be active, investigative learners, where they solve problems using their own strategies, as is required in RME (Dolk et al., 2002; Widjaja & Heck, 2003). In RME, learners need to be given space for exploration and constructing their own products and strategies (Dolk et al., 2002; Widjaja & Heck, 2003; Wijers et al., 2010).

The role of the teacher is also important in implementing RME principles (Doorman et al., 2008; Drijvers, 2013): he or she needs to guide discussions (Doorman et al., 2008; Gravemeijer, 2012), facilitate learners' learning, and address challenging questions from learners; encourage them to think critically (Widjaja & Heck, 2003); allow learners space to do construction; ensure that open exploratory activities are used; give learners the opportunity to share and discuss their work, and contemplate relationships between different approaches (Dolk et al., 2002). For teachers to profit from technology in the Mathematics classroom, they need to have knowledge about emerging teaching techniques in technology-rich classrooms (Drijvers et al., 2010a).

Widjaja and Heck (2003) and Zulkardi (2000) testify to the changing role of learners and teachers in RME towards more learner-centred learning. Interactivity in the classroom increases and learners become more responsible for their own learning. This changing role requires the support of school

management to achieve success (Widjaja & Heck, 2003). In-service courses for teacher education should address teachers' concerns and improve their knowledge and experience (Dolk et al., 2002). Teacher training, both in-service and pre-service, is essential since teachers are key figures in educational change (Zulkardi, 2000).

3.7 Chapter Summary

The chapter presented the review of the current literature with regard to the implementation of RME principles, aided by the use of ICT in Mathematics education. The themes of Mathematics, RME, ICT, methodology and role players were discussed. Table 3.1 lists the role of ICT in the RME approach to enhance teaching and learning.

Table 3.1 The role of Information and Communication Technology in the Realistic Mathematics Education Approach to Enhance Teaching and Learning

Author(s)	Title	Mathematics	RME	ICT	Methodology	Role players
(Andresen, 2007)	Introduction of a new construct: The conceptual tool "Flexibility."	Teaching experiments were used to investi-gate the use of laptops in teaching differential equations using model-ling.	The aim was to develop theory within the framework based on RME and related ideas.	Laptops	Case study	Upper secondary school Mathematics
(Bakker et al., 2003)	Design research on how IT may support the devel-opment of sym-bols and meaning in Mathematics education	The study focuses on how the use of IT in Mathematics education can help students reinvent mathematical concepts and representations. Different Mathematics topics were focused on. The challenge for Mathematics teachers is to develop Mathematics education that is in line with the conceptions of symbolising and developing meaning	RME is used as a background for the project.	The project investigated the contribution of ICT tools for the guided reinvention of mathematical concepts. It investigated the influence of ICT use on the students' process of symbolising. Different types of technological tools were used.	Design-based research was used. Each design-based research study consists of research cycles in which thought experiments and teaching experiments alternate. The article distinguishes between macro and micro cycles.	Early statistics education. Eighth grade class.
(Cahyono, 2012)	Virtualmatriks: A Conceptual Mathematization Process in Virtual Learning Environment.	In the Virtual Learning Environment, students interact with the teacher and fellow students. This interaction can enhance the learning of mathematical concepts and their application in real life. The model positively impacts on student learning.	The study aims to develop a model relating to a Virtual Learning Environment based on RME	Virtual Learning Environment	The research is conducted in stages. Interventions are designed and validated	Senior high school
(Dolk et al., 2002)	Using multimedia cases for	This project is to understand how	The project is based on the RME approach.	No ICT	The cyclical nature of the design-based	Pre-service Mathematics education

Author(s)	Title	Mathematics	RME	ICT	Methodology	Role players
	educating the primary school Mathematics teacher educator: A design study	multimedia video case studies support the professionalization of primary-school-Mathematics teachers. Multimedia cases are investigated which can assist in supporting teachers in learning to mathematize and didactize. Three aspects that are important in this project are reflection, narration and construction.	According to this research, a six-step framework is proposed that aims to help the participants develop narrative knowledge. This should foster a sound basis for teaching that is suited to RME.		research approach was suited for this project. It consisted of a design, trial, and revision phase of a course for pre-service Mathematics teachers.	teachers in the Netherlands.
(Domazet et al., 2013)	Mathematics Competence and International Mathematics Testing: Croatian Starting Point	The paper reviews the concept of compulsory Mathematics education and appropriate instruction methods. The paper reviews the concepts behind the social influences on the development of literacy in Mathematics education.	Links are made between the theoretical framework of PISA and RME. Mathematics education is no longer just about transferring mathematical knowledge, but about satisfying broader social goals in that.	No ICT	The paper maps past and current, and possible future, changes in the conceptualisation of school Mathematics and Mathematics teaching in Croatian compulsory education.	Croatian primary schools
(Doorman et al., 2013)	Design research in Mathematics education: the case of an ICT-rich learning arrangement for the concept of function	The concept of functions is an important but difficult topic in secondary school Mathematics curricula. It can be challenging for learners since the concept develops from calculation operations to objects. Various lessons promoted the conceptual	The theoretical framework for this study is made up of three components: the knowledge of the function concept, emergent modelling (which is a principle of RME) and theories on tool use.	The study dealt with the design and evaluation of an ICT-rich learning environment for Grade eight learners. The computer tool that was used is an applet called AlgebraArrows. This permits the construction and use of chains of operations and provides opportunities for	The methodology for the project was design-based research. It was carried out in a cyclic process of three design cycles, one pilot and two full cycles, each lasting for about one year.	Grade 8 learners (age group 13-14 year) in the Netherlands

Author(s)	Title	Mathematics	RME	ICT	Methodology	Role players
		development of functions. The final test data did not reveal whether all students improved with regard to the understanding of functions.		creating tables, graphs and formulae. The applet is used to provide tasks which the teacher can monitor.		
(Doorman et al., 2008)	Tool use and the development of the function concept: from repeated calculations to functional thinking	The study focuses on activities with computer tools that help students to construct mathematical representations and actions. They are given the opportunity to explore the representations. The concept of functions is explored. The transition from function machines to actual objects is represented in different ways.	One aspect that forms part of the theoretical framework is emergent modelling, which has its foundation in the theory of RME.	The study reports on a project where the following activities are combined: paper and pencil and computer tools as well as the careful guidance of the teacher. The tool that is used is an applet called AlgebraArrows. This which permits the construction and use of chains of operations and provides options for creating tables, graphs, and formulae.	The study makes use of design-based research. This involves the formulating, testing and filtering of conjectures, and designing instructional activities. The assumptions create a connection between the teaching experiments and the manner in which students can be supported in the development of the topic.	An initial small-scale pilot study took place. Thereafter the learning arrangement was tested and improved in two successive teaching experiments in Grade 8, with mid- and high-achieving 13–14-year-old learners.
(Doorman et al., 2007)	Problem solving as a challenge for Mathematics education in The Netherlands	The focus was on the implementation of a problem based curriculum, centred on the principles of RME and incorporated the use of technology.	The study was based on the principles of RME.	No ICT	The aim of the study was to investigate how Dutch high-achieving Grade four learners did problem-solving tasks. These tasks were specifically developed for the World Class Tests.	A total of 152 high-achieving fourth grade learners (9 – 10 year olds).
(Doorman et al., 2008)	Tool use supporting the learning and teaching of the Function concept	The focus is on the use of computer tools for grade 8 (13-14 years) students' acquisition of the concept of functions. The aim is to identify the relationship between the use of technological tools and	The study is based on the philosophy of RME	Computer tools	The research process involves a cyclical process of instructional design, teaching experiments and data analysis.	Grade 8 learners

Author(s)	Title	Mathematics	RME	ICT	Methodology	Role players
		the learning of Mathematics.				
(Drijvers, 2013)	Digital technology in mathematics education: Why it works (or doesn't)	The study shows that the success or failure of the integration of technology in Mathematics education relies on three factors: the design, the role of the teacher, and the educational context. The study identified factors that promote or hinder the successful integration of digital technology in Mathematics education.	RME is one of the theoretical perspectives that is investigated in this study.	The paper looks at the contribution of technology in Mathematics education from a more realistic perspective.	Six different cases from leading studies in the field were studied.	The role of the teacher is extremely important in the incorporation of technology in Mathematics education
(Drijvers et al., 2013)	Digital design: RME principles for designing online tasks	The design of tasks is important—this paper explores how the RME principles which are usually used for paper-and-pencil tasks can be used in a digital environment. Examples of tasks relating to Algebra, Calculus and Geometry are designed and presented in the Digital Mathematics Environment.	The paper investigates how the principles of RME can be used to design online tasks. They concluded that the principles of guided reinvention, didactical phenomenology, and emergent modelling can inform and guide digital design.	A Digital Mathematics Environment (DME) is used. The DME integrates a content management system, a learning management system and an authoring environment.	This is a theoretical paper in which different digital designs of tasks for Algebra, Calculus and Geometry are studied.	Grade 8 learners
(Drijvers & Doorman, 1996)	The graphics calculator in mathematics education	This study supports the notion that the graphics calculator can encourage the use of realistic contexts. The graphics calculator is used to integrate Algebra and Geometry and encourages the learner to create links	The theory of RME was used as point of departure for the study.	Graphics calculator	Developmental research.	Prevocational secondary education (16 year olds).

Author(s)	Title	Mathematics	RME	ICT	Methodology	Role players
		between the different aspects of Mathematics				
(Fauzan et al., 2013)	The Development of an RME-based Geometry Course for Indonesian Primary Schools	The study involved the use of an RME-based geometry course for teaching geometry at grade four in Indonesian primary school. The aim was to improve pupils' understanding, reasoning, activity, creativity, and motivation.	A Geometry course for Indonesian primary schools was based on the RME approach.		Design-based methodology was followed. The three stages, namely front-end analysis, the prototyping stage, and the assessment stage were conducted over a four year period.	Grade 4's at an Indonesian primary school
(Fitzallen & Watson, 2014)	Developing a Sequence of Learning Experiences in Statistics	This study focuses on the design of a range of learning experiences that promotes the understanding of statistical covariation as well as graph creation and interpretation skills. Based on the RME philosophy, the process should build students' reasoning gradually from the concrete to the abstract by making use of models, diagrams, and other imagery to reinforce reasoning.	RME is used as a guide for student learning that focuses on the development of abstract thinking.	The learning experiences are facilitated by the software package TinkerPlots: Dynamic Data Exploration.	Sequences of learning experiments are given to learners.	Secondary school
(Gordon et al., 2012)	Keyconet 2012 Literature Review: Key competence development in school education in Europe	This paper discusses issues such as the building of Mathematics competence, cooperative learning, problem solving grounded in real life examples, and metacognition.	Based on the principles of RME	No ICT	Literature review	Learners in general
(Gravemeijer,	Aiming for 21st	The article argues that	A reform approach to	The study focuses on	Literature review	Learners in general

Author(s)	Title	Mathematics	RME	ICT	Methodology	Role players
2012)	century skills	interactive, problem-centred Mathematics education will contribute to most 21st century skills.	Mathematics education is followed, where students have to reinvent Mathematics by themselves with the help of the teacher and the textbook. This links with the tradition of RME.	mathematical skills that are needed in a computerized environment. The article analyses the role computers and computerized appliances play as interfaces between the users and reality.		
(Gravemeijer & Doorman, 1999)	Context problems in realistic mathematics education: A calculus course as an example	In RME, context problems are intended to sustain a reinvention process that enables students to understand formal Mathematics. In this article, the authors argue that discrete functions and their graphs play an essential role as a midway between the context problems that have to be solved and the formal calculus that is developed.	The RME approach is used to design a Calculus course that relies on context problems.	No ICT	Theoretical article	Primary school Mathematics
(Gravemeijer, 1999)	How Emergent Models May Foster the Constitution of Formal Mathematics.	The article deals with the role that emergent models can play in establishing formal Mathematics. An instructional sequence is designed which deals with mental computation strategies for addition and subtraction up to 100. This elaborates on what is meant by emergent models and the role they play in	The study is based on instructional design theory for RME.	No ICT	The design heuristic, which is addressed, is part of an instructional design approach. This is part of developmental research.	Primary school

Author(s)	Title	Mathematics	RME	ICT	Methodology	Role players
		developing formal Mathematics.				
(Heck, 2009)	Modelling in cross-disciplinary authentic student research projects	The focus of the paper is on how the use of ICT can contribute to the realisation of research projects in Mathematics and Science education.	The RME principles of using relevant contexts and of considering Mathematics as a human activity are prevalent in this paper. This implies that students make sense by participating in the mathematical representation process.	The examples in this paper show that ICT can contribute to the recognition of true review. Students are given the opportunity to process, analyse and visualise data; to do calculations that are normally cumbersome; and to compare and report results from experiments, models, and theory.	Three examples of students' inquiry work are analysed.	Dutch secondary school
(Heck et al., 2007)	Applets for learning school algebra and calculus	The article explores in what ways applets can help student to develop algebraic understanding and skills. In this project, they try to extend existing technologies into concepts that work for school Mathematics. They also produce learning materials that is based on technology.	The RME approach was the key theoretical framework for the design of the software and of the instructional material.	The focus is on the development and use of Java applets for learning and practising Algebra and Calculus at secondary school level.	Developmental work during the GALOIS project.	Secondary school calculus students
(Heck et al., 2008)	Mathematica empowered applets for learning school algebra and calculus	The article describes the development and use of applets that are empowered by Mathematica. These are used for learning and practising Algebra and Calculus at secondary school level. The applets help the teacher to understand the learning process of	The RME approach was the key theoretical framework for the design of the computer programs and the accompanying lesson material.	The applets provide scoring facilities, are integrated in a virtual learning environment Here answers are stored, and can be edited by teachers to create their new tests and exercises.	Developmental work	Secondary school level

Author(s)	Title	Mathematics	RME	ICT	Methodology	Role players
		students and to make wise choices when constructing a lesson.				
(Kizito, 2012a)	Pretesting mathematical concepts with the mobile phone: Implications for curriculum design	This paper describes the characteristics of a diagnostic test which was used to determine initial student understanding of the basic Calculus concepts of the derivative and the integral before the start of their course. The aim of the project was to try to find out what the didactic functionalities of a mobile phone were and how these could be used to support the learning of Mathematics through distance education.	RME was used as the theoretical framework for this project. This testing was part of a larger project that examined what was required to adapt the instructional design viewpoint of Realistic Mathematics Education (RME) to the teaching and learning of Calculus via distance education.	The mobile phone was used as mode of delivery for the diagnostic test. The speed and ease of following the students' results was a positive aspect of the mobile phone, while issues of availability like cost and ease of use was an imitation.	Pilot study	The students were UNISA students who had completed their first semester in a first-year calculus course.
(Mousoulides et al., 2007)	From problem solving to modelling	In this paper, the concept of emergent modelling is explored from the field of problem solving. The importance of modelling is discussed from the existing literature.	The use of modelling problems can enhance student sense-making within a real context. It can bring real world situations into school Mathematics, which aids in fostering a positive attitude towards Mathematics.	No ICT	In essence, this study followed the structure of design-based research. The modelling approach involves a number of iterative cycles, in which students refine their results and improve their solutions.	The paper is a literature review that explores the emergence of modelling within the genre of problem solving research
(Perez, 2012)	Scaffolding Teachers' Construction of a Learning Trajectory for Mathematics Supported by ICT	The teachers indicated content areas that their learners found challenging—the first theme that was identified was Algebra, in particular the	The instructional theory of RME is used to develop a hypothetical learning trajectory (HLT) to develop class activities that reflect the notion that	This project explores how technology can be used to improve teachers' teaching practices and their students' learning of Mathematics. In	This is a developmental project that makes use of a methodology of co-design, a user-centred, collaborative approach to design-based research. It is suitable	Two Mathematics teachers at a lower secondary school in Sweden

Author(s)	Title	Mathematics	RME	ICT	Methodology	Role players
		distributive law. The teachers also indicated a need for their skills development with regard to mathematical representations. GeoGebra was used to show the connection between Algebra and Geometry. The project made use of a teacher-centred, collaborative approach where the researcher also used scaffolding to support the teachers in the project.	Mathematics is a human activity.	particular, GeoGebra is used to develop the teachers' capability relating to Mathematical representations.	to use in cases where expertise from different areas are needed, in this case Mathematics, pedagogy and technology.	
(Risnawati et al., 2014)	The development of realistic mathematics education-based blog at linear algebra course in UIN Suska Riau	The blog is related to Algebra courses so that the students can access courses they need anywhere and at any time. Students can review the effectiveness of blogs in supporting the learning process of Algebra.	The development of the RME-based blog was effective as learning media and as a source to improve students' understanding of Algebra.	Technology tools such as laptops, smartphones and tablets as well as social networks such as Facebook, Twitter and others are commonly used by students for personal and social purposes. This project focuses on the academic use of technology.	Development research.	Students studying Linear Algebra at university.
(Stols, 2012)	Does the use of technology make a difference in the geometric cognitive growth of pre-service mathematics teachers?	The study investigated to what extent pre-service Mathematics teachers' cognitive growth in Geometry improved in terms of the Van Hiele levels. The study found that the use of dynamic geometry software enhanced student	The design of the geometry course was loosely based on the five tenets developed by the Realistic Mathematics Education (RME) movement.	This investigation took place in a technology-enriched environment and compared these students' growth with that of students in a learning environment without any technological enrichment.	A quasi-experimental non-equivalent comparison group design was used. A control and experimental group was used in the study.	Third year BEd pre-service Mathematics teachers.

Author(s)	Title	Mathematics	RME	ICT	Methodology	Role players
		teachers' geometric visualisation, analysis and deduction, but not their ability to informally justify their reasoning and to understand the formal aspects of deduction.				
(van Loon-Hillen, van Gog, & Brand-Gruwel, 2012)	Effects of worked examples in a primary school mathematics curriculum.	For a three week period, the study tested the effects of relying more heavily on worked examples, as analogies for problem solving. The curriculum was used to teach both the control group and the experimental group. Although the results showed no significant di ffer performance or cognitive load; the experimental group achieved this level with significantly less acquisition time.	The existing curriculum was taught using a RME teaching approach.	No ICT.	The study made use of a quasi-experimental design.	Primary school Mathematics class, using the existing curriculum.
(Widjaja & Heck, 2003)	How a realistic mathematics education approach and microcomputer-based laboratory worked in lessons on graphing at an Indonesian junior high school	The project investigated the expectations, performance, and opinions of the pupils and the teacher with respect to graphing activities using a computer tool. The content that was used in this project was graphs with emphasis on interpretation and application.	The RME approach was used. The results of the project indicated that the pupils made remarkable progress in their performances that can be attributed to the approach that was used.	The project made use of Microcomputer-Based Laboratory (MBL). Computer tools were used for measuring data in practical work.	The research experiment was set in a classroom situation where the pupils made use of the designed classroom activities. Pre- and post-tests were used to investigate the effect of the ICT-rich RME-based instructions on the pupils' graphing skills.	One class with 13 – 14 year old pupils (Junior High School) used the materials and activities that were designed. This was done in an Indonesian classroom context.

Author(s)	Title	Mathematics	RME	ICT	Methodology	Role players
(Zaranis, 2013)	The use of Information and Communication Technologies in the first grade of primary school for teaching rectangles based in Realistic Mathematics Education	The purpose of the study was to investigate the impact of the intervention of Primary Shape Model (PSM) on the Geometry competence of the first grade learners, This model was based on the Van Hiele levels and also on the RME principles.	The educational software was designed following the principles of RME.	The first level of the Primary Shape Model (PSM) started with the educational software which was designed using Flash CS3 Professional Edition. Various computer activities were included in the following levels.	An experimental group, were taught rectangles with computers and a control group received only the traditional instruction relating to rectangles. Students in both groups were pre-tested and post-tested for their performance in Geometry.	First grade learners from ten primary schools in Athens and Crete.
(Zaranis, 2014c)	The use of ICT in the first grade of primary school for teaching circles, triangles, rectangles and squares	This study investigated whether ICT helps to improve first grade students' basic Geometry achievement regarding circles, triangles, rectangles and squares or not. The study examined and compared the effects of a new model which made use of both computer and non-computer activities for teaching circles, triangles, rectangles and squares separately.	Rather than following a traditional teaching method, the RME approach was used to present the material.	A story called 'The Family of Shapes' was presented to the students. Various geometrical shapes were presented in a fictional family setting where each member of the family represented a particular shape. The story was designed using Flash CS3 Professional Edition and presented with a video projector in the classroom.	There was an experimental and control group. The experimental group were taught about circles, triangles, rectangles and squares with the support of computers. The students in the control group were not exposed to the computer oriented curriculum. Students in both groups were given pre-tests and post-tests to establish their Geometry performance.	First grade learners from ten public primary schools in Athens and Crete.
(Zaranis, 2014a)	Geometry Teaching Through ICT in Primary School	Using computer based mathematical teaching to teach learners can significantly help in developing proper mathematical skills and cultivate deeper conceptual thinking than in traditional teaching. The Van Hiele model formed	The software that was designed and the students' activities that were developed for this study were inspired by the framework of RME.	Technology was used to create a rich learning environment, consistent with the modern era.	Experimental research	First grade pupils

Author(s)	Title	Mathematics	RME	ICT	Methodology	Role players
		part of the framework for the study.				
(Zaranis, 2014b)	The use of ICT in kindergarten for teaching addition based on realistic mathematics education	The study investigated the effect of ICT on learners' achievement in addition and counting. The researcher and the teachers developed a model which combines the principles of RME and the use if ICT in kindergarten. The main purpose of the model was to foster learners' mathematical concepts and skills about addition and to involve them in self-regulated learning. Some activities were computer activities and others not.	A comparison is made between learners who are taught by the traditional approach as opposed to those being taught by the principles of RME.	The learners were given software that consisted of a story and several activities, some with the use of computers and others without. The results showed that the use of computers had a positive effect on the learning of addition.	The learners were divided into two groups—the experimental group who were taught with computers and the control group who were not exposed to the computer oriented curriculum.	The participants were kindergarten learners in Crete.
(Zaranis et al., 2013)	Using mobile devices for teaching realistic mathematics in kindergarten education	The activities are based on the fundamental Mathematical concepts of kindergarten. These include counting and simple addition and subtraction. The paper investigated the use of ICT in preschool education, as well as the learners' knowledge	The project revealed that the teaching of RME with the use of educational software for tablets produces better learning outcomes than traditional teaching methods.	The purpose of the paper was to provide a better understanding of the characteristics and the effect of ICT and mLearning in preschool education. The results showed that the use of the tablet supported learning compared to the traditional teaching method without technology.	An instructional intervention.	The initial pre-tests were done with a small sample of kindergarten learners in a specific region in Greece.
(Boon, 2006)	Designing didactical tools and microworlds for mathematics educations	This paper reflects on the role of (solving) algorithms in Mathematics education with the aid of	The theoretical foundation of this work is based on the theory of RME. The main purpose of this theory	The paper distinguishes between three types of applets: applets that offer a virtual reality, those that	The paper reports on the design activities in the field of small didactical fields and java applets done at	Various studies were referred to in this paper.

Author(s)	Title	Mathematics	RME	ICT	Methodology	Role players
		technological tools.	is for students to accept the knowledge that they acquire as their own.	facilitate the use of models and applets that offer a mathematical microworld.	the Freudenthal Institute over a number of years.	
(Stephan & Akyuz, 2014)	An Instructional Design Collaborative in One Middle School	The researchers designed and implemented a HLT and instructional tasks to teach Integers	All tasks that were designed used the design principles of RME, where the focus is on the development of a sequence of tasks, rather than stand-alone problems.	No ICT	The researchers designed their own instructional sequence, which consisted of 5 phases.	One middle school in Florida, USA (12 – 14 year olds)
(Cobb et al., 2008)	Learning from and Adapting the theory of realistic mathematics education	Experiments were designed relating to the teaching and learning of statistical data analysis	Adaptations were made that built on the RME approach to the design of learning material.	No ICT	Design-based research	A US Middle school in a seventh-grade classroom with 29 twelve-year-old students
(Drijvers et al., 2010a)	The teacher and the tool: instrumental orchestrations in the technology-rich mathematics classroom	The project consisted of eight 50-min lessons, aimed at the development of the function concept. Teachers' choices regarding intervention were in line with their views on the teaching of Mathematics.	The design was guided by RME principles.	The technology used in this project was a Java applet called Algebra Arrows. The applet was embedded in an electronic learning environment called the Digital Mathematics Environment (DME). The applet allows for the construction and use of chains of operations and provides options for creating tables, graphs and formulae.	The intervention took place in three research cycles, each consisting of a (re)design phase, a teaching experiment phase and a data analysis phase.	Secondary education – eighth grade students
(Wijers et al., 2010)	Mobilemath: exploring mathematics outside the classroom	This project investigated the effect of a location-based game with hand held technology (HHT) on the engagement in mathematical activities.	The study's theoretical framework is formed by notions on engagement, and on authentic RME.	HHT offer options for topics, such as Algebra, graphing and geometry, and can be used outside the classroom, because of wireless	Explorative design-based research was used. Data were gathered by means of participatory observation, online storage of game data,	The project was tested in a pilot on three different secondary schools with 60 12–14-year-old students.

Author(s)	Title	Mathematics	RME	ICT	Methodology	Role players
		This was made possible through the design of a Geometry game called MobileMath. Almost all students recognised the Mathematics in the game and described it in mathematical terms.		communication. These mobile devices can engage students in meaningful learning, inside as well as outside of school.	an online survey and interviews with students and teachers.	
(Zulkardi, 2000)	Realistic Mathematics Education theory meets web technology	The reform in Indonesian Mathematics is discussed under the following three sections: The new goal is described as Maths literacy, the new theory as RME and the new content is a meaningful context. The reform of Mathematics from traditional teaching to problem solving is discussed.	Based on RME principles. RME is seen as an innovation in Mathematics education.	The main purpose of this study was to develop a web-based system which provides resources, training opportunities and communication tools for student teachers in Indonesia.	Theoretical paper	Various learners
(Freudenthal, 1968)	Why to Teach Mathematics So as to Be Useful	Mathematics as a human activity; real life and useful	Foundations for RME	none	Theoretical	Teachers, Learners
(Simon, 1995)	Reconstructing Mathematics pedagogy from a constructivist perspective	The data led to a model of how teachers make decisions with regard to tasks.	Discussion about HLT	none	The role of constructivist theory in Mathematics Education	Teachers and learners
(Treffers, 1993)	Wiskobas and Freudenthal Realistic Mathematics Education	Reality is the source of learning.	Discussion of Freudenthal's foundations of RME	none	Theoretical	Teachers, learners
(Freudenthal, 1971)	Geometry between the Devil and the Deep Sea	Foundations of what Mathematics is - essentially a human activity,	Foundations of RME	none	Theoretical	Teachers, learners

Author(s)	Title	Mathematics	RME	ICT	Methodology	Role players
(Freudenthal, 1991)	Revisiting Mathematics Education: China Lectures	The essence of RME	Foundations od RME	none	Theoretical	Teachers, learners
(Treffers, 1978)	Three dimensions: A Model of goal and theory description in Mathematics Instruction - The Wiskobas Project	Foundations of RME	Foundations of RME	none	Theoretical	Teachers, learners

Chapter Four

Participants' Needs and Perceptions with regard to Real Life Mathematics

4.1 Introduction

In this chapter I report on the data set that I obtained when doing the needs analysis individual interviews with the participants. Two factors were the driving force behind the necessity to do a needs analysis with the participants. Firstly, the systematic literature review exposed various aspects that should be further explored within the context of this study. Secondly, the needs analysis was driven by the research sub-question: *What support needs do teachers have in order to successfully implement the RME approach in their teaching practice?*

A closer analysis of the research sub-question revealed that there were various areas that had to be investigated in order to sufficiently answer the question that this study proposed. It was therefore necessary to meet with the participants in order to ascertain their perceptions and experiences relating to a number of aspects. These aspects include the following: which content areas they found problematic to teach; which content areas their learners found challenging; and the facets of the curriculum they had difficulty in relating to real life situations. I also obtained information about their perceptions and ideas about teaching Mathematics in a realistic way as well as their observations about the use of technology in Mathematics teaching and learning. These aspects would inform the design process of the mobile app (Chapter 5).

The way in which I obtained this information was by conducting individual interviews with four teacher-students. I designed an interview schedule (Addendum 2.6) based on the findings of the systematic literature review as well as the requirements needed to answer the research sub-question. I made use of purposive sampling to select the participants. These students all attended contact sessions at the Potchefstroom Campus of the NWU and agreed to meet with me and participate in this research. They also all complied with the criteria for selection, which were discussed in § 2.8.1. The individual meetings took place at a location and at a time convenient for the students and did not interfere with their teaching schedule in any way. Two meetings took place at my office at the Potchefstroom Campus of the NWU. The third meeting took place at one participant's school in Potchefstroom in the HOD's office and the fourth meeting took place at a coffee shop in Ventersdorp. I paid for the transport costs of the participants that needed to travel to Potchefstroom, and I provided all participants with refreshments to ensure that they were comfortable and not inconvenienced in any way.

I transcribed each interview and coded these interviews with the aid of ATLAS.ti™, a Computer Assisted Qualitative Data Analysis Software (CAQDAS) (ATLAS.ti, 2014). Data were coded in the same HU as the systematic literature review in ATLAS.ti™. I therefore used the existing codes and created additional codes where necessary in order to code and categorise the interviews as part of an integrated HU. The detailed codebook was discussed in Table 2.6 (§ 2.6.5). In order to ensure the trustworthiness of the coding, two peers who are experts in the fields of Mathematics education peer-coded one interview (§ 2.8.3.2) Cohen's Kappa was calculated as 0.99 which therefore validated this process. This calculation can be seen at Addendum 2.7. The systematic literature review revealed five main themes; four of these five themes were used to structure the analysis of the interview data. The four main themes used in this chapter were: Mathematics, RME, ICT and role players. In order to analyse the interviews, these themes were further refined as: participants' needs in term of Mathematics; the facets of RME that arose in the interviews; their thoughts and ideas about ICT; and finally the role players that played a part in the given context (Figure 4.1). Each of the said main themes has various sub-themes that will be further explored in this chapter.

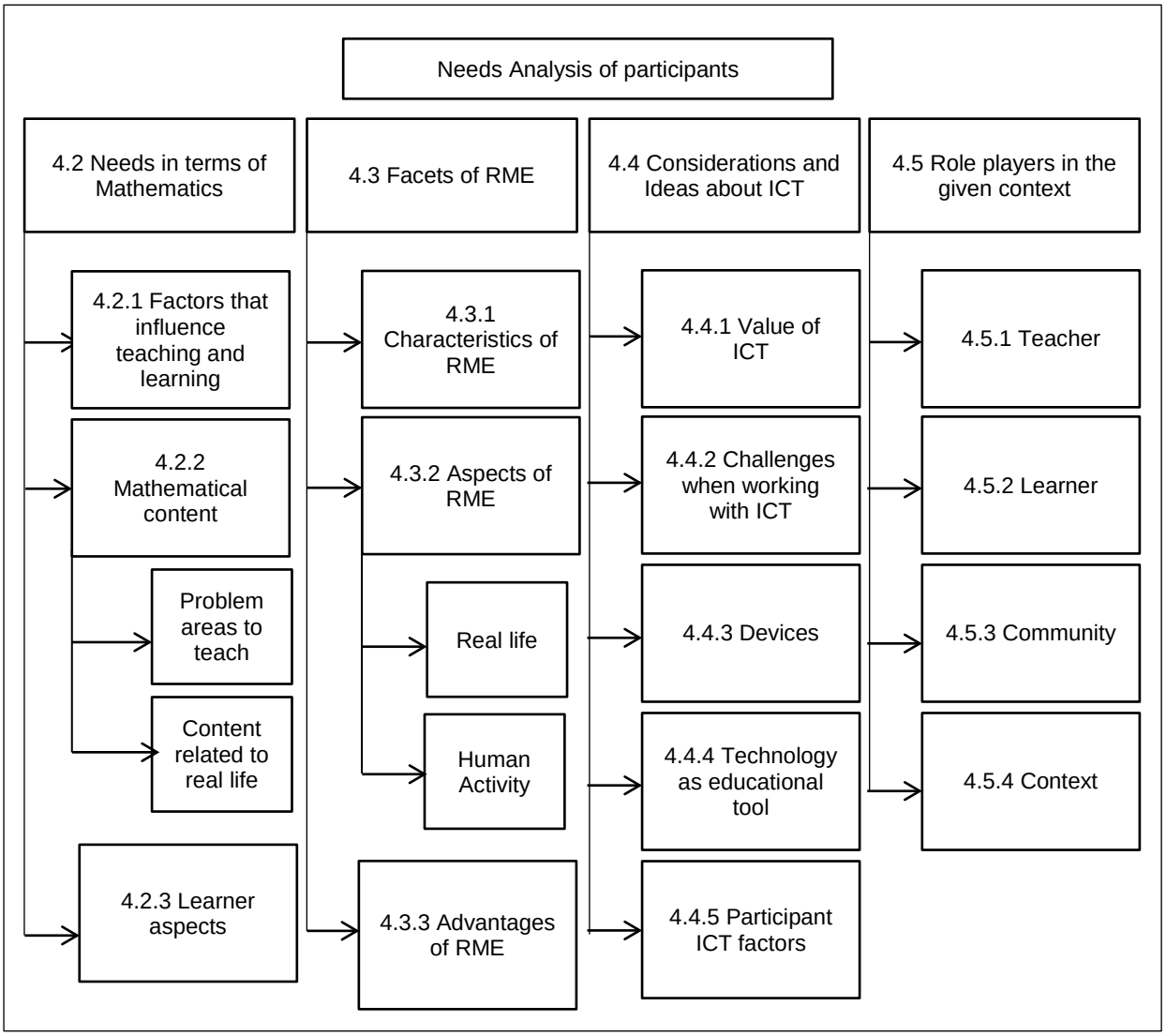


Figure 4.1: Advance Organiser of the Main Themes for the Analysis

Figure 4.2 illustrates the network view of the coding structure and themes that arose from this set of data, the four main themes being: Mathematics; RME; ICT; and finally role players. As discussed above, when analysing the data, these themes were made more specific in order to zoom in on the interview data. Due to the fact that the participants were asked to share their perceptions and experiences of aspects relating to these themes, I focused the discussion on their perceptions of these critical themes. This chapter reveals those support needs as well as other insights that the student-teachers had in terms of realising the RME approach in their classrooms.

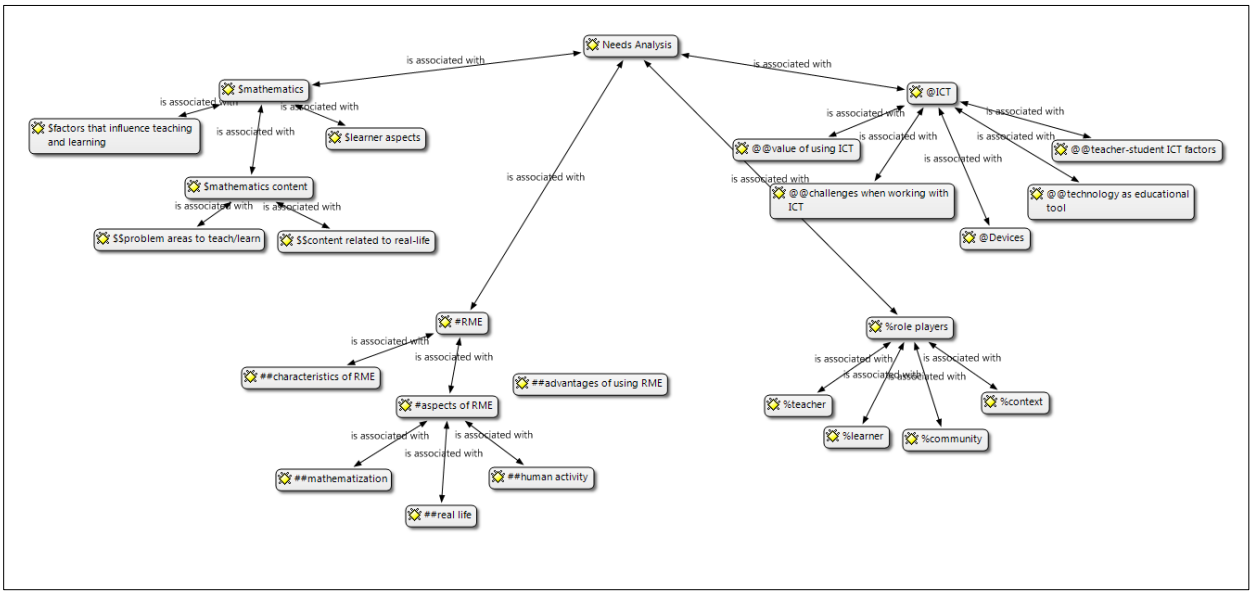


Figure 4.2: Coding Structure for the Needs Analysis Individual Interviews

4.2 Needs and Perceptions in Terms of Mathematics

The three sub-themes that emerged within this main theme are: factors that influence teaching and learning; Mathematics content—with the specific focus on areas that the participants found problematic to teach as well as content that can be related to real life; and learner aspects. Figure 4.3 indicates the participants' needs with regard to the Mathematics that they are teaching.

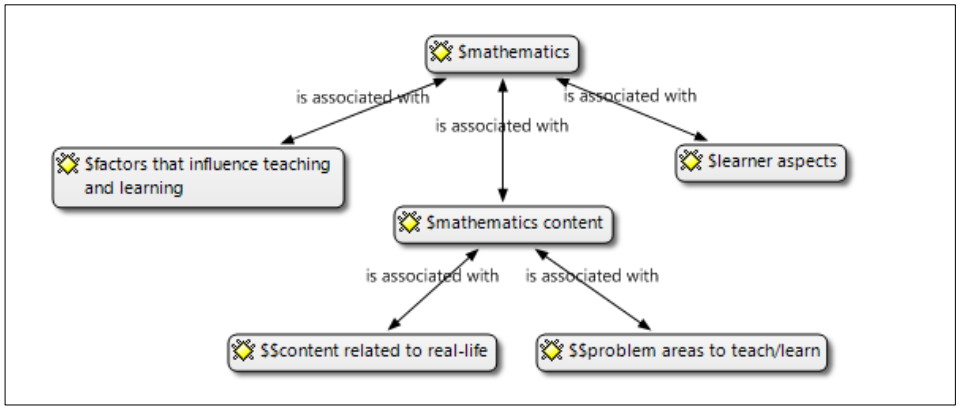


Figure 4.3: Coding Structure for the Participants' Needs in terms of Mathematics

4.2.1 Factors that Influence Teaching and Learning

One of the factors that clearly stood out in the interviews was the participants' aspirations to move forward. Not only are they currently studying to improve their qualifications, although they are employed full-time, but they expressed their desire to develop in other ways as well which could improve teaching and learning:

> *Then if they are not doing well, I used to think that maybe I need some sort of development (P59:23).*
>
> *So, I'm learning again (P49:3).*
>
> *If I'm given a chance everything is okay (P49:37).*
>
> *Yes, I am eager I am eager you know (P60:72).*

Another aspect that influences the teaching and learning of Mathematics is establishing good foundations of Mathematics from the Foundation Phase already:

> *I think it's because they lack a proper foundation, the problem starts at the foundation phase (P59:7).*

Not only does what takes place in the classroom impact on students' learning (Samuelsson, 2010, p. 61), but early Mathematics knowledge and skills are said to be the most important predictors for later achievement in Mathematics, and also for achievement in other content areas (Claessens & Engel, 2013, p. 1).

Immediate feedback to learners is an essential aspect of effective teaching and learning (Sancho-Vinuesa, Escudero-Viladoms, & Masla, 2013, p. 1). The participants also reported that immediate feedback to the learners was important:

> *It will help the learners, even with speed. Uh, marking, even that immediate feedback, because if you are given feedback immediately, you can try it again (P59:67).*

4.2.2 Mathematical Content

In this section, the data revealed two sub-themes relating to Mathematics content, namely: problem areas to teach and learn, and content related to real life. The first sub-theme revealed the following information about areas the learners found challenging and also facets of the curriculum that teachers found to be problematic. The areas of the curriculum that were most problematic for the participants to teach were: Financial Mathematics; Geometry; Probability; Algebra; Integers; Counting techniques; Graphs; Functions; the application of Calculus and Trigonometry. In some cases the content was problematic for the learners as well:

> *Learners do have a problem with algebra and integers in fact. If they do have a problem with integers, it is going to give them a problem when dealing with algebra (P59:4).*

In some cases the teachers found the content challenging as well:

> *Then, recently we have Euclidean geometry which has been introduced. Uhm, it's also a major problem, but I think the problem is not the learners but maybe the majority of teachers who are teaching that. They did not even do it during their times at school. It's something that was brought in as a crash program for teachers to be taught as well (P61:3).*

One of the key characteristics in RME is that learning strands should be intertwined (Drijvers & Doorman, 1996, p. 429). Learners ought to have a cohesive view of Mathematics and the way different topics are associated with each other and fit into one curriculum (Widjaja & Heck, 2003, p. 5). One participant expressed his desire to also learn progressively across grades and across the curriculum:

> *The thing is I started learning probability when I was teaching grade 12. After xxx left, then I took over. So I wanted to do it from grade 10. So this is how now I'm learning, right from grade 12. Because if you're teaching it at grade 12, there are some of the concepts that you don't know at the grade 11 and grade 10 level (P49:8-9).*

The content they found the easiest to relate to real life was Calculus, Probability, Integers, Data Handling, Measurement and Financial Mathematics:

> *Because we are working with money every day (P49:16).*
>
> *Through and through it every day life situations where you talk of borrowing, investments and other areas of activities (P61:8).*

A reason offered for poor learner performance in Mathematics was that learners dislike Mathematics because they are not learning relevant and significant content (Fauzan et al., 2013, p. 162). The study aims to make participants more aware of the need to relate the content that they teach in Mathematics to real life situations. I therefore asked them which areas they needed help with to relate the content to real life. The content areas where they needed help to relate the content to real life were: the application of Calculus; exponents, algebra, functions; integers; counting principles. Although the participants considered Financial Mathematics and probability as topics that can easily be related to real life situations, they also felt that they needed help to relate those two areas to real life situations.

4.2.3 Learner Aspects

One of the participants likes to use peer teaching in her classroom. She pointed out the benefits that her learners experienced:

> *Yes, the learners present and they present it so well, they present it so well. In the meantime they are also learning, you understand? They are also learning. So, I also tell some other learners they must also do it, because you gain more knowledge by just standing there and going home to prepare the topic to do it. They come prepared and then they do it so nicely (P60:85).*

Cahyono (2012) indicated that by interacting with each other, learners can enhance their understanding of mathematical concepts and the way in which they relate to real life (§ 3.2.7).

4.3 The Facets of Realistic Mathematics Education

Participants discussed various aspects that highlighted the facets of RME that are present in their teaching. This section includes a discussion on the characteristics of RME; certain aspects of RME; and the advantages of RME. Figure 4.4 indicates the coding structure that represents the framework for this discussion.

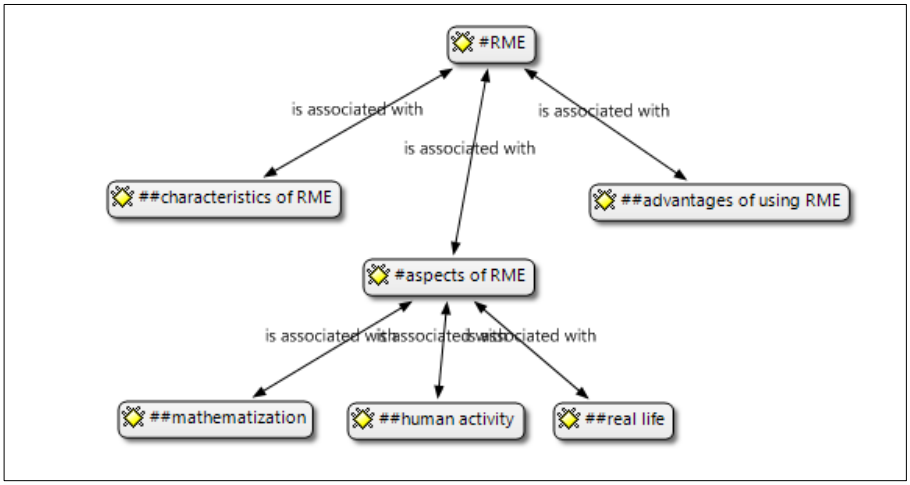


Figure 4.4: Coding Structure for the Facets of Realistic Mathematics Education as Highlighted by the Participants

4.3.1 Characteristics of Realistic Mathematics Education

Participants raised a number of factors that related to the characteristics of RME. As discussed in Chapter Three, the systematic literature review revealed various authors that have discussed the characteristics of RME. These characteristics include: the use of contextual problems; the use of models; student contributions are encouraged; interactivity; and the intertwining of learning strands (Drijvers & Doorman, 1996).

A crucial aspect of RME is when students are able to solve problems in realistic contexts which they deem as relevant (Boon, 2006). The participants expressed their views about the importance of referring to real life situations when teaching Mathematics:

> *Definitely, it is more important to relate it to more real life situations (P49:24).*
>
> *Because if you relate it to real life situations, it becomes easier for them to understand and they can also relate other things to the topic that you are teaching (P59:29).*

Context problems should be the mooring points for students to learn and reinvent Mathematics (Stephan & Akyuz, 2014). The value of using real life situations should therefore not be underestimated. Participants expressed their views about the importance of relating Mathematics to real life:

> *Because then we are linking the new work to something that they are familiar with (P60:26).*
>
> *Like, I said, some of the learners they don't know, when you're playing football or any other sports, Maths is applicable everywhere (P49:25).*
>
> *When they are sleeping, when they get up, Maths is applicable to them, every way, they don't know that, so they take it as something that is very special, they put it aside (P49:27).*

When designing learning activities, teachers should remember that the activity should encourage the students' work with Mathematics (Andresen, 2007). The teachers' role in the design of instructional material is crucial: they need to anticipate supportive mathematical imagery; create challenging assessment; use their mathematical knowledge to improve on the instructional sequence; and rework existing tasks (Stephan & Akyuz, 2014). Participants indicated a few examples of how real life situations can link with mathematical content:

> *In most cases that had to do with gradient. The gradient of that particular road is not evenly balanced. In mathematical terms that has to do with gradient (P61:7).*
>
> *It appears as if there is no link but calculus it also talks about gradient and so forth. That is real life situations (P61:11).*
>
> *Uhm, in Financial Mathematics as well. Through and through it every day life situation where you talk of borrowing, investments and other areas of activities (P61:8).*

Modelling activities should come about in situations and contexts that are experientially real to the students (Doorman et al., 2008). The problem situation should set models in motion that initially represent the problem situation, but have the potential to develop into a more general model (Bakker et al., 2003). One participant clearly emphasized this aspect of RME:

> *So, you cannot do Mathematics without modelling. And, once you talk about modelling, you are now talking about real life situations (P61:12).*

The starting point for learners' learning should be real contexts that are meaningful to them. These situations should be used as the basis for concept formation which leads to more formal Mathematics (Widjaja & Heck, 2003). One participant acknowledged the importance of choosing a context that is experientially real for those particular learners:

> *So it is difficult to relate this to somebody who doesn't know about the situation (P49:21).*

Interactivity is promoted when learners are more accountable for their own learning (Widjaja & Heck, 2003):

And especially sharing those ideas of Mathematics, it will help a lot (P59:57).

As was mentioned in § 3.3.2, the intertwining or integration of different strands or topics is an essential part of RME (Widjaja & Heck, 2003). Participants encourage this integration.

Because then we are linking the new work to something that they are familiar with (P60:26).

Another participant diagnosed a challenge that learners had with integration and also acknowledged the misconceptions that can develop due to a lack of integration:

We have a problem in functions, but the problem emanates from the fact that they don't understand for example exponential function—if learners don't understand how to solve exponential equations then when you go to functions it becomes very difficult (P61:1).

4.3.2 Aspects of Realistic Mathematics Education

The two aspects of RME that were evident in the discussion with the participants were that Mathematics is considered to be a human activity (Freudenthal, 1971); and that problem solving should be done in realistic contexts (Gravemeijer & Doorman, 1999). Freudenthal (1971) advocated that for Mathematics to be of human value, it needs to be useful, i.e. closely linked to reality. One participant used the example of children playing football, and how that could link to Mathematics:

Like, I said, some of the learners they don't know, when you're playing football or any other sports, Maths is applicable everywhere (P49:25).

As mentioned in § 4.3.1, it is important when implementing the principles of RME that we pose problems in realistic contexts, that learners recognise as relevant and real (Boon, 2006). One participant indicated that by following that teaching approach, finding relevant contexts could be challenging. He indicated that what might be familiar to learners in urban areas, might not be relevant to learners in rural areas. In the example that he used, he pointed out that not all learners were familiar with the Kentucky™ fast food franchise, and that teachers should be cautious when deciding on an appropriate context for the learners that were in their classrooms:

You see, most of our learners they are not, how can I say it, they are not familiar with the situations. Like for instance you cannot just ask somebody from the farms about Kentucky, you see? So it is difficult to relate this to somebody who doesn't know about the situation (P49:17-21).

One of the participants expressed his opinion that certain content areas were well-suited to real life situations:

Some of the topics, they come clear to real life situations (P49:13).

He as well as another participant also pointed out that not all areas were easy to connect to realistic contexts:

It's difficult to relate (P49:14).

Hmm, there are topics that we can relate to real life situations, the others they are a little bit more confusing or difficult to relate (P59:17).

A third participant pointed out examples of a realistic context that could be used to relate the content

area of Financial Mathematics to reality:

> *Uhm, in Financial Mathematics as well. Through and through it is an everyday life situation where you talk of borrowing, investments and other areas of activities (P61:8).*

The participants also expressed their ideas about which content areas they found easy to relate to real life. These were Calculus, Financial Maths, Probability, Integers, Data handling, and Functions.

4.3.3 Advantages of Realistic Mathematics Education

In Chapter Three, I pointed out various advantages that are expressed in the literature regarding the use of RME principles (§ 3.3.5.1). These include: improvement of learner motivation (Cahyono, 2012); a more positive attitude towards Mathematics (Fauzan et al., 2013); and interactivity and collaboration are promoted (Widjaja & Heck, 2003). A participant also suggested that by using real life examples, learner understanding could possibly be improved:

> *Because if you relate it to real life situations, it becomes easier for them to understand and they can also relate other things to the topic that you are teaching (P59:29).*

4.4 Ideas and Considerations about Information and Communication Technology

In this section I will discuss the considerations and ideas of the participants relating to ICT that were revealed in the individual interviews. I discuss this section as five main themes, namely the value of using ICT; challenges when working with ICT; various technology devices; the manner in which technology can function as an educational tool; and ICT factors that relate specifically to the participants. Figure 4.5 illustrates the network view of the coding structure for the participants' considerations and ideas about ICT.

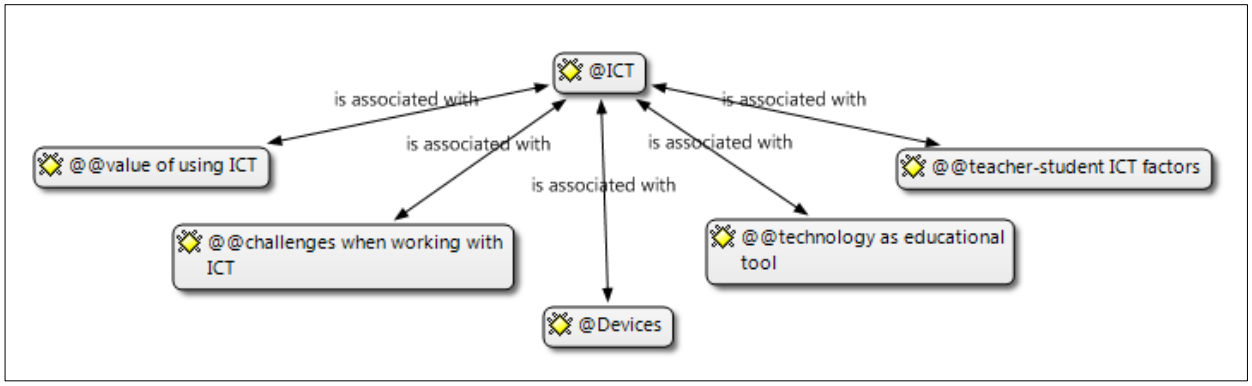


Figure 4.5: Coding Structure for the Participants' Considerations and Ideas about Information and Communication Technology

4.4.1 The Value of Using Information and Communication Technology

In relation to the integration of technology in teaching, Drijvers et al. (2010a) describe the role of the teacher as both critical and problematic (§ 3.4.1). The teacher's role is critical because the teacher's

approach to using technology has far-reaching effects in the classroom. It is problematic because teachers have difficulty in adapting their teaching strategies when working with technology. The participants pointed out positive aspects of using technology in their experience. This is a promising aspect since the teacher's role is so crucial in implementing the use of technology in the classroom. With reference to the Interactive Whiteboard (IWB), the participants said:

> *Because if you are prepared, it could save time. You don't have to write again. As long as you prepare in time, then you just put your preparation there and you go on, so a lot of time will be saved (P59:64).*
>
> *It would be much beneficial for example if I use the interactive whiteboard, almost all learners like it at our school. The whole generation of today, they can all use computers (P61:37).*

With reference to the overhead projector, they said the following:

> *It was easy to use them, rather than making use of the chalk. We could prepare in advance (P49:33-34).*
>
> *But all in all it's too beneficial to use technology. It also saves time, instead of you writing notes on the board for one hour, it's easier using the technology (P61:40).*

4.4.2 Challenges while Working with Information and Communication Technology

Participants expressed their concerns and challenges concerning the use and integration of ICT in their teaching of Mathematics. We should keep in mind that the literature review revealed that using technology in Mathematics education does not provide instant improvement (Drijvers, 2013). Challenges will therefore always exist. One of the participants expressed his concerns about the use of calculators in the lower grades:

> *What about these calculators? Because with the grade eight, let me say grade six, seven and eight, I think that using calculators is not good. Because they tend to...how can I put it—to lose those skills, calculating skills or basic skills of Mathematics. Even a simple problem, they want to use a calculator instead of doing it themselves (P59:68).*

When I asked him about using calculators in higher grades, the same participant said:

> *No, no, calculators at least with them, it saves time. Because there, they are doing more complicated things. For example with financial Maths, they have to use a calculator (P59:70).*

Another challenging area that the participants discussed related to proper training on the use of technology in the Mathematics classroom:

> *There was training, and then the department sent someone from them to come. But, it was not only for Maths (P60:33).*
>
> *Yes, they have the tablet. We have the interactive whiteboard, but we the teachers are still struggling to use the interactive whiteboard. Most especially for Maths. They say that somebody from the department will come and help us (P60:8).*

Further challenges that concerned the participants related to poor infrastructures and facilities:

> *At present we don't have technology devices at our school, but I think those devices if we can have those devices they are going to help us...because it will ease the burden (P59:31).*
>
> *Ja, last time when we were using the overhead projectors, but now the state of our school we cannot, we have so many projectors here, but they are not functioning (P49:34).*

General concerns that they expressed had to do with their experiences and perceptions of technology:

Yes, and my experience as well is that learners might be fascinated with the use of the computer and the overhead for one or two days, then they will lose interest. That is my experience (P61:26).

You can say cell phones are not designed for teaching and learning, it's just for communication purposes, so if you use them for teaching and learning, I don't think it's a good idea, unless if you want to communicate with them, send them homework, maybe the results, something like that (P59:37).

4.4.3 Devices

Two participants pointed out the benefit of working with an Interactive White Board (IWB):

Yes and I can also save my lessons. I can even use it next year for the other groups (P60:54).

It would be much beneficial for example if I use the interactive whiteboard, almost all learners like it at our school. The whole generation of today, they can all use computers (P61:37).

Because in our interactive whiteboards there are pre-recorded lesson (P60:58).

For me, to be able to create a suitable intervention tool, that all participants would have access to, it was necessary to establish which devices the student-teachers would be willing to use. They indicated that they were happy to use the following devices: IWBs, mobile phones, tablets, laptops and desktop computers. Since mLearning is a rapidly expanding field that provides new opportunities for education (Kizito, 2012a), and as the participants indicated their willingness to work with mobile devices, I therefore decided to design a mobile app as an intervention tool. Some of the advantages that mobile devices offer were discussed in § 3.4.3. These include the following: mobile devices offer the potential for stimulation, motivation, availability as well as ease of use (Zaranis et al., 2013). A detailed description of the mobile app is presented in Chapter Five.

4.4.4 Technology as an Educational Tool

The use of WhatsApp™ for groups of learners is said to have various advantages in the educational arena (Barhoumi, 2015, p. 223). The technical advantages of WhatsApp™ include: simple operation; low cost; availability; and immediacy (Bouhnik & Deshan, 2014, p. 217). One participant makes use of WhatsApp™ groups with her Mathematics classes and relies on the immediacy that it offers:

So, I usually take the answers, then I sometimes take a camera, and then I send it to them (P60:41).

The educational advantages include creating a pleasant environment and in-depth acquaintance with fellow students (Bouhnik & Deshan, 2014, p. 217). The academic advantages are accessibility of learning materials; teacher availability; and the extension of learning after class hours (Bouhnik & Deshan, 2014, p. 217). The participant confirms these aspects of teacher availability and extended learning:

We've got a group WhatsApp™. If there is maybe something that they don't understand, we chat (P60:36).

Yes, like I give them homework, somebody's stuck along the way they don't have any, you know they don't have many resources even to refer to. I will give them one or two tips on how to continue with the problems (P61:29).

Participants referred to other forms of technology that are used as educational tools. The participants

referred specifically to the use of web-based interactives and GeoGebra:

> *With a computer, I think there are other programmes which might help. When I was doing ACE in Potchefstroom, there are those programmes where you will be tested on a certain topic. Then the computer will immediately correct you and show you how to do this problem if you are doing it wrongly (P59:66).*
>
> *If you want to show a learner that the angle at the centre is double the angle at the circumference attended by the same chord for example, you can do it by technology (P61:16).*

Tablets are another form of mobile technology that is becoming a popular educational tool. The benefits of using tablets include: ease of use; anywhere, anytime learning; a variety of educational apps are available for use; teachers' workload can be reduced with digital collection and marking of tasks; and greater autonomy of learning (Churchill, Fox, & King, 2012, p. 252). Participants also referred to the use of tablets in the classroom:

> *Some of the questions, previous question papers are there with memos, and they can research, they can do whatever they want there. It's off-line at night, but during the day it's on-line (P60:14-15).*

4.4.5 Participants' Information and Communication Technology Factors

The participants in general are positive about and can benefit from the use of technology in various ways:

> *Yes, I am eager I am eager you know, to let them come and use that (60:72).*
>
> *I'd be willing to use it with other colleagues to discuss tasks related to Maths (P59:42).*
>
> *I'm using a computer at my home (P60:45).*

However, a number of challenges remain:

> *I don't have time for social media (P49:44).*
>
> *My email is not working, but I will try to fix it (P49:65).*

4.5 Role players

In this section I discuss four role players that emerged in both the literature and in the interview data. These are teachers, learners, the community and the context. Figure 4.6 illustrates the coding structure that I used for this data analysis.

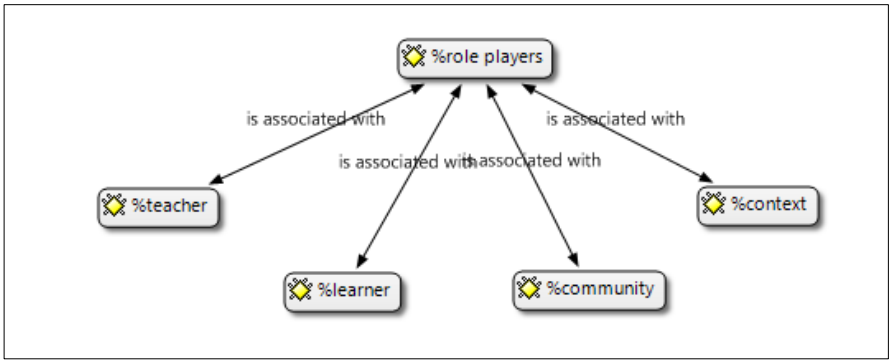


Figure 4.6: Coding Structure for the Role of Various Role players in this Study

Although teachers and learners could benefit from the use of technology, not all stakeholders share the optimism about how ICT can effectively be used in teaching and learning (Drijvers et al., 2010a; Heck et al., 2007). Widjaja and Heck (2003) and Zulkardi (2000) affirm the changing role of teachers and learners in the RME classroom. These issues can account for the increased pressure on the various role players in education. The participants expressed their opinions on the topic.

4.5.1 The Role of the Teacher

Various issues were raised that related to the role of the teacher. Firstly, the challenging task of a Mathematics teacher was emphasized:

> *Ja, so it's difficult to teach Maths, very difficult (P49:31).*

Yet the participants' enthusiasm was tangible:

> *I'm excited (P60:28).*

A common belief exists that teachers teach the way they were taught. Dunn and Dunn (1979, p. 241) rather believe that teachers teach the way they learned, and that their teaching style is directed by learning styles. One participant also raised this issue:

> *But how I think is also influenced by the way I was taught myself although I have changed a bit (P61:13).*

The participants also raised challenges that they experience in terms of time and professional development:

> *Time is a problem, because now I'm having extra classes with the grade elevens on Wednesdays and also on Saturdays. I don't have time for social media (P49:43-33).*
> *I feel that maybe if I can maybe have more knowledge (P60:31).*

4.5.2 The Role of the Learner

Since the study was more focused on the perceptions of the teachers, few aspects relating to learners were revealed despite social and contextual circumstances that can negatively influence learners' learning (Considine & Zappala, 2002, p. 130). One participant testified to the poor socio-economic status of their learners:

> *But, maybe only, let me say ten percent of our children have got a good background, they have got a satellite dish. So mostly they are from shacks, where they do not have electricity, you know... (P60:77).*

4.5.3 The Role of the Community

The role of the community should not be underestimated. Community demographics affect how students perform in Mathematics (Sackey, 2014, p. i). The participants shared their experiences of the community:

> *Yes, because I once asked them to watch that learning channel. Then I realized that most of our learners, 90% of them don't have that. Some they have the TV, but they don't have the*

dish, they can't afford to pay the dish every month (P60:78).

If learners have supportive relationships with their parents, it positively influences learner motivation (Wentzel, 1998, p. 206) and academic performance (Makgato & Mji, 2006, p. 261). Despite often being aware of this, parents are rarely involved in their children's schooling (Makgato & Mji, 2006, p. 261). The participants touched on this aspect in the discussion:

> *It goes back to what I said; there are so many factors within the community which are inherent on how this will work. I did not think that I had enough support from their parents. Because learners in their own, they have a lot of things they need to do (P61:30).*
>
> *Even their parents, they are worse than these learners (P49:29).*

4.5.4 The Role of the Context

There is a definite link between the quality of school facilities and student achievement (Uline & Tschannen-Moran, 2008, p. 55). Participants point out the challenges they experience with regard to the school facilities:

> *And some of the classes do not have electricity. That's another problem (P49:35).*
>
> *You have a situation here where there is no electricity in the community, electricity is gone. But, for me I think through my experience, I have tried both electronic and the traditional. I think they should all be used concurrently in such a way that for example after using the traditional one, you can move onto technology so that the learners can understand it better (P61:38).*
>
> *Unfortunately at the moment, because we have only one interactive whiteboard in one of the computer labs. So, any specific teacher who wants to use it will have to discuss it with other teachers when and so on. So, I have not agreed to having a class with them because of the congestion of the program of the curriculum. Because for example in grade 10 we have nine classes, grade 11 around seven, grade 12 we have close to two hundred learners. It has not been easy (P61:35).*

Howie (2003, p. 2) suggests various factors for the poor performance of learners in Mathematics in South Africa in particular. In relation to institutional issues these include "the lack of professional leadership; heavy teaching loads; overcrowded classrooms; poor communication between policy-makers and practitioners; and a lack of professional staff in the ministry of education." So too did the participants highlight aspects relating to the higher structures that they found problematic in their teaching practice:

> *For example with Euclidean geometry, the curriculum statement does not give the teacher the opportunity for learners to discover where these theorems are coming from (P61:15).*
>
> *Since the department said that every grade 12 teacher must have a laptop, but we are still waiting for them to give that to us, the laptop (P60:44).*
>
> *We even went to the workshop, but you know those workshops are only for three days. And we were doing so many things—we were doing geometry because they also introduced geometry. But we did it in our times—those theorems. Yes, all of us feel that it was not enough training, really it was not enough (P60:3-5).*

4.6 Chapter Summary

In this chapter I presented a detailed analysis of the interview data that were generated to establish

what needs the participants had in terms of teaching Mathematics in a more realistic way. I firstly explored the participants' needs in terms of Mathematics itself, and started by focusing on factors that influenced teaching and learning. An aspect that emerged that was very reassuring was the desire of the participants to move forward and improve as teachers. Their aspirations were evident not only in that they are currently improving their qualifications, but they also expressed the desire to be involved in professional development and training. Other aspects that they raised regarding factors that influence teaching and learning were twofold. They discussed the importance of establishing good Mathematical foundations when teaching and also mentioned the significance of providing learners with immediate feedback.

This chapter also presented a discussion on the problem areas that learners had, as well as challenges that teachers faced regarding the content of the Mathematics curriculum. Literature has pointed out that a possible reason for learners' poor performance in Mathematics is that the content is not presented as relevant and significant. I therefore explored areas that the participants had difficulty in making the content realistic, relevant and significant for their learners. This would help me in designing a relevant and appropriate intervention tool.

Since this study is deeply entrenched in the theory of RME, it was essential to touch on the characteristics of RME in this discussion of the data. All five characteristics of RME emerged in the data as participants discussed the importance of interactivity, modelling, integration between strands, the use of realistic contexts and learner participation. These characteristics all featured strongly in the discussion with the participants. Apart from the characteristics of RME, I further explored two aspects related to RME, namely that Mathematics is considered to be a human activity and the importance of real life examples in Mathematics teaching. This led to a discussion on the advantages of using real contexts to improve learner understanding, which literature has shown us.

Apart from Mathematics and RME specifically, the other main theme that underpins this study is the use of ICT in teaching and learning. This chapter also detailed the teacher-students' considerations and ideas about ICT. I discussed their ideas regarding the value of ICT and the critical role that teachers play in the integration of technology in teaching. I further explored the challenges raised by the participants. These included concerns about using calculators in the lower grades; the lack of technological training that the participants experienced; the poor infrastructures and facilities that they had to deal with which impacted on their use of ICT; and general experiences and perceptions that the participants shared about ICT. They also discussed their ideas and experiences about different technological devices such as the Interactive Whiteboard. In order to successfully incorporate technology in the intervention stage of this study, it was important for me to establish which technology they were most comfortable with. The participants shared their ideas on this topic and clearly indicated their preference for mobile technology. This informed my choice of technology for the design phase of the study. I also explored their ideas on technology as an educational tool. They shared their experiences of working with WhatsApp™ groups, GeoGebra, web-based interactives and tablets.

The final stage of this chapter dealt with the four role players that featured in the discussion with the participants, namely the teacher, learner, community and context. In the interviews, the participants reinforced the notion that teaching Mathematics was a challenging task, yet their enthusiasm to be Mathematics teachers was evident. They expressed their opinions on challenges they experienced in terms of teaching styles; time and professional development. The most dominant aspect relating to learners that featured was the effect of a poor socio-economic status on their learning. The role of the community and the context was also discussed and participants shared their views on aspects such as the crucial role that parents play in the children's education; the effect of poor school and home facilities on learner achievement and the influence that decisions by authorities had on teaching and learning.

Chapter Five presents the design of the intervention tool. In this chapter I discuss in detail the purpose of the mobile tool; and the design process. When discussing the purpose, I explored the theory of mLearning, the connection between technology and RME and also the context for which the tool was designed. In the discussion on the design process, I expanded on the theoretical aspects of the design process and also the technical facets of the design process.

As was discussed in § 2.4, upon completion of each stage of a design-based research project, researchers will perform systematic reflection and documentation to produce the required scientific yield in the form of theories or design principles (Plomp, 2007, p. 15). As is illustrated in Figure 2.6, upon completion of the needs and content analysis phase, I have produced a number of design principles relating to that phase. The design process led to the realisation of a number of principles that were established in the needs analysis phase. These principles are presented in Table 4.1.

Table 4.1 Design Principles for the Design of a Mobile App Based on the Principles of Realistic Mathematics Education Derived from Individual Interviews with Participants

Aspect	Implications
Determine participants' needs in terms of teaching Mathematics	• Establish what needs the participants have with regard to content areas (both teachers and learners) • Establish participants' needs in terms of making Mathematics realistic for the learners
Establish the participants' eagerness for being mentored	• Participants should indicate a desire to be involved in professional development • Participants must aspire to move forward
Adhere to the characteristics and principles of RME	• Meaningful, realistic contexts should be the foundation for students to reinvent the Mathematics • Student contributions should be encouraged • Incorporate the use of modelling • Make use of interactivity • Ensure the intertwining of content areas and concepts
Establish participants' views about ICT	• Participants should see the value of ICT • The challenges that participants experience with regard to ICT should be addressed as far as possible • Establish which devices the participants find most suitable

Aspect	Implications
	• Determine in what manner participants use technology as an educational tool
Determine which role players feature most prominently within the context	• Teachers are the key role players in integrating ICT in teaching and learning • Learners' learning is influenced by their socio-economic circumstances • Parents play an essential role in learners' academic performance • Facilities and higher authorities can be important within the given context

Chapter Five

The Design and Development of the Mobile Intervention Tool

5.1 Introduction

A purpose of this phase of the investigation was to establish the support needs of teachers in order to effectively implement the RME approach in their teaching practice. After doing individual interviews with students (Chapter Four), I established the areas in which they needed assistance, not only with their teaching, but also in making Mathematics more realistic for their learners. To address these areas of need, it was necessary to design an intervention tool for the teacher-students to use, based on the principles and characteristics of RME. Figure 5.1 illustrates the structure that will be followed in this chapter.

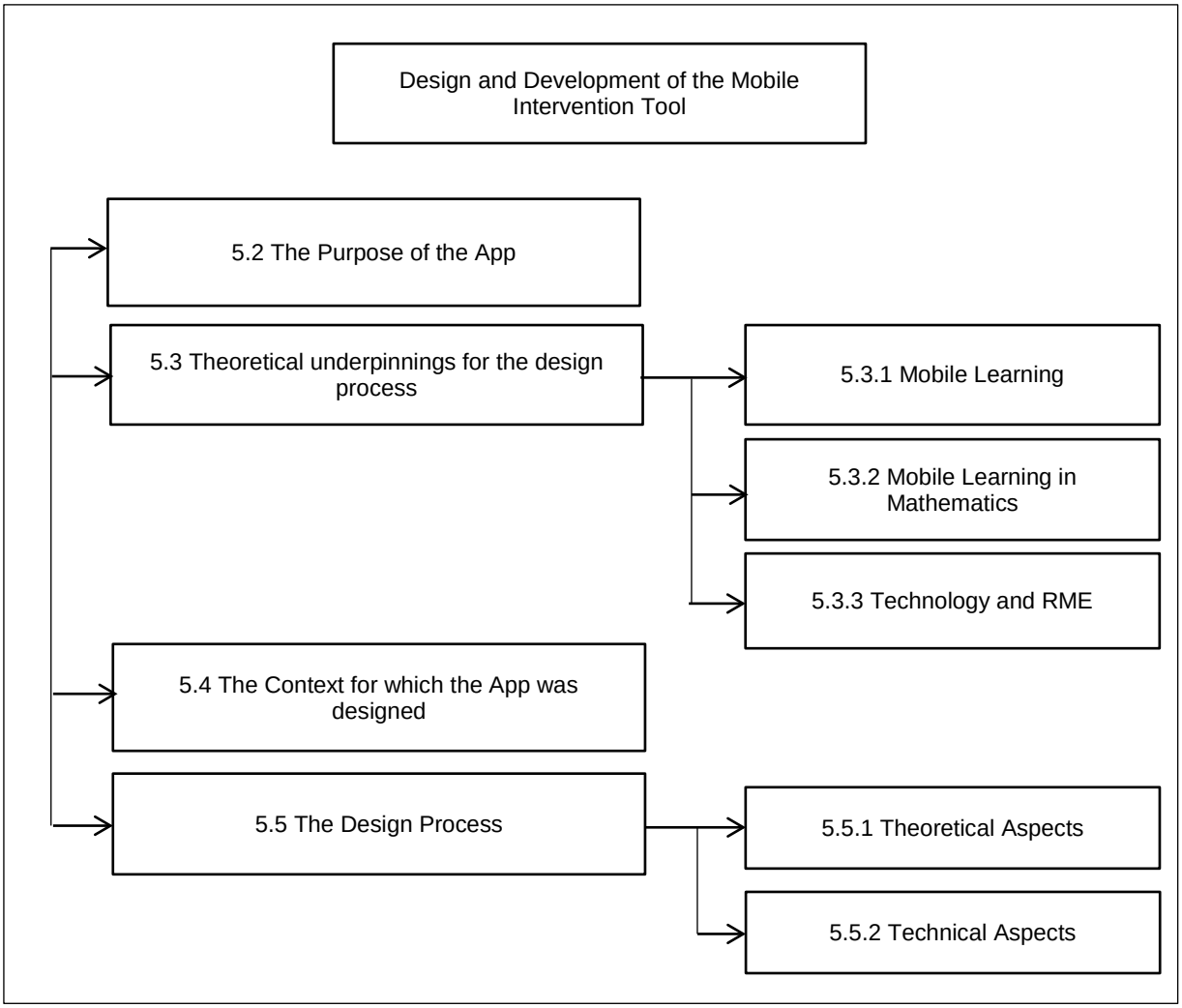


Figure 5.1: Advance Organiser for the Design and Development Process of the Intervention Tool

5.2 The Purpose of the Mobile Tool

MLearning has made a valuable contribution to the educational environment in recent years (Alzaza & Yaakub, 2010, p. 95). Mobile devices are stimulating platforms for learning due to their computing ability, their affordability and their ubiquitous nature (White & Martin, 2014, p. 95). The teacher-students in this study shared their perceptions, experiences and needs regarding the academic content and the effective implementation of the RME approach in their teaching practice (Chapter Four). They confirmed inter alia that they were excited about the use of mLearning to address the issues and needs that they faced in their teaching, specifically relating to content and linking school Mathematics to real life situations. I therefore deemed a mobile tool, in the form of an app, to be suitable for the teacher-students in order to present the mathematical content and principles of RME to them.

5.3 Theoretical Underpinnings for the Design Process

In this section I discussed some of the theoretical underpinnings that drove the design of the app. I investigated relevant literature relating to mLearning in general and probed aspects relating specifically to the design and use of mLearning in Mathematics. I also briefly mentioned the link between technology and RME. A detailed discussion of this literature concerned with ICT and RME was done in the systematic literature review in § 3.4.5.

5.3.1 Mobile Learning (mLearning)

Literature offers various definitions for mobile learning. MLearning is learning realised with the aid of a small, portable computing device, which typically involves the viewing of content or lessons in manageable chunks (McConatha, Praul, & Lynch, 2008, p. 15) as a specific type of learning model for using mobile technology (Cheon, Lee, Crooks, & Song, 2012, p. 1054). Sharples (2013, p. 2) provides various definitions for mobile learning. He highlights one definition that encompasses both perspectives of learners' mobility as well as learning with portable technology as that of O'Malley et al. (2003, p. 7), which states: "Any sort of learning that happens when the learner is not at a fixed, predetermined location, or learning that happens when the learner takes advantage of the learning opportunities offered by mobile technologies."

Mobile technology offers a new generation of learning to people of all ages without being bound by place and time (Alzaza & Yaakub, 2010, p. 95). mLearning is becoming more popular within formal education as its benefits are widely publicised as being cost effective and flexible in terms of time and location; it is a ubiquitous form of communication, and can serve as a study aid (Cheon et al., 2012, p. 1054). Cheon et al. (2012, p. 1054) summarise the characteristics of mobile devices into three aspects: the portability; instant connectivity; and context sensitivity.

Al-Emran, Elsherif, and Shaalan (2016, p. 94) list the benefits of mLearning for students: it enhances collaboration between learners with each other as well as with their educators; it facilitates team collaboration; the sharing of knowledge; and raising of technological awareness. MLearning assists students to create social interaction; it promotes collaborative learning, interactivity and instant feedback as well as collaboration between peers; it improves their knowledge structure; their learning achievements (Domingo & Garganté, 2016, p. 22; Mouza & Barrett-Greenly, 2015, p. 3) and motivation (Mouza & Barrett-Greenly, 2015, p. 3). Domingo and Garganté (2016, p. 22) further point out that students are more willing to engage when learning with mobile technology; their desire to accomplish educational tasks also increases with the use of mobile technology; and it helps learners to be more self-directed in their learning. Mobile communication in education is a solution with a selection of prospects and challenges (Kommers & Hooreman, 2009, p. 88).

mLearning applications have various educational benefits: they can be used as study aids; can be accessed from almost anywhere; and with the aid of location capabilities, students can use location-based information (Cheon et al., 2012, p. 1054). Mobile technology apps enrich higher education by extending traditional educational platforms and encouraging distance learning or settings outside of the classroom (Al-Emran et al., 2016, p. 94). Content applications that make use of personalized instruction can facilitate academic growth and self-efficacy among students (Mouza & Barrett-Greenly, 2015, p. 12). In their paper on the professional development of teachers, Mouza and Barrett-Greenly (2015, p. 13) note that professional development that is grounded in recognised best practices is essential in assisting teachers to obtain the knowledge and skills they need to fully benefit from mLearning.

The use of mLearning has become essential for higher education due to its numerous benefits (Al-Emran et al., 2016, p. 94). Increased affordability and functionality has made mLearning more attractive in education (Domingo & Garganté, 2016, p. 21). Jantjies and Joy (2015, p. 308) are of the opinion that mLearning apps used in subjects such as Physical Science and Mathematics require appropriate content that considers the users' challenges and needs.

In recent years, there has been a tremendous increase in the use of tablets and smartphones in the educational arena (Calder, 2015, p. 233). Tablet computers provide new potential for Mathematics learning, (Zhang, Trussell, Gallegos, & Asam, 2015, p. 32). Tablets offer the following benefits: they are light; they are portable; most tablets have a long battery life; and they offer a positive sensory experience to children. They also allow learners to do Mathematics tasks at their own pace; and provide immediate feedback (Zhang et al., 2015, p. 33). Immediate feedback is one of the aspects that teacher-students in this study identified as an essential aspect of teaching and learning in the individual interviews (§ 4.2.1).

5.3.2 Mobile Learning (mLearning) in Mathematics

The use of digital technology in Mathematics education allows for improved engagement with mathematical processes and concepts (Calder, 2015, p. 233). Specifically, mLearning affords new learning opportunities in Mathematics (Graven, 2011, p. 45). Mathematics apps are an effective way in which learners who struggle can be provided with instructional support (Zhang et al., 2015, p. 38). They allow students to solve more problems than when using paper and pencils; they are easy to use; and are generally able to actively engage students (Zhang et al., 2015, p. 38). Students' experiences with mobile devices are often more rewarding than teachers may realise (Lee & Kautz, 2015, p. 575). When learning Mathematics, students have the opportunity to explore and take risks with digital technology, which is not always otherwise possible (Calder, 2015, p. 238).

Cayton-Hodges, Feng, and Pan (2015, p. 3) report on a survey of Mathematics apps available in the Apple App Store. The intention of the survey was to learn what should and should not be done in the development of Mathematics assessment in the apps. They mention various aspects that should be considered in the design of Mathematics apps. There should be a rich and accurate representation of the mathematical content; the interactions around the mathematical content should be lavish and engaging; there should be an opportunity for users to reflect on tasks they have done; and hints, scaffolds and other tools are important and are typically presented on a need to know basis (Cayton-Hodges et al., 2015, p. 16).

Lee and Kautz (2015, pp. 576-577) present a framework for the design and implementation of mathematical tasks with the aid of mLearning. The framework consists of six steps, which do not have to take place chronologically. The purpose of the framework is to provide structure, specify components and indicate the relationships between technology-based task design and implementation. The steps are as follows: define the lesson objective; define the technological environment; define the types of investigations; design and develop the tasks; implementation; and enactment (Lee & Kautz, 2015, p. 577). Figure 5.2 illustrates the framework suggested by Lee and Kautz (2015) which focuses on the aspects that need to be considered and the actions that need to be taken when incorporating mobile technology into Mathematics teaching and learning.

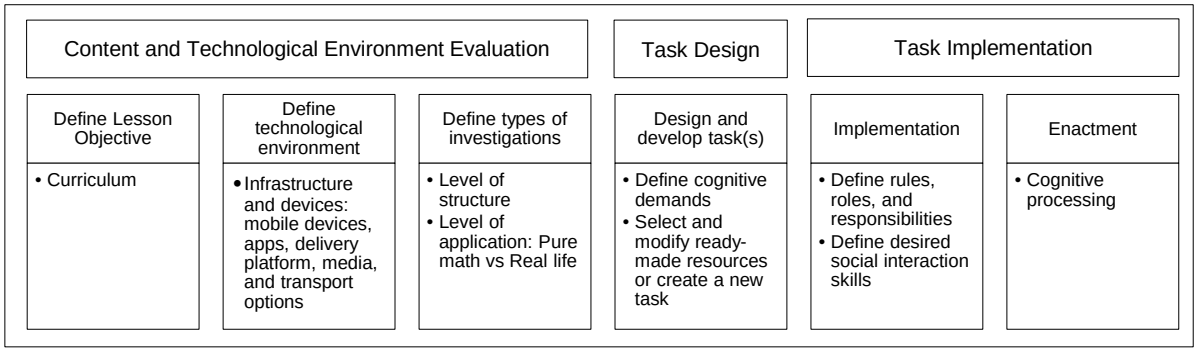


Figure 5.2: Framework for mLearning task design and implementation (Lee & Kautz, 2015, p. 577)

Students and teachers alike generally find apps engaging and motivating, and promote their inclusion in programmes. This might partly be due to the fact that learners today are captivated by digital media, and are able to use these to communicate, investigate and process ideas very effectively (Calder, 2015, p. 246).

5.3.3 The Relationship between Technology and Realistic Mathematics Education

Drijvers et al. (2010b, p. 89) describe the integration of technology into the teaching and learning of Mathematics as urgent. The fundamentals of how Mathematics should be taught are challenged by the incorporation of technology in Mathematics education (Olive et al., 2010, p. 138). Digital technology in Mathematics education has the capability to allow students to construct and engage with mathematical knowledge which allows for the subject to be entrenched in authentic contexts and sanctions the creation of meaning for the students (Bray, Oldham, & Tangney, 2015, p. 2489). Technology can be used as an effective facilitator in the implementation of the RME approach (Widjaja & Heck, 2003), and using technology in the RME approach produces better results than traditional teaching (Zaranis, 2013). As discussed in Chapter Three, it is essential that digital technology be incorporated in different contexts in a natural way (Drijvers, 2013) (§ 3.3.3.6). So too should the context be chosen in such a way that the technology can aid the process of mathematization (Boon, 2006). Technological environments have the prospect of reconnecting the learners with authentic contexts in which they can create meaning (Olive et al., 2010, p. 138). The use of technology also places the emphasis on the "practice and applications of mathematics through visualisation, manipulation, modelling, and the use of mathematics in complex situations" (Olive et al., 2010, p. 139).

5.4 The Context for which the App was designed

The app was designed to assist teacher-students with the challenge of making the mathematical content more realistic for their learners. The teacher-students are in-service teachers from both rural and urban environments. They are enrolled for the BEdHons programme in Mathematics education which implies that they either have a first degree or a certificate such as the Advanced Certificate in Education (ACE) with Mathematics as specialisation area. They all share the common desire to improve their qualifications. The teacher-students are older than the average student since they already have a prior qualification and are required to be employed full-time as teachers. They often have family commitments along with the commitments of full-time teaching and part-time studying. The majority of the students teach at disadvantaged schools and are women (Kok, 2009). The participants in this study agreed that they were willing to use technology to take part in the project, and all agreed that mobile technology was a suitable method of communicating and delivering material within the context of this study (§ 4.4.3). It was therefore appropriate within the context of this study to design an app as an intervention tool.

In the 2016 ECAR Educause survey performed at the North-West University, 68% of students were found to have android devices, 21% to have iOS, 8% to have a windows device and 3% to have other devices (Dahlstrom, Brooks, Pomerantz, & Reeves, 2016). It was therefore fitting to design the app for Android devices since the majority of students own an Android device.

5.5 The Design Process

The process of designing the app required a team effort. The team consisted of an experienced programmer that designed the app; myself, who featured as the content specialist and also the creator of the design documents; my promoter, an expert in the field of using technology in teaching and learning, and who also oversaw the design process; and my assistant promoter, an expert in the discipline of Mathematics as well as Mathematics education, with a keen interest in the use of technology in Mathematics education. Various theoretical and conceptual aspects as well as specific technical aspects guided the process. The app consists of two main sections: a content section and an activity section. In the content section the mathematical content is presented in a systematic way. The progression of the topic is done methodically to assist those users who are not completely au fait with the content. The second component in the app is an activity section, where the concepts that were presented in the content section are reinforced and also extended to new and challenging situations.

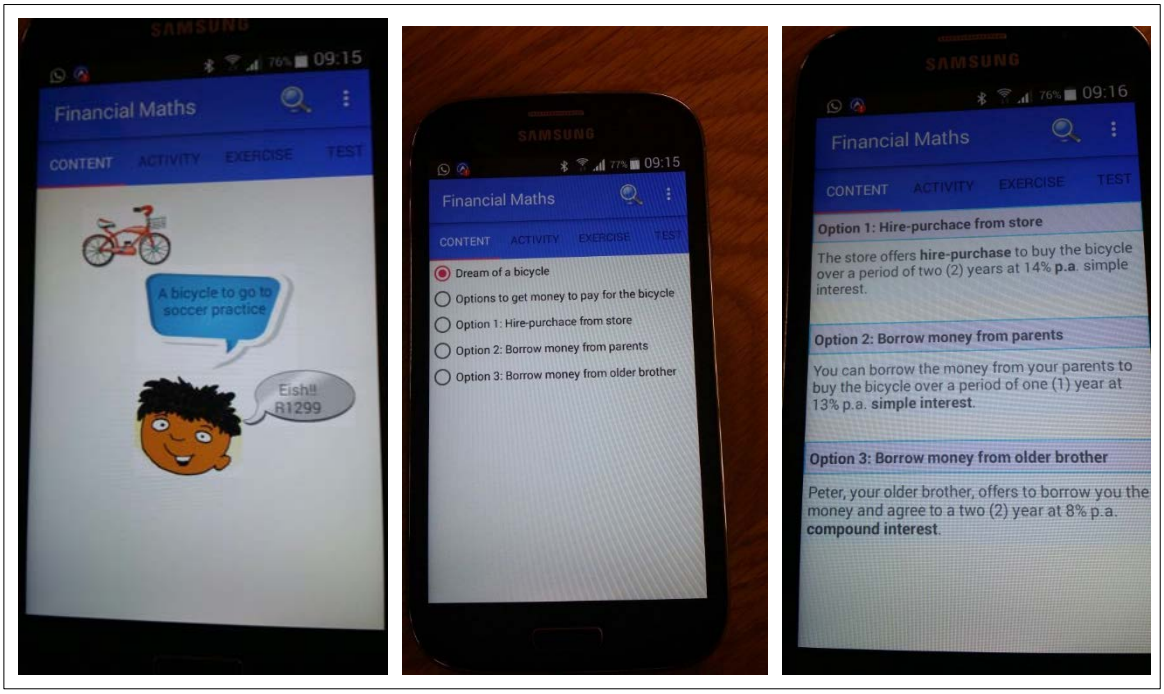
5.5.1 The Theoretical Aspects of Creating the App

McKenney and Reeves (2012, p. 78) state that the process of design-based research starts with an analysis of the existing problem in educational practice and is often supplemented by literature to establish a proper understanding of the problem. In Chapter Two, Figure 2.6 illustrates the design process specific to this study. A detailed discussion of the research design methodology is presented in Chapter Two. After the completion of the systematic literature review and the needs analysis interviews with the teacher-students, the first phase of the design process for this study was completed.

After the needs analysis interviews with the teacher-students had been done, the data were coded, classified and analysed. The detailed analysis was described in Chapter Four. Teacher-students shared their perceptions and experiences relating to what support needs they had in terms of effectively implementing the RME approach in their teaching practice. The data analysis revealed the content areas that the teacher-students and/or their learners had difficulty in understanding. The topics which all four respondents experienced as problem areas were Financial Mathematics; Geometry; Probability; Algebra; Integers; Counting techniques; Graphs; Functions; Calculus and Trigonometry (§ 4.2.2). The participants shared that they needed help in relating the following content

areas to real life: Calculus; Exponents; Algebra; Functions; Integers; Financial Mathematics; Probability; and Counting principles (§ 4.2.2). I selected to design and develop a mobile app for the content area of Financial Mathematics because this topic featured in both categories discussed above. This process encompasses the needs and content analysis stage of the design process as presented in Figure 2.6.

The second phase of the process (Figure 2.6) relates to the design development and evaluation phase. Solutions informed by existing design principles and innovations in technology comprise the second step in the process (Amiel & Reeves, 2008, p. 34). To implement this phase, I approached a programmer who is experienced in mobile app development to design and construct the intervention app. An initial discussion took place between my promoter, the programmer and I where we discussed the possibilities and requirements for designing the app. I then proceeded to draw up a first draft of the app design document (Addendum 5.1). This document provided the programmer with a broad idea of the content of the app. It also outlined the tasks, hints and instructions in detail, as well as some notes to the programmer on how I wanted the app to work. The initial draft of the app design document was perused by my promoter and sent to the programmer before he developed a first prototype of the app. Figure 5.3 illustrates the screenshots of the first prototype of the app.



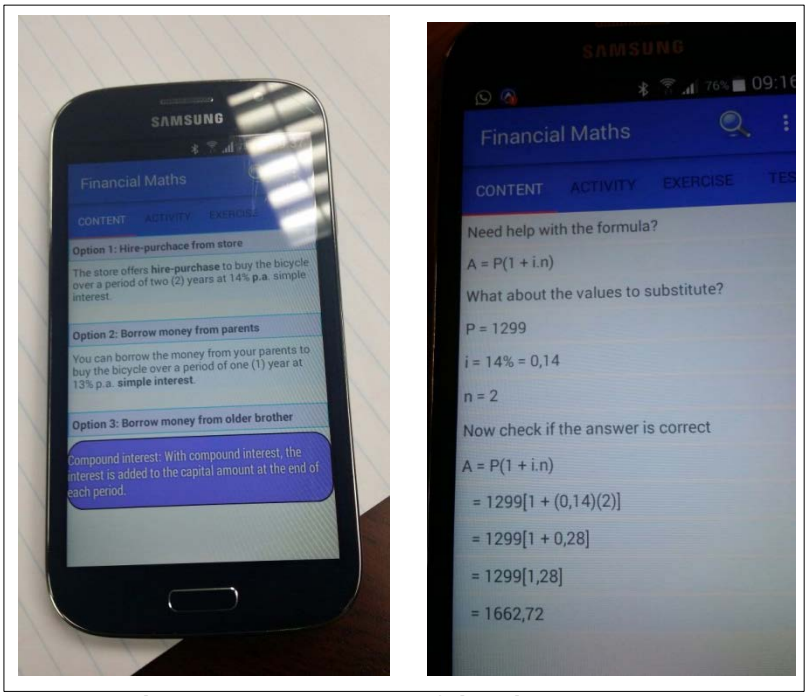


Figure 5.3: Images of the First Prototype

My promoter and I met with the programmer once the first prototype was designed. We discussed further expansion of the app and looked at possibilities that were viable within the parameters of the study. These factors were also discussed with the assistant promoter, who gave guidance particularly in the areas of the mathematical content and the principles and characteristics of the RME approach. After this discussion, I edited and extended the app design document for the further development of the app. A third meeting was arranged between the other three experts involved in the design of the app. At this meeting the second prototype was presented and discussed. The assistant promoter also provided input and feedback on the second prototype. Plomp (2007, p. 15) states that experts or users can be involved in the testing of the prototypes. In this case the first and second prototypes were tested by the promoters, me and the programmer.

The final evaluation and reflection stage forms the third step of the design-based research process (Figure 2.6). This involves iterative cycles of refinement of solutions that are implemented in practice (Amiel & Reeves, 2008, p. 34). To refine the app further, I asked an RME expert, who is extensively experienced and qualified in both Mathematics and Mathematics education to evaluate the third prototype of the app with the specific focus on the inclusion of the RME principles and characteristics. His evaluation revealed that the principles and characteristics were sufficiently included in the app. He further added that the app should achieve the purpose for which it was created, which is to assist teacher-students with making the teaching and learning of Mathematics more realistic for their learners. He suggested that the app be tested before being released for the teacher-students to use.

This suggestion was followed up and duly done—the app was further tested by two Mathematics education experts. Their main suggestion to improve the app was to expand on the initial problem situation, and make it more understandable to the user:

Put something in that sets the scenario more clearly. The start of the app is too vague.

Their evaluation also reinforced the need to create the app and confirmed its suitability for teacher-students:

> *It works very well because all the steps are clearly shown. It will help the teachers to teach their learners to follow logical steps in their calculations.*

The app included an activity that clearly showed the intertwining of the concepts of compound and simple interest, a key characteristic of RME (Widjaja & Heck, 2003). The Mathematics education experts found this appropriate and necessary:

> *I like this very much. By including this comparison of concepts you can see wow, it's quicker to use the compound interest formula, and yet you still get the same answer as when repeating the simple interest formula.*

The experts also suggested that the app be extended to other content areas and that it be made available to a wider range of teachers.

When designing the app, the principles and characteristics of RME played an essential role. As was discussed in detail in Chapter Three, the five characteristics of RME are: i) the use of contextual problems; ii) the use of models; iii) the use of student contribution; iv) interactivity (between pupil and between pupils and teacher); and v) intertwining of learning strands.

The context was selected so that it was meaningful and natural to the students. As is illustrated in Figure 5.4, in the app in the section on the content, the scenario of a school girl or boy who wants to buy a bicycle to attend sport practice was used as it is a common and natural phenomenon to school children.

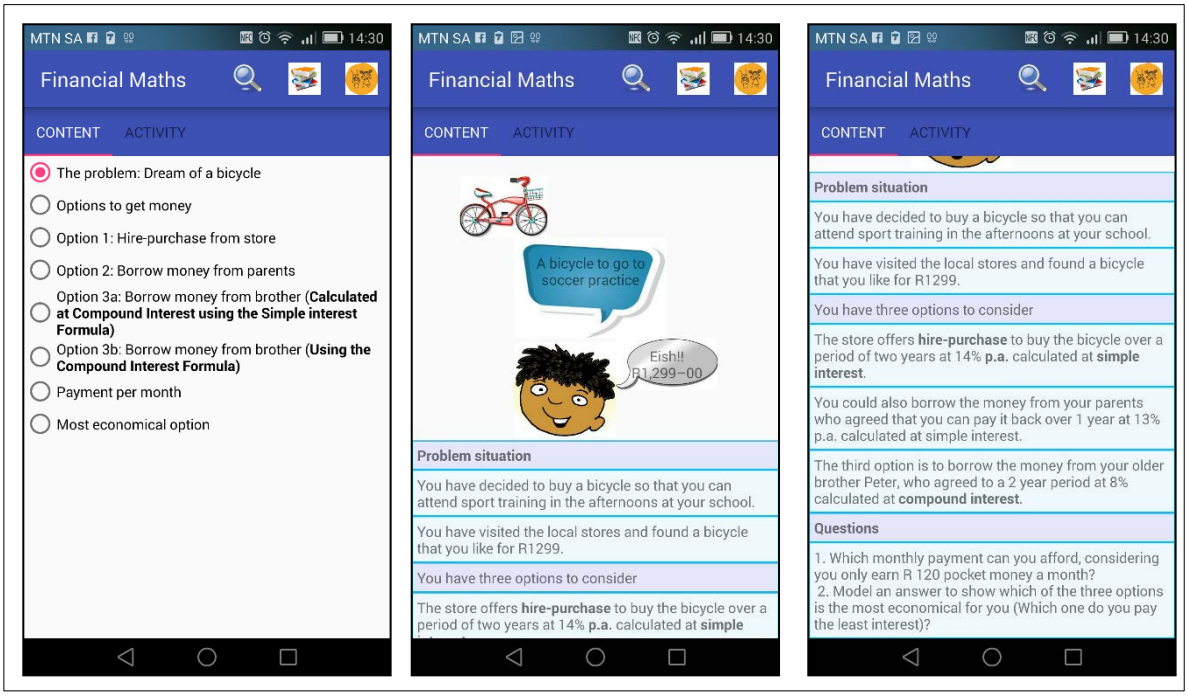


Figure 5.4: A Meaningful and Natural Context is Created

The context should allow students to develop intuitive notions that form the foundation of the concept (Stols, 2012). The authentic purchase option for the bicycle allows the students to establish and expand on the concept of Financial Mathematics. As Widjaja and Heck (2003) state (§ 3.3.2), the context should be the anchor for the formation and development of concepts. As Figure 5.5 illustrates, from the given context, the concepts of simple interest and compound interest develop in a natural manner.

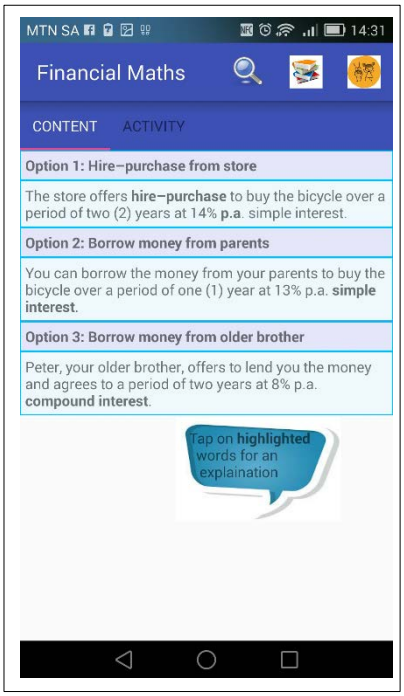


Figure 5.5: Simple Interest and Compound Interest Develop in a Natural Manner

The use of models is encouraged in RME (§ 3.3.2). Models in the form of equations (Figure 5.6) are used in this task and students are also asked to model an answer when comparing the three options for the loan. They are encouraged to use any type of model and are then given information to establish if their model contains the correct information.

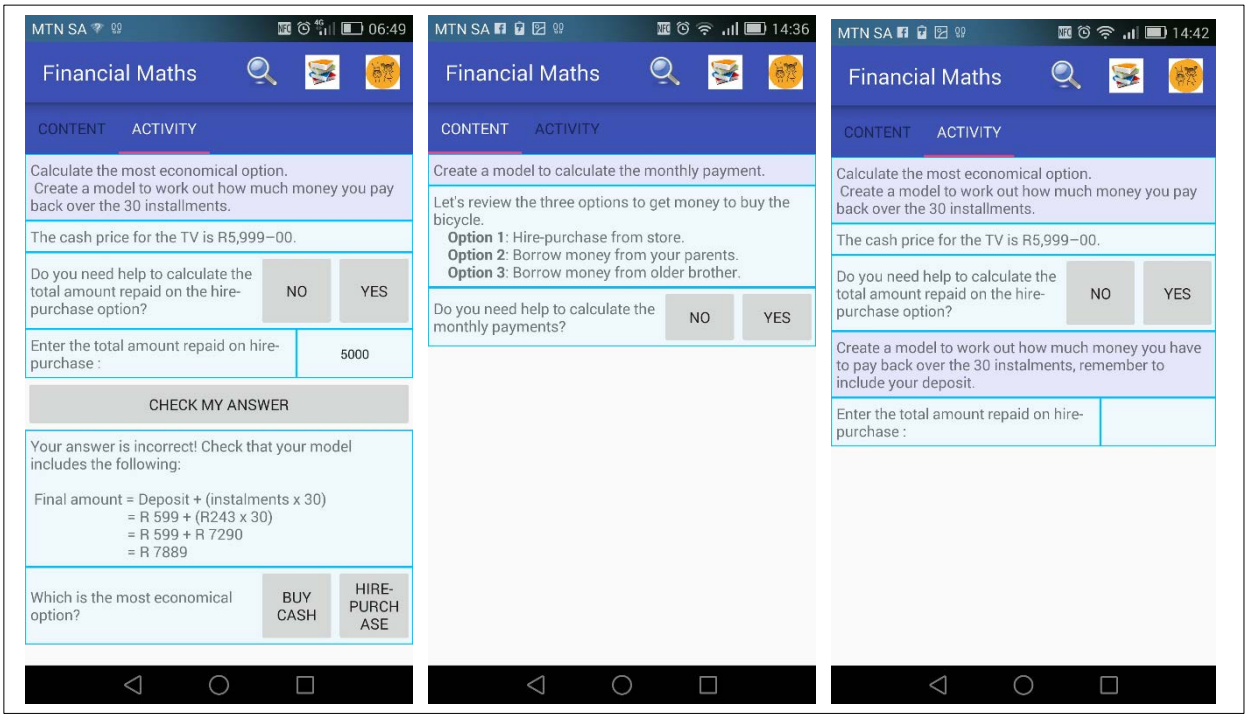


Figure 5.6: The Use of Models is Encouraged

The third characteristic of RME is that student contributions are encouraged. Technology assists in creating an environment in which learners can be actively involved, where they can discover, explore and visualise (Stols, 2012) (§ 3.4.2).

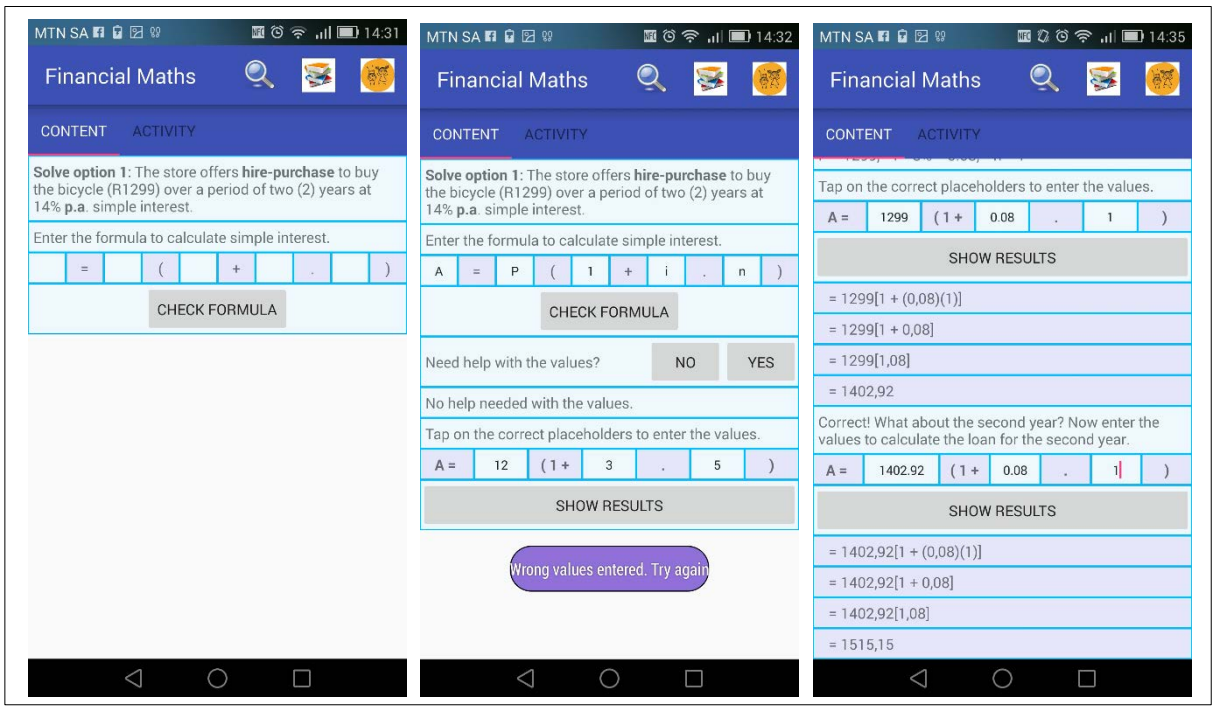


Figure 5.7: Student Contribution is Emphasised

Figure 5.7 illustrates that the app is designed with a strong emphasis on student contribution. Students are expected to enter their own values, do their own calculations and are then evaluated on

whether the solutions are correct or not. Figure 5.8 illustrates that guidance is given to redirect them if their solutions are incorrect, but otherwise they have the freedom to work on their own.

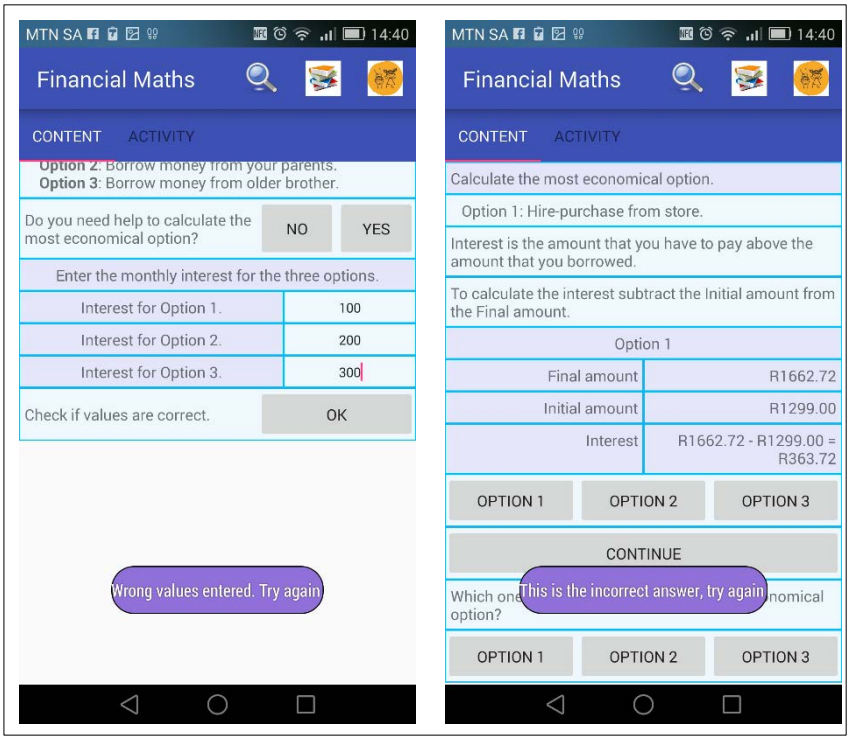


Figure 5.8: An Example of an Incorrect Entry

Since this app is used by teacher-students who are situated at various rural and urban areas in South Africa, interactivity is a challenging aspect to incorporate. During the activity in the app, teacher-students are given a task that requires of them to give advice as to the most economical option between buying a television for cash or on hire-purchase. After having an opportunity to model a solution and offer the most economical option, if their answer is incorrect, they are recommended to discuss these mathematical issues with a colleague. The screenshot on the left in Figure 5.9 illustrates this example. The screenshot on the right of Figure 5.9 indicates a situation where the answer was correctly calculated, but the teacher-students are given an opportunity to discuss possible reasons as to why that answer was correct.

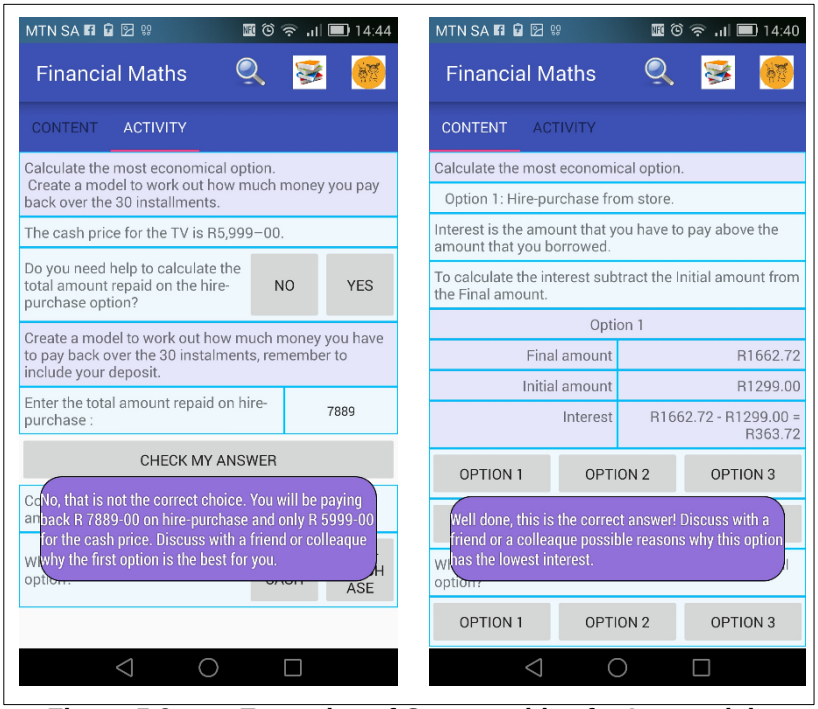


Figure 5.9: Examples of Opportunities for Interactivity

These examples allow for communication and collaboration with peers about the content presented in the app and activity. By discussing problem areas, it forces the teacher-students to reflect on the content, also a key principle of RME discussed by Widjaja and Heck (2003) in § 3.3.2. The aim of the app is to guide teacher-students on the implementation of the RME approach in their teaching. In a classroom situation the teacher-students will more easily be able to implement interactivity between themselves and the learners as well as between learners themselves.

The intertwining of learning strands is an essential part of RME. In this app, two types of integration have taken place. Firstly the content is integrated with a problem in the real world (Widjaja & Heck, 2003). Secondly the integration of the two concepts presented in the app is done by asking the teacher-students to calculate the compound interest option using the familiar formula for simple interest. This is illustrated in Figure 5.10. In so doing, the concepts are then presented alongside each other and the links and similarities, as well as the differences are highlighted.

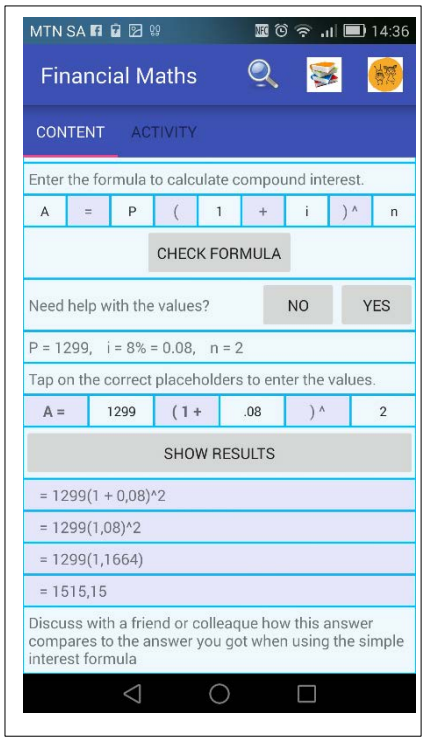


Figure 5.10: An Example of how Intertwining of Strands is Incorporated

A further area of connectivity is the link between the Financial Mathematics concepts and basic order in which operations are dealt with in Mathematics. In order to do correct calculations with the formulas, operations must be performed in the correct order to achieve the correct results.

Horizontal and vertical mathematization play an essential role in RME as is presented in § 3.3.3.2 with reference to the work of Treffers (1978) and Freudenthal (1991). In horizontal mathematization the student converts the problem situation into some form of Mathematics (Freudenthal, 1991). In the app, when teacher-students think about what formula to use, and what values to substitute, this conversion takes place. Vertical mathematization involves a higher level of abstraction within Mathematics, e.g. finding shortcuts and making connections between concepts (Freudenthal, 1991). Both these aspects are encouraged when teacher-students are asked to initially calculate the compound interest using repetition of the simple interest formula, and then compare that answer to the calculation done with the compound interest formula.

The process as suggested by Cahyono (2012) (§ 3.3.3.6) starts with a problem in the real world (the child wants to buy a bicycle to attend sport practice in the afternoons). The student will mathematize, reflect on the problem, do formalization (by using formulas and making comparisons) and then return to the real world (by solving the problem of which is the most economical option).

The principles of RME are discussed in detail in § 3.3.4 and include the following aspects: guided reinvention (otherwise known as progressive mathematization); emergent modelling and didactical phenomenology. These principles should be used in the design of tasks within the RME approach.

As suggested above, teacher-students are given the opportunity to develop their own Mathematics by inventing their own strategies to solve the problem. Should their strategy be incorrect, the app is designed in such a way that there is help if they require it, as can be seen in Figure 5.11. If they do not need help, they have freedom to solve the problem in their own way (Figure 5.11).

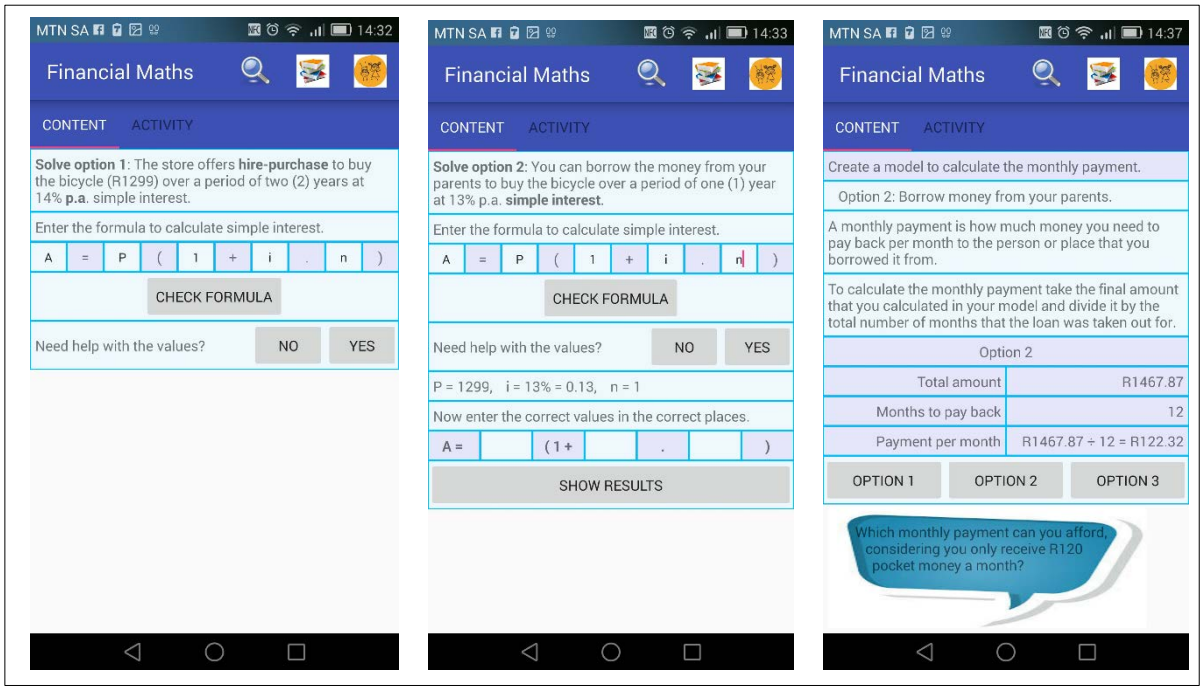


Figure 5.11: Assistance is Available Should Students Require It

This assistance is in line with the work of Drijvers et al. (2013) (§ 3.3.4.1) who suggests that the process requires guidance from the teacher to help direct the progression in a sensible manner.

Guided reinvention was touched upon in the discussion above about vertical and horizontal mathematization. Formal Mathematics develops out of the students' activities (Gravemeijer, 1999). The context plays an important role in the reinvention process. Students are not expected to reinvent everything themselves (Gravemeijer & Doorman, 1999) and it is essential that the teacher facilitates the teacher-students through the process. The hints and guidance in the app act as a means for me as the "teacher" to facilitate their progress. In Figure 5.12, I have illustrated two different hints that are available. The first screen indicates that certain terminology has been highlighted in the app. When teacher-students click on the bold words or terms, a definition or explanation of what the term means appears on the screen. The second screen illustrates a hint that appears when an incorrect calculation is made. The teacher-students are given guidance as to how the correct value can be calculated. As I mentioned in § 3.4.2, by providing hints and tips, teachers construct a trial and error approach which allows students to use different strategies to solve problems (Heck et al., 2008).

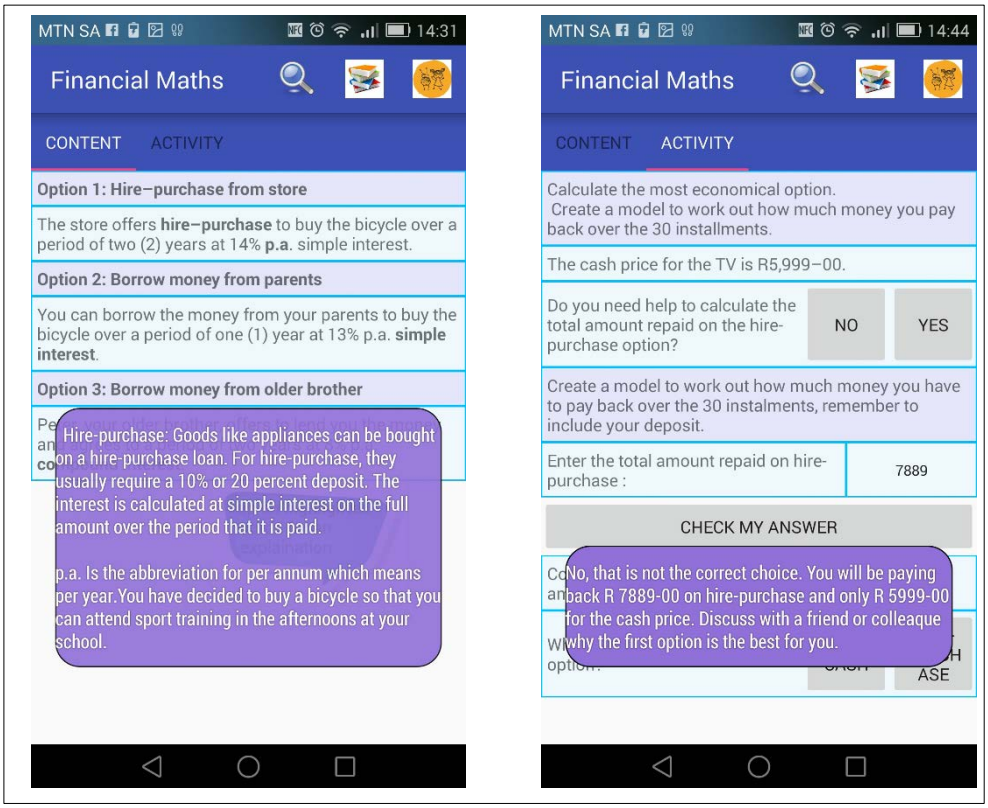


Figure 5.12: Hints relating to the Terminology and Calculations can be Accessed

In certain instances like in Figure 5.13 they are given three opportunities to make their own attempt at a problem, and only after three attempts, will there be any intervention in the form of a hint. This allows the teacher-students the opportunity to develop and try their own strategies first, before any intervention is offered, which is a key principle in RME.

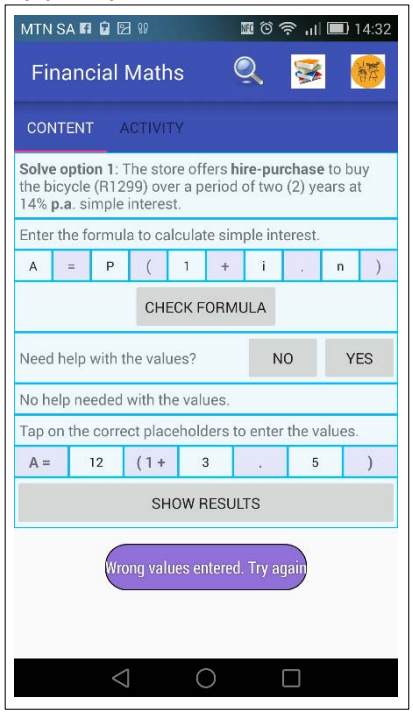


Figure 5.13: Three Opportunities are offered for Students' own Strategies

As Dolk et al. (2002) suggests, when training teachers, it is important that they experience mathematization and reinvention first handed when learning about the reinvention process. This was an important motivation for designing the app for teacher-students to use in order for them to learn about the RME approach and implement it in their classrooms.

In the discussion on guided reinvention in § 3.3.4.1, Drijvers (2013) points out that although ICT provides opportunities for reinvention, it may constrain students' prospects for exploration as a result of too much pre-planned guidance. I kept this in mind when designing the app and hence incorporated the prompts that first ask if the student needs help, rather than just offering the help, also allowing them three opportunities to offer a solution before any remediation or hints are given.

The second principle of RME is emergent modelling, which refers to the idea that models emerge from the activity of the students (Gravemeijer, 1999). The use of models is also discussed above as one of the characteristics of RME. An important aspect is that activities should be designed in such a way that students can shift from reasoning with models of informal activities to modelling of formal Mathematics (Bakker et al., 2003). I have tried to incorporate the modelling of informal activities through the use of the simple and compound interest formulas. In the activity, when teacher-students are asked to model an answer that would be the most economical purchase of a television, they have the opportunity to model the formal Mathematics. As discussed in § 3.3.4.2, models should be grounded in the context problems and not in intended Mathematics (Gravemeijer, 1999). In the app, the purpose of creating the model is to solve the problem in the given context, for example to give advice on which of the three options is the most economical when buying the bicycle, and also which option is the best when buying a television. The context demands the creation of the model, not the Mathematics.

The third principle of RME as was discussed in § 3.3.4.3 is didactical phenomenology. Meaningful phenomena that need to be structured and organised are presented to students and should be organised by the Mathematics (Drijvers, 2013). The phenomena of a person's personal finances, is the focus of the app. The scenario of buying a bicycle, and the choices offered when buying a television needs to be organised and structured with the aid of the mathematical content.

With reference to the aspects that should be considered in the design of Mathematics apps as suggested by Cayton-Hodges et al. (2015, p. 16) in § 5.3.2, I incorporated the following four aspects in the design process. Firstly, a rich and accurate representation of the mathematical content should be done. In the app I provided users with definitions of terms such as hire-purchase, simple interest, compound interest, the meaning of p.a.; I gave assistance with necessary simple and compound interest formulas and also made connections between different strands to reinforce the content. The second aspect discussed in § 5.3.2 was that the interactions around the content should be lavish and engaging. I provided numerous opportunities for the user to interact with the content at various levels.

I ensured that the user is constantly engaged by providing cues and links between screens and activities. When calculating the compound interest using the simple interest formula, I involve the users by saying the following: *Correct! What about the second year? Now enter the values to calculate the loan for the second year.* This is illustrated in Figure 5.14.

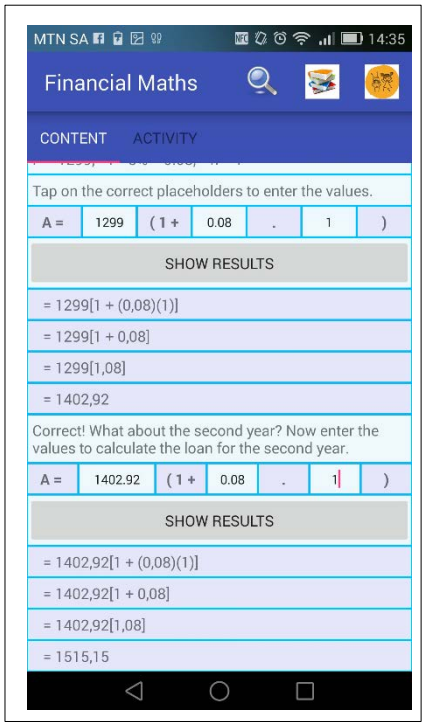


Figure 5.14: An Illustration of How Users are Encouraged to Engage with the Content

The third aspect discussed by Cayton-Hodges et al. (2015, p. 16) is that users should be given the opportunity to reflect on tasks. This aspect has also been incorporated in the Financial Mathematics app. In Figure 5.10, I encourage users to reflect on why two answers are the same and recommend that they discuss possibilities with a friend or colleague. Another example of this can be seen in the screenshot on the right of Figure 5.12, where users are again encouraged to reflect on a particular answer. The fourth aspect discussed in § 5.3.2 is that hints, scaffolds and other tools are important in the design of apps and that these should be presented on a need to know basis. Figure 5.11 and 5.12 both illustrate examples of hints and assistance that are available to the user, should they require these. The manner in which the app was structured and designed is evident of a process of scaffolding. Small portions of information and tasks are given one step at a time so that users can be guided through the process of learning the content and doing the activities.

5.5.2 The Technical Aspects of Creating the App

When creating the app, the design team followed nine distinct steps. Firstly, a clearly defined purpose of the mobile app was defined. This entailed a detailed exploration of what the app should be able to do; what its primary appeal would be; which concrete problem it would address; what aspect of life it

aimed to improve. I designed, re-evaluated and refined the design documents which sketched the standards, planning design, development and on-going evaluation of the project (Alessi & Trollip, 2001). The second step set the basis for the user interface. This entailed a visual conceptualisation of the main features as well as a rough layout and structure of the app. The designer created sketches of the proposed layout and structure of the app. Step three involved a search to establish if there were other similar apps available. This was necessary because there are over one million apps for Android and iOS already available. This search also assisted the designer with design inspiration for the Financial Mathematics app, and provided him with information on the technical requirements for the app. The designer downloaded three free analogous apps from the Google Play Store and studied their look and feel, and where applicable noted aspects that could be appropriate for the Financial Mathematics app.

In the fourth step, all the ideas and features came together as a clear picture of the prototype of the app. This process is known as wire framing (Lloyd & Dykes, 2011, p. 2501) and also entails the creation of a storyboard for the project. The storyboard provided a road map which illustrated the connections between the different screens and also how the user could navigate through the app. The result of the fourth step, the prototype and the storyboard, became the basis of the back-end structure for the fifth step. This step involved the sketching of the servers, app programming interface (APIs), and data diagrams as helpful references to the designer. These sketches provided self-explanatory diagrams that all could follow during the project. Step five also included modifications to the prototype and storyboard according to technical limitations. Because of the limited size of the app, no additional modifications to the prototype and storyboard were necessary.

The sixth step was an essential one. Friends, family, colleagues and experts were approached to review the prototype. They were requested to test the app and give feedback and identify flaws and dead-end links. Based on this feedback, the app was modified. This review step was done to finally specify the app concept before going into the design process. At this point I tested the app with the RME expert and two colleagues, who are Mathematics education experts (§ 5.5.1).

In step seven, individual screens were designed. The task was to create high-resolution versions of the prototype. Here, the designer included all comments from the prototype testers, so that the optimal user interface could be designed. Upon completion and implementation of the individual screens, the actual app concept was complete. All the graphics were inserted and I signed off on all text. The actual design was now implemented and made clickable. Step eight was to test the full design again and collect as much feedback from a variety of users. These new ideas and suggestions were used to refine the app. The designer effected the last changes to the layout.

The final step ensured that there was a consistent look and feel to the layout, and that it would perform reliably on different devices. The Financial Mathematics app was installed on various android devices

and was tested for functionality in a live environment. The final completed product was uploaded to Google Play Store and made available for the teacher-students to utilise and examine further.

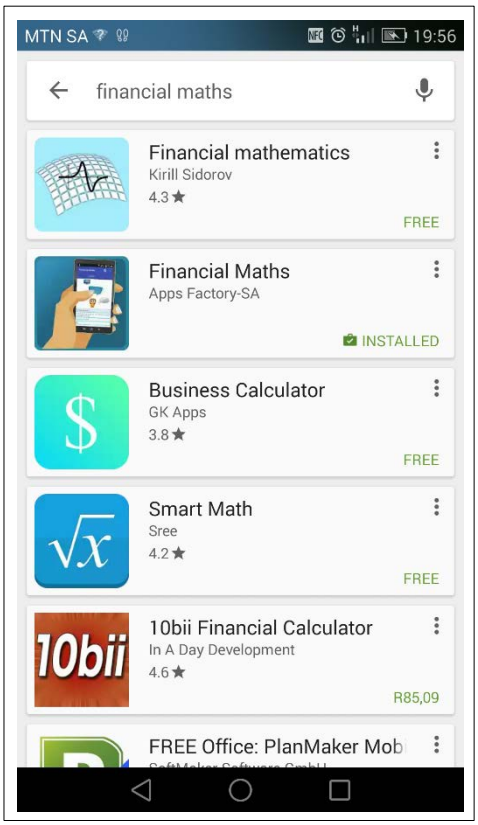


Figure 5.15: Illustration of the App in Google Play Store

Figure 5.15 illustrates the app which is available for free in Google Play Store. The app in question is the second from the top in the illustration. The next phase of this process was to present the completed app to the teacher-students in order to establish their experiences, perceptions and feedback regarding the app.

The nine steps that the design team followed in designing the Financial Mathematics app are loosely based on the framework that Lee and Kautz (2015, p. 577) suggest for the design and implementation of mathematical tasks in mLearning environments (§ 5.3.2).

Table 5.1: Comparison of the steps used in designing the Financial Mathematics app with the Design Framework of Lee and Kautz (2015, p. 577)

Design and implementation framework as suggested by Lee and Kautz (2015, p. 577)	Design and implementation steps of the Financial Mathematics app
Define Lesson Objective	1. A clearly defined purpose for the app was designed.
Define technological environment	2. A visual conceptualisation of the main features as well as a rough layout and structure of the app was designed. 3. A search was done to establish if there were other similar apps available.
Define types of investigations	4. Wire framing and the creation of a storyboard for the project.
Design and develop task(s)	5. The servers, app programming interface (APIs), and data diagrams were sketched. Modifications to the prototype and storyboard were done.

Design and implementation framework as suggested by Lee and Kautz (2015, p. 577)	Design and implementation steps of the Financial Mathematics app
	6. The review of the prototype by various parties. 7. Individual screens were designed.
Implementation	8. Testing of the full design again. As much feedback as possible from a variety of users was collected. 9. Ensuring a consistent look and feel of the layout, and that the app would perform reliably on different devices.
Enactment	The app was presented to the teacher-students for their feedback and perceptions.

Table 5.1 illustrates how the steps that were followed in the design process for this study was based on the design framework of (Lee & Kautz, 2015, p. 577).

The paper entitled *Design of a prototype Mobile Application to make mathematics education more realistic* written by the programmer, myself and my promoter underwent a peer review process and has been accepted for the proceedings of the 13th International Conference of Mobile Learning in Budapest from 10 -12 April 2017. The paper is available at Addendum 5.2.

5.6 Chapter Summary

This chapter presented a discussion of why a mobile app was suitable for this study and went on to sketch the background of what mLearning is. The associations between the use of technology and the RME approach were also presented. The chapter further explored the context of the teacher-students for which the app was designed. A detailed discussion of the process that was employed to design and test the app followed. In this discussion the theoretical aspects of the design were discussed: the various phases of design were discussed; the creation of the different prototypes was explored as well as the testing of the app; how the design of the app was related to the principles and characteristics of RME; as well as the technical aspects that were followed to design the app. In Chapter Six, the data generated from the focus group interviews were analysed. Teacher-students were introduced to the app and their evaluation and perceptions of the app were analysed in the following chapter.

As was discussed in § 2.4, throughout each stage of a design-based research project, researchers will reflect and document aspects to produce scientific output in the form of theories or design principles (Plomp, 2007, p. 15). As is illustrated in Figure 2.6, upon completion of the design, development and formative evaluation phase, I have produced a number of design principles relating to that design phase. The design process led to the awareness of a number of principles that were followed in the design process. These principles are presented in Table 5.2.

Table 5.2 Design Principles for the Design of a Mobile App Based on the Principles of RME Derived from the Design Process

Aspect	Implications
Thorough planning of the app	• A detailed design document should be created • The problem situation should be clearly sketched • The purpose of the app must be clearly defined from the start
Rigorous testing of the app throughout the design process	• The different prototypes should be rigorously tested by a range of parties (novices and experts)
The principles and characteristics of RME should be included	• The intertwining of concepts and strands should be incorporated • Relevant contextual problems must be used • The use of models ought to be encouraged • User interaction with the app and contributions should be encouraged • Users ought to be given the opportunity to invent their own strategies to solve problems
Pedagogical Aspects	• Mathematics content should be rich and accurately presented • Interactions with the content should be engaging • Users must be given the opportunity to reflect on tasks • Hints and scaffolds should be incorporated and used on request

Chapter Six

Participants' Observations and Experiences of the Mobile App

6.1 Introduction

In this chapter I report on the analysis of the data obtained from the focus group interviews. Ultimately I aim to produce guidelines for the effective use of technology in implementing the principles of RME in teaching practice. In order to create these guidelines, I need to establish how the participants' experienced using the app and what observations they made about the app. After having gained insight into the existing literature regarding the use of the RME approach assisted by technology in Mathematics education in the systematic literature review (Chapter Three) as well as by establishing the needs expressed by the participants in terms of applying the principles of this approach in their teaching (Chapter Four), I, together with a design team, designed a mobile app for the topic of Financial Mathematics. In the design of this app I took into account all aspects previously explored in the existing literature and needs analysis of the participants to create a suitable and rewarding intervention tool that could aid the participants of this study as well as other teacher-students in future to make their teaching of Mathematics more realistic.

The next stage of this study was to effectively produce the required guidelines that would answer the third research question: *What are the guidelines for the effective use of technology in implementing the RME approach in teaching practice*? In order to proceed to this stage, it was necessary for me to again meet with participants and present them with the completed app. Then I had to establish how they experienced that app, determine their observations about the app and take note of any critique they had to offer about the app. The plan to gain this information was to arrange a focus group interview with all four participants. However, it was not possible for one of the participants to meet with the other participants due to logistical reasons. I therefore decided to divide the group in two and have two focus group interviews. In each focus group interview, there were two participants and myself, who acted as the interviewer. I designed an interview schedule (Addendum 2.8) which was loosely based on the main themes of the systematic literature review. The focus group interviews took place in my office at the Potchefstroom Campus of the North-West University. I paid for the travel expenses of each student and provided them with refreshments to ensure that they were comfortable and that I did not cause them any inconvenience. These meetings took place outside of school hours, and did not disrupt the participants' teaching programme in any way.

I transcribed the recordings of the focus group interviews with the aid of ATLAS.ti™ (ATLAS.ti, 2014). As with the individual interviews, data were again coded in the same HU as the systematic literature review in ATLAS.ti™. I therefore used the existing codes and created three additional codes in order to code and categorise the focus group interviews as part of an integrated HU. The detailed

codebook was discussed in Table 2.6 (§ 2.6.5). I extended the codebook with three additional codes, which are presented in Table 6.1.

Table 6.1 Extension of the Codebook Table

Code	Description	Example
Usability of the app	Any comments made by the participants regarding the usefulness and ease of use of the app.	"The app helps to make the lesson more interesting" (P62:51).
Positive aspects of app	Any positive aspect that the participants raised in relation to the design, implementation or usability of the app.	"It is also going to motivate the ones who didn't get the correct solution" (P65:7).
Suggestions to improve app	Any suggestions that the participants had with regard to improving the app in relation to content or technical aspects.	"Maybe you consider specific topics for the slower learners, for those who are willing to learn you can give them more advanced content" (P65:22).

As with the individual interview data, the five themes that the systematic literature review revealed also formed the foundation of the structure for the analysis of the focus group interviews. As in Chapter Four, the four main themes used in this chapter were: Mathematics, RME, ICT and role players. I refined the themes further to suit the focus group interview data as: observations of the app in terms of Mathematics; the facets of RME observed in the app; considerations and ideas about ICT; and role players in the given context. Since the focus group interviews also focused on the evaluation of the app, I added the following theme: suggestions relating to the app. Figure 6.1 depicts the network view of the coding structure and themes that were used in the data set.

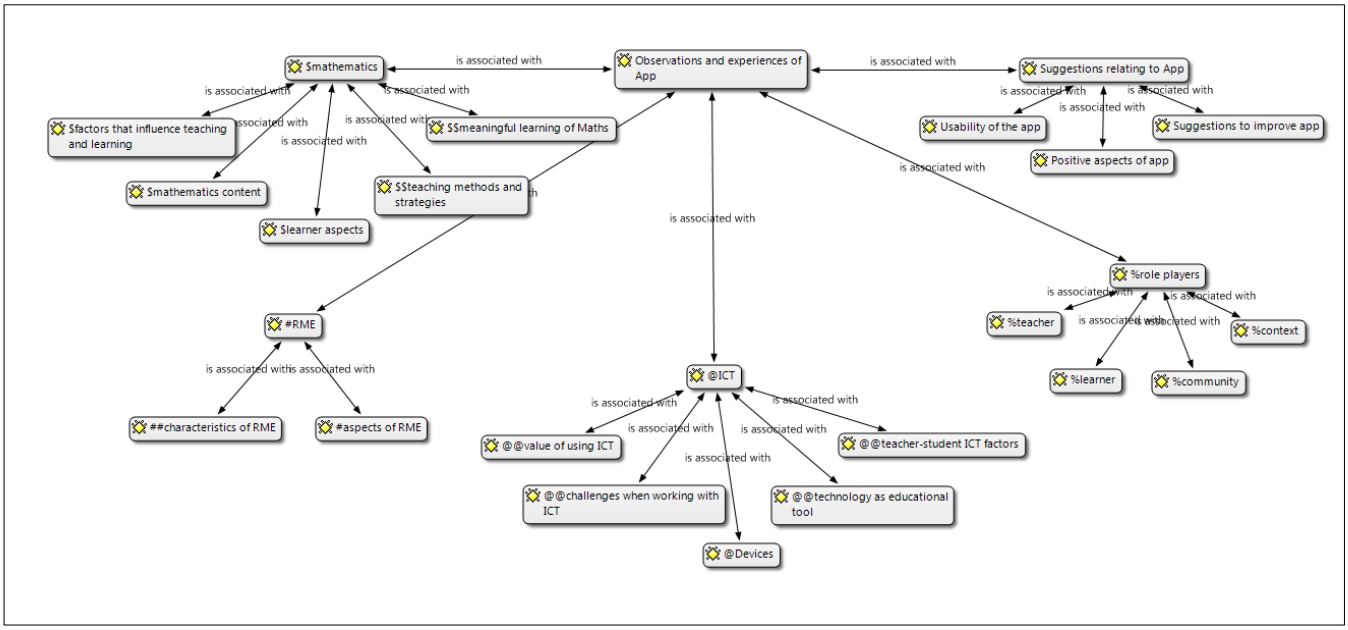


Figure 6.1: The Coding Structure for the Needs Analysis Individual Interviews

The themes as discussed above and a more refined structure for the analysis that follows in this chapter are presented in Figure 6.2.

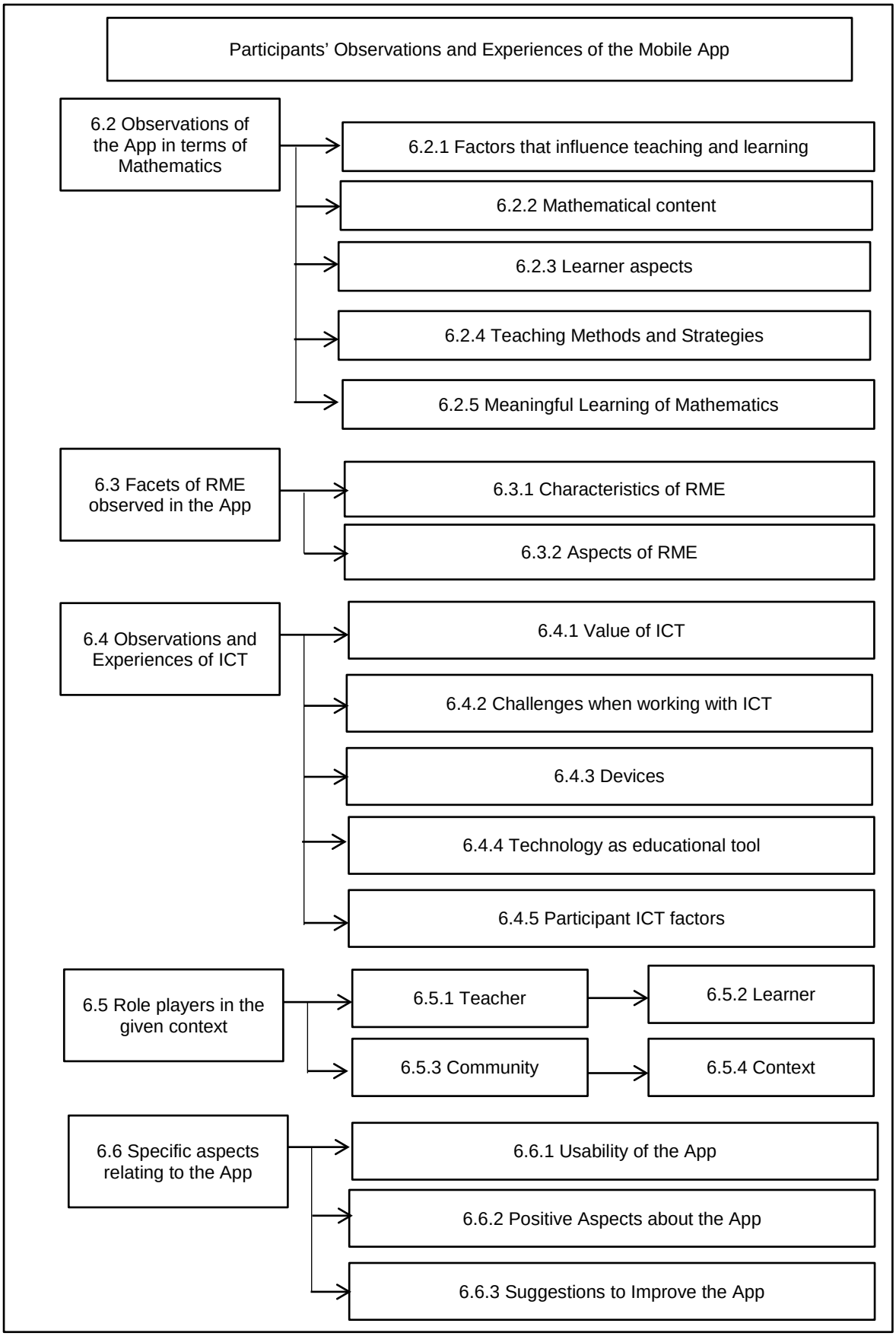


Figure 6.2: Advance Organiser of the Main Themes for the Analysis

6.2 Observations of the App in terms of Mathematics

There were five sub-themes that materialized within this theme, namely factors that influence teaching and learning; Mathematics content; learner aspects; teaching methods and strategies; and meaningful learning of Mathematics. Figure 6.3 displays the categories for the observations that the participants made in terms of Mathematics.

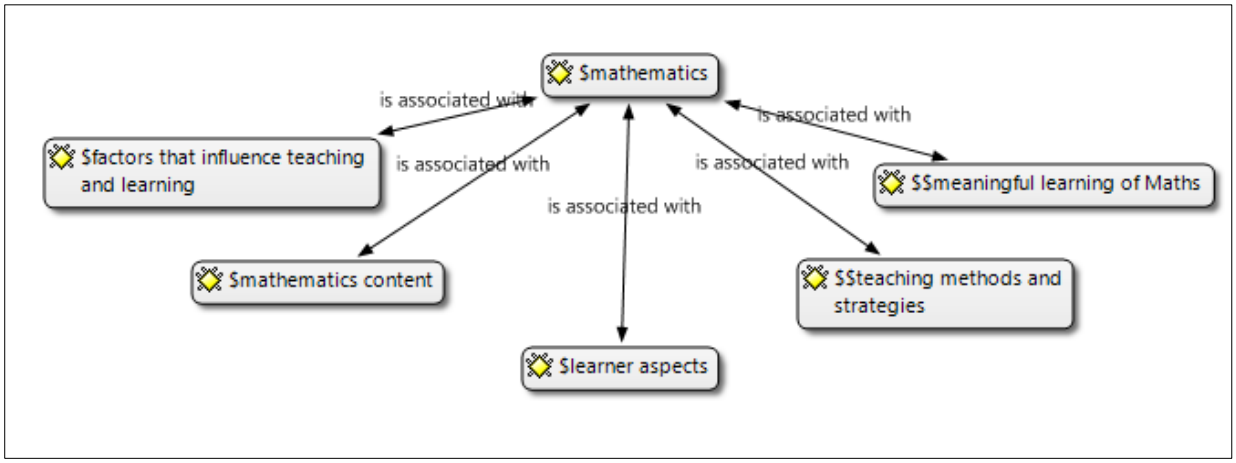


Figure 6.3: Coding Structure for the Participants' Observations of the App in terms of Mathematics

6.2.1 Factors that Influence Teaching and Learning

As discussed in § 4.2.1, immediate feedback is not only an essential aspect of effective teaching and learning, but was also an aspect that the participants found to be important for their learners. The participants reported that the app provided immediate feedback to the user:

> *And also the feedback—they will get immediate feedback. In the end it will help them to correct their sums. If they are getting it wrong, it will tell them— this is wrong. This is going to help the teacher in the class—getting feedback immediately. Even time is not going to be wasted (P65:19-20).*

> *I think it is good. I also think that it is going to challenge the users. Especially the part where they can choose if they want help or not. Most of them will choose the no[2]—they do not ask for help, but if they get it wrong, they will get feedback immediately from the app. Those slower ones will go for the yes. As time goes on, they will improve and will opt for the no. And then they will keep on improving (P65:2).*

In the systematic literature review (§ 3.2.1), it was also established that motivation can increase due to three factors: digital assignments; diagnostic testing; and the use of real life contexts. This was confirmed by the participants who said that:

> *The teachers and learners will be motivated to use this app (P62:22).*

> *There will be those that are so motivated by the app that they will just improve (P62:54).*

> *The first ones who have done a solution and find the solution to be correct, they will boast to the others who did not find a solution (P65:6).*

[2] "No" refers to clickable icons in the app where users are given the option to get help with formulas, values and methods or not. When users click on "No" it indicates that they do not need help. When they click on "Yes" it indicates that they do want help. In this case help is provided.

It will help them (P62:38).

They will love it! (P62:23).

When asked how they would use the app to improve teaching practice, the response was:

At any time when they feel like using it, while revising they can just use it (P62:37).

Those who are lazy to write and those whose handwriting is horrible—the app can help—it cuts all of that out (P62:52).

Those brighter ones will go for the no, they will want to do it for themselves. The slower learners will go for the yes, for the help. As time goes on they will also want to go for the no. So you are catering for both (P65:4).

It will also help those who battle to read. It will also improve their reading skills if they can hear and see at the same time (P65:41).

As discussed in § 3.2.1, technology allows learners to work creatively (Zaranis et al., 2013) and, when used correctly, improves problem solving skills and promotes critical thinking (Stols, 2012).

6.2.2 Mathematical Content

One of the participants raised an important issue in Mathematics education and that is mathematical content knowledge:

Based on the needs for the content, we must acquaint ourselves first with the content before we can use it here (P62:18).

Content knowledge is an essential component of what Ball et al. (2008, p. 395) refer to as mathematical knowledge for teaching (MKT). They define this as the "mathematical knowledge needed to carry out the work of teaching mathematics." Of relevance in this study as well is the incorporation of the teacher-students' knowledge of technology, as described by Mishra and Koehler (2006) in the TPACK framework (§ 1.3.1).

Further to the discussion on mathematical content, I asked the participants which sections of the curriculum they thought might benefit the most from an app such as this one. Their responses were: Calculus, Financial Maths, Probability, Geometry (specific mention was made of the theorem of Pythagoras), data handling, and graphs. The participants summarised it as follows:

The only one that will suffer will be Algebra. But the rest, even number patterns will be fine (P62:39).

Any section would be suitable (P65:35).

6.2.3 Learner Aspects

The theory of RME revolves around the construction of knowledge (Gravemeijer, 1999). Teachers should be sensitive to where their learners are (§ 3.2.7) and should design activities accordingly to provide them with the opportunity of constructing knowledge (Kizito, 2012a). Participants shared their ideas on how the app can meet the varying needs of different learners in the Mathematics classroom:

The weaker learners can be able to do it. If they can't then they click on the icon for help. It helps the weaker learners (P62:13).

There are learners who are so shy when they are in class they can work comfortably when they are alone. I think this will be so helpful especially for the weaker learners (P62:14).

A more advanced learner can do even more work. A slow learner you can give them more time (P65:21).

In this way you can attend to individual learners (P62:2).

The demotivated learners will also be motivated (P62:25).

It is also going to motivate the ones who didn't get the correct solution and even improve the pace at which they work (P65:7).

6.2.4 Teaching Methods and Strategies

Models come in many forms and are more than a visual representation of abstract Mathematics (Boon, 2006) (§ 3.2.5.1). Modelling ideas can be done in the form of words, diagrams and formulas (Muthen & Khoo, 1998). Participants discussed challenges they experienced with their learners with regard to the use of formulas:

So, you can even ask for the formula. Because sometimes they struggle to know which formula they should use (P62:3).

6.2.5 Meaningful Learning of Mathematics

As discussed in § 3.2.5.2, the RME approach can contribute towards a positive classroom climate (Fauzan et al., 2013). Participants mentioned the possibility of healthy competition between users when working with the app:

In fact there will be a competition (P65:5).

This endorses what was discussed in § 3.2.4 which says that one way in which meaningful learning can be encouraged is through playing games (Zaranis et al., 2013). Another participant said the following:

They like technology. They like to use cell phones. It's like playing games—they like playing games, wherever they are they play games. This is like playing a game (P62:25).

Participants suggested ways in which the app could contribute in their teaching:

My school has tablets and in my grade twelve Maths classes there are only twenty one learners, I am going to download the app to help them. What I like most is that the users can work on the Maths at home (P62:49).

This app can help with very large classes (P62:50).

When asked how this app can assist teachers to teach Mathematics in a meaningful way, the response was:

It can. It can. As a teacher...because each and every teacher wants their learners to be involved. You can get them involved if they have got this. You can communicate with them through this (P62:12).

6.3 Facets of Realistic Mathematics Education Observed in the Mobile App

The characteristics, principles and aspects of RME have featured strongly throughout this study. In the focus group interviews, participants also divulged their ideas on which facets of RME were apparent in the app. This theme was discussion with the sub-themes of characteristics and aspects of RME. Figure 6.4 illustrates the coding structure for this discussion.

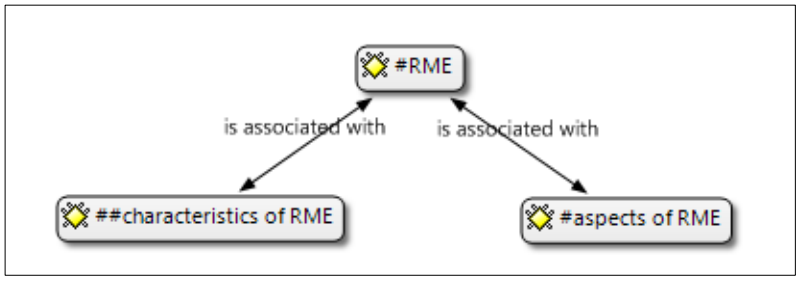


Figure 6.4: Coding Structure for the Facets of Realistic Mathematics Education Observed in the Mobile App

6.3.1 Characteristics of Realistic Mathematics Education

In § 3.3.2, I discussed the characteristics of RME in detail. These included: the use of contextual problems; the use of models; the use of student contribution; interactivity and intertwining of learning strands (Drijvers & Doorman, 1996). One of the questions in the focus group interview schedule was: *How does this app address the principles and characteristics of RME?* The participants shared their views of the characteristics of RME that they noticed in the app:

> *The topics are intertwined. They can bring topics together. They can see how this one is intertwined with the next topic (P62:11).*
>
> *And they are actively involved (P62:11).*
>
> *There will be collaboration, when they are together. They will ask each other: Who has done the sum first? (P65:17).*
>
> *I think all the characteristics of RME are there. There is collaboration; users will be actively involved for sure—they are doing it themselves. Intertwining of strands is there: like with simple and compound interest—showing that you can use simple interest to get to the compound interest formula (P65:18).*

6.3.2 Aspects of Realistic Mathematics Education

The emphasis on real life situations has been discussed throughout this study (Gravemeijer & Doorman, 1999). When evaluating the app, the participants also noted the link to real life:

> *You can do Maths with something that they know—make it real life (P65:46).*

The second aspect of RME that forms one of the cornerstones of the approach is that Mathematics is a human activity (Freudenthal, 1971). This aspect was discussed in detail in § 3.3.3.1. Participants shared their ideas on this aspect:

> *Anywhere, in a taxi, in a bus—wherever they are, they can do Maths (P65:10).*

6.4 Participants' Observations and Experiences of Information and Communication Technology in the Mobile App

In this section I will discuss the participants' observations and experiences of ICT in the mobile app. The five main themes are the same as those discussed in Chapter Four. These include: the value of ICT; challenges when working with ICT; the use of various devices; technology used as an educational tool; and factors of ICT that relate to the participants. Figure 6.5 illustrates the network view of the coding structure for the participants' observations and experiences of ICT in the app.

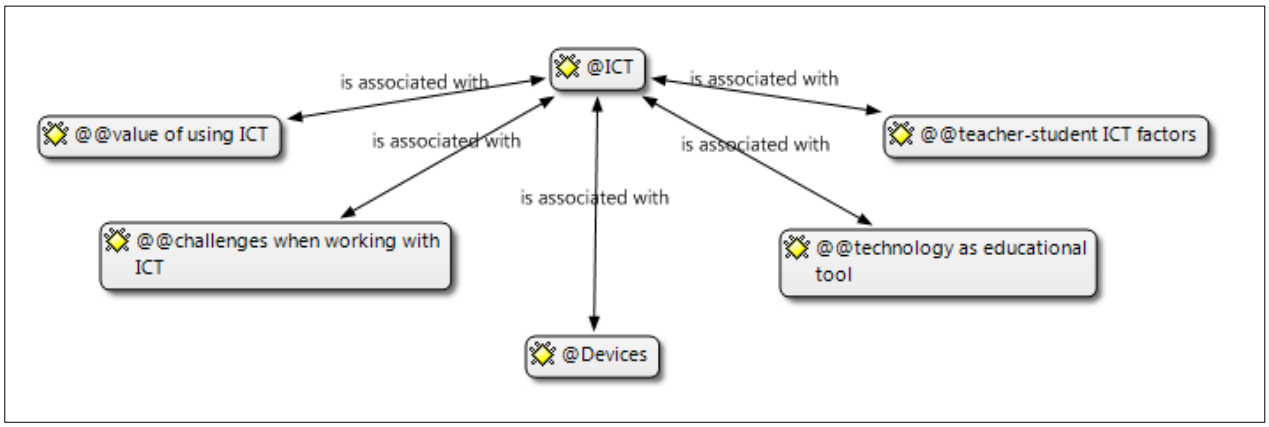


Figure 6.5: Coding Structure for the Participants' Observations and Experiences of Information and Communication Technology in the Mobile App

6.4.1 The Value of Using Information and Communication Technology

The systematic literature review detailed the value of using ICT in education (§ 3.4.1). More specifically, the benefits of mobile technology were discussed in § 5.3.1. Progressively more educators are using mobile technology for increased learner engagement; interaction and collaboration among peers; and as an extended time and place for learning (Kiger, Herro, & Prunty, 2012, p. 62). Participants also identified the value of using the app on a mobile device:

> *Because it gives the user a chance to do Mathematics anywhere, anytime. They will get immediate feedback (P65:9).*
>
> *Rather than going home, having to take out books and then learning—it can be done anywhere at any time (P65:10).*
>
> *They can afford to buy cell phones rather than computers. Even though that can't afford to buy computers they can buy cell phones (P62:8).*
>
> *It's more accessible to them and it's easy to carry, where they are and while they are relaxing they can learn while you are sitting outside (P62:9).*

6.4.2 Challenges while Working with Information and Communication Technology

Technology has proved to be a suitable way to facilitate RME in teaching Mathematics (§ 3.4.5), however, there are challenges when integrating ICT with the principles of RME (Drijvers et al., 2013). One of the recommendation in the work of Kizito (2012a) was that future studies should make use of

interviews to explore students' perceptions of mobile technology (§ 3.4.6). Participants in the current study were given the opportunity to express their views on mobile technology in two separate interviews. They discussed challenges they perceived when working with the mobile app:

> *It's a challenge for some teachers. The teachers must know how to download it. Without downloading it, I will be stuck (P62:16-17).*
>
> *To manage time can be a problem. If the users are used to doing Maths on the technology and they now have to write it out they will not know how much time they should spend on each question (P62:53).*
>
> *I think it would limit their basic skills—those learners will not have the basic Mathematics skill like calculating. When it comes to exam time, they will not be able to use this. They will use paper-and-pencil. If they are used to the technology—it can be a problem for them to write exams with paper-and-pencil (P65:48).*
>
> *I don't think they should only use this—in the end they should also write something. So, one should create opportunities for them to answer questions on paper-and-pencil as well—after learning the content, maybe use worksheets where they get to do the sums on paper (P65:49).*

In the last quotation above, participants expressed the concern of becoming too reliant on the app, and that written activities should also be incorporated into a lesson. This issue was also raised in § 3.4.6 where Zaranis (2014b) warns against tool dependency.

6.4.3 Devices

Various aspects of mLearning in Mathematics were discussed in § 5.3.2. One important aspect is that mLearning creates new opportunities for learning in Mathematics (Graven, 2011, p. 45). Participants mentioned their observations about various devices, particularly mobile devices in their contexts:

> *This generation like using these devices. They are interested in using cell phone, tablets. It will capture their minds. So they are focused, they will be focused (P65:34).*
>
> *Cell phones, there are many. Let's say more than 90% of them they have cell phones (P62:7).*

6.4.4 Technology as an Educational Tool

Mobile technologies such as the mobile phone and tablet computer have the potential to profoundly change how teaching and learning is conducted (Petocz & Manuguerra, 2011, p. 61). Mobile technology serves as an ideal means in which to facilitate change with regard to a new generation of students and their new ways of learning, yet offering the same content in new formats (Petocz & Manuguerra, 2011, p. 65). The participants in this study also saw the potential of mobile technology in teaching Mathematics:

> *They can teach themselves with this, while they are revising and doing this, they can check on themselves. Because the learners will be directly involved. They won't sit back and just listen to you saying everything. They will just type it in, rectify themselves. They are actively involved, and they can interact with others (P62:5-6).*

6.4.5 Participants' Information and Communication Technology factors

In the individual interviews, participants in general expressed their positive attitude towards working with technology in the Mathematics classroom. This stance was extended to the focus group interviews as well. Here it was also apparent that they felt that the Financial Mathematics app could aid the teaching process, and that users of the app would not have difficulty in navigating and using the technology:

> *I don't think the technology will hinder them. Let's take for example, all the Maths educators can use a calculator—so I don't think it will give them a problem. They will get used to smart phones. As long as they get used to using a calculator, they will get used to using the smart phone. Those educators that are not Maths educators, they struggle to work with a calculator (P65:24).*
>
> *If they can be showed how that is working, it won't be a problem. Even the calculators—how to use the calculator—we were never taught how to use a calculator, but in the end we know how to use the calculator. There are topics that we have to use a calculator, so we were never taught how to use a calculator (P65:25).*
>
> *It wasn't a problem to navigate. The fact is we own cell phones and we have been fiddling with the app. To work on this technology, is easy. But for a person who has got this for the first time it will be a challenge. Based on what we know about learners, most of them do have cell phones. I think they can use it (P62:15).*

The participants did mention possible challenges relating to technology use, especially for the older generation of teachers:

> *It will be easier for the new teachers because they know technology. The older teachers they might struggle a bit (P62:51).*

Despite the challenges, their aspiration to move forward is again evident (§ 4.2.1):

> *With the cell phone, nowadays we all have cell phones, we use cell phones and we are also encouraged to align ourselves with the current technology. So it shouldn't be a problem (P65:26).*

In the individual interview, one participant in particular expressed his views about mobile phones, where it was apparent that he did not see them as a tool for teaching Mathematics. He said the following:

> *About the cell phones, I don't think it's good...you can say cell phones are not designed for teaching and learning, it's just for communication purposes, so if you use them for teaching and learning, I don't think it's a good idea, unless if you want to communicate with them, send them homework, maybe the results, something like that (P59:36-37).*

When I presented the app to him and we proceeded to discuss its use in the Mathematics classroom, he said the following:

> *It will work. It is interesting...This is a new method of doing those things which are giving learners a problem. Nowadays people like this sort of things—they like using technology...This generation like using these devices. They are interested in using cell phone, tablets. It will capture their minds. So they are focused, they will be focused (P65:44-49).*

It was encouraging to notice a change of heart in this participant with regard to using mobile technology in the classroom. Another participant summarised his impression of the app in the following manner:

> *I am very impressed. In fact we are tired of chalkboards—it will be nice to use something new (P65:50).*

6.5 Role Players

One question in the focus group interview was: *Which role players will influence the use of an app such as this in the school situation?* One of the participants responded in the following way:

> *I think all of them play a role (P65:28).*

Figure 6.6 portrays the coding structure for the different role players in this study.

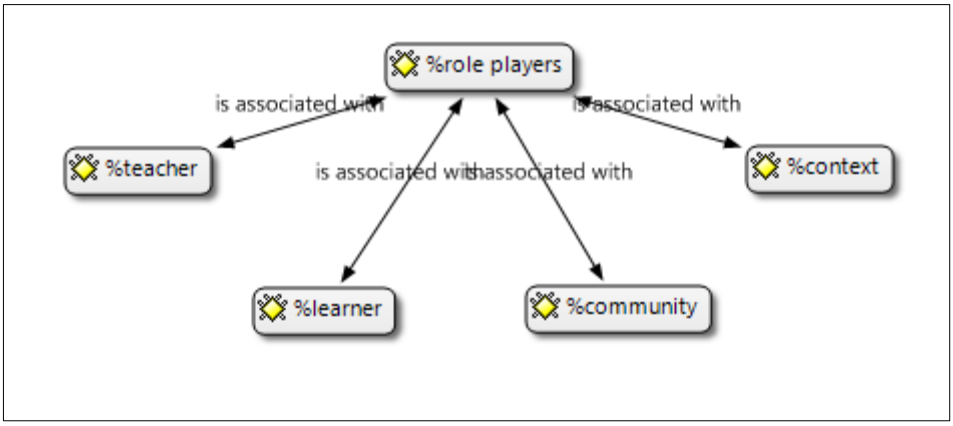


Figure 6.6: Coding Structure for Various Role players in the given Context

6.5.1 The Role of the Teacher

In the systematic literature review (§ 4.5), it was noted that not all stakeholders share the same optimism about using ICT in teaching and learning (Heck et al., 2007). The participants all suggested that the Mathematics teacher would be the greatest influence on the successful use of the app:

> *The Mathematics teacher is directly involved with this app. They must convince the parents; they must convince the Department that they need this app. The Mathematics teacher will influence its use (P62:26-27).*

> *The Mathematics teacher must encourage all the stakeholders to be involved in using the app. By showing them how possible, how easy and how economical it is to use it. But if you have no facts to support your statement then you will fail, the Department will fail to deliver this. But if there is a need and it can be shown to the stakeholders that if we apply this app—we save time, all learners are involved, all learners are interested. Maybe we should have one class where we use the traditional method and one classroom, use this technology and show the difference between them (P62:29).*

> *On the part of the educator—the educator, the Mathematic educators should take the initiative to show that this really works (P65:28).*

A participant also emphasised the key role that the teacher should play in the implementation of the mobile app in the classroom:

> *But, a teacher is a leader—they must always be in front of the learners. Teachers must learn more. They must be able to know it first before they can take it to the learners (P62:20).*

6.5.2 The Role of the Learner

A participant expressed her optimism that the learners would positively influence their parents with regard to the learners' use of the mobile app in the Mathematics classroom:

> *You just show the learners first. Then we call a parents' meeting. The learners would already have told their parents about the app. We will just show them how easy it will be (P62:28).*

6.5.3 The Role of the Community

When discussing the role of the community after having done the individual interviews, various factors relating to parents were raised (§ 4.5.3). An aspect that was evident was that parental support has a significant effect on learner motivation (Wentzel, 1998, p. 206) as well as academic performance (Makgato & Mji, 2006, p. 261). In the focus group interviews, parents also featured in the discussion:

> *Even their parents will be motivated to buy them cell phones and then even improve to buy more smartphones and install mathematical programs on the cell phones (P65:15).*
>
> *Because let's say for example parents—in terms of the cell phones—if we can tell the parents to buy cell phones for the learners, then parents can buy it. As long as they realise that it is effectively used and can benefit the learners, they are going to buy it (P65:28).*

6.5.4 The Role of the Context

School management and departmental issues featured in the focus group interviews, as they did in the individual interviews. Departmental characteristics have a definite impact on student learning (Vermunt, 2005, p. 205). Some of the aspects that are referred to as departmental characteristics include: course design and objectives, learning materials, and assessment procedures. The participants raised issues relating to departmental issues as well. Their concern was with role players such as the Heads of Department (HODs), Principals and the Department of Education:

> *In terms of the HODs and principals, they are the ones that should come up with the implementation and encourage the educators to implement this. So, if the Head of Department can encourage educators to implement and if the educator is prepared to implement—then from the Department to the parents—everything will run smoothly. Because the Department I think they should come up with programs where they design programs for the schools then the HOD's and the principals should check that those programs are implemented (P65:28).*

When asked if school principals would be happy for teachers to use an app such as this one, participants responded as follows:

> *Not at all. They like teachers to work. They think we are lazy (P62:30).*
>
> *Not at all. It involves money. This involves money. If for example you go to the principal and say that you want to use this app, you will work out the feasibility—how much will it cost—it will work out to be thousands. If the principal had never seen it being applied, it will not be easy to convince them. It was for the Maths teacher to convince. That's why I said if I bring*

the principal to class and show him how wonderful this app works, then maybe he can buy the story. Then we'll be fine (P62:30).

I think this is a question of whether the school has money to buy the technology. If the Department can provide the technology, then we can always use this technology. Just like textbooks—they provide textbooks, so they can provide mobile technology, then it will be simple (P65:27).

6.6 Specific Aspects Relating to the App

The codes that were added to the HU when coding the focus group interviews all fall into this category. Since the emphasis of the focus group interviews was on the evaluation of the app and to determine what observations and perceptions the participants had about the app, it was necessary to create codes that accounted for any suggestions the participants had with regard to the app. The sub-themes that fell into this category were: the usability of the app, positive aspects about the app and suggestions to improve the app. The coding structure for this section is illustrated by Figure 6.7.

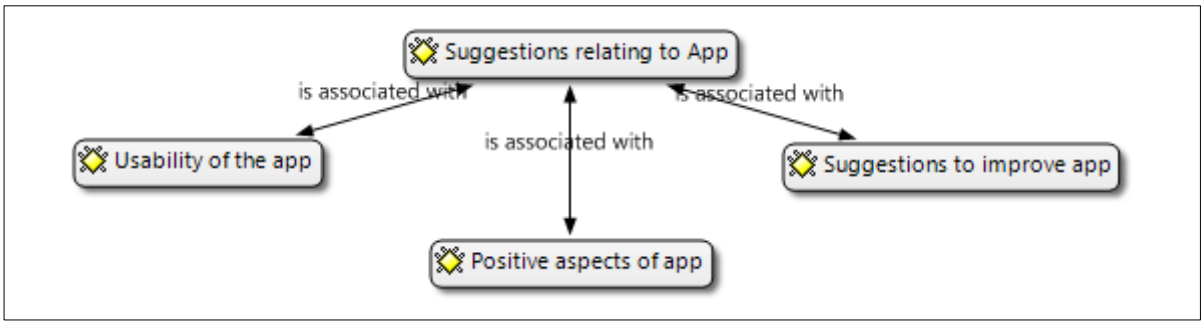


Figure 6.7: Coding Structure for Specific Suggestions relating to the App

6.6.1 Usability of the App

The usability of the app refers to the usefulness and ease of use of the app. Usability is often measured in terms of three aspects, namely: effectiveness, efficiency and satisfaction (Harrison, Flood, & Duce, 2013). These three aspects were all accounted for by the participants in this study. Effectiveness is illustrated by the following comments:

It will work. It is interesting (P65:30).

The app helps to make the lesson more interesting (P62:51).

Efficiency is illustrated by the comments that follow:

They can play with this and still arrive at the right answer (62:4).

They won't find it difficult to use this technology—they are hands on (P65:33).

Satisfaction is demonstrated by the following:

Ah, it's so easy! (P62:1).

This is excellent, excellent! (P65:31).

When asked if the app was easy to navigate, the participants replied:

> *It was for the first time, but it was not so difficult. But once used, I'm telling you it will come easy to them (P62:36).*

6.6.2 Positive aspects about the App

Two comments that the participants made related to design aspects:

> *The terminology helps them to differentiate between the two concepts (P65:1).*
>
> *It helps by giving them three chances to do sums (P65:3).*

The other observations were made concerning more general aspects:

> *This is a new method of doing those things which are giving learners a problem. Nowadays people like this sort of things—they like using technology (P65:32).*
>
> *I think this will help in other subject too—if teachers that teach other subjects can see this, they would also want to use such apps. This will help them too (P65:47).*

6.6.3 Suggestions to Improve the App

The suggestions offered by the participants in terms of improving the app can be divided into three categories. These categories are suggestions relating to pedagogical aspects; design aspects and general additional aspects to consider. Regarding pedagogical issues, participants suggested a greater variety of questions for two reasons: firstly to cater for slower learners and secondly to ensure the whole coverage of the curriculum.

> *Maybe you consider specific topics for the slower learners, for those who are willing to learn you can give them more advanced content (P65:22).*
>
> *I think putting all the questions that could be available—so that you can attempt each and every question that is possible. Otherwise if the question is not there and they are applying this—there will be a challenge, even if I don't know how to do it now. They will be a challenge; I can contact whoever to help me. So to improve it, let's have as many questions as possible, especially those in real life. Having all the questions will give an opportunity for interaction with other learners and with the teacher. Once they are loaded it will put a challenge to the user each and every time they open the phone. I think it can be improved in that way (P62:41).*

In § 5.5.1, I have illustrated how I made use of hints and scaffolds, which is encouraged by Cayton-Hodges et al. (2015) in the design of mobile Mathematics apps. Participants were reluctant to immediately offer such hints and would prefer that users have plenty of opportunity to try on their own before any such scaffolding takes place:

> *Like we already said, just to change where the learners have to choose between yes or no. They should be given more opportunity to do it on their own first. They should be forced to try the sums on their own first, and then the help can come. Twice or thrice is enough otherwise they might become demotivated (P65:36).*
>
> *Block it and then they can't access it anymore. Then they know they must try themselves. They must be able to think (P62:44).*

Concerning the issue of design, interactivity was an essential component that was taken into account in the app for two specific reasons. Firstly, interactivity is an essential characteristic of RME (Zulkardi,

2000) and secondly it is one the main attributes of mLearning (Domingo & Garganté, 2016, p. 22). Although some interactivity was built in to the app, the participants were keen to increase the levels of interactivity:

> *Secondly, I would say how can I show this to someone else—how can this be shown to a colleague, while I am working, so that other colleagues can help me (P62:41).*
>
> *They should be able to contact fellow users, like with WhatsApp™. I can say how do you do this? I couldn't get to the right answer, how do I tackle it? They can look into your steps and say no, you are using the wrong formula. Let's try this formula. By a friend, the learners feel so proud when they can help a fellow learner (P62:42).*

The final aspect to be discussed is the additional aspects that might be considered for inclusion in the app. Apps have the potential to offer affective engagement. This type of engagement is often influenced by the visual and interactive characteristics of the app (Calder, 2015, p. 245). This is a shortcoming that the design team also identified in the design of the Financial Mathematics app. Participants pointed out the following:

> *I think if you can make it to be audio-visual - that would be great (P65:37).*
>
> *There was also no sound. The sound can tell them what to do. Maybe make it an option for learners—those who want sound maybe have that option, those who don't want sound can switch it off (P65:38-39).*
>
> *I have seen that you have put some colourful pictures in there—even in the formula—maybe add some pictures but not too much. It should also depend on the grades—maybe in the senior and FET phases you can limit the images. In the lower grade you can include more pictures (P65:42-43).*
>
> *What about animation? It will be good especially for the grade fours, fives and sixes—because they like cartoons—you can add it there. If you can include animations that will be good for the lower grades (P65:44).*

A general comment which is encouraging for future development was:

> *It would be good if you could develop these apps for the whole curriculum—from foundation phase right through. Then they can get used to using it (P65:45).*

6.7 Chapter Summary

In this chapter, I presented a detailed analysis of the data that were generated at two focus group interviews with four participants in this study. The aim of the focus group interviews was to establish the participants' observations, opinions and experiences of the mobile app for Financial Mathematics. A number of familiar themes were apparent in this data. I firstly explored the participants' observations and experiences of the app in terms of Mathematics. I dealt with various factors that influenced the teaching and learning of Mathematics. The aspects that arose in this section were the importance of immediate feedback in the app; and the opportunity for increased motivation that mobile technology offers its users. Participants also offered various reasons as to why they felt that the mobile app could improve their teaching practice. In this category, I also explored the participants' ideas on mathematical content. Two important aspects transpired. Firstly participants discussed the importance of mathematical content knowledge, and secondly they revealed other sections of the curriculum that would possibly benefit from the design of similar apps. The third aspect of this section

addressed learner aspects. Participants discussed how the app met the varying needs of different learners. Concerning teaching methods and strategies, I discussed the modelling of formulas as an important teaching method. Finally, I discussed aspects relating to the meaningful learning of Mathematics. Features that were discussed included the notion of competition and games that the app evoked in the participants, and the importance thereof in contributing towards an environment in which meaningful learning could take place.

The second theme that I explored was that of RME. The facets of RME that the participants observed in the app were characterised into two components: characteristics of RME and aspects of RME. The participants were able to identify all the characteristics of RME, in particular collaboration and intertwining of strands. The other aspects of RME that were evident were the notion of Mathematics as a human activity, and that activities should be based on real life situations.

The third theme was the participants' considerations and ideas about ICT that they observed in the app. They were able to identify a number of positive aspects that the app offered in terms of ICT, and also pointed out some challenges. I explored the participants' opinions about different devices. The discussion centred on the use of mobile devices and the suitability thereof for the app. Participants also shared their thoughts on technology as an educational tool. I discussed factors that related specifically to the participants with regard to factors of ICT.

I discussed the data which related to the theme of role players, and considered the participants' views regarding the role of the teacher; the learner; the community and the context. They emphasised the importance of each of the role players and mentioned specific ways in which they could influence the use of the app. The final aspect that I discussed was suggestions that the participants made relating to the app. I discussed the usability of the app, where participants shared their ideas on the usefulness of the app. This was followed by a discussion of what they perceived to be positive about the app and also suggestions that they made to improve the app. I discussed these suggestions in three categories, namely: pedagogical issues, design aspects and as well as general suggestions.

By presenting the participants with the app and allowing them the opportunity the offer their observations and evaluation thereof, the first cycle of the final phase of the design-based research process had been completed. In figure 2.6 I illustrated this as the final evaluation cycle. I ended the preceding two chapters with design principles that were derived from the analysis of the interview data and the design process of the app. I will proceed to do the same in this chapter. Since this phase also completes a cycle (final evaluation) of the semi-summative evaluation phase (Figure 2.6), the outcome here too will be design principles, which this time will relate to the evaluation of the mobile app for Financial Mathematics.

Table 6.2 Design Principles for the Design of a Mobile App Based on the Principles of Realistic Mathematics Education Derived from Focus Group Interviews with Participants

Aspect	Design Principles
Participants' observations of the app in terms of Mathematics	• The immediate feedback that users are provided with is useful • Working with the app promotes motivation • The content knowledge of a teacher is essential in effectively implementing the app • All areas of the curriculum are suitable for such an app • A variety of learners with varying needs can be catered for by the app • The app can promote the classroom climate through the use of competitions and games
Facets of RME observed in the app	• The app is able to present content based on the characteristics and aspects of RME
Considerations concerning ICT in the app	• Mobile technology is a suitable means of presenting the app • Teachers should guard against using the app only, there should be written activities as well • Teachers are required to have basic technology skills such as being able to download the app from Goole App Store • The older generation of teachers may need more assistance with ICT than the younger generation
The effect of role players on the use of the app	• The Mathematics teacher is the most prominent role player in the implementation of the app • Parents play an important role in the successful use of the app • Institutional and departmental issues might hinder the implementation of the app in teaching practice
Suggestions form participants about the app	• The app should comply with the aspects that define usability of the app, namely effectiveness, efficiency and satisfaction • The app should be extended to other subject areas • The app should present a variety of questions to cater for slow learners and cover the full spectrum of the curriculum • Hints and scaffolds should be used wisely and on request, after allowing the user sufficient opportunities to make their own attempt on activities • More interactivity should be incorporated. Consider aspects that can aid interaction outside of the app such as WhatsApp™ • The app should be presented with sound and more visual aspects—less text should be used • Animation should be considered for future designs

Chapter Seven

Guidelines for the Effective use of Technology in Implementing the Realistic Mathematics Education Approach in Teaching Practice

7.1 Introduction

The study was underpinned by the main research question: *What are the guidelines for the effective use of technology in implementing the RME approach in teaching practice?* This chapter addresses this research question and presents a voluptuous overview of the research journey. I start with a detailed overview of the preceding chapters and also address the research questions. I also present the envisaged guidelines for the effective use of technology in implementing the RME approach in teaching practice. I discuss the contribution that this study has made to the field of Mathematics education and will also mention the limitations of the study. Future questions that arose from the study are presented and I also share my reflections of the research journey which I undertook.

7.2 Summary of Chapters

In this section a summary of the first six chapters will follow.

7.2.1 Chapter One: Orientation to the Research Journey

Chapter One provided an orientation for this research and presented a detailed plan of the research report that followed. It sketched the problem situation that needed to be investigated and provided a motivation for doing the research. I explored the existing literature and pointed out the gaps in that literature. A literature review concerning the main themes of the study was presented. The discussion of the literature revolved around three themes: RME; the use of technology in Mathematics education; and the relationship between RME and technology. I then presented the research questions and the purpose of doing this research. An overview of the research design and methodology was discussed and issues such as the population; participant selection; methods of data generation; and analysis were mentioned. I detailed a review of the methodology of a qualitative design-based research study and discussed the ethical implications thereof. This chapter also provided a list of terminology for clarification and mentioned the contribution that the study might make towards the field of Mathematics education.

7.2.2 Chapter Two: Research Design and Methodology

In Chapter Two the detailed research design and methodology was presented. Upon closer inspection of the research questions, the research approach and method was selected and presented in detail. This chapter provided the roadmap of the research process that was to follow. I discussed the theoretical underpinnings of a qualitative design-based research study from an interpretivist perspective. After exploring the literature concerned with design-based research, I discussed the various phases of this design-based research process applicable specifically to this study. The necessary ethical considerations were documented. In Chapter Two I presented a comprehensive documentation of the systematic literature review process that I followed. This discussion included a description of the document search process; the criteria according to which the documents were selected; the measures followed to assure quality assessment of the documents and a description of how the data were analysed with the aid of ATLAS.ti™. In this chapter I also presented the codebook relevant for this study. I indicated the manner in which validity and reliability issues were adhered to and discussed the limitations of the systematic literature review.

The chapter continued to explain the processes and methods followed in the first qualitative strand—the needs analysis. Here I discussed the criteria for the selection of the participants and the selection of the site. The data for the needs analysis were in the form of individual semi-structured interviews with the teacher-students. Data were analysed with the aid of ATLAS.ti™ in the same HU as the systematic literature review. The next phase of the research that I reported on in this chapter was the second qualitative strand in which I determined the teacher-students' perceptions of the intervention tool, which took the form of a mobile app. In this section I also discussed the participant and site selection. Data were collected through two focus group interviews and the analysis was also done with the aid of ATLAS.ti™ in the same HU. I finalised the chapter with an illustration of how validity and reliability were ensured in the qualitative inquiry and gave a description of the role that I played as researcher.

7.2.3 Chapter Three: Systematic Literature Review

Chapter Three provided a report of the systematic literature review of the current literature that pertained to the use of ICT in the implementation of the RME approach in Mathematics education. The systematic literature review revealed five themes: Mathematics; RME; ICT; methodology; and role players. The discussion on Mathematics revolved around seven aspects. These aspects were as follows: the factors that influence teaching and learning were discussed; the impact of RME and technology on Mathematics education; the influence of different teaching approaches on teaching practice; ways in which meaningful learning could be established and promoted; different teaching methods and strategies, which included the use of models; the effect of RME and technology on mathematical content; and various aspects that relate specifically to learners such as learner understanding and learner reasoning.

The second theme was that of RME. I first discussed the theory of RME, followed by the characteristics, aspects and principles of RME. The five characteristics of RME featured throughout the study as did the three principles of guided reinvention, emergent modelling and didactical phenomenology. Specific aspects that related to RME were also explored such as the implementation of tasks in RME; the use of HLTs; and the use of real life contexts. The chapter continued with a report on the advantages, disadvantages and recommendations for RME.

The third theme of this chapter was ICT. I explored the value of using ICT in Mathematics education as well as the challenges that go with its use and implementation. I also discussed different ICT tools and devices that were apparent in the systematic literature review. The discussion on devices concentrated mainly on mobile tools such as smartphones, tablets, and calculators. I extended this discussion on ICT by exploring virtual learning environments and the connection between ICT and the RME approach to teaching Mathematics.

The next theme that was discussed was methodology. The emphasis was on design-based research specifically since it featured as the chosen methodology for this study. The final theme that was explored in this chapter was role players. In this chapter I discussed only two sub-themes for role players, namely the role of teachers and the role of learners. In further thematic analyses in this study, I extended these to other relevant role players.

7.2.4 Chapter Four: Participants' Needs and Perceptions with regard to Real Life Mathematics

In Chapter Four I reported on the data that were gathered through the individual interviews with the teacher-students. The purpose of gathering this data was to assemble sufficient information to answer the research question: *What support needs do teachers have in order to successfully implement the RME approach in their teaching practice?* The data were coded in ATLAS.ti™ in the same HU as the systematic literature review, which meant that the same codes as well as some additional codes were used for this process. Four of the five main themes from the systematic literature review were addressed in the analysis and were discussed. These themes were Mathematics, RME, ICT and role players. I centred the discussion on the teacher-students' needs and perceptions in terms of Mathematics and divided the category into three sub-themes, namely factors that influence teaching and learning; mathematical content and learner aspects. The focus in the section on mathematical content was particularly on problem areas that the teacher-students had in terms of teaching the mathematical content and also areas of the content which in their opinions was related to real life.

The discussion continued to the theme of RME. Here, as with the systematic literature review, I looked at the characteristics, aspects and advantages of using RME as expressed by the teacher-

students in the interviews. In the section on characteristics, the analysis centred on student contributions; interactivity; the use of context problems; modelling; and the integration of different strands of the curriculum. The next theme that was discussed was ICT. Five sub-themes for ICT emerged in Chapter Four. These included the value of using ICT; challenges experienced when working with ICT; devices; technology as an educational tool; and teacher-student ICT factors. The latter two themes (and codes) were introduced at this point to cater for the data relating to the teacher-students' needs and perceptions of ICT in Mathematics education. In the section on technology as an educational tool, I explored factors such as the teacher-students' use of WhatsApp™ in their teaching as well as other educational tools such as web-based interactives and GeoGebra. The final theme that was discussed in this chapter was the various role players. In this chapter it was necessary to discuss the data under four sub-themes, namely the roles of the teacher; learner; community; and context. In the section on the role of the community, the role of parents featured strongly as a factor that influences learner motivation. Under the role of the context I explored the teacher-students' opinions about the importance of facilities at schools as well as departmental factors that play a role in effective education. The final stage of this chapter was a summary of the design principles relevant to the data obtained from the individual interviews.

7.2.5 Chapter Five: The Design and Development of the Mobile Intervention Tool

Chapter Five encompassed a report on the design phase of this study. The intervention tool, which was in the form of a mobile app on Financial Mathematics, was designed to assist teacher-students with incorporating the characteristics of RME into their teaching practice. This chapter detailed the process that was followed in the design of that app. Firstly I addressed the purpose of designing the app. I explored the theoretical underpinnings for the design process and reported on current literature relating to mLearning. I concentrated on mLearning in Mathematics specifically; and again considered the connection between technology and RME, in particular the role that technology played in this approach to teaching Mathematics. I briefly sketched the context for which the app was designed by reporting on the teacher-students' contexts. I then continued to explain the process that was followed with regard to the design of the app. A team consisting of myself, a programmer; my promoter, an expert in the field of using technology in teaching and learning; and my assistant promoter, an expert in Mathematics and Mathematics education, were involved in the design process. I discussed the theoretical aspects of designing the app and also explained how the design process proceeded according to the phases of a design-based research methodology. The various stages of prototyping were discussed, as well as how the app was designed based on the characteristics and principles of the RME approach. The final part of this chapter explored the technical aspects of creating the app. Here nine specific steps were followed and these were discussed in this section. I also drew the comparison of the nine steps that we followed with the five design steps suggested in literature. The chapter ended with a summary of the design principles that were relevant for this design process.

7.2.6 Chapter Six: Participants' Observations and Experiences of the Mobile App

In Chapter Six I reported on the data that were obtained from the focus group interviews in which teacher-students were introduced to the app. Their observations and experiences of the app were recorded and analysed. The data were analysed in ATLAS.ti™ as an integrated data set in the same HU unit as all the other data in this study. The chapter presented this data analysis under five themes. The first four themes were familiar themes that arose in the systematic literature review and were also used in the data analysis presented in Chapter Four. These themes were Mathematics, RME, ICT and role players. An additional theme was created for the discussion of the data which centred on specific aspects relating to the app. Additional codes were also created to code and analyse this data. The subsequent extension of the codebook was presented in this chapter.

In the first theme, the focus was on observations about Mathematics that the teacher-students made in the app. The discussion was presented in five sub-themes: factors that influence teaching and learning; mathematical content; learner aspects; teaching methods and strategies; and meaningful learning of Mathematics. Factors such as the importance of immediate feedback; the role of teachers' mathematical content knowledge; modelling; and the creation of a positive classroom climate were discussed. The second theme that was discussed was facets of RME that the teacher-students observed in the app. Here I discussed two aspects that arose from the data, namely the characteristics of RME and also aspects of RME. Teacher-students' observations and experiences of ICT in the app were discussed next. The same five sub-themes were discussed here that were discussed in Chapter Four. These were the value of ICT; challenges when working with ICT; devices; technology as an educational tool and teacher-student ICT factors. Of importance here was a discussion on the benefits of using mobile technology specifically and also the challenges involved with the older generation with regard to adopting new technology. An interesting comparison was drawn between opinions that one teacher-student had about ICT before using the app compared to after having used the app. The teacher-students' aspiration to move forward was also discussed in the section.

The same role players featured in the discussion of the focus group data as was discussed in Chapter Four. Issues relating to the roles of teachers, learners, the community and the context were discussed. Once again, the role of parents as well as departmental issues was presented in the discussion. The final theme of this chapter was that of specific aspects relating to the app. Three sub-themes were explored. Firstly, the usability of the app, where I mentioned how teacher-students experienced the ease of use of the app, was discussed. I continued the discussion by investigating positive aspects that they experienced when working with the app and also suggestions that they made to improve the app. The discussion on suggestions was categorised into three divisions, where I discussed their suggestions in terms of pedagogical aspects; design aspects and general aspect that need to be considered for future design. This chapter concluded with a summary of design principles that were gathered from the focus group interviews.

7.3 Addressing the Research Questions

In order to effectively answer the main research question of this study, two sub-questions were posed to gain insight on the main question. These sub-questions were:

1. How can the use of the RME approach, facilitated by technology, enhance teaching practice?
2. What support needs do teachers have in order to effectively implement the RME approach in their teaching practice?

The first sub-question of this study: *How can the use of the RME approach, facilitated by technology, enhance teaching practice?* was addressed in Chapter Three the culmination of the systematic literature review. This chapter presented the current literature that existed relating to the effect of the RME approach, facilitated by technology, on teaching practice. The three most dominant themes that were discussed in this chapter were Mathematics, RME and ICT. The discussion illustrated the interaction between these three themes and discussed what literature revealed about how the RME approach, assisted by technology, could enhance teaching practice

The second sub-question of the study: *What support needs do teachers have in order to effectively implement the RME approach in their teaching practice?* was addressed in Chapter Four, the analysis of the individual interviews with the teacher-students. This chapter presented the analysis of data that revolved around the needs analysis of the teacher-students. The data were analysed with ATLAS.ti™ and revealed the teacher-students' needs in terms of teaching Mathematics. The content areas that they found problematic to teach as well as the areas of the curriculum that their learners found challenging was revealed. They also disclosed which areas of the curriculum they found difficult to relate to real life. The discussion also divulged aspects relating to the theory of RME that were important to the teacher-students, and they shared their views about using ICT in their teaching of Mathematics.

In order to address the main research question which guided the research: *What are the guidelines for the effective use of technology in implementing the RME approach in teaching practice?* I created an inventory of all the design principles that have emerged from this study. These design principles have been re-categorised according to the five main themes that were prevalent throughout the study: Mathematics, RME, ICT, role players, and aspects relating to the app. Table 7.1 presents a summary of the guidelines.

Table 7.1 Guidelines for the Effective use of Technology in Implementing the Realistic Mathematics Education Approach in Teaching Practice

Prevalent themes of this study	Guidelines for the effective use of technology in implementing the RME approach in teaching practice
Mathematics	• Establish what needs the teacher-students have with regard to content areas (both teachers and learners) • Mathematics content should be rich and accurately presented

Prevalent themes of this study	Guidelines for the effective use of technology in implementing the RME approach in teaching practice
	• Interactions with the content should be engaging • Users must be given the opportunity to reflect on tasks • Hints and scaffolds should be used wisely and on request, after allowing the user sufficient opportunities to make their own attempt on activities • The immediate feedback that users are provided with is useful • Working with the app promotes motivation • The content knowledge of a teacher is essential in effectively implementing the app • All areas of the curriculum are suitable for such an app • A variety of learners with varying needs can be catered for by the app • The app can promote the classroom climate through the use of competitions and games
RME	• Meaningful, realistic contexts should be the foundation for students to reinvent the Mathematics • Establish teacher-students' needs in terms of making Mathematics realistic for the learners • User contributions should be encouraged • Incorporate the use of modelling • Make use of interactivity • Ensure the intertwining of content areas and concepts • Users ought to be given the opportunity to invent their own strategies to solve problems
ICT	• Teacher-students should see the value of ICT when using such an intervention tool • The challenges that teacher-students experience with regard to ICT should be addressed as far as possible • Establish which devices the teacher-students find most suitable • Determine in what manner teacher-students use technology as an educational tool • Mobile technology is a suitable means of presenting the app • Teachers should guard against only using the app, there should be written activities as well • Teachers are required to have basic technology skills such as being able to download the app from Goole App Store • The older generation of teachers may need more assistance with ICT than the younger generation
Role players	• Teachers are the key role players in integrating ICT in teaching and learning • Learners' learning is influenced by their socio-economic circumstances • Parents play an essential role in learners' academic performance • Facilities and higher authorities can be important within the given context • The Mathematics teacher is the most prominent role player in the implementation of the app • Parents play an important role in the successful use of the app • Institutional and departmental issues might hinder the implementation of the app in teaching practice • Participants should indicate a desire to be involved in professional development • Participants must aspire to move forward
Aspects relating to the app	• A detailed design document should be created • The problem situation should be clearly sketched • The purpose of the app must be clearly defined from the start • The different prototypes should be rigorously tested by a range of parties (novices and experts) • User interaction with the app should be encouraged • The app should comply with the aspects that define usability of the app, namely effectiveness, efficiency and satisfaction • The app should be extended to other subject areas • The app should present a variety of questions to cater for slow learners and cover the full spectrum of the curriculum • More interactivity should be incorporated. Consider aspects that can aid interaction outside of the app such as WhatsApp™ • The app should be presented with sound and more visual aspects—less text should be used • Animation should be considered for future designs

The guidelines relating to Mathematics centre around three main categories: mathematical content, teaching and learning strategies and classroom ethos. The majority of the guidelines refer to mathematical content with the focus on rich and accurate presentation of the content; content that is engaging; and the importance of teachers' mathematical content knowledge. With regard to strategies—factors such as scaffolding, hints and immediate feedback were found to be useful tools in this study. Classroom ethos also featured in the guidelines. This refers to factors such as motivation and classroom climate. The study revealed how these factors could be promoted by the app.

The guidelines that centred on the theme of RME focused on the characteristics of RME. The importance of incorporating realistic contexts to allow for students to reinvent Mathematics is an important guideline. The use of modelling; interactivity and ensuring the intertwining of strands was essential in this study. With regard to ICT, the guidelines were varied, yet an important characteristic of the ICT guidelines was: taking into account the teacher-students view of different aspects of ICT. These included what they saw as challenges with ICT; the value they perceived of ICT; the devices and tools they felt comfortable with as well as the skills they had and were required to have to effectively use the technology.

The guidelines concerned with role players again centred on teachers, learners and contexts. The focus was on guidelines relating to teachers since they featured as the key role players in this study. Guidelines relating to parents and institutional matters were important factors when considering the implementation of the app. The final set of guidelines related to aspects relating to the app. The guidelines here were categorised into four categories. Various guidelines dealt with the phases of design, which included defining a definite purpose, proper planning and rigorous testing of the app. Some guidelines concerned with pedagogical issues were raised such as using different questions for a variety of learners and also for full coverage of the curriculum. Various guidelines relating to the usability of the app were revealed, and finally a number of guidelines were suggested for future development, such as including sound and animation.

The involvement of the NWU in the effective use of technology in implementing the RME approach in teaching practice is at the level of expanding the app to a wider range of teacher-students with more apps that cover the entire curriculum. Another guideline that has implications for the NWU is to expand the project to other subject areas other than Mathematics. This has cost implications that the university would have to consider. Either way, the NWU has a responsibility to assist in improving the Mathematical content knowledge of the teacher-students.

At the level of the lecturer, her or his involvement would be quite extensive in effectively using technology to implement RME in teaching practice. Most of the guidelines related to Mathematics would be the lecturer's responsibility, such as establishing the teacher-students' needs; creating content that is rich and accurately presented; designing the app to provide immediate feedback; and incorporating aspects such as competitions and games in the app to promote a positive classroom

climate. In the guidelines relating to RME the responsibility rests on the lecturer and the teacher-students. It is the lecturer's responsibility to teach and implement the characteristics of RME in any intervention tool that is designed for the teacher-students, and it would be the teacher-students' responsibility to then implement those characteristics in their own classrooms for their learners.

In the guidelines relating to ICT, the responsibility is also shared mainly between the lecturer and the teacher-students. The lecturer needs to establish various aspects about the teacher-students in order to design the intervention such as establishing what device is most suitable, and in what manner the teacher-students are willing to use technology as an educational tool. The teacher-students on the other hand initially need to acquire certain skills, if they do not already have those, like being able to download an app. Thereafter, when implementing the technology in their teaching, they need to guard against relying too heavily on technology in the classroom and should be able to balance written activities and exercises with the use of technology.

Teachers feature most strongly in the guidelines about role players. They feature as the key role players in implementing the RME approach in their teaching practice. Concerning the aspects relating to the app, it is the lecturer's responsibility to plan, create and test a well-defined tool for intervention. The lecturer should also ensure sufficient and appropriate questioning to suit all learners.

With reference to Figure 2.6, the final phase of a design-based research project is the semi-summative evaluation phase. As discussed in § 6.7, the teacher-students' evaluation of the app (Chapter Six) accounted for the final evaluation stage. The final stage of the entire process is the generation of guidelines (Figure 2.6). The guidelines presented above constitute the completion of the design-based research process for this study.

In § 2.4 I discussed that design-based research should generate evidence-based claims about learning that addresses theoretical issues and generate theoretical knowledge in the field, rather than show that a particular design works (Barab & Squire, 2004, p. 6). In this study, the guidelines that have been produced are able to provide specific strategies that should be followed when using technology to implement the RME approach in teaching. The guidelines are related specifically to themes that were dominant throughout the study. I have also discussed where the responsibility to carry out these guidelines lie, namely with the NWU, the lecturer or the teacher-student. Therefore, these guidelines have contributed to the theoretical knowledge in the field, and certainly did address relevant theoretical issues.

7.4 Contribution of the Study

The study contributes to the subject area and discipline of Mathematics in the following manner:

- The systematic literature review proved to be an invaluable tool in providing clarity and insight into the RME approach, as facilitated by technology, in enhancing teaching practice.
- The study contributed towards the discipline of Mathematics by providing insight into the needs that teacher-students have with regard to making Mathematics more realistic for their learners.
- The initial evaluators of the app found it to be practical, suitable and meaningful in the area of Mathematics education. They also felt it could contribute to a wider range of students than those involved in the study and recognised the contribution it could make if designed for other content areas as well.
- Teacher-students were given the opportunity for a robust exchange of ideas that contributed to enhance and enrich the study.
- The teacher-students were exceptionally positive about the app and its potential for a wider range of Mathematics teachers as well as their learners.
- The teacher-students found their participation in the study rewarding and shared how much they had learned in the process.
- The guidelines that were produced are practical, hands-on and meaningful pointers that can be used for future developments in the area of enhancing Mathematics education with technology.
- The design principles that were produced in this study have the potential to inform lecturers, instructional designers and curriculum experts in the design and adaptation of course material for students.
- Teacher-students and their learners have benefitted from the app since the mathematical content was presented in a relevant and realistic way.
- The study has the potential to be the start of a larger project in the Faculty of Education Sciences at the NWU where student-teachers can be supported in their Mathematics teaching practice through mobile technology.
- This thesis will contribute to the research output of the Research Focus Area Self Directed Learning (SDL) at the NWU.

7.5 Limitations of the Study

The limitations of this study relate at two levels, a methodological level as well as a practical level. Firstly, methodologically this study was a qualitative study that was seated in the interpretivist paradigm. This meant that it accounted for one quadrant or perspective in Figure 2.2 by Burrell and Morgan (1994, p. 22) on paradigms for the analysis of social theory (§ 2.2.2). The study might have been richer if the perspectives of the teacher-students could have been quantified by including a quantitative component to the study. This was my first attempt at a qualitative study, which could have impeded on its success; however, issues of trustworthiness proved the validity of the study.

On a practical level, the second area that might have limited this study was the small number of respondents that were used. The possibility exists that if more participants had taken part in the study, the results might have been different. However, data saturation was achieved with the participants in this study, which implied that it was not necessary to extend the study to any more participants (§ 2.8.3.2).

7.6 Future Questions

The study revealed the following future research questions:

- What are the most important factors that hinder Mathematics teachers from implementing technology in their teaching of Mathematics?
- What effect does teachers' mathematical content knowledge have on the implementation of the RME approach in their teaching?
- What effect does teachers' technological content knowledge have on the implementation of the RME approach in their teaching?
- How can parental support enhance learners' experience of Mathematics?
- How should hints and scaffolds be used in the design of a mobile intervention tool to effectively guide users through mathematical content?
- How does the interaction between users of a mobile tool influence their academic achievement in Mathematics?
- How do teachers incorporate the use of a mobile tool in the teaching of Mathematics?

7.7 Reflections on my Research Journey

I am fervent about teaching, in particular teaching Mathematics. My journey began as a high school Mathematics teacher, where I gained invaluable experience on the teaching and learning of Mathematics. There I developed a desire to assist other teachers in their efforts to make Mathematics more meaningful for their learners. I advanced to a lecturing position at the NWU, and here have the privilege of training and assisting Mathematics teachers.

As a researcher, my first experience with research was with quantitative research. Because of my love for working with numbers and statistics, this was the obvious choice. I enjoyed doing quantitative research and found it fairly easy to do. However, being a person, who likes to be challenged, I decided that I wanted to learn more about qualitative research. The research question for my PhD study steered me in the direction of qualitative research and gave me the opportunity to explore my interest with this type of research. I found the process challenging but rewarding.

Firstly, doing a systematic literature review was a fascinating process. That which appeared to be cumbersome in the beginning, turned out to be a highly structured frame according to which I could analyse the rest of the data in the study. This structure that the systematic literature review provided was familiar and comforting to me, the mathematician. I found the idea of having to do interviews daunting since I had no experience in that field. Yet I quickly felt comfortable for two reasons: I was discussing issues of Mathematics education that are crucial to me, and I was discussing them with people that I care for, my Mathematics teacher-students. I quickly got the gist of acting as an interviewer and enjoyed hearing my students' ideas on and opinions of aspects important to all of us.

The design process of the app was also a completely new experience for me. Although I have been involved in creating and designing course material at secondary school and tertiary level, I had no experience in designing material to be implemented by a technological tool. Since my second passion after Mathematics is technology, this was an exciting and stimulating process. The excellent support that I received from my promoter and the programmer helped to make the design process smooth and enjoyable. So much so that I would love to extend this study to many more apps that cover the entire Mathematics curriculum.

Experiencing the feedback from the teacher-students after introducing them to the app was a fulfilling experience. Because I am fully aware of their circumstances and contexts, being able to provide them with a tool that can assist them in their jobs was gratifying. Their reaction to the app was so positive, and they too could see the potential thereof for their daunting task as Mathematics teachers as well as for the learners that they are responsible for. It was also satisfying to notice a positive change in one of the teacher-students' perception of technology use in the Mathematics classroom.

By doing a qualitative study, I was able to experience the human side of Mathematics and technology, which in my Master's dissertation was much more quantitated. Although I had the desire to learn more about qualitative research, I was apprehensive as to how I would cope with the process. The experience exceeded my greatest expectations; I found the journey of doing qualitative research fascinating, rewarding, enriching and enlightening. I would encourage any quantitative researcher to take the plunge and also give qualitative research a fair chance.

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ETHICS APPROVAL CERTIFICATE OF STUDY

Based on approval by the **Ethics Committee of the Faculty of Education Sciences (ESREC)** on **15/11/2016** after being reviewed at the meeting held on **26/11/2015**, the North-West University Institutional Research Ethics Regulatory Committee (NWU-IRERC) hereby **approves** your study as indicated below. This implies that the NWU-IRERC grants its permission that, provided the special conditions specified below are met and pending any other authorisation that may be necessary, the study may be initiated, using the ethics number below.

Study title: Guidelines for effective technology facilitation of Realistic Mathematics Education to enhance teaching practice.

Study Leader/Supervisor: Prof AS Blignaut
Research team: DJ Laubsher & Prof HD Nieuwoudt

Ethics number:

N	W	U	-	HS	-	2	0	1	4	-	0	2	6	7
Institution				Study Number						Year				Status

Status: S = Submission; R = Re-Submission; P = Provisional Authorisation; A = Authorisation

Application Type: N/A

Commencement date: 2016-11-15 **Expiry date**: 2017-11-15 **Risk:** Minimal

Special conditions of the approval (if applicable):

- Translation of the informed consent document to the languages applicable to the study participants should be submitted to the ESREC (if applicable).
- Any research at governmental or private institutions, permission must still be obtained from relevant authorities and provided to the ESREC. Ethics approval is required BEFORE approval can be obtained from these authorities.

General conditions:

While this ethics approval is subject to all declarations, undertakings and agreements incorporated and signed in the application form, please note the following:

- *The study leader (principle investigator) must report in the prescribed format to the NWU-IRERC via ESREC:*
  - *annually (or as otherwise requested) on the progress of the study, and upon completion of the project*
  - *without any delay in case of any adverse event (or any matter that interrupts sound ethical principles) during the course of the project.*
  - *Annually a number of projects may be randomly selected for an external audit.*
- *The approval applies strictly to the proposal as stipulated in the application form. Would any changes to the proposal be deemed necessary during the course of the study, the study leader must apply for approval of these changes at the ESREC. Would there be deviated from the study proposal without the necessary approval of such changes, the ethics approval is immediately and automatically forfeited.*
- *The date of approval indicates the first date that the project may be started. Would the project have to continue after the expiry date, a new application must be made to the NWU-IRERC via ESREC and new approval received before or on the expiry date.*
- *In the interest of ethical responsibility the NWU-IRERC and ESREC retains the right to:*
  - *request access to any information or data at any time during the course or after completion of the study;*
  - *to ask further questions, seek additional information, require further modification or monitor the conduct of your research or the informed consent process.*
  - *withdraw or postpone approval if:*
    - *any unethical principles or practices of the project are revealed or suspected,*
    - *it becomes apparent that any relevant information was withheld from the ESREC or that information has been false or misrepresented,*
    - *the required annual report and reporting of adverse events was not done timely and accurately,*
    - *new institutional rules, national legislation or international conventions deem it necessary.*
- *ESREC can be contacted for further information or any report templates via Erna.Conradie@nwu.ac.za or 018 299 4656*

The IRERC would like to remain at your service as scientist and researcher, and wishes you well with your project. Please do not hesitate to contact the IRERC or ESREC for any further enquiries or requests for assistance.

Yours sincerely

Prof LA Du Plessis

Digitally signed by Prof LA Du Plessis
Date: 2016.11.18 08:18:49 +02'00'

Prof Linda du Plessis

Chair NWU Institutional Research Ethics Regulatory Committee (IRERC)

Addendum 2.2

NORTH-WEST UNIVERSITY
YUNIBESITI YA BOKONE-BOPHIRIMA
NOORDWES-UNIVERSITEIT
POTCHEFSTROOM CAMPUS

Faculty of Education Sciences

Private Bag X6001
POTCHEFSTROOM, 2520
South Africa
Tel (018) 299 4585
Fax (018) 285 2080
E-Mail: dorothy.laubscher@nwu.ac.za
1 January 2016

Mathematics Education student
School for Natural Sciences and Technology Education
North West University
Potchefstroom
South Africa

Dear Sir / Madam

Permission to participate in research

I, Dorothy Laubscher, am enrolled for a PhD study at the Faculty of Education Sciences, Potchefstroom Campus of the North-West University. I am in the process of conducting research on the teaching and learning of Mathematics. The title of my thesis is: *Guidelines for effective technology facilitation of Realistic Mathematics Education to enhance teaching practice.*

You have been selected as part of a sample of participants to take part in the above-mentioned research project about the use of technology to make Mathematics teaching and learning more realistic. The project will involve an informal discussion with the researcher regarding your needs in the mathematics classroom in relation to making Mathematics content more realistic for the learners. Different technologies, like Facebook, WhatsApp™, Interactive Whiteboards etc. will be used to present content to you based on your teaching needs, as expressed in the informal discussion. Thereafter, interviews about your experience and opinions regarding the topic will be conducted. I hereby request your participation in this project.

I pledge to maintain the professional and research ethical codes. This signifies that:
- Your participation in this research remains voluntary and you may, at any time, withdraw from the research without any penalty
- Your personal information will at all times be treated as confidential
- No demands will be made on your academic teaching programme
- Should you be interested, the research findings will be made available to you and your school.

The Ethical clearance number for this study is NWU-HS-2014-0267.

Could you please provide me with your written consent by filling in the section on the next page. Your participation is voluntary. Your input and opinions are greatly appreciated!

Thank you for your willingness to complete this research.

Yours sincerely

Dorothy Laubscher
PhD Candidate, North-West University
Student number: 10218343
Tel: 018 – 299 4585
Email: dorothy.laubscher@nwu.ac.za

Seugnet Blignaut
Promotor
Tel: 016 - 910 3514
Email: Seugnet.Blignaut@nwu.ac.za

Informed Consent to participate in Research Project:

Guidelines for effective technology facilitation of Realistic Mathematics Education to enhance teaching practice

LETTER OF PERMISSION: RESEARCH PARTICIPANT

I, _____, (name and surname)

a student from the North West University hereby give my permission to participate in the above mentioned research project. I am aware that my participation in this study remains voluntary and that I, at any time, may withdraw from the research. I understand that if I do not wish to participate in this research, it will not be held against me, as participation is voluntary. I also understand that all personal information will be treated as confidential by the researchers.

_____

Signature

_____

Date

Addendum 2.3

Search Document

Number of Search	Date of Search	Database	Keywords	Hits	
1	27/01/2015	Scopus	(realistic* or RME) and (teach* or learn* or educat*) and math* and (tech* or ict or educational tech*)	355	
2	27/01/2015	EBSCOHost	(realistic* or RME) and (teach* or learn* or educat*) and math* and (tech* or ict or educational tech*)	644	
3	27/01/2015	Web of Science	(realistic* or RME) and (teach* or learn* or educat*) and math* and (tech* or ict or educational tech*)	34	
4	27/01/2015	Science Direct	(realistic* or RME) and (teach* or learn* or educat*) and math* and (tech* or ict or educational tech*)	61446	
5	27/01/2015	MathSciNet	(realistic* or RME) and (teach* or learn* or educat*) and math* and (tech* or ict or educational tech*)	118	
6	27/01/2015	Google Scholar	(realistic* or RME) and (teach* or learn* or educat*) and math* and (tech* or ict or educational tech*)	169	
7	03/02/2015	Scopus	(realistic* or RME) and (teach* or learn* or educat*) and math* and (tech* or ict or educat* tech*)	167 (selected 36 according to titles)	
8	03/02/2015	EBSCOHost	(realistic* or RME) and (teach* or learn* or educat*) and math* and (tech* or ict or educat* tech*)	534 (selected 44 according to titles)	
9	04/02/2015	Web of Science	(realistic* or RME) and (teach* or learn* or educat*) and math* and (tech* or ict or educat* tech*)	17 (8 selected according to title)	
10	04/02/2015	Science Direct	(realistic* or RME) and (teach* or learn* or educat*) and math* and (tech* or ict or educat* tech*)	34017	
11	04/02/2015	MathSciNet	(realistic* or RME) and (teach* or learn* or educat*) and math* and (tech* or ict or educat* tech*)	72 (2 selected according to titles)	
12	04/02/2015	Google Scholar	(realistic* or RME) and (teach* or learn* or educat*) and math* and (tech* or ict or educat* tech*)	136 (70 selected according to titles)	
13	05/02/2015	Science Direct	(realistic* or RME) and (teach* or learn* or educat*) and math* and (tech* or ict or educat* tech*)	Applied filters on publication titles 1084 (164 selected according to titles)	
14	16/05/2015	MathSciNet	Author map: Doorman, L M	1 publication	0 relevant
15			Author map: Drijvers, P	0	0
16			Author map: Gravemeijer, KPE	0	0
17			Author map: Heck, A	5	0
18			Author map: Boon, P	5	0
19			Author map: Zaranis, N	0	0
20	16/05/2015	Science Direct	Author map: Doorman, L M	0	0
21			Author map: Drijvers, P	0	0

22			Author map: Gravemeijer, KPE	0	0
23			Author map: Heck, A	15	0
24			Author map: Boon, P	4	0
25			Author map: Zaranis, N	0	0
26	16/05/2015	Scopus	Author map: Doorman, L M	1	0
27			Author map: Drijvers, P	23	23
28			Author map: Gravemeijer, KPE	20	20
29			Author map: Heck, A	9	9
30			Author map: Boon, P	4	4
31			Author map: Zaranis, N	4	4
32	19/05/2015	Google Scholar	Author map: Doorman, L M	2	1
33			Author map: Drijvers, P	111	77
34			Author map: Gravemeijer, KPE	146	78
35			Author map: Heck, A	123	23
36			Author map: Boon, P	16	0
37			Author map: Zaranis, N	20	13
38	25/05/2015	EBSCOHost	Author map: Doorman, L M	1	1
39			Author map: Drijvers, P	30	30
40			Author map: Gravemeijer, KPE	23	23
41			Author map: Heck, A	13	13
42			Author map: Boon, P	0	0
43			Author map: Zaranis, N	0	0
44	25/05/2015	Web of Science	Author map: Doorman, L M	0	0
45			Author map: Drijvers, P	5	5
46			Author map: Gravemeijer, KPE	8	8
47			Author map: Heck, A	0	0
48			Author map: Boon, P	0	0
49			Author map: Zaranis, N	0	0

No	Author	Title	Year Published	RME	Realistic	ICT	matics/teacher/edu	thematics Educat	Include	Comments
	EBSCO									
1	Bellamy, John S.; Mativo, John M	A Different Angle for Teaching Math	2010		x		x	x		
2	Diao, Su-Meng; Liu, Yun; Zeng, Qing-An; Luo, Gui-Xun;	A novel opinion dynamics model bas	2014			x	x			
3	Stephan, Michelle; Akyuz, Didem.	A Proposed Instructional Theory for I	2012	x			x	x		
4	Hagedorn, Linda Serra; Purnamasari, Agustina Veny.	A Realistic Look at STEM and the Ro	2012				x			
5	Morgan, Pamela J.; Cleave-Hogg, Doreen; Desousa, Su	Applying theory to practice in underg	2006			x				
6	Krumpe, Norm. Mathematics Teacher.	Creating Three-Dimensional Scenes	2005		x		x			
7	Zandieh, Michelle; Rasmussen, Chris.	Defining as a mathematical activity: A	2010	x			x	x		
8	Yang, Jian; Wang, Yongtian.	Design of vascular networks: A math	2013		x		x			
9	van Loon-Hillen, Nelleke; van Gog, Tamara; Brand-Gru	Effects of worked examples in a prim	2012	x		x	x	x		
10	Kuang-Nan Lin.	Experience and Ideas on Intermediat	2010				x			
11	Cotton, Samuel E.; Calkins, Celeste M	Give Students an Engineering Edge:	2011		x		x			
12	Lim, Cher Ping.	Global Citizenship Education, Schoo	2008							
13	Player-Koro, Catarina. Learning	Hype, hope and ICT in teacher educa	2013		x	x	x	x		
14	Johnson, Estrella; Caughman, John; Fredericks, Julie;	Implementing inquiry-oriented curricu	2013				x	x		
15	Sorgo, Andrej; Verckovnik, Tatjana; Kocijancic, Slavko	Information and Communication Tec	2010		x	x				
16	Andresen, Mette.	Introduction of a new construct: The	2007	x		x	x	x		
17	Hadjerrouit, Said.	Learner-Centered Web-Based Instru	2005			x				
18	Bell, Irene Teresa; Gibson, Ken.	Mathematical knowledge of technolo	2009		x	x	x	x		
19	Inoue, Noriyuki.	Minimalism as a Guiding Principle: L	2008		x		x	x		Include
20	Hughes, Charles E.; Stapleton, Christopher B.; Hughes,	Mixed Reality in Education, Entertain	2005		x	x				
21	Akbari Torkestani, Javad; Meybodi, Mohammad Reza.	Mobility-based multicast routing algo	2010		x	x	x	x		
22	Quale, Andreas.	On the use of a standard spreadshee	2012			x				
23	Ärlebäck, Jonas Bergman.	On the use of Realistic Fermi proble	2009		x		x	x		
24	Cobb, George W.	One Possible Frame for Thinking abo	2007		x	x	x	x		
25	Berlin, Donna F.; White, Arthur L	Preservice Mathematics and Science	2010				x	x		
26	Bencze, John Lawrence.	Private Profit, Science, and Science	2008		x	x				
27	Fessakis, G.; Gouli, E.; Mavroudi, E	Problem solving by 5–6 years old kin	2013			x				IWB
28	van Putten, Cornelis M.; van den Brom-Snijders, Petra	Progressive Mathematization of Long	2005	x			x	x		
29	Sparrow, Len.	Real and Relevant Mathematics: IS I	2008		x		x	x		Practical suggestions
30	Belardinelli, Paolo; Ortiz, Erick; Braun, Christoph.	Source Activity Correlation Effects o	2012		x		x			
31	Depaepe, Fien; De Corte, Erik; Verschaffel, Lieven.	Teachers' approaches towards word	2010		x	x	x	x		
32	Johnson, Estrella.	Teachers' mathematical activity in in	2013				x	x		
33	Andrews, Paul; Sayers, Judy.	Teaching Linear Equations: Case Stu	2012		x	x	x	x		
34	Wedelin, D.; Adawi,	Teaching Mathematical Modelling an	2014		x		x	x		
35	Perrenet, Jacob; Adan, Ivo.	The Academic Merits of Modelling in	2010				x	x		
36	Fagnant, Annick.	The use of mathematical symbolism	2005	x			x	x		
37	Kump, Barbara; Moskaliuk, Johannes; Dennerlein, Seb	Tracing knowledge co-evolution in a	2013		x	x	x	x		
38	Kim, Younghoon; Shim, Kyuseok.	TWILITE: A recommendation system	2014			x				
39	Lazakidou, Georgia; Retalis, Symeon.	Using computer supported collaborat	2010			x	x	x		
40	Robinson, Carol L	Using Sport to Engage and Motivate	2012		x	x	x	x		
41	Guryan, Jonathan; Jacob, Brian; Klopfer, Eric; Groff, Je	Using Technology to Explore Social	2008		x	x	x	x		
42	Sheppard, Ian.	Using TOOL-MAKING to contextualis	2009		x		x	x		Geom
43	Cahyono, Adi Nur.	Virtualmatriks: A Conceptual Mathem	2012	x		x	x	x		
44	Stohlmann, Micah.	YouTube Incorporated with Mathema	2012		x	x	x	x		
	GOOGLE SCHOLAR									
45	H Barnes	The theory of Realistic Mathematics	2005	x			x	x		
46	D Tirosh, T Wood	Tools and processes in mathematics	2008	x		x	x	x		RME Ch 12
47	PF Conway, FC Sloane	International trends in post-primary m	2005							RME Ch 3 (3.3)(3.5)Mic
48	P Drijvers	Digital technology in mathematics ed	2012		x	x	x	x		
49	N Zaranis, M Kalogiannakis, S Papadakis	Using mobile devices for teaching re	2013	x		x	x	x		
50	K Beswick	Putting context in context: an examin	2011		x		x	x		
51	RN Kizito	Pretesting mathematical concepts wi	2011	x		x	x	x		Distance
52	M Wu, D Zhang	An overview of the mathematics curr	2006	x			x	x		
53	T Dykstra	High performance and success in ed	2006	x			x	x		
54	LM Doorman, PHM Drijvers, KPE Gravemeijer	Design research in mathematics edu	2013	x		x	x	x		NB
55	M Artigue	Didactical design in mathematics edu	2009	x	x		x	x		
56	T de Jong, P Hendrikse, H van der Meij	Learning Mathematics Through Inqui	2010	x			x	x		NB

57	N Zaranis	The use of Information and Commun	2013	x		x	x	x			NB
58	TA Le	Applying realistic mathematics educa	2007	x			x	x			NB dissertation
59	KPE Gravemeijer	Transforming mathematics educatior	2014	x			x	x			NB Ch in book
60	K Gravemeijer	Aiming for 21st century skills	2012	x		x	x	x			HLT
61	P Dowling	Recontextualization in Mathematics t	2014	x			x	x			
62	A Heck, P Boon, C Bokhove	Applets for learning school algebra a	2007	x		x	x	x			
63	C Karagiannidis, P Politis, I Karasavvidis	Research on e-Learning and ICT in Education		x		x	x	x			Ch in book
64	M Perez	Scaffolding Teachers' Construction c	2012	x		x	x	x			
65	N Zaranis	The use of ICT in the first grade of pr	2014	x		x	x	x			
66	C Karagiannidis, P Politis, I Karasavvidis	Research on e-Learning and ICT in E	2014	x		x	x	x			
67	N Zaranis	Geometry Teaching Through ICT in P	2014	x		x	x	x			
68	N Zaranis	The use of ICT in kindergarten for tea	2014	x		x	x	x			
69	P Drijvers, P Boon, M Doorman	Digital design: RME principles for designing online tasks		x		x	x	x			NB
70	P Drijvers	Digital technology in mathematics ed	2013	x		x	x	x			
71	HK Slettenhaar, PAM Kommers	An adaptive teacher in an ever-chan	2010	x			x	x			
72	B Jaworski	Reflective Practitioner in Mathematic	2014	x			x	x			Reflection
73	N. Zaranis	The use of ICT in the 1ste drade of p	2013	x			x	x			
74	G Stols	Does the use of technology make a d	2012	x		x	x	x			
75	B Council, B Darussalam	BMDP, 647 Bootstrapping, 648 Borb	2013	x			x	x			
76	D Pitta-Pantazi	Number Teaching and Learning	2014	x		x	x	x			
77	B Carroll	An investigation into the teaching an	2011	x	x	x	x	x			Thesis
78		REVIEW OF MATHEMATICS IN POS	2005	x			x	x			
79	SW Siyepu, T Mtonjeni	GEOMETRICAL CONCEPTS IN REAL-LIFE CONTEXT: A (		x			x	x			
80	T Lowrie	Rural and Remote Mathematics Edu	2014	x			x	x			
81	P Dickinson, F Eade, L Akitt, J Monaghan	BSRLM Proceedings: Vol 25 No 3 La	2005	x			x	x			Trialling RME in english secondary schools
82	K Gravemeijer	Number Lines in Mathematics Educa	2014	x			x	x			
83	M Swan	Design research in mathematics edu	2014		x		x	x			
84	N MOUSOULIDES, B SRIRAMAN, C CHRISTOU	From problem solving to modelling	2007	x		x	x	x			
85	Z Zulkardi	Innovation and exemplary practices i	2013	x			x	x			
86	S Dimarco	Mathematics in the middle: challengi	2010	x			x	x			
87	J Gordon, O Rey, A Siewiorek, M Vivitsou	KeyCoNet 2012 Literature Review: K	2012	x		x	x	x			
88	A Heck, P Boon, W van Velthoven	Mathematica empowered applets for	2008	x		x	x	x			
89	S Patsiomitou	The development of students geome	2008		x	x	x	x			Geom. Sketchpad-book
90	P. Dickson, F. Eade	Looking at the workplace through ma	2010	x			x	x			Using RME with low to middle attaining pupils
91	MFNL Abdullah, P Iannone	Analysis of classroom interaction from the combined view of self-regulating strategies and discourse analysis: What can we learn?									
92	G Priestnall, E Brown, M Sharples, G Polmear	Augmenting the field experience: A s	2010		x	x	x	x			
93	SICG Martins	An approach to teach Calculus/Math	2013			x	x	x			Thesis
94	N Fitzallen, J Watson	Developing a Sequence of Learning	2014	x		x	x	x			Ch in book
95	M Domazet, B Baranović, J Matić - Sociologija i prostor	Mathematics Competence and Interr	2013	x		x	x	x			
96	M Doorman, P Drijvers, K Gravemeijer, P Boon	Tool use and the development of the	2012	x		x	x	x			
97	P Cobb, K Jackson, C Dunlap	Design research: An analysis and cri	2014	x			x	x			
98	M YERUSHALMY, J AINLEY, M DOORMAN, C KIERAN	Announcement and call for papers		x			x	x			
99	E Dunphy, T Dooley, G Shiel	Mathematics in Early Childhood and	2014	x			x	x			
100	A Heck	Modelling in cross-disciplinary authe	2010	x		x	x	x			
101	M Ndlovu, A Mji	Pedagogical implications of students	2012	x		x	x	x			
102	SH Kim	Preferred contexts for mathematical	2006	x		x	x	x			M Thesis
103	R Beatty, J Moss	Connecting grade 4 students from di	2006	x		x	x	x			
104	W Yung	Thoughts and practice of a Hong Kor	2011	x		x	x	x			Thesis
105	M Van den Heuvel-Panhuizen	Didactical Phenomenology (Freuden	2014	x			x	x			
106	J Tolboom	The potential of a classroom network	2012	x		x	x	x			Thesis
107	R Risnawati, K Annisa, R Elviza	THE DEVELOPMENT OF REALISTI	2014	x		x	x	x			
108	NC Verhoef	Lesson study: the effect on teachers'	2011	x			x	x			
109	J Threlfall	18 The 'gifted and talented'	2010	x			x	x			
110	O Olande	Graphical artefacts: Taxonomy of stu	2014				x	x			
111	A Hansen, D Drews, J Dudgeon, F Lawton, L Surtees	Children's Errors in Mathematics	2014	x		x	x	x			
112	FUNY ICRIEMS	ICRIEMS 2014 PROCEEDING-FRON	2014	x			x	x			
113	WC Council	School Information	2012			x	x	x			
114	C Headlam	An investigation into children's under	2013	x		x	x	x			NB 5 design principles of RME, Thesis
	SCIENCE DIRECT										

115	Dermot Donnelly, Oliver McGarr, John O'Reilly	A framework for teachers' integration	2011			x	x	x			
116	Christina Hadjithoma, Yiasemina Karagiorgi	The use of ICT in primary schools wi	2009			x	x	x			
117	Joni de Almeida Amorim, Izabel de Moraes Sarmento R	Defining the design parameters of a	2011			x					
118	Liisa Ilomäki, Pirkko Rantanen	Intensive use of ICT in school: Devel	2007			x					
119	Athanassios Jimoyiannis	Designing and implementing an integ	2010			x					
120	Adem Kilicman, Munther A. Hassan, S.K. Said Husain	Teaching and Learning using Mather	2010		x	x	x	x			
121	Shiang-Kwei Wang, Hui-Yin Hsu, Thomas C. Reeves, D	Professional development to enhanc	2014			x					DBR
122	Robertas Jucevičius, Irena Patašienė, Martynas Patašiu	Digital Dimension of Smart City: Criti	2014			x					
123	Eugenia M.W. Ng	Using a mixed research method to ev	2014			x	x	x			
124	Heikki Ruismäki, Riikka-Liisa Salomaa, Inkeri Ruokonen	Minerva Plaza – A New Technology-E	2015			x	x				
125	Hannah Slay, Ingrid Siebörger, Cheryl Hodgkinson-Willi	Interactive whiteboards: Real beauty	2008			x	x				
126	J.C. Aczel, S.R. Peake, P. Hardy	Designing capacity-building in e-lear	2008			x	x				
127	Gulcan Numanoglu, Safak Bayir	Evaluation of information and commu	2009			x	x				
128	Simon Marginson	Higher Education in the Global Know	2010			x					
129	E. Makri-Botsari, F. Paraskeva	Exploring the personality characteris	2010			x					
130	Ton Mooij	Contextual learning theory: Concrete	2007			x	x				
131	Paul F. Burke, Peter J. Aubusson, Sandra R. Schuck, J	How do early career teachers value d	2015				x				
132	Ya-Ting Carolyn Yang	Virtual CEOs: A blended approach to	2015			x	x				
133	Matthew Kearney, Kevin Burden, Tapan Rai	Investigating teachers' adoption of si	2015			x	x				
134	Norlidah Alias, SaedahSiraj, Mohd Nazri Abdul Rahmar	Research and Trends in the Studies	2013			x					
135	Peggy A. Ertmer, Anne Ottenbreit-Leftwich	Removing obstacles to the pedagogi	2013			x	x				
136	Danakorn Nincarean, Mohamad Bilal Alia, Noor Dayana	Mobile Augmented Reality: The Pote	2013			x	x				
137	Omid AskariSichani, Mahdi Jalili	Influence maximization of informed a	2015			x					
138	Xiao-Ning Shen, Xin Yao	Mathematical modeling and multi-obj	2015			x		x			
139	Heng Hou	What makes an online community of	2015			x	x				
140	Xiaodong Wei, Dongdong Weng, Yue Liu, Yongtian Wa	Teaching based on augmented realit	2015			x	x				
141	Alan R.R. de Freitas, Peter J. Fleming, Frederico G. Gu	Aggregation Trees for visualization a	2015								
142	Tassos A. Mikropoulos, Antonis Natsis	Educational virtual environments: A t	2011			x	x	x			
143	Fabiane Barreto Vavassori Benitti	Exploring the educational potential o	2012			x					
144	Nikleia Eteokleous	Evaluating computer technology inte	2008			x	x				
145	Theodore J. Kopcha, Anne Ottenbreit-Leftwich, Jiyoon J	Examining the TPACK framework thr	2014			x	x				
146	Oscar Ardaiz-Villanueva, Xabier Nicuesa-Chacón, Osca	Evaluation of computer tools for idea	2011			x	x				
147	Chau Kien Tsong, Toh Seong Chong, Zarina Samsudin	Tangible Multimedia: A Case Study f	2012		x	x	x				
148	Nico Rutten, Wouter R. van Joolingen, Jan T. van der V	The learning effects of computer sim	2012			x	x				
149	José A. González-Martínez, Miguel L. Bote-Lorenzo, Ed	Cloud computing and education: A st	2105			x	x				
150	Francisco Campuzano, Teresa Garcia-Valverde, Juan A	Generation of human computational	2015			x					
151	Robert S. Savage, Ozlem Erten, Philip Abrami, Geoffrey	ABRACADABRA in the hands of teac	2010			x	x				
152	Radu Lucian Olteanu, Crinela Dumitrescu, Gabriel Gorg	Related aspects to the impact of virtu	2009			x	x				
153	Maja Ćukušić, Nikša Alfirević, Andrina Granić, Željko G	Effects of attitudes and behaviours o	2010			x	x				
154	Monique Volman	A variety of roles for a new type of te	2005			x	x				
155	Gráinne Conole, Maarten de Laat, Teresa Dillon, Jonath	'Disruptive technologies', 'pedagogic	2008			x					
156	Richard F. Schmid, Robert M. Bernard, Eugene Borokh	The effects of technology use in post	2014			x	x				
157	Eline Vanassche, Geert Kelchtermans	Teacher educators' professionalism i	2014				x				
158	Florence Martin, Jeffrey Ertzberger	Here and now mobile learning: An ex	2013			x	x				
159	Alexis Taylor	Perceptions of prospective entrants t	2006				x				
160	Mohd Hishamuddin Abdul Rahman, Noraffandy Yahaya	Open Wonderland: A Potential 3-D M	2013			x	x				
161	Romiţă B. Iucu, Elena Marin	Authentic Learning in Adult Educatio	2014				x				
162	Fariba Mirzaei, Fatin Aliah Phang, Hamidreza Kashefi	Assessing and Improving Reflective	2014								
163	Tosti H.C. Chiang, Stephen J.H. Yang, Gwo-Jen Hwang	Students' online interactive patterns	2014				x				
164	Yusuf Aydin	The Effects of Problem Based Appro	2014		x		x	x			
165	Peter Sommerauer, Oliver Müller	Augmented reality in informal learnin	2014			x	x	x			
166	Charoula Angeli	Transforming a teacher education m	2005			x	x				
167	Peggy A. Ertmer, Anne T. Ottenbreit-Leftwich, Olgun Sa	Teacher beliefs and technology integ	2012			x	x				
168	Jonathan M. Vitale, Michael I. Swart, John B. Black	Integrating intuitive and novel ground	2014				x	x			
169	Patrick Suppes, Tie Liang, Elizabeth E. Macken, Daniel	Positive technological and negative p	2014			x	x	x			
170	Dogan Ozgen, Bahadir Gulsun	Combining possibilistic linear progra	2014				x				
171	Anke Tigchelaar, Jan D. Vermunt, Niels Brouwer	Patterns of development in second-c	2014				x				
172	Funda Dağ, Levent Durdu, Serpil Gerdan	Evaluation of Educational Authoring	2014			x	x				
173	Elena C. Daschmann, Thomas Goetz, Robert H. Stupni	Exploring the antecedents of boredo	2014				x				

174	Peter van Rosmalen	A narrative literature review of games	2014			x	x	x			
175	Franziska Bokhorst, Johannes Moskaliuk, Ulrike Cress	How patterns support computer-med	2014			x					
176	Emma Nuraihan Mior Ibrahim, Norazlinawati Walid	Trust Contributing Factors in M-Learn	2014			x	x				
177	Yalın Kılıç Türel	An interactive whiteboard student su	2011			x	x				
178	Hyeonjin Kim, Michael J. Hannafin	Developing situated knowledge abou	2011			x	x				
179	Patrick Buckley, John Garvey, Fergal McGrath	A case study on using prediction ma	2011			x	x				
180	Răzvan-Lucian Andronic, Anca-Olga Andronic, Elena D	Opinions about Distance Learning in	2012				x				
181	Virginie März, Geert Kelchtermans	Sense-making and structure in teach	2013				x	x			
182	Thomas M. Connolly, Elizabeth A. Boyle, Ewan MacArth	A systematic literature review of emp	2012			x	x				
183	Ecaterina Sarah Frăsineanu	Approach to Learning Process: Supe	2013				x				
184	Dan Wang, Manman Gao	Educational equality or social mobilit	2013				x				
185	Yusuf Zorlu, Oktay Baykara, Fulya Zorlu	The Views of Pre-service Classroom	2013			x	x				
186	J. Strobel, J. Wang, N.R. Weber, M. Dyehouse	The role of authenticity in design-bas	2013				x				
187	Sandra Y. Okita, Selen Turkay, Mihwa Kim, Yumiko Mut	Learning by teaching with virtual pee	2013			x	x				
188	Jennifer L. Eastwood, Troy D. Sadler	Teachers' implementation of a game	2013			x	x				
189	Hsin-Kai Wu, Silvia Wen-Yu Lee, Hsin-Yi Chang, Jyh-C	Current status, opportunities and cha	2013			x	x				
190	Wu-Yuin Hwang, Shih-Shin Hu	Analysis of peer learning behaviors u	2013			x	x	x			
191	Rafał Wojciechowski, Wojciech Cellary	Evaluation of learners' attitude towar	2013			x	x				
192	Orna Schatz-Oppenheimer, Nurit Dvir	From ugly duckling to swan: Stories c	2014				x				
193	Tesha Sengupta-Irving, Elizabeth Redman, Noel Enyed	Re-storying practice: Using stories at	2013				x	x			
194	Francesco Buccafurri, Vincenzo Daniele Foti, Gianluca	Bridge analysis in a Social Internetw	2013			x	x				
195	Donghee Yvette Wohn, Nicole B. Ellison, M. Laeeq Kha	The role of social media in shaping fi	2013			x	x				
196	Tracy Packiam Alloway, John Horton, Ross G. Alloway,	Social networking sites and cognitive	2013			x	x				
197	Fang-Ying Yang, Chun-Yen Chang, Wan-Ru Chien, Yu-	Tracking learners' visual attention du	2013			x	x				
198	Gerald Matua Amandu, Joshua Kanaabi Muliira, Dennis	Using Moodle E-learning Platform to	2013			x	x				
199	Didem İnel, Ali Günay Balım	Concept Cartoons Assisted Problem	2013			x	x				
200	N. Barbalios, I. Ioannidou, P. Tzionas, S. Paraskeuopou	A model supported interactive virtual	2013			x	x				
201	Jian-Wei Lin, Yuan-Cheng Lai	Online formative assessments with s	2013			x	x				
202	Zahira Merchant, Ernest T. Goetz, Lauren Cifuentes, W	Effectiveness of virtual reality-based	2014			x	x				
203	Gwen Noteborn, Amber Dailey-Hebert, Katerina Bohle C	Essential knowledge for academic pe	2014			x	x				
204	Gary Cheng	Exploring students' learning styles in	2014			x	x				
205	Keith R. Bujak, Iulian Radu, Richard Catrambone, Blair	A psychological perspective on augm	2013		x	x	x	x			
206	Matthew H. Schneps, Jonathan Ruel, Gerhard Sonnert,	Conceptualizing astronomical scale:	2014			x	x				
207	Carina Girvan, Brendan Tangney, Timothy Savage	SLurtles: Supporting constructionist l	2013			x	x				
208	Tina Seidel, Geraldine Blomberg, Alexander Renkl	Instructional strategies for using vide	2013			x	x				
209	Mansureh Kebritchi, Atsusi "2c" Hirumi	Examining the pedagogical foundatio	2008			x	x				
210	Ani Grubišić, Slavomir Stankov, Marko Rosić, Branko Ž	Controlled experiment replication in e	2009			x	x				
211	Siu Cheung Kong	The development of a cognitive tool	2008			x	x	x			
212	Lee Tien Tien, Kamisah Osman	Pedagogical Agents in Interactive Mu	2010			x	x				
213	Marina Papastergiou	Digital Game-Based Learning in high	2009			x	x				
214	Thomas Li-Ping Tang, M. Jill Austin	Students' perceptions of teaching tec	2009			x	x				
215	Joey Offer, Beth Bos	The design and application of techno	2009			x	x	x			
216	MariCarmen González-Videgaray, Gregorio Hernández-	Learning objects in theory and practi	2009			x	x				
217	Luca Chittaro, Roberto Ranon	Web3D technologies in learning, edu	2007			x	x				
218	Kyriaki Anagnostopoulou, Vassilia Hatzinikita, Vasilia C	PISA And Biology School Textbooks	2012			x	x				
219	Zeynep Yildiz, Belkis Karabiyik	The Implementation of a Lesson Plan	2012				x	x			
220	Anke Tigchelaar, Jan D. Vermunt, Niels Brouwer	Patterns of development in second-c	2012				x				
221	Erin Peters Burton	Student work products as a teaching	2013				x				
222	Ana-Maria Bliuc, Robert A. Ellis, Peter Goodyear, Leanı	A blended learning Approach to teac	2011			x	x				
223	Ece Özdoğan	Play, mathematic and mathematical	2011		x		x	x			
224	Mansoureh Takaffoli, Farzad Sangi, Justin Fagnan, Osr	Community Evolution Mining in Dyna	2011			x					
225	Hokyoung Ryu, David Parsons	Risky business or sharing the load? -	2012			x	x				
226	Sahin Oruç, Burcu Teymuroglu	The effects of using cartoon in teach	2011			x	x				
227	Fengfeng Ke	Computer-game-based tutoring of ma	2011			x	x	x			
228	Carina González-González, Francisco Blanco-Izquierdo	Designing social videogames for edu	2012			x	x				
229	Elif Seyitoğlu, Kübra Akkaya, Cemalettin Yıldız, Selahat	Students' views about activities deve	2005				x	x			
230	Roselainy Abdul Rahman, Yudariah Mohammad Yusof,	Improving the Teaching of Engineeri	2012				x	x			
231	Jin-Young Kim	A study on learners' perceptional typ	2012			x	x				
232	Robin Kay, Ilona Kletskin	Evaluating the use of problem-based	2012			x	x	x			

233	Petr Sládek, Tomáš Miléř, Renáta Benárová	How to increase students' interest in	2011		x	x	x				
234	Sare Şengül, Yasemin Katranci	Problem Solving and Problem Posing	2012				x	x			
235	Scott J. Warren, Mary Jo Dondlinger, Julie McLeod, Chr	Opening The Door: An evaluation of	2012			x	x				
236	Devrim Üzel, Emine Özdemir	The Effects of Problem-Based E-Lea	2012			x	x	x			
237	Minchi C. Kim, Michael J. Hannafin	Scaffolding problem solving in techn	2011			x	x				
238	Neil Marshall, Chantal Buteau, Daniel H. Jarvis, Zsolt L	Do mathematicians integrate comput	2012			x	x	x			
239	Leslie M. Miller, Ching-I. Chang, Shu Wang, Margaret E	Learning and motivational impacts of	2011			x	x				
240	Nicolae Ioana, Simona Mioara Marin	Education, Excitement and Concern	2012				x				
241	Ibrahim Ahmad, Azizah Jaafar	Computer Games: Implementation in	2012			x	x				
242	Levent Bayram	Use of Online Video Cases in Teach	2012			x	x				
243	Sapna Cheryan, Andrew N. Meltzoff, Saenam Kim	Classrooms matter: The design of vi	2011			x	x				
244	Philip G. Butcher, Sally E. Jordan	A comparison of human and comput	2010			x	x				
245	Barney Dalgarno, Andrea G. Bishop, William Adlong, D	Effectiveness of a Virtual Laboratory	2009			x	x				
246	Winnie Wing-mui So, Jacky Wai-cheong Pow, Vincent H	The interactive use of a video databa	2009			x	x				
247	F. Paraskeva, H. Bouta, Aik. Papagianni	Individual characteristics and compu	2008			x	x				
248	Karen D. Könings, Saskia Brand-Gruwel, Jeroen J.G. va	Teachers' perspectives on innovatio	2007				x				
249	Şengül S. Anagün, Cansu Filik İşcen	Pre-service teachers' perceptions an	2010			x	x				
250	Rohani Ahmad Tarmizi, Mohd Ariff Ahmad Tarmizi, Nur	Problem-based learning: engaging st	2010				x				
251	Aleandra da Silva Figueira-Sampaio, Eliane Elias Ferre	A constructivist computational tool to	2009			x	x				
252	Veljko Potkonjak, Miomir Vukobratović, Kosta Jovanovi	Virtual Mechatronic/Robotic laborato	2010			x	x				
253	Iwan Kartiko, Manolya Kavakli, Ken Cheng	Learning science in a virtual reality a	2010			x	x				
254	Cher Ping Lim	Global citizenship education, school	2008			x	x				
255	Yusuf Koc, Deniz Peker, Aslihan Osmanoglu	Supporting teacher professional dev	2009			x	x				
256	Chun-Yi Lee, Ming-Puu Chen	A computer game as a context for no	2009			x	x	x			
257	Fien Depaepe, Erik De Corte, Lieven Verschaffel	Teachers' approaches towards word	2010		x	x	x	x			
258	Peter Giovanni Araña Sangcap	Mathematics-related Beliefs of Filipin	2010				x	x			
259	Elinda Ai-Lim Lee, Kok Wai Wong, Chun Che Fung	How does desktop virtual reality enh	2010			x	x				
260	David A. Georgina, Charles C. Hosford	Higher education faculty perceptions	2009			x	x				
261	Wei Wang, Xiaohong Guan, Beizhan Wang, Yaping Wa	A novel mobility model based on sen	2010			x					
262	Maja Wrzesien, Mariano Alcañiz Raya	Learning in serious virtual worlds: Ev	2010			x	x				
263	Evangelos Triantafillou, Elissavet Georgiadou, Anastas	The design and evaluation of a comp	2008			x	x				
264	Jung-Chuan Yen, Chun-Yi Lee	Exploring problem solving patterns a	2011			x	x				
265	Martin Ebner, Andreas Holzinger	Successful implementation of user-c	2007			x	x				
266	B. Slof, G. Erkens, P.A. Kirschner, J. Janssen, C. Phiel	Fostering complex learning-task perf	2010			x	x				
267	Teresa Monahan, Gavin McArdle, Michela Bertolotto	Virtual reality for collaborative e-learn	2008			x	x				
268	Brian Marsh, Nick Mitchell, Peter Adamczyk	Interactive video technology: Enhanc	2010			x	x				
269	Nancy Dalgarno, Lynda Colgan	Supporting novice elementary mathe	2007			x	x	x			
270	Mustafa Koc	Let's make a movie: Investigating pr	2011			x	x				
271	Seher Mandaci Şahin, Adnan Baki	A new model to assess mathematica	2010				x	x			
272	Georgia Lazakidou, Symeon Retalis	Using computer supported collaborat	2010			x	x	x			
	Scopus										
273	Zaranis, N	The use of ICT in kindergarten fo	2015	x	x	x	x	x			
274	Nicholas, Z	The use of Information and Comm	2013	x	x	x	x	x			
275	Player-Koro, C.	Hype, hope and ICT in teacher ed	2013			x	x	x			
276	Sakinah, M., Purwanto, S.E.	Increasing students' mathematica	2013	x	x	x	x	x			
277	Mustafa, H.M.H., Al-Hamadi, A., Al-Ghamdi, S.A.,	On assessment of students' acad	2013			x	x				
278	Bratianu, C., Orzea, I.	Knowledge strategies in using so	2012			x	x				
279	Panagouli, D., Priovolou, M.	Let's go to the movies! learning n	2012	x	x	x	x	x			
280	Stohlmann, M.	Youtube incorporated with mathe	2012	x	x	x	x	x			
281	Robinson, C.L.	Using sport to engage and motiva	2012		x	x	x	x			
282	Maktabi, H., Tohidi, H., Seyedaliakbar, S.M.	An application of linear programm	2011				x	x			
283	Csíkos, C., Kelemen, R., Verschaffel, L.	Fifth-grade students' approaches	2011				x	x			
284	Tims, H., Corbett, K.S., Turner III, G.E., Hall, D.E.	Technology enabled projects for	2011			x	x				
285	Hussain, D.M.A., Javed, M.A., Ahmed, Z.	Visualization through knowledge	2011			x					
286	Mustafa, H.M., Badran, S.M., Al-Hamadi, A., Al-Sc	On mathematical modeling of coc	2011			x	x	x			
287	Yoon, C., Dreyfus, T., Thomas, M.O.J.	How high is the tramping track? M	2010	x	x	x	x	x			
288	Perrenet, J., Adan, I.	The academic merits of modelling	2010				x	x			
289	Verleger, M., Diefes-Dux, H.	Facilitating teaching and research	2010		x	x	x	x			
290	Valle, C., Rosser, S., Murray, J., Newstetter, W., J	InTEL: Presenting realistic exerci	2010		x	x	x	x			

291	Berlin, D.F., White, A.L.	Preservice mathematics and scie	2010				x	x		
292	Kruse, S., Nielsen, K., Troelsen, R.	Becoming a teaching scholar	2009				x	x		
293	Mihăescu, M.C.	Using mathematics for data traffic	2009			x	x	x		
294	De Almeida, A.M., Pascoal, M., Fernandes, J., Per	Experiential learning in Science:	2009			x	x			
295	Corsaro, S., De Angelis, P.L., Guarracino, M., Mar	Kremm: An e-learning system for	2009		x	x	x	x		
296	Bell, I.T., Gibson, K.	Mathematical knowledge of techr	2009			x	x	x		
297	Lim, C.P.	Global citizenship education, sch	2008			x	x	x		
298	Guryan, J., Jacob, B., Klopfer, E., Groff, J.	Using Technology to Explore Soc	2008			x	x			
299	Yue, J.	Spatial visualization by realistic 3	2007			x				
300	Depaepe, F., De Corte, E., Verschaffel, L.	Unraveling the culture of the matt	2007		x	x	x	x		
301	Straesser, R.	Didactics of mathematics: More th	2007				x	x		
302	Doorman, M., Drijvers, P., Dekker, T., van den He	Problem solving as a challenge fo	2007	x	x	x	x	x		
303	López-Morteo, G., Galaviz-Férman, M., López, G.,	The use of interactive instructors	2007		x	x	x	x		
304	Howard, W.E., Williams, R., Prantil, V.	Work in progress - Incorporating	2007		x	x	x	x		
305	Anthony, L., Yang, J., Koedinger, K.R.	Towards the application of a hand	2006				x	x		
306	Rautopuro, J., Pöntinen, S., Kukkonen, J.	Towards the information society -	2006			x	x			
307	Fagnant, A.	The use of mathematical symboli	2005	x	x		x	x		
308	Van Putten, C.M., Van Den Brom-Snijders, P.A., E	Progressive mathematization of l	2005	x	x		x	x		
	Web of Science and MathSciNet									
309	Rutherford, Teomara, Farkas, George, Duncan, G	A Randomized Trial of an Elemer	2014		x	x	x	x		
310	Player-Koro, Catarina	hope and ICT in teacher educatio	2013			x	x	x		
311	Berlin, D.F., White, A.L.	Preservice mathematics and scie	2010				x	x		
312	Guryan, J., Jacob, B., Klopfer, E., Groff, J.	Using technology to explore socia	2008			x				
313	Cobb, George W.	One Possible Frame for Thinking	2007			x	x	x		
314	fagnant, A.	The use of mathematical symboli	2005	x	x		x	x		
315	Van Putten, C.M., Van Den Brom-Snijders, P.A., E	Progressive mathematization of l	2005	x	x		x	x		
316	Maag, M	The effectiveness of an interactiv	2004			x	x			
317	Anita, Sebastian; Arnautu, Viorel; Capasso, Vince	An introduction to optimal control	2011			x	x	x		
318	Gu, Edward Y. L.	A journey from robot to digital hur	2013			x	x	x		

Total	No	Author	Title	Year Published	Link	Relevance		RME	Realistic	ICT	matics/teacher/edu	thematics Educati	Reason for exclusion	Include	Comments
1	9	van Loon-Hillen, Nelleke; van Gog, Tamara; Brand-Gruwel, Saskia.	Effects of worked examples in a primary school mathematics curriculum.	2012				x		x	x	x			
2	16	Andresen, Mette.	Introduction of a new construct: The conceptual tool "Flexibility,".	2007				x		x	x	x			
3	43	Cahyono, Adi Nur.	Virtualmatriks: A Conceptual Mathematization Process in Virtual Learning Environment.	2012				x		x	x	x			
4	46	Tirosh D, Wood T	Tools and processes in mathematics teacher education	2008				x		x	x	x			RME Ch 12
5	49	Zaranis N, Kalogiannakis M, Papadakis S	Using mobile devices for teaching realistic mathematics in kindergarten education	2013				x		x	x	x			
6	51	Kizito RN	Pretesting mathematical concepts with the mobile phone: Implications for curriculum design	2011				x		x	x	x			Distance
7	54	Doorman LM, Drijvers PHM, Gravemeijer KPE	Design research in mathematics education: the case of an ict-rich learning arrangement for the concept of function	2013				x		x	x	x			NB
8	274	Zaranis N	The use of Information and Communication Technologies in the first grade of primary school for teaching rectangles based in Realisti	2013				x		x	x	x			NB
9	60	Gravemeijer K	Aiming for 21st century skills	2012				x		x	x	x			HLT
10	62	Heck A, Boon P, Bokhove C	Applets for learning school algebra and calculus	2007				x		x	x	x			
11	64	Perez M	Scaffolding Teachers' Construction of a Learning Trajectory for Mathematics Supported by ICT	2012				x		x	x	x			
12	65	Zaranis N	The use of ICT in the first grade of primary school for teaching circles, triangles, rectangles and squares	2014				x		x	x	x			
13	66	Karagiannidis C, Politis P, Karasavvidis I	Research on e-Learning and ICT in Education: Technological, Pedagogical and Instructional Perspectives	2014				x		x	x	x			
14	67	Zaranis N	Geometry Teaching Through ICT in Primary School	2014				x		x	x	x			
15	69	Drijvers P, Boon P, Doorman M	Digital design: RME principles for designing online tasks					x		x	x	x			NB
16	70	Drijvers P	Digital technology in mathematics education: why it works (or doesn't)	2013				x		x	x	x			
17	74	Stols G	Does the use of technology make a difference in the geometric cognitive growth of pre-service mathematics teachers?	2012				x		x	x	x			
18	76	Pitta-Pantazi D	Number Teaching and Learning	2014				x		x	x	x			
19	77	Carroll B	An investigation into the teaching and learning of applications of mathematics in senior-cycle schools in Ireland	2011				x	x	x	x	x			Thesis
20	84	Mousoulides N, Sriraman B, Christou C	From problem solving to modelling	2007				x		x	x	x			
21	87	Gordon J, Rey O, Siewiorek A, Vivitsou M	KeyCoNet 2012 Literature Review: Key competence development in school education in Europe	2012				x		x	x	x			
22	88	Heck A, Boon P, van Velthoven W	Mathematica empowered applets for learning school algebra and calculus	2008				x		x	x	x			
23	94	Fitzallen N, Watson J	Developing a Sequence of Learning Experiences in Statistics	2014				x		x	x	x			Ch in book
24	95	Domazet M, Baranović B, Matić - Sociologija i prostor J	Mathematics Competence and International Mathematics Testing: Croatian Starting Point	2013				x		x	x	x			
25	96	Doorman M, Drijvers P, Gravemeijer K, Boon P	Tool use and the development of the function concept: from repeated calculations to functional thinking	2012				x		x	x	x			
26	100	Heck A	Modelling in cross-disciplinary authentic student research projects	2010				x		x	x	x			
27	101	Ndlovu M, Mji A	Pedagogical implications of students' misconceptions about deductive geometric proof	2012				x		x	x	x			
28	102	Kim SH	Preferred contexts for mathematical literacy of Korean grade 8-10 learners	2006				x		x	x	x			M Thesis
29	103	Beatty R, Moss J	Connecting grade 4 students from diverse urban classrooms: virtual collaboration to solve generalizing problems	2006				x		x	x	x			
30	104	Yung W	Thoughts and practice of a Hong Kong teacher in mathematics alternative assessment via concerns-based adoption model	2011				x		x	x	x			Thesis
31	106	Tolboom J	The potential of a classroom network to support teacher feedback	2012				x		x	x	x			Thesis
32	107	Risnawati R, Annisa K, Elviza R	THE DEVELOPMENT OF REALISTIC MATHEMATICS EDUCATION-BASED BLOG AT LINEAR ALGEBRA COURSE IN UIN SUSK	2014				x		x	x	x			
33	111	Hansen A, Drews D, Dudgeon J, Lawton F, Surtees L	Children's Errors in Mathematics	2014				x		x	x	x			
34	114	Headlam C	An investigation into children's understanding of the order of operations	2013				x		x	x	x			NB 5 design principles of RME, Thesis
35	273	Zaranis, N	The use of ICT in kindergarten for teaching addition based on realistic mathematics education	2015				x	x	x	x	x			
36	276	Sakinah, M., Purwanto, S.E.	Increasing students' mathematical creative thinking abilities through realistic mathematics education using ICT and deduct	2013				x	x	x	x	x			
37	279	Panagouli, D., Priovolou, M.	Let's go to the movies! learning math through creativity and role playing	2012				x	x	x	x	x			
38	280	Stohlmann, M.	Youtube incorporated with mathematical modelling activities: Benefits, concerns, and future research opportunities	2012				x	x	x	x	x			
39	287	Yoon, C., Dreyfus, T., Thomas, M.O.J.	How high is the tramping track? Mathematising and applying in a calculus model-eliciting activity	2010				x	x	x	x	x			
40	302	Doorman, M., Drijvers, P., Dekker, T., van den Heuvel-Panhuize	Problem solving as a challenge for mathematics education in The Netherlands	2007				x	x	x	x	x			

Total	No	Author	Title	Year Published	RME	ICT	matics/teacher/edu	thematics Educati	Reason for exclusion	Include	Comments
1	16	Andresen, Mette.	Introduction of a new construct: The conceptual tool "Flexibility.".	2007	x	x	x	x			
2	103	Beatty R, Moss J	Connecting grade 4 students from diverse urban classrooms: virtual collaboration to solve generalizing problems	2006	x	x	x	x			
3	43	Cahyono, Adi Nur.	Virtualmatriks: A Conceptual Mathematization Process in Virtual Learning Environment.	2012	x	x	x	x			
4	77	Carroll B	An investigation into the teaching and learning of applications of mathematics in senior-cycle schools in Ireland	2011	x	x	x	x			Thesis
5	95	Domazet M, Baranović B, Matić - Sociologija i prostor J	Mathematics Competence and International Mathematics Testing: Croatian Starting Point	2013	x	x	x	x			
6	54	Doorman LM, Drijvers PHM, Gravemeijer KPE	Design research in mathematics education: the case of an ict-rich learning arrangement for the concept of function	2013	x	x	x	x			NB
7	96	Doorman M, Drijvers P, Gravemeijer K, Boon P	Tool use and the development of the function concept: from repeated calculations to functional thinking	2012	x	x	x	x			
8	302	Doorman, M., Drijvers, P., Dekker, T., van den Heuvel-Panhui	Problem solving as a challenge for mathematics education in The Netherlands	2007	x	x	x	x			
9	70	Drijvers P	Digital technology in mathematics education: why it works (or doesn't)	2013	x	x	x	x			
10	69	Drijvers P, Boon P, Doorman M	Digital design: RME principles for designing online tasks		x	x	x	x			NB
11	94	Fitzallen N, Watson J	Developing a Sequence of Learning Experiences in Statistics	2014	x	x	x	x			Ch in book
12	87	Gordon J, Rey O, Siewiorek A, Vivitsou M	KeyCoNet 2012 Literature Review: Key competence development in school education in Europe	2012	x	x	x	x			
13	60	Gravemeijer K	Aiming for 21st century skills	2012	x	x	x	x			HLT
14	111	Hansen A, Drews D, Dudgeon J, Lawton F, Surtees L	Children's Errors in Mathematics	2014	x	x	x	x			
15	114	Headlam C	An investigation into children's understanding of the order of operations	2013	x	x	x	x			NB 5 design principles of RME, Thesis
16	100	Heck A	Modelling in cross-disciplinary authentic student research projects	2010	x	x	x	x			
17	62	Heck A, Boon P, Bokhove C	Applets for learning school algebra and calculus	2007	x	x	x	x			
18	88	Heck A, Boon P, van Velthoven W	Mathematica empowered applets for learning school algebra and calculus	2008	x	x	x	x			
19	66	Karagiannidis C, Politis P, Karasavvidis I	Research on e-Learning and ICT in Education: Technological, Pedagogical and Instructional Perspectives	2014	x	x	x	x			
20	102	Kim SH	Preferred contexts for mathematical literacy of Korean grade 8-10 learners	2006	x	x	x	x			M Thesis
21	51	Kizito RN	Pretesting mathematical concepts with the mobile phone: Implications for curriculum design	2011	x	x	x	x			Distance
22	84	Mousoulides N, Sriraman B, Christou C	From problem solving to modelling	2007	x	x	x	x			
23	101	Ndlovu M, Mji A	Pedagogical implications of students' misconceptions about deductive geometric proof	2012	x	x	x	x			
24	279	Panagouli, D., Priovolou, M.	Let's go to the movies! learning math through creativity and role playing	2012	x	x	x	x			
25	64	Perez M	Scaffolding Teachers' Construction of a Learning Trajectory for Mathematics Supported by ICT	2012	x	x	x	x			
26	76	Pitta-Pantazi D	Number Teaching and Learning	2014	x	x	x	x			
27	107	Risnawati R, Annisa K, Elviza R	THE DEVELOPMENT OF REALISTIC MATHEMATICS EDUCATION-BASED BLOG AT LINEAR ALGEBRA COUR	2014	x	x	x	x			
28	276	Sakinah, M., Purwanto, S.E.	Increasing students' mathematical creative thinking abilities through realistic mathematics education us	2013	x	x	x	x			
29	280	Stohlmann, M.	Youtube incorporated with mathematical modelling activities: Benefits, concerns, and future research o	2012	x	x	x	x			
30	74	Stols G	Does the use of technology make a difference in the geometric cognitive growth of pre-service mathematics teacher	2012	x	x	x	x			
31	46	Tirosh D, Wood T	Tools and processes in mathematics teacher education	2008	x	x	x	x			RME Ch 12
32	106	Tolboom J	The potential of a classroom network to support teacher feedback	2012	x	x	x	x			Thesis
33	9	van Loon-Hillen, Nelleke; van Gog, Tamara; Brand-Gruwel, Saskia.	Effects of worked examples in a primary school mathematics curriculum.	2012	x	x	x	x			
34	287	Yoon, C., Dreyfus, T., Thomas, M.O.J.	How high is the tramping track? Mathematising and applying in a calculus model-eliciting activity	2010	x	x	x	x			
35	104	Yung W	Thoughts and practice of a Hong Kong teacher in mathematics alternative assessment via concerns-based adoption	2011	x	x	x	x			Thesis
36	65	Zaranis N	The use of ICT in the first grade of primary school for teaching circles, triangles, rectangles and squares	2014	x	x	x	x			
37	67	Zaranis N	Geometry Teaching Through ICT in Primary School	2014	x	x	x	x			
38	49	Zaranis N, Kalogiannakis M, Papadakis S	Using mobile devices for teaching realistic mathematics in kindergarten education	2013	x	x	x	x			
39	273	Zaranis, N	The use of ICT in kindergarten for teaching addition based on realistic mathematics education	2015	x	x	x	x			
40	274	Zaranis, N	The use of Information and Communication Technologies in the first grade of primary school for teachir	2013	x	x	x	x			

No	Author	Title	ar Publish	RME	ICT	Mathemat	Mathemat
	Google Scholar: Doorman and Drijvers						
1	LM Doorman, PHM Drijvers, KPE Gravemeijer	Design research in mathematics education: the case of an ict-rich learning a	2013	x	x	x	x
2	P Drijvers	Students encountering obstacles using a CAS	2000		x	x	x
3	C Kieran, P Drijvers	The co-emergence of machine techniques, paper-and-pencil techniques, an	2006		x	x	x
4	P Drijvers, M Doorman, P Boon, H Reed…	The teacher and the tool: instrumental orchestrations in the technology-rich	2010		x	x	x
5	P Drijvers, L Trouche	From artifacts to instruments: A theoretical framework behind the orchestra	2008		x	x	x
6	P Drijvers, C Kieran, MA Mariotti, J Ainley	Integrating technology into mathematics education: Theoretical perspectives	2010		x	x	x
7	M Van den Heuvel-Panhuizen, P Drijvers	Realistic mathematics education	2014	x		x	
8	P Drijvers - Zentralblatt für Didaktik der Mathema	Learning mathematics in a computer algebra environment: obstacles are op	2002		x	x	x
9	P Drijvers, K Gravemeijer	Computer algebra as an instrument: Examples of algebraic schemes	2005		x	x	x
10	L Trouche, P Drijvers	Handheld technology for mathematics education: Flashback into the future	2010		x	x	x
11	P Drijvers, M Doorman	The graphics calculator in mathematics education	1996	x	x	x	x
12	HC Reed, P Drijvers, PA Kirschner	Effects of attitudes and behaviours on learning mathematics with computer t	2010		x	x	x
13	M Doorman, P Drijvers, T Dekker	Problem solving as a challenge for mathematics education in The Netherlan	2007	x		x	
14	P Drijvers, O van Herwaarden	Instrumentation of ICT tools: the case of algebra in a computer algebra envi	2000		x	x	x
15	P Drijvers, M Doorman, P Boon	Instrumental orchestration: theory and practice	2009		x	x	x
16	P Drijvers	Learning Algebra in a Computer Algebra Environment.	2004		x	x	x
17	L Trouche, P Drijvers, G Gueudet	Technology-driven developments and policy implications for mathematics ed	2013	x	x	x	x
18	C Bokhove, P Drijvers	Digital tools for algebra education: Criteria and evaluation	2010		x	x	x
19	M Wijers, V Jonker, P Drijvers	MobileMath: exploring mathematics outside the classroom	2010		x	x	x
20	P Drijvers	Teachers transforming resources into orchestrations	2012		x	x	x
21	P Drijvers, A Goddijn, M Kindt	Algebra education: exploring topics and themes	2010			x	x
22	C Bokhove, P Drijvers	Symbol sense behavior in digital activities	, 2010		x	x	x
23	P Drijvers	The concept of parameter in a computer algebra environment	2001		x	x	x
24	P Drijvers, M Doorman, P Boon	Tool use in a technology-rich learning arrangement for the concept of functio	2007		x	x	x
25	M Doorman, P Drijvers, K Gravemeijer, P Boon	Tool use and the development of the function concept: from repeated calcul	2012		x	x	x
26	P Drijvers, JD Godino, V Font, L Trouche	One episode, two lenses	2013			x	x
27	C Bokhove, P Drijvers	Effects of a digital intervention on the development of algebraic expertise	2012		x	x	x
28	P Drijvers, M Doorman, P Boon, S Gisbergen	Teachers using technology: Orchestrations and profiles	2009		x	x	x
29	B Barzel, P Drijvers, M Maschietto	Tools and technologies in mathematical didactics	2006		x	x	x
30	P Drijvers, HG Weigand	The role of handheld technology in the mathematics classroom	2010		x	x	x
31	P Drijvers	Digital technology in mathematics education: Why it works (or doesn't)	2012		x	x	x
32	P Drijvers, A Verweij, E van Winsen	Mathematics lessons with DERIVE	1997		x	x	x
33	P Drijvers, M Doorman, P Boon, HC Reed	The teacher and the tool: whole-class teaching behavior in the technology-ri	2010		x	x	x
34	C Bokhove, P Drijvers	Effects of feedback in an online algebra intervention	2012		x	x	x
35	P Drijvers, P Boon, M Van Reeuwijk	Algebra and technology	2010		x	x	x
36	P Boon, P Drijvers	Chaining operations to get insight in expressions and functions	2005		x	x	x
37	P Drijvers	Secondary algebra education	2010		x	x	x
38	P Drijvers, S Tacoma, A Besamusca, M Doorma	Digital resources inviting changes in mid-adopting teachers' practices and o	2013		x	x	x
39	A Bakker, M Doorman, P Drijvers	] Design research on how IT may support the development of symbols and r	2003	x	x	x	x
40	P Drijvers	Context, abstractie en vaardigheid in schoolalgebra	2006				
41	C Bokhove, P Drijvers	Effects of feedback conditions for an online algebra tool	2011		x	x	x

42	M Doorman, P Drijvers	Algebra in function	2010		x	x	x
43	C Kieran, A Boileau, D Tanguay, P Drijvers	Design researchers' documentational genesis in a study on equivalence of a	2013			x	x
44	P Drijvers	From" Work-and-Walk-By" to" Sherpa-at-Work".	2011		x	x	x
45	A Jupri, P Drijvers	Difficulties in initial algebra learning in Indonesia	2014			x	x
46	P Drijvers, S Tacoma, A Besamusca	Digital technology and mid-adopting teachers' professional development: a c	2014		x	x	x
47	P Drijvers, P Boon, M Doorman, C Bokhove, S T	Digital design: RME principles for designing online tasks	2013	x	x	x	x
48	P Drijvers	You Never Forget Your First Love... The TI-92 in Teacher Education.	1997		x	x	x
49	AWM Dress, H Melenk, BK Waits, P Drijvers	Applications of computer algebra	2003		x	x	x
50	P Drijvers, T Dekker, M Wijers	Patterns and Formulas	2010			x	x
51	P Drijvers, MA Mariotti, J Olive, AI Sacristán	Introduction to Section 2	2010		x	x	x
52	L Trouche, P Drijvers	Webbing and orchestration. Two interrelated views on digital tools in mather	2014		x	x	x
53	P Drijvers	TOOLS AND TECHNOLOGIES IN DIATHENIATICAL DIDACTICS			x	x	x
54	P Drijvers, J Monaghan, M Thomas, L Trouche	Use of technology in secondary mathematics	2014		x	x	x
55	P DRIJVERS, J BÉLANGER, C CORRIVEAU	Objectives of the internship					
56	A Clark-Wilson, P Drijvers	HOW TO TEACH MATHEMATICS WITH TECHNOLOGY: THE EDUMATICS			x	x	x
57	M Zeller, B Barzel, P Drijvers	First Report			x	x	x
58	P Drijvers	7 Learning algebra in			x	x	x
59	P Dowling, T Dreyfus, P Drijvers	Reviewers for Volume 12	2010				
60	G Roorda, P Vos, P Drijvers	Graphing calculator supported instrumentation schemes for the concept of c	2014		x	x	x
61	A Dowker, P Dowling, P Drijvers	Reviewers for Volume 10	2008				
62	P Drijvers, M Doorman, P Kirschner	The Effect of Online Tasks for Algebra on Student Achievement in Grade 8	2014		x	x	x
63	S Tacoma, P Drijvers, P Boon	The Use of a Digital Environment to Improve First Year Science Students' S	2011		x	x	
64	A Jupri, P Drijvers	The impact of a technology-rich intervention on grade 7 students' skills in ini	2014		x	x	x
65	P Drijvers, P Boon, M Doorman	RME principles for designing online tasks		x	x	x	x
66	B Combes, P Drijvers, S Edirisinghi, S Essuman	Members of the group			x	x	x
67	P Kop, F Janssen, P Drijvers	HOW EXPERTS GRAPH FORMULAS			x	x	x
68	P Kop, F Janssen, P Drijvers, M Veenman	IDENTIFYING A FRAMEWORK FOR GRAPHING FORMULAS FROM EXPERT STRATEGIES				x	x
69	B Barzel, P Drijvers, M Maschietto, L Trouche	WORKING GROUP 9 Tools and technology in mathematical didactics			x	x	x
70	P Drijvers	Digital technology in mathematics education: why it works (or doesn't)	2013		x	x	x
71	B Barzel, P Drijvers	Matthias Zeller	2009		x	x	x
72	L Ball, R Brown, P Drijvers, D Driver, P Flynn, K	The Working Group on CAS and Algebra			x	x	x
73	P Drijvers	Digital technology in mathematics education	2014		x	x	x
74	H Doerr, A Dowker, P Drijvers	Reviewers for Volume 7	2005				
75	T Dana-Picard, I Kidron, D Zeitoun, P Drijvers	WORKING GROUP 9. Tools and technologies in mathematical didactics 1331			x	x	x
76	M Doorman, P Boon, P Drijvers, S van Gisberge	Tooluse supporting the learning and teaching of the Function concept		x	x	x	x
77	P Drijvers, B Barzel	Equations with Technology: Different Tools, Different Views.	2012		x	x	x
78	T Dana-Picard, I Kidron, D Zeitoun, P Drijvers	WORKING GROUP 9 Tools and technology in mathematical didactics			x	x	x
	Google Scholar: Gravemeijer						
79	M Stephan, K McClain, K Gravemeijer	Participating in classroom mathematical practices	2011			x	x
80	J Van den Akker, K Gravemeijer, S McKenney	Educational design research	2006				
81	K Gravemeijer -	Educational development and developmental research in mathematics educ	1994			x	x
82	K Gravemeijer	How emergent models may foster the constitution of formal mathematics	1999			x	x
83	K Gravemeijer, P Cobb	Design research from a learning design perspective	2006				
84	K Gravemeijer, M Doorman	Context problems in realistic mathematics education: A calculus course as a	1999 -	x		x	x

85	P Cobb, K McClain, K Gravemeijer	Learning about statistical covariation	2003			x	x
86	J Van den Akker, K Gravemeijer	Introducing educational design research	2006				
87	K Gravemeijer	Developmental research as a research method	1998				
88	K Gravemeijer	Solving word problems: A case of modelling?	1997			x	x
89	M Doorman, P Boon, H Reed, K Gravemeijer	The teacher and the tool: instrumental orchestrations in the technology-rich	2010		x	x	x
90	K Gravemeijer, J Terwel	Hans Freudenthal: a mathematician on didactics and curriculum theory	, 2000	x		x	x
91	P Cobb, K Gravemeijer	Experimenting to support and understand learning processes	2008				
92	KP Gravemeijer	Symbolizing, modeling and tool use in mathematics education	2002			x	x
93	K McClain, P Cobb, K Gravemeijer	Supporting students' ways of reasoning about data	2000			x	x
94	K Gravemeijer, M Stephan	Emergent models as an instructional design heuristic	2002	x		x	x
95	P Drijvers, K Gravemeijer	Computer algebra as an instrument: Examples of algebraic schemes	2005		x	x	x
96	K Gravemeijer	Preamble: from models to modeling	2002			x	x
97	K Gravemeijer, D van Eerde	Design research as a means for building a knowledge base for teachers and	2009			x	x
98	K McClain, P Cobb, K Gravemeijer, B Estes	Developing Mathematical Reasoning within the Context of Measurement.	1999			x	x
99	K Gravemeijer	Emergent modelling as a precursor to mathematical modelling	2007	x		x	x
100	KP Gravemeijer	From a different perspective: Building on students' informal knowledge	1998	x		x	x
101	K Gravemeijer, J Bowers, M Stephan	Chapter 4: A Hypothetical Learning Trajectory on Measurement and Flexible	2003				
102	M Stephan, P Cobb, K Gravemeijer	Chapter 5: Coordinating social and individual analyses: Learning as particip	2003			x	
103	M Dolk, J den Hertog, K Gravemeijer	Using multimedia cases for educating the primary school mathematics teach	2002		x	x	x
104	K Gravemeijer	Creating opportunities for students to reinvent mathematics	2004			x	x
105	A Bakker, KPE Gravemeijer	An historical phenomenology of mean and median	2006	x		x	x
106	K Gravemeijer	Emergent modeling as the basis for an instructional sequence on data analy	2002			x	x
107	L Slangen, H van Keulen, K Gravemeijer	What pupils can learn from working with robotic direct manipulation environ	2011		x	x	x
108	S Van Gisbergen, K Gravemeijer	Tool use in a technology-rich learning arrangement for the concept of functio	2007		x	x	x
109	M Doorman, P Drijvers, K Gravemeijer, P Boon	Tool use and the development of the function concept: from repeated calcul	2012		x	x	x
110	K Gravemeijer, A Bakker	Design research and design heuristics in statistics education	2006			x	x
111	R Taconis, S Bolhuis, K Gravemeijer	Some key issues in creating inquiry-based instructional practices that aim at	2013				
112	P Cobb, E Yackel, K Gravemeijer, A Sfard	A journey in mathematics education research: Insights from the work of Pau	2010			x	x
113	M Wijers, SF Akkerman, KPE Gravemeijer	Proportional reasoning in the laboratory: An intervention study in vocational	2014			x	x
114	K Gravemeijer	Mathematics education and the information society	2013		x	x	x
115	S Palha, R Dekker, K Gravemeijer	Developing shift problems to foster geometrical proof and understanding	2013			x	x
116	S Van Gisbergen, K Gravemeijer	Tool use and conceptual development: an example of a form-function shift	2009		x	x	x
117	K Gravemeijer	Emergent modeling and iterative processes of design and improvement in m	2007		x	x	x
118	KPE Gravemeijer	Transforming mathematics education: The role of textbooks and teachers	2014			x	x
119	K Hoogland, A Bakker, J Koning, K Gravemeijer	Comparing students' results on word problems with their results on image-ri	2012			x	x
120	M van Uum, K Gravemeijer	Characteristics of effective professional development for primary science an	2011		x		
121	RK Wagner, K Gravemeijer	Instructional models in domains and professions	2008			x	x
122	LKJ Baartman, K Gravemeijer	Science and technology education for the future	2011		x		
123	K Gravemeijer	How concrete is concrete?	2014			x	x
124	K Gravemeijer	] Revisiting 'Mathematics education revisited'	2005				
125	A Fauzan, T Plomp, K Gravemeijer	The Development of an RME-based Geometry Course for Indonesian Prima	2013	x	x	x	x
126	K Gravemeijer	Aiming for 21st century skills	2012				
127	R Taconis, H van Keulen, K Gravemeijer	Promoting science and technology in primary education: a review of integrat	2014		x		
128	D Hoek, K Gravemeijer	Changes of interaction during the development of a mathematical learning e	2011			x	x

129	K Gravemeijer, A Rampal -	Mathematics Curriculum Development	2015			x	x
130	H de Beer, M van Eijck, K Gravemeijer	On Teaching Primary Calculus: First Step in a Design Research and some Issues				x	x
131	G Koeno	Competencies in and for the 21st Century	2014				
132	H de Beer, M van Eijck, K Gravemeijer	Design Principles for Teaching Primary Calculus			x	x	x
133	K Fuson, S Goodchild, K Gravemeijer	Reviewers for Volume 6	2004			x	x
134	H de Beer, K Gravemeijer, M van Eijck	Discrete and continuous reasoning about change in primary school classroo	2015			x	x
135	K GRAVEMEIJER	headings: explanation'or understanding'(Bruner 1994). Here one may think	2012				
136	N Figueiredo, F van Galen, K Gravemeijer	Educational Designer	2009			x	x
137	S Palha, R Dekker, K Gravemeijer	THE EFFECT OF SHIFT-PROBLEM LESSONS IN THE MATHEMATICS CL	2014			x	x
138	S Goodchild, M Goos, K Gravemeijer	Reviewers for Volume 11	2009			x	x
139	J van Keulen, K Gravemeijer	Preparing Teachers to Teach Robotics in Primary Schools	2011		x	x	x
140	A Sfard, E Yackel, K Gravemeijer, P Cobb	A Journey in Mathematics Education Research	2011			x	x
141	K Gravemeijer	INTRODUCTION TO THE PME PLENARY PANEL,'SCHOOL MATHEMATIC	2007			x	x
142	R Taconis, S Bolhuis, K Gravemeijer	CREATING A CULTURE OF INQUIRY IN THE CLASSROOM WHILE FOST	2015				
143	J Evans, S Friel, D Geary, K Gravemeijer	Reviewers for Volume 5	2003			x	x
144	M Goos, A Graeber, K Gravemeijer	Reviewers for Volume 8	2006			x	x
145	L Gordon-Calvert, K Gravemeijer	Reviewers for Volume 9	2007			x	x
146	M Doorman, P Boon, S van Gisbergen, K Graver	WORKING GROUP 9. Tools and technologies in mathematical didactics 1331			x	x	x
147	L Baartman, K Gravemeijer, E de Bruijn	Nurses' and Technicians' Communication and Learning at the Boundary	2013		x		
148	M van den Heuvel-Panhuizen, K Gravemeijer	Freudenthal Institute, University of Utrecht, Utrecht, The Netherlands		x		x	x
149	K Gravemeijer, J Bowers, M Stephan	] Continuing the Design Research Cycle: A Revised Measurement and Arithmetic Sequence				x	x
150	P Boon, P Drijvers, S van Gisbergen, K Graveme	Tooluse supporting the learning and teaching of the Function concept			x	x	x
151	K Gravemeijer	Computer algebra as an instrument: examples of algebraic schemes	2009		x	x	x
152	K Gravemeijer, J Bowers, M Stephan	Chapter 6: Continuing the Design Research Cycle: A Revised Measurement	2003			x	x
153	M Doorman, P Boon, S van Gisbergen, K Graver	WORKING GROUP 9 Tools and technology in mathematical didactics			x	x	x
154	K Gravemeijer	Introduction to Section II	2002		x	x	x
155	K Gravemeijer	Number Lines in Mathematics Education	2014			x	x
156	R Goldstone, L Gomez, K Gravemeijer	List of Guest Reviewers—Volume 15	2006			x	x
	Google Scholar: Heck						
157	A Heck, E Kedzierska, T Ellermeijer	Design and implementation of an integrated computer working environment	2009		x	x	x
158	YB Widjaja, A Heck	How a realistic mathematics education approach and microcomputer-based	2003	x	x	x	x
159	T Ellermeijer, A Heck	Differences between the use of mathematical entities in mathematics and ph	2002			x	x
160	A Heck, A Holleman	Walk like a mathematician: an example of authentic education	2003			x	x
161	A Heck	Coach: an environment where mathematics meets science and technology	1999		x	x	x
162	A Heck	Bringing reality into the classroom	2009			x	x
163	E van de Vrie, A Heck	The Role of Digital, Formative Testing in e-Learning for Mathematics: A Cas	2012			x	x
164	A Heck, P Uylings	Capturing the real world in the classroom	2006			x	x
165	A Heck	How a Realistic Mathematics Education Approach andMicrocomputer-Based	2003	x	x	x	x
166	A Heck	A versatile environment for active learners and teachers	2002		x	x	x
167	C Bokhove, G Koolstra, P Boon, A Heck	Towards an integrated learning environment for mathematics	2007			x	x
168	A Heck, H Houwing, C de Beurs	An e-Class in Action: Experiences with ICT-Intensive Teaching and Learning	2009		x	x	x
169	A Heck	Modelling in cross-disciplinary authentic student research projects	2010		x	x	x
170	T Ellermeijer, A Heck	Walk like a physicist; an example of authentic education	2003				
171	AM Cohen, A Heck	Applied Computer Algebra: Experience from Cam Design	1994		x	x	x

172	A Heck, B Bruidegom	Bridging between contexts and concepts: How data video and computer mo	2007		x	x	x
173	A Heck, P Boon, C Bokhove	Applets for learning school algebra and calculus	2007		x	x	x
174	Ö Demir, A Heck	A new learning trajectory for trigonometric functions	2013			x	x
175	JH Davenport, AJP Heck	An overview of computer algebra	1992		x	x	x
176	H Cuypers, H van der Kooij, E van de Vrie, A He	The Role of Digital, Formative Testing in e-Learning for Mathematics: A Cas	2012		x	x	x
177	A Heck, P Boon, W van Velthoven	Mathematica empowered applets for learning school algebra and calculus	2008		x	x	x
178	A Heck	Use of ICT for acquiring, practicing and assessing algebraic expertise			x	x	x
179	A Heck	] Een GeoGebra-ondersteunde benadering van sinus en cosinus	2012		x	x	x
	Google Scholar: Zaranis						
180	N Zaranis	The influence of ICT on the numeracy achievement of Greek kindergarten c	2011		x	x	x
181	G Dimakos, N Zaranis	The influence of the Geometer's Sketchpad on the geometry achievement o	2010		x	x	x
182	N Zaranis, M Kalogiannakis, S Papadakis	Using mobile devices for teaching realistic mathematics in kindergarten edu	2013	x	x	x	x
183	M Geller, SD Brandão, M Kalogiannakis, N Zarar	ICT AND OTHER RESOURCES FOR TEACHING/LEARNING SCIENCE	2011		x		
184	N Zaranis	The use of Information and Communication Technologies in the first grade c	2013	x	x	x	x
185	M Kalogiannakis, N Zaranis	Preschool science education with the use of ICT: a case study	2012		x		
186	N Zaranis, G Baralis, E Skordialos - Kosuke Kan	Comparing the effectiveness of using ICT for teaching addition in the first grade stud	x		x	x	x
187	N Zaranis	The use of ICT in Preschool Education for the teaching of Triangles	2013		x	x	x
188	N Zaranis, V Oikonomidis	Profiling the attitudes of Greek kindergarten teachers towards computers	2013		x	x	x
189	N Zaranis	The use of ICT in the first grade of primary school for teaching circles, triang	2014		x	x	x
190	N Zaranis, V Oikonomidis	The main factors of the attitudes of Greek kindergarten teachers towards inf	2014		x		
191	N Zaranis	Geometry Teaching Through ICT in Primary School	2014		x	x	x
192	N Zaranis	The use of ICT in kindergarten for teaching addition based on realistic math	2014	x	x	x	x
	Scopus: Boon						
193	Albers, W., Boon, P.C., Kallenberg, W.C.M.	The asymptotic behavior of tests for normal means based on a variance pre	2015				
194	Albers, W., Boon, P.C., Kallenberg, W.C.M.	Size and power of pretest procedures					
195	Boon, P.C., Roes, K.C.B	Design and analysis issues for crossover designs in phase I clinical studies	1999				
196	Albers, W., Boon, P.C., Kallenberg, W.C.M.	Testing equality of two normal means using a variance pre-test	1998				
	Scopus: Drijvers						
197	Jupri, A., Drijvers, P., van den Heuvel-Panhuizen	Difficulties in initial algebra learning in Indonesia	2014			x	x
198	Jupri, A., Drijvers, P., Van den Heuvel-Panhuizen	Student difficulties in solving equations from an operational and a structural	2014			x	x
199	Drijvers, P., Doorman, M., Kirschner, P., Hoogve	The effect of online tasks for algebra on student achievement in grade 8	2014		x	x	x
200	Trouche, L., Drijvers, P.	Webbing and orchestration. Two interrelated views on digital tools in mather	2014		x	x	x
201	Kieran, C., Boileau, A., Tanguay, D., Drijvers, P	Design researchers' documentational genesis in a study on equivalence of a	2013			x	x
202	Drijvers, P., Tacoma, S., Besamusca, A., Doorm	Digital resources inviting changes in mid-adopting teachers' practices and o	2013		x		
203	van Stiphout, I., Drijvers, P., Gravemeijer, K.	The development of students' algebraic proficiency	2013			x	x
204	Drijvers, P., Godino, J.D., Font, V., Trouche, L	One episode, two lenses: A reflective analysis of student learning with comp	2012		x	x	x
205	Doorman, M., Drijvers, P., Gravemeijer, K., Boon	Tool use and the development of the function concept: From repeated calcu	2012		x	x	x
206	Bokhove, C., Drijvers, P.	Effects of feedback in an online algebra intervention	2012		x	x	x
207	Bokhove, C., Drijvers, P	ffects of a digital intervention on the development of algebraic expertise	2012		x	x	x
208	Drijvers, P., Weigand, H.-G.	The role of handheld technology in the mathematics classroom	2010		x	x	x
209	Wijers, M., Jonker, V., Drijvers, P.	MobileMath: Exploring mathematics outside the classroom	2010		x	x	x
210	Reed, H.C., Drijvers, P., Kirschner, P.A.	Effects of attitudes and behaviours on learning mathematics with computer t	2010		x	x	x
211	Gravemeijer, K., Doorman, M., Drijvers, P.	Symbolising and the development of meaning in computer-supported algebr	2010		x	x	x
212	Drijvers, P., Doorman, M., Boon, P., Reed, H., G	The teacher and the tool: Instrumental orchestrations in the technology-rich	2010		x	x	x

213	Trouche, L., Drijvers, P.	Handheld technology for mathematics education: Flashback into the future	2010		x	x	x
214	Bokhove, C., Drijvers, P.	Digital tools for Algebra education: Criteria and evaluation	2010		x	x	x
215	Doorman, M., Drijvers, P., Dekker, T., van den H	Problem solving as a challenge for mathematics education in The Netherlan	2007			x	x
216	Kieran, C., Drijvers, P.	The co-emergence of machine techniques, paper-and-pencil techniques, an	2006			x	x
217	Drijvers, P.	Learning mathematics in a computer algebra environment: Obstacles are op	2002			x	x
218	Drijvers, P.	Students encountering obstacees using a cas	2000		x	x	x
219	Drijvers, P., Doorman, M.	The graphics calculator in mathematics education	1996		x	x	x
	Scopus Gravemeijer						
220	Palha, S., Dekker, R., Gravemeijer, K.	THE EFFECT OF SHIFT-PROBLEM LESSONS IN THE MATHEMATICS CL	2014			x	x
221	Kock, Z.-J., Taconis, R., Bolhuis, S., Gravemeije	CREATING A CULTURE OF INQUIRY IN THE CLASSROOM WHILE FOST	2014			x	x
222	Bakker, A., Groenveld, D., Wijers, M., Akkerman	Proportional reasoning in the laboratory: An intervention study in vocational	2014		x	x	x
223	Gresnigt, R., Taconis, R., van Keulen, H., Grave	Promoting science and technology in primary education: a review of integrat	2014			x	x

	No	Author	Title	ar Publis	RME	ICT	Mathemat	Mathemat	Include	Comments
		Google Scholar: Doorman and Drijvers								
1	1	LM Doorman, PHM Drijvers, K	Design research in mathematics education: the case of an ict-rich learning arrangement for the conc	2013	x	x	x	x		
2	11	P Drijvers, M Doorman	The graphics calculator in mathematics education	1996	x	x	x	x		
3	17	L Trouche, P Drijvers, G Gueud	Technology-driven developments and policy implications for mathematics education	2013	x	x	x	x		
4	39	A Bakker, M Doorman, P Drijve	Design research on how IT may support the development of symbols and meaning in mathematics	2003	x	x	x	x		
5	65	P Drijvers, P Boon, M Doorma	Digital design: RME principles for designing online tasks	2013	x	x	x	x		
6	76	M Doorman, P Boon, P Drijvers	Tooluse supporting the learning and teaching of the Function concept		x	x	x	x		
		Google Scholar: Gravemeijer								
7	125	A Fauzan, T Plomp, K Gravem	The Development of an RME-based Geometry Course for Indonesian Primary Schools	2013	x	x	x	x		
		Google Scholar: Heck								
8	158	YB Widjaja, A Heck	How a realistic mathematics education approach and microcomputer-based laboratory worked in les	2003	x	x	x	x		
		Google Scholar: Zaranis								
9	182	N Zaranis, M Kalogiannakis, S	Using mobile devices for teaching realistic mathematics in kindergarten education	2013	x	x	x	x		
10	184	N Zaranis	The use of Information and Communication Technologies in the first grade of primary school for teac	2013	x	x	x	x		
11	186	N Zaranis, G Baralis, E Skordia	Comparing the effectiveness of using ICT for teaching addition in the first grade students based on Realistic		x	x	x	x		
12	192	N Zaranis	The use of ICT in kindergarten for teaching addition based on realistic mathematics education	2014	x	x	x	x		
		Scopus Gravemeijer								
13	237	Dolk, M., Hertog, J.D., Gravem	Using multimedia cases for educating the primary school mathematics teacher educator: A design s	2002	x	x	x	x		
14	240	Gravemeijer, K., Doorman, M.	Context problems in realistic mathematics education: A calculus course as an example	1999	x	x	x	x		
		Scopus: Zaranis								
		Ebsco: Doorman								
15	276	Doorman, Michiel; Drijvers, Pa	Problem solving as a challenge for mathematics education in The Netherlands	2007	x	x	x	x		
		Ebsco: Gravemeijer								
16	311	Gravemeijer, Koeno.	How Emergent Models May Foster the Constitution of Formal Mathematics.	1999	x	x	x	x		
		Articles already selected from other searches								

Total	No	Author	Title	Year Published	Link	Relevance		RME	Realistic	ICT	matics/teacher/edu	thematics Educati	Reason for exclusion	Include	Comments
1	9	van Loon-Hillen, Nelleke; van Gog, Tamara; Brand-Gruwel, Saskia.	Effects of worked examples in a primary school mathematics curriculum.	2012				x		x	x	x			
2	16	Andresen, Mette.	Introduction of a new construct: The conceptual tool "Flexibility.".	2007				x		x	x	x			
3	43	Cahyono, Adi Nur.	Virtualmatriks: A Conceptual Mathematization Process in Virtual Learning Environment.	2012				x		x	x	x			
4	46	Tirosh D, Wood T	Tools and processes in mathematics teacher education	2008				x		x	x	x			RME Ch 12
5	49	Zaranis N, Kalogiannakis M, Papadakis S	Using mobile devices for teaching realistic mathematics in kindergarten education	2013				x		x	x	x			
6	51	Kizito RN	Pretesting mathematical concepts with the mobile phone: Implications for curriculum design	2011				x		x	x	x			Distance
7	54	Doorman LM, Drijvers PHM, Gravemeijer KPE	Design research in mathematics education: the case of an ict-rich learning arrangement for the	2013				x		x	x	x			NB
8	60	Gravemeijer K	Aiming for 21st century skills	2012				x		x	x	x			HLT
9	62	Heck A, Boon P, Bokhove C	Applets for learning school algebra and calculus	2007				x		x	x	x			
10	64	Perez M	Scaffolding Teachers' Construction of a Learning Trajectory for Mathematics Supported by ICT	2012				x		x	x	x			
11	65	Zaranis N	The use of ICT in the first grade of primary school for teaching circles, triangles, rectangles and	2014				x		x	x	x			
12	66	Karagiannidis C, Politis P, Karasavvidis I	Research on e-Learning and ICT in Education: Technological, Pedagogical and Instructional P	2014				x		x	x	x			
13	67	Zaranis N	Geometry Teaching Through ICT in Primary School	2014				x		x	x	x			
14	69	Drijvers P, Boon P, Doorman M	Digital design: RME principles for designing online tasks					x		x	x	x			NB
15	70	Drijvers P	Digital technology in mathematics education: why it works (or doesn't)	2013				x		x	x	x			
16	74	Stols G	Does the use of technology make a difference in the geometric cognitive growth of pre-service	2012				x		x	x	x			
17	76	Pitta-Pantazi D	Number Teaching and Learning	2014				x		x	x	x			
18	77	Carroll B	An investigation into the teaching and learning of applications of mathematics in senior-cycle s	2011				x	x	x	x	x			Thesis
19	84	Mousoulides N, Sriraman B, Christou C	From problem solving to modelling	2007				x		x	x	x			
20	87	Gordon J, Rey O, Siewiorek A, Vivitsou M	KeyCoNet 2012 Literature Review: Key competence development in school education in Europ	2012				x		x	x	x			
21	88	Heck A, Boon P, van Velthoven W	Mathematica empowered applets for learning school algebra and calculus	2008				x		x	x	x			
22	94	Fitzallen N, Watson J	Developing a Sequence of Learning Experiences in Statistics	2014				x		x	x	x			Ch in book
23	95	Domazet M, Baranović B, Matić - Sociologija i prostor J	Mathematics Competence and International Mathematics Testing: Croatian Starting Point	2013				x		x	x	x			
24	96	Doorman M, Drijvers P, Gravemeijer K, Boon P	Tool use and the development of the function concept: from repeated calculations to functional	2012				x		x	x	x			
25	100	Heck A	Modelling in cross-disciplinary authentic student research projects	2010				x		x	x	x			
26	101	Ndlovu M, Mji A	Pedagogical implications of students' misconceptions about deductive geometric proof	2012				x		x	x	x			
27	102	Kim SH	Preferred contexts for mathematical literacy of Korean grade 8-10 learners	2006				x		x	x	x			M Thesis
28	103	Beatty R, Moss J	Connecting grade 4 students from diverse urban classrooms: virtual collaboration to solve gene	2006				x		x	x	x			
29	104	Yung W	Thoughts and practice of a Hong Kong teacher in mathematics alternative assessment via con	2011				x		x	x	x			Thesis
30	106	Tolboom J	The potential of a classroom network to support teacher feedback	2012				x		x	x	x			Thesis
31	107	Risnawati R, Annisa K, Elviza R	THE DEVELOPMENT OF REALISTIC MATHEMATICS EDUCATION-BASED BLOG AT LINE	2014				x		x	x	x			
32	111	Hansen A, Drews D, Dudgeon J, Lawton F, Surtees L	Children's Errors in Mathematics	2014				x		x	x	x			
33	114	Headlam C	An investigation into children's understanding of the order of operations	2013				x		x	x	x			NB 5 design principles of RME, Thesis
34	273	Zaranis, N	The use of ICT in kindergarten for teaching addition based on realistic mathematics ed	2015				x	x	x	x	x			
35	274	Zaranis, N	The use of Information and Communication Technologies in the first grade of primary s	2013				x	x	x	x	x			
36	276	Sakinah, M., Purwanto, S.E.	Increasing students' mathematical creative thinking abilities through realistic mathema	2013				x	x	x	x	x			
37	279	Panagouli, D., Priovolou, M.	Let's go to the movies! learning math through creativity and role playing	2012				x	x	x	x	x			
38	280	Stohlmann, M.	Youtube incorporated with mathematical modelling activities: Benefits, concerns, and f	2012				x	x	x	x	x			
39	287	Yoon, C., Dreyfus, T., Thomas, M.O.J.	How high is the tramping track? Mathematising and applying in a calculus model-elicit	2010				x	x	x	x	x			
40	302	Doorman, M., Drijvers, P., Dekker, T., van den Heuvel-Panhui	Problem solving as a challenge for mathematics education in The Netherlands	2007				x	x	x	x	x			
		Author search													
1	11	P Drijvers, M Doorman	The graphics calculator in mathematics education	1996				x	x	x	x	x			
2	17	L Trouche, P Drijvers, G Gueudet	Technology-driven developments and policy implications for mathematics education	2013				x	x	x	x	x			
3	39	A Bakker, M Doorman, P Drijvers	] Design research on how IT may support the development of symbols and meaning in	2003				x	x	x	x	x			
4	65	P Drijvers, P Boon, M Doorman	RME principles for designing online tasks					x	x	x	x	x			
5	76	M Doorman, P Boon, P Drijvers, S van Gisbergen	Tooluse supporting the learning and teaching of the Function concept					x	x	x	x	x			
6	125	A Fauzan, T Plomp, K Gravemeijer	The Development of an RME-based Geometry Course for Indonesian Primary Schools	2013				x	x	x	x	x			
7	158	YB Widjaja, A Heck	How a realistic mathematics education approach and microcomputer-based laboratory	2003				x	x	x	x	x			
8	237	Dolk, M., Hertog, J.D., Gravemeijer, K.	Using multimedia cases for educating the primary school mathematics teacher educat	2002				x	x	x	x	x			
9	240	Gravemeijer, K., Doorman, M.	Context problems in realistic mathematics education: A calculus course as an example	1999				x	x	x	x	x			
10	311	Gravemeijer, Koeno.	How Emergent Models May Foster the Constitution of Formal Mathematics.	1999				x	x	x	x	x			

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1	16	Andresen, Mette.	Introduction of a new construct: The conceptual tool "Flexibility.".	2007					x		x	x	x			
2	39	Bakker,A Doorman,M Drijvers, P	Design research on how IT may support the development of symbols and meaning in r	2003					x	x	x	x	x			
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4	43	Cahyono, Adi Nur.	Virtualmatriks: A Conceptual Mathematization Process in Virtual Learning Environment.	2012					x		x	x	x			
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7	95	Domazet M, Baranović B, Matić - Sociologija i prostor J	Mathematics Competence and International Mathematics Testing: Croatian Starting Point	2013					x		x	x	x			
8	54	Doorman LM, Drijvers PHM, Gravemeijer KPE	Design research in mathematics education: the case of an ict-rich learning arrangement for the	2013					x		x	x	x			NB
9	96	Doorman M, Drijvers P, Gravemeijer K, Boon P	Tool use and the development of the function concept: from repeated calculations to functional	2012					x		x	x	x			
10	302	Doorman, M., Drijvers, P., Dekker, T., van den Heuvel-Panhui	Problem solving as a challenge for mathematics education in The Netherlands	2007					x	x	x	x	x			
11	76	Doorman,M Boon,P Drijvers,P van Gisbergen, S	Tooluse supporting the learning and teaching of the Function concept						x	x	x	x	x			
12	70	Drijvers P	Digital technology in mathematics education: why it works (or doesn't)	2013					x		x	x	x			
13	69	Drijvers P, Boon P, Doorman M	Digital design: RME principles for designing online tasks						x		x	x	x			NB
14	11	Drijvers, P, Doorman, M	The graphics calculator in mathematics education	1996					x	x	x	x	x			
15	65	Drijvers,P Boon,P Doorman, M	RME principles for designing online tasks						x	x	x	x	x			
16	125	Fauzan,A Plomp,T Gravemeijer,K	The Development of an RME-based Geometry Course for Indonesian Primary Schools	2013					x	x	x	x	x			
17	94	Fitzallen N, Watson J	Developing a Sequence of Learning Experiences in Statistics	2014					x		x	x	x			Ch in book
18	87	Gordon J, Rey O, Siewiorek A, Vivitsou M	KeyCoNet 2012 Literature Review: Key competence development in school education in Europ	2012					x		x	x	x			
19	60	Gravemeijer K	Aiming for 21st century skills	2012					x		x	x	x			HLT
20	240	Gravemeijer, K., Doorman, M.	Context problems in realistic mathematics education: A calculus course as an example	1999					x	x	x	x	x			
21	311	Gravemeijer, Koeno.	How Emergent Models May Foster the Constitution of Formal Mathematics.	1999					x	x	x	x	x			
22	111	Hansen A, Drews D, Dudgeon J, Lawton F, Surtees L	Children's Errors in Mathematics	2014					x		x	x	x			
23	114	Headlam C	An investigation into children's understanding of the order of operations	2013					x		x	x	x			NB 5 design principles of RME, Thesis
24	100	Heck A	Modelling in cross-disciplinary authentic student research projects	2010					x		x	x	x			
25	62	Heck A, Boon P, Bokhove C	Applets for learning school algebra and calculus	2007					x		x	x	x			
26	88	Heck A, Boon P, van Velthoven W	Mathematica empowered applets for learning school algebra and calculus	2008					x		x	x	x			
27	63	Karagiannidis C, Politis P, Karasavvidis I	Research on e-Learning and ICT in Education						x		x	x	x			Ch in book
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33	64	Perez M	Scaffolding Teachers' Construction of a Learning Trajectory for Mathematics Supported by ICT	2012					x		x	x	x			
34	76	Pitta-Pantazi D	Number Teaching and Learning	2014					x		x	x	x			
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36	276	Sakinah, M., Purwanto, S.E.	Increasing students' mathematical creative thinking abilities through realistic mathema	2013					x	x	x	x	x			
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38	74	Stols G	Does the use of technology make a difference in the geometric cognitive growth of pre-service t	2012					x		x	x	x			
39	46	Tirosh D, Wood T	Tools and processes in mathematics teacher education	2008					x		x	x	x			RME Ch 12
40	106	Tolboom J	The potential of a classroom network to support teacher feedback	2012					x		x	x	x			Thesis
41	17	Trouche,L Drijvers, P Gueudet, G	Technology-driven developments and policy implications for mathematics education	2013					x	x	x	x	x			
42	9	van Loon-Hillen, Nelleke; van Gog, Tamara; Brand-Gruwel, Saskia.	Effects of worked examples in a primary school mathematics curriculum.	2012					x		x	x	x			
43	158	Widjaja,YB Heck,A	How a realistic mathematics education approach and microcomputer-based laboratory	2003					x	x	x	x	x			
44	287	Yoon, C., Dreyfus, T., Thomas, M.O.J.	How high is the tramping track? Mathematising and applying in a calculus model-elicit	2010					x	x	x	x	x			
45	104	Yung W	Thoughts and practice of a Hong Kong teacher in mathematics alternative assessment via con	2011					x		x	x	x			Thesis
46	57	Zaranis N	The use of Information and Communication Technologies in the first grade of primary school fo	2013					x		x	x	x			NB
47	65	Zaranis N	The use of ICT in the first grade of primary school for teaching circles, triangles, rectangles and	2014					x		x	x	x			
48	67	Zaranis N	Geometry Teaching Through ICT in Primary School	2014					x		x	x	x			
49	68	Zaranis N	The use of ICT in kindergarten for teaching addition based on realistic mathematics education	2014					x		x	x	x			
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9	76	Doorman,M Boon,P Drijvers,P van Gisbergen, S	Tooluse supporting the learning and teaching of the Function concept					x	x	x	x	x			
10	70	Drijvers P	Digital technology in mathematics education: why it works (or doesn't)	2013				x		x	x	x			
11	69	Drijvers P, Boon P, Doorman M	Digital design: RME principles for designing online tasks					x		x	x	x			NB
12	11	Drijvers, P, Doorman, M	The graphics calculator in mathematics education	1996				x	x	x	x	x			
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22	51	Kizito RN	Pretesting mathematical concepts with the mobile phone: Implications for curriculum design	2011				x		x	x	x			Distance
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26	74	Stols G	Does the use of technology make a difference in the geometric cognitive growth of pre-service	2012				x		x	x	x			
27	9	van Loon-Hillen, Nelleke; van Gog, Tamara; Brand-Gruwel, Saskia.	Effects of worked examples in a primary school mathematics curriculum.	2012				x		x	x	x			
28	158	Widjaja,YB Heck,A	How a realistic mathematics education approach and microcomputer-based laboratory	2003				x	x	x	x	x			
29	57	Zaranis N	The use of Information and Communication Technologies in the first grade of primary school fo	2013				x		x	x	x			NB
30	65	Zaranis N	The use of ICT in the first grade of primary school for teaching circles, triangles, rectangles and	2014				x		x	x	x			
31	67	Zaranis N	Geometry Teaching Through ICT in Primary School	2014				x		x	x	x			
32	68	Zaranis N	The use of ICT in kindergarten for teaching addition based on realistic mathematics education	2014				x		x	x	x			
33	49	Zaranis N, Kalogiannakis M, Papadakis S	Using mobile devices for teaching realistic mathematics in kindergarten education	2013				x		x	x	x			
34	HP	Boon, P	Designing Didactical Tools And Microworlds For Mathematics Educations	2006				x		x	x	x			
35	HP	Stephan, M Akyuz, D	An Instructional Design Collaborative in One Middle School	2013				x		x	x	x			
36	HP	Cobb, P Zhao, Q	Learning from and Adapting the theory of realistic mathematics education	2008				x		x	x	x			
37	HP	Drijvers, P Doorman M Boon, P	The teacher and the tool: instrumental orchestrations in the technology-rich mathematics class	2010				x		x	x	x			
38	HP	Wijers, M Jonker, V	MobileMath: exploring mathematics outside the classroom	2010				x		x	x	x			
39	HP	Zulkardi Webb	Realistic Mathematics Education Theory Meets Web Technology	2000				x		x	x	x			

HP = handpicked

Total	No	Author	Title	Year Published	Link	Relevance		RME	Realistic	ICT	matics/teacher/edu	thematics Educati	Reason for exclusion	Include	Comments
1	16	Andresen, Mette.	Introduction of a new construct: The conceptual tool "Flexibility.".	2007				x		x	x	x			
2	39	Bakker,A Doorman,M Drijvers, P	Design research on how IT may support the development of symbols and meaning in r	2003				x	x	x	x	x			
3	43	Cahyono, Adi Nur.	Virtualmatriks: A Conceptual Mathematization Process in Virtual Learning Environment.	2012				x		x	x	x			
4	237	Dolk, M., Hertog, J.D., Gravemeijer, K.	Using multimedia cases for educating the primary school mathematics teacher educat	2002				x	x	x	x	x			
5	95	Domazet M, Baranović B, Matić - Sociologija i prostor J	Mathematics Competence and International Mathematics Testing: Croatian Starting Point	2013				x		x	x	x			
6	54	Doorman LM, Drijvers PHM, Gravemeijer KPE	Design research in mathematics education: the case of an ict-rich learning arrangement for the	2013				x		x	x	x			NB
7	96	Doorman M, Drijvers P, Gravemeijer K, Boon P	Tool use and the development of the function concept: from repeated calculations to functional	2012				x		x	x	x			
8	302	Doorman, M., Drijvers, P., Dekker, T., van den Heuvel-Panhui	Problem solving as a challenge for mathematics education in The Netherlands	2007				x	x	x	x	x			
9	76	Doorman,M Boon,P Drijvers,P van Gisbergen, S	Tooluse supporting the learning and teaching of the Function concept					x	x	x	x	x			
10	70	Drijvers P	Digital technology in mathematics education: why it works (or doesn't)	2013				x		x	x	x			
11	69	Drijvers P, Boon P, Doorman M	Digital design: RME principles for designing online tasks					x		x	x	x			NB
12	11	Drijvers, P, Doorman, M	The graphics calculator in mathematics education	1996				x	x	x	x	x			
13	125	Fauzan,A Plomp,T Gravemeijer,K	The Development of an RME-based Geometry Course for Indonesian Primary Schools	2013				x	x	x	x	x			
14	94	Fitzallen N, Watson J	Developing a Sequence of Learning Experiences in Statistics	2014				x		x	x	x			Ch in book
15	87	Gordon J, Rey O, Siewiorek A, Vivitsou M	KeyCoNet 2012 Literature Review: Key competence development in school education in Europ	2012				x		x	x	x			
16	60	Gravemeijer K	Aiming for 21st century skills	2012				x		x	x	x			HLT
17	240	Gravemeijer, K., Doorman, M.	Context problems in realistic mathematics education: A calculus course as an example	1999				x	x	x	x	x			
18	311	Gravemeijer, Koeno.	How Emergent Models May Foster the Constitution of Formal Mathematics.	1999				x	x	x	x	x			
19	100	Heck A	Modelling in cross-disciplinary authentic student research projects	2010				x		x	x	x			
20	62	Heck A, Boon P, Bokhove C	Applets for learning school algebra and calculus	2007				x		x	x	x			
21	88	Heck A, Boon P, van Velthoven W	Mathematica empowered applets for learning school algebra and calculus	2008				x		x	x	x			
22	51	Kizito RN	Pretesting mathematical concepts with the mobile phone: Implications for curriculum design	2011				x		x	x	x			Distance
23	84	Mousoulides N, Sriraman B, Christou C	From problem solving to modelling	2007				x		x	x	x			
24	64	Perez M	Scaffolding Teachers' Construction of a Learning Trajectory for Mathematics Supported by ICT	2012				x		x	x	x			
25	107	Risnawati R, Annisa K, Elviza R	THE DEVELOPMENT OF REALISTIC MATHEMATICS EDUCATION-BASED BLOG AT LINE	2014				x		x	x	x			
26	74	Stols G	Does the use of technology make a difference in the geometric cognitive growth of pre-service	2012				x		x	x	x			
27	9	van Loon-Hillen, Nelleke; van Gog, Tamara; Brand-Gruwel, Saskia.	Effects of worked examples in a primary school mathematics curriculum.	2012				x		x	x	x			
28	158	Widjaja,YB Heck,A	How a realistic mathematics education approach and microcomputer-based laboratory	2003				x	x	x	x	x			
29	57	Zaranis N	The use of Information and Communication Technologies in the first grade of primary school fo	2013				x		x	x	x			NB
30	65	Zaranis N	The use of ICT in the first grade of primary school for teaching circles, triangles, rectangles and	2014				x		x	x	x			
31	67	Zaranis N	Geometry Teaching Through ICT in Primary School	2014				x		x	x	x			
32	68	Zaranis N	The use of ICT in kindergarten for teaching addition based on realistic mathematics education	2014				x		x	x	x			
33	49	Zaranis N, Kalogiannakis M, Papadakis S	Using mobile devices for teaching realistic mathematics in kindergarten education	2013				x		x	x	x			
34	HP	Boon, P	Designing Didactical Tools And Microworlds For Mathematics Educations	2006				x		x	x	x			
35	HP	Stephan, M Akyuz, D	An Instructional Design Collaborative in One Middle School	2013				x		x	x	x			
36	HP	Cobb, P Zhao, Q	Learning from and Adapting the theory of realistic mathematics education	2008				x		x	x	x			
37	HP	Drijvers, P Doorman M Boon, P	The teacher and the tool: instrumental orchestrations in the technology-rich mathematics class	2010				x		x	x	x			
38	HP	Wijers, M Jonker, V	MobileMath: exploring mathematics outside the classroom	2010				x		x	x	x			
39	HP	Zulkardi Webb	Realistic Mathematics Education Theory Meets Web Technology	2000				x		x	x	x			
40	Key	Freudenthal, Hans	Why to Teach Mathematics So as to Be Useful	1968											
41	Key	Simon, Martin	Reconstructing Mathematics pedagogy from a constructivist perspective	1995											
42	Key	Treffers, A	Wiskobas and Freudenthal Realistic Mathematics Education	1993											
43	Key	Freudenthal, Hans	Geometry between the Devil and the Deep Sea	1971											
44	Key	Freudenthal, Hans	Revisiting Mathematics Education: China Lectures	2010											
45	Key	Treffers, A	Three dimensions: A Model of goal and theory description in Mathematics Instruction - The Wis	1978											

HP = handpicked

Cohen's Kappa for the Systematic Literature Review

		Peer: Agree	Peer: Disagree
Researcher	Agree	123	1
	Disagree	1	0

$$p = \frac{(123 + 0)}{125}$$

= 0.984

$$p_e = \frac{\left(\frac{124}{125} \times \frac{124}{125}\right) + \left(\frac{1}{125} \times \frac{1}{125}\right)}{125}$$

$$= \frac{0{,}984064 + 0{,}000064}{125}$$

= 0,007873024

Cohen's Kappa $= \frac{p - p_e}{1 - p_e}$

$$= \frac{0{,}984 - 0{,}007873024}{1 - 0.007873024}$$

$$= \frac{0{,}976126976}{0{,}992126976}$$

= 0,98387

≈ 1

This calculation was confirmed at http://graphpad.com/quickcalcs/kappa1.cfm

QuickCalcs

1. Select category 2. Choose calculator 3. Enter data **4. View results**

	A	B	Total
A	123	1	**124**
B	1	0	**1**
Total	**124**	**1**	**125**

Number of observed agreements: 123 (98.40% of the observations)
Number of agreements expected by chance: 123.0 (98.41% of the observations)

Kappa= -0.008
SE of kappa = 0.006
95% confidence interval: From -0.019 to 0.003
The strength of agreement is worse than what you expect to see by chance alone.

Addendum 2.6

Questions for the individual interview

1. What sections of the curriculum are most problematic for you to teach?
2. How often are you able to relate the content you are teaching to real-life situations?
3. Which sections of work do you think will be easiest to link to real-life situations?
4. In which sections of the curriculum do you need help to link the content to real-life situations?
5. Do you think it is important to relate your teaching to real-life situations? Explain your answer.
6. How do you feel about using technology in your teaching?
7. How do you feel about using mobile technology such as WhatsApp™ and SMS to discuss with me and other colleagues, possible ways of making the mathematics content more realistic for you learners?
8. How do you feel about being part of a WhatsApp™ group to discuss issues of making the mathematics more realistic?
9. What experience do you have with Facebook?
10. How do feel about using Facebook to discuss issues of making mathematics more realistic for your learners?
11. What experience do you have regarding interactive whiteboards?
12. How do you feel about using the interactive whiteboard to discuss issues of making mathematics more realistic for your learners?
13. What type of technology would you be most comfortable with for this project?

1. Watter afdelings van die kurrikulum is vir u die mees problematies om te behandel?
2. Hoe dikwels is dit moontlik vir u om die inhoud wat u onderrig te verbind aan alledaagse lewensituasies?
3. Watter dele van die werk dink u is die maklikste om te verbind aan alledaagse lewensituasies?
4. Met watter dele van die kurrikulum het u hulp nodig om die inhoud te verbind met alledaagse lewensituasies?
5. Dink u dit is belangrik om u onderrig te verbind aan werklike lewensituasies? Motiveer jou antwoord.
6. Hoe voel u daaroor om tegnologie te gebruik gedurende u onderrig?
7. Hoe voel u daaroor om mobiele tegnologie soos Whatsapp en SMS'e te gebruik om met my, en u ander kollegas, die moontlike wyses waarop wiskunde-inhoud meer realisties vir u leerders gemaak kan word, te bespreek?

8. Hoe voel u daaroor om deel te wees van ‘n Whatsapp-groep wat die kwessies rondom wyses waarop wiskunde-inhoud meer realisties vir u leerders gemaak kan word, te bespreek?

9. Watter tipe ondervinding het u met Facebook?

10. Hoe voel u daaroor om Facebook te gebruik om die kwessies rondom wyses waarop wiskunde-inhoud meer realisties vir u leerders gemaak kan word, te bespreek?

11. Watter tipe ondervinding het u m.b.t. die interaktiewe witborde?

12. Hoe voel u daaroor om die interaktiewe witborde te gebruik om kwessies rondom wyses waarop wiskunde-inhoud meer realisties vir u leerders gemaak kan word, te bespreek

13. Watter tipe tegnolgie sal u die mees gemaklikste mee wees om te gebruik tydens hierdie projek?

Cohen's Kappa for Individual Interviews

		Peers: Agree	Peers: Disagree
Researcher	Agree	207	1
	Disagree	1	0

$$p = \frac{(207 + 0)}{209}$$

= 0.990430622

$$p_e = \frac{\left(\frac{208}{209} \times \frac{208}{209}\right) + \left(\frac{1}{209} \times \frac{1}{209}\right)}{209}$$

$$= \frac{0{,}990453515 + 0{,}000022893}{209}$$

= 0,004738902

Cohen's Kappa $= \frac{p - p_e}{1 - p_e}$

$$= \frac{0{,}990430622 - 0{,}004738902}{1 - 0.004738902}$$

$$= \frac{0{,}9856172}{0{,}995261098}$$

= 0,990

≈ 1

This calculation was confirmed at http://graphpad.com/quickcalcs/kappa1.cfm

QuickCalcs

1. Select category 2. Choose calculator 3. Enter data **4. View results**

	A	B	Total
A	207	1	**208**
B	1	0	**1**
Total	**208**	**1**	**209**

Number of observed agreements: 207 (99.04% of the observations)
Number of agreements expected by chance: 207.0 (99.05% of the observations)

Kappa= -0.005
SE of kappa = 0.003
95% confidence interval: From -0.011 to 0.002
The strength of agreement is worse than what you expect to see by chance alone.

Addendum 2.8

Focus Group Interview Schedule

1. Why is mobile technology suitable to present Mathematics content in a realistic way?

2. How does this app address the principles and characteristics of RME?

3. How can this app assist teachers to teach Mathematics in a meaningful way?

4. What knowledge of technology would a teacher need to effectively use the app?

5. Which role-players will influence the use of an app such as this in the school situation?

6. How did you experience using this app?

7. How would you use this app to improve your teaching practice?

8. Which sections of the curriculum would benefit most from an app such as this one?

9. What can be done to improve this app?

Addendum 5.1

App Design Document: Financial Maths

Content

Problem situation

Sipho has decided to to buy a bicycle so that he can attend soccer training in the afternoons at his school. Sipho visited the local stores and found a bicycle that he likes for R1299.

He has three options to consider. The store offers **hire-purchase** to buy the bicycle over a period of two years at 14% **p.a**. **simple interest**, (his dad will need to do the hire-purchase for him). He could also borrow the money from his dad who agreed that he can pay it back over 1 year at 13% p.a. simple interest. The third option is to borrow the money from his older brother Peter, who agreed to a 2 year period at 8% **compound interest**.

Questions:

1. Which is the best option for Sipho, considering he only earns R 150 pocket money a month? Motivate your answer.

2. Model an answer to show which of the three options is the most economical for Sipho (Which one does he pay the least interest)? Motivate.

Definitions: Click on the bold words in the problem to reveal their meanings

Interest	When money is loaned, interest is charged for being able to use the money. When you invest money with a bank, you will be paid interest.
p.a.	Is the abbreviation for per annum, it means per year.
Hire-purchase	Goods like appliances can be bought on a hire-purchase loan. For hire-purchase, they usually require a 10% or 20% deposit. The interest is calculated at simple interest on the full loan amount over the period that it is repaid.
Simple Interest	Simple interest is calculated on the original amount of a loan or investment.

Formula to calculate simple interest

Simple interest is calculated with the following formula:

$A = P(1 + i.n)$ where

A is the total or accumulated amount.

For investments, A is the amount invested plus the interest earned.

For a loan, A is the total amount you have to pay back, the amount you borrowed plus the interest

P is the amount borrowed or invested.

i is the simple interest rate per annum (per year) (p.a.)

Remember that if the interest rate is 10% p.a. then

i = 10%

$i = \frac{10}{100}$

$i = 0{,}1$

n is the number of years that the money has been invested or borrowed.

Compound Interest

With compound interest, the interest is added to the capital amount at the end of each period.

Compound interest is interest on interest.

Long-term loans such as home loans make use of compound interest.

Formula to calculate compound interest

Compound interest is calculated with the following formula:

$A = P(1 + i)^n$ where

A is the future value of the money invested or borrowed. It includes the principal amount and the interest earned.

P is the initial (principal) amount invested or borrowed

i is the interest rate per annum, compounded annually

n is the number of years for which the money has been invested or borrowed.

Hints to answer Question 1:

Do you know how what to do for the hire-purchase option?

Hint 1a: **Calculate the simple interest** for hire-purchase option

Need help with the formula?

Yes	**No**
$A = P(1 + i.n)$	If the formula is correct, go on to the next question. If something is incorrect, correct that as follows: If P is not correct, give the hint: P is the amount borrowed or invested If i is not correct, give the hint: i is the simple interest rate p.a. If n is not correct, give the following hint: n is the number of years that the money has been invested or borrowed.

Fill in the values you need for the formula

P = ☐	If incorrect then say: P = 1299 this is the amount you need to borrow
$i = \square\% = \square$	If incorrect, then say $i = 14\% = 0{,}14$ the interest rate is 14% per annum (per year)
n = □	If incorrect, they say n = 2 because you will pay back the money over a period of 2 years

Now fill these values into the formula

$$A = P(1 + i.n)$$

= □[□ + (□)(□)]

If the correct value (see below) is entered in the block ✓

= 1299[1 + (0,14)(2)]

If the incorrect value is entered x . If the answer is incorrect three times, then give the correct answer as follows: P = 1299 this is the amount you need to borrow

$i = 14\% = 0{,}14$ the interest rate is 14% per annum (per year)

n = 2 because you will pay back the money over a period of 2 years

= □[1 +□]

If the correct value (see below) is entered in the block ✓

= 1299[1 + 0,28]

If the incorrect value is entered x . If the answer is incorrect three times, then give the correct answer as follows: P = 1299 this is the amount you need to borrow

0,14 x 2 = 0,28

= □[□]

If the correct value (see below) is entered in the block ✓

= 1299[1,28]

If the incorrect value is entered x . If the answer is incorrect three times, then give the correct answer as follows: P = 1299 this is the amount you need to borrow

1 + 0,28 = 1,28

= R □

If the correct value (see below) is entered in the block ✓

= R 1662,72

If the incorrect value is entered x . If the answer is incorrect three times, then give the correct answer as follows: 1299 x 1,28 = 1662,72

Do you know how what to do for the loan from dad option?

Hint 2a: **Calculate the simple interest** for the loan from dad option

Need help with the formula?

Yes	**No**
$A = P(1 + i.n)$	If the formula is correct, go on to the next question. If something is incorrect, correct that as follows: If P is not correct, give the hint: P is the amount borrowed or invested If i is not correct, give the hint: i is the simple interest rate p.a. If n is not correct, give the following hint: n is the number of years that the money has been invested or borrowed.

What about the values to substitute?

P = □ If incorrect then say: P = 1299 this is the amount you need to borrow

$i = \square\% = \square$ If incorrect, then say $i = 13\% = 0{,}13$ the interest rate is 13% per annum (per year)

n = □ If incorrect, they say n = 1 because you will pay back the money over a period of 1 year

Now do the sum

$A = P(1 + i.n)$

= □[□ + (□)(□)]

If the correct value (see below) is entered in the block ✓

= 1299[1 + (0,13)(1)]

If the incorrect value is entered x . If the answer is incorrect three times, then give the correct answer as follows: P = 1299 this is the amount you need to borrow

$i = 13\% = 0{,}13$ the interest rate is 13% per annum (per year)

n = 1 because you will pay back the money over a period of 1 year

= □[1 +□]

If the correct value (see below) is entered in the block ✓

= 1299[1 + 0,13]

If the incorrect value is entered x . If the answer is incorrect three times, then give the correct answer as follows: P = 1299 this is the amount you need to borrow

0,13 x 1 = 0,13

= □[□]

If the correct value (see below) is entered in the block ✓

= 1299[1,13]

If the incorrect value is entered x . If the answer is incorrect three times, then give the correct answer as follows: P = 1299 this is the amount you need to borrow

1 + 0,13 = 1,13

= R □

If the correct value (see below) is entered in the block ✓

= R 1467,87

If the incorrect value is entered x . If the answer is incorrect three times, then give the correct answer as follows: 1299 x 1,13 = 1467,87

Did you know that compound interest means that the interest is added to the capital amount at the end of each year? Keep this in mind and work out the compound interest using the simple interest formula.

Hint 3a: Loan at the end of the first year:

$A = P(1 + i.n)$

= □[1 + (□)(□)]

If the correct value (see below) is entered in the block ✓

= 1299[1 + (0,08)(1)]

If the incorrect value is entered x . If the answer is incorrect three times, then give the correct answer as follows: P = 1299 this is the amount you need to borrow

$i = 8\% = 0{,}08$ the interest rate is 8% per annum (per year)

n = 1 Because you are working out the simple interest one year at a time

= □[1 + □]

If the correct value (see below) is entered in the block ✓

= 1299[1 + 0,08]

If the incorrect value is entered x . If the answer is incorrect three times, then give the correct answer as follows: P = 1299 this is the amount you need to borrow

0,08 x 1 = 0,08

= □[□]

If the correct value (see below) is entered in the block ✓

= 1299[1,08]

If the incorrect value is entered x . If the answer is incorrect three times, then give the correct answer as follows: P = 1299 this is the amount you need to borrow

1 + 0,08 = 1,08

= R □

If the correct value (see below) is entered in the block ✓

= R 1402,92

If the incorrect value is entered x . If the answer is incorrect three times, then give the correct answer as follows: 1299 x 1,08 = 1402,92

What about the second year?

$A = P(1 + i.n)$

= □[1 + (□)(□)]

If the correct value (see below) is entered in the block ✓

= 1402,92[1 + (0,08)(1)]

If the incorrect value is entered x . If the answer is incorrect three times, then give the correct answer as follows: P = 1402,92 the interest from year one was added to the amount you borrowed.

$i = 8\% = 0{,}08$ the interest rate is 8% per annum (per year)

n = 1 Because you are working out the simple interest one year at a time

= □[1 + □]

If the correct value (see below) is entered in the block ✓

= 1402,92[1 + 0,08]

If the incorrect value is entered x . If the answer is incorrect three times, then give the correct answer as follows: P = 1402,92 the interest from year one was added to the amount you borrowed.

0,08 x 1 = 0,08

= □[□]

If the correct value (see below) is entered in the block ✓

= 1402,92[1,08]

If the incorrect value is entered x . If the answer is incorrect three times, then give the correct answer as follows: P = 1402,92 the interest from year one was added to the amount you borrowed.

1 + 0,08 = 1,08

= R □

If the correct value (see below) is entered in the block ✓

= R 1515,15

If the incorrect value is entered x . If the answer is incorrect three times, then give the correct answer as follows: 1402,92 x 1,08 = 1515,15

Now do the calculation for the loan from Peter. Do you need help?

Hint 4a: **Calculate the compound interest** for loan option from older brother Peter

Need help with the formula?

Yes	**No**
$A = P(1 + i)^n$	If the formula is correct, go on to the next question. If something is incorrect, correct that as follows: If A is not correct, give the hint: A is the future value of the money invested or borrowed. It includes the principal amount and the interest earned. If P is not correct, give the hint:

	P is the initial (principal) amount invested or borrowed If i is not correct, give the hint: i is the interest rate per annum, compounded annually If n is not correct, give the following hint n is the number of years for which the money has been invested or borrowed.

What about the values to substitute?

P = ☐ If incorrect then say: P = 1299 this is the amount you need to borrow

$i = \square\% = \square$ If incorrect, then say $i = 8\% = 0{,}08$ the interest rate is 8% compounded annually

n = □ If incorrect, they say n = 2 because you will pay back the money over a period of 2 years

Now check if your answer is correct

$A = P(1 + i)^n$

$A = \square(1 + \square)^{\square}$

If the correct value (see below) is entered in the block ✓

$= 1299(1 + 0{,}08)^2$

If the incorrect value is entered x . If the answer is incorrect three times, then give the correct answer as follows: P = 1299 this is the amount you need to borrow

$i = 8\% = 0{,}08$ the interest rate is 8% per annum (per year)

n = 2 because you will pay back the money over a period of 2 years

= □(□)$^{□}$

If the correct value (see below) is entered in the block ✓

= 1299$(1{,}08)^2$

If the incorrect value is entered x . If the answer is incorrect three times, then give the correct answer as follows: P = 1299 is the amount you need to borrow.

1 + 0,08 = 1,08

= □(□)

If the correct value (see below) is entered in the block ✓

= 1299(1,1664)

If the incorrect value is entered x . If the answer is incorrect three times, then give the correct answer as follows: P = 1299 is the amount you need to borrow.

$(1{,}08)^2$ = 1,1664

= R □

If the correct value (see below) is entered in the block ✓

= R 1515,15

If the incorrect value is entered x . If the answer is incorrect three times, then give the correct answer as follows:

1299 x 1,1664 = 1515,15

Now compare the two answers you got for this option (using the simple interest formula and using the compound interest formula. What do you notice?

To answer Question 1: Payment per month:

Not sure how to work out which option is best?

Hint 5a: Calculate the monthly payment for each option

I don't know what a monthly payment is ....

Hint 5b: A monthly payment is how much money needs to be paid back per month to the person or place that it was borrowed from.

How do I calculate a monthly payment?

Hint 5c: Take the final amount from your calculation and divide it by the total number of months that the loan was taken out for.

Does your model have this information for the hire-purchase option?

Hire-purchase option

Payment per month = □

If the correct value (see below) is entered in the block ✓

= R 69,28 per month

If the incorrect value is entered x . If the answer is incorrect three times, then give the correct answer as follows:

Payment per month = R 1662,72 ÷ 24

= R 69,28 per month

Does your model have this information for the loan from dad option?

Loan from dad option

Payment per month = □

If the correct value (see below) is entered in the block ✓

= R 122,32 per month

If the incorrect value is entered x . If the answer is incorrect three times, then give the correct answer as follows:

Payment per month = R 1467,87 ÷ 12

= R 122,32 per month

Does your model have the following information for the loan from Peter option?

Loan from Peter

Payment per month = □

If the correct value (see below) is entered in the block ✓

= R 63,13 per month

If the incorrect value is entered x . If the answer is incorrect three times, then give the correct answer as follows:

Payment per month = R1515,15 ÷ 24

= R 63,13 per month

Which is the cheapest option in term of monthly payments? □

If answer is Peter ✓

If any other answer x . If the answer is incorrect three times, then give the correct answer as Peter.

To answer Question 2: The most economical option (the least interest paid):

Hire-purchase option: Interest = □

If answer is R353,72 ✓

If answer is incorrect then x

Interest = Final amount – Initial amount

Interest = R 1662,72 – R 1299

= R 353,72

Loan from dad option: Interest = □

If answer is R 168,87 ✓

If answer is incorrect then x

Interest = Final amount – Initial amount

Interest = R 1467,87 – R 1299

= R 168,87

Loan from Peter: Interest = □

If answer is R 216,15 ✓

If answer is incorrect then x

Interest = Final amount – Initial amount

Interest = R 1515, 15 – R 1299

= R 216, 15

Most economical option is □

If answer is loan from dad ✓

If answer is incorrect three times, then give the correct answer as loan from dad

Activity



TV for sale

Cash Price R 5 999

OR

Hire-purchase

Deposit R599 and only R 243 per month

(30 instalments)

1. Sarah needs to decide which option is the most economical for her. Create a model to give her advice and help her to decide.

2. What is the interest rate that they are charging for the hire-purchase?

Question1:

Do you need help to start? yes □ no □

If they say yes, then say: Create a model to work out how much money Sarah pays back over the 30 instalments, remember to include her deposit.

After that say:

Hire-purchase final amount = □

If they say no then: Hire-purchase final amount = □

If they say R 7889 then say – well done, that is correct

If they have a different answer, then say: Your answer is incorrect. Check that your model includes the following:

Final amount = Deposit + (monthly instalment x 30)

= R 599 + (R 243 x 30)

= 599 + 7290

= R 7889

What is your advice to Sarah? Choose the most economical option for her.

Cash Price of R 5999 □ OR Hire-purchase with deposit R 599 and R 243 monthly □

If they choose Cash Price then say – Well done, that is the correct choice

If they choose Hire-purchase then say: No that is not the correct choice. She will be paying back R 7889 on hire-purchase and only R 5 999 for the cash price. Discuss with a colleague why the first option is the best for Sarah.

Question 2

Do you need help to work out what the interest rate is?

yes □ no □

If yes, then say: remember that Hire-purchase means Simple Interest and then say: Which formula do I use?

□ = □(□ + □.□)

If no then say: Which formula do I use?

□ = □(□ + □. □)

The answer should be: $A = P(1 + i.n)$

If the formula is correct, go on to the next question. If something is incorrect, correct that as follows:

If P is not correct, give the hint:

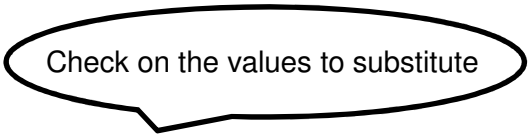
P is the amount borrowed or invested

If i is not correct, give the hint:

i is the simple interest rate p.a.

If n is not correct, give the following hint:

n is the number of years that the money has been invested or borrowed.



P = □

A = □ x □

= □

n = □ years

The answers should be:

P = 5400

If P is incorrect three times, then say: Do you know why it is R 5400 and not R 5999? Because there was a deposit of R 599

$A = 243 \times 30$

= 7290

If anything above is incorrect three times, then display the correct answer

$$n = 2\frac{1}{2}$$

If n is incorrect three times, then say: What happened to 30 months? It was converted to years.

Now check your answer

$$A = P(1 + i.n)$$

$$= \square[\square + (\square)(\square)]$$

If the correct values (see below) are entered in the block then say ✓

$$7290 = 5400[1 + (i)(2\frac{1}{2})]$$

If the incorrect values are entered then say x . If the answer is incorrect three times, then give the correct answer as follows:

$$7290 = 5400[1 + 2{,}5i]$$

$$7290 = 5400 + 13500i$$

$$7290 - 5400 = 13500i$$

$$1890 = 13500i$$

$$i = \frac{1890}{13500}$$

$$i = 0{,}14 = 14\%$$

What is the interest rate that they are charging for the hire-purchase?

$$i = \square\%$$

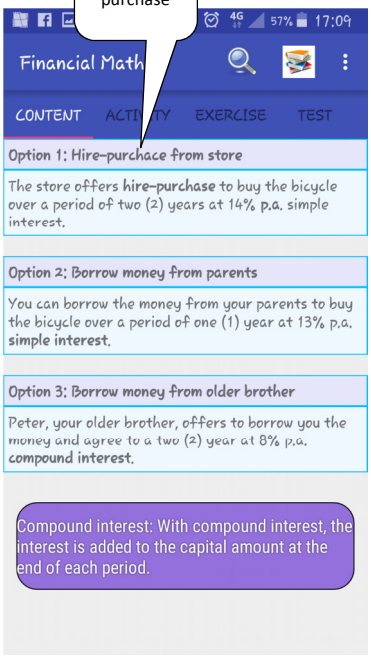
The answer should be 14%. If correct then say: Well done, that is correct.

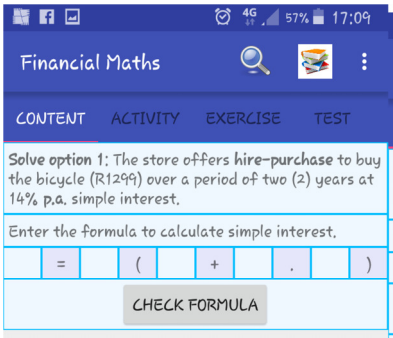
If the answer is incorrect three times, then display the correct answer.

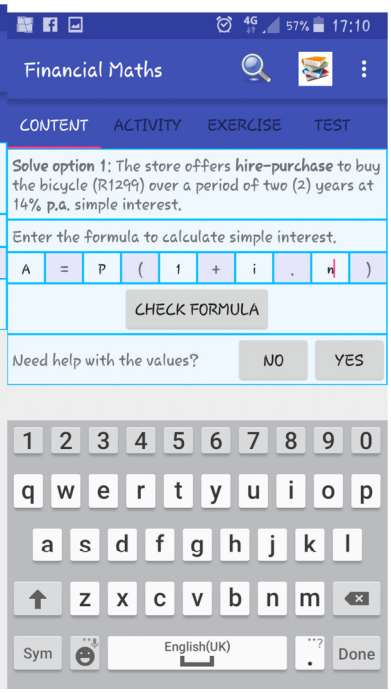
Changes on 10/03/16

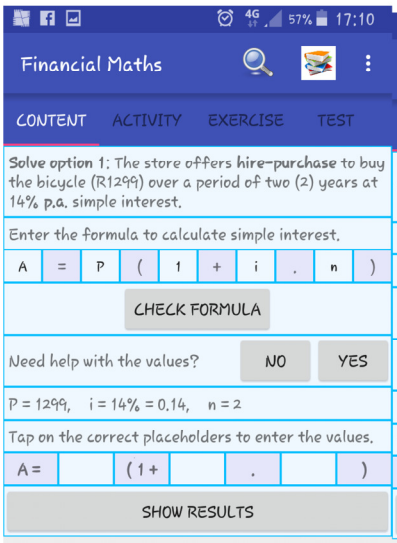


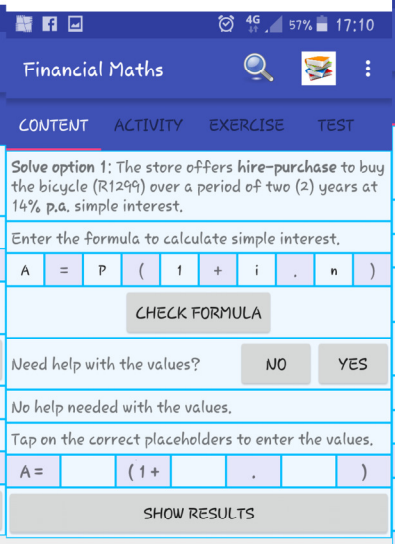
Please replace purchace with purchase

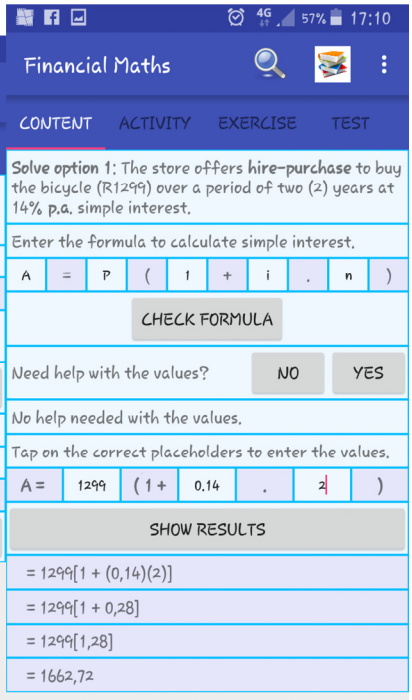


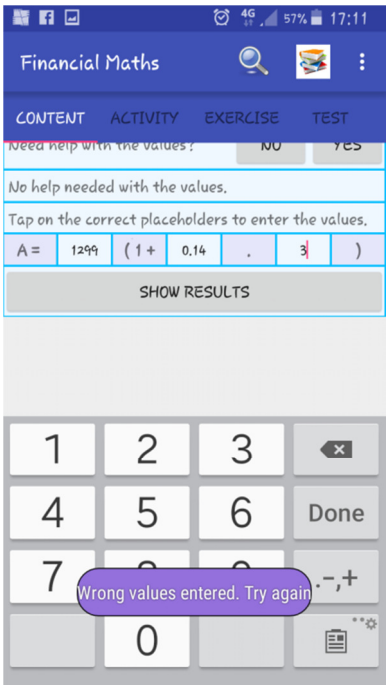


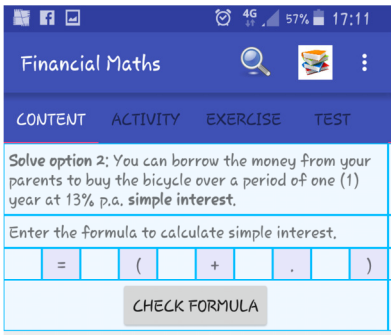


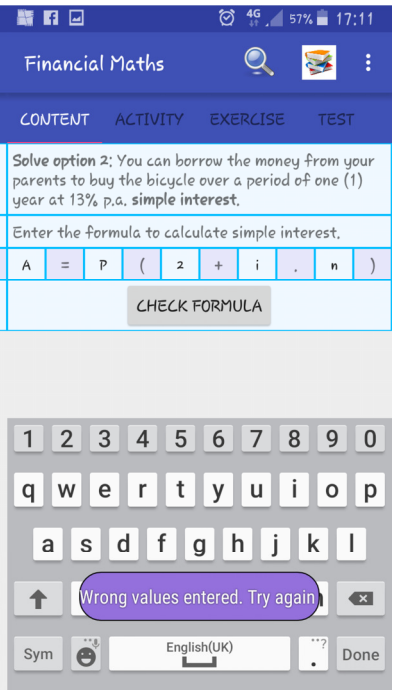


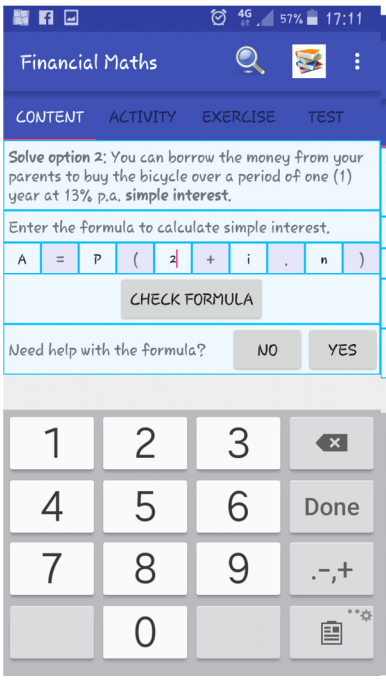


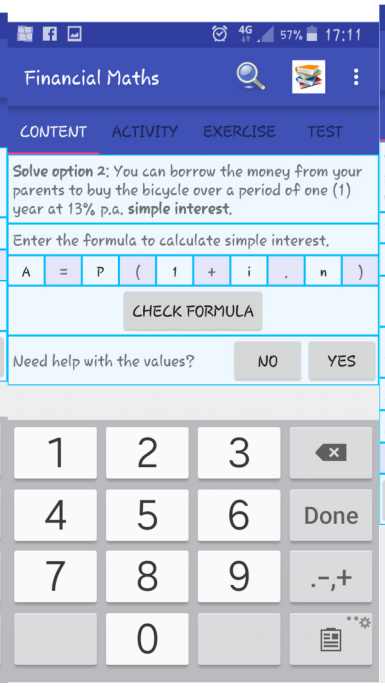


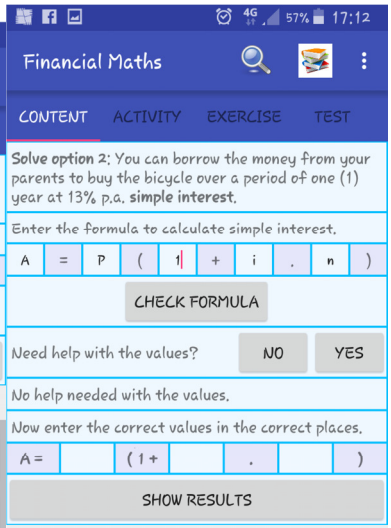

17:11
Financial Maths
CONTENT
ACTIVITY
EXERCISE
TEST
No help needed with the values.
Tap on the correct placeholders to enter the values.
A =
1299
(1 +
0.14
.
3
)
SHOW RESULTS
1
2
3
4
5
6
Done
7
.-,+
Wrong values entered. Try again
0

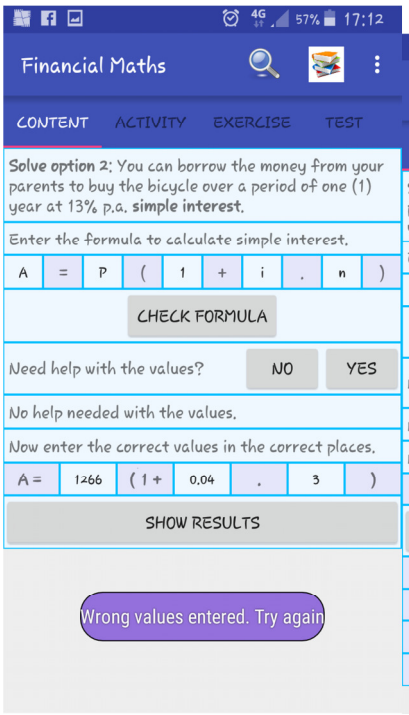

17:11
Financial Maths
CONTENT
ACTIVITY
EXERCISE
TEST
Solve option 2: You can borrow the money from your parents to buy the bicycle over a period of one (1) year at 13% p.a. simple interest.
Enter the formula to calculate simple interest.
=
(
+
.
)
CHECK FORMULA

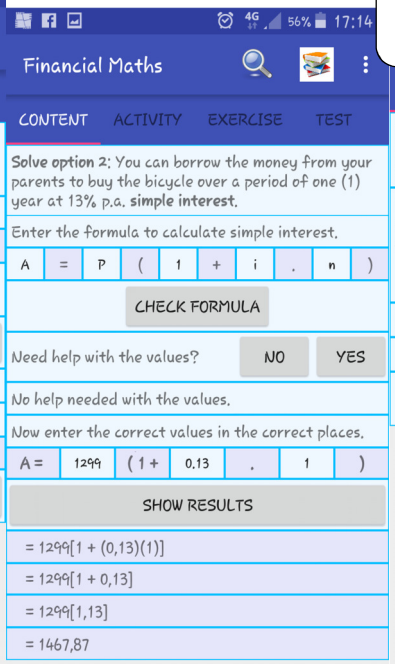

17:11
Financial Maths
CONTENT
ACTIVITY
EXERCISE
TEST
Solve option 2: You can borrow the money from your parents to buy the bicycle over a period of one (1) year at 13% p.a. simple interest.
Enter the formula to calculate simple interest.
A
=
P
(
2
+
i
.
n
)
CHECK FORMULA
1 2 3 4 5 6 7 8 9 0
q w e r t y u i o p
a s d f g h j k l
Wrong values entered. Try again
Sym
English(UK)
Done

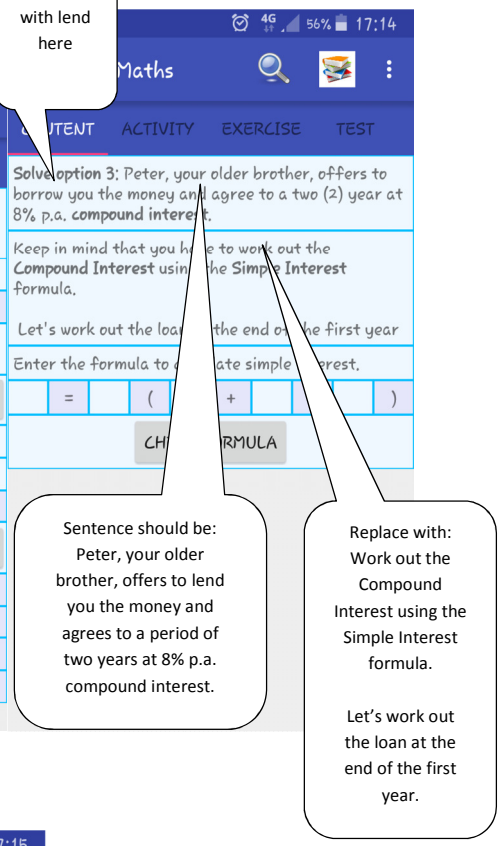

17:11
Financial Maths
CONTENT
ACTIVITY
EXERCISE
TEST
Solve option 2: You can borrow the money from your parents to buy the bicycle over a period of one (1) year at 13% p.a. simple interest.
Enter the formula to calculate simple interest.
A
=
P
(
2
+
i
.
n
)
CHECK FORMULA
Need help with the formula?
NO
YES
1
2
3
4
5
6
Done
7
8
9
.-,+
0

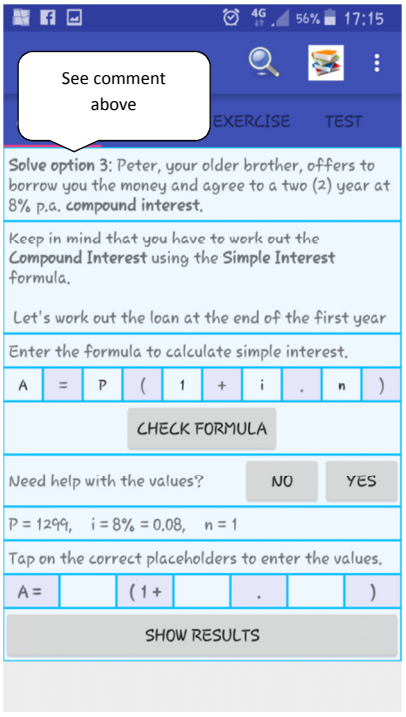

17:11
Financial Maths
CONTENT
ACTIVITY
EXERCISE
TEST
Solve option 2: You can borrow the money from your parents to buy the bicycle over a period of one (1) year at 13% p.a. simple interest.
Enter the formula to calculate simple interest.
A
=
P
(
1
+
i
.
n
)
CHECK FORMULA
Need help with the values?
NO
YES
1
2
3
4
5
6
Done
7
8
9
.-,+
0

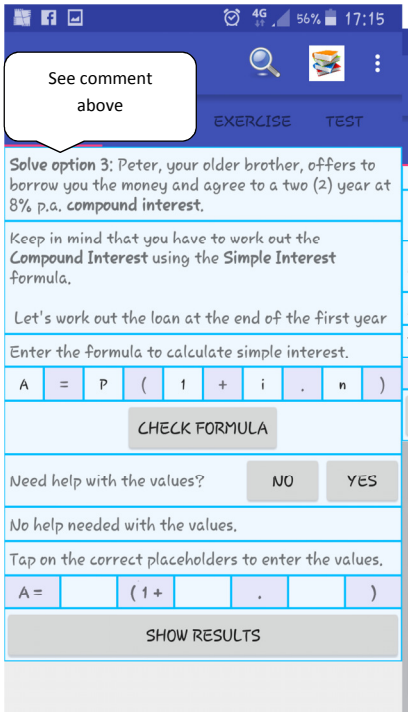

17:12
Financial Maths
CONTENT
ACTIVITY
EXERCISE
TEST
Solve option 2: You can borrow the money from your parents to buy the bicycle over a period of one (1) year at 13% p.a. simple interest.
Enter the formula to calculate simple interest.
A
=
P
(
1
+
i
.
n
)
CHECK FORMULA
Need help with the values?
NO
YES
No help needed with the values.
Now enter the correct values in the correct places.
A =
(1 +
.
)
SHOW RESULTS

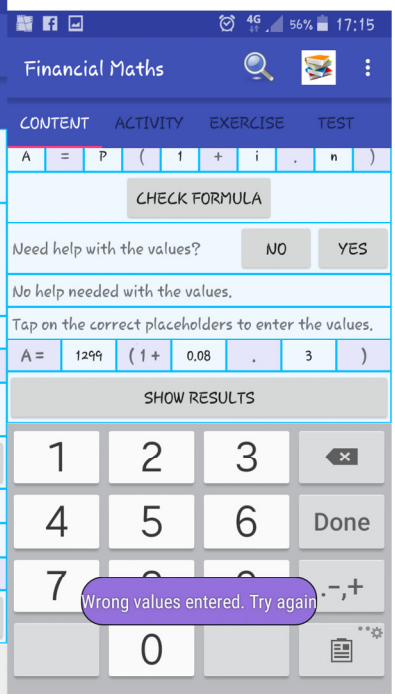

Financial Maths
CONTENT ACTIVITY EXERCISE TEST
Solve option 2: You can borrow the money from your parents to buy the bicycle over a period of one (1) year at 13% p.a. simple interest.
Enter the formula to calculate simple interest.
A = P (1 + i . n)
CHECK FORMULA
Need help with the values? NO YES
No help needed with the values.
Now enter the correct values in the correct places.
A = 1266 (1 + 0.04 . 3)
SHOW RESULTS
Wrong values entered. Try again

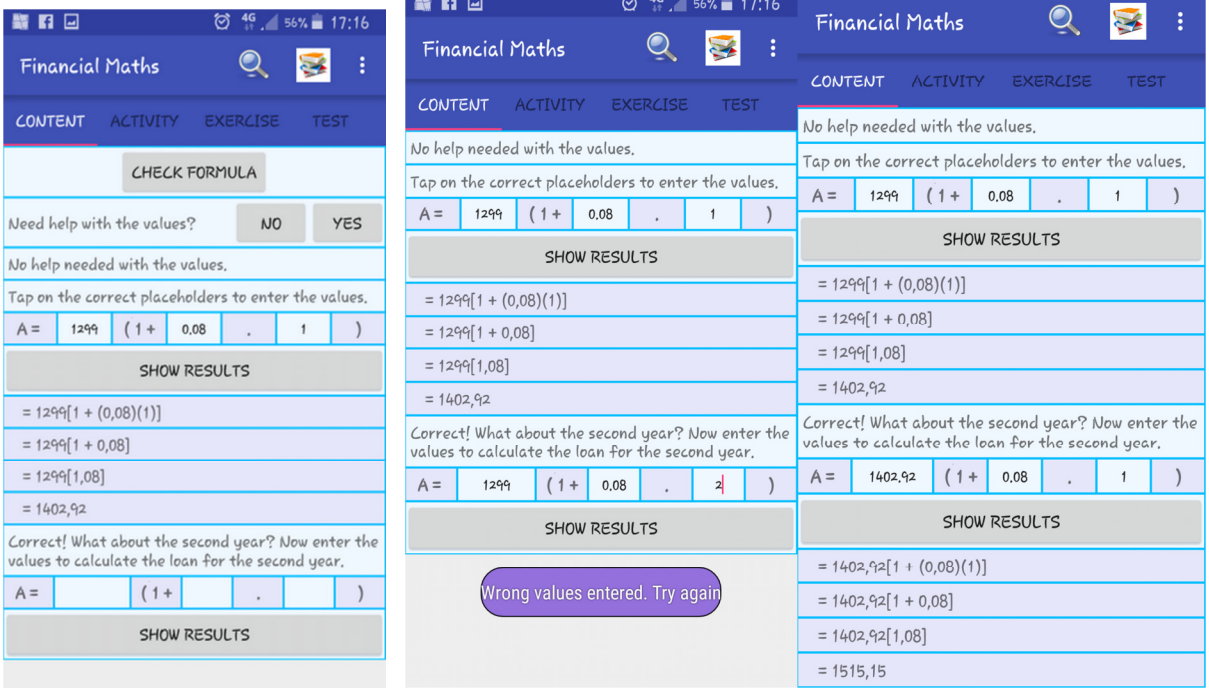

Financial Maths
CONTENT ACTIVITY EXERCISE TEST
Solve option 2: You can borrow the money from your parents to buy the bicycle over a period of one (1) year at 13% p.a. simple interest.
Enter the formula to calculate simple interest.
A = P (1 + i . n)
CHECK FORMULA
Need help with the values? NO YES
No help needed with the values.
Now enter the correct values in the correct places.
A = 1299 (1 + 0.13 . 1)
SHOW RESULTS
= 1299[1 + (0,13)(1)]
= 1299[1 + 0,13]
= 1299[1,13]
= 1467,87

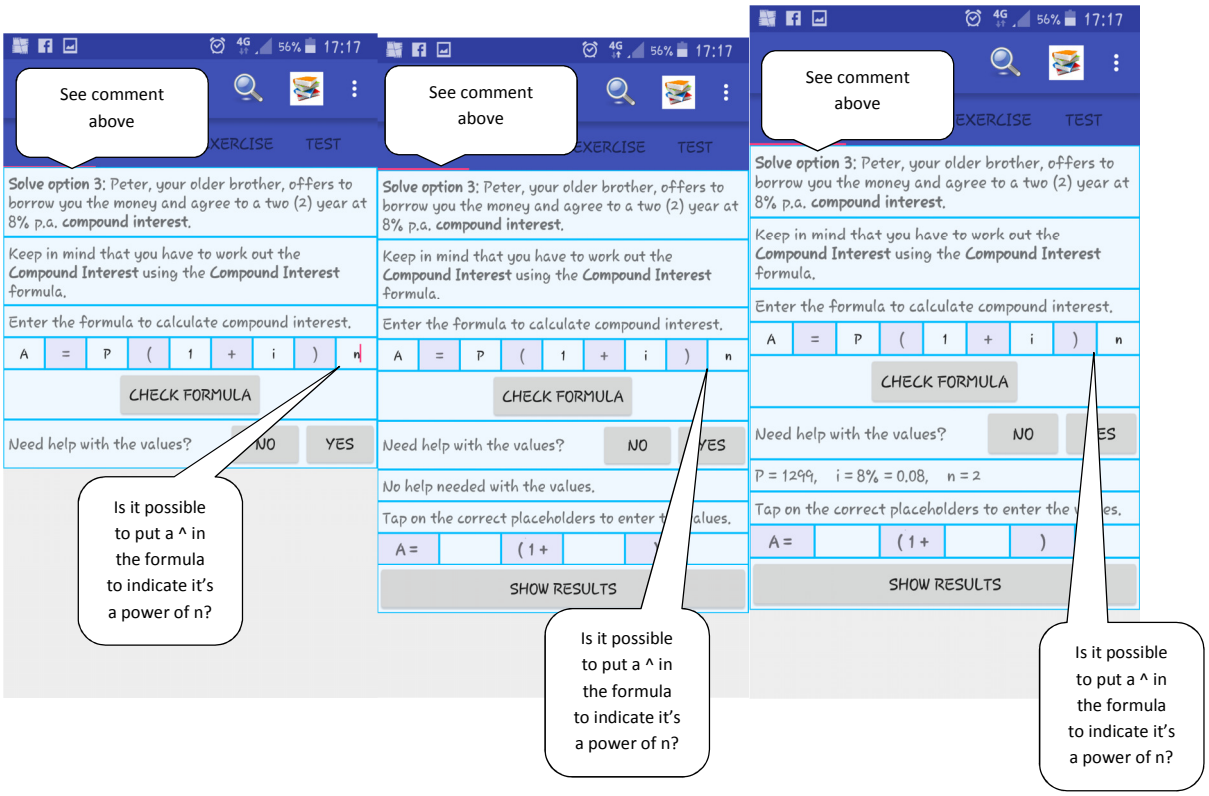

Please replace borrow with lend here
CONTENT ACTIVITY EXERCISE TEST
Solve option 3: Peter, your older brother, offers to borrow you the money and agree to a two (2) year at 8% p.a. compound interest.
Keep in mind that you have to work out the Compound Interest using the Simple Interest formula.
Let's work out the loan at the end of the first year
Enter the formula to calculate simple interest.
Sentence should be: Peter, your older brother, offers to lend you the money and agrees to a period of two years at 8% p.a. compound interest.
Replace with: Work out the Compound Interest using the Simple Interest formula.
Let's work out the loan at the end of the first year.

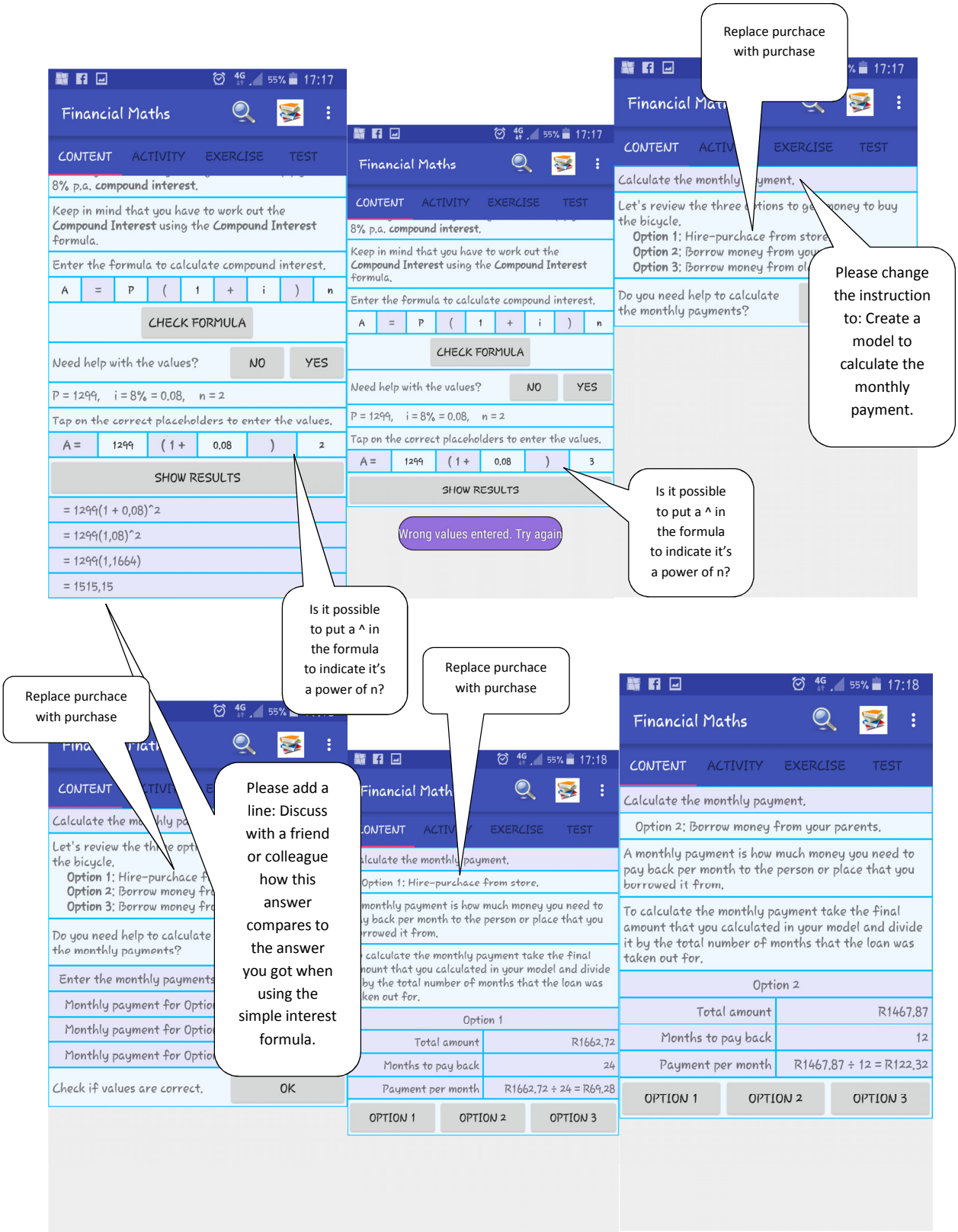

See comment above
Solve option 3: Peter, your older brother, offers to borrow you the money and agree to a two (2) year at 8% p.a. compound interest.
Keep in mind that you have to work out the Compound Interest using the Simple Interest formula.
Let's work out the loan at the end of the first year
Enter the formula to calculate simple interest.
A = P (1 + i . n)
CHECK FORMULA
Need help with the values? NO YES
P = 1299, i = 8% = 0.08, n = 1
Tap on the correct placeholders to enter the values.
A = (1 + .)
SHOW RESULTS

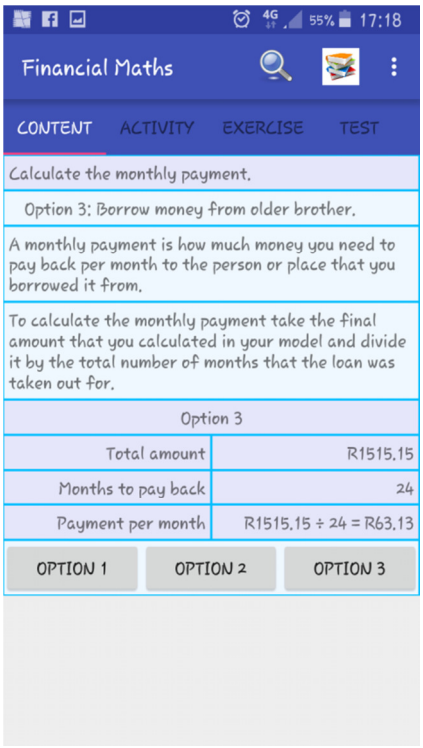

See comment above
Solve option 3: Peter, your older brother, offers to borrow you the money and agree to a two (2) year at 8% p.a. compound interest.
Keep in mind that you have to work out the Compound Interest using the Simple Interest formula.
Let's work out the loan at the end of the first year
Enter the formula to calculate simple interest.
A = P (1 + i . n)
CHECK FORMULA
Need help with the values? NO YES
No help needed with the values.
Tap on the correct placeholders to enter the values.
A = (1 + .)
SHOW RESULTS

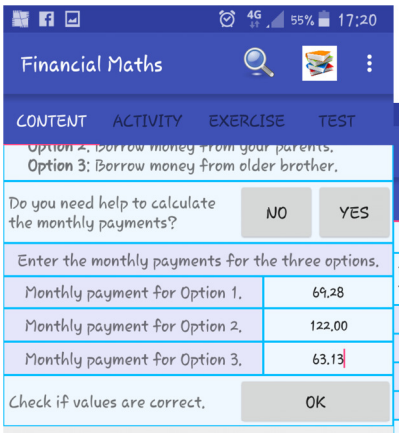

Financial Maths
CONTENT ACTIVITY EXERCISE TEST
A = P (1 + i . n)
CHECK FORMULA
Need help with the values? NO YES
No help needed with the values.
Tap on the correct placeholders to enter the values.
A = 1299 (1 + 0.08 . 3)
SHOW RESULTS
1 2 3
4 5 6 Done
7 .-,+
0
Wrong values entered. Try again

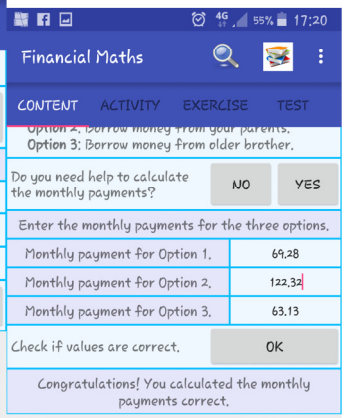

Financial Maths
CONTENT ACTIVITY EXERCISE TEST
CHECK FORMULA
Need help with the values? NO YES
No help needed with the values.
Tap on the correct placeholders to enter the values.
A = 1299 (1 + 0.08 . 1)
SHOW RESULTS
= 1299[1 + (0,08)(1)]
= 1299[1 + 0,08]
= 1299[1,08]
= 1402,92
Correct! What about the second year? Now enter the values to calculate the loan for the second year.
A = 1299 (1 + 0.08 . 2)
Wrong values entered. Try again
A = 1402,92 (1 + 0.08 . 1)
= 1402,92[1 + (0,08)(1)]
= 1402,92[1 + 0,08]
= 1402,92[1,08]
= 1515,15

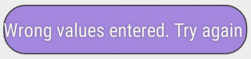

See comment above
See comment above
See comment above
Solve option 3: Peter, your older brother, offers to borrow you the money and agree to a two (2) year at 8% p.a. compound interest.
Keep in mind that you have to work out the Compound Interest using the Compound Interest formula.
Enter the formula to calculate compound interest.
A = P (1 + i) n
CHECK FORMULA
Need help with the values? NO YES
P = 1299, i = 8% = 0.08, n = 2
Is it possible to put a ^ in the formula to indicate it's a power of n?
Is it possible to put a ^ in the formula to indicate it's a power of n?
Is it possible to put a ^ in the formula to indicate it's a power of n?

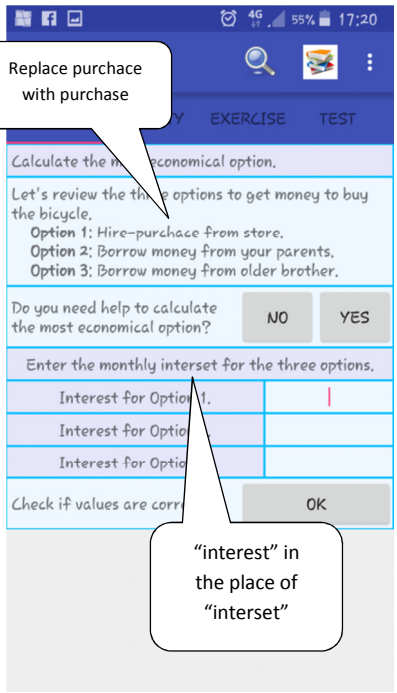

Financial Maths
CONTENT ACTIVITY EXERCISE TEST
8% p.a. compound interest.
Keep in mind that you have to work out the Compound Interest using the Compound Interest formula.
Enter the formula to calculate compound interest.
A = P (1 + i) n
CHECK FORMULA
Need help with the values? NO YES
P = 1299, i = 8% = 0.08, n = 2
Tap on the correct placeholders to enter the values.
A = 1299 (1 + 0.08) 2
SHOW RESULTS
= 1299(1 + 0,08)^2
= 1299(1,08)^2
= 1299(1,1664)
= 1515,15
Wrong values entered. Try again
Calculate the monthly payment.
Let's review the three options to get money to buy the bicycle.
Option 1: Hire-purchase from store
Option 2: Borrow money from your parents
Option 3: Borrow money from old
Do you need help to calculate the monthly payments?
Option 1: Hire-purchase from store.
A monthly payment is how much money you need to pay back per month to the person or place that you borrowed it from.
To calculate the monthly payment take the final amount that you calculated in your model and divide it by the total number of months that the loan was taken out for.
Option 1
Total amount R1662.72
Months to pay back 24
Payment per month R1662.72 ÷ 24 = R69.28
Option 2
Total amount R1467.87
Months to pay back 12
Payment per month R1467.87 ÷ 12 = R122.32
OPTION 1 OPTION 2 OPTION 3
Enter the monthly payments
Monthly payment for Option
Check if values are correct. OK
Replace purchace with purchase
Please change the instruction to: Create a model to calculate the monthly payment.
Is it possible to put a ^ in the formula to indicate it's a power of n?
Please add a line: Discuss with a friend or colleague how this answer compares to the answer you got when using the simple interest formula.





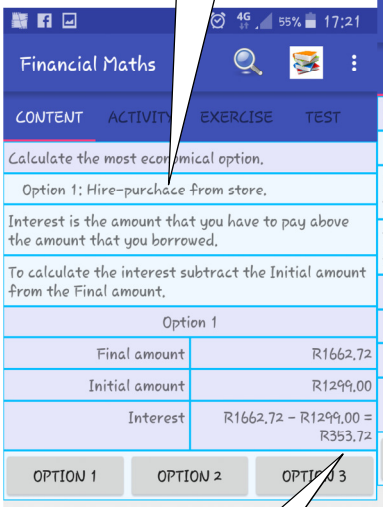






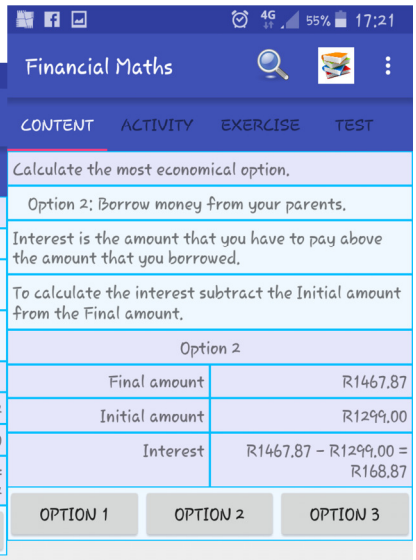
Replace purchace with purchase

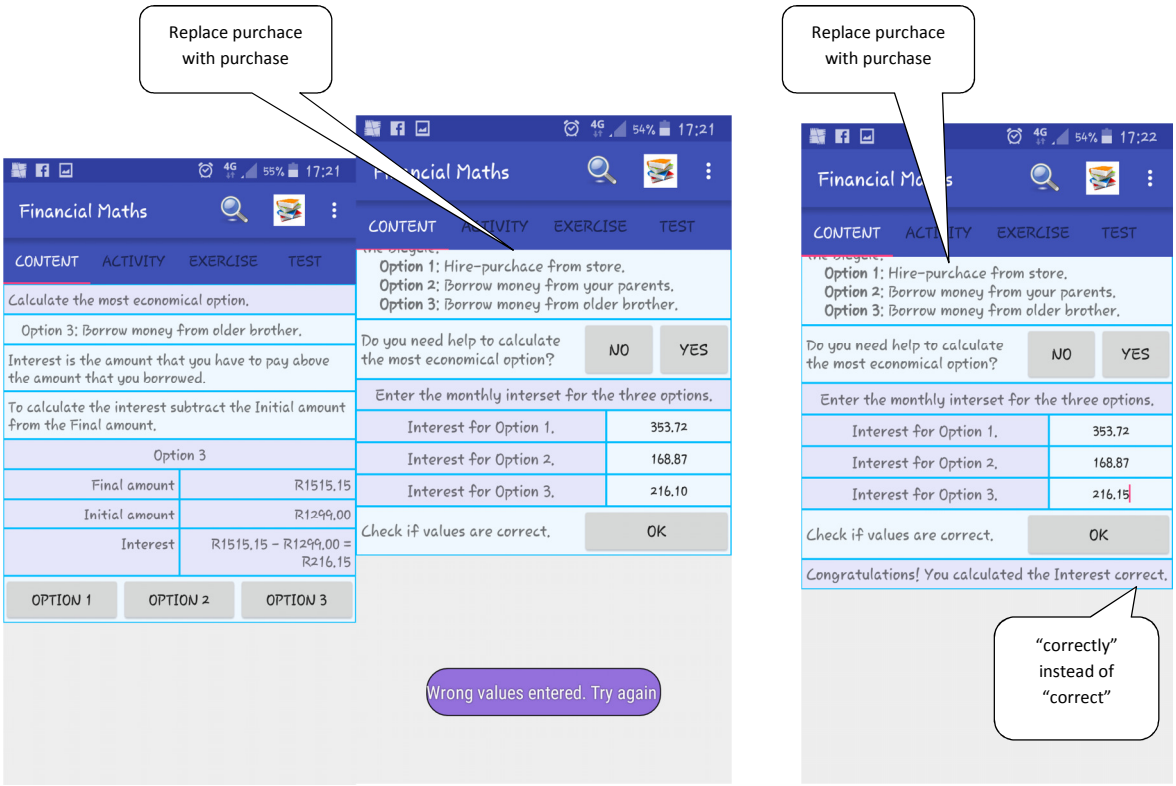
"interest" in the place of "interset"



Replace purchace with purchase

The answer should be R363,72





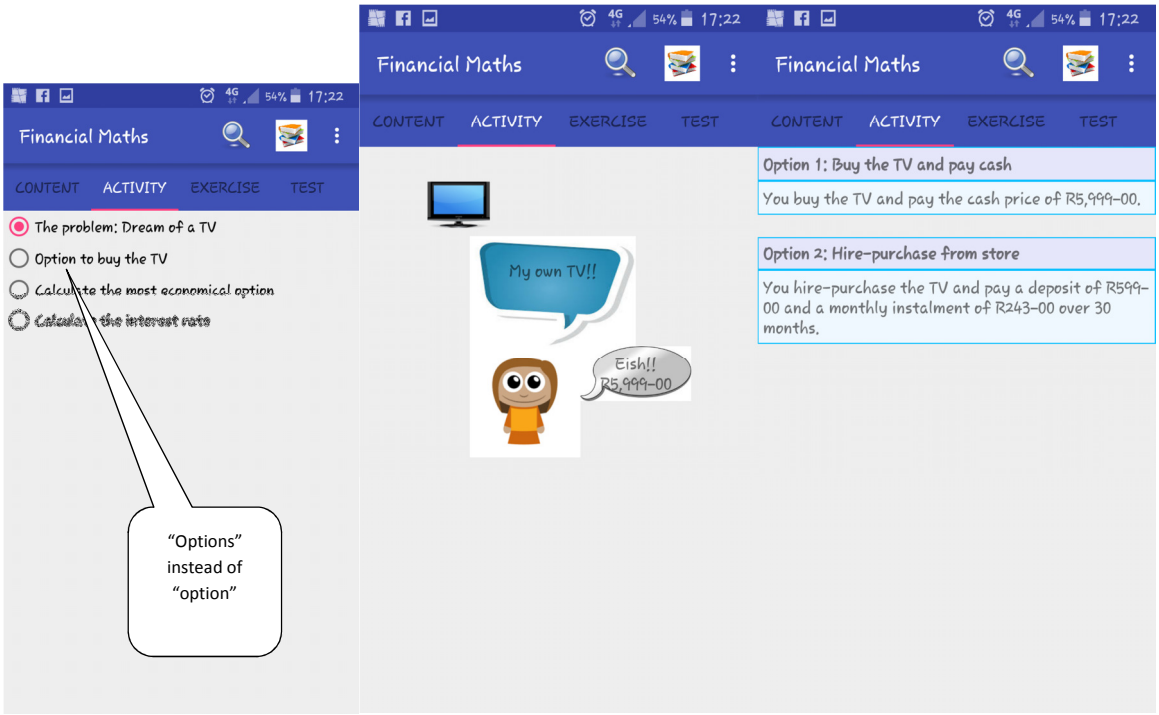
After this screen we need to answer the question:

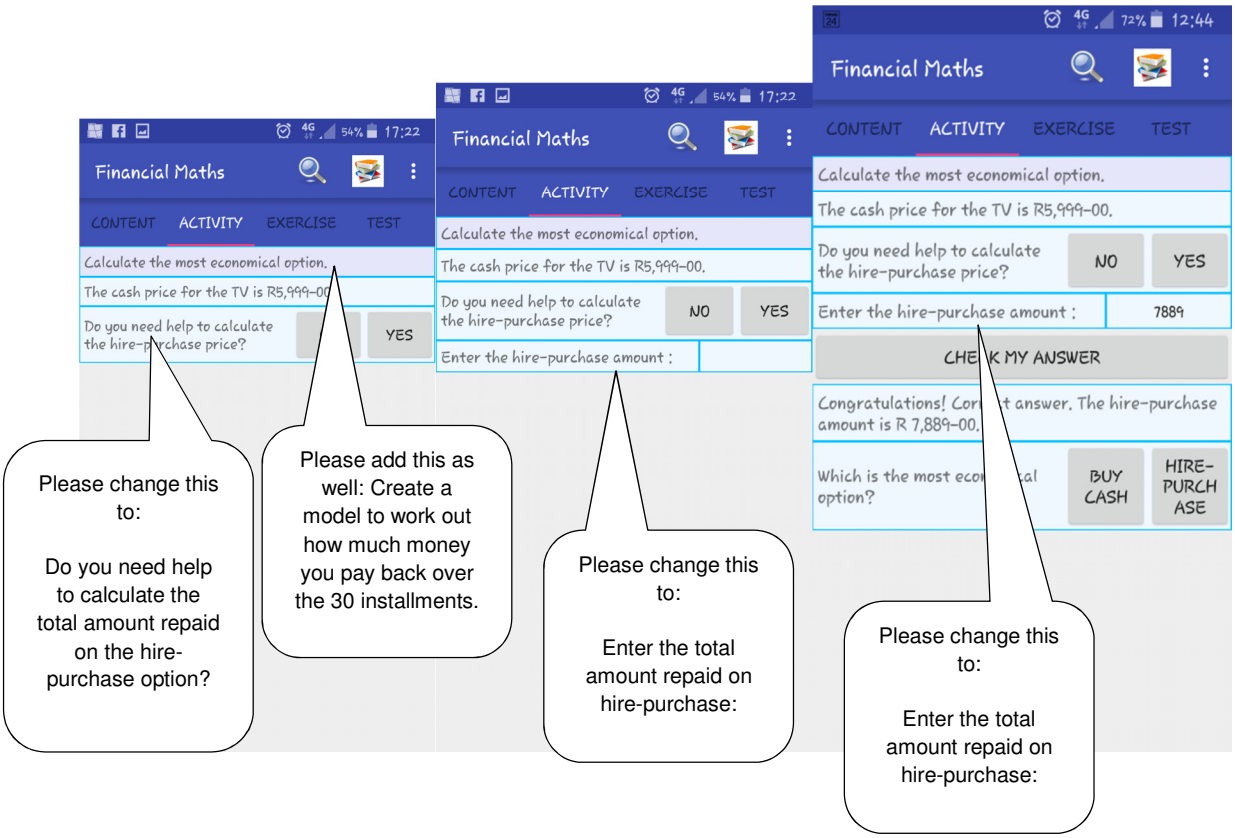
Which is the most economical option? Option 1 Option 2 Option 3

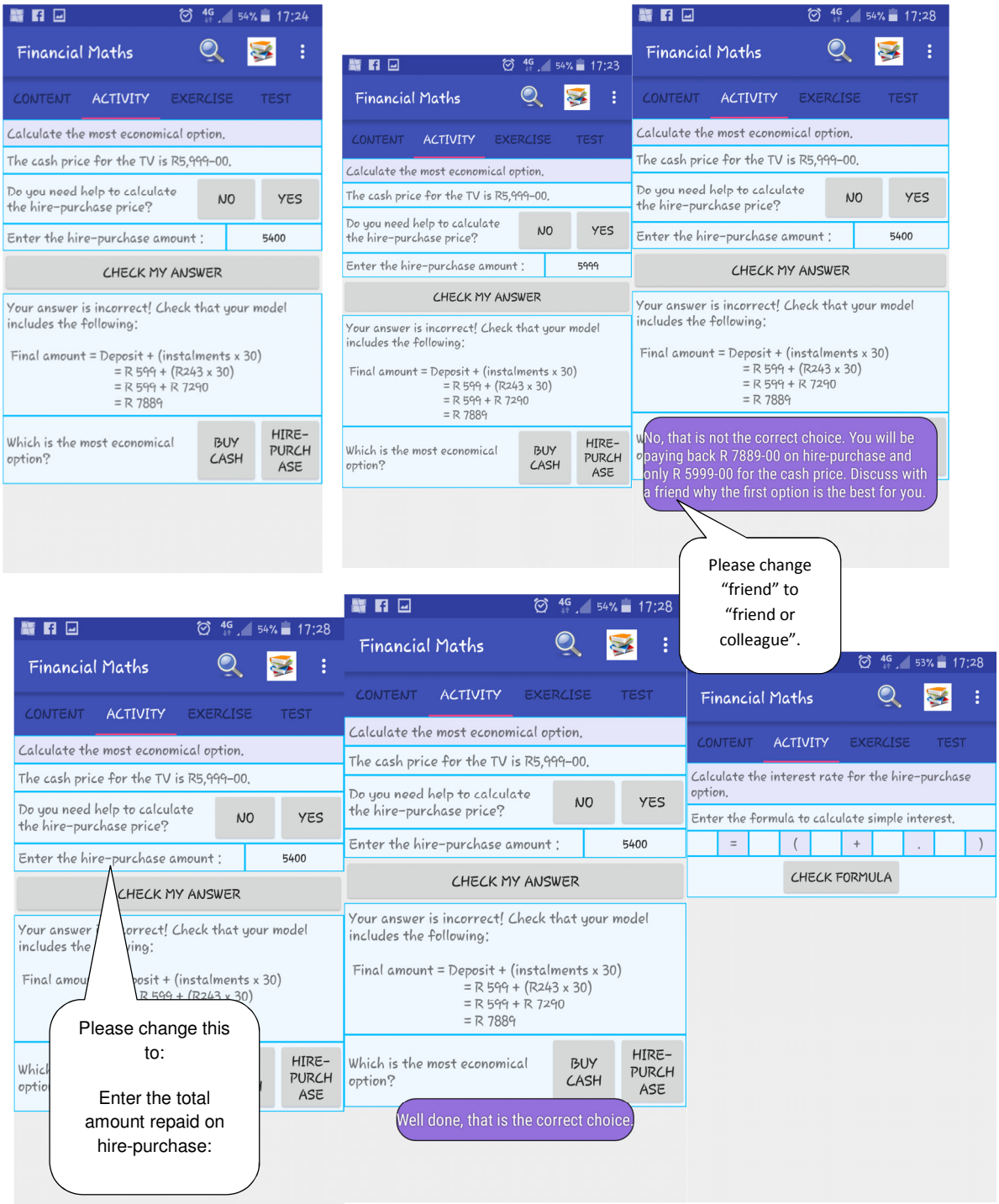
If the answer is Option 1 then say: This is the incorrect answer, try again.

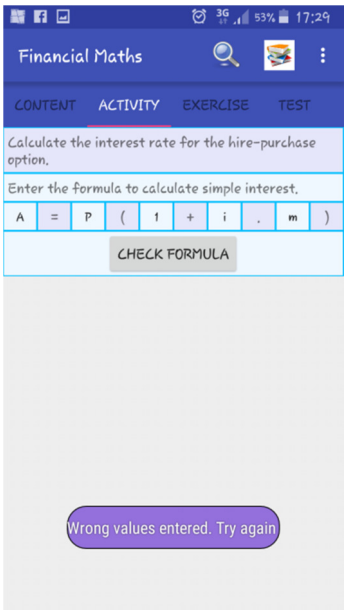
If the answer is Option 2 then say: Well done, this is the correct answer! Discuss with a friend or colleague possible reasons why this option has the lowest interest.

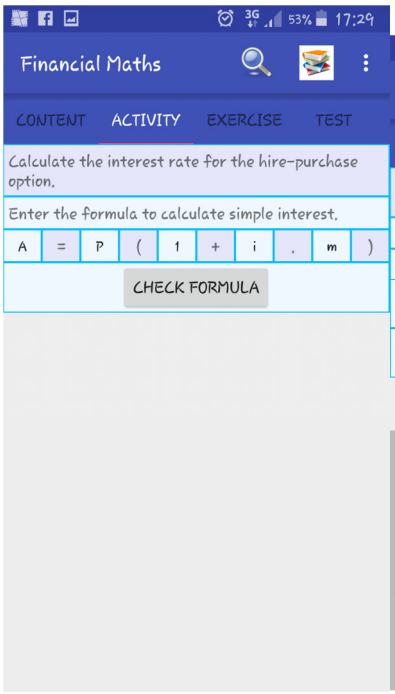
If the answer is Option 3 then say: This is the incorrect answer, try again.

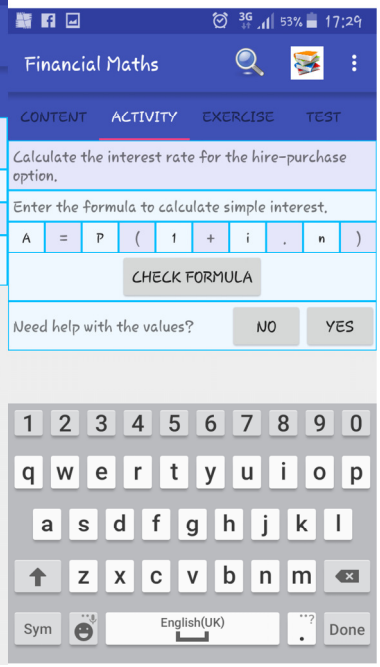

Financial Maths
CONTENT
ACTIVITY
EXERCISE
TEST
The problem: Dream of a TV
Option to buy the TV
Calculate the most economical option
Calculate the interest rate
"Options" instead of "option"
My own TV!!
Eish!! R5,999-00
Option 1: Buy the TV and pay cash
You buy the TV and pay the cash price of R5,999-00.
Option 2: Hire-purchase from store
You hire-purchase the TV and pay a deposit of R599-00 and a monthly instalment of R243-00 over 30 months.

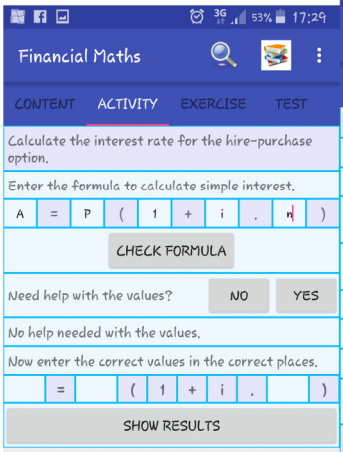

Financial Maths
CONTENT
ACTIVITY
EXERCISE
TEST
Calculate the most economical option.
The cash price for the TV is R5,999-00.
Do you need help to calculate the hire-purchase price?
NO
YES
Enter the hire-purchase amount :
7889
CHECK MY ANSWER
Congratulations! Correct answer. The hire-purchase amount is R 7,889-00.
Which is the most economical option?
BUY CASH
HIRE-PURCHASE
Please change this to:
Do you need help to calculate the total amount repaid on the hire-purchase option?
Please add this as well: Create a model to work out how much money you pay back over the 30 installments.
Please change this to:
Enter the total amount repaid on hire-purchase:
Please change this to:
Enter the total amount repaid on hire-purchase:

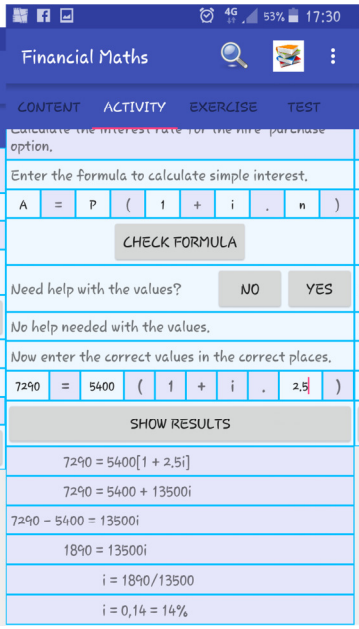

Financial Maths
CONTENT
ACTIVITY
EXERCISE
TEST
Calculate the most economical option.
The cash price for the TV is R5,999-00.
Do you need help to calculate the hire-purchase price?
NO
YES
Enter the hire-purchase amount :
5400
CHECK MY ANSWER
Your answer is incorrect! Check that your model includes the following:
Final amount = Deposit + (instalments x 30)
= R 599 + (R243 x 30)
= R 599 + R 7290
= R 7889
Which is the most economical option?
BUY CASH
HIRE-PURCHASE
No, that is not the correct choice. You will be paying back R 7889-00 on hire-purchase and only R 5999-00 for the cash price. Discuss with a friend why the first option is the best for you.
Please change "friend" to "friend or colleague".
Please change this to:
Enter the total amount repaid on hire-purchase:
Well done, that is the correct choice.
Calculate the interest rate for the hire-purchase option.
Enter the formula to calculate simple interest.
CHECK FORMULA

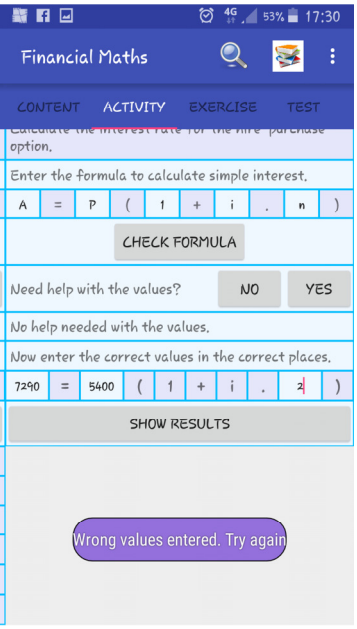

Financial Maths
CONTENT ACTIVITY EXERCISE TEST
Calculate the interest rate for the hire-purchase option.
Enter the formula to calculate simple interest.
A = P (1 + i . m)
CHECK FORMULA
Wrong values entered. Try again


Financial Maths
CONTENT ACTIVITY EXERCISE TEST
Calculate the interest rate for the hire-purchase option.
Enter the formula to calculate simple interest.
A = P (1 + i . m)
CHECK FORMULA


Financial Maths
CONTENT ACTIVITY EXERCISE TEST
Calculate the interest rate for the hire-purchase option.
Enter the formula to calculate simple interest.
A = P (1 + i . n)
CHECK FORMULA
Need help with the values? NO YES
1 2 3 4 5 6 7 8 9 0
q w e r t y u i o p
a s d f g h j k l
z x c v b n m
Sym English(UK) Done


Financial Maths
CONTENT ACTIVITY EXERCISE TEST
Calculate the interest rate for the hire-purchase option.
Enter the formula to calculate simple interest.
A = P (1 + i . n)
CHECK FORMULA
Need help with the values? NO YES
No help needed with the values.
Now enter the correct values in the correct places.
= (1 + i .)
SHOW RESULTS


Financial Maths
CONTENT ACTIVITY EXERCISE TEST
Enter the formula to calculate simple interest.
A = P (1 + i . n)
CHECK FORMULA
Need help with the values? NO YES
No help needed with the values.
Now enter the correct values in the correct places.
7290 = 5400 (1 + i . 2,5)
SHOW RESULTS
7290 = 5400[1 + 2,5i]
7290 = 5400 + 13500i
7290 – 5400 = 13500i
1890 = 13500i
i = 1890/13500
i = 0,14 = 14%


Financial Maths
CONTENT ACTIVITY EXERCISE TEST
Enter the formula to calculate simple interest.
A = P (1 + i . n)
CHECK FORMULA
Need help with the values? NO YES
No help needed with the values.
Now enter the correct values in the correct places.
7290 = 5400 (1 + i . 2)
SHOW RESULTS
Wrong values entered. Try again

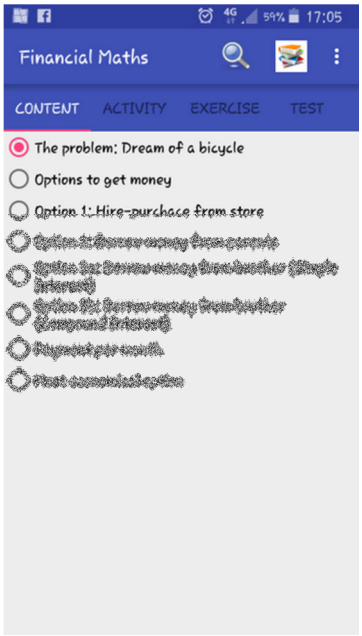
Changes to app 10-05-16

Can we introduce the content section with the detailed problem and questions that need to be solved? Perhaps have a screen that sets the problem situation and asks both questions before any other information is given.

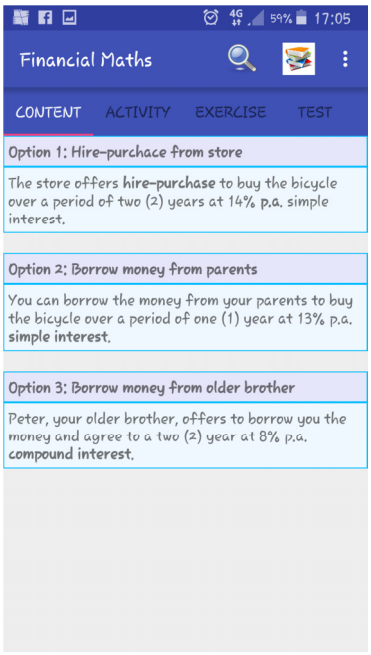
Problem situation
You have decided to buy a bicycle so that you can attend sport training in the afternoons at your school. You have visited the local stores and found a bicycle that you like for R1299. You have three options to consider. The store offers **hire-purchase** to buy the bicycle over a period of two years at 14% **p.a**. calculated at **simple interest**. You could also borrow the money from your parents who agreed that you can pay it back over 1 year at 13% p.a. calculated at simple interest. The third option is to borrow the money from your older brother Peter, who agreed to a 2 year period at 8% calculated at **compound interest**.

Questions:

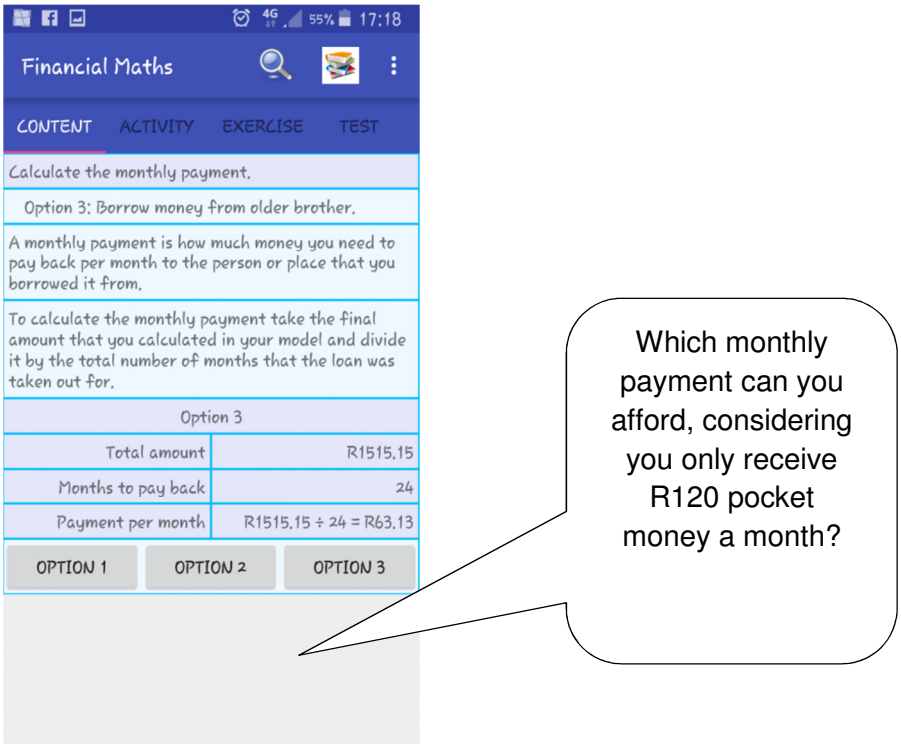
1. Which monthly payment can you afford, considering you only earn R 120 pocket money a month?
2. Model an answer to show which of the three options is the most economical for you (Which one do you pay the least interest)?

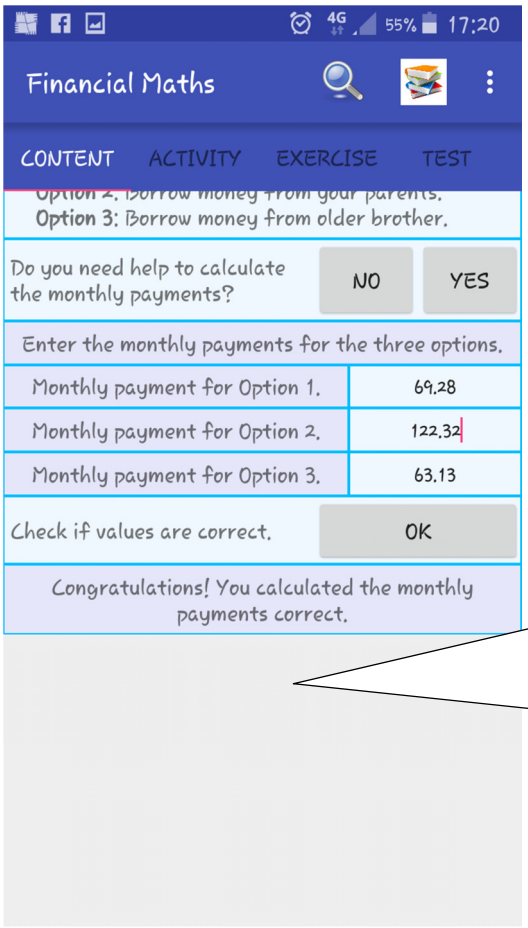


They will not know that the highlighted words can be defined. Can we have a hint pop up or appear or something, so that they know to **tap** on the bold words.



When they are asked to work out the monthly payment and if they say they need help to work out the monthly payment, then the three options' monthly payments can be seen. After the last one has been clicked, they must be asked the question again:





The question must also be asked if they entered the monthly payments themselves as well.

Which monthly payment can you afford, considering you only receive R120 pocket money a month?

Can we introduce the activity section with the detailed problem and questions that need to be solved? Perhaps have a screen that sets the problem situation and asks both questions.

You want to buy a new TV. The following options are available for you:

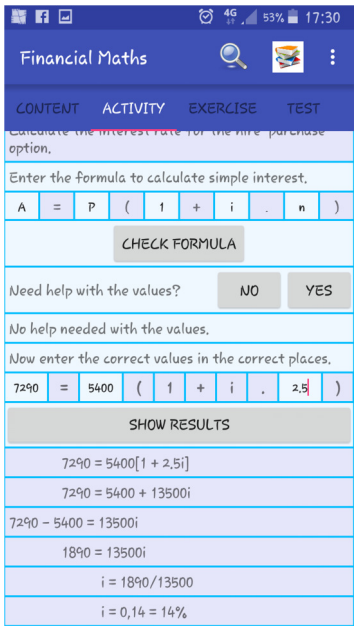
Cash Price R 5 999

OR

Hire-purchase: Deposit R599 and only R 243 per month (30 installments)

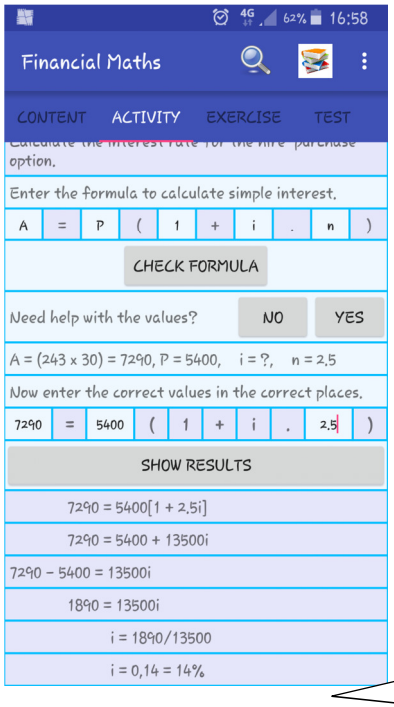
1. You must decide which option is the most economical. Create a model to help make the decision.

2. What is the interest rate that the store is charging for the hire-purchase?



When they calculate the interest rate correctly, can we include a final line that says:

Well done, you have correctly calculated the interest rate to be 14%



Also if they needed help with the values, and then correctly enter the values and calculate the interest rate, can we include a final line that says:

Well done, you have correctly calculated the interest rate to be 14%

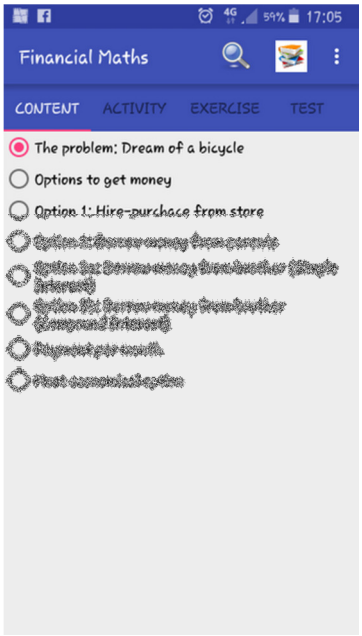
Changes to app 10-05-16

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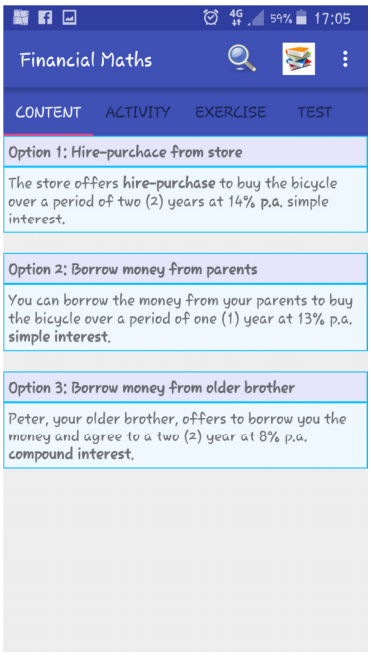
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Questions:

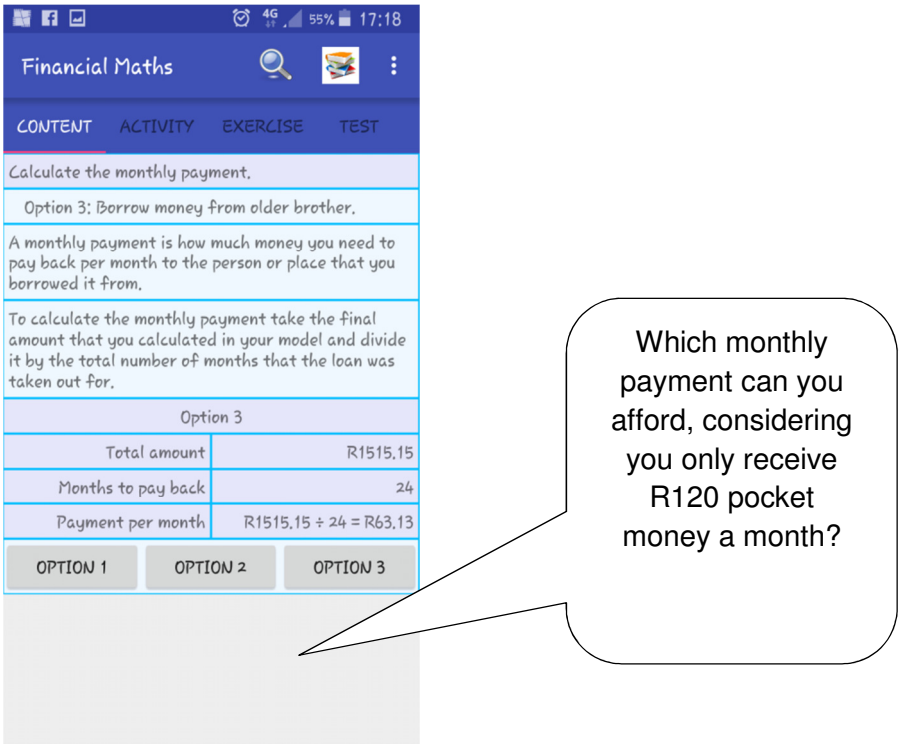
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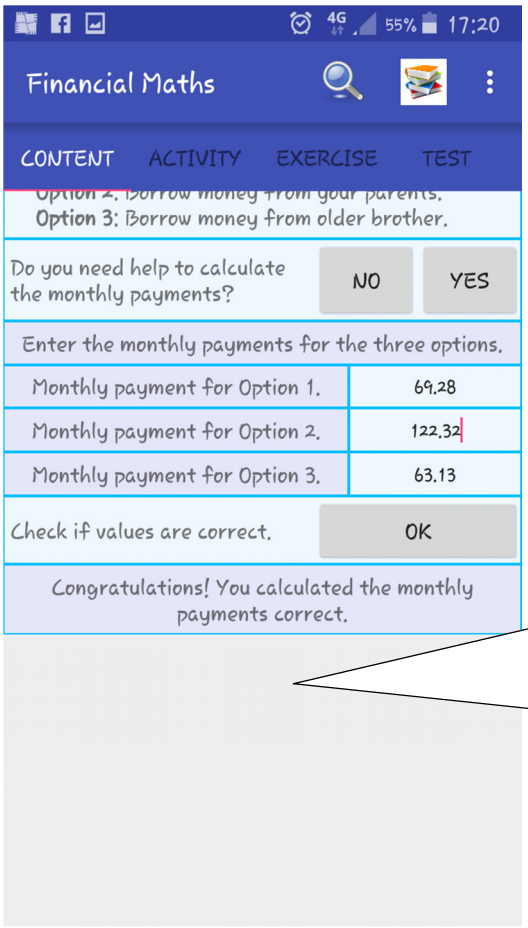


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The question must also be asked if they entered the monthly payments themselves as well.

Which monthly payment can you afford, considering you only receive R120 pocket money a month?

Can we introduce the activity section with the detailed problem and questions that need to be solved? Perhaps have a screen that sets the problem situation and asks both questions.

You want to buy a new TV. The following options are available for you:

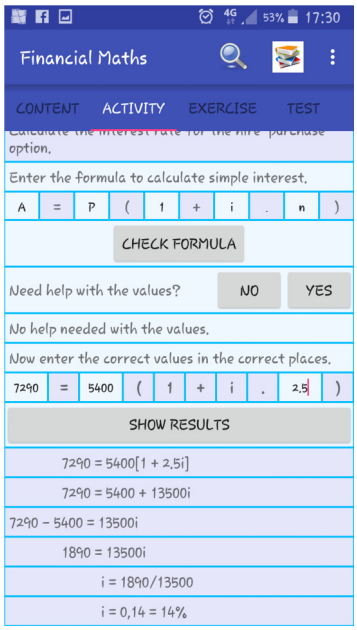
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OR

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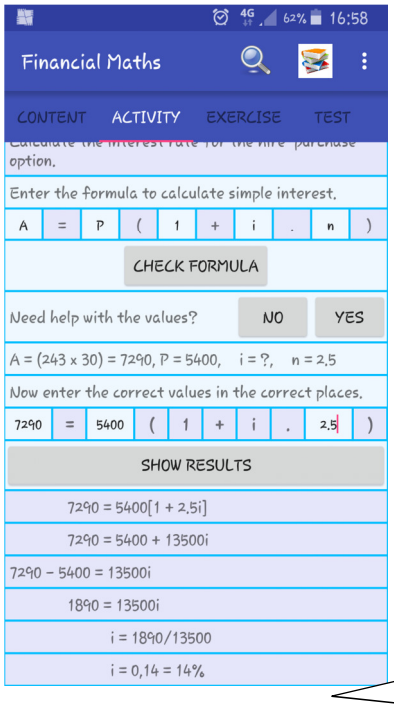
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Also if they needed help with the values, and then correctly enter the values and calculate the interest rate, can we include a final line that says:

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DESIGN OF A PROTOTYPE MOBILE APPLICATION TO MAKE MATHEMATICS EDUCATION MORE REALISTIC

Dawid B. Jordaan[1], Dorothy J. Laubscher[2] & A. Seugnet Blignaut[1]
[1]*TELIT-SA, North-West University, Vanderbijlpark Campus, South Africa,*
[2]*SDL, North-West University, Potchefstroom Campus, South Africa*

ABSTRACT

To enter the world of work, students require skills which include flexibility, critical thinking, problem solving, collaboration and communication. The use of mobile technologies which are specifically created for a context could stimulate motivation in students to recognise the relevance of Mathematics in the real world. South Africa in particular is a suitable location to use mobile learning across distance due to its viability, affordability and availability as opposed to web-based technology. The design team developed an app for financial Mathematics based on the principles of realistic mathematics education (RME) for mathematics teachers to use with their students. The paper describes the design process and general usability aspects of the formative evaluation of the prototype app for financial Mathematics. The authors suggest that a graphic artist becomes part of the design team to enhance the visual attractiveness of the app.

KEYWORDS

Realistic Mathematics Education (RME); mobile applications; Distance Education (DE); Android; interactivity; user interface.

1. INTRODUCTION

The rapidly changing information society influences the roles of learners and teachers (Gravemeijer, 2012). Future jobs require what Gravemeijer (2012) refers to as 21st century skills, which include "flexibility, critical thinking, problem solving, collaboration, and communication." Gravemeijer (2012) emphasizes the importance of addressing these demands by investigating educational practices that may nurture these goals. Heck et al. (2007) are of the opinion that the use of mobile technology could increase motivation as well as performance in learners. Also, the use of real-life contexts stimulate motivation because the relevance of Mathematics can be seen in the real world (Risnawati et al., 2014).

Various reasons are offered for poor performance in Mathematics, which include: when topics are taught (Fauzan et al., 2013), inaccurate learning material, inadequate or outdated teaching methods, poor forms of assessment, students' anxiety about Mathematics (Widjaja and Heck, 2003); and students' negative attitudes towards Mathematics (Widjaja and Heck, 2003).

Mobile learning is a field that is rapidly expanding and it affords new possibilities to improve learning, especially in a distance education (DE) environment (Kizito, 2012). South Africa in particular is a suitable location to use mobile learning in distance education due to its viability and affordability as opposed to web-based technology (Kizito, 2012). The portability of mobile technology allows learners to work creatively and collaboratively (Zaranis et al., 2013). When used correctly, technology could promote critical thinking, improve problem solving skills and facilitate collaboration (Stols, 2012). Further research is required to determine how the mobile phone can effectively be used in testing in DE (Kizito, 2012).

This paper investigates the need for, and the design and development of, a mobile application which would assist Mathematics teachers in making their teaching and their students' learning more realistic in terms of the context in which the learning takes place.

2. LITERATURE REVIEW

2.1 Realistic Mathematics Education

Mathematics teachers are challenged to develop Mathematics education that is in line with the dynamic conceptions of symbolizing and the development of meaning (Bakker et al., 2003). The role of the teacher is essential to understand how to teach effectively with technology (Drijvers, 2012). Digital tasks should be designed in such a way that students are encouraged to develop their own Mathematics. The teacher should guide the process and should know when to further explore a topic or concept and when to cease to investigate a topic (Drijvers et al., 2013).

The traditional approach to teaching Mathematics still dominate in many classrooms today (Fauzan et al., 2013). This approach is characterized by teachers who actively explain material and provide examples and exercises, while learners listen, write and perform the tasks the teacher requires (Gravemeijer, 2012, Widjaja and Heck, 2003). The social norms of traditional classrooms dictate that the teachers' answers are always correct and that students should follow given procedures to reach correct answers, which are more important than reasoning (Gravemeijer, 2012). Learners seldom have the opportunity to understand the rationale behind algorithms that are taught to them (Risnawati et al., 2014). Such a traditional approach to teaching Mathematics is held responsible for the poor quality of Mathematics Education and students' negative attitude towards Mathematics (Fauzan et al., 2013, Widjaja and Heck, 2003). Another consequence of traditional teaching is that when students solve word problems, high level cognitive and metacognitive processes are often absent (Mousoulides et al., 2007).

Reform from the traditional approach demands that curricula, teaching materials and assessment need to be adapted (Zulkardi, 2000). Freudenthal (1968), often regarded as the father of Realistic Mathematics Education (RME), opposes the traditional idea that the end result of the work of mathematicians is the starting point for Mathematics Education (Gravemeijer and Doorman, 1999). RME is regarded as a domain-specific instruction theory for the teaching and learning of Mathematics (Drijvers et al., 2013). The RME-based teaching and learning process promotes learner-centred learning (Fauzan et al., 2013). Problem solving through modelling leads to the design of activities that allow for students to deal with non-routine problem situations that demand the development of important mathematical ideas which can be extended, explored and refined in other problem situations (Mousoulides et al., 2007). Technological tools create new prospects for problem solving in Mathematics (Doorman et al., 2007). A corner stone of RME is that students are encouraged to not only receive information, but also question and process information (Widjaja and Heck, 2003), actively participate in the educational process and develop mathematical tools and insights (Drijvers et al., 2013).

2.2 Mobile learning

Mobile technology offers a new generation of learning to students of all ages without being bound by place and time (Alzaza and Yaakub, 2010). Mobile learning (mLearning) is becoming more popular within formal education as its benefits offer cost-efficiency, portability; instant connectivity; and context sensitivity. Mobile learning assists students to create social interaction; it promotes collaborative learning, interactivity and instant feedback as well as collaboration between peers; it improves their knowledge structure; their learning achievements and motivation (Mouza and Barrett-Greenly, 2015). Domingo and Garganté (2016) further point out that students are more willing to engage when learning with mobile technology; their desire to accomplish educational tasks also increases with the use of mobile technology, and it helps learners to become more self-directed in their learning. Mobile communication in education is a solution with a selection of prospects and challenges (Kommers and Hooreman, 2009).

mLearning applications have various educational benefits: they can be used as study aids; can be accessed from almost anywhere; and with the aid of location capabilities, students can use location-based information (Cheon et al., 2012). Mobile technology applications supplement higher education by extending traditional educational platforms and encouraging distance learning or using settings outside of the classroom (Al-Emran et al., 2016). Content applications that make use of personalized instruction can facilitate academic growth and self-efficacy among students (Mouza and Barrett-Greenly, 2015).

3. METHOD

This research relates to the first phase of a four-phased qualitative design-based research process where four teacher-students in the Open Distance Education (ODL) program of the North-West University (NWU), Potchefstroom Campus participated in individual interviews. They shared their perceptions and experiences on support teachers' need in order to effectively implement the RME principles in their teaching practice. The teacher-students confirmed that their own students inter alia (i) had problems with certain areas of Mathematics, (ii) suffered challenges with the integration of the content across the curriculum, (iii) were excited about the affordances that mobile learning could bring to Mathematics education, and (iv) experienced challenges to link school Mathematics to real life problems, e.g. handling their own finances. The researchers decided to zoom in on the aspect of financial Mathematics for the content aspect of the mobile application (app) because all the participants mentioned this area as troublesome to their students.

The context in which the app will be used should form the anchor for the development of the content (Widjaja and Heck, 2003). The app was specifically designed for teacher-students enrolled in an open distance learning (ODL) teacher professional development programme through the Unit of Open Distance Learning (UoDL) at the North-West University at the Potchefstroom Campus. They were all established Mathematics teachers at semi-rural schools in the North-West Province, South Africa, and enrolled for a BEd Honours programme. The design of the application was based on RME principles: (i) guided reinvention (or progressive mathematisation); (ii) emergent modelling; and (iii) didactical phenomenology (Andresen, 2007). The process of guided reinvention began with a real-life problem which was then mathematised (Gravemeijer and Doorman, 1999). The aim with the app was to guide these teacher students to become comfortable with using technology in teaching and learning, as well as to enable them to interact with their learners on RME.

3.1 Creating the app

Software engineering (SWE) is the application of *engineering* to the design, development, implementation, testing and maintenance of *software* in a systematic method. The objective of software engineering is to produce software systems to customers in a cost effective way. Once installed these systems should display characteristics such as efficiency, reliability, maintainability, robustness, portability and so on (Sommerville, 2016). The software process is a set of activities and associate results which produce a software product. Software specification, development, validation and evolution are fundamental process activities common to all software processes.

Software development models are processes or methodologies used for the development of software projects. There are many development models that have been developed in order to achieve different objectives. The models specify the various stages of the process and the order in which they are carried out (Sommerville, 2016). Examples of software development models are: Evolutionary Prototyping Model; Spiral Method (SDM); Iterative and Incremental Method; Extreme programming (Agile development); Waterfall model; Prototype model; Rapid application development model (RAD) and so on (Whitten and Bentley, 2007). Choosing the right model for developing a software product or application is very important as the model determine where and when the development and testing processes are carried out.

For the development of our app, which we name *Financial Maths* App, we decided to use the Prototype Model because of the advantages that this model offer: Users are actively involved in the development; Since a working model of the system is provided, the users get a better understanding of the system being developed; Errors can be detected much earlier; Rapid user feedback is available leading to better solutions; Missing functionality can be identified easily; and Confusing or difficult functions can be identified (Mohammed et al., 2010). The basic idea is that instead of freezing the requirements (as required by many other models) before a design or coding can proceed, a prototype is built to understand the requirements based on the currently known requirements. By using this prototype, the client can get an "actual feel" of the system, since the interactions with the prototype can enable the client to better understand the requirements of the desired system. The prototype is usually not a complete system and many of the details are not built into the prototype. The goal is to provide a system with overall functionality. Figure 1 depicts the phases in the development of a prototype.

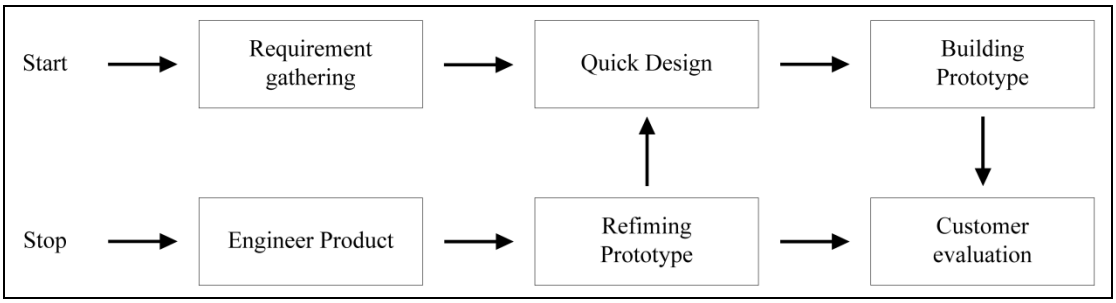


Figure 1. Prototyping model (Sommerville, 2016)

3.2 Creating the app

Background. While the first author of this paper was the designer of the app, the second author was the content specialist and also responsible for the various design documents, and the third author was the expert on the use of technology in teaching and learning. Together they formed a tight project team. They obtained ethics clearance for this research from the Ethics Committee of the North-West University (NWU-HS-2014-0267).

Requirement (Step 1). The first step in creating an app was to develop a clearly defined purpose for the mobile app: we had to determine what the app should be able to do; what its primary appeal would be; which concrete problem it was going to address; and what part of life it aimed to improve. Detail design documents were created which outlined the standards, planning design, development, and ongoing evaluation of the project (Fleisch and Schöer, 2014).

Quick Design (Step 2). Next the foundation of the user interface was laid. This step visually conceptualized the main features and a rough layout and structure of the app. Sketches for the proposed layout and structure of the app were drawn which assisted the team to understand the journey better Figure 2 is an example of such a sketch.

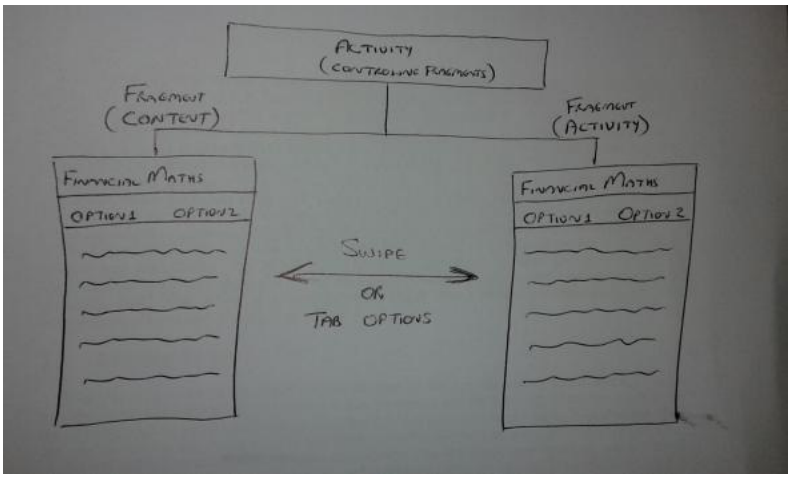


Figure 2. Example of layout sketch (own scetch)

Building Prototype (Step 3). At this stage all the ideas and features culminated as a clear picture of the structure and a storyboard for the project was created. The storyboard provided a road map which illustrated the connections between the different screens and how the user could navigate through the app. The storyboard formed the foundation for the first version of the prototype.

Customer Evaluation (Step 4). The next step involved the testing of the prototype. Friends, family, colleagues, and experts all helped to review the prototype. They were requested to test run the app and give sincere feedback and identify flaws and dead-end links. If possible to monitor how they used the app, one should take note of their actions as these would provide important feedback for user interface (UI) and user experience (UX) evaluation (Molich and Nielsen, 1990). Based on their feedback, the prototype was modified. The aim was to finally specify the app concept before going into the design process. For the

Financial Maths app, the content author tested the app with two of her Mathematics colleagues who are Mathematics lecturers, as well as with a RME expert.

Refining Prototype (Step 5). Next, individual screen content were designed. The task was to create high-resolution versions of the prototype. All comments from the prototype testers were included in order to design the most suitable user interface. With the screen designs completed and implemented, the actual app concept was complete, all the graphics were inserted, and all text was signed off—the actual design was now implemented and made clickable.

Repeat (Step 2 to 5). The next step was to test the full design once more and collect as much feedback as possible from a variety of users. The new ideas and comments were used to refine the app. A consistent look and feel of the layout was assured and we ensured that it would perform reliably on different devices.

Deployment (Step 6). As the vast majority of the intended students used Android devices (Dahlstrom et al., 2016), the brief was to develop the *Financial Maths app* for Android devices (smart phones and tablets). The *Financial Maths* app was consequently installed on Android devices and tested for functionality in a live environment.

4. USABILITY OF THE APP

4.1 The user interface

Figure 1 illustrates two scenarios that are realistic and relevant to students at school, from which the concepts of simple and compound interest could develop. Students are not expected to reinvent all the content themselves, and should be guided by the teacher who allows them the opportunity to reflect on their invented strategies (Gravemeijer and Doorman, 1999). Drijvers et al. (2013) suggest that guidance from the teacher will help the progression in a sensible manner, as is suggested in Figure 3.

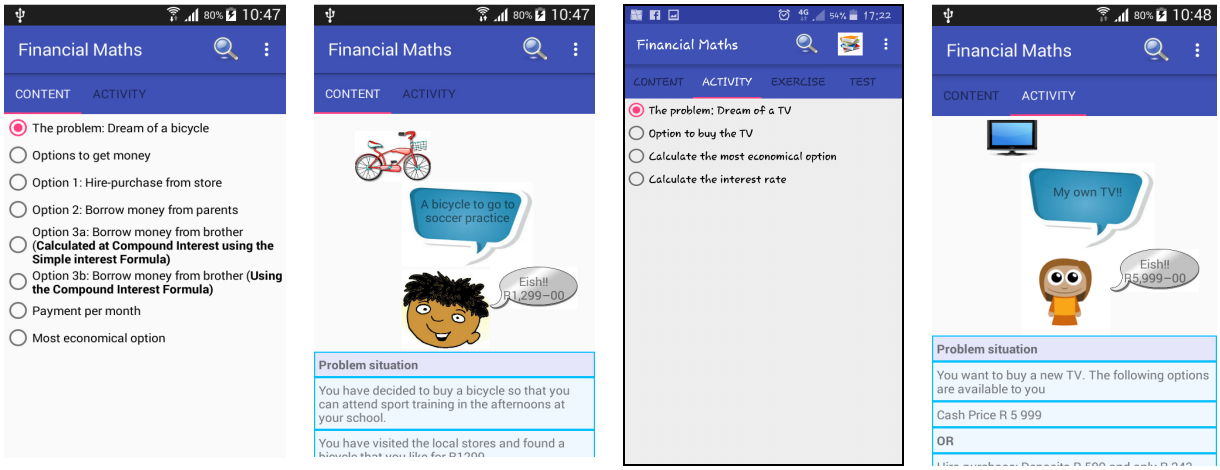


Figure 3. A context that is meaningful and natural is created

Emergent modelling refers to the development of models from the student's activity which leads to the emergence of formal Mathematics (Gravemeijer and Doorman, 1999). The use of models is encouraged as a bridge between what is abstract to students and what is real (Dolk et al., 2002). Figure 4 illustrates how the use of models has been encouraged in the app. The models too are grounded in the contextual problem and are not derived from the intended Mathematics (Gravemeijer, 1999). The last screenshot in Figure 4 illustrates that students are in various instances given three opportunities to attempt to develop their own strategies to solve a problem before a hint is given.

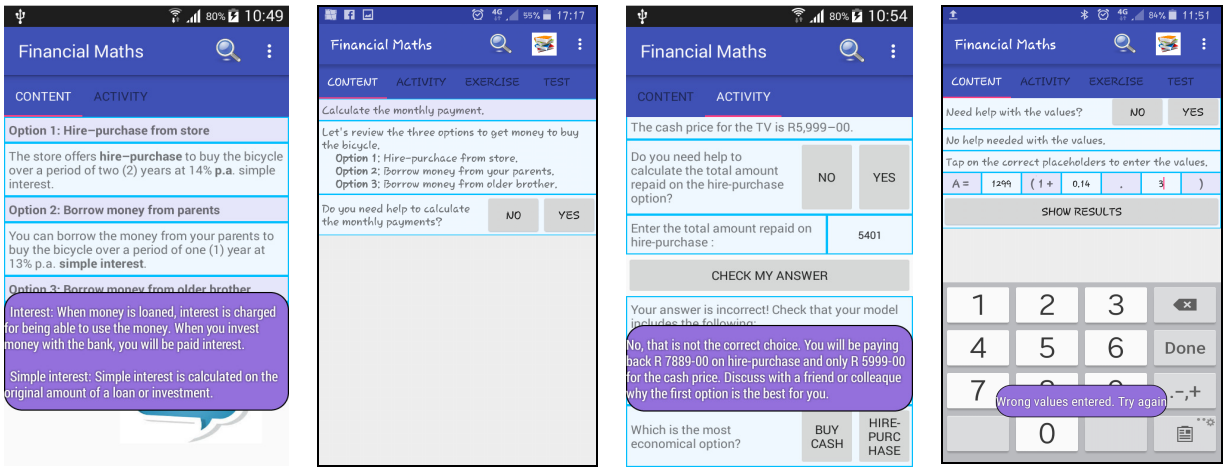


Figure 4. Hints and tips are provided should students require them

The third principle that was incorporated in the design of the app was didactical phenomenology, which deals with the idea of how mathematical structures can assist in organising phenomena in real life (Zulkardi, 2000). Students are confronted with situations that need to be organised, and in that way students can build concepts (Bakker et al., 2003). The phenomena of personal finances, hire-purchase and interest rates are relevant real-life notions that can be structured with mathematical content (Figure 5).

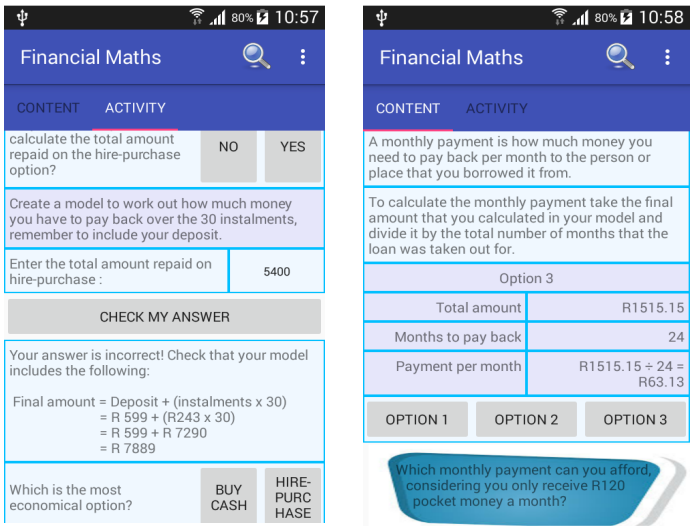


Figure 5. An illustration of the use of models

4.2 General usability of the app

The initial design document provided the design team with a broad idea of the content of the app. In the drafts to follow, more particulars are given with regard to the detail of when hints and tips should appear; how many attempts students can make on a solution before receiving help; and words of encouragement when the solutions are correctly worked out. Molich and Nielsen (1990) are of the opinion that general usability evaluations should comprise not more than ten members, and the design team followed this suggestion. The three team members perused and tested the first two prototypes and suggested changes in terms of usability. An RME expert evaluated the third prototype of the app and focused specifically on the inclusion of the RME principles. His evaluation revealed that all the principles of RME were sufficiently addressed in the app and that it should achieve the purpose for which it was designed, namely to make

teachers' teaching of Mathematics, and learners' learning experiences more realistic, relevant and experientially real. He suggested that the app be thoroughly tested before being released for student use.

Two more experts in the field of Mathematics education tested the app. Their main suggestion for improving the app was to make the problem situation more visible at the start of the app:

> *Put something in that sets the scenario more clearly. The start of the app is too vague.*

Their evaluation also included comments on why they felt the app was suited to the purpose for which it was created:

> *It works very well because all the steps are clearly shown. It will help the teachers to teach their learners to follow logical steps in their calculations.*

The app included an activity that clearly showed the intertwining of the concepts of compound and simple interest, a key characteristic of RME (Widjaja and Heck, 2003). The Mathematics education experts found this appropriate and necessary:

> *I like this very much. By including this comparison of concepts you can see wow, it's quicker to use the compound interest formula, and yet you still get the same answer as when repeating the simple interest formula.*

They also suggested that the app be extended to other content areas and that it be made available to a wider range of teachers.

5. CONCLUSIONS AND RECOMMENDATIONS

The design of this app has the potential to be extended into a larger project where a variety of problem content areas in Mathematics, at different grade levels, could be addressed. These apps could be made available to any Mathematics teacher who needs assistance with both content as well as the notion of incorporating relevant, real life contexts in their Mathematics teaching.

A comment made by a Mathematics education specialist relates to the need for the development of users of such apps in terms of the integration of technology into their every teaching and learning:

> *We had a bigger problem working the phone that doing the sums. Let's hope that the students that will be using this app are comfortable with the technology.*

Although one of the points of criticism from the subject experts was that the problem scenario was not well articulated, the design team is of the opinion that a more graphic approach would work better. The context and content should be presented with the aid of more graphics and less words.

ACKNOWLEDGEMENT

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I hereby declare that I have edited the language of the PhD thesis

Guidelines for effective technology facilitation of Realistic Mathematics Education to enhance teaching practice

by

DJ Laubscher, student number 10218343

submitted for the degree
Doctor Philosophiae in Mathematics Education
at the Potchefstroom Campus of the North-West University

The responsibility to accept recommendations and effect changes remains with the author.

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17 November 2016

Systematic literature review relating to the use if ICT to facilitate the principles of realistic mathematics education to enhance teaching practice

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Addendum 9

Original ATLAS.ti project:systematic literature review relating to the use if ict to facilitate the principles of realistic mathematics education to enhance teaching practice.atlcb

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General

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Statistics:

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Primary Documents

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Memos (0)

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Introduction of a new construct: The conceptual tool "Flexibility"
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1
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Abstract
This paper presents a new construct: the conceptual tool 'Flexibility'. The construct was a result of an attempt to extract experiences of teaching and learning with the use of laptops. It was further developed and refined on the basis of four small-scale teaching experiments. The teaching experiments, being part of a development project in upper secondary school mathematics, investigated the use of laptops for teaching differential equations from a modeling point of view. The research was double-aimed: one objective was to conclude the project with some recommendations for the design of teaching, in the form of guidelines suitable for a wider dissemination amongst upper secondary mathematics teachers. The other main aim was to draw on the project's experiences for theory development within a framework based on Realistic Mathematics Education (RME) and related ideas. The construction of 'Flexibility' served both these aims.
Keywords: computer environments; flexibility; instrumental genesis; technology; teaching and learning; laptops; differential equations; Realistic mathematics education (RME)
1. Background
In Denmark, a recent reform of the structure and the curriculum in upper secondary school encompassed introducing the use of CAS in mathematics, chemistry and physics. Teaching with CAS was not required to follow authorized plans or materials: the design of teaching sequences, planning and preparation of extra teaching materials etc. were still the individual teacher's responsibilities. Thereby, the reform made new demands on the teachers' professional development: it was far from obvious how the use of CAS should be integrated in the individual teacher's repertoire of teaching instruments
2
. A metropolitan area six-year school development
project, titled 'World Class Math and Science'
3
. partly served as a precursor for the reform,

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2
For a discussion of the generation of CAS use as the teacher's teaching instrument, see (Andresen 2006 p 265-275)
3
www.matnatverdensklasse.dk TMME, vol4, no.2, p.231
although the government did not sponsor the project. Part A of the development project
4
encompassed experiments with laptop classes in upper secondary school mathematics.
I was employed at the Danish University of Education to do the research in part A of the development project for my Ph.D. in mathematics education. Following the standpoint described in Wittmann (1998) and Lesh & Sriraman (2005), that mathematics education is a design science, I believe that research and development has to be linked to teaching and learning practice at crucial points. Improvement of teaching practice must be merged with the progress of the whole field of educational research and reversely, research progress linked to the development of teaching practice. So, the research for the Ph.D. project became double-aimed: one objective was to conclude the World Class project with some recommendations for the design of teaching, in the form of guidelines suitable for a wider dissemination amongst upper secondary mathematics teachers. The other main aim was to draw on the project's experiences for theory development within the theoretical framework based on Realistic Mathematics Education (RME) and related ideas. My construction of 'Flexibility' served both these aims.
1.1 Objectives of the construction
With this background, the initial goal of the research was to extract knowledge from the project's teaching experiments in a way that made this knowledge useful. More precisely, the aim was to identify, articulate and conceptualise the participants' shared experiences of improved learning and, subsequently, to turn these experiences into a form that would:
- Allow teachers to take them into account for improvement of their own teaching
- Serve as a contribution to math education theory
In yearly evaluation interviews, participants from the World Class A project repeatedly had expressed a shared experience of improved learning: in the interviews, a number of students and teachers described their feelings of getting "a better overview" and "a deeper understanding", apparently as a result of the use of laptops. These remarks were developed further with explanations like "because you can easily get series of graphs", "you do not get stuck in technical details", "it is easy to see examples" or "in stead of remembering a lot of techniques you are allowed to concentrate on the ideas". Such answers are in accordance with some expectations to CAS (and computer) use, widespread amongst math teachers. Teachers, like many researchers
5
, believe that besides to favour visualisations, the CAS routines can be
incorporated into the design of teaching as shortcuts, to facilitate students' focusing on ideas, structures and conceptions.
Hence, in the research project the challenge was to critically sort out examples of fruitful support of students' learning from the general feeling of "flow in the classroom". So the objective turned out to be pointing out important elements of learning activities, for teachers to aim at in their future design and preparation of teaching sequences and single lessons. These elements of learning activities are important, according to two criteria:
1. The activity should promote the student's actual work with mathematics in an observable way

4
See (Andresen 2006 chapter 3, p 21-39) for a detailed description
5
See for example (Drijvers, 2003, p 92). Hypothetically, the work with CAS may offer the students a shortcut to reification or a shortcut to working with "objects" as if the processes were reified. TMME, vol4, no.2, p.232
2. Arguments based on the theoretical framework should support the claim that the activity could promote the student's learning
The mental actions 'change of perspective' and 'change of representation' turned out to be a common denominator for the important elements in focus of my interest. They gave inspiration to my choice of the term 'flexibility' to denote the new construct in the following definition.
1.2 Definition: Flexibility of mathematical conceptions
Definition
The flexibility of a mathematical conception constructed by a person is the
designation of all the changes of perspective and all the changes between different representations the person can manage within this conception.
(Andresen, 2006, p. 136):
In this definition of the conceptual tool flexibility, the term 'change of perspective' means change between different facets of the mathematical conception in question, regarded as the student's construct
6
. My selection of a number of complementary pairs of perspectives intended
to make flexibility an operational tool. Two main considerations determined my choice of pairs, in accordance with (1.) and (2.) above:
- Changes within the pairs of perspective should be recognisable for the teacher (or any observer)
- The changes should be pivots for the students' learning process.
Within the notion of flexibility, the term 'representation' is used in the sense of the media of expression or communication. The objective of the construction was not to categorise all mental changes. The overall aim was to offer teachers a few building blocks in the form of simple design heuristics. My design of these building blocks should be appropriately based on the research to ensure that the use of them was likely to support the students' learning.
Neither did the construction aim to establish any one-to-one correspondence between every mathematical conception and its exposure within each perspective. Supposedly, it is clear that any mathematical conception can be expressed in each perspective in more than one way, and it may be exposed in more than one way in each representation. The perspectives and representations, referred to in the definition, are presented in a later paragraph in this paper. Before that, the next paragraph tells about the theoretical basis for the concept of flexibility.

6
According to L. P. Steffe and P. W. Thompson (Steffe & Thompson 2000 p 268-269), the experience of students' learning allows the researcher to inquire the students' mathematical realities. These realities are called students' mathematics and by (partly) knowing them, it is possible for the researchers to construct a model of students' mathematics, called mathematics of the students. Students' mathematics, which the students have constructed as a result of their interactions in their physical and sociocultural milieu, is indicated by what the students say and do when they engage in mathematical activity. In contrast, mathematics of the students is part of the shared knowledge in the classroom, compatible with the educational goals. TMME, vol4, no.2, p.233
2. Basic ideas beyond the concept of 'flexibility'
Literature studies played an important role for the development and refinement of my earliest idea of 'flexibility'. This paper only presents three main ideas from the framework that forms the basis for my definition of 'flexibility'
7
: i) a specific dynamical approach to concept formation
that combines main heuristics of Seymour Papert and Jean Piaget, introduced by Edith Ackermann, ii) vertical and horizontal mathematising in the RME sense realised in Koeno Gravemeijer's four level model and iii) the French theory of Instrumental Genesis.
2.1. A dynamical approach to concept formation
To facilitate cognitive growth, Edith Ackermann presents the idea of a bi-directional interplay between "diving in" and "stepping back" in (Ackermann 1990 p 6). Ackermann refers to both Jean Piaget and Seymour Papert
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as constructivists who see children as the builders of their own
cognitive tools, as well as builders of their external realities. Both Piaget and Papert consider knowledge as a personal experience to be constructed. Further, they both acknowledge adaptation as
.. the ability to maintain a balance between stability and change, closure and
openness, continuity and diversity, or, in Piaget's words, between assimilation and accommodation. (...) The main difference is that Piaget's interest was mainly in the construction of internal stability whereas Papert is more interested in the dynamics of change. (Ackermann, 1990, p.4)
The main point in Ackermann's description of Papert's view is: diving into situations rather than looking at them from a distance, connectedness rather than separation are powerful means of gaining understanding: becoming one with the phenomenon under study is a key to learning.

Ackermann's description of Piaget's view can be summarised like this: the way children progressively become detached from the world of concrete objects and local contingencies is closely related to their gradually becoming able to mentally manipulate symbolic objects within the realm of hypothetical worlds, so that rules and invariants are means of interpreting and organizing the world, and abstract and formal thinking are the most powerful way to handle complex environments.
Ackermann states that her own perspective is
..an integration of the above views. Along with Piaget I view separateness through progressive decentration as a necessary step toward reaching a deeper understanding. I see constructing invariants as the flipside of generating variation. (..) I share Papert's idea that diving into unknown situations, at the cost of experiencing a momentary sense of loss, is a crucial part of learning. (..) My claim is, that both "diving in" and "stepping back" are equally important in getting such a cognitive dance going. (Ackermann 1990 p 6)

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The complete theoretical framework is presented and discussed in (Andresen 2006, chapter 5)
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In a footnote, Ackermann says: "Describing the difference between Piaget and Papert has been useful for me, and might be of general interest for the reasons mentioned in the text. It is through working directly with both thinkers (first, at the Piaget Institute, and currently at MIT) that I became progressively convinced of the need for integrating structural and differential approaches in describing human development." (Ackermann 1990 p 27) *TMME*, vol4, no.2, p.234
Ackermann's idea of a 'cognitive dance' gave me the inspiration to construct 'flexibility' as a means to capture the dynamics of concept formation rather than static descriptions of cognitive structures. My method was to identify and group changes of perspective and changes between representations.
2.2. Vertical and horizontal mathematising in RME
Within the paradigm of RME, guided reinvention (or progressive mathematising), didactical phenomenology and emergent models are the three key heuristics for the design of teaching. In contrast to the cognitive theories of concept formation, 'modelling' was acknowledged to be an issue of interest in the World Class project's teaching culture. Moreover, a 'guided reinvention' design is to a great extend in accordance with the prevailing norms for good teaching. So it seemed reasonable to include RME into the framework in the following way. The four-levelmodel of activity intends to capture the way students' thinking evolves (Gravemeijer & Stephan 2002 pp 159-160):
Fig 1. Levels of activity (Gravemeijer and Stephan, 2002, p. 159)
In the model, the activities at the level of task setting take place in a situation that is experientially real for the students. This enables the students to reason with a model of the problem but still think about acting in the situation. For experienced students an experientially real situation may also be mathematical. At the next, referential, level the students act with a model that is meaningful because it signifies an experientially real activity for them. At the general level the type of activity has changed since the students' attention has shifted from the contextual meaning to the mathematical relations involved. The students' activities are no longer dependent on situation-specific imagery. Finally, the students will no longer need the support of a model for their more formal mathematical reasoning, which constitutes the fourth level of activity.
So, progressive mathematising implies the students' raising up through the four levels. The levelraising does not happen as a total, one-directional movement: the student may in one context of content be at one level and in another context at another level. Besides, the raising from one level to another happens over time, where the student switches between two levels in both directions several times. The levels, further, have different character. It follows that in terms of RME, *TMME*, vol4, no.2, p.235
concept formation by modelling can be expressed as changes between levels: horisontal mathematising happens as changes from situational, task setting level to referential level when models emerge, and vertical mathematising happens as changes from the level of model of (referential) - to the level of model for (general) and to further up to the formal level. In the description below, the first three levels in the four-level-model are considered as perspectives on the mathematical conceptions in question. Level-raising is considered as the result of bidirectional changes between these perspectives, which are included in 'flexibility'.
2.3. The French theory of Instrumental Genesis
The French theory of instrumental genesis is based on the idea that an artefact, for example a CAS calculator, does not in itself serve as a tool. It becomes useful, and then denoted an 'instrument', only after the user's formation of (one or more) mental utilisation scheme(s). Such utilisation schemes connect the artefact with conceptual knowledge and understanding of the way it may be used to solve a given task. The utilisation schemes contribute to the formation of instrumented action schemes. So, an instrument consists of the tool, for example a laptop with the CAS software Derive, the student's mental utilisation schemes and the task or problem to be solved. (Drijvers, 2003, pp. 96-97).
The process in which the artefact becomes an instrument is called 'instrumental genesis'. The process proceeds through activities in
The two-sided relationship between tool and learner as a process in which the tool in a manner of speaking shapes the thinking of the learner, but also is shaped by his thinking. (Drijvers & Gravemeijer, 2005, p. 190).
The two directions of the process can be linked to the construction of epistemic and pragmatic knowledge, respectively. The distinction between construction of epistemic and pragmatic knowledge is reflected in the definition of flexibility, which discerns between pairs of perspectives with relation to the construction of epistemic knowledge and, partly corresponding, pairs with relation to pragmatic knowledge.
This outline of the theory of instrumental genesis reveals the underlying French framework: the scheme concept, encompassing utilisation schemes and instrumented action schemes, was introduced by Vergnaud (Trouche, 2005, p.149). Since the mental utilisation schemes are not directly accessible for study and analysis, the concept of 'instrumented techniques' is of special interest: instrumented techniques are the external, visible and manifest parts of the instrumented action scheme. Still, an instrumented technique involves conceptual elements, since the technique reflects the schemes. This leads me to the following two crucial conclusions:
- The study of a student's development and use of instrumented techniques is useful to enlighten the student's development of those instrumented action schemes, to which the techniques relate
- Development of mathematical conceptions cannot be studied if use of technology is considered separate from the student's other activities *TMME*, vol4, no.2, p.236
The first point stresses the importance of empirical studies of students' work. The second point opposes my research to the standpoint, that teaching may be performed independently of what tools the students have at their disposal. It is also in contrast to the view, that the influence of for example computer use can be overlooked as if it were just a matter of 'digitising the pencil case'. This is in line with what Jean-Baptiste Lagrange stresses in (Lagrange, 2005, pp. 131-132):
The traditional opposition of concepts and skills should be tempered by recognising a technical dimension in mathematical activity, which is not reducible to skills. A cause of misunderstanding is that, at certain moments, a technique can take the form of a skill.
Besides the very grouping of the pairs of perspectives, the construction of the 'tool - object' pair was another result of the impact of the theory of instrumental genesis. The pair composed by a 'tool' perspective on a mathematical conception and an 'object' perspective on the same conception is realised for example in problem-solving settings. The tool - object pair corresponds to Anna Sfard's process - object duality (Sfard 1991). The tool perspective on mathematical conceptions is opposed to the 'pure skill' viewpoint and it implies the technical dimension that Lagrange refers to in the above quotation. So, within the notion of perspectives, the term tool denotes mathematical processes, carried out by the student with his instrument to serve a concrete purpose. It should be remarked that the purpose of the activity makes the difference between the tool perspective and Sfard's process perspective, not the instrument. Activities that aim at construction of pragmatic knowledge, involves the tool perspective on the mathematical conception in question. If the purpose is construction of epistemic knowledge, the process perspective is involved. This tool perspective on mathematical conceptions is in accordance with Régine Douady's definition of concept as a tool:
We say that a concept is a tool when the interest is focused on its use for solving a problem. A tool is involved in a specific context, by somebody, at a given time. A given tool may be adapted to several problems, several tools may be adapted to a given problem. (Douady, 1991, p. 115)
Mathematical activities, then, must be considered from a tool perspective when they are considered as parts or elements of a technique in the above mentioned sense. To take the corresponding object perspective, genesis of the instrument is requested. Therefore, the instrumental genesis is in a crucial way linked to and maybe a prerequisite for the change to object perspective.
2.4. Perspectives and representations, referred to in the definition of flexibility
The pairs of perspectives form three groups: perspectives intrinsic to mathematics, and perspectives relevant for construction of epistemic- and pragmatic knowledge respectively. As it was mentioned above, the distinction between epistemic and pragmatic knowledge intended to take into account that the conceptual tool flexibility should resonate with the theory of instrumental genesis. All the changes between perspectives and between representations are bidirectional. Each pair of perspectives can be developed reflexively during the bi-directional changes. *TMME*, vol4, no.2, p.237
In the notion of flexibility the term representation means representation system or communication media in a functional sense. So, the representations function partly as the media for externalisations of internal conceptual systems (Lesh and Doerr, 2000, p. 364; Mousoulides, Sriraman & Christou, 2007) and resembling the use of the term in the KOM-report (Niss, 2002). There is no sharp distinction between the four representations. Especially, the technical representation in many cases widely overlaps with the others.
2.4.1. Perspectives intrinsic to mathematics
Local and global position
Local position and global position are intrinsic mathematical perspectives. For example, changes between them occur when a single member of a group of objects is picked out for closer examination or if a single object is incorporated into a collection or family of objects. Using CAS, this change is easily realised for example by the use of facilities as substitution and copy - paste.
General - specific
General status and specific status are intrinsic mathematical perspectives, which concern the domain of validity. Inductive reasoning is linked to changes from specific to general perspective, whereas deductive reasoning is linked to changes from general to specific perspective. The use of CAS allows a quick and easy generating of specific objects from general expressions or formulas.
Analytic- constructive
The analytic and constructive perspectives are well-known phases in working with geometry. Examples are measuring a given thing and construction of a figure with a given measure, respectively. Especially, changes between these two perspectives are of relevance in sequences of modelling at a functional level.
2.4.2. Perspectives linked to construction of epistemic knowledge
The process - object duality.
The process perspective is operational and the object perspective is structural. The distinction and the connection between these two perspectives are encompassed by the reification theory. In contrast with the viewpoint of reification theory, the students' mental activities captured in the notion of flexibility, though, may go in both directions and thereby support the development of both perspectives.
Situated - decontextualised
The situated perspective is used to capture concrete aspects of problems, and of handling actual problems and challenges. The decontextualised perspective is the result of extracting rules, aiming at internal stability, which can occur through abstraction. Changes between situated and decontextualised perspective are linked to reflections, but not necessarily to modelling at the functional level. Therefore, this pair of perspectives relates to formation of epistemic rather than pragmatic knowledge. Changes from situated to decontextualised perspective correspond to the raising from referential to general level in the four-level model described above. *TMME*, vol4, no.2, p.238
2.4.3. Perspectives linked to construction of pragmatic knowledge
The tool - object duality
The tool perspective
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focuses on the use of a mathematical conception for solving a problem in a specific context, with some specific aids. The corresponding object perspective means the distant overview-perspective on the tool. Flexibility in this case encompasses the student's distant overview over a collection of mathematical tools, besides his capability to change from the distant over-view perspective into a tool perspective in the actual context.
Model - reality
The changes between reality and model perspective result in level-raising from the first to the second level in Gravemeijer's four-level model. In a number of cases, reality is expressed in natural language, and changes from reality to model then happens in connection with a change of representation from natural- to formal language.
Model of - model for
The changes between model of and model for perspective result in raising from the referential to the general level in Gravemeijer's four-level model. The change is exemplified in the case of calculus by K. Gravemeijer and M. Doorman (Gravemeijer & Doorman, 1999, p.111).
2.5. Representations
The term representation is used in a very broad sense. Consequently, changes between the four representations may occur in different contexts, and happen at a variety of levels. As part of the students' modelling and problem solving activities, the communication media may for example change at a functional level. In contrast, in other cases a single change to natural language may involve cognitive challenges like interpretation of a graph or ascribing meaning to a symbol.
The four representations do not intend to form any classification of media for expression of mathematical conceptions. In line with this, computer language is included as a representation on its own although it overlaps with the other representations, because changes to and from computer language is one part of the instrumental genesis.
Analytic representation (formal language)
Analytic representation includes formal expressions and formulas, algebraic expressions and symbols. Changes to analytic representation are related with symbolising whereas changes from analytic representation are connected with interpretation of symbols and the ascribing of meaning to symbols. The changes may cover more or less complex actions: For example, a shift to formal language may imply a routine translation from the graph of a linear function into the corresponding formal expression. Or it may describe the level raise from situated to formal level or from referential to general level. During modelling processes, the changes to formal language often happens stepwise, passing stages of partly formalised in-between-expressions.

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Here, the term tool is used in a broad meaning, not synonymous with artifact in contrast to instrument like in the theory of instrumental genesis *TMME*, vol4, no.2, p.239
Graphical representation
Graphic representation includes graphs, curves, diagrams and tables, drawings etc. and explanatory gesticulation. Changes to and from graphic representation may be the result of for example the student's construction of a spatial conception of a curve that corresponds to the formal expression of a mathematical conception. Though, the curved line in the calculator's window, created by one press on the button is regarded as a graphic representation of the

conception in question as well.
Natural language
The representation natural language encompasses spoken as well as written expressions and includes talk, explanations and negotiations, some texts written by the students or the by teacher, and some textbook texts.
The teacher's assessment has an impact on the communication in the classroom: depending on the social norms in the classroom, the students' can learn standard phrases of explanation by rote. The teacher's assessment of the student's understandings often relies on the student's capability to handle the content "in his or her own words". Standard phrases resembling own words, of course, can still be learned by root. Nevertheless, in the notion of flexibility such standard phrases are considered natural language. Natural language can overlap with formal language when technical terms are used, for example in group discussions.
The changes to natural language in many cases cover complex processes of interpretation and construction of meaning.
Technical representation (computer language)
Changes to technical representation encompass the translation from other representations into the version, adapted to the computer or calculator in question. Changes to computer language span from simple routines where students choose a well known instrument, to complex processes of instrumental genesis. Expressions in the other representations and changes between them are mediated in computer language: graphs, formal expressions and natural language can all be expressed in technical language too. In some cases, further, the software allows simultaneously use of formal language and graphic representation.
3. Methods and modes of the research
The initial construction of flexibility was based on teaching experiences, a preparatory classroom study and literature studies
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from the World Class A project referred to in the evaluation interviews, and partly second-hand ones, collected over years in informal talk and discussions with teacher-friends and colleagues.
Subsequently, I carried out an empirical study which aimed to
- Identify signs which could indicate flexibility in the students' mathematical conceptions
- Inquire how the teaching, the task, the teacher's questions etc. provoked the students to demonstrate flexibility
- Interpret the role of flexibility for the students' further working with mathematics
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The research design, the empirical studies and inquiries and their relations and roles in the design are presented and discussed in (Andresen, 2006, chapter 6) *TMME*, vol4, no.2, p.240
The outcome of the study appeared at different levels. For example, one of the overall conclusions was that flexibility of the students' conceptions in general was prerequisite for acting with competence in the KOM sense (Niss, 2002). In several concrete cases I concluded that the use of certain software commands enhanced the flexibility of the students' mathematical conceptions. Finally, at the level of analysis I concluded that the conceptual tool flexibility was useful to throw light on the students' process of getting used to computer use.
Four small-scale teaching experiments served as cases for the empirical study. The participating teachers designed teaching sequences on differential equations for the experiments based on shared written materials in the form of a booklet, prepared by a group of teachers from the development project (Hjersing, N., Hammershøj, P. and Jørgensen, B. 2004). For the teachers, the aim of the teaching experiments was to develop good practices of teaching differential equations from a dynamical systems point of view. The booklet focused on modeling with a problem solving approach.
Data from the teaching experiments had the form of field notes and film recordings from classroom observations, students written reports, group interviews with teachers and students, and teaching materials. I observed fifty lessons spread over the four classes with 22, 12, 10 and 6 lessons during the winter 2003-2004. The lessons were chosen, restricted by practical circumstances. The four teachers who had voluntarily agreed were very helpful and obliging supportive. Thanks to them, the students were also helpful and open-minded.
I observed almost all lessons on the subject differential equations, taught by one of the booklet's teacher-authors. The goals were: i) to study the authors' overall intensions with the modelling approach to the subject, ii) to qualify my reactions on the booklet to the authors and iii) to inquire the use of laptops in class room teaching. In one other class, my observations focused on the modelling aspect. In the last two classes, I observed students' group work on project tasks. It was my hope that the dialogue and negotiations between the students in the groups would reveal signs of their learning process at a closer and more personal level, compared to the classroom observations.
Data were studied during interpretative analysis, taking social and psychological perspectives into account. The analysis followed the approach, designed by Paul Cobb et al. to meet the following three criteria (Cobb, Stephan, McClain & Gravemeijer 2001 p 116):
- Enable documentation of the collective mathematical development of the classroom community over the extended periods of time covered by instructional sequences
- Enable documentation of the developing mathematical reasoning of individual students as they participate in the practices of the classroom community
- Result in analyses that feed back to inform the improvement of the instructional designs.
4. Episode from a case study
The rest of this paper presents an excerpt from one of the Ph.D. project's nine cases. The objective of this part of the research was to inquire into the hypothesis: the construct 'flexibility' may capture important elements of learning activities, like they were described in the first parts of this paper. The excerpt is the larger part of one (the second) of the case's three episodes. It intends to illustrate the analyses of data following the three main aims, listed above. *My* analysis *TMME*, vol4, no.2, p.241
of the episode mostly concentrates on the changes between model of perspective and model for perspective and on the inquiry of the model for perspective as it appears in two groups of students' work. Obviously, other perspectives and representations are involved and might have been chosen as the object of analysis as well. In the thesis, other cases concentrate on other perspectives and representations
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4.1. Case 8, chemical reactions
The theme of case 8 was changes between model of perspective and model for perspective. This case encompassed episodes 8.1 to 8.3. The main aim of the analyses of the case was to inquire the model for perspective as it appeared in some of the students work. Two groups of students (group 9 and group 11) in two different schools worked with the same project task. The project task concerned with exploration of differential equations' models of the rate of chemical reactions of order zero, one and two. The episodes in case 8 were based on the project task, the written reports from the two groups of students, a transcription of film recordings of one of the group's (group 9) work with the project task during one lesson, and my field notes from the same lesson. The project task took the model of chemical reactions as its starting point
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. After a few
introductory statements concerning the technical handling of the amounts and concentrations of the compounds, for example in the case of precipitation of a compound, the model of the reaction rate was introduced in the task:
Fig 2. Excerpt from page 51 in (Hjersing et al. 2004), (author's translation)
The task's design intended to encourage the student's explorations of this model, aiming to see special cases and recognise them as models of certain reactions of order zero, one and two. The students in the two groups were taught these topics in chemistry in advance. In their study program, they all combined high-level chemistry and high-level mathematics. In the textbook the special cases of the model were deduced from the general expression and then treated mathematically. In that sense, the project task dealt with changes between model of perspective with reference to the chemical setting and model for perspective with reference to the more general model in the mathematical setting. In the following, only episode 8.2 is considered.

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See the overview over Cases and episodes p. 202-204 in (Andresen 2006)
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For an introduction to the rates of chemical reactions and equilibrium see H. F. Holtzclaw, W.R. Robinson and W.H. Nebergall (1984): General chemistry, D.C. Heath and Company, USA pp 407-449 or P.W. Atkins (1990): Physical Chemistry. Oxford University Press pp775-810
Theoretically, the rate of chemical reactions in general is expressed as:
The rate of combination of two or more chemical compounds is proportional to the product of their concentrations
The relation is expressed like this:

[ ] [ ] [ ]
dc x y z
Rate k A B C
dt
= = ∓
(1.1)
That is, the rate of production/consumption of C is proportional to the concentration of the reactants ([A], [B], [C]) raised to the power of x, y and z respectively. The degree of the exponent denotes the order of the reaction. *TMME*, vol4, no.2, p.242
4.2. Episode 8.2: group 9 answering question 3
When the episode took place, two students from group 9 worked with the chemical reactions project whereas the third member of the group was absent. At this time the students had passed the first one and a half page of the task's text, where the reaction rate was modelled for reactions of the different orders. The two students worked with page 53 in the booklet:
Fig3. Page 53 from the booklet (Hjersing et al. 2004), author's translation
Irreversible second order reactions
Consider the irreversible reaction A+ B X Y → + .
One molecule A and one molecule B combine to one molecule of each of the compounds X and Y. The rate of consumption of A and B equals the production rate of X and Y.
We have inquired second order reactions in the simple cases, where the initial concentrations of the reactants were equal. But what happens, if [A] does not equal [B] from the beginning, or if some X or Y is already produced?
This time, we will model the production of X. The differential equation, mentioned earlier, now changes into:

dx
kab
dt
=
(1.6)
We now want an expression on the right side, which only depends on the immediate [X], and on some initial values of [A] and [B]. So the goal is to express a and b as functions of x.
The immediate concentration of a equals: The initial concentration of a (a0) minus the 'new' x. The 'new' x equals the actual concentration of x minus the initial concentration of x (x0).
Similar to [B], so all in all gives (1.6) the equation:

0 0 0 0
(( ))(( ))
dx
k a x x b x x
dt
= − − − −
(1.7)

0 0 0 0
( )( )
dx
k a x x b x x
dt
= + − + −
(1.8)
Example
2 mol silver nitrate AgNO3 is mixed with 3 mol hydrochloric acid HCl. White silver chloride precipitates and the reaction runs completely. In this case there is ½ mol silver chloride when the reaction starts. The rate constant k is 1.
$AgNO_3 + HCl \rightarrow AgCl + HNO_3$

Based on the text we state the following:
k = 1
a0 = 2
b0 = 3
x0 = ½
Then, the differential equation is:
1 1
1(2)(3)
2 2
dx
x x
dt
= + − + −
(1.9)
Task3
Find the equilibrium points, where the rate equals zero for the differential equation in (1.8) and explain, what this means in practice *TMME*, vol4, no.2, p.243
In terms of flexibility, (1.1) gives a model for perspective since the level of the equation is general, whereas the perspective of (1.9) is model of because the level is referential – it refers to the actual experiment in the task.
Therefore, (1.8) can give both perspectives on the equation, depending on which of the two others it relates to:
(1.8) in model of perspective with regard to (1.1)
The model (1.8) is a general application of (1.1) to the case of a second order irreversible reaction with two reactants.
Fig 4. Explanation of (1.8)
So, with regard to (1.1) (1.8) serves as a model at referential level, referring to second order reactions. The perspective of (1.8), then, is model of whereas the perspective of (1.1) is model for.
(1.8) in model for perspective with regard to (1.9)
The model (1.9) is an application of (1.8) to the case of the reaction between Ag
+
and Cl

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where
a stands for the concentration of Ag
+
(or AgNO3) and b stands for the concentration of Cl
-
(or
HCl). x stands for the concentration of AgCl and dx/dt is the formation rate of AgCl. Since (1.9)
is concerning with the actual formation of silver chloride, (1.9) is even more concrete than (1.8).
So, (1.8) is a more general and decontextualised equation than (1.9). Consequently, in this case
(1.8) serves to give the model for perspective and (1.9) serves to give the model of perspective
on the equation.
The example and task 3 (in fig 3.) both aim to make the students change in both directions
between the general mathematical model (1.8.) and the model of the actual reaction (1.9).
Therefore, in terms of flexibility, they encourage to changes between model for and model of
perspective. These changes happened stepwise in the actual case.
In the written report, the answer from group 9 to task 3 (fig 3.) was:
In (1.8),
[X] = x
equals the concentration of the product,
[ ] 0 0
A a a x x = = + − ( )
equals the concentration of one of the reactants, and
[ ] 0 0
B = = + − b b x x ( )
equals the concentration of the other reactant. 0
a
is the
initial concentration of the compound a, similar for b and x. TMME, vol4, no.2, p.244
Fig 5. The students' answer
Apparently, the students linked the notion of equilibrium points with the roots of the polynomial.
In their text, they interpreted the term equilibrium point in natural language in three (slightly)
different ways, namely as
1) Points of the reaction where the reaction rate equals zero
2) Points of the reaction where the concentration is unchanged
3) A point where the reaction stops

In all three interpretations the equilibrium points were seen in a model of perspective. The
students made a change to formal language when they turned to the equation #35 (fig 5.) and
simultaneously to graphic representation, which was used in parallel (mixed in between to
analytic expressions) with the formal language.
- Where does the rate for the differential equation equal 0? Where the rate equals zero are
the equilibrium points.
To make the differential equation equal zero, one of the 3 factors in the differential equation
has to equal zero.

It is seen from the graph, that the equilibrium points are 2,5 and 3,5.
This means that the concentrations are unchanged in these points. From a mathematical
point of view, both these points are useful, but in practice only 2,5 is of relevance because
the reaction will stop here. TMME, vol4, no.2, p.245
The students explicitly interpreted the actual values of the roots as concentrations. This
interpretation revealed that they could handle the changes between equilibrium points in the
model perspective (referential level) to the concentrations in the reality perspective. Later on, the
students changed to a model for perspective on the equilibrium points, where they interpreted the
equilibriums as the roots of the polynomial. They referred to the distinction between referential
level (that is, model of perspective) and general level (that is, model for perspective) with the
remark "From a mathematical point of view" in contrast to practice, which was understood to be
at the referential level, that is, in a model of perspective.
By the students interpretation of the equilibrium point as a point where the reaction stops and
where the rate equals zero, they linked the conception of equilibrium closely to the conception of
rate. From the excerpt of text from the report (fig 5.), it is uncertain to say whether the students'
chemical conception of equilibrium was a dynamic or a static one. The third point, (3), above
and the following excerpts from the recordings of the students' work with task 3 may suggest
that the students perceived the conception of chemical equilibrium in a static way even if the
conception of rate was well described in their report's introduction (mentioned in the previous
episode
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). The students tended to confuse the rate and the rate constant:
S2: We look for the equilibrium point. Rate and rate constant is not the same, isn't it?
S1: The rate…
S1: Let us just…
S2: The rate, is it k or is that simply the rate constant
S1: It is the rate constant
S2: Then I guess we have to isolate k
In this way, then, the students were tempted to completely reduce the complexity of the problem.
The dialogue revealed that they might doubt whether the equations (1.1) (fig 2.) , (1.8) and (1.9)
(both fig 3.) modelled the reaction rate mathematically (general level) as well as chemically
(referential level). According to my interpretation, hence, what confused them was the
combination of mathematical and chemical models. Combining them is the very issue of
changing between model for perspective and model of perspective in this case.
The dialog continued:
S1: No, because the rate is zero
S2: Then we simply have to write it…
S1: I do not understand what we are supposed to do. (reads) "The equilibrium points where the
rate is zero"
S2: Where equal amounts are being produced. Where the amount left is the same as what is
produced, isn't it?

Here, S2 referred to the chemical conception of dynamical equilibrium. In the next remark, S1
interpreted this involvement of a chemical conception as a change from (mathematical) model
perspective to (chemical) reality perspective, but S2 maintained the model perspective by
referring to theory:

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Episode 8.1 page 251-252 in (Andresen 2006) TMME, vol4, no.2, p.246
S1: This is explaining what it means in real world, isn't it?
S2: But didn't we learn theory about this in chemistry?
S1: I do not understand what they… when the rate is zero, is it simply the rate constant?
The students called the teacher who referred to page 16-17 in the booklet, where the equilibrium
solutions were considered:
Fig 6. Excerpt from (Hjersing et al. 2004, p.16), author's translation
S1: (reads) the right hand side of the equation is a polynomial of second degree, find the roots,..
T: So what is this?
S1: I am reading… Equilibrium solutions – that is what we are looking for
T: Equilibrium solutions, what is it then?
S2: It was negative and positive second degree
The problem had now been transformed to the simpler one of finding the roots of a polynomial
of second degree. This problem was expressed in graphic representation, natural and formal
language simultaneously. The problem was seen in a model for perspective
14
as far as it had to be
solved in a purely technical context with no connection with the chemical model.
When the dialogue continued, the teacher tried to establish the connection between the simpler
problem and the question of equilibrium solutions to the differential equation. In terms of
flexibility, the teacher insisted on keeping the changes between model for perspective and model
of perspective in focus of interest. Shortly after, the episode was concluded with the teacher
having accomplished the guidance of the students. The students went to the computer room for
doing some graphs.
4.3. Conclusion of case 8
The changes between model of and model for perspective were in this case realised as changes
between a mathematically formalised chemical model and a more general mathematical model. It
was understood in the project task's text that the students knew the formalised chemical models
in advance. So, the students were supposed in advance to manage changes between the chemical
reality, in the form of proceedings of the actual reactions, and the formalised chemical models. In
the episodes, the changes between the two perspectives went in both directions: the report from
group 11 for the major part took the model for perspective (illustrated in episode 8.1), but the

14
This situation illustrates a tool perspective on the solving of quadratic equation
1
dP P
k P
dt N
( )
= - - − | |
( )
(…)
The right side of the equation is a polynomial of second degree in P with the roots
P t( ) 0 =

P t N ( ) =

It is positive between the roots and negative outside. TMME, vol4, no.2, p.247
task's questions provoked changes to and fro a model of perspective and further interchanges
between model and reality perspective (it was demonstrated in episode 8.3). The report from
group 9 divided into parts that took the model of perspective and other parts that took a model
for perspective. The links between the two perspectives, though, were not always clearly
revealed (it was shown in episode 8.3).
Episode 8.2 demonstrated how the teacher guided the students to change between model of and
model for perspective. The episode pointed to the important role of the guidance from the
teacher, as well as from the task's questions.
During the complete case, changes between reality and model perspectives and between model
of and model for perspectives were realised on the relation, modelled by the differential
equations (1.1), (1.8) and (1.9). The same relation was expressed in all the four different
representations. According to the definition of flexibility, the individual student's flexibility of
the conception of this mathematical relation (e.g. the differential equations model), in this
context, was developed to the extent that he or she was able to manage these changes. Working
with the tasks in the case seemed to support this flexibility since the individual student was more
likely to manage the changes after the teaching sequence.
5. Conclusion
To conclude this paper three questions are discussed: what contribution does the construction of
flexibility add to the field, are the promises, criteria and requests from the first part of the paper
kept, and what questions does this paper not answer?
5.1. What is new?
The introduction of a new construct like flexibility with its definition and technical notion to the
field of math education, which may seem almost overloaded with a diversity of notions and
terms already, has to be justified. One important argument is that the introduction of flexibility
summarises, connects and simplifies key elements of well established and acknowledged theory.
The issue of visualisation, for example, is included in flexibility in terms of change to graphic
representation. The relation between the levels in Gravemeijer's model is another example. The
novel idea beyond the construction of flexibility is to focus on the dynamics as a common
denominator, which leads to consider all these key elements in a new light and in a new
combination.
The case illustrates how flexibility intends to serve as a tool for clarification: The case's task
presented a rich and complex structure of models at different levels, and links between them. The
task was suitable for the students' exploration of the relation, modelled by the differential
equations, exactly because it was sufficiently complex to offer possibilities of open ended
inquiries. During the case, one of the students' main difficulties concerned the changes between
the levels of the model, represented in the three equations. Another main difficulty was the
transformation of the problem to the simpler one of finding the roots of a second degree
polynomial, followed by interpretation of the result. These main difficulties are closely related to
the main learning potentials of the task, so attempts to avoid them by omitting parts of them for
simplification would be of little use. The analysis of the case demonstrates how these main issues
of the task can be interpreted in terms of flexibility. My claim is that the teacher could strengthen TMME, vol4, no.2, p.248
the guidance of the students and make it more explicit without giving the answers, if he or she
had such an interpretation in mind.
The case also illustrates that the apparent conflict of having the equation (1.8) at two different
levels is cancelled when focus, in terms of flexibility, is on the dynamics of changing between
levels rather than on placing conceptions at the levels.

5.2. Does flexibility keep the promises?
According to the initial goal of the research mentioned in this paper, flexibility should serve as a
tool for teachers to take the project's experiences into account for improvement of their own
teaching. The case gives an example of flexibility's potentials for improved guidance, according
to my interpretation. To meet the goal, though, more focused teaching experiments, based on an
elaborated description of the single elements of flexibility would be the next step.
The second request, which concerns the contribution to math education theory, was already
discussed in a previous paragraph. The objectives of the construction set two demands on
flexibility: signs of flexibility should be observable, and claims of its relevance should be
theoretically founded. The construction of flexibility intended to capture important learning
activities which promote the student's actual work with mathematics in an observable way. The
case demonstrates that the changes of perspective and changes between different representations
are observable. The presentation in this paper of three basic ideas beyond the concept of
flexibility serves to justify
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the claim, that the changes promote the student's learning.
5.3. What is left?
It follows from the research design that the effect was not tested on teaching designs, which aim
at development of flexibility of the mathematical conceptions in question. A number of
guidelines are presented to conclude the Ph.D. thesis. The guidelines are meant for teachers who
want to aim at flexibility in the design of their teaching. A large scale inquiry of the effect could
include teaching materials and teaching designs based on these guidelines, pre- and after tests
and qualitative evaluation. The guidelines are (Andresen, 2006, pp. 294-295):

Teaching that aims to support flexibility in the mathematical conceptions with and without the use of CAS should be based on the following principles:
- The design of tasks and problems ensures that changes of perspective go in both directions in all the pairs of perspectives that form "flexibility":
1. Local – global
2. General – specific
3. Analytic- constructive
4. Process – object
5. Situated – decontextualised
6. Tool – object
7. Model – reality
8. Model of – model for
- The design of tasks and problems ensures that changes between representations go in both directions between graphic representation, analytic representation (or formal language), natural language and technical representation (or computer language)

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Further justification builds on the discussions all through chapters 5 and 7 in (Andresen, 2006). TMME, vol4, no.2, p.249
- Expressive work and explorative work with mathematical models are both important. The teaching is designed to ensure that the students over time are encouraged to model a number of key conceptions including all the four levels situated, referential, general and formal.
- A diversity of strategies is not only accepted, but appreciated in the classroom. The students are encouraged to try out ideas and techniques. Results, ideas and strategies are discussed and negotiated with open minds in the classroom.
The relations between flexibility and the shortcuts mentioned in the first part of this paper, and the role of flexibility in the instrumental genesis, apparently, are issues in focus of interest for the continued work with development and refinement of the construct flexibility. It should be remarked that although the project took the use of laptops as its starting point, flexibility is not restricted to mathematical conceptions within a computer environment.
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P 2: 10 Doorman Boon Tooluse supporting.pdf

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Codes (13)###design of activities, ###guided reinvention, ##real life, #RME, $$$modelling, $$maths strategies, $factors that influence teaching and learning, $mathematics, $mathematics content, %role players, @@ICT systems: virtual learning environment (VLE), @Devices, @Tools

Memos (0)

Document body:

Tooluse supporting the learning and teaching of the Function concept
Paper for ISDDE2008
Michiel Doorman
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+
, Paul Drijvers
+
, Sjef van Gisbergen
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1. Introduction
Tool use is indispensable both in daily life as well as in doing and learning mathematics. Research suggests a close relationship between tool use, cognitive development and social practice. The TOOlUse-project at the Freudenthal Institute focuses on the use of computer tools for grade 8 (13-14 years) students' acquisition of the mathematical concept of function, in an effort to identify the relationship between the use of technological tools and the learning of mathematics
1
. The main research question aims at understanding how computer tools can be integrated in an instructional sequence on the function concept, so that their use fosters learning. This is studied in terms of the notion of instrumental genesis (Kieran & Drijvers, 2006, Trouche, 2004) and domainspecific theories on the teaching and learning of mathematics within the philosophy of realistic mathematics education (Freudenthal, 1991). The research setup includes a cyclic process of instructional design, teaching experiments and data-analysis (Gravemeijer, 2004).
2. Theoretical framework
We investigated an instructional approach for the function concept that aims at a learning process in which the mathematics is build upon everyday intuitions. This is also in line with the objective of realistic mathematics education (RME), where instructional design is aimed at creating optimal opportunities for the emergence of formal mathematical knowledge from situation specific reasoning. In order to achieve this, contextual problems are cast into situations that are experientially real for the students. During the process of teaching and learning students can preserve the connection between the mathematical concepts and what is described by these concepts. Students' final understanding of the formal mathematics should stay connected with their understanding of these experientially real, everyday-life phenomena (Freudenthal, 1991).
Well-chosen context problems offer students opportunities to develop informal, highly context-specific models and solving strategies (Doorman et al., 2007). These informal solving procedures then may become subject of formalization and generalization to constitute a process of further abstraction, in RME dubbed as: progressive mathematization. The instructional designer tries to construct a set of problems that can

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The project is supported by the Netherlands Organisation for Scientific Research (NWO) with grant number 4121-04-123. The project's website is http://www.fi.uu.nl/tooluse.
'lead to a series of processes that together result in the reinvention of the mathematics aimed at.
Research on the design of primary-school RME sequences has shown that the concept of emergent models can function as a powerful design heuristic (Gravemeijer, 1999). Context problems are selected, offering the students the opportunity to develop situation-specific representations and strategies. Students' reasoning gives rise to (informal) models during classroom discussions and subsequent activities. In this sense, the models emerge from the activity of the students. Even if the models are not actually invented by the students, great care is taken to approximate students' inventions as closely as possible by choosing models that link up with their learning history and with their contributions. Another important criterion is in the potential of the models to support mathematization towards the intended concepts. The idea is to look for models that can be generalized and formalized to develop into entities of their own, which then can become models for mathematical reasoning (e.g. Doorman, 2007; Gravemeijer, 2004; Rasmussen & Blumenfeld, 2007).
An additional theoretical foundation of the research we discuss here is the use of computer minitools in mathematics education. In general, tools influence the process of students' mathematical sense making. Cobb (1999) illustrated this by describing the interplay between students' ways of symbolizing and mathematical reasoning. In relation to this he referred to the notion of affordances (Cobb, ibid). The computer minitools afforded the students' reasoning on statistical problems. Affordances are not properties of tools that exist independently of users, but are achievements of users in activity. As a consequence, instructional designers should take into account how students might act and reason with the tools as they participate in a sequence of mathematical practices.
We adopted these ideas for integrating a computer tool in the instructional materials for the teaching and learning of the function concept. The tool was meant to create opportunities for students to develop understanding of the input-output relationship and the related dynamics in graphs and tables.
The instructional design: AlgebraArrows
In the design phase of the project, teaching materials and a computer tool were developed. The computer, AlgebraArrows (AA), primarily supports the construction of input-output chains of operations (Boon, 2008). These chains can be applied to single numerical values as well as to variables. In the latter case, tables, formulas and (dot) graphs can be shown. The chains can be extended, linked, compared and compressed. The applet is embedded in an electronic learning environment, allowing students to save final situations of each activity and to continue working in any location with internet access. The researchers, as well as the teacher, are able to monitor the answers of the students with all-class result overviews.
We expected the starting function image for most of the students to be the local calculation procedure. This could be applied to one single input number at the time and then result in an output number. The instructional design aimed at the transition from these calculations to reasoning with a formula and its graphs and tables. The goal of the learning arrangement with the computer tool was to keep the chaining of calculations connected with formulas during the whole instructional sequence. The sequence started 2with three open-ended activities (mobile phone offers, a quadrilateral and breaking distance) to reveal the students' thinking and to provide for strategies that could be used by the teacher to introduce the computer tool AlgebraArrows. Both the purpose of and the representations in the tool were object of discussion (see Figure 1).

Figure 1 Poster of students' calculations and a ´living operation chain´

The chain of operations in the AA applet, applied to a single numerical value, signifies the function as an input-output assignment signifying a list of calculations. This is an important step for the function concept, because it supports the importance of identifying the relevant variables and their dependencies. The step is certainly not trivial for the students. During a pilot experiment we noticed that, even after a few activities with AA, many students wanted to translate a formula like 2·x + 100 into a chain starting with input 2.
In Figure 2, two chains concerning different mobile phone offers invite a global investigation of properties. The technique of putting numerical values in the input window allows for comparison of function values. A variable input allows for a more global comparison, in which graphs, tables and formulas can be involved.

3Figure 2 An input – output assignment
The applet allows not only a change of input value, but also the study of the covariation by means of tables and graphs. The chain in Figure 3 represents the cost of one particular mobile phone offer. The dynamics of the co-variation can be investigated by techniques of substituting different values, scrolling through the table, tracing the graph and studying the formula. The task for the students is to find the change of cost per minute change of the input variable – the number of minutes of phone calls.
Figure 3
4The computer tool AA was supposed to support students' investigations of many cases. The investigations by pairs of students were expected to foster a trial-and-improveprocess and negotiation of meaning. During the investigations they could find patterns and develop strategies in dealing with dynamic input-output dependencies. Consequently, AA was expected to afford a means to structure inquiry about these relationships through the construction and modification of arrow chains, graphs and tables, and the generation and comparison of argumentations.
The conjectures concerning the role of AA in the instructional sequence were investigated with a design research method of iterative phases of instructional design, teaching experiments and data analyses.
Results
The teaching experiments took place in three classes at three different schools. Video recordings were made of classroom teaching and group work (17 hours), and screen videos of three pairs of students working with the computer were captured (6 hours). Students' final answers on the activities with AA were saved on a central server (41 pairs). We collected written work of the students as well as the results of a written assessment at the end of the experiment. Data analysis was carried out by qualitative analyses and coding of the data using the video analysis tool ATLAS.ti.
We started the analysis by watching and annotating the work of a pair of students (Lisa and Romy). We had screen videos (with audio) of the three lessons with the computer tool. These screen videos were imported in ATLAS.ti. Initially, the clipping was guided by exercises as units of analysis. We described each exercise-clip with the comment-tool and added analytical notes. With these notes we wrote down a storyline of Lisa and Romy and their learning process with respect to the learning environment and our prior expectations. The clips were coded with the number of the activity as well as with their names. These codes could be used to browse through the data and to combine it with other data sources. We also imported and coded photocopies of written work and of screenshots of the computer activities of all the students which were saved on the central server.
We expected the two students to initially use the calculation chains in their reasoning about input-output-dependencies, and that this reasoning would change to using tables and graphs as basic structuring elements. These expectations were based upon the idea of emergent modeling, which is a characteristic of realistic mathematics education. One of the key aspects of emergent modeling is the development of a model and the ways of reasoning with the model through a sequence of manifestations (Gravemeijer, 2004)
In the teaching experiment we provided the students with one tool for a sequence of ten lessons. The manifestation of the chain itself did not change, but the function of the representation changed. As a result, we didn't see a clear development through a number of manifestations of a model, whereas the focus of reasoning with the model shifted from individual calculations to dependency relationships. Consequently, we noted in the work of Lisa and Romy that they reasoned with calculation chains during all the activities. Moreover, their reasoning changed from referring to individual calculations to characterizing dynamic processes. This storyline was discussed among the project 5members to try to understand these findings and the relation to the instructional sequence. During the discussions we successively clarified and assessed possible conjectures with the storyline and the references to the clips.
As a result of this analysis the idea arose that we could better use the principle of a form-function-shift (Saxe, 2002) for describing what happened in the learning process of Lisa and Romy. The arrow chain initially built on the notion of recording the structure of a series of calculations. The form-function-shift took place when the two students started using this record of calculations as a tool for investigating dependencyrelationships. Simultaneously, as the investigations with arrow chains built on the imagery of calculations with successive operations, constructing and modifying an arrow chain signified organizing a dependency-relationship. Calculation-routines turned into explorations, during which new narratives about relationships were constructed. The articulation of this shift seemed helpful in labeling different phases in the learning process with codes.
We conjectured that in this form-function-shift the goal of the activities for the students shifted from calculating a specific value to reasoning about relationships, and the chain is used as a means for scaffolding and communicating ways of reasoning (emerging mathematical goals: Saxe, 2002). Next, based upon the analysis of the work of Lisa and Romy, we identified two illustrative and rather similar open-ended activities, one in the beginning of the sequence and one in the end. For these activities we characterized different solution strategies with the notion of the form-function-shift and constructed four codes for these characterizations (from "a record for calculations" to "as a tool for reasoning"). All the screenshots of the students were coded in order to try to get quantitative support for the emerging conjecture. During the coding process we discussed a few cases for fine-tuning the code-definitions and the interpretations of the codes.
The result of this second analysis offered evidence for the conjecture of a formfunction-shift in the students' reasoning with chains of calculations in the computer tool. We found that with the activity in the beginning 25 (out of 41) pairs of students used the chain as a record for calculating a specific value, while the activity at the end of the sequence showed that 24 (out of 34) pairs used the chain as a tool for reasoning. However, during the coding we also observed that with the first activity students sometimes tried to use AA as a tool for reasoning, but were not able to really operate it sufficiently. This observation brought us back to a prior notion of the theoretical framework that tool-use development is closely related to a process of instrumental genesis.
Further research is needed to merge the notions of emergent modeling and formfunction-shift with the notion of instrumental genesis to be able to better understand the interplay between tool use and mathematics learning.
Moreover, in the classroom learning processes, we noticed that students initially reasoned and wrote about mathematical and kinematical notions with a tentative language and inscriptions that were not as precise as the notions aimed at. This seems similar to Goldin's description of three main stages in the development of representational systems: (i) an inventive and semiotic stage, (ii) structural development and establishment of relationships, and (iii) an autonomous stage (Goldin, 2003). During the inventive stage, students tentatively used inscriptions and language to communicate their developing ideas. In the autonomous stage the system can function flexibly in new contexts. We observed that the teacher played an important role during this process. The teacher guided 6whole class discussions and alternated between evoking students to present their solutions and focusing to the mathematics needed for what is to come (Doorman, 2007; Sherin, 2002).
Redesign and further research should also reveal the criteria by which the teacher can make decisions during this balancing act of guiding students through a process for developing the function concept.
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P 3: 11 Drijvers Digital technology.pdf

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Codes (11)##real life, $$mathematics education, $factors that influence teaching and learning, $mathematics, $mathematics content, %role players, &pedagogy, &TPCK, @@ICT systems: virtual learning environment (VLE), @@teacher-student ICT factors, @Devices

Memos (0)

Document body:

Drijvers, P. (2013). Digital technology in mathematics education: why it works (or doesn't). PNA, 8(1), 1-20.
DIGITAL TECHNOLOGY IN MATHEMATICS EDUCATION: WHY IT WORKS (OR DOESN'T)
Paul Drijvers
The integration of digital technology confronts teachers, educators and researchers with many questions. What is the potential of ICT for learning and teaching, and which factors are decisive in making it work in the mathematics classroom? To investigate these questions, six cases from leading studies in the field are described, and decisive success factors are identified. This leads to the conclusion that crucial factors for the success of digital technology in mathematics education include the design of the digital tool and corresponding tasks exploiting the tool's pedagogical potential, the role of the teacher and the educational context.
Keywords: Didactical function; Digital technology; Instrumentation
La tecnología digital en educación matemática: por qué funciona (o no)
La integración de la tecnología digital enfrenta a profesores, formadores de profesores e investigadores a muchas preguntas. ¿Cuál es el potencial de las TIC en el aprendizaje y la enseñanza, y qué factores son determinantes al trabajar en clase de matemáticas? Para investigar estas cuestiones, se describen seis casos de estudio prominentes en el área, y se identifican los factores decisivos para el éxito. Esto lleva a la conclusión de que los factores cruciales para el éxito de la tecnología digital en la educación matemática incluyen el diseño de la herramienta digital y de las tareas apropiadas que exploren el potencial pedagógico de la herramienta, el papel del profesor y el contexto educativo.
Términos clave: Función didáctica; Instrumentación; Tecnología digital
For over two decades, many stakeholders have highlighted the potential of digital technologies for mathematics education. The National Council of Teachers of Mathematics, for example, in its position statement claims that "Technology is an essential tool for learning mathematics in the 21st century, and all schools must ensure that all their students have access to technology" (National Council of Teachers of Mathematics [NCTM], 2008). ICMI devoted two studies to the 2 P. Drijvers PNA 8(1)
integration of ICT in mathematics education, the second one expressing that "…adigital technologies were becoming ever more ubiquitous and their influence touching most, if not all, education systems" (Hoyles & Lagrange, 2010, p. 2).
However, the integration of digital technology still confronts teachers, educators and researchers with many questions. What exactly is the potential of ICT for learning and teaching, how to exploit this potential in mathematics education, does digital technology really work, why does it work, which factors are decisive in making it work or preventing it from working? What does a quarter of a century of educational research and development have to offer here?
Of course, these questions are not clearly articulated. What do we mean by "it works"? Does this mean that the use of digital technology improves student learning, invites deeper learning, motivated learning, more efficient or more effective learning? Does it mean that ICT empowers teachers to better teach mathematics? And, concerning the effect of educational research, do studies on digital technology work in the sense that they provide answers to these questions, or do they just help the researcher to better understand the phenomenon, and as such contribute only indirectly to improving mathematics education? My interpretation of "why it works" in the title of this contribution includes both learning and teaching, and also refers to learning on the part of the researcher.
In this paper I will explore the question of "why digital technology works or does not" by briefly revisiting a number of leading studies in the field, that are paradigmatic for a theme, approach, method, or type of results. For each of these studies, the focus is on what they offer on identifying decisive factors for learning, teaching and research progress. As such, this contribution reports on a concise and somewhat personal journey throughor a helicopter flight overthe landscape of research studies on technology in mathematics education.
FRAMEWORK FOR CASE DESCRIPTION
How to decide which studies to include in this retrospective and even somewhat historical paper? Even if somewhat subjective and personal arguments cannot be completely ignored, the case selection is based on a number of criteria. A first criterion for including a study or a set of studies is that it really contributes to the field, by providing a new perspective, a new direction or is paradigmatic for a new approach to the topic. An indication for this is that the study is frequently quoted and has led to follow-up studies. A second criterion for inclusion is that the study under consideration contributes to theoretical development in the field of integrating technology in mathematics education, and as such promotes thought in this domain. A third and final criterion for the set of cases presented in this paper as a whole, is variation. Variation does not only apply to theoretical perspectives, but also to the mathematical topic addressed in the study, the type of technological tools used, and the pedagogical functionality of the digital tech-Digital Technology in Mathematics Education: Why it Works (or doesn't) 3 PNA 8(1)
nology. Concerning this functionality, we use an adapted version of the model by Drijvers, Boon, and Van Reeuwijk (2010) which distinguishes three main didactical functionalities for digital technology: (a) the tool function for doing mathematics, which refers to outsourcing work that could also be done by hand; (b) the function of learning environment for practicing skills; and (c) the function of learning environment for fostering the development of conceptual understanding (see Figure 1). Even if these three functionalities are neither exhaustive nor mutually exclusive, they may help to position the pedagogical type of use of the

technology involved. In general, the third function is the most challenging one to
exploit.
Figure 1. Didactical functions of technology in mathematics education
CASE DESCRIPTIONS
In this section we will discuss the selected studies in a short frame. To do this in
a way that does justice to them and in the meanwhile serves the purpose of this
paper, we first present a global description of each case, including the mathematical topic, the digital tool and the type of tool use. Next, I will explain what is
crucial and new in the study, and why I decided to include it. Then the theoretical
perspective is addressed. Each case description is closed by a reflection on
whether digital technology worked well for the student, the teacher or the researcher, and which factors may explain the success or failure.
Case 1: Concept-First Resequencing by Heid (1988)
The first case description concerns a study reported by Heid (1988), which is
considered as one of the first leading studies into the use of digital technology in
mathematics education. The study addresses the resequencing of a calculus
course for first-year university students in business, architecture and life sciences
using computer algebra, table tools and graphing tools that were used for concept
development (branch c, Figure 1). The digital technology allowed for a "conceptfirst" approach, which means that calculus concepts were extensively taught,
whereas the computational skills were treated only briefly at the end of the
Didactical functions
of technology in
math education
Do mathematics
(a)
Learn mathematics
Practice skills
(b)
Develop concepts
(c) 4 P. Drijvers
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course. The results were remarkable in that the students in the experimental
group, who attended the resequenced, technology-intensive course, outperformed
the control group, who attended a traditional course, on conceptual tasks in the
final test, and also did nearly as well on the computational tasks that had to be
carried out by hand. The subjects in the experimental group reported that the use
of the computer took over the calculational work, made them feel confident
about their work and helped them to concentrate on the global problem-solving
process.
One of the reasons to discuss the Heid study here is that it is paradigmatic in
its approach in that its results form a first "proof of existence": indeed, it seems
possible to use digital technology as a lever to reorganize a course and to successfully apply a concept-first approach, using digital technology in the pedagogical function of enhancing concept development, without a loss of student
achievement on by-hand skills.
From a theoretical perspective, Heid's notion of resequencing seems closely
related to Pea's distinction of ICT as amplifier and as (re-)organizer (Pea, 1987).
The former refers to the amplification of possibilities, for example by investigating many cases of similar situations at high speed. The latter refers to the ICT
tool functioning as organizer or reorganizer, thereby affecting the organization
and the character of the curriculum. In the light of that time's thinking on the role
of digital tools to empower children to make their own constructions (Papert,
1980), the organizing function of digital technology was often considered more
interesting than the amplification.
So did technology work in this case? Yes, it did at the level of learning: The
final test results of the experimental group turned out to be very satisfying. And
yes, it also worked at a more theoretical level, as the notions of resequencing and
concept-first approach were operationalised and made concrete. Now why did it
work, which factors might explain these positive results? Even if nowadays we
would not consider the digital technology available in 1988 as very sophisticated,
I would guess that at the time the approach was new and motivating to the students, and the representations offered by the technology did indeed invite conceptual development. Decisive, however, I believe was the fact that the researcher herself designed and delivered the resequenced course. I conjecture that she
was very aware of the opportunities and constraints of the digital technology, and
was skilled in carefully designing activities in which the opportunities were exploited, and in teaching the course in a way that benefitted from this. Whether the
course, if delivered by another teacher, would have been equally successful, is
something we will never know.
Case 2: Handheld Graphing Technology
The second case description concerns the rise of handheld graphing technology
in the 1990s. For several reasons, graphing calculators became quite popular
among students, teachers and educators at that time (for an overview, see Digital Technology in Mathematics Education: Why it Works (or doesn't) 5
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Trouche & Drijvers, 2010). Teaching materials were designed that made extensive use of these devices and researchers investigated the benefits of this type of
technology-rich activities (Burrill et al., 2002). Very much in line with the work
by Heid (see Case 1), the focus of much of this work is on the pedagogical function of concept development. The main idea seems to be that students' curiosity
and motivation can be stimulated by the confrontation with dynamic phenomena
that invite mathematical reasoning, in many cases concerning the relationships
between multiple representations of the same mathematical object. In many cases
this mathematical object is a function, but examples involving other topics, such
as statistics, can also be found.
As an example, Figure 2 shows two graphing calculator screens which students set up to explore the effect of changes in the formula of the linear functions
Y1 and Y2 on the graph of the product function Y3. This naturally leads to questions about properties of the product function and the relationship with properties
of the two components (Doorman, Drijvers, & Kindt, 1994).
Figure 2. Exploring the product of two linear functions
A paradigmatic study in this field is done by Doerr and Zangor (2000). The researchers report on a small-scale qualitative study, in which 15-17 year old precalculus students study the concept of function using a graphing calculator, with
a focus on the pedagogical tool functionality of concept development (branch c,
Figure 1). The authors identify five modes of tool use, namely computation,
transformation, data collection and analysis, visualisation, and checking. The results show that the teacher was crucial in establishing and reinforcing these
modes of tool use, for example by setting up whole-class discussions "around"
the projected screen of the graphing calculator, to develop shared meaning and
avoid a too individual development of tool use and mathematical insight. The researchers stress that using digital technology in mathematics teaching is not independent from the educational context and the mathematical practices in the
classroom in particular.
The main reason to discuss the Doerr and Zangor study here is that it highlights the importance of the educational context in studies on the effect of digital
technology, and the crucial role of the teacher in particular. The relevance of the
educational context has later been elaborated in the notion of pedagogical map by
Pierce and Stacey (2010). Concerning the teacher, she establishes a culture of 6 P. Drijvers
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discussing graphing calculator output in a format that is close to what is called a
"Discuss-the-Screen orchestration" in Drijvers, Doorman, Boon, Reed, and
Gravemeijer (2010) and by these means contributes to the co-construction of a
shared repertoire of ways to use the graphing device.
From a theoretical perspective, Doerr and Zangor (2000) use frameworks on
learning as the co-construction of meaning, and that the "features of a tool are
not something in and of themselves, but rather are constituted by the actions and
activities of people" (p. 146). Even if this may sound somewhat trivial nowadays,
during the period of initial enthusiasm these were important insights with consequences for the role of the teacher, who led the process of sharing and coconstruction, particularly in the case of personal, private technology.
So did technology work in this case? Doerr and Zangor did not assess learning outcomes, but it seems that the students developed a rich and meaningful
repertoire of ways to use the graphing calculator for their mathematical work.
Why did this work, which factors might explain these findings? My interpretation is that the use of digital tools for exploratory activities which target conceptual development is not self-evident, as it is hard for students, without the mathematical background that we as teachers have, to "see" the mathematics behind
the phenomena under consideration. It is here where the teacher comes in, and
where the study becomes very informative for both teachers and researchers. In
this case, I believe that the fact that the teacher herself was skilled in using the
graphing calculator, was aware of its limitations, and was willing to explicitly
pay attention to the co-construction of a shared and meaningful repertoire of tool
techniques explains the results. As in the Heid study described in Case 1, the role
of the teacher seems to be an important factor. The issue of how to deal with private, handheld technology is very relevant nowadays, as many students have
smart phones with sophisticated mathematical applications, and again, teachers
are faced with the danger of too individually constructed techniques and insights.
Case 3: Instrumental Genesis
By the end of the previous century, French researchers who were working on the
integration of computer algebra and dynamic geometry in secondary mathematics education felt the need to go beyond the then current theoretical views. Even
if they still experimented with explorative tasks, such as finding the number of
zeros at the end of K (Trouche & Drijvers, 2010), a theoretical perspective was
needed that would do justice to the complex interaction between techniques to
use the digital technology, conventional paper-and-pencil work and conceptual
understanding. This led to the development of the instrumental genesis framework, or the instrumental approach to tool use (Artigue, 2002; Guin & Trouche,
1999; Lagrange, 2000). Even if there are different streams within instrumentation
theory (Monaghan, 2005), it is widely recognized that the core of this approach is
the idea that the co-emergence of mental schemes and tool techniques while
working with digital technology is essential for learning. This co-emergence is Digital Technology in Mathematics Education: Why it Works (or doesn't) 7
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the process of instrumental genesis. The tool techniques involved have both a
pragmatic meaning (they allow the student to use the tool for the intended task)
and an epistemic meaning, in that they contribute to the students' understanding.
Rather than exploration, the reconciliation of digital tool use, paper-and-pencil
use, and conceptual understanding is stressed (Kieran & Drijvers, 2006).
A paradigmatic study in this field is the one by Drijvers (2003) on the use of
handheld computer algebra for the learning of the concept of parameter. Four
classes of 14-15 year old students worked on activities using a handheld computer algebra device both in its role of mathematical tool and for conceptual development (branches a & c, Figure 1) to develop the notion of parameter as a "super-variable" that defines classes of functions and that can, depending on the
situation, play the different roles that "ordinary" variables play as well. The results of the study include detailed analyses and descriptions of techniques that
students use, and the corresponding expected mental scheme development. Figure 3 provides a schematic summary of such an analysis for the case of solving
parametric equations in a computer algebra environment (Drijvers, Godino, Font,
& Trouche, 2012).
Figure 3. Conceptual elements related to the application of the solve command
The main reason to discuss this study here is that by providing elaborated examples it contributes to a concrete and operationalised view on the schemes and
techniques that are at the heart of the instrumental approach. The study shows
that the instrumental approach is a fruitful perspective that can provide tangible
guidelines for both the design of student materials and the analysis of student behaviour.
From a theoretical perspective, apart from the concretisation of the notions of
schemes and techniques, the author integrated this with a more general view on
mathematics education, namely the theory of realistic mathematics education
(Freudenthal, 1991). The two perspectives seemed to be complementary and both
provided relevant guidelines for design and analysis.
An equation should contain an = sign
Indicate the unknown to solve
Notice the scope of
the square root sign
"solve with respect to x" = "express x in terms of b"
Word /letter vars?
A solution can be
an expression 8 P. Drijvers
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So did technology work in this case? No and yes. The conclusions on the
learning effects of the intervention are not very clear. Even if the students learned
much about the concept of parameter, their work still showed weaknesses both in
the use of the tool and in the understanding of the mathematics. This suggests an
incomplete instrumental genesis. Factors that may explain these findings are: (a)
the difficulty of the mathematical subject for students of this age, (b) the complexity of the computer algebra tool, and (c) the efforts and skills needed by the
teachers to not only go through their personal process of instrumental genesis,
but also to facilitate the students' instrumental genesis by their way of teaching.
The latter aspect was addressed more explicitly later in the notion of instrumental
orchestration (Drijvers & Trouche, 2008; Trouche, 2004). The study did work in
the sense that it contributed to the researchers' understanding of the complexity
of integrating sophisticated digital technologies in teaching relatively young students. The close intertwinement of the students' cognitive schemes and the techniques for using the digital technology is identified as a decisive factor in the
learning outcomes of technology-rich mathematics education.
Case 4: Online Applications
With the growing availability and bandwidth of internet, researchers became interested in the potential of online interactive applications or applets for mathematics education. The advantages of online content include access without local
software installation, ease of distribution and updating for developers, and permanent availability for users as long as the internet is accessible.
Many studies investigate this potential. For example, Boon (2009) explores
the opportunities for teaching 3D geometry using online applets. Doorman,
Drijvers, Gravemeijer, Boon, and Reed (2012) describe a teaching experiment in
grade 8 focusing on the concept of function using an applet called AlgebraArrows
1
for building chains of operations. Apart from an instrumental perspective
(see Case 3), the theoretical framework includes domain-specific theories on reification, realistic mathematics education and emergent modelling. The applet is
used for concept development (branch c, Figure 1). A third example is the study
by Bokhove, who focuses on acquiring, practicing and assessing algebraic skills
(Bokhove, 2011; Bokhove & Drijvers, 2012). His teaching experiments took
place in grade 12 classes and made use of applets that offer means to manipulate
algebraic expressions and equations
2
. The theoretical framework in this case included notions from algebra pedagogy such as symbol sense, which is expected
to support skill mastery, but also elements from educational science on assessment and on feedback. In contrast to the studies described so far, the role of the
digital tool in Bokhove's work includes the environment to practice skills

1
See http://www.fisme.science.uu.nl/tooluse/en/
2
See http://www.algebrametinzicht.nl/ Digital Technology in Mathematics Education: Why it Works (or doesn't) 9
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(branch b, Figure 1), which might be the easiest role, even if the design of appropriate feedback is an issue to tackle.
As a paradigmatic design research study in this field, let us now describe the
work done by Bakker in somewhat more detail (Bakker, 2004; Bakker & Gravemeijer, 2006; Bakker & Hoffmann, 2005). Bakker investigated early statistical
reasoning of students in Grades 7 and 8. In one of the tasks, students investigate
data from life spans of two brands of batteries while using applets to design and
explore useful representations and symbolizations (see Figure 4). Clearly, the
digital tools' pedagogical functionality is on concept development once more
(branch c, Figure 1). The design of the hypothetical learning trajectory and the
student materials was informed by the development of statistics in history. In his
analysis of student data, Bakker uses Peirce's (1931-1935) notions of diagrammatic reasoning and hypostatic abstraction to underpin his conclusion that the
teaching sequence, including the role of digital tools, invited students' reasoning
about a frequency distribution as an object-like entity, as became manifest when
they started to speak about the "bump" to describe the drawings at Figure 4's
right hand side.
Figure 4. Applets for investigating a small set of statistical data
The main reasons to discuss Bakker's work here are not only the originality of
the dedicated digital tools which meet new ideas on statistical reasoning and statistics education, and which were designed in collaboration with others (Cobb,
McClain, & Gravemeijer, 2003), but also the rich relationships with the different
resources and approaches, such as the historical perspective, to inform the design.

From a theoretical perspective, it is interesting to notice that even if technology plays an important role in Bakker's study, the design and analysis are driven
by theoretical perspectives from outside the frame of research on the use of technology in mathematics education, but rather from the world of mathematics pedagogy and beyond. I believe that this is a meaningful and promising approach.
On the one hand, as researchers we should benefit from specific results and theories from studies on the use of digital tools in mathematics education. On the 10 P. Drijvers
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other hand, we should not forget to involve theories on mathematics education
and educational science in general.
So did technology work in Bakker's case? Yes, it did in the sense that the author clearly reports on conceptual development by the students involved in the
study. Why did this work, which decisive factors might explain these findings? I
believe that an important lesson to be learnt from this study is that design research on the use of digital technology in mathematics education should not limit
itself to the study of the tools alone, but should include the tasks, and their embedding in teaching as a whole, in order to understand what works and why it
works. In this case, I would guess it is the combination of the digital tools, the
tasks and activities, but also the whole-class discussions, the paper-and-pencil
work, the established mathematical practices, in short the educational context as
a whole, that explains the result. A second lesson to learn for us as researchers is
that a theoretical framework which integrates different perspectives can be very
powerful for generating interesting and relevant research results.
Case 5: Mobile Mathematics
Research on the use of mobile technology in mathematics education is in its early
stages but its importance is rapidly growing. It is evident that mobile technology
and smart phones in particular are very popular among students and more and
more wide-spread. Wireless Internet access allows for the use of mobile applications (also called MIDLETS, Mobile Information Device applications), SMS and
email services offer communication and collaboration opportunities, GPS facilities allow for geographical and geometrical activities and the tool's mobile and
handheld characteristics invite out-of-school activities, for example the gathering
of real-life data that inform biology or chemistry lessons (Daher, 2010).
As a paradigmatic example, I now address the MobileMath pilot study carried out by Wijers, Jonker, and Drijvers (2010). In this study, the tool consisted
of a mobile phone with GPS facilities and a "native" application, designed for the
purpose of this game, which generated the view on the game situation and arranged communication with other teams' devices. The mathematical topic involved is geometry: teams of Grades 7 and 8 students used the GPS and the application to play an outdoor game on constructing parallelograms (including
rectangles and squares), and could eventually destroy other groups' geometrical
shapes. This so-called MobileMath game aims at making students experience
properties of geometrical figures in a lively, embodied game context. While playing the game, students look at the map to imagine where they want to make a
shape, walk to the location for the first vertex to enter this location in the mobile
device, which generates a dot on the map, walk to the location of the second vertex of their imagined shape which provides a line on the screen connecting the
first vertex with the current (moving) location, etc. The map in Figure 5 shows
some student constructions. The deconstruction option brought extra challenge
and competition into the game. From the data the researchers conclude that Mo-Digital Technology in Mathematics Education: Why it Works (or doesn't) 11
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bileMath adds a geometrical dimension to the world, transforming it into a game
board. MobileMath also invites mathematical activity, such as the (re)discovery
and use of characteristics of squares, rectangles and parallelograms, and taking
notice of geometrical aspects of the world.
Figure 5. Map of students' parallelogram constructions using GPS
One reason to discuss this study here is that the digital toolthe modern smart
phone with GPS facilities rather than an "old school" computeracts in multiple
ways, and its use includes all branches of the diagram displayed in Figure 1. The
device enables the exploration of properties of quadrilaterals (branch c). It also
allows for practicing the construction of parallelograms, which meets branch b.
And finally, the tool also functions as an environment to outsource the mathematical work, in this case the drawing of the shapes (branch a).
As seems to be the case in other studies on the integration of mobile technology in mathematics education, the theoretical perspective used by Wijers, Jonker,
and Drijvers (2010) is different from the frameworks common in most research
on technology in mathematics education. It is closely associated with frameworks from studies on serious gaming, and focuses (a) on student engagement
and (b) on task authenticity. Enhancing student engagement is seen as an important potential of educational games. In the MobileMath study, student engagement is stimulated by the game's hybrid reality character: On the mobile device's screen, students see the map of the reality in which they are walking, as
well as the virtual geometrical shapes they are creating. Hybrid reality games are
seen as beneficial for student engagement. In addition to this, the authors refer to
Prensky (2001) for a model on heuristics for the design of engaging games,
which include clear rules and goals, outcome and feedback, conflict, challenge
and competition, and interaction. Concerning task authenticity, the authors claim
that the effectiveness of learning activities can be enhanced if the tasks are authentic and realistic. In line with the framework of Realistic Mathematics Education, realistic means that problem situations presented in learning activities 12 P. Drijvers
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should be "experientially" real to students and have meaningful, authentic problem situations as starting points, so that students experience the game's activity
as making sense.
So did the digital technology work in this case? As far as engagement and
authenticity are concerned, the answer is "yes". The researchers report that the
students were engaged in the game and experienced it as challenging. Apparently, the game factor, in combination with the possible attractiveness of the digital
device, works out well. A second factor might be the outdoor and physical character of the game, which students may experience as a welcome change from
regular classroom teaching. What is not clear yet, however, is whether these effects will persist if this type of activity were to become more common. Also, the
study presented here has a small-scale pilot character and would certainly need
further replication.
Case 6: Teachers' Practices and Professional Development
If we recapitulate the previous cases, in all but the last one the teachers' practices
and experiences were identified as an important factor explaining why digital
technology worked or why it did not. Therefore, this final case focuses on the
role of the teacher, teaching practices and teachers' professional development.
One of the first studies focusing on teachers' practices and professional development was the one by Ruthven and Hennessy (2002). In this study and in
subsequent work (e.g., Ruthven, 2007) crucial factors are identified that affect
teachers integrating digital technology in their teaching. In relation to the instrumental genesis model, Trouche developed the notion of instrumental orchestration to stress the relevance of teaching practices (Trouche, 2004). Case studies
based on these models describe teachers' practices in relation to their opinions
and beliefs (Drijvers, 2012; Drijvers, Doorman, Boon, Reed, & Gravemeijer,
2010; Pierce & Ball, 2009). Another model on teachers' professional knowledge
is Technological Pedagogical Content Knowledge (TPACK), which became
widespread but is also criticized (Graham, 2011; Koehler, Mishra, & Yahya,
2007; Voogt, Fisser, ParejaRoblin, Tondeur, & Van Braak, 2013).
In addressing the questions of how to prepare teachers for technology-rich
teaching and how to enhance their professional development in this field, in line
with the work done by Wenger (1998) on communities of practice, it is suggested
that the participation in a community of teachers who co-design and use resources for teaching, can contribute to this (e.g., see Fuglestad, 2007; Jaworski
2006). Digital technology in such an enterprise acts on two levels: first, the professional development concerns its use in mathematics education, and second,
digital technology may support the community's work by offering online and virtual facilities for exchange. Digital technology is both the subject at stake and the Digital Technology in Mathematics Education: Why it Works (or doesn't) 13
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vehicle to address it. Efforts have been done to exploit digital technology's potential for teachers' professional development by designing online courses
3
.
As a paradigmatic design research study in this field, let us now describe the
work done by Sabra (2011) in somewhat more detail. In his PhD dissertation,
Sabra describes two case studies of teachers' collaborative process of professional development in detail. In the first case ten teachers in the same school collaborate on the design of a final assessment training session and a mathematics investigation task while integrating the use of TI Nspire in their teaching. The second
case study concerns a project in which eleven teachers, all members of the Sesamath community from all over France, collaboratively design resources on the
concept of function that are part of the course manual. The analysis shows that
the two communities develop in quite different ways, but that in both developments some critical incidentscalled documentary incidents in the thesisare
decisive. The digital tools in this case include web facilities for collaborative
work, file exchange and communication; the role they play for the participating
teachers is best characterized by branch a in Figure 1, the role of a tool for doing
mathematics, or rather a tool for collaborating on the design of mathematical resources.
The main reason to discuss this study here is that its rich data including interviews, blogs and observations and its fine-grained data analysis provide a detailed insight in how communities of teachers may work (or may not) and how
technology may support this.
From a theoretical perspective, Sabra uses the notion of documentational
genesis as a main concept. Figure 6, taken from Gueudet and Trouche (2009),
shows how this is analogue to the notion of instrumental genesis, but now addressing the level of teachers using and designing digital resources. The interesting point here, in my opinion, is that a similar framework is applied to and elaborated for different situations and different levels of technology integration.

3
E.g., see http://www.edumatics.eu/ 14 P. Drijvers
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Figure 6. Schematic representation of a documentational genesis (Gueudet &
Trouche, 2009)
So did technology work in the Sabra study? Maybe the answer is different for the
two cases that are described. In the case of the team of teachers within the same
school, it seems that the digital technology does not have so much to offer, and
that the professional interest of the community members does not invite a real
engagement in an effective collaboration. As a result, one can wonder whether
the targeted professional development really took place, and whether the community really contributed to it. In the second case of the teachers all over France,
the analysis shows a very lively process of collaboration, which is clearly afforded by the digital technology and would not have been possible without it. Similar
to the other cases described in this paper, it seems that decisive factors that explain the phenomena go beyond the straightforward point of the available technology. My impression is that for a school team of teachers, collaboration is far
from self-evident, whereas teachers who volunteer for a role in the Sesamath project share a professional interest to engage in a virtual community and in a shared
process of distant collaboration. This, I would conjecture, might be the main explanation for the different results in the two cases Sabra describes.
CONCLUSION
Theslightly provocativequestion raised in the title of the paper is why digital
technology in mathematics education works or does not. The underlying aim was
to identify factors that promote or hinder the successful integration of digital
technology in mathematics education. The analysis of the six cases described in
this paper show that the integration of technology in mathematics education is a
subtle question, and that success and failure occur at levels of learning, teaching
and research. In spite of this complexity, three factors emerge as decisive and
crucial: the design, the role of the teacher, and the educational context.
The first factor concerns design. Cases 1, 3, and 4 reveal the crucial role of
design. This concerns not only the design of the digital technology involved, but Digital Technology in Mathematics Education: Why it Works (or doesn't) 15
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also the design of corresponding tasks and activities, and the design of lessons
and teaching in general, three design levels that are of course interrelated. In
terms of the instrumentalgenesis model, the criterion for appropriate design is
that it enhances the co-emergence of technical mastery to use the digital technology for solving mathematical tasks, and the genesis of mental schemes that include the conceptual understanding of the mathematics at stake. As a prerequisite, the pedagogical or didactical functionality in which the digital tool is
incorporated (see Figure 1) should match with the tool's characteristics and affordances. Finally, even if the digital technology´s affordances and constraints
are important design factors, the main guidelines and design heuristics should
come from pedagogical and didactical considerations rather than being guided by
the technology´s limitations or properties.
The second factor concerns the role of the teacher, which is crucial in Cases
1, 2, and 6. The integration of technology in mathematics education is not a panacea that reduces the importance of the teacher. Rather, the teacher has to orchestrate learning, for example by synthesizing the results of technology-rich activities, highlighting fruitful tool techniques, and relating the experiences within the
technological environment to paper-and-pencil skills or to other mathematical
activities. To be able to do so, a process of professional development is required,
which includes the teacher's own instrumental genesis, or, in terms of the
TPACK model, the development of his technological and pedagogical content
knowledge. Case 6 suggests that technology can help the teacher to advance on
this, together with colleagues in technology-supported collaboration. What seems
to be an open question is how the role of the teacher changes if we consider the
use of technology in out-of-school learning, gaming, and other forms of informal
education (see Case 5).
The third and final factor concerns the educational context, which includes
mathematical practices and the elements of the pedagogical map designed by
Pierce and Stacey (2010). Case 2 reveals how important it is that the use of digital technology is embedded in an educational context that is coherent and in
which the work with technology is integrated in a natural way. Case 5, the MobileMath example, shows that taking into account the educational context includes attention for important aspects such as student motivation and engagement. Another factor that is not so much elaborated in the case descriptions but is
important to mention here, is assessment, which should be in line with the students' activities with technology; not doing so would suggest that in the end the
use of digital technology is not important. Finally, the use of digital technology
may lead to an extension of the educational context towards out-of-school settings, as exemplified in Case 5.
The three factors identified above may seem very trivial, and to a certain extend they are quite straightforward indeed; however, their importance, I believe,
can hardly be overestimated and to really take them into account in educational
practice is far from trivial. 16 P. Drijvers
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DISCUSSION
Let me first acknowledge that the study presented here clearly has its limitations.
The discussion of the studies addressed cannot be but somewhat superficial in the
frame of this paper. Also, the number of studies is small, and the choice of the
studies included is not neutral. This being said, I do believe the article provides
avery roughmap of the landscape of research studies on technology in mathematics education and reveals some trends in the domain over the previous decades.
So what trends can be seen in retrospective? Globally speaking, a first trend
to identify is that from optimism on student learning in the early studies towards
a more realistic and nuanced view, the latter acknowledging the subtlety of the
relationships between the use of digital technology, the student's thinking, and
his paper-and-pencil work. A second trend is the focus not only on learning but
also on teaching. The importance of the teacher is widely recognized and models
such as TPACK, instrumental orchestration and the pedagogical map help to understand what is different in teaching with technology and to investigate how
teachers can engage in a process of professional development. The third and final
trend I would like to mention here concerns theoretical development. Whereas
many early studies mainly use theoretical views that are specific for and dedicated to the use of digital technology (e.g., Pea's notions of amplifier and reorganizer in the Heid study), recent studies often include more general theories on mathematics education or learning in general, and also combine different theoretical
perspectives (e.g., see the work by Bakker, using Pierce, RME, and other theoretical views).
To close off this discussion, I would like to express my strong belief that
these theoretical developments are crucial for the advancements in the field. The
studies addressed in this paper show strong relationships between the theoretical
frameworks, the digital tools and the mathematical topics (Kieran & Drijvers,
2012). We now have a myriad of theoretical approaches available in our work,
including very specific theories on the use of technology in mathematics education, domain-specific instruction theories, and very general views on teaching
and learning. One of the challenges in our work, therefore, is to combine and
contrast the lenses each of these approaches offer (Drijvers, Godino, Font, &
Trouche, 2012). The notion of networking theories (Bikner-Ahsbahs & Prediger,
2010) provides a good starting point that may help to better understand the role
of digital technology in mathematics education and, as a consequence, to contribute to the learning and teaching of the topic.
ACKNOWLEDGEMENT
I thank Arthur Bakker, Vincent Jonker, Carolyn Kieran, Hussein Sabra and Luc
Trouche for their helpful comments on the draft version of this paper. Digital Technology in Mathematics Education: Why it Works (or doesn't) 17
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Codes (16)###design of activities, ###didactical phenomenology, ###emergent modelling, ###guided reinvention, ##principles of RME, ##real life, #RME, $$$approaches to teaching Mathematics, $$$modelling, $$mathematics education, $factors that influence teaching and learning, $mathematics, %role players, @@ICT systems: virtual learning environment (VLE), @Devices, @Tools

Memos (0)

Document body:

Digital design:
RME principles for designing online tasks
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Abstract: The theory of Realistic Mathematics Education provides principles that can be applied to task design. In this paper, we investigate how these principles apply to the design of online tasks. To do so, we present examples of tasks on algebra, calculus and geometry designed in the Digital Mathematics Environment. As a result, we conclude that the principles of guided reinvention, didactical phenomenology, and emergent modeling can inform and guide digital design, but that some aspects work out differently compared to the design of paper-and-pencil tasks.
Keywords: Didactical Phenomenology, Digital Design, Digital Mathematics Environment, Emergent Modeling, Guided Reinvention, Realistic Mathematics Education
Introduction
Task design is widely recognized as an important, but complex and subtle activity. Based on the experience of skilled designers, design guidelines and heuristics have been identified (e.g., see Watson & Mason, 2006). Whereas the design and use of digital content nowadays plays an increasingly important role in mathematics education, most of these design principles are based on and applied to the design of paper-and-pencil tasks. The question, therefore, is how such design principles apply to digital design, and how the rich experience and knowledge in the field of designing paper-and-pencil tasks can be transferred to the case of digital design.
To address this question, we limit ourselves to three principles that emerge from the theory of Realistic Mathematics Education and may inform design: guided reinvention, didactical phenomenology, and emergent modeling. We revisit some tasks designed in the Freudenthal Institute's Digital Mathematics Environment in the field of algebra, calculus and geometry education.
RME Design Principles
Realistic Mathematics Education (RME) is a domain-specific instruction theory for the teaching and learning of mathematics. According to RME, mathematics should be seen as an activity (Freudenthal, 1973), and students, rather than being receivers of ready-made mathematics, should be active participants in the educational process, in which they develop mathematical tools and insights by themselves. This point of departure led to the following RME principles that inform task design: guided reinvention, didactical phenomenology, and emergent modeling (Freudenthal, 1973; Gravemeijer, 1994; Van den Heuvel-Panhuizen, 1996).
According to the principle of guided reinvention, students should be given the opportunity to experience a process similar to that by which a given mathematical topic was invented. Even if this primarily is a teaching principle, it has consequences for task design: tasks - or sets of tasks - should invite students to develop 'their own' mathematics. This process, however, needs guidance from the teacher, to help to further develop sensible directions, to leave 'dead-end streets' and to ascertain convergence towards shared knowledge according to the standards within the mathematical community.
Didactical phenomenology concerns the relation between the thought object -the 'noeumenon'– and the phenomenon -the 'phainomenon'– from the perspective of teaching and learning. In particular, it addresses the question how mathematical 'thought objects' can help in organizing and structuring phenomena in reality.
The challenge for the task designer, then, is to find such meaningful phenomena that beg to be organized and structured by the targeted mathematical knowledge.
According to the emergent modeling perspective, a model may play different roles during different phases of activity. Initially, a model is context-specific: it refers to a meaningful problem situation that is experientially real for the student, and is a model of that situation. Then, through working with the model, it gradually acquires a more generic character and develops into a model for mathematical reasoning that is possible because of the development of new mathematical objects in a more abstract framework of mathematical relations that the model starts to refer to. This notion is elaborated into a four-level structure that represents levels of mathematical activity (Gravemeijer, 1994). For the task designer, the challenge is to find suitable situations that ask for the development of such models, and allow for a process of progressive abstraction.
While applying these RME principles to the design of paper-and-pencil tasks is not straightforward, their use for the design of digital content is even more challenging. Although early research on the use of graphing calculators identified opportunities for an RME-based teaching approach (Drijvers & Doorman, 1996), later studies describe the tension between RME principles and the integration of computer algebra software (Drijvers, 2000). Apparently, the match between RME and ICT is not self-evident.
The Digital Mathematics Environment
As technology for teaching mathematics the Freudenthal Institute's Digital Mathematics Environment (DME) is used. The DME integrates a content management system, a learning management system and an authoring environment. The content consists of online modules in the form of Java applets. The learning management system offers means to distribute content among students and to monitor student progress.
The authoring tool is the DME's design environment. Authors, such as teachers, text book authors, or educators, can use the tool for adapting existing online modules or for designing new ones, based on existing materials and basic tools such as graphing and equation editing facilities. While designing, the author can split up the screens in different windows, add applications and tasks, and design feedback. Knowledge of the underlying Java programming language is not required; rather, an intuitive and mathematical interface makes the digital design accessible to a wide audience (Figure 1).

Figure 1 The Digital Mathematics Environment Authoring Tool

Examples of the results of digital design using the DME's authoring tool can be found at www.fi.uu.nl/dwo/demo/en. In this paper, we will briefly discuss some examples from algebra, calculus and geometry from the perspective of the above RME principles.
Digital Tasks for Algebra
As an example of the design of digital tasks for algebra, we consider the work by Bokhove, available at www.fi.uu.nl/dwo/voho/. The digital tasks focus on solving polynomial equations. However, the online units go beyond procedural practice and also focus on the development of symbol sense and strategic skills (Bokhove & Drijvers, 2010, 2012). Crucial factors in the design are the sequencing of the tasks (with sometimes thought-provoking equations) and the design of feedback, and its timing and fading in particular.

Figure 2 Solving an equation with unexpected difficulties (Bokhove & Drijvers, 2010, 2012)

Figure 2 shows an exemplary equation in line 1, and a part of a student's work in lines 2-4. The feedback just refers to the algebraic equivalence of the subsequent equations the student enters; no feedback is provided on the problem solving strategy. This can be seen as a manifestation of the guided reinvention heuristic: this combination of task and tool provides the opportunity for students to reinvent efficient ways of solving equations. Eventually, students can continue entering equivalent equations without coming closer to the solution, but this exploration space is supposed to elicit a wish for efficiency.
From the didactical phenomenology perspective, this task may seem quite poor: what is the phenomenon at stake that would motivate students to engage in mathematics? In defense of this, one can argue that the target group of this online module consists of students in grade 12, who were to do the national examination soon, and who were familiar with the 'world of polynomial equations'. This familiarity makes that these tasks can be appropriate for the mathematization of the field and the development of new problem solving strategies.
Data shows that students work easily with this type of online tasks (Tacoma, Drijvers, & Boon, 2011). Instrumental genesis, the process of developing schemes to use tools to solve the tasks (Artigue, 2002), was an issue to a much lesser extent than in the case of computer algebra (Drijvers, 2002; Drijvers et al., 2012). Apparently, using CAS puts higher demands on instrumental genesis, and this is something to take into account as a designer.
Digital Tasks on Functions and Calculus
As an example of the design of digital tasks for (pre-)calculus we consider the online module Function and Arrow Chain, available at www.fi.uu.nl/dwo/prootool/en. The digital tasks focus on the development of conceptual understanding of the notion of functions, where a function is seen as an input-output assignment, as a dynamic process of co-variation, and as a mathematical object with different representations (Doorman et al., 2012). A crucial factor in the design is sequencing the tasks so these function models emerge in a natural way of increasing complexity and abstraction.

A function as an input-output assignment: braking distance as a function of velocity
Investigation of the co-variation of velocity and braking distance through tracing the graph
Investigation of a family of functions, representing braking distances for three different vehicles

Figure 3 Different views on function in the Function and Arrow Chain module

Figure 3 shows some screens from this module, in which the function gradually develops from a numerical input-output engine, to a process of co-variation and, finally, a mathematical object that is part of a family of functions that can be compared. This sequence of screen shots reflects the emergent modeling heuristic: the context, in this case on of a vehicle's braking distance as a function of its velocity, leads to function models of increasing complexity and abstraction. The digital tool supports this development by means of offering techniques of increasing richness and an increasing repertoire of connected function representations. Of course, this approach requires a context that is suitable for this emergent modeling process. Finding such a context is a question of didactical phenomenology.
In an online calculus course for university freshmen, the co-variation idea is supported by a Geogebra applet for tracing graphs (www.fi.uu.nl/dwo/sk/en/). However, in the context of this intensive and short-period remedial course, the emergent modeling design heuristics was exploited to a lesser extent than was the case for the Function and Arrow Chain course.
Digital Tasks for Geometry
An example of the design of digital tasks for geometry is the module available at www.fi.uu.nl/dwo/dpict/en/. The digital tasks focus on exploring, discovering, and proving properties of bisectors, altitudes and medians in triangles. The latter aspect, the proving, provides a particular design challenge: how to design tasks that offer support and guidance for a proof, but leave room for reinvention?

Figure 4 Proving that three medians of a triangle intersect in one point

While reading the proving task shown in Figure 4, it should be noted that Dutch grade 8 students, the target group for this module, have little experience in proving, and, therefore, need strong guidance: the structure of the proof and the corresponding sequence of images is suggested to the students. Still, writing down the final argument in the reasoning, as is requested in task c, is very difficult to them. The design principle of guided reinvention is easier to apply to tasks in which students explore the properties of bisectors, altitudes and medians in triangles: the dragging options of the dynamic geometry system, in this case Geogebra, in collaboration with DME's feedback, provide a strong learning environment in which students can really experience the geometrical situation and discover the targeted properties.
In the case of this online module, classroom observations show that attention needs to be paid to students' and teachers' instrumental genesis: the interplay between Geogebra and the DME in this module is powerful but it may also be demanding and initially complex to novice users.
Conclusion
In this short paper, we set out to investigate the application of the RME principles of guided reinvention, didactical phenomenology, and emergent modeling for the case of the design of digital tasks in the online DME. To do so, we considered three exemplary tasks. The first example, the task on solving a relatively complex equation, shows that guided reinvention in this case concerns the development of new problem solving strategies, invited by tasks that cannot be solved with the strategies available so far. The didactical phenomenology heuristic here does not lead to the use of real life contexts, but rather takes the world of polynomial equations as a point of departure. In the second example, a real life context -in this case the stopping distance situation- does form the starting point. Emergent modeling heuristics are manifest in the gradual abstraction of the students' view on function. The third example on geometrical proof again takes a guided

reinvention perspective, in which the designers chose strong guidance. Again, the didactical phenomenology here does not lead to a real life problem situation, but to a problem in the world of geometry, that is expected to be experientially real to the students.
What can we conclude about the three RME principles and their application to digital design? The principle of guided reinvention seems to apply well to digital design. ICT offers opportunities for exploration and investigation, and in this way for reinvention. The design choice to confront students with unexpected examples can be seen as a way to invite reinvention as well. In general, this guided reinvention approach might suffer from constraints of the technology that may limit the students' exploration space, such as requirements for input formats and styles, and pre-designed tools that may incorporate too much guidance.
The didactical phenomenology heuristics is also valuable for digital design, but it seems that the phenomena that play a central role in the task do not necessarily come from real life: ICT already forms a meaningful 'world' on its own for the student, in addition to the world of mathematics. Having a real life context as an entrance to these two worlds may lead to cognitive overload. This is a question to be considered carefully in the design process.
Emergent modeling can be a fruitful design heuristics for digital design. As in the case of paper-and-pencil design, the models need to lend themselves for further development towards increasing mathematical abstraction and complexity. Specific for the case of digital design is that these emerging models need to be supported by the digital tools available, for example by an increasing repertoire of representations and techniques in the digital environment, or by increasing options to dynamically use these representations, connect them, and switch between them (Duval, 2006).
As an overall conclusion, this brief exploration of the issue suggests that the three RME principles are valuable for digital design, even if some appropriation is needed compared to the design of paper-and-pencil tasks, such as taking into account the constraints of the digital tools and the fact that the technological environment forms an additional 'world' to the student. The transfer of skills, developed in the ICT environment, to paper-and-pencil, for example, may need specific attention, as well as the instrumental genesis involved in the learning process.
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Codes (12)###guided reinvention, ##characteristics of RME, ##real life, #RME, $$$modelling, $$mathematics education, $factors that influence teaching and learning, $mathematics, $mathematics content, @@challenges when working with ICT, @@value of using ICT, @Devices

Memos (0)

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JOURNAL OF MATHEMATICAL BEHAVIOR 15, 425-440 (1996)
The Graphics Calculator in
Mathematics Education
PAUL DRIJVERS
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This article covers a project conducted by the Freudenthal Institute from August 1991 to September 1994 entitled "The graphics calculator in mathematics education." The theory of realistic mathematics education was taken as the point of departure for formulating the hypotheses. The developmental research design was used. Observation of the students' behavior during the experimental lessons supports the premise that the graphics calculator can stimulate the use of realistic contexts, the exploratory and dynamic approach to mathematics, a more integrated view of mathematics, and a more flexible behavior in problem solving.
[See Figure 1 on p. 4261
INTRODUCTION
Teacher:
Johan:
Alex:
Teacher:
Alex:
Erik Jan:
Teacher:
Erik Jan:
How can you shift the graph of y = x2 two places to the right?
y = x2 + 2 (see Figure 1). He enters this and presses GRAPH.) Oh no! y = x2
+ 2?, GRAPH . that's not right either.
y = x2 + 4x + 4, no, that's not right, the graph shifted two places to the left.
So then it is y = x2 - 4x + 4. That's right.
and three places to the right?
(After trying this out on the graphics calculator): y = x2 - 6x + 9. (Rewriting
the formulas gives y = (x - 2)² and y = (x - 3)2.)
And if you have y = x2 + 2– and you shift that three places to the right?
What do you think?
First complete the square, that'11 give you y = (x + 1)2 - 1, and then three
places to the right, so -3 in parentheses, that's y = (x - 2)² - 1.
This conversation took place recently during a mathematics lesson in Class 4
HAVO. L All of the students in this class were supplied with a graphics calculator
for the duration of the project, and by this point they were quite familiar with it.
It is clear that the direct feedback given by such a calculator can help one to
discover the connections between formulas and graphs.
Correspondence should be sent to Paul Drijvers, Freudenthal Institute, Research Group on Mathematics and Computer Science Education, Tilbcrdreef 4, 3561 GG Utrecht, The Netherlands.
'Fourth class of prevocational secondary education (approximately 16-year-old students).
425 426 DRIJVERS AND DOORMAN
Figure 1.
The development of the graphics calculator has also given rise to a lively
discussion among researchers of mathematics education. How should such a
machine-one that reduces the drawing of graphs to the push of a button-be
integrated into mathematics education? What is the potential of this new technology and what difficulties can one expect to find?
International research in this area has resulted in a stream of publications. An
interesting overview is presented by Wilson and Krapfl (1994). Darien Lauten,
Graham, and Ferrini-Mundi (1994) considered the influence of the use of the
graphics calculator on the understanding of basic calculus concepts. Goldenberg
(1988) investigated the problems that may confront students when working with
a graphics calculator. Ruthven (1992) indicated the potential value of the graphics calculator for mathematics education. Monaghan (1993) investigated the
relation between using the graphics calculator and developing insight. A calculus
textbook in which the graphics calculator plays a major role was developed by
Demana, Waits, and Clemens (1992).
The Netherlands is not lagging behind in such research. During the period
between 1991 and 1994, the Freudenthal Institute, supported by a grant from the
Ministry of Education and Science, conducted a research project entitled "The
graphics calculator in mathematics education." This project focused on secondary education, in particular on the role of the graphics calculator in Math B in
preparation for subsequent exact studies in Class 5 VWO." The decision to focus
2Fifth class of preuniversity education (approximately 17.year-old students). THE GRAPHICS CALCULATOR IN MATHEMATICS EDUCATION 427
on Math B has to do with the fact that the curriculum for this subject in particular
is now under discussion. In addition, it was felt that the graphics calculator
would have a more profound influence in Math B than in Math A. After all, Math
A, which is oriented toward further studies in the humanities, focuses primarily
on the application of mathematics. Formulas used in these applications are an
attempt to represent reality and one can often suffice with approximate answers
due to the model character inherent here (see de Lange, 1987). The use of the
graphics calculator does not affect the fundamentals of Math A. In Math B, on
the other hand, exact answers are generally required. The numerical results
provided by the graphics calculator at times seem to be at odds with this precision. The traditional analysis of functions will change due to the graphics calculator. Graphs are the starting point of further investigations instead of the final
result of the analysis. Different questions come up from the graphical representation that is provided by the machine.
Extensive experimentation using the graphics calculator was conducted at two
different schools in the framework of this project. The students were provided
with a calculator during each lesson. In a later stage of the project they were also
permitted to take their calculator home. On the whole, the students' usual math
textbook was followed during these lessons, but specific units developed by the
research team in which the graphics calculator is used more explicitly were also
part of the program. These lessons in particular, in which this experimental
material was presented, were closely observed. The opening paragraph of this
article is a fragment of one such observational report. Similar observations form
the basis of the theoretical side of this project: providing answers to the research
questions that were formulated at the outset of the project. The results of the
project are published in a research report under the same title (Doorman, Drijvers, & Kindt, 1994). The intent of this article is to offer an impression of the
project and of the most significant results.
The following section first describes the characteristics of realistic mathematics education and how potential contribution by the graphics calculator to this
type of mathematics education is viewed. This leads to the first hypotheses that
form the starting point for the research project at hand. The third section deals
with how the project was conducted. The research method-called developmental research-is described briefly. The development of experimental instructional material is of great importance within this type of research. The choice of
topics for the experimental student material is mentioned here as well. In the
fourth section, a fragment of the developed material is examined more closely.
Data from the observations, described in the fifth section, provide an illustration
of the way in which students dealt with this material. A reflection on the example
given and on the starting points of the research in general is found in the sixth
section. Finally, the last section briefly examines further developments that will
take place in the near future. 428 DRIJVERS AND DOORMAN
REALISTIC MATHEMATICS EDUCATION
AND GRAPHICS CALCULATORS
The domain-specific instruction theory for realistic mathematics education has
acquired a considerable following in The Netherlands during recent decades.
According to this theory, realistic applications should play an important role in
the learning process right from the start. Solution procedures are (re)constructed
by the students themselves through using problems that have meaning in their
reality. Some of the important characteristics of realistic mathematics education are:
l A variety of solution strategies.
l A high degree of student input.
l Use of informal strategies and informal knowledge.
l Footholds for reflection.
l Stimulus for raising the level, for generalization, and for formalization.
For an extensive discussion on the theory of realistic mathematics education, see
Freudenthal (199 1) and Treffers (1987).
How do these general starting points of realistic mathematics education translate to the specific situation of implementing the graphics calculator in math
class? In the course of the research project, this instruction theory became crystallized in five hypotheses.
Realistic Contexts
Mathematical models used in realistic applications often contain "unsightly"
numbers or formulas. For the sake of manageability, reality is sometimes twisted
in order to assure a smooth problem. With the arrival of the graphics calculator,
there is no longer any need to tarnish the realistic quality of the problem. The
machine, after all, takes over the time-consuming technical work, leaving the
student free to concentrate on the process of mathematization, the solution strategy, and the drawing of reasonable conclusions.
The first hypothesis is, therefore:
Using the graphics calculator enables the attention to shift from purely algorithmic
operations to translating realistic problems into a mathematical model and interpreting the results.
In the unit "Optimalization" (Drijvers & Kindt, 1992), for instance, Snellius'
law is mentioned (see also Kindt, 1993). The speeds at which light travels
through the two media are extremely high. By using the graphics calculator,
where a scale model can easily be adapted, this should no longer be a drawback. THE GRAPHICS CALCULATOR IN MATHEMATICS EDUCATION 429
Exploration
Thanks to its direct feedback, the graphics calculator can offer opportunities for
exploratory activities. Even during the initial phase of familiarization, a problem
can often already be investigated graphically. Insight into the structure of a
formula can be acquired through the graphics calculator. Inventory and classification activities can lead to discoveries that then, through reflection and generalization, result in interesting mathematical theorems. This contrasts with the
traditional method, in which definitions and theorems are stated at the beginning
of the learning path in the expectation that insight will be acquired through
repeated application. The second hypotheses is as follows:
Use of the graphics calculator stimulates the posing of new questions and the
generalization of problems. For the student, this involves a broadening of the
mathematical field of vision and a change of attitude with respect to mathematics
from one of "passive performance" to that of "active investigation."
The exploratory potential of the graphics calculator is addressed, for example,
on the worksheet on classifying cubic functions (Drijvers, 1993). The students'
assignment is quite open: Based on the various graphs, they are to try and catalog
all the possible cubic functions. The exploration that is here indispensable is
made possible thanks to the graphics calculator.
Integration
Use of the graphics calculator contributes to the integration of the two classical
components of mathematics: algebra and geometry. Operations with algebraic
expressions via graphic (or geometrical) illustration of functions on the screen
can by this means be performed in a great variety of ways. Algebraic laws and
rules can thereby be discovered and then tested graphically. This leads to a more
visual form of algebra instruction. The class conversation at the beginning of this
article is such an example. It was recorded during a lesson on the unit entitled

"The Algebra of Graphs" (Doorman & Drijvers, 1993).
Conversely, geometrically shaded assignments may evoke algebraic activities.
Examples of such assignments are: "draw regular five-pointed star" or "make a
spiral curve." The student is thus asked to design formulas (input for the ma-
chine) in order to achieve the desired geometric result (output). Hypothesis 3 can
be stated:
Use of the graphics calculator encourages the integration of geometric and alge-
braic activities and continually stimulates the student to create links between the
various aspects of mathematics. 430 DRIJVERS AND DOORMAN
Dynamics
The graphics calculator has a number of dynamic aspects. First, one can quickly
and effectively follow the results of alterations in the problem. The influence of a
given parameter in the formula can easily be made clear by using graphics.
Another dynamic aspect is the ability to trace a graph or curve with the cursor,
while reading off the constantly changing coordinates on the screen. In the unit
entitled "Movements in the plane" (Doorman, Drijvers, & Kindt, 1994a, 1994b),
for instance, the speed of the valve of a turning bicycle wheel is made visible. A
third dynamic aspect is the ability to zoom in and out of the graph. This enables one
to continually alter one's frame of reference from global to local and vice versa.
The fourth hypothesis is, therefore:
The graphics calculator is ideal for visually illustrating the changing behavior of
magnitudes in their mutual relation and, therefore, encourages a dynamic manner
of observing analytical models.
This aspect is examined more closely in van den Brink and Doorman (1994).
Flexibility
Due to the advent of the graphics calculator, the repertoire of techniques and
skills a student must master will markedly change. Freehand drawing of a graph
based on a strictly prescribed analysis of functions-a much practiced skill until
now-will hardly matter. On the other hand, skills such as estimation, reading
graphs, and successive approximation will increase in significance.
One of the first problems students face when making one or more graphs on
the screen is the limitations of the screen itself. The students will have to figure
out on their own the relevant domain (or range) of the variables in the problem at
hand, based on the context of the mathematical model. Ruthven (1992) used the
term trial and improve for this type of technique.
Finally, we should point out that the graphics calculator produces numerical
approximations in its results, even in situations where an exact result is desired.
The student is therefore required to maintain a critical attitude with respect to the
machine's numerical productions.
All in all, we may expect that, due to the influence of the graphics calculator, a shift
in emphasis will occur away from "rigid techniques" and toward a more flexible
solution procedure, in which a critical attitude is developed with respect to numeri-
cal results.
The five points just elucidated form a strong link with the starting points of
realistic mathematics education. The terms exploration and dynamics, by the
way, are frequently used in discussions about the use of the graphics calculator
and computer algebra in education. The development of this terminology in THE GRAPHICS CALCULATOR IN MATHEMATICS EDUCATION 431
didactic practice, however, is not nearly as self-evident and, in a number of
publications, downright disappointing.
DEVELOPMENTAL RESEARCH
The project was conducted according to the method called developmental
research. This involves a cyclical process of consideration and testing, an alter-
nation of thought experiment and educational experiment. In the thought experi-
ment, the designer imagines how the educational process might take place. In the
educational experiment the thought experiment is then tested in actuality. The
curriculum being tested, however, is not simply left unchanged. The experience
acquired is immediately used to adjust the curriculum or to develop new educa-
tional activities. In fact, the new curriculum is created through the experiments:
Development and testing go hand in hand.
The results of the present developmental research consist of experimental
instructional material, observations of the lessons, and reflections based on these
observations. An attempt at answering the research questions, starting from the
observations and the realistic view of education, is found in the theoretical
reflection. For a more extensive description of developmental research-not
connected with the graphics calculator-see Gravemeijer (1994). Examples of
developmental research that focuses on realistic mathematics education are de-
scribed by van Reeuwijk (1992) and de Lange, Burrill, Romberg, and van
Reeuwijk (1993).
As mentioned earlier, the development of student materials plays an important
role in developmental research. Four topics have been selected for units to be
developed, on the basis of the hypotheses already listed and also, of course, on
the basis of less concrete intuition. The following are brief descriptions of each
topic.
Differentiation
Differential calculus plays a crucial part in the mathematics curriculum. The
graphics calculator is an excellent means of approximating differential quotients
and of drawing estimations of graphs of slope functions. This would seem to
offer possibilities for exploratory activities and for supporting concept formation.
Optimalization
Problems of optimalization appear in all sorts of situations and disciplines. The
problem (often originating in reality) must first be translated into mathematical
terms. Then, during the solution stage, a variety of solution procedures often
prove valuable. Here lies the potential for forging links with geometry.
The Algebra of Graphs
The algebra of graphs involves the link between operating with graphs and 432 DRIJVERS AND DOORMAN
operating with formulas. The graphics calculator's ability to quickly investigate
on a graph the results of alterations in the formula is an important aspect here. In
this way, algebraic rules and properties can be linked with graphic characteristics
and, therefore, can become more "real."
Movements in the Plane
The dynamic potential of the graphics calculator can optimally be applied to the
topic "Parametric Curves," where time is often the parameter. Here, as with
"Optimalization," realistic problems and links with geometry play an important
role.
EXAMPLE: ROLLING COINS
What does instructional material involving the graphics calculator actually look
like? A decision was made in this project not to let the machine stand too much in
the spotlight. The point, after all, is still mathematics. The students must often
(to the disappointment of some) put their calculator down and concentrate on the
algebraic calculations or geometrical reasoning.
The following fragment from the unit entitled "Movements in the plane" is
used here as an example. Classroom experience using this material offers an
impression of the ways in which the students made use of the graphics calcula-
tor's potential.
In Figure 2, two guilders lie next to one another. The left guilder is fixed. P is the
point on the edge of the right guilder where it touches the left guilder. The right
guilder is now going to roll around the edge of the left guilder without sliding. The
question is, what path will P describe.
The first assignment contains a concrete orientation. This leads to the choos-
ing of an appropriate coordinate system that is shown in Figure 3, and to the
constructing of the motion equations. Then the students draw the curve, creating
Figure 2. THE GRAPHICS CALCULATOR IN MATHEMATICS EDUCATION 433
a picture of the cardioid, after which they investigate the speed with which the
point passes through the cardioid. By using the graphics calculator to animate the
curve, it becomes clear that this speed is not constant. The speed vector is
calculated using differential calculus. Geometrical observations complement the
assignments on the speed of the movement. Then variations in the assignment are
presented.
1. As the right guilder starts rolling, point Q is found halfway between the center
of the rolling guilder and the origin.
a. How do these motion equations differ with respect to those of P?
b. Draw Q's path.
c. Is Q's average speed the same as P's? Expiain.
Now the experiment becomes more difficult to conduct. We tape a coin
called a rijksdaalder on top of the rolling guilder that, due to its greater
diameter, also covers part of the fixed guilder.
2. The point R is found at (- i/z, 0) at the moment the right guilder starts rolling.
a. How can you change the motion equations of P's path in order to get R's
path?
b. In what way does R's path look different from P's path?
In this assignment, the distance from the moving point to the center of the guilder
has been changed. In the following assignment this is generalized and the stu-
dents must investigate a variety of cases. They are free to classify the graphs they
find as they choose. The focus here is on their own investigation.
3. Assume that a point P lies at a distance a from the center of the rolling guilder.
The radius of the guilder we will still call 1.
Investigate how the form of the path described by P depends on the value of a.
Pay attention to extreme cases as well. Give a brief summary of your results.
OBSERVATIONS IN THE CLASSROOM
How did this assignment actually work in the classroom? The experiment took
place in a Math B group of 18-year-olds in the sixth class of preuniversity 434 DRIJVERS ANDDOORMAN
education. These students had already worked extensively with the graphics
calculator during the previous school year. But we were somewhat concerned:
Would the students' skill with the machine still be at their fingertips? And how
would new students, who had never worked with the machine, adapt to the
situation?
Although some students reacted by giggling, they did start experimenting with
two actual guilders. Later, that proved to be of functional assistance. In particu-
lar, when they began to suspect that P was turning twice as fast around the center
of the rolling coin as that same center was turning around the fixed guilder, many
students reached for the actual coins to verify that suspicion.
At the point in the school year that this assignment was given, circular
movements having a variety of angle speeds and radii had already been handled
extensively. Constructing the motion equations, therefore, was not particularly
problematic. Moreover, the graphics calculator immediately revealed any errors
by drawing an "impossible" curve. This led, on the whole, to an improvement in
the motion equations. For instance, a number of students entered the following:
x(t) = 2 cos t - 1 + cos (2t)
y(t) = sin t + sin (2t).
The drawing in Figure 4 immediately revealed what was wrong.
This now can easily be improved! The students then entered:
x(t) = 2 cos t - 1 + cos (2t + T)
y(t) = 2 sin t + sin (2t + 7-).
These equations resulted in the curve shown in Figure 5 on the screen of the
graphics calculator.
The matter of varying the distance between P and the center of the rolling
guilder was also dealt with later on. The following fragment, taken from an
observational report, gives an impression of this work.
1
Figure 4. THEGRAPHICSCALCULATORINMATHEMATICSEDUCATION 435
Femke is having trouble with Problem 1. She has entered:
x(t) = 2cost - 1 + 5 cos (4t + rr)
y(t) = 2sint + f sin (4t + 7r)
"
.because it's turning twice as fast if it's twice as small."
A cloudlike drawing that is shown in Figure 6 had appeared on her screen.
In a sense she is right, as she is imagining a smaller coin, a dime for instance,
rolling over a guilder. She is running ahead of the spirograph curves. In this case,
however, the rolling coin remains a guilder, but the point whose path is being
followed no longer lies on the edge.
In Problem 2 the observer asks Femke: "And if you now made the factor 14 even
larger?" "Then the loop would get larger," she replies, "and the outside, too."
Problem 3 no longer presents any difficulties. Some students even enter the
parametric equations using a variable:
x(t) = 2cost - 1 + a cos (2t + Tr)
y(t) = 2sint + a sin (2t + 7r).
Interesting drawings result from these so-called Pascal curves (see Figure 7,
p. 436).
During the last 5 minutes of class, the results of Problem 3 are written in a row
on the board (see Table 1).
One student comments on the "dent": "When a is small, the dent is also small,
and when a gets bigger, the dent grows too."
Where the loop is concerned, the teacher asks: "Does it matter how much
greater a is than i?"
Figure 6. 436 DRIJVERS AND DOORMAN
Figure 7.
Student: When a is close to 1, it's small; if a is big, it's big.
Teacher: So does the small one ever catch up with the big one?
Student: No, they'll overlap, they'll get close together.
This brought the unit on the cardioid to a satisfying close.
Thanks to the graphics calculator, the topic "Curves" worked well. The stu-
dents watched the curves being created in a dynamic manner. The path became a
starting point for further study, rather than the static end product of a vast ritual of
investigation into intersections of the axes, horizontal and vertical tangents, and
so on.
TABLE 1
Classification of the Pascal Curves
a
o-+0
1-I
O<a<I
Cr>I+

Shape
circle
cardioid
'dent'
loop
REFLECTION AND CONCLUSIONS
Let us glance back at the example presented and at the accompanying observations in the classroom. The five hypotheses in Section 2 serve as the guideline for this. Along with each of the hypotheses, the results of the project on the whole are also summarized briefly. THE GRAPHICS CALCULATOR IN MATHEMATICS EDUCATION 431
Realistic Contexts
The assignment using two guilders worked well. The students were able to get a good picture of the problem. The variations on the original problem also seemed to come across quite naturally to the students. For that matter, the graphics calculator contributed little here. By choosing the radius of the guilder as the unit of length, "unattractive" coefficients could be avoided. So the strength of the graphics calculator in dealing with "unsightly" numbers was not put to use here. The general picture regarding this point is quite positive. Realistic data often lead to complex formulas with unattractive coefficients. The graphics calculator can then overcome the diffculties inherent in the calculating and drawing. True-to-life applications can by this means come within the students' reach. This is particularly desirable in the case of Math A. An additional advantage is the fact that attention can be paid to higher skills, such as constructing the mathematical model and choosing which strategy to use for solving the problem.
One aspect that repeatedly emerges when realistic contexts are used together with the graphics calculator is the difference between the various languages involved: normal language, mathematical language, and machine language. Students, especially at first, do become confused by this. They are being asked to function, as it were, on three fronts. Helping them to handle this flexibly will require the necessary didactic attention.
Exploration
In the example given of the cardioid, the graphics calculator does, indeed, support the exploratory activities. Particularly in the case where the distance from P to the center is varied, the graphics calculator performs a crucial task. The students arrive at a classification that they themselves have developed through their own experience.
On the whole, the project's results on this point are encouraging. The potential of the graphics calculator evidently stimulates many students to engage in informative exploratory activities. The mathematical level of the result, however, tends to be rather varied. Although such exploration can lead to excellent discoveries and insights, the results are at times disappointing. This has to do, on the one hand, with the fact that drawing conclusions from investigative activities places heavy demands on the students. On the other hand, it is also true that students in Math B often have not yet been confronted with open problems, and that an investigative attitude must also be acquired.
Nonetheless, it would seem that this potential for engaging in exploratory activities is one of the most important trump cards of the graphics calculator.
Integration
Laying links between algebra and geometry was brought out fairly well in the lessons on the cardioid. During the investigation into the speed of the point, the graphic image was placed next to the algebraic one. The geometric method, in 438 DRIJVERS AND DOORMAN
which speed vectors are added up, was difficult for the students. Table 1 presents a rather nice classification of the examples that were generated with the aid of the graphics calculator.
On the whole, this integration did not occur without a struggle. Working in different areas and from different viewpoints can produce difficulties, but new insights and an expansion of concepts as well. At first, the students tended to regard the various areas as separate entities and choose one single approach. The graphics calculator, on the other hand, offered them the opportunity to focus more on the entwining of these areas. In the end, the students generally appeared to accept this and to appreciate it.
Dynamics
The topic entitled "Curves in parametric equations" is, of course, particularly suited for a dynamic approach, and the graphics calculator is an excellent aid here. The students in the experimental lesson saw the point on the plane shift and the speed of the movement become visible. The parametric equations that describe the movements on the plane really came to life in this way. A second dynamic aspect that emerged from the observed lessons was the direct feedback the machine offered the students. This feedback made is possible for the students to learn from any errors and to immediately make improvements.
On the whole, the experiences encountered in this project confirmed the idea that the graphics calculator would provide the students with the advantage of a global graphic representation, an initial conceptualization of the problem. The machine removed the students' inhibitions to making a quick sketch. A dynamic aspect here is the fact that the user can see how ,a graph (or a curve or a group of graphs) is created. When the picture is finished, TRACE offers the opportunity to follow the graph point by point and to read the changing coordinates on the screen. The graphics calculator's direct feedback encourages the students to reflect on what they have done. This alternation between experimentation and reflection is an important aspect of concept formation in the realistic approach to mathematics.
Flexibility
The skills a student is expected to have mastered in order to work through the series of problems on the cardioid are different from those required in the traditional textbooks. Drawing the curve, for instance, is something that these students no longer needed to do by hand. On the other hand, it is quite important that the student be able to extract an indication of the exact maximum value from the approximation given by the graphics calculator. Moreover, it is assumed that the student will be able to link the consequences of the alterations in the motion equations with the changes in the curve.
On the whole, the expectation that the arrival of the graphics calculator would cause a shift of emphasis from rigid techniques to a more flexible solution THE GRAPHICS CALCULATOR IN MATHEMATICS EDUCATION 439
procedure was indeed confirmed. In addition, a critical attitude must be developed with regard to numerical results. The importance of mastering routine operations will decline and the necessity of developing a flexible attitude for working with the machine will increase. The students will, as it were, become more mathematically nimble.
AND FURTHER. . .
The research project did not, of course, exclusively confirm expectations and it is not now over and done with. Although, indeed, the experiences on the whole were quite positive, there was insufficient investigation into, in particular, the influence of the graphics calculator on the subject-matter content. This has to do with certain preconditions for the project at hand: The students participating in the experiment were not given permission to use the graphics calculator on their final exam. As a result, it was impossible to diverge too far from the normal program and its traditional approach. A side effect was that the students regarded using the graphics calculator as something of a noncommittal event.
In the subsequent project-which has already begun-this will change.
Based on the realization that the graphics calculator can only attain a satisfactory place in the mathematics curriculum if the machine also plays a role on the exams, an experimental exam project has been set up. The two test schools involved in this project have obtained a divergent regulation regarding the mathematics exams, one that permits use of the graphics calculator. Obviously, these exams will differ from the traditional ones. The first such experimental exams for VWO Math A and Math B are administered in 1996.
Further introduction of the graphics calculator into mathematics education will depend partly on developments of the new curricula for the upper levels of secondary education. In our opinion, it is quite clear that the availability of the graphics calculator will influence the actual content of the units to be developed. Routine analysis of functions, for instance, will diminish in importance.
We expect that the exam experiment involving the graphics calculator will contribute to concrete changes in the subject of mathematics in the upper levels of secondary education. The desirability of introducing the graphics calculator on a national level, particularly considering developments in other countries, is, in our opinion, quite clear, and is scheduled for 1998.
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Memos (0)

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Chapter 8 Chapter
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The Development of an RME-based
Geometry Course for Indonesian
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8. The development of an RME-based
geometry course for Indonesian primary
schools
Ahmad Fauzan, Tjeerd Plomp & Koeno Gravemeijer
Abstract
The aim of this study was to develop and implement a valid, practical, and effective RME-based

geometry course for Indonesian primary schools using design research approach. The research activities were divided into three stages namely front-end analysis, prototyping stage, and assessment stage that were conducted in a four year period. The focus of the chapter is to present detail and rational regarding the three stages. The result of the study was a high quality RME-based geometry course for teaching geometry at grade 4 in Indonesian primary school consisted of teacher's guide and student book. In the products lies the local instructional theory for teaching geometry that was effective for improving pupils' understanding, reasoning, activity, creativity, and motivation.

1. Introduction to the problem

The study reported in this chapter has been conducted in the period 1998 - 2002. At that time the quality of mathematics education in Indonesia, especially in primary and secondary education, was considered poor (see Soedjadi, 1992, 2000), whilst the mathematics learning and teaching process in the classrooms was dominated by the traditional method (see Somerset, 1997; Marsigit, 2000). The traditional way of teaching had a negative influence on the pupils' attitudes towards mathematics which means that most pupils did not like to learn mathematics, and that some of them were even afraid of mathematics (Marpaung, 1995, 2001). This study aimed to explore whether another approach to mathematics education could address these shortcomings by developing and implementing an exemplary course viz. Realistic Mathematics Education (RME)-based geometry course, for teaching and learning the topic Area and Perimeter at Grade 4 in Indonesian primary schools.

The focus of the study was to develop and implement a valid, practical, and effective RMEbased geometry course by applying design research approach. These processes were guided by the main research question:

What are the characteristics of a valid, practical and effective RME-based geometry course for learning and teaching the topic Area and Perimeter at Grade 4 in Indonesian primary school?

2. Context of study: mathematics education in Indonesia

The general goals of mathematics education in Indonesian primary education were phrased as follows

• Preparing the pupils to be able to deal with the dynamic changes of world situations effectively and efficiently through practical works based upon logical reasoning, rational and critical thinking, caution and honesty.

• Preparing pupils to be able to use mathematics and mathematical reasoning in their everyday life and in studying other sciences.

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After reading the lofty goals, questions arose as to why the quality of mathematics education in Indonesian primary schools was still poor. The following paragraphs will discuss some primary causes of poor quality of mathematics education in Indonesian primary schools.

The first cause was that the lofty goals have become blurred when they came into practice. The specific instructional objectives from Grade 1 till Grade 6 were still dominated by remembering facts and concepts and reproducing them verbally, studying computational aspects, and applying formulas.

The second reason was poor quality of mathematics textbooks. In the textbooks, many abstract concepts were introduced without paying much attention to aspects such as logic, reasoning, and understanding (Soedjadi, 2000). The topics that were taught seem far removed from pupils' daily life. Even the teachers themselves sometimes did not know the usefulness of the topics they have taught.

A third reason for poor quality of mathematics education in Indonesian primary schools was related to teachers. Most teachers preferred a traditional approach in teaching mathematics. In general, the climate in Indonesian classrooms (see Fauzan, 2000, 2001; Fauzan, Siettenhaar & Plomp, 2002a, 2002b; Somerset: 1997), was similar to that in several African countries as was summarised by De Feiter and Van den Akker (1995) and Ottevanger (2001) as follows:

• 'chalk and talk' is the preferred teaching style;
• emphasis on factual knowledge;
• lack of learning questioning;
• only correct answers are accepted and acted upon;
• whole-class activities of writing/there is no practical work carried out.

The impact of the situations described above was that most students dislike learning mathematics because they were not learning the mathematics they need. They also did not have the opportunity to learn significant mathematics, and lack commitment towards or were not engaged in existing curricula. The other impact was that the students' achievements in mathematics were poor from year to year (see Fauzan 2002). The poor performance of Indonesian students could also be seen from the Trends in International of Mathematics and Science Study (TIMSS) report (Mullis et al., 2000). Related to geometry, some findings indicated that geometry tended to be the most difficult among the mathematics topics not only for students but also for teachers (see Fauzan, 1996,1998; Herawati, 1994; Amin, 1995). The poor performance of the students in geometry and their negative attitude toward geometry became the big challenges for this study. These issues together with the fundamental problems mentioned in section 1 lead to the following questions.

• How to design a high quality course that could promote not only pupil learning but also pupil's attitude in learning mathematics?
• How to support teachers in implementing the course?

The questions, which became the main focus in this study, were addressed in a research project aimed at developing and implementing an RME-based geometry course.

3. Design research as the research approach

This study built upon two "schools of thought" of design research. The first one emerges in the context of more general design and development questions (see Van den Akker, 1999; Van den Akker & Plomp, 1993; Plomp, 2009; Richey & Nelson, 1996). The second one developed within the area of mathematics education by mathematics educators in the Freudenthal Institute (FI), The Netherlands (see Freudenthal, 1991; Gravemeijer, 1994a, 1994b, 1999). In the following part we will characterize how we perceived and used design research in this study.

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What is design research?

According to Van den Akker & Plomp (1993), design research is characterized by its twofold purpose:

• Development of prototypical products (curriculum documents and materials), including empirical evidence of their quality.
• Generating methodological directions for the design and evaluation of such products.

This study was about development and implementation of an RME-based geometry course that fits the first purpose.

Richey & Nelson (1996) and Van den Akker (1999) distinguish two types of design research. These are summarized by Nieveen et al. (2006) as development studies and validation studies.

• Validation studies have a focus on designing learning environments or trajectories with the purpose to develop and validate theories about the process of learning and how learning environments can be designed. Validation studies aim at advancing learning and instruction theories, such as Realistic Mathematics Education (Gravemeijer & Cobb, 2006).
• Development studies aim at design principles for developing innovative interventions that are relevant for educational practice. "Development studies integrate state-of-the-art knowledge from prior research in the design process and fine-tune educational innovations based on piloting in the field. … By unpacking the design process, design principles that can inform future development and implementation decisions are derived." (Nieveen et al., 2006: 153).

Considering that this study was aiming at developing a high quality RME-based geometry course, it may be categorized as development study type of design research. But as this research also aimed at validating whether the constructivist approach of Realistic Mathematics Education could be successfully applied in the context of Indonesian mathematics education, this research was a validation study type of design research as well.

Important activities in design research are its cyclic nature (of analysis design, development, implementation, evaluation and reflection) and the use of formative evaluation as a key activity to establish evidence of product quality and to generate guidelines for product improvement (Ottevanger, 2001). Related to this, Nieveen (1997) and Van den Akker (1999) mentioned three main stages or phases in design research (see also Plomp, 2009), which are front-end analysis/ preliminary investigation; prototype phase, and assessment phase, consisting of summative evaluation of the final product. Throughout all these activities, a systematic reflection on the development methodology has to take place to produce design principles (Plomp, 2009).

Following these activities and the work of Nieveen (1997) and Ottevanger (2001), the study was divided into three stages namely front-end analysis, prototyping stage, and assessment stage.

During the prototyping stage, the design research approach proposed by Freudenthal Institute (see Gravemeijer 1999; and Gravemeijer & Cobb, 2006) was applied. This approach was followed in developing the content of RME-based geometry course, especially in designing the instructional sequences. Freudenthal (1991, p. 161), in relation to the development of RME, defines design research as:

"Experiencing a cyclic process of development and research so consciously, and reporting on it so candidly that it justifies, and that this experience can be transmitted to others to become like their own experiences"

Gravemeijer (1999; and see also Gravemeijer & Cobb, 2006) states that in this approach researchers direct their attention to developing instructional sequences in learning mathematics. To do so, they start with thought experiments, thinking about the learning route that will be

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passed through by pupils. By reflecting on the results of instruction experiments in which the results of the thought experiments are tried out, they continue with the next thought experiment. Researchers in this approach have a long term learning process in mind. In this long-term process, the subsequent of thought and instruction experiments are connected. This situation leads to the description that development can be seen as a cumulative cyclic process, as it is shown in Figure 1.

Figure 1: Design research as a cumulative approach (source: Gravemeijer, 1999)

The cycles of the thought and instruction experiment described above indicate the activities carried out on a daily basis developing a learning sequence. For example, the second thought experiment is conducted based on the results of the first instruction experiment. The results of this thought experiments are tested through the second instruction experiment on the next day. This process is continued until the instructional sequences, consisting of a number of lessons for teaching a mathematics topic that work well, are developed. The instructional sequences are called local instructional theories. Gravemeijer and Cobb (2006) call the daily basis of the development a micro cycle and the development of the whole learning sequences a macro cycle (see Figure 2).

Figure 2: Micro and macro cycles of design research (Gravemeijer & Cobb, 2006)

Our study combined the two approaches in design research, as is summarized in Table 1.

thought exp.
thought exp.
thought exp.
thought exp.
thought exp.
instruction exp.
instruction exp.
instruction exp.
instruction exp.

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Table 1: The summary of design research

Type of research	Design research (development and validation studies).
Main focus	Design and research on the RME-based geometry course and testing of the characteristics.
Aims	To develop a high quality RME-based geometry course that was suitable for learning and teaching the topic Area and Perimeter at Grade 4 in Indonesian primary schools.
Expected results	1. A high quality RME-based geometry course. 2. Lesson learned about: • characteristics of a high quality RME-based geometry course; • development process of the RME-based geometry course; • implementation process (how teachers teach in the classrooms and how pupils learn); • the improvement on pupils' understanding, reasoning, activity, creativity, and motivation; • the local instructional theory for learning and teaching the topic Area and Perimeter.

A high quality RME-based geometry course referred to three quality criteria namely validity, practicality and effectiveness (Nieveen, 1997; 1999).

• Validity refers to the extent that the design of the intervention include "state of the art knowledge" (content validity) and the various components of the intervention are consistently linked to each other (construct validity).
• Practicality refers to the extent that users (teachers and pupils) and other experts consider the intervention as appealing and usable in normal conditions.
• Effectiveness refers to the extent that the experiences and outcomes from the intervention are consistent with the intended aims.

The stages of the research

As mentioned in the previous section, the design research was conducted in three stages (see Figure 3). These processes were realized in a four-year research, which included three fieldwork periods in Indonesian primary schools.

Figure 3: The general research design

In the design, the prototyping stage is presented in two cycles to indicate that there were two consecutive prototypes of the RME-based geometry course that were developed during this stage. The cycles in the design also include the formative evaluations that were conducted in each stage of the study.

Front-end Analysis
◄
◄
◄
Prototyping stage
Assessment stage

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4. The results of the study

Following the phases described in Figure 1, the focus, research activities, data collection techniques, and the results of the study in each stage will be elaborated in the next sections.

Front-end analysis

The purpose of front-end analysis is to get a picture of the starting point and the potential end points of the course. The work done in this stage included context and problem analysis, literature review, and analysis of available and promising examples.

The context and problem analysis was conducted by doing several activities such as review of Indonesia curriculum documents and related research results, classroom observations,

interview with principals, the fourth grade teachers and some pupils from six primary schools (3 schools in Padang, West Sumatra and 3 schools in Surabaya, East Java). The schools have been chosen purposively by considering the quality of the schools (high, middle, and low). The summary of these activities has been described in section 2.
The literature review on the RME theory gave the pedagogical direction to design the first draft of the RME-based geometry course on the topic Area and Perimeter for pupils at Grade 4 (pupils around 10 years old). As it was designed based on the RME approach, it needed to reflect the three key principles of RME: guided reinvention, didactical phenomenology and emergent models (Gravemeijer, 1994a, 1999).
The guided reinvention principle was applied to sequence the learning trajectory so that pupils could learn the topic Area and Perimeter as intended based on the RME point of view. The second RME principle was realized by using contextual problems as starting point so that the pupils would experience the process of horizontal and vertical mathematization (see Gravemeijer, 1999). The contextual problems would also facilitate the pupils to use their own models (model of) to solve the problems until one of them emerges into a formal way to solve a mathematical problem (model for).
Based on context analysis (Indonesia) and review of literature and related documents (see Fauzan 2002), it was decided to design an RME-based geometry course with the vision was to broaden the concept of Area and Perimeter. The rationale for broadening of the concepts was that when we talk about Area and Perimeter in our daily life, we are not only dealing with regular shapes such as squares, rectangles or triangles, but also irregular shapes or surfaces of 3-dimentional objects such as cakes, lands, and tiles.
The vision of the RME-based geometry course was elaborated further by broadening the concepts of Area and Perimeter through the following aspects.
• Relating Area and Perimeter to other "magnitudes"
The concepts Area and Perimeter are frequently involved in our daily activities. For that reason, it is considered to be important to relate them to other magnitudes such as costs, weight, paint, rice field, cake and fence. By relating the concepts to other magnitudes, it gives the pupils the opportunity to learn the concepts in a more meaningful way.
• Introducing the exchange of measurement units as a counting strategy
In most literature on traditional mathematics the square is introduced as the only measurement unit. However, in reality we use various non-square measurement units, such as rectangle or triangle tiles. So that, introducing the exchange of measurement units as a counting strategy would be useful for helping pupils to understand area.
• Investigating the relation between Area and Perimeter
There is a strong belief that Area and Perimeter are directly proportional to each other (Gravemeijer, 1992), in which people think that the bigger the perimeters the bigger the areas, or vice versa. To prevent pupils from this confusion, in the RME-based geometry course the concepts of Area and Perimeter are taught consecutively. 167
• Connecting measurement units to reality
This aspect of broadening the concept of area is to make the pupils aware that many objects in their real life can be used as a measurement unit. Moreover, relating the measurement units such as cm
2
, m
2
, and km
2
cm to reality (for examples, the sizes of: the
thumb nails, the surface of the tables, the forests) will help the pupils to understand the idea of the relative sizes of those measurement units as well as the relationship between one measurement unit and the others.
• Making pupils aware of the model-character of the concept (approximation, neglecting irregularities)
Teaching the topic Area and Perimeter in traditional mathematics causes pupils to think that areas of the rectangular shapes are always the product of two lengths and that learning the topic Area and Perimeter is identical with applying the formulas (see Romberg, 1997). In reality we mostly deal with irregular shapes. It means that we need to teach pupils about the idea of approximation regarding Area and Perimeter, in order to make them aware that the measurement is never exact.
• Integrating some geometry activities
In traditional mathematics, the geometry activities for learning the topic area are dominated by counting grids and applying the formulas. In the RME-based geometry course some geometry activities are involved such as re-shaping: cutting a figure into pieces and reallocating these pieces to get another shape, and tessellation: arranging tiles in various ways. Re-shaping is considered as an important activity in the RME-based geometry course because it not only makes it easier for pupils to find areas of various geometry shapes but also makes them aware of the conservation of area. Meanwhile, tessellation will make the pupils aware of the possibilities of compensation. Gravemeijer (1992) argues that the tessellations are just like an excursion in geometry, and at the same time it makes the pupils realize that area units do not have to be squares.
• Involving reallotment problems
Reallotment is a concept in which the area of a shape remains the same when it is reshaped. By working on the reallotment problems, for examples the problems about tessellations, pupils will better understand that: a shape can be seen as the sum of other shapes or as a portion of another shape; a shape can also be arranged to form a different shape by cutting and pasting. The concept of reallotment will also help pupils to realize that the objects that have the same areas can have various shapes.
Based on the vision and the review of related documents, such as the realistic geometry textbooks developed from the Wiskobas project in The Netherlands (see Gravemeijer, 1994a; Klein, 1999; Treffers, 1991), the paper entitled Reallotment written by Gravemeijer (1992), and the book with the same title used in the project Mathematics in Context (MiC) in the USA (see NSF, 1997), the hypothetical learning trajectory in the RME-based geometry course was designed. The detail of the hypothetical learning trajectory can be found in Fauzan (2002).
Prototyping stage
A prototype is a preliminary version or a model of all or a part of a system before full commitment is made to develop it (Smith, 1991). According to Nieveen (1997) the term "develop" in this definition refers to the construction of the final product. So, the prototypes are all products that are designed before the final product will be constructed and fully implemented in practice.
As applying the RME principles into Indonesian context needed some adjustments, the prototyping approach was used in this study because this approach gives the opportunity to 168 develop an RME-based geometry course fitting the Indonesian context (see Goodrum, Dorsey & Schwen, 1993; Nieveen, 1997; Shneiderman, 1992; Tessmer, 1994). Two prototypes of the RME-based geometry course for learning and teaching the topic Area and Perimeter were developed in this stage namely, prototype 1 and prototype 2. The latter was built upon the experiences in prototype 1. Each prototype consisted of a student book and teacher guide for ten lesson periods (one lesson takes 70 minutes). The way that the prototypes were developed followed the design research approach of Freudenthal Institute (see section 3).
Prototype 1 of the RME-based geometry course
It was important to determine that the draft course would fit not only the Indonesian curriculum, but also the ideas underlying RME. For this purpose, prototype 1 was reviewed by two Dutch RME experts (in RME and geometry) and three Indonesian subject matter experts (in mathematics curriculum and also familiar with RME).
The main focus of developing prototype 1 was to reach a valid RME-based geometry course. This activity was guided by the next research question:
What are the characteristics of a valid RME-based geometry course for learning and teaching the topic Area and Perimeter at Grade 4 in Indonesian primary schools?
The investigation into the content validity of the RME-based geometry course was focused on the following issues:
• Does the content of the RME-based geometry course include the subjects/topics that are supposed to be taught for topic Area and Perimeter?
• Does the content of the RME-based geometry course reflect the RME's key principles?
• Does the RME-based geometry course reflect the RME's teaching and learning principles?
• Does the RME-based geometry course reflect the important aspects of realistic geometry? (see De Moor, 1994).
Meanwhile, the construct validity or the internal consistency of the RME-based geometry course dealt with the following questions.
• Is the content of the RME-based geometry course sequenced properly?
• Is the content well chosen to meet the objectives/goals described in the beginning of each lesson?
Research activities in answering the questions above are summarized in Table 2. 169
Table 2: The summary of evaluation of prototype 1

Focus of evaluation (What?)	The purposes of valuation (Why?)	Evaluation activities (Method (How?))	Instruments
Validity of the RMEbased geometry course, focused on the content and construct validity.	To test the characteristics of the RME-based geometry course whether they meet the criteria mentioned in the questions above.	Interview and discussion with the Dutch RME experts and Indonesian subject matter experts, classroom observations, analyzing pupil's portfolios.	Interview guidelines, observation scheme.
The learning and teaching process using the RME-based geometry course.	To assess whether the hypothetical learning trajectory works as intended.	Classroom observations in two schools and interviews with pupils and teachers from those schools.	Observation scheme, interview guidelines.

Based on the results of experts' reviews, it was concluded that the RME experts approved of the content and the construct validity of the RME-based geometry course as well as the hypothetical learning trajectory for learning and teaching the topic Area and Perimeter. Nevertheless, the RME experts recommended that some contextual problems should be improved in order to strengthen the conjectured learning trajectory.
After the revision process, prototype 1 was implemented in two primary schools in Indonesia. The reason to choose the two schools was because they had different conditions. The pupils from school 1 were very heterogeneous in mathematical ability, while the pupils in school 2 were rather homogeneous. It was assumed that the variations between the schools would enrich the results of the classroom experiments. Another reason was the willingness of the two schools, especially the teachers and principals, for collaboration.
Some problems emerged during classroom experiments such as:
• Some pupils could not finish working on the given contextual problems because of several problems regarding their negative attitude (such as very dependent to the teacher and not used to working in-group) and the time constraint.
• There were some contexts that were not used by the pupils when they were solving some contextual problems. This was because the statement in those problems did not guide the pupils to use the contexts. See Fauzan (2002) for more details.
Prototype 2 of the geometry course-based RME
The results of the classroom experiments as described in the previous section led to the development and implementation of prototype 2 of the RME-based geometry course. The main focus of the development and implementation of prototype 2 was to investigate the validity and the practicality of the RME-based geometry course. The validity was re-investigated at this stage because the results of the development and implementation of prototype 1 showed that the validity of the RME-based geometry course needed to be improved. The research question for this stage of the study was:
What are the characteristics of a valid and practical Geometry course-based RME for learning and teaching the topic Area and Perimeter at Grade 4 in Indonesian primary schools?
As mentioned by Van den Akker (1999), practicality refers to the extent that users (and other experts) consider the intervention as appealing and usable under normal conditions. It means 170 that the RME-based geometry course should meet the needs and wishes of pupils and teachers at Grade 4 in Indonesian primary schools. For this purpose, some questions were formulated to investigate the practicality of the RME-based geometry course, for examples:
• Has the RME-based geometry course potential for developing student's understanding?
• Is the student book easy to use?
• Does the learning process progress as intended?
• Do teachers use the teacher guide as intended?
Research activities in developing and implementing prototype 2 of the RME-based geometry course consisted of two cycles, namely expert reviews and classroom experiments as has been elaborated in Table 3.
Table 3: The summary of evaluation of prototype 2

Focus of evaluation (What?)	The purposes of

evaluation (Why?)
Evaluation activities
(Method (How?)) Instruments
Validity of the
RME-based
geometry
course
(continued).
To test the characteristics of
the RME-based geometry
course to see whether they
meet the criteria (see the
questions in section 5a).
Interview and discussion
with two Dutch RME
experts and three
Indonesian subject matter
experts, classroom
observations.
Interview
guidelines,
observation
scheme.
Practicality of
the RMEbased
geometry
course.
To check whether the
learning and teaching
process using the RMEbased geometry course
meet the criteria (see the
questions before Table 3).
Interview and discussion
with the Dutch RME
experts, Indonesian subject
matter experts, teachers,
principals, supervisors and
pupils, and classroom
observations in two
schools.
Interview
guidelines,
observation
scheme.

Prototype 2 of the RME-based geometry course was implemented (by the researcher as a teacher) in two primary schools from two different provinces, after the revision process. The main reasons for choosing the two schools were similar to those when implementing prototype 1. Moreover, the cultures of the two places were different, especially in working habits, so this would probably enrich the results of the research.

The results from the experts' validation showed that prototype 2 of the RME-based course met the criteria of the content and construct validity. However, the experts did have suggestions such as:

• add the contextual problems that would show the idea of approximation and that the results of measurements are never exact;
• relate Area and Perimeter to other magnitudes, reshaping, adding and subtracting area.

The findings from the classroom experiments also indicated that the conjectured learning trajectory for the topic Area and Perimeter worked as intended for most pupils. Based on these results it was concluded that the RME-based geometry course met the criteria of the content and construct validity. It means that the local instructional theory designed in the RME-based geometry course was valid for learning and teaching the topic Area and Perimeter. 171

The experts, supervisors, and principals agreed that prototype 2 of the RME-based geometry course had potential to develop pupils' understanding, reasoning, activity, creativity and motivation. They also agreed that the RME-based geometry course would be usable and useful for teaching the topic Area and Perimeter. Based on the classroom observations, it was observed that the RME-based geometry course could be used by the pupils and teachers as intended. The results from the interviews with the pupils and teachers also indicated that the student book and teacher guide were easy to use. Based on these results it could be concluded that prototype 2 of the RME-based geometry course reached the criteria of the practicality.

At this stage of the study, the practicality of the RME-based geometry course was only evaluated in two schools in which the researcher acted as teacher. Therefore, the course needs to be evaluated in more schools in the assessment stage by involving teachers in the implementation processes, to generate more evidence about its practicality. In addition, the effectiveness of the RME-based geometry course had also to be investigated in order to show its effect on the pupils' learning.

The assessment stage

After the development and implementation of two prototypes of the RME-based geometry course during the prototyping stage, this study moved to the last phase called the assessment stage. In this stage the final version of the RME-based geometry course was implemented in five Indonesian primary schools in order to gain more insights in the practicality and effectiveness. The term further insights means two things: first, the number of schools for the classroom experiments would increase, and five additional classroom teachers would be involved in the implementation to find out whether they could teach the RME-based geometry course. Second, this assessment stage would lead to an opportunity to further evaluate the conjectured learning trajectory.

In preparing the teachers to be able to implement the RME-based geometry course, a series of workshop-based doing mathematics activities (see Gravemeijer, 1994a; Treffers, 1991) and reflections after the teaching practices have been conducted (see also Fauzan, 2002). The research question for the assessment stage was:

What are the characteristics of a practical and effective RME-based geometry course for learning and teaching the topic Area and Perimeter at Grade 4 in Indonesian primary schools?

The RME-based geometry course reached the effectiveness criteria if it would have a positive impact on the pupils and teachers in learning and teaching the topic Area and Perimeter. These criteria were measured using four of five levels of effectiveness mentioned by Kirckpatrick (1987) and Guskey (1999, 2000) namely participants' reaction, participants' learning, participants' use of new knowledge and skills, and impact (the learning outcomes). The participants in this study were the pupils and teachers. The level impact to organization was not used to assess the effectiveness of the RME-based geometry course because it was not applicable to this study. The levels of effectiveness in this study were elaborated upon by posing the following questions:

• Participants' (pupils and teachers) reactions:
Did they like the RME-based geometry course?
Was their time well spent?
Was the RME-based geometry course useful for them?
• Participant's learning:
Did the teachers and pupils acquire the intended RME knowledge? 172
• Participant's use of new knowledge and skills:
Did the teachers and pupils effectively apply the RME knowledge and skills?
• Pupils' learning outcomes:
Were the pupils more confident as learners than before?
What was the impact of the RME-based geometry course on the pupils' performance and achievement?

The learning outcomes in this study consisted of pupils' achievement and reasoning, pupils' motivation, activity, and creativity. In RME pupils learn mathematics based on activities they experience in their daily life; pupils have a big opportunity to construct their knowledge by themselves, etc. These conditions led to a hypothesis that the RME-based geometry course would increase pupils' achievement. Moreover, in solving a contextual problem pupils are always stimulated (written in the teacher guide) to explain "what do they do and why?" This requirement was assumed to have potential to promote pupils' reasoning.

Research activities in the assessment stage are summarized in Table 4.

Table 4: The summary of evaluation of final version

Focus of evaluation
(What?)
The purposes of
evaluation (Why?)
Evaluation
activities
(Method (How?)) Instruments
Practicality of the RMEbased geometry course
(continued).
To gain more insight
into whether the
learning and teaching
process using the
RME-based geometry
course meet the
criteria.
Interview with
teachers and pupils,
and classroom
observations in five
schools.
Interview
guidelines,
observation
scheme.
Effectiveness of the
RME-based geometry
course.
To gain more insight
regarding the impact
of the RME-based
geometry course on
the pupils for each
level of effectiveness.
Interview with
teachers and pupils,
classroom
observations, pretest and post-test,
and analyzing
pupil's portfolios.
Interview
guidelines,
observation
scheme, test
and assessment
materials, and
questionnaire.
Teachers ability to
implement the Geometry
course-based RME.
To gain information as
to whether the
teachers could
implement the RMEbased geometry
course as intended.
Classroom
observation and
interviews with
teachers.
Observation
scheme,
interview
guideline.

As mentioned before, the final version of the RME-based geometry course (comprising 10 lesson periods, one lesson takes 70 minutes) was implemented in five primary schools (involving five teachers and seven classes). The results of the assessment stage can be summarized as follows (see also Fauzan 2002):

1. The pupils could use the student book without any difficulties and they could learn the topic Area and Perimeter as intended according to the RME point of view. The teacher guide was useful for the teachers in implementing the RME-based geometry course. Three teachers 173 said that the teacher guide was easy to use, while one teacher suggested that the teacher guide should be elaborated in more detail (note: she used to teach lower grades pupils before). From these findings it can be concluded that the RME-based geometry course met the criteria of the practicality. The practical RME-based geometry course can be characterized as follows:
• The RME-based geometry course had the potential to develop pupils' understanding, reasoning, activity, creativity and motivation in learning the topic Area and Perimeter. These results can be seen on the example of pupils' works below, when finding the areas of rectangle, parallelogram, and trapezoid using their own ideas (the pupils did not know the formulas yet).

The results on the first row show that the pupils used combination of counting and reallotment strategies to find the areas. On the second row, the pupil used global reallotment first, then used counting strategy. There are two important findings showed in the example. The first, the hypothetical learning trajectory worked well (the context used by the pupils to find various strategies). The second, the RME-based geometry course had stimulated pupils' creativity and reasoning in finding the area of a shape.
• The teaching learning process using the RME-based geometry course fostered pupilcentered learning.
• The pupils could use the student book without any difficulties, and they could also learn the topic Area and Perimeter (using the student book) as intended according to the RME point of view.
• The teacher guide was useful for and easy to use by the teachers.
• The RME-based geometry course met the criteria of the effectiveness as it resulted in

some positive impacts on the pupils. The positive impacts of the RME-based geometry course on the pupils are characterized as follows:
- The pupils liked the RME-based geometry course. They said that the RME-based geometry course was useful and gave them more confidence as the learners.
- Most pupils acquired the intended RME knowledge which then enabled them to find out for themselves several concepts included in the RME-based geometry course. They could also use the new knowledge and skills that they had acquired from one lesson in the next lessons.
- The pupils' attitudes in learning mathematics developed in a positive direction. The pupils became less dependent and did engage actively in the learning and teaching processes. They also became more motivated to find different strategies in solving the contextual problems. Their reasoning developed from being very
Another pupil used
global reallotment to
find that the area of the
trapezoid was 24.□ 174
weak at the beginning to being able to reason mathematically by the end of the classroom experiments.
- The pupils actively engaged in the learning process and they also creatively found various solutions in solving the contextual problems in the student book.
- The pupils' achievements (in the experimental classes) in the post-tests were significantly higher than those in the pre-tests. The pupils' achievements in the experimental classrooms were significantly higher than the achievements of the pupils in Grade 4 and 5 who had been taught the topic Area and Perimeter using the traditional method.
- A significant difference was found between the motivation of the pupils before and after they had been taught the RME-based geometry course, especially on the aspect self-concept of mathematical ability (see Blöte, 1993)
2. The teachers liked the RME-based geometry course, and in general they could implement the Geometry course-based RME as intended, although sometimes they still used the traditional way of teaching. It was also observed that at some occasions the teachers could not fully apply the RME knowledge and skills that they had gained from the training probably because they were not yet used to the RME approach.
5. Conclusion
After a four-year study of the development and implementation of the RME-based geometry course at Grade 4 in Indonesian primary schools using design research approach, some conclusions can be drawn (see also Fauzan, 2002):
a. The RME-based geometry course for pupils at Grade 4 in Indonesian primary schools met the criteria of validity (content and construct validity), practicality, and effectiveness. It suggests that the learning trajectory designed in the RME-based geometry course can be used as a local instructional theory for learning and teaching the topic Area and Perimeter.
b. The RME approach could be utilized in Indonesian primary schools. Further, the RME approach could address some problems mentioned earlier in this chapter, especially in changing the classroom climate and providing guidelines in how to develop and implement a good quality course material for teaching mathematics.
c. The two "schools of thought" of design research that were combined in this study resulted a high quality RME-based geometry course and the local instructional theory aimed for.
The results outlined above indicated that the RME approach could be utilized in Indonesian primary schools. Further, the RME approach could address some problems mentioned earlier in this chapter, especially in changing the classroom climate and providing guidelines in how to develop and implement a good quality curriculum material for teaching mathematics.
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NOLEINE FITZALLEN & JANE WATSON

DEVELOPING A SEQUENCE OF LEARNING EXPERIENCES IN STATISTICS

Planning meaningful learning experiences in a discipline such as statistics, leads to a search for starting points both from a general learning perspective and from the developmental pathways known to contribute to understanding the content chosen. This chapter focuses on the design of a sequence of learning experiences that promotes the development of understanding of statistical covariation and associated graph creation and interpretation skills, facilitated by the software package TinkerPlots: Dynamic Data Exploration (Konold & Miller, 2005). The objective is to demonstrate the importance of the various structural perspectives that contribute to the planning of meaningful learning experiences in relation to learning about statistical covariation as an end point of a learning sequence that takes a developmental approach to introduce students to a variety of graph types. The staged introduction of graph construction and interpretation activities that align with expectations from the curriculum for a variety of graph types with increasing complexity and sample size, offer promise for the most successful learning outcomes for students (Fitzallen, 2012a). The software package chosen for implementation offers affordances that enhance the learning of additional statistical concepts as a consequence of taking an inquiry-based learning perspective.

BACKGROUND PLANNING

Connections to Real World Contexts

Using authentic contexts and data is particularly important in statistics education, as suggested by Gould, Kreuter, and Palmer (2006); Fitzallen and Watson (2010); and Langrall, Nisbet, Mooney, and Jansem (2011). It is also considered important to use real data from a context that students can relate to, particularly for primary school students. The Reality in Mathematics Education (RME) instructional design theory developed by Wiskobas and Freudenthal provides an overarching organiser for structuring statistical investigations that make connections to real world problems. RME is a guide for student learning that focuses on the development of abstract thinking (Stephan, 2009; Treffers, 1993). It allows students to be active participants in the learning process and promotes the development of learning experiences that start with rich contexts that provide real-life sources of mathematics. "The contexts thus serve not only as a source but as an area of application as well" (HeuvelPanhuizen, 2009, p. 12). Although the context underlying a statistical question needs N. FITZALLEN & J. WATSON
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to be meaningful and motivating to provide real data for analysis, students do not actually have to have experienced the situation but must be able to make thoughtful connections to the context. In addition, active participation is promoted when students make artefacts that are meaningful and useful to them (Bakker, 2004; Stephan, 2009). The RME philosophy also suggests instruction should build students' reasoning gradually from the concrete to the abstract, through the use of manipulatives, diagrams, and other imagery to reinforce students' reasoning. It accommodates the use of graphing software packages because they place an emphasis on student construction of models and symbols to facilitate abstract thinking. Within the RME structure instruction is intentionally designed so that students reorganise their thinking progressively toward more abstract ideas (Stephan, 2009) and then relate those ideas back to the practical context of the situation in order to make sense of the statistical information in that context (Fitzallen & Watson, 2012). Such practices encapsulate a holistic view of learning and actively engage students in the learning process at a deeper level (Ramsden, 2003), thus addressing students' needs and facilitating successful realisation of educational goals and purposes.

Statistical Investigations

In order to focus on the relevant mathematical ideas of covariation, graph creation, and graph interpretation within the broader structure of RME, the Model of Statistical Investigation developed by Watson (2009) detailed in Figure 1, is an essential element of learning sequences. Watson's model starts a statistical investigation with a question. The question sets the scene for an investigation and establishes the context from which the data collected are related. The Data Collection step provides the data that can be represented in a number of forms - numerical, pictorial, or graphical. The data are often reduced using statistical calculations of measures of centre and spread or graphical representations such as a scatterplot. At the final step of Inference, those representations or others are then used to make conjectures about other populations that recognise the uncertainty of the conclusions drawn. Uncertainty emerges from the Chance step of the model. Underpinning all the steps in Watson's statistical investigation model is the concept of variation.

The Data Analysis and Probability Standard of the Principles and Standards for School Mathematics (2000) puts an emphasis on developing statistical reasoning skills and calls for students to formulate questions and collect, organise, and display relevant data to answer questions; to select and use appropriate statistical methods to analyse data; and to develop and evaluate inferences and predictions that are based on data (National Council of Teachers of Mathematics [NCTM], 2000). Similarly Wild and Pfannkuch (1999) offer an extensive four-dimensional model that includes behaviours associated with constructing graphs within an inquiry-based learning perspective. The model relates to the way statisticians work and think statistically and applies to the way in which students engage in statistical investigations. It includes four dimensions: DEVELOPING A SEQUENCE OF LEARNING EXPERIENCES IN STATISTICS
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Dimension 1: The investigative cycle
Dimension 2: Types of thinking
Dimension 3: The interrogative cycle
Dimension 4: Dispositions

Dimension 1 is related to the thinking processes employed when working through a statistical investigation. This involves planning an investigation, collecting data, analysing data, and drawing conclusions. Dimension 2 is related to the types of problem solving strategies applied when working through a statistical problem. Wild and Pfannkuch (1999) posit that the types of thinking in this dimension are "the foundations on which statistical thinking rests" (p. 227). Dimension 3 adopts a cyclical process of data interrogation that involves thinking critically about the data in order to distil and encapsulate ideas and information. Dimension 4 includes the personal qualities, dispositions, and habits of mind employed when working with data. Dimension 2: Types of Thinking of the Statistical Thinking Model is particularly useful when considering the way in which students work with data when creating graphs (Pfannkuch & Wild, 2004; Wild & Pfannkuch, 1999). The type of thinking most relevant to this chapter is transnumeration. Pfannkuch and Wild (2004) describe transnumeration as changing data representations to engender understanding, capturing the characteristics of a real situation, and communicating messages in data. The notion of transnumeration is extremely important as new technologies that incorporate interactive and dynamic commands as a way of working within software environments foster the manipulation of both data and data representations.

Curriculum Frameworks

In 2007, the American Statistical Association (ASA) commissioned development of the guidelines and curriculum framework detailed in the GAISE Report (Franklin et al., 2007). To support teachers to incorporate statistical activities into learning programs the GAISE Report curriculum framework utilises four problem solving processes that acknowledge the role of variation in statistics: Formulate questions, Figure 1. Model of Statistical Investigation (Reproduced from Watson, 2009, p. 91).N. FITZALLEN & J. WATSON
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Collect data, Analyse data, and Interpret results. It provides a developmental sequence of learning experiences that incorporates all four problem-solving steps across three levels of development. Although implicitly embedded in all levels of the framework when recommendations are made for the application of scatterplots, covariation, or "covariability" as it is termed in the framework, is included as a point of focus in Level B of the developmental framework. It is also applied in Level A when scatterplots are used to "observe association between two variables (p. 31)" and "look for patterns and trends" (p. 33) in data. At Level B the use of scatterplots is recommended for "measuring the strength of association between two quantitative variables" (p. 48) and "modelling linear association" (p. 51). Similar recommendations are made in Level C.

In the Australian context, The Australian Curriculum: Mathematics (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2012a) introduces the components of a statistical investigation throughout the curriculum, with graphing important at all year levels and the use of digital technologies first mentioned at Year 3. Mention of a "range of graphical displays" occurs in most years but consideration of scatterplots is not included until Year 10. This does not exclude the use of scatterplots to develop understanding of covariation with younger students long before scatterplots are mentioned explicitly.

Development of Understanding

Although curriculum documents identify the content to be covered at particular stages in students' education, they do not provide explicit information about the sequence in which the concepts should be introduced to students and how the concepts may be developed within a learning program. To supplement this deficiency, Watson and Fitzallen (2010) provide hierarchical sequences of graph creation and graph interpretation that have the potential to contribute to the development of sequences of learning experiences. Their graph creation hierarchy involves four cycles of development. It begins with introducing the use of graphs for displaying data about one attribute in Cycle 1 [G1 in Figure 2], progresses to using graphs to summarise data in Cycle 2, which includes the use of graphs for comparing data sets and exploring the relationship between two attributes [G2]. Alternatively after Cycle 1, large data sets [G3] may be used for extending notions of graphing established in Cycle 1. Informal Decision Making for Graphs [G4] in Cycle 3 provides the opportunity to move from the construction and creation of graphs to using them purposefully to answer questions about the data. A time for consolidation and application is offered in Cycle 4 Graph Interpretation [G5]. The graph creation hierarchy can be used to determine students' level of understanding of statistical concepts, such as distribution, covariation, and informal inference. It is applicable in its entirety or parts thereof, to all year levels of the compulsory years of schooling and beyond when students are required to use graphical representations to interpret data. DEVELOPING A SEQUENCE OF LEARNING EXPERIENCES IN STATISTICS
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Figure 2. Map of cycles and levels of learning in the Watson & Fitzallen (2010) graph creation and interpretation hierarchy.

The Concept of Graph [G1] (Figure 3) in the Watson and Fitzallen (2010) graph creation and interpretation hierarchy includes the basic elements of a graph: Attribute, Data, Variation, and Scale. It suggests an order that describes the process of developing an understanding of the characteristics of a graph and the different levels of understanding that may be realised when drawing parts of a graph together to tell a story about the data. This reflects the findings of research of the importance of developing the ability to see data as an aggregate and gaining a global perspective (Ben-Zvi, Garfield, & Zieffler, 2006). At the relational stage, "students can create a meaningful picture/diagram with appropriate scale and tell the story of variation in the data and what it means for the attribute displayed" ((Watson & Fitzallen, 2010, p. 59). As a foundation for graph creation, The Concept of Graph [G1] underpins all the following cycles. An example of a graph created in this level is provided in Figure 4. The graph is a stacked dot plot created in TinkerPlots.

Figure 3. Developmental sequence for the basic Concept of Graph [G1]
(Reproduced from Watson & Fitzallen, 2010, p. 59).

Figure 4. Stacked dot plot displaying the data for one attribute (Belly Button Height).

The second cycle of graph creation includes two sub-cycles: The Concept of Graph for Multiple Attributes [G2] (Figure 5) and The Concept of Graph for Large Data Sets [G3] (Figure 7). The Concept of Graph for Multiple Attributes [G2] emphasises the different types of attributes, introduces a variety of graphs, and establishes N. FITZALLEN & J. WATSON
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the need for two dimensional representations to accommodate the measurement of more than one attribute for each case. At the relational level, "students can recognise among the graph types they understand, the one appropriate for particular associations between attributes and apply it to tell the story of the data" (Watson & Fitzallen, 2010, p. 59). An example of a graph created in this level is provided in Figure 6.

Figure 5. Developmental sequence for the Concept of Graph for Multiple Attributes [G2]
(Reproduced from Watson & Fitzallen, 2010, p. 60).

Figure 6. Scatterplot displaying the covariation of two attributes (Foot Length and Belly Button Height. The main cluster of data has been circled using the pencil tool.

The Concept of Graph for Large Data Sets [G3] (Figure 7) introduces large data sets, more complex graphical representations, and data reduction methods. At the relational stage, "students have the ability to select the appropriate graph for a particular data set and attribute" and establish an "integrated understanding of the various representations" used to interpret large data sets (p. 60). Figure 8 shows a graph that displays the relationship between two attributes - Gender and Belly Button Height. The Hat Plot tool in TinkerPlots has been used to summarise the data. The middle 50% of the data is covered by the "crown" of the hat plot. The brims to the left and right of the crown cover 25% of the data each. The Hat Plot is a precursor of the box plot. It is less complex and more accessible for younger students (Watson, Fitzallen, Wilson, & Creed, 2009).DEVELOPING A SEQUENCE OF LEARNING EXPERIENCES IN STATISTICS
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Figure 7. Developmental sequence for the Concept of Graph for Large Data Sets [G3]
(Reproduced from Watson & Fitzallen, 2010, p. 60).

Figure 8. Graph displaying a large data set for two attributes
(Gender and Belly Button Height).

The third level - Informal Decision Making for Graphs (Figure 9) potentially takes the graph creation process one step further and leads to students making informal inferences. It involves making decisions about creating and choosing appropriate graphs for a particular purpose and recognising the potential different representations have in assisting to draw conclusions and make informal inferences beyond the data.

At the relational level, "all elements need to be integrated for completely justified conclusions to be reached" (Watson & Fitzallen, 2010, p. 61).
*The term informal is used here to distinguish the hierarchy from one that would involve formal statistical tests.
Figure 9. Development sequence for Informal Decision Making for Graphs [G4] (Reproduced from Watson & Fitzallen, 2010, p. 61).N. FITZALLEN & J. WATSON
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Figure 10 provides an example of a graph created in Level 4. It is a scatter plot displaying the covariation of two attributes, Height and Belly Button Height. The annotations on the graph were made with the pencil tool in TinkerPlots. The two horizontal lines at the top and bottom of the graph indicate the regions of the graph where the data would be considered outliers. The circled data are also considered outliers. The zigzag line that tracks the trend of the bulk of the data in the graph is used to indicate the relationship between the two attributes. The fluctuation in the line indicates that although there is a general trend between the two attributes there is recognition that there is variation evident within the trend (Fitzallen, 2012a). The stack of data points to the right of the graph indicate that there are cases in the data set that do not have data associated with at least one of the attributes.
Figure 10. Scatter plot annotated using the pencil tool in TinkerPlots.
Watson and Fitzallen (2010) stress the importance of providing time for students to consolidate their understanding of graph creation and the role different graph types play in the data analysis process. Not only do students need to understand the elements of a graph but also they need to make connections to the context of the data to make meaningful conclusions and justifications. These aspects of learning also apply to another hierarchical sequence - Graph Interpretation [G5] (Figure 11). It is flexible enough to be applied as a further cycle of learning after consolidation of any of the Concept of Graph cycles and attends to the factors, such as outliers, that potentially impact on the shape of a graph. It also includes a questioning attitude element, which is required for critical thinking and development of statistical literacy skills necessary to interpret unusual or potentially misleading graphs so often encountered in the media (Watson, 2006). The complexity of graph interpretation tasks is determined by which cycle of Concept of Graph applies to the data and questions asked.DEVELOPING A SEQUENCE OF LEARNING EXPERIENCES IN STATISTICS
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Figure 11. Developmental sequence for Graph Interpretation [G5] (Reproduced from Watson & Fitzallen, 2010, p. 61).
Data Analysis Software
Educational technologies provide many options for displaying data in graphical representations and improving student learning about statistical concepts (e.g., BenZvi, 2006; Fitzallen & Watson, 2010; Shaughnessy, 2007). It is therefore pertinent to consider the graphical representations a particular software package affords when designing a sequence of learning experiences that promotes understanding of statistical concepts. The sequence of learning experiences developed in this chapter takes into consideration the innovative software package, TinkerPlots: Dynamic Data Exploration (Konold & Miller, 2005). TinkerPlots is employed as a tool for students to prepare graphs in order to demonstrate, support, and display their understanding of covariation, graph creation and graph interpretation. As a constructivist software created for students rather than statisticians, the software provides freedom for students to display their understanding in many ways (Fitzallen & Watson, 2010; Watson & Donne, 2009) and also to interpret presented graphs to identify covariation (Fitzallen, 2012b; Konold & Khalil, 2003). The potential to use TinkerPlots to make informal inferences has also been demonstrated (Watson & Donne, 2008).
Discussion of Background Planning
Incorporating all the notions explored in this section in a sequence of learning experiences is complex. To optimise learning opportunities it is important to embed learning of statistical concepts within real world contexts (Stephan, 2009; Treffers, 1993) and explore statistical questions via an inquiry-based learning process (Watson, 2009; Wild & Pfannkuch, 1999). Aligning learning outcomes with curriculum frameworks (ACARA, 2012a; Franklin et al., 2007; NCTM, 2000) is also necessary to provide the opportunity for the development of understanding of statistical concepts within software learning environments (Watson & Fitzallen, 2010).
As the targeted learning outcome of the sequence of learning experiences in this chapter is the development of understanding of covariation, it is also necessary to incorporate pedagogies that address the learning of that statistical concept. The notions in this section could accommodate other learning outcomes such as variation N. FITZALLEN & J. WATSON
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and measures of centre. Regardless of the targeted learning outcome it is necessary to ensure any sequence of learning is informed by the curriculum, and previous research on the development of the statistical skills and ideas. These aspects of learning about covariation are addressed in the next section where an example of a sequence of learning experiences for covariation is presented.
SEQUENCE OF LEARNING EXPERIENCES
To address the specific statistical concept of covariation in a sequence of learning experiences, the fundamental notions of a teaching experiment based on using "growing samples" is applied (Ben-Zvi, 2006; Cobb & Gravemeijer, 2004; Cobb, McClain, & Gravemeijer, 2003). The lessons gradually introduce increasing sample sizes from the same population, reflecting the hierarchical framework for graphs in Figures 3 and 5. Each lesson requires students to make sense of the data as well as learn about the features of TinkerPlots that potentially support or influence their descriptions of the data and explanations of the conclusions drawn about the sample. Data sets of large and small sample size are compared and contrasted to explore the stability of variation and spread in the varying samples (Ben-Zvi). The lessons, based on explicit consideration of variation and Watson's (2009) investigation model in Figure 1, provide the opportunity for students to think about how certain they are about their conjectures and conclusions drawn. In addition the lessons allow students to use their knowledge of the context to support their thinking and explain how the context of the data influenced their thinking, as advocated by Watson (2006) and supported by the RME instructional design theory (Treffers, 1993).
The sequence of learning experiences developed is made up of seven lessons. In Lesson 1, the intention is to get students to generate data by recording information about attributes familiar to them. Measurements such as height, belly-button height, and foot length are gathered easily by individual students and teachers can then gather large data sets from the CensusAtSchool data base using the random sampler accessed from the education section of the Australian Bureau of Statistics website. (www.abs.gov.au/censusatschool). The data in Lessons 2 and 3 are generated by the students' class. Lesson 4 uses data generated from a second class (Class B) either at the same school or another school. In Lesson 5 the students are introduced to a data set (Class C) generated by the Australian CensusAtSchool database. The Class C data are used in conjunction with the Class B data to compare the two classes for a number of different attributes. Looking at the three data sets (Class A, B, C) collectively in order to compare classes, thinking becomes more abstract by Lesson 6. The focus in Lesson 7 is on comparing the results from interrogating a large data set (n = 200) generated by the CensusAtSchool database and comparing the results with those gleaned from the smaller data sets.
Although each lesson has a particular focus, the aim is to build students' knowledge of and experience with data analysis and then understanding of DEVELOPING A SEQUENCE OF LEARNING EXPERIENCES IN STATISTICS
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covariation by continually constructing various graph types and using a number of different data sets of different sample size in TinkerPlots. Throughout the seven lessons connections to the context of the data are made explicit by asking students to justify the conclusions drawn and the hypotheses made based on the context of the data and the questions explored.
The progression from small to large sample size and the connections to Watson and Fitzallen's (2010) hierarchy of graph creation and interpretation across the seven lessons are represented in Figure 12. In Lesson 1 the fundamentals of the Concept of Graph [G1] are established. By Lesson 7 there is the expectation that students will have encountered large data sets and constructed graphs to explore the relationship between two attributes, thereby, operating at Level 5 [G5] of the hierarchy to identify outliers, make informal inferences based on the graphical representations of the data, and express an understanding of covariation.
Figure 12. Change in sample size and lesson focus across the sequence of lessons.
It should be noted that the sequence of learning experiences does not detail when and how students learn about using TinkerPlots. Instructional advice about using the software package is not explicitly addressed because it has the potential to draw the focus away from developing an understanding of covariation to highlighting the technical issues associated with using TinkerPlots. Developing an understanding of using TinkerPlots is, however, implicitly embedded in the sequence of learning experiences through the use of multiple graph types - starting with stacked dot plots to display data for one attribute - and the need to construct various graph types and reconstruct them for multiple data sets. Constructing graphs noted in each of the lessons will facilitate beginners' use of TinkerPlots together with building their knowledge of covariation as they progress though the seven lessons. The details for each of the lessons are presented in Table 1 (Appendix 1).N. FITZALLEN & J. WATSON
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Also worth mentioning is the issue that the seven lessons require students to develop an understanding of statistical concepts other than covariation. This is a consequence of adopting developmental and inquiry-based learning approaches. These approaches promote learning that makes the connections among multiple concepts instead of learning about individual concepts separately without the opportunity to acknowledge the connections among different statistical ideas.
CONCLUSION
The development of the practical learning sequence is embedded in an overarching organiser for structuring investigations undertaken by students because investigations are at the heart of statistics education (Watson, 2009; Wild & Pfannkuch, 1999). A related model for statistical investigations is then introduced and links to the curriculum and research are considered. In particular a framework for graphing is explored because of its intimate links to understanding covariation at the introductory level, which in turn leads to an acknowledgement of the contribution of increasing sample sizes.
Making strong connections to the curriculum is vital when planning learning sequences, not only from the perspective of covering all the required mathematical content at the appropriate time but also from the perspective of including the General Capabilities, which are "key dimension[s] of the Australian Curriculum, [and] are addressed explicitly in the content of the learning areas" (ACARA, 2012b, p. 3). The prominence of the General Capabilities in the Australian Curriculum places an emphasis on providing an integrated approach to learning that encompasses the development of seven capabilities that include critical and creative thinking, information and communication technology (ICT) capability, and numeracy. The sequence of lessons described in this chapter encompasses these three capabilities comprehensively by utilising a statistical investigative approach (critical and creative thinking) to focus on the development of understanding of covariation (mathematical content/numeracy) through the application of TinkerPlots (ICT capability). This has implications for the future implementation of The Australian Curriculum: Mathematics (2012a) as it provides a model that teachers could use to foster the integrated and interconnected mode of learning advocated by the curriculum.
Although the purpose of designing the sequence of lessons was to ensure a developmental approach was taken to the introduction and application of covariation for students, the approach taken could be applied to other content areas in the Statistics and Probability content strand of The Australian Curriculum: Mathematics (ACARA, 2012a). The use of increasing sample size and application of various graphical representations across the sequence of lessons would accommodate an explicit emphasis on other specific statistics content such as measures of centre as well as the big ideas of statistics such as distribution, variation, expectation, randomness, and informal inference.DEVELOPING A SEQUENCE OF LEARNING EXPERIENCES IN STATISTICS
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APPPENDIX 1
Table 1. Sequence of Lessons
Lesson Sample Size, Graph Framework, and Lesson Focus
1 Data generated by individual students – G1, G5
Using the features of TinkerPlots and creating stacked dot plots with bins on the horizontal axis initially, and changing the size of the bins. Creating horizontal and vertical value bar graphs. Describing the case-based representation of data. Making connections between the data in the plot and the data on the cards. Accessing reference lines. Exploring variation in an attribute. Discussing the variation in a small sample. Conjecturing about the likelihood of other samples of similar size having the same or different values based on the context of the data. Establishing how the context of the data influenced the data by considering how and from whom the data were collected.
2 Data generated by a class – Class A (n=21) – G1, G5
Constructing graphs. Creating stacked dot plots and making the transition from the axis split into bins to using a continuous scale. Determining the range of the data. Describing variation and spread. Comparing values of data points. Using the mode as a reference point to describe the variation evident. Sorting data. Exploring the variation in other attributes. Comparing the spread and variation evident in the graphs and suggesting reasons for difference based on the context of the data. Hypothesising about where data points would be placed in the graph if new data were collected.
3 Data generated by two classes – Class A (n=21) Class B (n=21) – G1, G3, G4, G5
Constructing various graph types. Creating split stacked dot plots with bins and continuous scales on the horizontal axis. Comparing groups. Identifying outliers. Accessing hat plots. Discussing the differences between the two data sets. Using the context of the data to justify decisions about outliers and support conclusions drawn from the graphs about the data.
4 Data generated by two classes – Class A (n=21) Class B (n=21) – G2, G3, G4, G5
Constructing various graph types for various attributes. Creating scatterplots to display covariation. Describing the spread of the data. Identifying outliers. Removing outliers and then describing the changes to the spread and distribution of the data. Using hat plots to compare groups when comparing the data from the two classes. Using the context of the data to justify decisions about outliers and support conclusions drawn from the graphs about the data. Identifying covariation.
(Continued)N. FITZALLEN & J. WATSON
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Table 1. (Continued)
Lesson Sample Size, Graph Framework, and Lesson Focus
5 Data generated by Class B (n=21) and the CensusAtSchool sample (Year 5/6 students) – Class C (n=21) – G2, G3, G4, G5
Constructing various graph types for various attributes. Creating scatterplots. Identifying the trend. Hypothesising about the placement of additional data points based on the context of the data. Making connections between the data points and value of the data points using reference lines. Describing the relationship between two attributes to explore covariation. Comparing Class B and Class C for various attributes and using the context of the data to justify decisions.
6 Data from Class A (n=21), Class B (n=21), and Class C (n=21) – G2, G3, G4, G5
Looking collectively at plots from the three data sets and comparing classes using a variety of graph types. Identifying similarities and differences in the spread, distribution, mean, mode, median, and hat plots. Making suggestions for causes of variation from an understanding of the context. Thinking about the messages in the data and making inferences from the data based on the context of the data.
7 CensusAtSchool sample (Year 5/6 students) – (n=200) – G2, G3, G4, G5
Comparing data sets, particularly the differences among small and large data sets for the same attributes. Making inferences from the data. Justifying statements about covariation and the relationships among attributes.

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Key competence development in
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Gordon, J., Rey, O., Siewiorek, A., Vivitsou, M.,
& von Reis Saari, J.
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Introduction
This literature review outlines the key issues in relation to the definition and implementation of key competences in school education in Europe, at the classroom/school as well as systemic level. In order to focus the areas explored by our network's team of researchers, a matrix was formulated and agreed upon by all contributors to the literature review (see annex 1). As is evident from the matrix, the key focus is on analyzing implementation policies and practice, which constitutes the network's core remit. The review examines the various interrelated dimensions of innovative learning environments (formal and non-formal), curriculum design, teacher training, pedagogy and socio-economic issues, across a wide range of sources including scientific reviews and books, reports from EU and other international organizations, as well as national key reports about specific relevant initiatives.
The literature review is divided into three sections, each with a specific focus. The first section written by the University of Helsinki examines the implementation of key competences across school education in Europe, from the point of view of practice. The classroom and non-formal learning environments are therefore considered in relation to technology enhanced as well as collaborative and multidisciplinary learning, and the key competences of teachers. The authors consider the approaches to implementation from the national, regional and school levels.
The second section of the review is written by the European Institute of Education and Social Policy (EIESP) and analyzes approaches to the implementation of key competences in school education at systemic level, focusing on curriculum design and implementation and socio-economic aspects, as well as taking a closer look at some of the literature on transversal and cross-curricular key competences.
The final section written by l'Institut Français de l'Éducation (IFE), belonging to l'Ecole Normale Supérieure de Lyon (ENS) provides a specific example of how key competence development has been framed in French-speaking countries, considering particularly the definition and integration of competences in schools in France, and the assessment approaches used.
This general literature review [D3.1 (a)] addressing a variety of issues related to key competence development in schools in Europe is complemented by another review [D3.1 (b) by D. Pepper] specifically on approaches used for the assessment of key competences. Considering the importance of assessment for the successful implementation of key competences in school education, it was decided this topic deserved separate attention.
This current version of the general literature review is in its first draft and will be updated yearly throughout the project's lifetime. The aim of this first version was to cover the bulk of topics detailed in the matrix, and be guided on the basis of these initial findings as to which areas we would like to investigate further in 2013 and 2014. One such area which we plan to explore in next year's update is the extent to which countries use research and evidence to inform the implementation of policy in this area. When such an evidence-based approach is used, what are the outcomes and where evaluations exist what do we learn from them? The intention is to analyze international influences, such as how the latest PISA student data has impacted on various countries' formulation of key competence policy, also looking closely at
4
a very pertinent OECD working paper
1
(published as this first draft of the literature review
was being finalized) concerning how diverse stakeholders at various levels are involved in curriculum innovations in different countries. In the forthcoming updates to the literature review the intention is to better integrate the results of the various sections, and to include more examples of relevant literature emanating from countries represented within the network.
Defining competence – use of terminology
In an attempt to clarify the concept of 'competence', Weinert (2001) relates the term to the Greek notion of arete, meaning excellence, in the sense of being the best; also with the Latin term virtus, a kind of moral excellence, while it is generally understood as being concerned with 'what people can do rather than what they know'. The concept applies to individuals, social groups or institutions, and the words 'competence' and 'competency', or its plural form 'competencies', are often used interchangeably. This use neglects the large variety of meanings of 'competence', that can be captured by the terms 'ability', 'aptitude', 'capability', 'effectiveness' and 'skill'. Resulting from this overgeneralisation, the notion of competence, and its plural 'competences', has been replaced by the narrower version of 'competency', or the plural form 'competencies' recently. The latter denote discrete skills and activities that

individuals can perform (Allan, 2011).
The origin of the discussion on competence can be traced back to 1996 (Dabrowski et al., 2011) when the Council of Europe defined active citizenship as the capacity to accept responsibilities, participate in group decisions, resolve conflicts in a non-violent manner, and play a part in running and improving democratic institutions. In this way, competences are introduced in the rhetoric of official documents with a view to cover essential socio-political aspects in the life of the European citizen. The delineation of 'critical judgment' expresses the need to provide an argument-based 'opposing' force to information dissemination by the mass media and advertisers. Apparently, this policy aims to empower citizens with cognitive tools enabling knowledge expansion beyond the limits of information manipulation and propaganda of the media.
Recent studies and research demonstrate that there is a broad range of terminology used in the 27 member states linked to philosophies of education, history, approaches to learning and to curriculum development, recent reform processes and international influences whether research (e.g. DeSeCo) or policy "learning" through EU and other funded programmes, etc.
2 We have included below 2 boxes with quotes from Hoskins & Deakin Crick (2010) with their suggested definitions of competence and key competences (our bold). Similar terms (once translated into English for the purposes of comparative work) may not cover the same content and approach. Different terms do not necessarily indicate different philosophies, approaches and content.
The level of influence of the vocational sector and the labour market is variable among the countries, but may be significant.

1
OECD (2012) Bringing About Curriculum Innovations: Implicit approaches in the OECD AREA, OECD Education Working Papers No. 82
2
J. Gordon et al (2009)
5
"What is a Competence?
3
A competence refers to a complex combination of knowledge, skills, understanding, values, attitudes and desire which lead to effective, embodied human action in the world in a particular domain. One's achievement at work, in personal relationships or in civil society are not based simply on the accumulation of second hand knowledge stored as data, but as a combination of this knowledge with skills, values, attitudes, desires and motivation and its application in a particular human setting at a particular point in a trajectory in time. Competence implies a sense of agency, action and value.
To understand competences, the spotlight is on the accomplishment of 'real world tasks' and on a multiplicity of ways of knowing – for example, knowing how to do something; knowing oneself and one's desires, or knowing why something is important, as well as knowing about something. This is similar to Delors' 4 pillars of learning developed for UNESCO: "learning to live together, learning to know, learning to do and learning to be."(Delors, 1996). Importantly, competences are expressed in action and by definition are embedded in narratives and shaped by values – this action, or way of doing something is more important or desirable than that one because it leads to a particular end. Just as a competence is recognised in the context of the real world, the development of competences is also based in real world experiences and takes into account the full spectrum of learning opportunities (informal, nonformal and formal learning) throughout the life span. Perhaps the most thorough recent exploration of the concept was undertaken by OECD in DeSeCo1. Drawing on this work, the term competence was defined by Rychen and Salagnick (2003, p. 43) as:
the ability to successfully meet complex demands in a particular context through the mobilisation of psychosocial prerequisites (including cognitive and non-cognitive aspects) and as
the internal mental structures in the sense of abilities, dispositions or resources embedded in the individual" in interaction with a "specific real world task or demand."
Source: Hoskins & Deakin Crick (2010)
"What are key competences?
4
The DeSeCo programme identified four analytical elements of key competences: they are multifunctional; they are transversal across social fields; they refer to a higher order of mental complexity which includes an active, reflective and responsible approach to life; and they are multi-dimensional, incorporating know-how, analytical, critical, creative and communication skills, as well as common sense. Within this project, a number of OECD countries were asked to list which competences they considered to be key competences. Four groups were frequently mentioned in the country reports: (i) Social Competences / Cooperation; (ii) Literacies/Intelligent and applicable knowledge: (iii) Learning Competences/Lifelong Learning; and (iv) Communication Competences (Trier, 2002)."
Source: Hoskins & Deakin Crick (2010)

3
Hoskins, B & Deakin Crick R., (2010)
4
Ibid
6
In terms of the logic that underpins competence development, A. Tiana et al (p308) refer to Basil Bernstein's work in the 1990s on the difference between the economic and social logics in the term competence with the social logic focusing on the democratic perspective of social development while the economic logic focuses on human capital. They quote from Bernstein on the characterisation of the social logic of competence:
a) The anticipation of a universal democracy of acquisition: all persons are intrinsically competent and they are all in possession of common procedures, so there is no deficit as such;
b) The individual is active and creative in building a valid world of meanings and practices: in this case, differences emerge between persons, but not deficit;
c) The individual is self-regulating with a positive evolution, this development does not overtake via formal instruction: official socialisers become suspect, because the acquisition of these procedures constitutes a tacit, invisible act that is not subject to public regulation;
d) A critical and sceptical view of hierarchical relations, derived from the fact that, in certain theories, the role of socialisers should be limited to facilitation, adaptation and organisation of context: from this perspective, competence theories smack somewhat of emancipation." (Bernstein, 1996). P308
They go on to quote Angel Pérez Gómez (Pérez Gómez, 1983) asserting in what has become a classic Spanish text on pedagogy, "the student's learning in the classroom is much more complex than the learning gained from laboratory experiments. It is a situational learning, contextualised according to the structure of academic tasks and the ecological climate created through the negotiation of the social group' (Tiana et al 2011 p309).
And the purpose of key competences:
Adequate skills and competence are crucial to participate in working life, but also in social and civic life. They are the basis of community cohesion, based on democracy, mutual understanding, respect for diversity and active citizenship. Creativity, openness and interpersonal competences are also necessary for personal fulfillment and happiness.
Source: New Skills for New Jobs. Action Now, EU 2010
Competence frameworks and levels
The European Reference Framework and other international frameworks
The European Reference Framework of Key Competences for Lifelong Learning (OJEU, 2006) defines key competences as knowledge, skills and attitudes applied appropriately to a given context (Pepper, 2011). The Framework identifies eight key competences as necessary for personal fulfilment, active citizenship, social inclusion and employment; communication in the mother tongue; communication in foreign languages; mathematical competence and basic competences in science and technology; digital competence; learning to learn; social and civic competences; sense of initiative and entrepreneurship; and cultural awareness and expression. Each has a concise definition of its scope and all emphasise critical thinking, creativity, initiative, problem solving, risk assessment, decision taking,
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communication and constructive management of feelings. The latter are also known as 'transversal skills'.
Seemingly, the need to interpret the concept of 'competence' leads to its fragmentation instead: key competences, competencies and transversal skills appear on stage. Moreover, considering the feature of cultural diversity that characterises Europe, a unifying pattern for a pan-European consistent application of the term constitutes a utopia rather than an achievable goal. To add to the blurriness of the situation, the central decision to leave further interpretation to Member States depending upon the specific contexts of their education systems acknowledges the incompatibility of a 'melting pot' ideology with the pluri-cultural nature of Europe.
The discussion surrounding key competences in Europe is related to a wider, international push for educators, policy makers, citizens, and learners to come to terms with the changing living and working ecologies of this 21
st
century. These frameworks share overlapping
themes.
The transversal skills outlined in the European Reference Framework are in particular closely related to the recommendations set by other organisations, as well as scholars. The OECD, for example, has focused on three broad areas: The capability to use tools (including language) interactively, the capability to interact in heterogeneous groups, and the capability to act autonomously (OECD, 2005). The organization frames these competencies as a response to the demands of modern life.
Tapio Varis has emphasized the importance of critical thinking, problem solving capacity, improvements in expression communication and interaction and civic participation and active citizenship, in the UNESCO framework of competences for the 21
st
century, further stating
that "Instruction must offer tools for finding out the causes and effects of different phenomena and for drawing conclusions, which at its best leads to growth into active, critical and mediacritical world citizens" (Varis, 2011, p. 10). In the UNESCO report, ICT Transforming Education (2010), technological development is seen as integral in the development of new skills and ways of learning.
The World Bank has focused on Technology literacy, information management, communication, working in teams entrepreneurialism, global awareness, civic engagement, and problem solving (Wagner, 2012). Similarly the industry funded Assessment and Teaching of 21
st
Century Skills (AT21CS) consortium advocates for four categories: ways of thinking, ways of working, tools for working, and living in the world.
Wagner (2010) argues that the set of core competences every student must master before the end of high school is:
1. Critical thinking and problem solving (the ability to ask the right questions)
2. Collaboration across networks and leading by influence
3. Agility and adaptability
4. Initiative and entrepreneurialism
5. Accessing and analysing information
8
6. Effective written and oral communication
7. Curiosity and imagination
Critical scholars such as Gardner (2011) and Jenkins (2006) call for a focus both on creativity and critical thinking.
Themes that underlie all these frameworks, include increased collaborative, student driven, and technologically enabled learning, the cultivation of creative thinking, as well as fostering students' problem solving abilities and meta-cognitive resources, which have all been major themes in education science and scholarship.
National sets and frameworks
The European Reference Framework of key competences is generic in so far as it addresses all audiences of learners, children and adults, no matter their age or stage in their learning journey. This raises questions about its adaptability to different audiences. In the member states approaches skills, competences, key competences, etc. are grouped in many different ways. The table annexed maps the terminology used by the different member states with examples of how it is used in policy and curriculum. As much as possible the terms used have been given in the language of the country (see Annex 2).
An issue for reflection is what is meant by a "framework";
- In qualification terms it generally includes a notion of reference levels and progression routes. For example in the case of a National Qualification Framework (NQF) or a European Qualification Framework (EQF).
- Set of competences to be attained - by when?
- Set of competences with stages appropriate to ages/grades/levels and/or objectives.
- Foundation that underpins the curriculum (that may be otherwise subject based) or runs transversally across it.
This issue should be raised because of its likely implications for implementation and assessment.
The following table summarises the scope of the key competences as they are defined and used in different countries:
Types of approaches: Examples of countries:
Sets of broad cross-curricular skills or competences that are designed to ensure continuity across the formal learning system from early years education to higher education
None of the EU Member States.
Goals and/or cross curricular competences or skills that cover the whole school system
Austria (social and personal competences)
9
(primary education to the end of secondary school)
Belgium (NL), Bulgaria, Czech Republic, Estonia, Finland, Ireland (3 sets but similar), Sweden (early years, school and extracurricular activities), UK: Wales (3 to 19 curriculum), Northern Ireland, Scotland
Cross-curricular competences that address specifically compulsory or basic education only or mainly
France, Belgium (FR), Luxembourg (crosscurricular key competences currently in the process of being implemented do not yet

address also the upper secondary cycle).
Sets of key competences or skills or aims that
address specific levels of the school systems
(primary, lower secondary and upper
secondary or basic/primary education and
upper secondary)
Cyprus, Germany, Netherlands, Portugal,
Slovenia (VET only), Spain,
UK (England)
Sets of key competences / cross-curricular
key competences that are valid for students
and for teachers
None of the EU Members States
Source: Gordon et al. (2009)
Levels
A further issue raised by the notion of framework is that of levels or reference levels. The different frameworks in Europe (the Key Competences Framework, EQF, Common European Framework of Reference for Languages, the Common European Principles for Teacher Competences and Qualifications) contain two different approaches. Frameworks that serve for validation purposes identify reference levels and are linked to assessment systems (whether formal or through the validation of non-formal and informal learning). This is the case at EU level for EQF and the CEFRL. The other type describes learning outcomes desired at the end of a period of learning or discernible in an individual's professional qualification. These competences are still under review in terms of if and how to measure them. This is the case for the framework of key competences, the teacher competences as well as the Tuning generic competences that were developed for for higher education. They all contain elements of cross-curricular key competences whether or not they are made explicit
5
.
Some of the country approaches include an adaptation appropriate to the age of the pupils though not all are specific about how this aspect is implemented (see Annex 2). A useful example of an approach to social and emotional learning competences that were fully developed across a set of levels is contained in the report prepared in 2006 for the Welsh Assembly Government, Developing a Universal Toolkit for Measuring the Impact of Provision on the Outcomes Achieved by Young People for Extending Entitlement and Learning

5
Gordon et al. Op cit.
10
Pathways 14-19
6
(AlphaPlus 2006). A table showing an example of competences and levels from this report is in Annex 3.
The issue for further examination and implementation is about if and how countries build into their curricula age or grade appropriate adaptations of the skills/competences. Theoretically choices could move towards a level approach where pupils in the same year may have attained different levels for certain competences or an age-linked approach. There is an input and an output element to this discussion: whether competences are broken down by expectations in each school year/grade or whether pupils' attainment of the competence is assessed against a set of reference levels that is not totally based on the year/grade. This question is relevant for the transversal key competences in particular rather than for those that tend to be linked to a school discipline. In terms of implementation, the need to adapt the EU key competences to be acquired by all pupils by the end of, for example, compulsory education, is an area for further reflection.

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AlphaPlus Consultancy Ltd (2006)
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Section 1: Key competences in practice
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1. Key Competences for Europe
The interconnectedness sustained by technological advancement in the 21st century allows for the creation of a trans-European educational space mobilized by networks of institutions and experts (Lawn & Grek, 2012). These transnational flows of people, ideas and practices across borders also generate the necessity for a unifying pattern of principles targeting a consensus in terms of educational goals, teacher accreditation and comparable learning outcomes. Toward this end, the concept of competence has been embedded in the rhetoric of official documents and influenced the curricula of teacher education since the early 1990s, thus mapping out standards and frameworks across Europe.
The implementation of these intentions, however, is far from unproblematic. Research findings conducted by the Council of Europe indicate that, in many parts of the continent, initial teacher education did not adequately prepare beginning teachers to cope with the diversity they met in their classrooms (Council of Europe, 2008). On the contrary, the application of competence-based standards acted as enabler of stratifications encouraging a rigid over a complex type of thinking. According to the survey results, existing standards firmly entrench teachers' novice and incompetent identities rather than promote student teachers' thinking (Roy, 2003).
The problem that arises then (through our reflections upon Allan's (2011) considerations of the ways teachers integrate competences to diversity) addresses the multiple appearances of the concept. Does a competence-oriented pedagogy embrace educational needs across member-states? Does such targeting cater for a 21st century pedagogy committed to performance and values rather than being dissociated from the latter?
In this chapter, in view of the complexity associated with defining competences as referred to in the introduction of this literature review, we understand 'competence' as a generic term which denotes the knowledge, skills and attitudes that enable learners to cope with the requirements generated by rapid socio-technological changes in the new era.
1.1. Challenges to the situated competence-based learning perspective
Defining learning in terms of competences underscores the importance of considering knowledge in action and highlights the need for tying up knowledge acquisition with task resolution. In this respect, the development of key competences requires situated learning, or learning linked to a specific context and to concrete tasks to acquire the necessary competence.
The situated learning perspective emerged during the 1980s, when social scientists began to analyse cognitive processes as aspects of interaction, and some cognitive scientists to consider the social arrangements of learning as fundamental in determining what is learnt. Suchman (1987) argued that the cognitive concept of action, a process governed by plans with adaptations to unanticipated aspects of situations, is inferior to an account in which plans constitute resources but do not determine the course of situated action. Therefore, the perspective for the organisation of action as an emergent property of interactions between actors, and between actors and the action environment was established (Suchman, 1987; Lave et al., 1991). Analysis of problem solving in everyday settings indicated that problems involving mathematical reasoning were better understood as emerging from interactions between people and resources than as products of mental operations with and on symbolic
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representations. Hutchins' (2000) studies of reasoning and representational practices by navy ship-navigation teams and of remembering by the system of people and technological resources in an airplane cockpit constitute influential analyses of cognitive processes as aspects of social practice.
The application of the situated competence-based model to practice, however, resists the consistency and systematic approach required for a unifying European educational space. In a discussion on the findings of an OECD report concerning a survey conducted in 17 EU countries in 2009, Pepper (2011, pp. 340-41) reveals that few of the participating countries appeared to have either specifically defined the terms or developed clear assessment policies in relation to them. As Pepper (2011, p. 341) argues, difficulties in assessing competences arise from unpacking learning outcomes. This, in turn, is linked to a multiplicity of interconnected factors, such as the need to address knowledge, skills and attitudes, the application of competences to a range of real or authentic contexts and the threshold at which a certain level of competence is acquired.
Another challenge relates to the specification of key competences in sufficient detail to plan and assess learning on the one hand. On the other, this specification process should not exceed beyond the level where competence development is reduced to a series of procedural tasks that are completed without full appreciation of underlying concepts.
The subsequent implication of varying interpretations of competence between and within countries couples with varying unpacking of learning outcomes in relation to competences. To add to these, the consideration of the previously discussed dissociation of teachers from their own value system for the benefit of performance, as expressed by Allan (2011) is in agreement with Pepper's (2011) underpinning the need to prioritise concept understanding over procedural knowledge.
The issue of delineating the optimal conditions for learning key competence arises.
According to Tiana et al (2011, p. 318), a new professional culture based on a shared desire of all educational agents to create the most favourable learning conditions is essential for the future of education.
1.2. Toward a new school culture
Educational research emphasizes the need to understand the learners' active cognitive operations (Dewey, 1938), strategies, stages of conceptual development and the nature of the experiential processes of assimilation and accommodation. Perception-conception and action are seen to mutually interact (Dewey, 1896: coordination). A social perspective emphasizes the view that the environment includes (often physically but always conceptually) other people with whom the learner participates in activity systems (Vygotsky, 1978; Wertsch, 1991). The individual and society are mutually interchanging: 'culture is the capacity for constantly expanding the range and accuracy of one's perception of meanings' (Dewey, 1916, p. 123).
At a European level, therefore, the emergence of a culture supportive of the reframing and reorientation of a competence framework would need to incorporate:
☐ integrated curricular development
☑ changes in practices towards environments that facilitate learning
☐ coordinated teaching, and
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☐ teaching models leading to tasks aiming to generate competence and experiences through different forms of cultural expression (e.g., knowledge, values, rules etc.)
While inspired in part by a radical shift in the structure of labour and technology, the Key Competences relate to long-standing educational theories, as we shall discuss further in the following section. However, we also propose that in order to encourage the development of transversal skills, the cross-disciplinary competences that are echoed as well in other frameworks, learners, teachers and communities should be supported in developing new concepts through innovative, non-traditional avenues and venues in which learning can take place.
Considering the Knowledge, Skills, Attitudes, Values, and Ethics (KSAVE) model (Binkley et al., 2012) that conceptualises 21

st
century skills as a framework of four overarching
categories, in the following sections we discuss ways of thinking, and ways and tools for
working that can be part of educational curricula across Europe. This integration allows for
the necessary conditions to emerge so that teaching and learning environments contribute to
the development of competences aiming for improved life and career opportunities at the
individual level. And, at the collective level, for improved well-being through active
citizenship, and an enhanced personal and social responsibility.
1.3. Pedagogy for competence development and the importance of transfer
One of the underlying aims of Key Competence development is fostering life-long learning,
the capability of students to use what they have learned in schools, but more importantly, to
continue learning throughout the lifespan. This is intrinsically liked to the idea of transfer
(Thorndike and Woodworth, 1901), that is, the capability of students to transfer knowledge,
skills, and meta-cognitive skills to new situations, and ultimately to the "real" world outside of
school. This is a long-term goal of educational theorists, which educational science has
been actively addressing. The key outcomes of this research emphasize the need for
contextualised learning, constructivist approaches that combine student experimentation with
explicit content teaching, and dynamic assessment—aspects of what Bransford and
Schwartz (1999) have called Preparation for Future Learning.
Pedagogy, therefore, that would aim for competence development would also enhance
learner efforts for meaningful experiences through goal-directed, active, authentic and
collaborative tasks (Jonassen et al., 2008). These would tie with flexible national curricula
that consider project-based learning as an organic part along with systems of evaluation
catering for the needs of innovative learning environments. The teaching methods, then,
required for the development of key-competences should be oriented towards
interdisciplinary, cross-subject teaching, and team oriented learning blended
withindividualised approaches and project-based work. (Gordon, Halasz, Krawczyk et al.,
2009, p. 162).
In this way, as Gordon, Halasz, Krawczyk et al. (2009, p. 227) argue, effective pedagogy in
competence development can become an attainable goal, given that it is life-long oriented;
promotes active engagement; fosters individual and social processes and outcomes;
depends on the learning of all those who support the learning of others; and is built upon
consistent frameworks that focus on convergent aims.
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2. Key competences and transversal skills: examples from
education research
2.1. Key competences
The framework of key competences and transversal skills underscores the need for complex,
collaborative and trans-disciplinary learning environments. In this section, brief examples of
such environments are given relating to each of the eight key-competences, as well as
examples of particular schools and curricular reforms that have fostered the development of
key competences. Within Europe and worldwide, there is an established wealth of research
to draw upon regarding the development of key competences. While emerging technologies
present exciting ways to transform the teaching of key competences, and broaden students'
educational experiences, methods for developing key competences should also draw from
established programs that have shown to be successful over time. Thus it should be noted
that barriers to implementation, addressed again below, are more likely to be those of
administration, evaluation, and fiscal resources; many schools are already engaged in
exemplary teaching of Key Competences.
The key themes in teaching Key Competences for the twenty-first century relate to creating
meaningful education based on real problems and engagement, interdisciplinary
environments, and the explicit teaching of self-management (meta-cognition and selfregulatory skills).
Communication in the mother tongue
Using location-based technologies and the story-telling approach, Hansen and colleagues
(2012) investigate the development of drama spanning from pure art pieces to structured
learning experiences. According to the researchers (Hansen et al, 2012), the evidence of the
whole operation indicates the educational value of the technology-based intervention that,
rather than focusing on the use of technology, uses the technology to transform an
educational experience. Moreover, the application of drama techniques opens up
perspectives for mother language teaching and learning.
Communication in foreign (multiple) languages
Within European educational systems, the issue of education for competence in multiple
languages moves past the idea of foreign languages. It includes language education in
nations that have a plurality of official and unofficial languages. Multilingualism is a key goal
for maintaining national languages, and in many respects shares the pedagogy of first
language learning.
Students can be engaged in multiple language learning through experiencing its usefulness.
Travel and study abroad are not economically feasible for all students, however, the ability
for technology to facilitate intercultural communication where travel is not possible is
promising. Fisher, Evans and Esch (2004) report on a program that encouraged the
structured use of communications technology to enhance student learning of French,
involving students in Belgium, England, France and Senegal. This program encouraged
young people to learn autonomously and communicate freely, but with scaffolding and
stimulated activities to inspire their work. Not only did students have the opportunity to
communicate with native speakers, the research suggested they developed selfmanagement skills, collaborative skills, and intercultural learning. Thus, an environment was
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created that was also conducive to the development of other key competences and
transversal skills.
Foreign language learning
Mobile learning has been used to support a range of learning activities, for example
language learning using text messaging (Cavus & Ibrahim, 2009; Kennedy & Levy, 2008; Lu,
2008).
Math competence and basic competences in science and technology
Games have been widely implemented in teaching children math. For example, Moreno &
Durán (2004) conducted a study about discovery-based learning through computer game.
Other examples of successful integration of games into math curriculum could be found in
the literature, such as, Clements and Sarama (2007) and Kebritchi, Hirumi, and Bai (2010).
Another key example for mathematics is the Realistic Mathematics Education curriculum,
developed in the Netherlands by Fruedenthal and colleagues to address the problem of
transfer (Fruedenthal, 1968) and discussed further below. This shares the theme of offering
students with real problems that promote transfer, problem solving skills and engagement.
Similarly, Schwartz and Moore (1998) find students better understand and are able to
transfer their statistical learning when it is grounded first in their own experimentation.
Grounded in several different fields of learning science, student experimentation with
meaningful problems combined with explicit instruction has been shown to be a strong model
for competence development. The idea of engaging students in inquiry and experimentation
in mathematics and science is further discussed in the section on innovative learning
environments below.
Digital competence
Digital competence moves past the idea of simply being able to consume digital media, but
to have the critical thinking and creative skills that citizens need in order to be effective
receivers and creators in the digital realm.
Zahn, Pea, Hesse and Rosen (2010, p. 410) describe students developing websites for an
online history museum while developing skills in critical thinking, creativity and collaboration.
Trans-disciplinary, inquiry based learning can foster key competences and content learning
at the same time that transversal skills are developed.
It is important to note the digital competence increasingly is important to general literacy, as
well as to social and civic competence, discussed further below.
Learning to learn
Uncertain economic outlooks and the need for flexibility in working life have inspired a
European focus on life-long learning. True lifelong learning is centred on the idea of learning
how to learn, that is, the problem solving skills, self-regulation, self-efficacy and creative
flexible thinking needed to master new tasks and environments.
Blending a problem-based estimation scenario in a mobile-device-supported cooperative
learning environment, Lan and colleagues (2010) examined the ways elementary students'
cooperative skills and estimation strategies improved. This creative intervention benefited the
development of metacognition knowledge of the students' estimation strategies.
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Ultimately, learning to learn comes back to the idea of transfer (Bransford & Schwartz,
1999), a key theme in the education research discussed here, and one that points towards
certain modes of education: constructivist and experiential surely, but also grounded in
explicit teaching and teaching of self-management.
Social and civic competences
Social and civic competences can on one hand be seen as building the capacity for young
people to function in a pluralistic society where collaboration and cooperation are important.
An example of such a project is when digital storytelling techniques on mobile devices were
applied in a project that involved guiding visits to cultural heritage sites (Lombardo &
Damiano, 2012). The experience confirmed the importance of the approach in the emotional
engagement of the audience. This indicates the presence of aspects of embodied cognition
on the one hand; on the other, the potential of mobile technologies to contribute to the
establishment of social bonds among the members of a group is made visible.
Another important aspect of social and civic competences is the potential for citizens to
contest policy, resist influence and develop their sense of agency (or self-efficacy, see
Bandura, 2000) as democratic citizens. This is a difficult goal for traditional some structures.
In this report, innovative environments that foster creativity and agency, and technology
learning that fosters critical literacy, both discussed further below, are also seen as key
elements of social and civic competences.
Sense of initiative and entrepreneurship
Initiative and entrepreneurship are dependent on students' development not only of skill, but
of a sense of agency. That is, they are highly dependent on students' self-efficacy and
perseverance in difficult tasks. It is not only individual self-efficacy that is important, but
collective self-efficacy as well as Bandura (2001, pp. 17–18) argues:
"As globalization reaches ever deeper into people's lives, a strong sense of collective
efficacy to make transnational systems work for them becomes critical to furthering
their common interests…The magnitude of human problems also undermines
perceived efficacy to find effective solutions for them. Worldwide problems of growing
magnitude instil a sense of paralysis that there is little people can do to reduce such
problems. Global effects are the products of local actions"
Self-efficacy and collective efficacy are theorized as being developed through mastery
experiences as well as social modeling. The 21
st
Century classroom, through providing
open-ended challenges and scaffolding collaborative work needs to serve as a testlaboratory for students in developing their self-efficacy and developing initiative and
entrepreneurship competence.
Young people will "… have to cultivate multiple competencies to meet the ever-changing
occupational demands and roles" (Bandura, 2001, p. 11). To turn other competences into
initiative and entrepreneurship action, students need to be armed with self-efficacy.
Cultural awareness and expression
New techniques, such as collaborative digital storytelling on mobile devices have been
applied also to cultural and creative education. As discussed previously in the social
competences section, in a project that involved guiding visits to cultural heritage sites
(Lombardo & Damiano, 2012), project researchers used a location-based character-enacted
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storytelling technique. Through this use of a character-based movie, a memory of the
learning experience emerged. Such approaches integrate technology with cultural and
creative education. Collaborative projects designed for sharing, such as digital storytelling,
also open the possibility of creating educational venues for international learning, as stories
are shared between and created across cultures.
2.2. Transversal skills
Critical thinking, creativity and collaboration
Transversal skills like critical thinking and creativity need to be situated within the
environment of the 21
st
Century. As Hennessy et al. (2005, p. 179) note, critical thinking can
be lost in the deluge of information available on the Internet. Lewis et al. (2010, p. 7) argue
that in order to be prepared as critical consumers and users of modern media and
technology, youth also need to understand how to be producers and creators of it:
By together questioning texts and situations, conceptualizing problems, designing
solutions, building artefacts, redesigning, re-conceptualizing and reinterpreting,
people generate forms of public knowledge that in turn provide conceptual and
relational support for further interacting and learning.
To be creative and critical in the digital media environment, however, depends on a high
level of technological competency, and self-efficacy with learning new media and mastering
new tools. The argument is that in order to be competent, productive and critical in this
century, one should endeavour to have a basic understanding of the languages and
processes that underlie our daily lives today. This implies a basic understanding of computer
coding and algorithmic thinking (Goode and Margolis, 2010).
One example of designing technology environments for education that enable the
development of transversal skills in the MoViE platform from Finland, which provides the
capability of sharing and remixing of videos easily online for digital storytelling, designed to
facilitate collaborative learning (Multisilta, 2012), and used successfully for digital storytelling
by students at both the elementary and middle school ages (Tuomi & Multsilta, 2012). The
ability of these new tools to make use of the collaborative potential of Web 2.0, and allow for
active participation and critical thinking through remixing of media (Jenkins et al., 2006),
increases the potential of digital storytelling to promote learning for the 21
st
century. In
addition, critical thinking (Figg, McCartney, & Gonsoulin, 2009) and increased collaboration
(Sadik, 2008) have been outcomes of early studies of digital storytelling in the classroom.
Problem solving
The development of an ability to solve problems and make decisions is very important
learning outcome of education. Problem solving could be developed in problem-based
learning. This learning theory is a student-centred learning approach helping learners to
acquire and develop the knowledge, skills and capabilities needed to solve problems
effectively (Engel, 1997). The problem based learning aims to prepare students to encounter
ill-structured problems normally encountered in real life. Such problems are usually complex
and can have multiple solutions. The main principles of problem based learning -
contextuality, collaboration and experientialism (Boud & Feletti, 1991) are fruitful to utilize in
elementary, primary and higher education.
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Self-management
Self-regulation is important to problem solving, directing attention, and managing emotion

(Malmivuori, 2006, pp. 158-160). As such, it underlies other key competences, particularly learning to learn. The goal of fostering a sense of initiative and entrepreneurship is really a goal about providing students with the skills and self-regulation to become independent, creative and critical contributors. There are many different ways of discussing and evaluating self-regulation, or, for example self-control and executive function, but the evidence suggests that the existence of a unified underlying concept (Duckworth & Kern, 2011).
Strategies for self-regulation can be learned. There are examples of students using ICT to develop self-awareness of their self-regulation with regards to their work, and support in changing existing patterns of behaviour (i.e., Niemi et al. 2003). Such skills are increasingly necessary in complex, individualised learning environments.
2.3. School-level spaces for teaching key competences and transversal skills:
real-world examples
Self-learning is an important key competence and it can be structured as in Burnfoot Community School in Hawick (Scotland) that has been working with personal learning plans since 1999. Often the students are encouraged and advised to undertake efforts to learn by themselves. When they gather some experience, the next step is to give them the possibility to practise self-assessment and peer assessment. This kind of experience is very important for the preparation for lifelong learning. It also contributes directly to the development of the learning to learn competence (Gordon et al., 2009). Very popular and effective method to teach key competence is project-based education (Ravitz, Hixson, English, & Mergendoller, 2012). This method is strongly encouraged, for example, by the Austrian Ministry of Education.
There have been many studies carried out, investigating the effectiveness of project-based learning. For example, Boaler (2002b) compared student mathematics achievement in two similar British secondary schools, one using traditional instruction and the other using project-based instruction. After three years, students in the project-based-learning school significantly outperformed the traditional-school students in mathematics skills as well as conceptual and applied knowledge. Furthermore, Thomas (2000) found some evidence that project-based learning is effective for teaching problem solving and decision making.
Sherman Oaks Elementary, a school in San Jose, California is an example school that teaches by project-based learning. It features a dual immersion English/Spanish language program and project-based learning culminating in widely attended student exhibitions. According to the school Web site, the Sherman Oaks program focuses on "strong academics built around real-life learning" and has "significantly raised student achievement levels." Also Applied Learning Academy, Fort Worth, Texas (a year-round campus and a school of choice) gives students an opportunity to learn through real-world experience in an environment that "fosters critical thinking, creativity, and collaboration with Fort Worth's arts, business, and scientific communities." Students create portfolios, design high tech projects, collaborate with a local theatre company, and produce and field test documents for the school district. The school Web site reports TAAS (Texas state test) scores that are considerably above average - as well as positive attendance rates.
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In Sweden, the advent of friskolar or independent, non-fee paying schools coincided with an increase in schools built around innovative and specialist curricula. While this has not been uniformly positive (especially with regards to equity, discussed further below), it has also created notable successes. One such school is the private Marina Laroverket in Danderyd, Sweden, which was ranked the best school in the Stockholm region by Sveriges Kommuner och Landsting (the Swedish Local Authority administration, SKL) most recently in 2011 due to academic performance and retention of students. The school, which does not charge tuition, combines vocational with academic programs and sends students to study aboard a sailing ship four weeks at a time, during which they focus on learning subjects such as astronomy and marine biology, but also ways of working and self-management. While this example is not scalable, and its finances are dependent on the local economy, it is one example of a local success in creating an interdisciplinary learning environment that engages students in the learning of Key Competences.
Many examples of using innovative pedagogy to develop key competences without the use of ICT can be found in the area of vocational education. In Germany, efforts are being undertaken to increase in-company training, thereby promoting the acquisition of key competences such as the capacity to work in a team, responsibility, self-organisation and linking content-related knowledge with the ability to apply it (Gordon et al., 2009).
2.4. Examples of nation-wide curricular reform for teaching key-competences
While many of the examples in this section stress interdisciplinary examples, an established example of success within mathematics is the Realistic Mathematics Education (RME) developed in the Netherlands by Hans Freudenthal and colleagues to address the problem of transfer in mathematics education (Fruedenthal, 1968). The curriculum fosters co-operative learning (e.g. Terwel, Herfs, Mertens and Perrenet, 1994), as well as engagement with problem solving grounded in realistic examples. This nation-wide reform now has an established record (Vos, 2010) that speaks directly to the building of mathematics competence, but also addresses issues of ways of working, cooperation, problem solving and metacognition. Most importantly, it is designed to promote the transfer of learning. Transfer of learning, as discussed previously in this report (section 1.6, p. 12), entails the application of knowledge, skills and meta-cognitive abilities to real-life situations and, therefore, by no means constitutes a straightforward process. Nevertheless, these good practices indicate that interdisciplinary tasks implemented within a cross-curricular framework can be the vehicle for competence and transversal skills development. According to the Analysis of the Mapping of Key Competency Frameworks (Pirrie, 2004), competence-oriented cross-curricular reform has been an educational goal for nations across Europe aiming, among others, to upgrade the status of competence at all curricular levels (e.g. in Romania, Hungary and Ireland), to use related benchmarks and indicators (in Germany and Malta) and to pilot new standards for reading, languages, mathematics and sciences (e.g., in Austria and Cyprus).
Beyond the complexity inherent at the pedagogical level, as the same Analysis (Pirrie, 2004, p. 9) stresses, large-scale curricular innovations should consider implications arising from contextual factors as well. These can include established European states' practices of subject-dependent teacher training programmes; lack of flexibility in school timetables; the degree of autonomy at school level and the lack of a universal definition of 'cross-curricular' competences.
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Another pedagogically related theme in the development of curricula for key competences is the innovation in learning environments that must take place. However, it should also be noted that innovative contexts are not enough. The examples above also emphasize the importance of explicitly teaching ways of working and self-management skills, that is, of modelling for students how to learn and inquire independently. In the following section, there is a more in-depth exploration of innovative learning environments for teaching and learning key competences and transversal skills and this theme of intertwining innovative environments with explicit teaching and scaffolding remains.
3. Formal and non-formal learning spaces and applications for key competences
As de Corte (2010) cites the US attempts to define '21st century skills': 'today's students to be prepared for tomorrow's workplace (. . .) need learning environments that allow them to explore real-life situations'. Therefore, new learning environments are needed that support development of problem solving, critical thinking, creativity, initiative, problem solving, risk assessment and decision taking.
3.1. Learning theories behind interactive learning environments for formal and non-formal learning
There are several learning theories that can be used to explain how participants learn in interactive learning environments which could be used for developing Key Competences such us problem solving, critical thinking, creativity, initiative, problem solving, risk assessment, decision taking and constructive management of emotions. By interactive learning environments we mean learning environments that supports active learning i.e. learning with simulation and games, learning with mobile devices and with mobile social media.
First, in interactive learning environments that aim to develop Key Competences, learning is no longer a process of knowledge transfer from the expert to the novice but learners need to construct the knowledge themselves by interacting with the environment. Constructivist learning theories posit that knowledge is built by the learner, not supplied by the teacher, thus constructivism is one of theories that provide a strong rationale for using interactive learning environments to support learning (Kriz, 2010). Constructivism focuses on the process of knowledge construction and the development of reflexive awareness of that process (Bednar, Cunningham, Duffy, & Perry, 1992). Learners must individually discover and transform complex information, check new information against old rules and revise them when they no longer work. Learning also must be situated in a rich context, reflective of realworld contexts for this constructive process to occur and skills are developed through working on the problem, i.e., through authentic activity. This approach represents the concepts of "active" or "autonomous" learning (Johnson, Johnson, & Smith, 1999). It is also argued that interactive learning environments, through linking knowledge and doing, support the idea of learning-by-doing (Barab, Hay, Barnett, & Squire, 2001). Schank (2005) argues that learning-by-doing is always more effective than learning-by-telling and that the former is best accomplished through complex, high fidelity simulations that engage learners at the highest possible level.
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The "active" instructional approaches propose supporting or complementing traditional teaching methods with active learning experiences such as role-playing, simulations, selfpaced or team-based exercises, and other types of open-ended problems requiring critical or creative thinking. In addition, students are more highly motivated by interactive learning environments such us games than by more traditional instructional presentations (Lepper & Henderlong, 2000; Garris, Ahlers, & Driskell, 2002).
Further, interactive learning environments provide a meaningful environment for problembased learning (McFarlane, Sparrowhawk, & Heald, 2002) therefore educators have developed learning environments that support complex problem solving (Suomala, 1999).
Problem solving can be associated with discovery learning because interactive learning environments allow students to discover new rules and ideas rather than memorizing the material that have been presented to them. Discovery learning is one of the instruction models based on constructivism; it is an approach to instruction through which students interact with their environment by exploring and manipulating objects, wrestling with questions and controversies, or performing experiments (Rieber, 2000). The idea behind discovery learning is that students are more likely to remember concepts that they discover on their own (de Jong & van Joolingen, 1998). In many forms of interactive learning environments, students learn by doing, by trying new strategies and by making mistakes. They construct the knowledge internally by immersing themselves into the learning environments. Furthermore, in these environments, participants collaborate with each other and group work helps them to share and develop alternative viewpoints. In interactive learning environments, collaborative learning is often implemented, which supports the use of effective discursive learning methods (make explicit, discuss, reason, and reflect, convince) while allowing for the acquisition of essential social and communication skills (Dillenbourg, Baker, Blaye, & O'Malley, 1995). The main idea of collaborative learning is that collaborative knowledge construction, co-ordination of different perspectives, and shared evaluation of group activities enable a group to create something that goes beyond what any one individual could achieve alone (Bereiter, 2002; Stahl, 2003).
Therefore, learning in these environments is not the lonely act of an individual but a matter of being initiated into the practices of a community (Lave & Wenger, 1991). The learning process is seen as mediated in a social context in situated learning (Winn, 2002) and the socio-cultural approach. In a social context physical artefacts (or tools) are a good facilitator for learning new concepts, as they give a shared starting point and potentially show the student new ways to proceed (Wenger, 1999; Wertsch, 1991, 1998).
Developing Key Competences through interactive learning environments could be also associated with other learning theories for instance activity theory (Kuutti, 1996). These environments have been characterized also as a form of experiential learning (Kolb, 1984) because the process of knowledge creation relies on the transformation of self-experience (Haapsalo & Hyvönen, 2001). For example, according to Gredler (1996) educational games are experiential exercises. They offer here-and-now concrete experiences to validate and test abstract concepts presented in the gaming environment. Such concrete experiences are the heart of this approach in which knowledge is constructed, not transmitted, as a result of experiencing and interacting with the environment (Kebritchi & Hirumi, 2008). This is a key theme in the idea of meaningful transfer (Bansford & Schwartz, 2001), that formal learning be grounded in experience—particularly in experimentation.
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3.2. Examples of formal and non-formal learning environments
Games have been widely implemented in teaching children math. For example, Moreno & Durán (2004) conducted a study about discovery-based learning through computer game and verbal guidance for teaching addition and subtraction of integers. Clements & Sarama (2007) described training of early childhood mathematic skills through different drill-andpractice type mini-games. Further, Barendregt, Emanuelsson & Lindström (2009) study showed evidence for improving early arithmetic skills for preschool children through drill and practice game that uses a specifically designed tactile interface. There has been also an interest for investigating how commercial games can enhance learning (Keskitalo, Pyykkö, & Ruokamo; 2011). In addition, research on social networking tools such us Facebook is booming, for example on educational usage of Facebook (Mazman & Usluel; 2010) or on using the Facebook group as a learning management system (Wang et al.; 2012).
3.3. Barriers for using interactive learning environments at schools
In spite of many positive benefits of technology enhanced learning environments, these environments have been also criticized (Buckingham & Scanlon, 2002). In short, the barriers for using these learning environments in an educational setting include: time scheduling, physical setting, class expectations, teacher background, genre knowledge, technical problems, experience with group work, teacher preparation, class size and prioritizing (Egenfeldt-Nielsen, 2004).
Two major negative sides of using technology-enhanced learning environments are:
1. costs involved, because of the costs poorer schools and students can end up being disadvantaged;
2. students, and sometimes teachers, can get too immersed in the technology aspect, rather than the subject content.
There have been a lot of studies carried out, investigating if use of computers improve learning but the results are mixed. First, little evidence has accumulated of the positive impact of ICT on student learning outcomes in schools (Dillenbourg, 2000). Some of the negative factors that have been mentioned in the literature (Deryakulu, Buyukozturk, Karadeniz, & Olkun, 2009) which influence the use of technology enhanced environments are lack of appreciation from colleagues, the difficulty of classroom management in computer labs, inappropriate teaching materials (e.g. textbooks, software), large class sizes and the rapidly changing nature of ICT. Due to the rapidly changing nature of ICT teachers need to continuously update their knowledge. All of these issues affect negatively the ICT teachers' effectiveness and efficiency in their classes. However, the positive side of the ICT in education has to be mentioned, for example one advantage is the unprecedented possibilities computers present for active or research-oriented learning (Jakkola, 2012; Veermans, de Jong, & van Joolingen, 2006).
3.4. Mobile technology for interactive learning environments
With the increasing popularity of mobile devices, non-formal learning (learning through games or mobile devices) has attracted the interest of educators and researchers. The most

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important benefit offered by mobile technologies is that they can provide anytime and anywhere functionality. Therefore, mobile devices can support the great amount of learning that occurs during the many activities of everyday life and learning that occurs spontaneously outside of school. These devices enable learning that occurs across time and place as learners apply what they learn in one environment to developments in another (Sharples, Taylor & Vavoula, 2005; 2007).
3.5. Developing science and mathematics competences with social and civic engagement
There is also the potential for schools, particularly with the advent of technology, to connect learners to authentic experiences of scientific discovery. Citizen science has successfully engaged the public, including youth in mapping invasive species (Jordan et al., 2012), identifying new planets, (Fischer et al., 2012), and perhaps most surprisingly, discovering a new algorithm for folding protein structures through engaging players in an online game (Khatib et al., 2011). Such opportunities present additional ways for learner engagement in real, original science and discovery. Table 1 below illustrates some 'Real Science' examples for competence development.
Table 1: REAL SCIENCE FOR DEVELOPING COMPETENCE and CIVIC ENGAGEMENT: Examples of new environments for involving students in real science research
Zooniverse An online collection of some of the world's biggest citizen science initiatives, created by the Citizen Science Alliance in cooperation with many universities and other organizations. Includes extensive information for teachers, and opportunities for laboratory as well as observational science.
FoldIt Supported by the University of Washington, this online game helps scientists solve problems relating to protein folding, linked ultimately to treatments for AIDS, Cancer and Alzheimer's.
CamClicker Teaches animal observations skills while students work on real tasks to help with data collection, one of the many collaborative programs from the Cornell University ornithology labs
Evolution MegaLab A citizen science project hosted by the Open University monitoring evolution of organisms in Europe, which resources for teachers
BeeID A project that monitored the population of pollinators in the British Isles using Flikr and Facebook
Planet Hunters Using data from the NASA Kepler mission to, among other things, discover new plants.
Part of the Zooniverse
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Biology and ecology are particularly fertile areas for this sort of participatory science. Researchers in the United Kingdom, for example, used online social networks such as Facebook and Flikr to successfully improve the monitoring of the bee population in the British Isles (Stafford et al., 2010). In fields such as ornithology, participatory data gathering has a long history, and is becoming only more important with the advent of web-enabled communities of citizen scientists (Sullivan et al., 2009). The use of citizens for data collection is expanding and changing the way scientists think of data collection (Silvertown, 2009). Many of these programs welcome youth and provide outreach and support for teachers to engage with their students in these real scientific endeavours.
The growing capabilities of technology also increase the ability for schools to make use of such community activities. "Currently, mobile devices include GPS, compass and accelerometers that can produce contextual data for applications. This kind of contextual data can also enrich future mobile learning applications." (Multisilta, 2012, p. 284). Such technology will increasingly enable learners to take advantage of out of class, in the field, science. It can also be used to link students up with real-world science problems and to become participants in meaningful research.
In mathematics learning, online communities implicitly teach collaborative and social skills while members engage in a task of common interest. For example, the NRich mathematics website hosted by the University of Cambridge provides basic to complex mathematical problems, as well as moderation by mathematics students that supports peer-learning by scaffolding, encouraging students to persevere in problem solving rather than simply ask for an answer (Jared, 2010), thus fostering transversal skills such as problem solving, initiative and self-efficacy in addition to mathematics learning. Curated by faculty and students of the University of Cambridge, the NRich program is also an example of a new kind of university partnership with educators and learners.
These are collaborative, interactive, technology enabled activities present a new significant change in the ways that young people can engage with science and learning that reaches outside the classroom. Not only do such approaches increase engagement by linking inquiry learning with real world problems, but they are interdisciplinary by nature, having the potential to combine elements of digital, math and science competences with problem solving, critical thinking, and initiative.
3.6. Innovative technology learning for participatory literacy and civic competence
Digital competence is one of the seven key competences, but it is also a vehicle for 21 st
century literacy skills and developing civic competence. Many students are already adept consumers of technology. Some youth have also taken hobbies, such as game design to advanced levels. Kafai and Peppler (2011, p. 89) write regarding participant driven learning communities that "Educators should be especially interested in DIY [do it yourself] communities given the amount of time youth voluntarily spend in intense learning as they tackle highly technical practices, including film editing, robotics, and writing novels among a host of other activities across various DIY networks". Such informal networks, such as those surrounding the program language Scratch used, among other things, for game development, are driven by student initiative, and fertile ground for the development of key competences and transversal skills like creativity and problem solving.
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These activities are important because they give students more than a set of digital skills. They are opportunities for students to create, foster dispositions towards innovation and entrepreneurship as well as developing technology literacy. This is an important issue for critical citizenship. The argument is being made that traditional conceptions of computer competence are not longer enough Kafai and Peppler argue that, "we should be as equally concerned with "opening the black box" of digital technologies as we are about media ownership and controls issues... Such understandings are crucial for today's citizenship, as more aspects of life have moved into the digital domain" (Kafai & Peppler, 2011, p. 112). The challenge for schools systems is to facilitate access for all students.
Thus, the online environments listed in Table 2 should not be conceptualised as vehicles for students to develop technical skills only. Rather, they are free, openly available resources for promoting in young people the ability to create media, and also understand how media is created so that they can both contribute actively, and be critical consumers and citizens.
Table 2.: CREATIVE and CRITICAL TECHNOLOGY COMPETENCE: Examples of tools and communities for learning programming and creating media online
Hacksaurus.org A tool for remixing and examining the underlying design of websites aimed at teenagers
School of Webcraft From Peer to Peer University at p2pu.org, webbased, free peer learning on a variety of programming languages
Scratch.mit.edu A powerful visual programming tool, also used for creating art and games with a related community for educators, ScratchEd
Gamestarmechanic.com A website for making and designing games
Processing.org Supports programming in the visual arts
This participatory involvement in technology (Jenkins et al., 2006) has not been realized widely, yet it will increasingly become a part of participatory civic competence. Schools need to foster the capabilities of students to take part in creative, entrepreneurial work with technology, and doing so may necessitate the development of rigorous computer science curricula for all students (i.e., Goode and Margolis, 2010).
3.7. Hands on learning environments
Experiential learning is used to motivate and engage students. However, such environments are also important in developing interdisciplinary key-competences, such as innovative capacity and civic competence, as well as transversal skills like collaboration, problem solving, and self-management.
Example of one hands on program the explicitly aims to teach those transversal skills is the FIRST Lego league, popular throughout Europe and beginning with programs for students as young as six. Simple robotics clubs and competitions provide opportunities for students to collaborate, create, and use technology to solve problems. Some programs aimed at increasing student engagement have targeted even younger students (Cejka et al., 2006). In Belgium, Denis and Hubert (2001) analysed the collaboration of small groups of learners, 10 years of age, working on Lego robotics projects. They find students developing collaborative
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skills, as well as 'strategic competencies', which they relate specifically to metacognition (Denis & Hubert, 2001, p. 467) along with skills such as computer programming.
At the secondary level, programs such as robotics clubs, offer the opportunity to manage team finances, engage in high level engineering, and access mentorship from community professionals, opportunities rarely available in a typical school. Furthermore, by engaging students in real problem solving within a social environment, students develop strong bonds and have the opportunity to develop key competences such as initiative, as well as transversal skills like cooperation, leadership and self-management (Saari, 2012).
Participants also seem to be showing increased understanding and interest in technology and science related fields (Welch and Huffman, 2011, Mellchior et al., 2005). The competitions are characterised by the depth of their difficulty and their open-ended nature. Such programs are also example of frameworks in which schools and local industry can create mutually enriching partnerships to foster key competences. However, they are not available to all students.
3.8. When and where should learning take place?
Jenkins et al., noting the promise of these afterschool clubs recommend that 'Afterschool programs should be a site of experimentation and innovation, a place where educators catch up with the changing culture and teach new subjects that expand children's understanding of the world' (Jenkins et al., 2006, p. 59).
Livingstone (2012, p. 21) describes the changes that occurring as our conceptions of where learning takes place expand: "...we are witnessing some genuinely new learning opportunities, centring on possibilities of child-oriented digital creativity and on collaborative communication with those who share similarly specialist or rich forms of interest and expertise." (Livingstone, 2012, p. 21). These environments not only teach specific skills, perhaps most importantly they are sites for learning transversal skills, developing creativity and the potential of offering complex and productive environments for learners. Their value might in part be that they are not connected to a particular time or place. This makes administering and ensuring equal access to such environments a challenge for policy makers and school leaders.
3.9. The difficulties of equitable implementation revisited: Administration, resources and the problem of equal access to innovative environments
The alternative environments discussed in this section are largely freely available programs and resources. Elsewhere we have mentioned the availability of open-source materials. However, accessing these implies not only the availability of computers or mobile devices, but of knowledgeable teachers who have the time in their curriculum to make use of these resources. Affluent schools are already doing so; the challenge to policy makers is whether or not to support less affluent schools in doing so.
The problem with these innovative environments is that it would be easy to leave students to access them, or not, according to their own interest, or lack of interest. But this also implies leaving student exposure dependent on a student's own resources—or lack of resources. Such an approach promises to widen the divide between students.
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A curriculum-integrated approach, however, could enhance equal access, as, compared to after-school programs, integration has a higher potential for systemic impact (Repenning, 2012, p. 39). The United States National Science Foundation Innovative Technology Experiences for Students and Teachers (ITEST) program enabled the application of a gamedesign computational thinking (CT) strategy to middle school curricula in school sites in Alaska, California, Georgia, Ohio, S. Dakota, Texas and Wyoming. During implementation, over 10,000 games and simulations were created by students, which linked with the initial study aim for increased exposure and maximised participation. From a pedagogical point of view, scaffolding was found to be the main common factor supporting motivational levels and skills across different school contexts, gender and ethnicity (Repenning, 2012, p. 40).
Implementation of such an approach is likely to be local. Within Europe, the difference between schools varies greatly from country to country (Green, 2011). Thus an equitable implementation is more possible in some locations versus others. Even in comparatively equal societies such as Sweden, however, a proliferation of innovation, such as occurred with the advent of publically funded free-schools primarily at the upper-secondary level, also coincided with a rise in equality (Skolverket, 2003), perhaps as navigating the educational choices, and the quality of new programs became more taxing for families with less resources. Since then, Swedish policy makers have instigated tighter regulation.
For the sake of both innovation and equal access, schools should offer students the skills needed to take advantage of the exponential growth in participatory possibilities online and in their communities.
4. Key competences for teachers
4.1. Teachers, school culture and leadership are central
The research above highlights some of the methods research has revealed to be promising in promoting learners' development of key competences for the 21st century. The emphasis is on complex, open-ended learning environments, enabled by evolving technology and taking place in and outside schools. Rather than replacing teachers, this new ecology suggested here places new demands on teachers. In turn, the implication is that teachers will need meaningful support and training. Just as youth are asked to increase their skills in collaborative learning, networking and digital literacy, so their teachers must do the same.
4.2. Teacher Networks and Collaborative Learning
Collaborative learning is also important for the success of teachers. Programs such as the Helsinki Media Centre (Niemi et al., 2012) provide opportunities for teachers to share resources. This is particularly important because 'teaching with technology requires teachers to expand their knowledge of pedagogical practices across multiple aspects of the planning, implementation, and evaluation processes' (Ertmer & Ottenbreit-Leftwich, 2010, p. 260).

This learning is an on-going process; learning about technology is equivalent to asking teachers to hit a moving target. Teachers will never have "complete" knowledge about the tools available, as they are always in a state of flux' (Ertmer & Ottenbreit-Leftwich, 2010, pp. 260-261). Some schools, such as the KoulunMestarin Koulu in Espoo, Finland, have
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embraced the idea of learning from students as well using students' expertise to facilitate technological integration.
4.3. Challenges for teachers
Over the past decades, research has repeatedly shown that individual teachers exercise important influence over the meaningful adoption of new policies and technologies. Prescriptive policies advocating, for example, the use of specific technological tools will not produce change in the classroom. Due to top-down measures, in some countries teachers can feel pressured to adopt technology, the use of which can seem forced, rather than coming out of a true pedagogical need (Hennessy et al., 2005, p. 170). Teachers may be more likely to adopt and integrate ICT if they see it as adding value rather than 'bells and whistles' (Hennessy et al., 2005, p. 172), that is, rather than being used primarily for appearance, or to be able to check a regulatory box.
In order to realize the transformation of the classroom suggested by key competences, then, particular attention must be played to supporting teachers as they work towards fostering a new complex ecosystem in their classrooms. Smith et al. (2012) considered the impact of technology on teachers' pedagogies to range from traditional+ (adding a new element to an existing classroom structure) to transformational. It is a transformational pedagogy that would include the sorts of activities indicated by the key competencies, learning activities that would use the new opportunities provided by technology to reach across the traditions bounds of the school and take advantage of open-ended learning in the community and the wider world. However, keeping the adoption of new technology and programs motivated by pedagogically is a key factor in the success of the model: '...the unavoidable challenges and disappointments with classroom technology have not been felt as keenly here: keeping the tools subservient to the teaching mission has kept "the horse before the cart" (Smith et al., 2012, p. 491).
Another challenge for teachers is the evaluation in these complex, individualised environments. There may be a mismatch between the instructional method, and outcomes as defined by local governments (Livingstone, 2012, p. 17). There is also the difficulty of monitoring students in ICT environments that have not been designed to do so (Hennessy et al., 2005, p. 171). New forms of evaluation may be more time consuming as well.
4.4. Key competences for Teachers
Investigations into the adoption of technology into education has been shown to be highly teacher dependent. It is likely that the successful adoption of technologically enabled key competences will follow similar patterns. While online communities may provide students space for independent and individualised learning, not all students will access these resources without the intervention of teachers as mentors. Some of the activities mentioned above that are most likely to promote the interdisciplinary and transversal skills are highly teacher dependent.
Just as students are encouraged to develop key competences, so must teachers be supported in developing competences and transversal skills. Somekh (2008) suggests giving teachers time to play with technology as a way of encouraging exploration and growth. Programs such as the European Project Teacher Education on Robotics-Enhanced
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Constructivist Pedagogical Methods (TERECoP), engage students in the same open-ended learning opportunities recommended for students (See e.g., Alimisis 2009). In Taiwan, courses in Lego robotics were shown to increase pre-service teachers' self-efficacy, (Liu et al., 2010) and important factor in adopting new practices into the classroom.
It is important to note that what researchers find is not the skill of the teacher that is particularly important, but the relevance of the tool to the teacher pedagogically: 'This highlights the requirement for teachers to understand fully the affordances of a technology, not necessarily aiming for higher but for more purposeful technology uptake.' (Mama & Hennessy, 2010, p. 274). Simply demanding use does not result in transformation. It takes a supportive school culture.
4.5. Cultural and organizational support in schools
Successful transformations, such as those described by (Smith et al., 2012, p. 491) are dependent on a culture of support and resources. "We have argued that any facility supporting such collaborations is by its very nature as much a social system as any kind of technical system. Intentionally engaging in the support and furthering of the social dynamic is productive, sustaining participation and continuation of participants over time, and sustaining a welcoming and supportive environment for individuals or working groups to explore new ways of teaching and working.'
Indeed, in an extensive research project in Finnish schools, Niemi et al. (2012) found that the success was dependent on the school culture, particularly one that valued sharing, teamwork and risk-taking. That is, the environment of the teachers reflected the environment sought for students, one that combines collaboration and the initiative of entrepreneurship. Thus it is important to focus on the support and environment in which the integration of new classroom structures occurs: 'The wider implications entail a shift away from a technologically-driven model of ICT integration towards one based on teacher involvement. This means an emphasis on developing and sharing pedagogic expertise concerning ICT use in subject teaching and learning...'(Hennessy et al., 2005, p. 187). Ultimately, for students to access, though school, the complex and open-ended learning environments that seem poised to best promote the transversal skills outlined above, teachers need support in continuous, iterative learning and the time and space to engage in peer-to-peer networking in order to take advantage of emerging programs and learning opportunities for their students. Part of the support for the development of school and teacher level competences can be furthered through innovations in the relationship of universities and education research with teachers and schools, discussed in the next section.
5. Using resources effectively
5.1. Taking a long view of technology investment: mobile technology, open source and freely available programs
While currently facing barriers to implementation, in the long run, mobile technology may be a more affordable solution for educational innovations as smartphones decrease in cost. Already, developing countries are looking towards mobile technologies as an efficient way of fostering access to learning (Haßler, Hennessy, Lord, Cross, Jackson, & Simpson , 2011).
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While technology without context will not foster Key Competences, increasingly students without access to the interactive online world will be at increasing disadvantage. Mobile devices, cheap and more replaceable elements like interactive whiteboards and desktop computers, are a rapidly evolving alternative.
Technology innovation need not be expensive. If teachers are given the support and time, open source alternatives (Haßler & Jackson, 2010) exist for nearly all commercially available programs and operating systems. Tools like GeoGebra (Hohenwarter, 2006), an opensource program for mathematics that surpasses commercially available tools, also provide gateways into students developing technology competences, as they have the opportunity to contribute to their development.
Finally, and again depending on the availability of hardware, free tools for video conferencing, video editing, chatting and so on provide opportunities for students to develop language competences, but also civic competence and gain intercultural experiences. Curating such activities, however, is not an easy task, and teachers need the time and resources, and perhaps most importantly the space created by alternative assessments, to engage their students in these environments.
5.2. Fostering complex learning environments means funding educational projects with multidisciplinary goals
In the previous sections, examples of learning environments that have been shown to address key competences and transversal skills have been given. One of the overarching themes emerging from education science gathered here is that these learning environments reach several key educational goals simultaneously. For example, the French language learning program researched by Fisher, Evans and Esch (2004) referred to previously did not only give students access to peer learning with native speakers, it also was a site for learning autonomous working skills (independence and self-management) as well as developing civic competences like intercultural understanding. Such programs suggest that funding should not be given for each competence. Rather it suggests the need to support schools and teachers in developing integrated approaches. While many of the examples given here are made possible with technology, the use of technology does not simplify the teacher's role in designing these learning environments, scaffolding student progress, and evaluating complex outcomes. Rather, these suggestions imply that the role of the teacher is more demanding, and resources must be allocated accordingly.
5.3. Ensuring equal access to new educational environments
Some of the learning environments that look most promising for the development of key competences are either expensive, or based in spaces outside of school. As such, funding structures must work to extend these opportunities to less advantaged students. As Livingstone (2011, p. 20) argues, ' ...new opportunities, especially if they rely on out of school resources, generate new inequalities. Only publicly funded institutions, schools especially, but also youth and community centres, can work to make this fairer.'
Secondly, teachers at all schools, not only well-resourced schools, must be supported in accessing resources such as networks and grants that will enable them to grow their curriculum in innovative ways. However, this is also problematic, as discussed above. More active teachers at better-resourced schools are more likely to have the time and skills to be
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successful in grant applications. Less active teachers may be in need of the most help, and yet they are the least likely to access resources. Thus, there is the potential of funding to exacerbate existing inequalities (Green, 2011). As the Swedish case of free-schools demonstrates, the proliferation of innovation can coincide with increasing inequality (Skolverket, 2003) even in a comparatively equal society such as Sweden.
An example of coordinated efforts of local and educational authorities to encourage active participation in innovative learning environments are the cases of Kirkkonummi schools in Helsinki, Finland, and the 2
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and 6
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Primary schools in Artemis, Attica, Greece. In these areas, fund allocation aims to increase access and thus benefit less technologically savvy schools, a shared vision and goal for local authorities, school administration and teachers. The integration plan is still in progress and, therefore, it is left to implementation and project evaluation to show in what ways and to what extent technology-enhanced tasks and activities promote key-competence development.
Seemingly, shared vision and technology are significant aspects in mobilizing change processes inherent in a competence-based pedagogical framework. School change, however, requires the synergy of multiple resources to be a tangible goal. The findings of a study on the integration of Key Competences in six Normal Schools in New Zealand (Boyd and Watson, 2006) suggest that only a realistic, multi-dimensional competence-based plan can facilitate change, i.e., one that links new with established school practices; that combines real-world with virtual environments aiming for authentic learning experiences; that allows for teacher integrated planning, experimentation and reflection; and that enables students to connect their in- school with out-of-school lives. Embedded within change management, then, is the management of both human (i.e., teachers and students) and non-human resources (e.g., funds, time, materials, space) between and across schools at the local, regional and national level.
6. Implementation of learning programmes for key competences
The teaching methods outlined for the production of key-competences can be described as being oriented towards interdisciplinary, cross-subject teaching, team oriented learning, individualised approaches (e.g., individual study plans) and project-based work. (Gordon, Halasz, Krawczyk et al., 2009, p. 162). These approaches are already used by many educators, but widespread implementation will mean a change in culture for many schools, particularly schools in challenging environments. Such approaches demand flexibility at the national, local, and classroom levels. Furthermore, to realise the potential of local partnerships, implementation plans must recognize the complexity of the school context. That is, top down, one-size fits all approaches are unlikely to work, and furthermore they would limit the potential of schools to take advantage of their local contexts. Rather, a leadership environment must be created at the national, local, and school level where diversity and innovation can be supported. What is needed is not a one-time reform followed by stasis. Rather, it is to develop the habit of perpetual revolution: a system structure that encourages constant innovation.
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6.1. National, regional, and institution level dimensions
Such an approach implies a flexible national curriculum (Gordon, Halasz, Krawczyk et al. 2009, p. 221), as is the case in Finland, and systems of evaluation that take complex, project-based learning into account. At the regional and local level, successful implementations have demanded collaborations between administration, universities, and schools. Innovative environments require collaboration and support in order to flourish, as studies in Finland regarding the pedagogical integration of technology have shown (Niemi et al., 2012).
International examples of successful, collaborative projects for promoting key competences include the River City project, based at Harvard University in the United States. An interdisciplinary, inquiry based project in an online, collaborative environment, the project implementation required researchers to modify the program for different school environments, and develop on-going relationships with educators and administrators (Clarke & Dede, 2009). Detailing the development of a new, rigorous and inquiry based learning curriculum in Los Angeles, California, Goode and Margolis (2011) describe an iterative process, dependent on strong relationships between researchers, administrators and teachers that was highly dependent on individual school environments.
In another example of networked researcher and educator collaborations, in Finland, Mylläri, Åhlberg and Dillon (2010) researched a network developed through the Environment and School Initiatives programme and as part of a United Nations Educational, Scientific and Cultural Organization (UNESCO) initiative to promote education for sustainable development. Using online collaboration tools to support educators and best practices, the research concluded that a centralized, "expert centralisation" (p. 384) of a collaborative network was beneficial. Community building was also seen as critical for long-term benefits of the programme (p. 386).
Indicative of the need for policies of educational and research environment integration in cooperation with the surrounding community is the case of Linnaeus University in Sweden (Tågerud, 2010). Linnaeus, being the offspring of the Kalmar and Växjö University merge, as Tågerud (2010, p. 57) discusses, would need to maintain the holistic view and strategic goals by combining positive experiences from "parent" institutions. Of these, Kalmar University had adopted a connective approach at an inter-institutional level in the process of redefining pedagogical competence and providing a description of the expected documentation. To achieve this end, discussions held at Kalmar were used; also the model for Lund University's Faculty of Engineering; the formulations from Uppsala University; and the definition

developed in the Mälardalen University project along with the inspiration day held in November 2008 at Kalmar University within the framework of the project "Strategic Development of Pedagogical competence". Building upon foundational work toward the establishment of its own policies and principles, then, Sweden's youngest University bears the culture developed within and across some of the country's institutions with regard to the development of its teachers' competences.
Competence, however, changes over time and generates the need for encompassing adaptation and development strategies that supersede the formal education level and combine joint local, regional and trans-sectorial initiatives. In the Netherlands (ECBO, 2008), for example, learning and working 'infoshops' are implemented at local/regional level aiming to provide 'life-long learning' incentives to the employed, unemployed and those receiving social security benefits. The programme is the outcome of collaboration between the
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Ministries of Education and Social Affairs and involves the Employment Service, benefits agencies and municipal councils. The 'infoshops' are intended for anyone with questions relating to study, career advice, competence testing, etc. Initiatives at sector level known as 'career projects' (e.g., for construction workers, the painting and decorating trade etc.) serve similar purposes.
Evidently, the multi-level interaction of implementation strategies for competence development, as discussed in Gordon et al. (2009), is a complex process that involves not only horizontal exchange of actors at the macro-, meso-, and micro-level. Of equal significance, is also a vertical collaboration, between and among policy makers, curricula planners, administrators, university and schoolteachers, and life-long learning organizations.
6.2. Scalability
The studies mentioned here are examples of implementations of new, complex curricula that meet some of the criteria of educating for key competences. They involve long-term, iterative collaborations between administrators, educators, and researchers. Such designs do not lend themselves to traditional methods of education research, nor are they easily spread from a top down approach. Rather, they demand a shift in the relationship between universities, administrations, and schools. One framework for such partnerships can be found in the literature surround design-based research in education, which Sandoval and Bell (2004, p. 201) describe as an ideal method for studying new technologies and complex learning environments within the ecologies of authentic, real-life classrooms. Furthermore, it is a `` methodology designed by and for educators that seeks to increase the impact, transfer, and translation of education research into improved practice.'' (Anderson & Shattuck, 2012, p. 167). However, it requires an on-going relationship between the researchers and the researched.
The River City project mentioned above, for example, firmly situates the program with design-based research, noting a design that shifted ownership of the project from researchers and developers to users, and established strong bonds between researchers and schools (Clarke & Dede, 2009). However, the researchers and developers are positioned here as key agents and disseminators of curricular innovation. As the work by Mylläri, Åhlberg and Dillon (2010), also suggests, universities and education researchers, then, are also important as agents for scalability of innovative curricula and pedagogical practices.
Just as key-competences requires school systems and teachers to re-think the evaluation of students, so does it require researchers to reconsider school-university partnerships as well as the aims and requirements of education science within the academy. Developments of key competences also imply an increased tolerance for innovative learning environments, including schools learning to take advantage of informal and non-traditional learning spaces.
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Section 2: Key competences in policy
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1. Approaches to implementation at system(ic) level:

1.1 Curriculum design and implementation (frameworks, etc.)

1.1.1 The different country approaches

This section draws essentially on the work of the Gordon et el (2009) report on key competences and provides a background about the approaches used by EU member states with some examples of the content they put behind the terms in use and also makes use of data collected for Leney et al 2008 on the Shift to Learning Outcomes.

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The data comes from
the country fiches prepared for the study. The document in Annex 1 is a "mapping" of major categories of terminology about competences and skills used in the EU 27.
Although it is not easy to establish a typology of the policy formulations regarding key competences, it is possible to propose a rough classification by taking the dominant approach in each country as a criterion. It is illustrated in the table below and then more detailed explanations are given.

Functional approaches:
Mostly skills or competence-based
Cyprus, Germany, Ireland, Lithuania, Poland,
Slovenia, and United Kingdom.
Mostly subject-based Bulgaria, Italy, Malta and Portugal
Thematic approaches

Mostly through major issues of society Denmark and Slovakia
Mostly through developing personal qualities Austria (primary school), the Czech Republic, Greece, Hungary and Luxembourg
Goals and principles based Finland, Latvia, the Netherlands and Sweden
(see Annex 3)
Mixed approach (functional and thematic) Belgium, Estonia and France.
Source: G. Halasz & A. Michel (2011)
The acquisition of key competences by every young person is one of the long term objectives of the Updated strategic framework for European cooperation. The focus is on acquiring

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Gordon et al. Op cit. and Leney et al (208)
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learning outcomes for professional and private life in a perspective of lifelong and life-wide learning. Emphasis is placed on the quality of teachers and on the autonomy of schools which should be open to civil society and enterprises and put in place appropriate approaches to quality assurance.
For the countries that specifically used the term "competences", there is, firstly, general agreement that it is about the application of knowledge and skills and secondly that it includes knowledge, skills and attitudes. In Portugal competence relates to the integrated development of skills and attitudes conducive to the use of knowledge in different situations (familiar or not). That is the third aspect included in definitions. Furthermore in Portugal the notion of Competências essenciais (essential competencies) refers to the body of general and subject specific knowledge that is considered indispensable for all citizens in today's society. In particular, it is essential to identify the knowledge which enables pupils to develop their understanding of the nature of each subject and its processes, as well as a positive attitude towards intellectual activity and the practical work it entails. In the Maltese system, 'Competence' is the proven ability to use knowledge, skills and other abilities to perform a function against a given standard in work or study situations and in professional and/or personal development. This definition introduces a fourth notion, that of a standard to be reached. In Italy, knowledge and abilities are defined that each pupil will turn into personal competencies with the help of the school. They are indicated for each subject and cycles of primary education. This is perhaps a fifth notion that competence is individual.
The notion of key competences (AT, BG, CY, CZ, DK, LT, MT, NL, RO, SK, Slgenerally refers to subject-independent competences which are seen as providing a "core" or basic set (as in Spain) or a "foundation" (as in Belgiium-Fr, France and Luxembourg with the "socle" of competences). In France the "socle" includes both the discipline based and cross-curricular aspects. The French "socle" emphasises that it is the basis on which lifelong learning can be constructed which is an important element in terms of the purpose of the reforms and in line with the EU policy thrust. The Danish example of the definition of key competences is interesting as it makes explicit their role "acting as axes of rotation, activating professional competences and serving as a pre-requisite for the acquisition of professional competences". In Slovenia key competences include: Learning to learn, Social skills, ICT, Planning and developing career, Entrepreneurship, Environmental responsibility, Safety at work.
Cross-curricular key competences are explicitly defined in the German system as general (subject-independent) competencies essential in order to operate effectively at personal and professional level. They are not limited to cognitive abilities and represent complex operational competencies. They:
- are required for and supported by different subjects and subject areas,
- help solve complex, holistic tasks in real-life contexts,
- can be transferred to new situations not covered by the curriculum,
- can be characterised as general abilities.
In Austria, the term, dynamic skills (Dynamische Fertigkeiten) refer to subject-independent transversal competences which introduces a nice sense of interaction and development. In the Netherlands they are referred to as core objectives (which relate to subjects) and general objectives (which are cross-curricular). We come back to the reference to objectives later in the section.
Taking one country as an illustration, Estonia clearly differentiates between general and the domain-specific competence and cross-curricular themes. General competence
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(üldpädevus) is used in the national curriculum and consists of four competences (learning, activity, value and self-definition competence) while domain-specific competence (valdkonnapädevus) consists of seven that should cover each aspect that a person has with the world surrounding him/her as well as her/himself. Finally the compulsory cross-curricular themes (kohustuslikud läbivad teemad) are not taught as a separate subject but have to be covered while learning other subjects.
Reform in Hungary with the inclusion of key competences in the primary school curriculum demonstrates in an interesting manner one possible expression of cross-curricular key competences. Most of them are expressed as capabilities and the notion of values is included.
Hungary: the strategy of the ministry contained a list of specific key competence to be developed in primary school:
☐ the capability of using various learning techniques
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☐ the capability of intelligent learning
☐ the capability to apply knowledge
☐ instrumental competences (like communication, mathematics or ICT-related competences)
☐ social competences
☐ value orientation (the capability to understand and use norms and values) "
It is interesting to compare that list to the notion of essential used in Portugal: "it is essential to identify the knowledge which enables pupils to develop their understanding of the nature of each subject and its processes, as well as a positive attitude towards intellectual activity and the practical work it entails" in so far as the two approaches appear to differ on the importance placed on content or process competence.
In Finland (similarly to Estonia) the cross-curricular aspect is through themes that are educational challenges with social significance. At the same time, they are current statements on values. In practical terms, cross-curricular themes are policies that structure the upper secondary school's operational culture and priority areas that cross subject boundaries and integrate education. They deal with issues concerning the way of life as a whole. For the upper grades of basic education, the cross-curricular themes are:
☐ Growth of the person
☐ Cultural identity and internationalism
☐ Media skills and communication
☐ Participatory citizenship and entrepreneurship
☐ Responsibility for the environment, well-being and a sustainable future
☐ Safety and traffic competences
☐ Technology and the individual
The above examples all refer either to the idea of a foundation which underpins the curriculum, or are intended as threads running transversally across it.

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This notion is considered to be close to "learning to learn".
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The Lithuanian example is interestingly different as recent reforms, while stressing the need to develop citizenship, entrepreneurship, digital competences, learning to learn and cultural awareness competences, also state that to ensure the quality of education development the content of education need to be related to new competencies of an individual. The orientation is towards the development of general abilities, values, the provision of the necessary competencies based not so much on the transfer of knowledge, as on their analysis, critical assessment and practical application; such competencies shall relate the contents of education to actual life, actual problems and their solutions.
The question is whether those countries using the term "skill" are close to the above notions or to the definition of Chisholm (2005): a skill is "an ability, usually learned, and acquired through training to perform actions which achieve a desired outcome". In Hungary the term "kompetencia" is a frequent synonym of "képesség" which might be translated as aptitude or skill. In Austria, the term basic skills (Grundfertigkeiten) is used to convey the notion in its broadest sense. The terms skills, core skills and key skills are the common ones in use in the four countries of the UK and in Ireland all of which are very similar in the choice of skills to highlight in the schools' curriculum. While Ireland has developed a separate (though similar) framework for each age group (primary, junior secondary and senior secondary), Wales has one framework from age 3 years to 19. The skills embedded into the curriculum contain many elements present in the cross-curricular key competences in the EU framework including the aim that young people should become "full, active and responsible" members of society (see below Scotland). The ideas contained in learning to learn, entrepreneurship and interpersonal and civic competences in the EU framework are present in these different formulations of core/key skills. The emphasis is quite strong on personal "capabilities", a term used specifically in the Northern Ireland curriculum which has echoes of the Chisholm definition.
Like the French "socle", the Northern Irish curriculum puts explicit emphasis on the development of skills and capabilities for lifelong learning as well as for contributing effectively to society. These whole curriculum skills and capabilities consist of the CrossCurricular Skills (Communication, Using Maths and Using ICT) and Thinking Skills and Personal Capabilities - TSPCs - (including Managing Information, Thinking, Problem Solving and Decision-Making, Being Creative, Working with Others, and Self-Management). These skills are embedded and infused throughout the revised Northern Ireland Curriculum at each key stage and it is intended that pupils have opportunities to acquire, develop and demonstrate them in all areas of the curriculum. The notion of capabilities is linked here to a set of skills with quite a functional orientation while it is linked to a set of competences in the case of Hungary (see above) with a more transversal focus.
The Irish Key Skills Frameworks:
Primary - the ability to: Question, Analyse, Investigate, Think critically, Solve problems
Junior secondary: Interact effectively with others: Communication and literacy, Numeracy, Manipulative skills, Information technology, Thinking and learning, Problem solving, Personal and interpersonal, Social
Key Skills Framework senior cycle: Information processing, Critical and creative thinking, Working with others, Communicating, Being personally effective.
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Scotland – Core Skills:
Core Skills describe the broad, transferable skills that people need to be full, active and responsible members of society. The core skills include:
☐ Communication
☐ Numeracy
☐ Problem solving
☐ Using information technology
☐ Working with others.
One country that explicitly mentions cross-curricular, transversal skills is Poland for primary education. They reflect the cross-curricular elements of the EU framework and include:
1. planning, organising and assessing pupil's learning, taking responsibility for one's learning process,
2. effective communication in various circumstances, presenting one's viewpoint and acknowledging the views of others, proper use of language, preparation for public presentations,
3. effective cooperation within a group, building interpersonal relations, taking individual and collective decisions,
4. solving problems in a creative way,
5. searching for, ranging and using information from various sources, effective usage of ICT,
6. using the acquired knowledge in practice an creating of necessary experiences and patterns of behaviour,
7. developing personal interest, acquiring methods of negotiating and solving conflicts and social problems.
A third group of countries define goals and objectives. Flanders defines development (Ontwikkelingsdoelen) and final (Eindtermen) objectives. Whereas developmental objectives are minimum objectives which the education authorities consider desirable for a specific pupil population, the final objectives are objectives with regard to knowledge, insight, skills and attitudes, which the educational authorities consider necessary and attainable for a specific pupil population. Similarly the Swedish system is geared towards the idea of 'steering through goals' which are decided at the central level though it is the responsibility of the decentralised authorities (such as the municipalities) to fulfil these goals. There are goals to strive for (quality of education process) and the goals to be attained (learning outcomes). Amongst the goals, the knowledge, skills and attitudes corresponding to particular key competences are found. The goals are not ranked in terms of importance and hierarchy and present a range of capabilities or qualities or skills or competences that is broad and developmental. These goals cover all aspects of education from pre-school through to the end of secondary education. (See Annex 1 for the list.)
Another system in which the competences are implicit in the goals is Latvia which has goals common for all general secondary education and which, among others, aim to encourage the development of the student's personality and of their physical and mental capacities, and to develop their understanding of health as a condition for the quality of life; They also aim to encourage the development of positive, critical and socially active attitudes, and to develop
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understanding of rights and obligations of Latvian citizens; to develop the ability to study independently and improve knowledge as well as create motivation for lifelong learning and a purposeful career. The latter group are very similar to cross-curricular key competences mentioned earlier in the section.
In Hungary the National Core Curriculum also refers to basic goals which include the development of key competencies as well as a number of other areas. It also contains the principles which underpin and structure the NCC. The NCC published in 2003 also contained a thesaurus which defined the notion of "competence-based" and the relation between "competence" and "knowledge". According to this, "The 'competence based approach means a commitment that determines the taxonomy of the curriculum. In the background of this approach there is a theory of personality which considers the competences (personal, cognitive, social and special competences) as the main components of the personality... [This approach] links the competences to specific activities and tasks that are achieved by the human being: someone is competent in relation with an activity if he/she is capable to achieve the tasks related with this activity." This was the basis for restructuring the NCC so that instead of the element of national culture ("knowledge") it was focusing on specific developmental tasks that are supposed to develop specific competences.
Two illustrations of a values base are included here while bearing in mind that these are policy statements; the mechanisms and tools of implementation may produce outcomes other than those expressed.
In Finland, for example the values base is very strong and clearly stated.
Students will be educated in tolerance and international co-operation. Upper secondary

school instruction is based on respect for life and human rights. The educational ideal of the upper secondary school comprises the pursuit of truth, humaneness and justice. General upper secondary education must promote open democracy, equality and well-being. Students are seen to be the constructors of their own learning, competence and views of the world. Instruction must take into account the fact that human beings observe and analyse reality using all their senses. Educational work will place emphasis on co-operation, encouraging interaction and honesty. The aim is for students to know their rights and responsibilities and to grow to assume adult responsibility for their own choices and actions. During their upper secondary school years, students must be provided with experiences of how to shape the future through joint decisions and efforts. Upper secondary school instruction must encourage students to recognise conflicts between stated values and reality and to ponder critically the disadvantages and opportunities of Finnish society and international development. During their upper secondary school years, students must become able to form a structured understanding of basic civil rights in Finland, the Nordic countries and the European Union, their meaning in practical terms and ways to uphold and promote them. Upper secondary school must highlight the principles of sustainable development and provide capabilities to face the challenges posed by the changing world. The basic values of upper secondary school are consolidated by the cross-curricular themes set out in Section 5.2, which are value-based positions on current challenges in education and schooling.
England: values underpinning the curriculum:
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Education should reflect the enduring values that contribute to personal development and equality of opportunity for all, a healthy and just democracy, a productive economy, and sustainable development. These include values relating to:
- the self, recognising that we are unique human beings capable of spiritual, moral, intellectual and physical growth and development
- relationships as fundamental to the development and fulfilment of ourselves and others, and to the good of the community. We value others for themselves, not only for what they have or what they can do for us
- the diversity in our society, where truth, freedom, justice, human rights, the rule of law and collective effort are valued for the common good. We value families, including families of different kinds, as sources of love and support for all their members, and as the basis of a society in which people care for others. We also value the contributions made to our society by a diverse range of people, cultures and heritages
- the environment, both natural and shaped by humanity, as the basis of life and a source of wonder and inspiration which needs to be protected.
At the same time, education must enable us to respond positively to the opportunities and challenges of the rapidly changing world in which we live and work. In particular, we need to be prepared to engage as individuals, parents, workers and citizens with economic, social and cultural change, including the continued globalisation of the economy and society, with new work and leisure patterns and with the rapid expansion of communications technologies.
Some issues arising:
The different approaches in the EU countries cover a broad variety of skills, competences, values, themes, etc. but the following aspects are less or not at all present:
- The need for students to understand about how one learns best, that is one's individual learning patterns and preferences, is not well developed.
- There is little reference to the need to develop a sense of self and of understanding of self. The main references are the interpersonal skills/competences with a strong emphasis on interactions with others, social aspects, etc. This is well-covered in the psychological literature but not well adapted into learning frameworks.
- Few of the definitions and usages refer to values, to the values with which children and young people will relate to each other and to the world around them.
- The relationship to the environment in both senses: the wider world around and issues of the survival of the planet are dealt with in a limited manner.
Country Example: Scotland
This example is presented since the curriculum is built out of the overarching framework of aims and principles of the Curriculum for Excellence which aims to achieve a transformation in education in Scotland by providing a coherent, more flexible and enriched curriculum from aged 3 to 18 and includes the totality of experiences which are planned for children and young people through their education, wherever they are being educated. The purpose is encapsulated in the four capacities - to enable each child or young person to be a successful learner, a confident individual, a responsible citizen and an effective contributor. To reach these outcomes it is recognised that children need to be safe, healthy,
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achieving, nurtured, active, included and respected and responsible which is illustrated in the following wheel. This is an example of taking a holistic perspective of children as foundational to the learning activities and content. It links formal education with health and well-being outcomes and to this end support has been developed practitioners in all sectors and services and in local authorities.
http://www.educationscotland.gov.uk/thecurriculum/whatiscurriculumforexcellence/index.asp
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1.1.2 How have approaches been designed?
This section presents 3 examples of how key competences have been introduced into national systems: Spain, Poland and France. The Spanish and Polish examples refer to articles published in the European Journal of Education, vol. 46, No. 3, September 2011 on Key Competences in Europe. They present different contexts and approaches. The article on Spain illustrates a government led approach through a core curriculum while the Polish example illustrates the influence of international bodies on the school curriculum in transition countries. The French example refers to articles published in the Administration et Education, the journal of the French Association Française des Acteurs de l'Education, No. 2 2012 that focused on l'Ecole du Socle (the foundation of competences). It is an example of an iterative process over quite a long period between expert groups and government steering.
SPAIN
A. Tiana, J. Moya & F. Luengo, "Implementing key competences in basic education: reflections on curriculum design and development in Spain" (EJE, 3/2011)
The authors observe that in Spain the reference to competences, understood as a way of defining educational intentions, was introduced almost simultaneously in the curricula of basic, compulsory and higher education (Bolivar & Moya, 2007). In the case of school education the Recommendation of the European Parliament and of the Council of 18 December 2006 on key competences for lifelong learning was very important whilst for higher education the key influence was the Bologna Process. Tiana et al point out that in both cases, "the incorporation of competences seeks to underline that the knowledge acquired at school or university cannot be easily transferred outside the educational context and that there is a lack of systematic work to develop basic skills that permit reaction to the
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changing situations of life and work which require complex responses". The Organic Education Act (LOE, 2006) was being drafted, debated and passed at the same time as the preparation of the European Recommendation, hence the reason that Spain was one of the first EU member states to include a reference to key competences in their legislation. Prior to this, the notion of competence had been used in vocational training in Spain in the 1990s and was given a legal basis through the Organic Qualifications and Vocational Training Act of 2002 (LOCFP, 2002) in which professional competence was defined as "the set of knowledge and skills that a professional activity to be exercised in accordance with production and employment demands", whereas a professional qualification was defined as "the set of professional competences with significance for employment that can be acquired through modular or other kinds of training and through working experience".
Consequently key competences were also included in the law in reference to student assessment, diagnostic evaluation and qualification at the end of Compulsory Secondary Education so that competences served both as a reference to assess educational results and included in grade promotion and certification on completion of compulsory education. However the legislation did not specify what these key competences should be. This was carried out subsequently and gave rise to the establishemnt of the enseñanzas mínimas (core curriculum).
enseñanzas mínimas
"the Spanish education system is highly de-centralised and curricula are drawn up on a shared basis. Initially, the central government sets out the basic aspects of the curriculum, in other words, it determines the objectives, key competences, contents and evaluation criteria for each level, cycle or year and subject. These elements, thus defined, are known as enseñanzas mínimas, a misleading expression, which many interpret as 'minimum education' that all students must acquire, although this is not its meaning. In reality, they represent a core curriculum which must be respected throughout the State. Subsequently, the Autonomous Communities established the curricula for each level, cycle, year, area and subject, which must necessarily include the enseñanzas mínimas defined by the Government nationwide (this is the true meaning of minimum). These curricula then serve as a reference for schools in every Autonomous Community. On the basis of these curricula, schools and teachers draft their respective projects for each grade, area or subject. This curricular development process, carried out in several successive phases, ensures that both Government and Autonomous Community responsibilities are respected, giving schools their margin of autonomy and teachers their academic freedom. With this cascaded decision system, where each decision is derived from a previous one and included in it, the initial curricular design is given shape and permitted to grow in depth and breadth, whilst avoiding the many obstacles that may arise in its path and allowing adaptation to the specific environment through which it flows.
Once the law had been approved, the government defined the key competences that all students must develop by the end of compulsory education for primary and compulsory secondary education in Royal Decrees. Given that this approach was very new, it was decided to proceed "cautiously and gradually". The Royal Decrees provided a general framework with the description, purpose and distinctive aspects of each key competence, as well as the development level which should be reached by all students on completion of their basic education. They include how each knowledge areas in primary education and subject in compulsory secondary education contributes. This was a break with standard practice in
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which curricula established objectives, contents, methodology and evaluation criteria for each area and subject.
At the school level the authors point out that the curriculum developed by the school will be the set of experiences offered to its students which is qualitatively different from curricular design and cannot be "reduced to the decisions adopted by public authorities when selecting learning (key competences, objectives and evaluation criteria) and the related cultural elements (curricular areas and contents). Curricular design conditions but does not determine the school curriculum, as there is considerable scope for freedom and responsibility to transform curricular design into real curricula." Hence in this context the importance of school autonomy and education practices if the construction of an integrated curriculum is to be adapted to the specific conditions and characteristics of each school which become organisations run on a project basis. Thus the authors also insist that this is fundamentally about how they manage their own resources and activities on the basis of their own projects. The importance of schools becoming learning associations is underlined and the authors emphasise that "The integrated development of the curriculum to whatever level a school can attain has a single purpose: to increase learning opportunities and improve the results of each and every student."
POLAND
M. Dabrowski & J. Wisniewski, "Translating key competences into the school curriculum: lessons from the Polish experience" (EJE, 3/2011)
This article which focuses on general education in primary and secondary schools is set in the context of the transition process in Poland in which the challenges to education and learning have been compounded by major economic and social changes and the democratisation of structures and processes. It was written by authors who played a significant role in this period. The article underlines the complexity of the reform process and how the introduction of outcomes based approaches and key competences have affected mindsets and attitudes of all the functions and actors in the education system (governance, the role of principals and teachers, teacher training, parents, etc.). "Polish schools in the second half of the 20th century were institutions where encyclopaedism, teacher-oriented classrooms, and lack of autonomy and democratic procedures prevailed."
Though the process of education reform was at the outset limited it had a great symbolic value for Polish society as it was assumed that "values such as democracy, freedom of the individual, respect for the law, and tolerance should form the basis of the education system. Promotion of these values could be achieved by including them in the curricula and in the way schools operated, including organisational and teaching culture. These aspects of school operations also created the conditions for developing key competences." Lack of funds and the economic situation created a major obstacle which included reducing the number f teachers and the class hours, etc. while maintaining all subjects and this led to the need to establish minimum curriculum requirements which turned out to be the first step in developing a core curriculum.
The debate on key competences started in Poland due to the results of the International Adult Literacy Survey (IALS) in 1994 which caused dismay and led to an initial list of competences being developed for Polish schools in 1996. They were: preparing for team work, preparing for oral and written communication, preparing for using the latest information media, problem-solving skills, and preparing for continuous learning, retraining and re-
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skilling. This debate about using competences at work and in social interactions also led to reflection on how to measure competences and evaluate their acquisition. Thus the IALS results and the debate were the "beginning of a new paradigm to define goals for the education system' and was made concrete in the Kreator project launched in 1995 with EU Phare funding. It aimed at developing a method for including specific key competences in the teaching of individual subjects and applying adequate methods. The competences defined were: planning, organising and assessing one's learning; successful communication in various situations; successful teamwork; constructive problem solving; and proficient use of computer and Internet technology. This project, which unfortunately only lasted as long as the EU funding, also looked at the organisation of lessons in order to better support pupils' acquisition of knowledge and development of competences. See box below:
1. Commitment: During the first stage of the lesson, its topic is clearly presented, which motivates students. It is important to accurately formulate the goals and instructions and create an atmosphere that is conducive to commit students to problem solving. The organisational framework of the class is established at this stage (e.g. students are divided into groups).
2. Researching: Students individually analyse the tasks. This stage is devoted to discussion, analysis, negotiations, and relating to previous experience and acquired knowledge. Hypotheses are made and doubts are voiced. Teachers become observers and listeners; they verify how much knowledge and experience students contribute.
3. Processing: At this stage of the lesson, knowledge gained at the previous stage is organised and used in a creative manner. Students present their proposals as to how to

solve the problem they have faced. The success of this stage is subject to cooperation by students, their inventiveness and commitment. They deepen their understanding of the problem, assimilate knowledge and learn how to use it.
4. Presentation: At this stage, representatives of individual groups report on the results of their work to a wider audience, be it a class or group. When presenting the results of group work, students can compare the solutions that have been applied to a given problem and results achieved by other teams.
5. Reflection: The closing stage of the lesson is indispensable and very demanding on the teacher and students. Students conduct self-assessment, define what and how they have learnt, the purpose of the working methods adopted and how they could continue the work and use the acquired experience. Students and the teacher attempt to answer the following questions: What have we done? What have we achieved? What could we have achieved? What were the principles for our learning? How can we use the experience?
The Council of Europe also had an influence in Poland through the results of its symposium on Key Competences for Europe in 1996 which defined five sets of key competencies with which schools should "equip" young Europeans (Council of Europe, 1996). See box below:
1. Political and social competencies such as the capacity to accept responsibilities, to participate in group decisions, to resolve conflicts in a non-violent manner, and to play a part in running and improving democratic institutions.
2. Competencies relating to life in a multicultural society, such as accepting differences, respecting others and the capacity to live with people of other cultures, languages and religions.
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3. Competencies relating to the mastery of oral and written communication, which are essential for work and social life to the point that those who lack them are henceforward threatened with social exclusion. In this same register of communication, the mastery of more than one language is taking on growing importance.
4. Competencies associated with the emergence of the information society. The mastery of these technologies, the understanding of their applications, strengths and weaknesses, and the capacity for critical judgement with regard to information disseminated by the mass media and advertisers.
5. The capacity to learn throughout life as the basis of lifelong learning in both occupational contexts and individual and social life.
This informed the work of a group of experts set up in Poland by the Ministry of Education and which led to Regulation no. 8 on Core Curricula for Obligatory General Education Subjects (1997), which was the first of its kind in Polish legislation on education.
"Its philosophy and structure differed greatly from previous solutions. The education process was divided into two- or three-year stages which replaced the traditional division of curriculum into one-year cycles. The curriculum was divided into 21 subject areas, including those which were totally new for Polish schools, such as media education, health education, reading education and information education. It is clear that some of these subject areas were closely related to cross-curricular key competences." One important aspect was that the document addressed schools and not individual teachers. However a change of government led to its implantation being delayed till 1999 and then modified going back to instructions for individual teachers.
FRANCE
C. Lelièvre, "La difficile (et chaotique) histoire de la mise en place du socle", Administration et Education (2/2012).
The author traces the history of the reflection in France about a common foundation (socle commun) or a minimal knowledge (savoir minimal) or a shared basic culture (culture commune de base) that should be acquired by all pupils back to the legislation in 1959 making 16 years the end of compulsory schooling for children who had reached the age of 6 years old in 1959. This measure led throughout the following period to a discussion not only about the structure and organisation of secondary schools to respond to the longer compulsory schooling but also to its content and whether different streams should address the needs of what were considered to be the most competent pupils, the average pupils and those in difficulty. This led during the presidency of Giscard d'Estaing in the second half of the 1970s to a fierce debate about the meaning of a minimum common knowledge for all pupils with the word "minimum" creating the heat of the debate.
Lelièvre traces the iterative reflections from that period on between a succession of expert groups and ministers of education. He draws attention to the fact that for a long period, 30 years, the debate about the common or minimal knowledge that pupils should have acquired by the end of compulsory education was regularly buried due to its sensitivity. Under the presidency of François Mitterand, Pierre Bourdieu was commissioned to draft a report on the "fundamental principles for the future of the education system" in 1984 that recommended a "fundamental and obligatory core of knowledge and skills that all citizens should possess.
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Ten years later in 1994 the philosopher, Luc Ferry in his role as chair of the National Council of Programmes submitted a report recommending a "common foundation of knowledge and competences, including practical and reflexive competences" that should be for all pupils regardless of their level and capacity. In 1999 the sociologists Marie Duru-Bellat and François Dubet undertook a consultation in secondary schools concluding on the need for serious reflection on the competences and knowledge that should form the common foundation for a generation. But it was not until the commission on the national debate, led by Claude Thélot in 2005 that led to legislation establishing a "socle commun de connaissances et de competences". A further advance in this period was the confirmation that competences are all equally important for young people and cannot be compensated for, i.e. acquiring one but not another is not acceptable. The final stage was in 2005 under the newly formed High Council on Education which added to the 5 competences already defined (mastery of French; a foreign language; basic competence in mathematics and a scientific and technological culture; mastery of information and communication technology; and a humanistic culture) two others based on the EU framework of competences: civic and social competence; and autonomy and initiative.
This article demonstrates interestingly the importance of the history and culture of an education system and the importance of debates being embedded in the culture of the system. It is also an interesting example of role played by French experts nominated by successive ministers to examine what were seen as critical issues but plagued by sensitive debates.
1.2. Contributions from outside the formal education system:
This section looks both at non formal environments and the contribution of NGOs to the implementation of key competences (that are implemented in formal environments). The first example is ISSA, an international NGO that has had considerable influence in early childhood education and care policy and the professional development of teachers and educators over the last decade in a number of central and eastern EU member states as well as candidate and partner countries.
International Step by Step Association http://www.issa.nl/
The International Step by Step Association (ISSA) is a spin-off from the Open Society Institute (OSI) from which it receives support. They connect professionals and organisations working in the field of early childhood development and education, promoting equal access to quality education and care for all children, especially in the early years of their lives. ISSA was established in the Netherlands in 1999 and are now present in about 30 countries, including in Central and Eastern Europe where they implement the Step by Step Programme initiated by the OSI in 1994 and other programmes. Members of the ISSA network work closely with local and national educational authorities and other key stakeholders in the countries. Their services include professional development programmes for staff in preschool and primary-school.
Competent educators of the 21st Century (Step by Step, 2010) defines what they consider to be quality pedagogy for early years education and care,. It was developed with the support of a group of international experts in this field, to further the ISSA vision: "With support from family and community, every child reaches his or her full potential and develops the skills necessary for being a successful and active member of a democratic knowledge society'.
ISSA promotes:
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- Equal access to quality education and care opportunities;
- Child-centred, individualised teaching and learning, combining high-level instruction with support for the needs of each child;
- Development of skills and dispositions for lifelong learning and participation in a democracy; (Our bold)
- Recognition of educators' many roles as facilitators, guides, and role models in the learning process and as active members of their communities;
- Family involvement in children's development and education;
- Community engagement in public education;
- Respect for diversity, inclusive practices, and culturally appropriate learning environments and methods; and
- Self-improvement and on-going professional development
ISSA's quality approach takes as its foundations the United Nations Convention of the Rights of the Child as well as other major international documents and inspiration such as the Council of Europe programme, Building a Europe for and with Children and the OECD Starting Strong reports. They consider that what constitutes high quality in education is a "complex and multi-faceted matter, including structural conditions (e.g. group size, adult/child ratio, etc.), as well as process factors (e.g. sensitivity of the adult, quality of interactions and curricula, etc.)" that should support professional autonomy. They first adopted standards to help practitioners their everyday performance through the principles of learner-centred, interactive pedagogy, the development of life-long learning skills, and cooperation with families and communities. They are widely used in their programmes and in working with authorities and stakeholders. A practice of mentoring to assist educators in improving their performance based on the standards has been adopted. Continuous improvement and integration of new findings, such as neuroscience, has led to the development of Competent educators of the 21st Century which consists of seven focus areas that reflect ISSA's main beliefs about quality pedagogy and identify ways to aspire to excellence:
1. Interactions
2. Family and Community
3. Inclusion, Diversity and Values of Democracy
4. Assessment and Planning
5. Teaching Strategies
6. Learning Environment
7. Professional Development
The seven areas promote "practices guided by humanistic and socio-constructivist principles, emphasizing developmentally appropriate practices, an individualised approach, and the idea that learning occurs in interaction, and is a dialogue between children and adults, as well as between children, which is marked by respecting each other, stimulating and giving autonomy to the learner, and assuming that children are competent and full citizens even while they need support from adults". This supports the development of lifelong learning competences, including:
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- interpersonal and civic competences,
- awareness about environmental issues and sustainable development,
- intercultural understanding,
- entrepreneurship, and
- ICT skills.
An important aspect is that teachers and educators are seen as having an important role in helping children to develop self-esteem, self-confidence, learning skills, the disposition for living and working with others and respect for diversity. ISSA considers that they therefore need space for their own development on both a personal and pedagogical level to be able to model and share with children.
Bertelsmann Stiftung: Selbsevaluation in Schulen
http://www.bertelsmann-stiftung.de/cps/rde/xchg/SID-A71F7D30-C0E78958/bst/hs.xsl/5278.htm
The second example in this section is from a self-evaluation tool built by the Bertelsmann Foundation for use by schools. The Bertelsmann Foundation has developed SEIS (Selbsevaluation in Schulen), which is a self-evaluation instrument for schools that can be administered through a computer or in a pen and paper format. It was originally developed in the mid-nineties in the context of the Bertelsmann Award for Innovative Systems when they began to identify through the process what good school quality implied. This led in 2003 to developing the software for delivering the self-evaluation. Several thousand schools use SEIS in different Länder of Germany. (Gordon et al, 2009-2010) The data obtained is for the school in a perspective of school development and improvement. There are specific questionnaires for students of different ages (e.g. up to grades 6 and grade 7 and above), teachers, support staff and parents. The data belongs to the school and is confidential but comparisons can be made with aggregated data from other schools. Bertelsmann Foundation also manages a programme called 'Good and Healthy Schools' (Anschub.de) and for these schools there are extra questions pertaining to that project. The types of outcomes expressed as skills that are contained in the student survey are of interest for this literature review. This example (with the items re-organised into categories selected by the author) is taken from the questionnaire for grade 7 and above students.
Personal well-being - setting personal goals for improvement
- explain ideas well in writing and orally
Relationships Relationship with teachers
- support from teachers
- encouragement
- promoting
- pay attention
Feeling that the class has taught to:
- work well with others, respecting & paying attention;
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participating in group work; solving tasks, confidence, do groupwork
- get along with others
Impact of
environment: school,
community or home
(in this case school)
Feeling that the class has taught to:
- to recognise strengths & weaknesses
- learn from mistakes
- solve a task
- recognise what I am good at & what needs improving
Perception that school has helped to:
- learn new things
- organise time
- be healthy
- protect the environment
- think about what I see in media

- solve problems in different ways
- explain thought process when solving a problem
- make decisions
Satisfaction with going to school:
- overall satisfaction
- feeling that school promotes talent
- feeling that school treats students fairly
- satisfaction with teaching & learning methods
- satisfaction with what needs to be done to achieve academically
- pupils assess lessons
- school recognised good effort & achievement
- school provides healthy and nutritious meal plans
- active promotion of health is important at school
Activities and agency School has helped:
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- act as leaders in a group
- solve disagreements
Participation with extra-curricular activities at school
- Use of school campus beyond class time
Student opinion is considered on decision that affect students
- Feeling safe in school
- school is welcoming and friendly
- existence of counsellors & support services for parents
- student council can influence decisions regarding school life & work
- students are involved in planning of school life
Support:
- Teachers effectively deal with bullying and bad treatment by other students; inappropriate behaviour
- students with personal problems receive help & support
- someone to turn to in school for assistance
Source: Gordon et al, 2009-2010
Learning for Well-being
http://www.eiesp.org/site/pages/view/80-learning-for-well-being-conference-february.html
In 2009, co n ve n e d b y th e U n i ve rs a l Ed u cati o n Fo u n d ati o n , a group of foundations established the 'Learning for Well-being' Consortium of Foundations in Europe
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to articulate a new vision of Learning for Well-being. The Consortium participates in Learning for Well-being, which is a global initiative of partners that work together to develop and support the process of how each child and young person learns to realize their unique potential in harmony with others and the environment. In this way, it contributes to an inclusive society where more and more children and youth learn how to live joyful, meaningful and healthy lives. During 2010 to early 2012 the Consortium developed and launched, Learning for Well-being; a Policy Priority for Children and Youth in Europe: A Process for Change. In the wake of the financial crisis, new stimulus has been given to the challenge of formulating policies that take us beyond the economic imperatives, and aim to increase Well-being for All. The policy glossary represents a cross-sector perspective developed through a consultative process involving a broad range of stakeholders. To ensure integration of state-of-the-art research and policy, there were consultations with OECD, the Council of Europe, the EU and many experts, foundations,

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The current members are: Bertelsmann Stiftung, Evens Foundation, Fondation Roi Baudouin, Freudenberg Stiftung, Guerrand-Hermès Foundation for Peace, Fondation M, Robert Bosch Stiftung and Universal Education Foundation.
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youth organizations, government and non-government organizations across education, health, social affairs, children's rights, media, ICT, family, etc. Very briefly stated, the policy glossary identified five priority principles as foundational for a Learning for Well-being approach:
1. Need for all of society to contribute
2. Sectors working together in integrated, proactive, multi-dimensional approaches
3. Children and Youth are part of the policy process
4. Learning for Well-being: A Process Approach
• Take the child's perspective
• Encourage expression of each child's unique potential
• Focus on strengths and inner differences
• Emphasize the nature and quality of relationships
• Be holistic
5. Measuring what matters
Learning for Well-being has developed, including through working with research partners, a conceptual framework and a first set of core capacities (as well as a pilot Voice of Children survey):
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Core Capacities:
Unfolding Unique Potential ☑ Awareness of one's life purpose
☐ Awareness of sacredness of life - mindfulness towards all beings
☑ Appreciation and gratitude for one's self
☐ Autonomy (to know oneself as independent & unique)
☑ Awareness of one's particular gifts & contributions
☑ Being at peace with one's way of being
Appreciating/Understanding Inner Diversity
☑ Being appreciated and respected
☐ Awareness of one's learning processes/needs
☑ Knowing about one's inner world
☐ Understanding the uniqueness of others
Relationships/Communication ☑ Self-esteem/self-worth
☐ •Knowing one's communication needs/processes
☐ •Knowing one's feelings/beliefs
☐ •Empathy
☑ •Curiosity
☐ •Playfulness
Participation/Engagement ☒ Capacity to choose (sense of agency)
☐ Self-motivation
☐ Capacity to plan/organize
☐ Connecting to inner resources (creativity, intuition, somatic experiences)
☐ Knowing how to engage/disengage using your own limits and boundaries
☐ Knowing one's responsibility/duty
☐ Awareness of one's contribution/role in the current circumstances
Systems Perspective/Selforganization
☐ Understanding interdependence & interconnectedness
☐ Knowing oneself as a whole system: knowledge of mind-body connection
☐ Sensory awareness (inputs of all kinds)
☐ Seeing/recognizing patterns - macro/micro
☐ Universal sense of belonging and connectedness
Others ☒ Consciousness (being awake/mindful)
☑ Knowing how to care for one's physicality
☑ Relaxation (all levels)
☑ Coping with stress, difficulties, and trauma
☑ Emotional self-regulation
Adapted from Lippman & O'Toole, 2011
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2. Socio-economic issues:
2.1. Schools as learning communities to combat disadvantage
The arguments in the literature on education and disadvantage tend to focus on:
- Education as a route out of poverty whether for adult learners of young people in terms of supporting future employment
- The role of schools in creating a learning community embedded in the local community, welcoming, supportive and accepting diversity.
- Some literature, either related to neurosciences or to social and emotional learning, indirectly addresses the disadvantage issues by emphasising the conditions under which children can or cannot learn - if you are hungry, afraid, in pain, etc. your learning will be less effective.
However, the research literature on disadvantage does not address directly the issue of competence development. The articles taken from the INCLUDE-ED project, focus strongly on the participation and engagement of pupils and their families/communities in developing the educational project of the school and learning communities.
Three of the articles in this section were based on the work of the INCLUD-ED project funded by FP6 from 2006-2011, Strategies for inclusion and social cohesion from education in Europe. The project website is a rich source of data for this theme and for the role of local communities: (http://creaub.info/included/results/ - see Project 2 on effective educational practices).
P .Melgar, R. Larena, L. Ruiz & S. Rammel, "How to Move from Power-based to Dialogic Relations? Lessons from Roma Women" (EJE, 2/2011).
The emphasis in this article is on transforming power-based relations between the education professionals and Roma parents into dialogic ones, by which they are referring to dynamic relations between stakeholders where dialogue and exchange become the foundation of cooperation. Thus in the case of the INCLUDE-ED project sites (parents http://creaub.info/included) proposals put forward by the professionals did not take priority over those of the Roma parents. Instead the aim was to undertake a collective approach to understanding what will improve the quality of education. Recognising Roma culture and contributions was very important and ensuring that their voices, experiences, skills and knowledge of participate in the decision-making spaces. In parallel to co-creating strategies for Roma children to succeed in developing their capacities in school, the authors underline the importance of implementing effective strategies for Roma women to enter the labour market in appropriate areas that use their skills and the effect this has overtime on reducing poverty. The project also supported family education classes, such as literacy or courses so that Roma mothers could work as canteens supervisors and observed the importance of the presence of Roma mothers in schools for the continuing presence of girls. In this article the emphasis is not so much on specific key competences as such but on the importance of creating a learning community within schools that involves families in ways that are empowering and change the way that schools "look at" children living in poverty and vice versa, the experience of the school by the pupil and their families.
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INCLUDE-ED focused on developing Successful Education Actions in schools where there are a high percentage of children living in poverty, with a migrant or minority status and/or other forms of disadvantage. The outcomes of the project funded under FP6 focus on the improvements experienced in schools after five years of implementation of the SEAs. School widely reduced early school leaving, improved pupil's academic results and eradicated conflicts fostering social cohesion in the community. The project team observed that successful actions do not require additional resources but rather a reorganization of the existing ones in the classroom and in the school with a focus on nurturing learning communities. The interest for the key competences network is in the partnerships created between schools, authorities and families to better support children's learning.
R. Valls & M. Padrós, "Using Dialogic Research to Overcome Poverty: from principles to action" (EJE, 2/2011)
This article which draws on the experience and data of the INCLUDE-ED programme, focuses on using the Critical Communicative Methodology inaction research to contribute to alleviating effects of poverty and disadvantage taking account of the fact that there is an increasing agreement among policy-makers that it is crucial to include the voices of those who are living in poverty in order to fight exclusion most effectively. The article argues that egalitarian dialogue, successful actions' approach and informing effective policies facilitate the engagement between researchers, end-users, and all the other involved stakeholders and therefore the move from principles to action in the process of overcoming poverty in Europe. Education is seen to be a major player in that process. The authors argue that for education to be an effective tool in fighting poverty, one must include all voices, especially those of people who are more socially vulnerable, stressing the impact of collaboration within the whole community and between schools and their contexts, valuing the potential of all students and families to support and collaborate with teachers.
An example of the dialogic process in Spain is examined especially as it led the stakeholders and community members to ask the researchers to extend the same procedure to transform the entire neighbourhood and to an agreement between the City Council of Albacete and the Research Centre on Theories and Practices that Overcome Inequalities (CREA) that coordinates INCLUD-ED to ground an URBAN programme3 (2007-2013). This time the transformation was planned and undertaken "with" the neighbours in the same critical communicative procedure that had been previously applied with success in the school.
J. Diez, S. Gatt & S. Racionero, "Placing Immigrant and Minority Family and Community Members at the School's centre: the role of community participation" (EJE, 2/2011)
This article takes as its starting point that schools alone cannot reverse the high rates of school failure in the poorest communities in Europe; they need the contributions of the entire community. Coordination between families, the larger community, and the school has proven crucial to enhance student learning and achievement, especially for minority and disadvantaged families. However, families from such backgrounds often participate in their schools only peripherally because the schools take a "tourist" approach, call parents to inform them about school projects and teachers' programmes, or consult them about decisions to be made by professionals, rather than engaging them in their children's education. In contrast, the INCLUD-ED project studied schools across Europe whose students are culturally diverse and from low SES backgrounds; here, the communities are deeply involved in the schools and the students do well academically. This article focused on three strategies used by these successful schools to engage immigrant and minority community members in more active,
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decisive, and intellectual ways and thus have greater impact on the school and the students' learning:
1. Overcoming Intentions, Ensuring Participation: The article reports that staff in schools where a real effort was made to ensure the participation of minority and immigrant

families in the governance structures had a positive impact on the school life, helping them reach out to more families who could better understand how the school worked, thus improving home-school coordination; it also provided positive role models for the children.
2. Creating Informal Spaces for Dialogue and Participation where all parents are encouraged to participate in discussing issues related to the school and their children's learning.
3. From Folkloric to Intellectual Contributors: A crucial element identified was when teachers had high expectations of the families no matter their literacy status, capacity to speak the language of the country, etc..
4. Though the schools studied use different practices, the three strategies were found to contribute to a transformative result in all schools: moving minority and disadvantaged families from the periphery of school participation to the centre.
Early childhood education and care in promoting educational attainment, including the social development of children from disadvantaged backgrounds and in fostering social inclusion (EU-Australia joint study).
This study is currently underway, led by the Sofreco and so any comments and references are preliminary to its finalisation and approval. The combined influence of schools and the environment do not start with compulsory education. In early childhood education and care, and focusing here in particular on the pre-school phase (generally about age 3 - 6 years or to the start of compulsory education), multiple factors affect children in vulnerable situations
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and how they learn. This study is underway for DG EAC and it includes an extensive literature review and case studies which will be completed in the autumn. Preliminary findings emphasise that poverty is one of several risk factors facing poor children, as well as readiness of kindergartens to accept diverse children; the level of preschool attendance; the quality of kindergartens; the nurturing capacity or educators and the extent of their outreach to parents; the level of parental behaviour, education and language mastery; prenatal exposure to alcohol, tobacco and depression leading to high infant mortality rates, low birth weight and special needs in infants.
2.2. School in its local environment
The first article included is about creating the conditions where the whole community (parents and school) will participate in establishing a solid basis on which children's learning can take place. As work on the social determinants of health has demonstrated, health, physical emotional and mental have an effect on learning.

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This notion covers a broad range of situations such as children living in poverty; children from low-income immigrant backgrounds; children from illiterate or poorly educated parents; children from second language backgrounds; children from marginalised rural areas; and specific groups such as Roma children. It also in some case covers children with a physical, sensory or learning disability.
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A. Flecha, R. Garcia & R. Rudd, "Using Health Literacy in School to overcome inequalities" (EJE, 2/2011)
This article is focusing on the notion of cultural intelligence and improving the health literacy of adults from vulnerable groups. It is based on the key idea that "Health literacy interventions could be broader in scope and include the cultural intelligence of adults from vulnerable groups throughout the life cycle of the programmes: from design, through organisation, to implementation." It entails residents in the community being perceived as capable of entering into dialogue with professionals and making decisions about the life they want for themselves and for their children in relation to health. The article, drawn from the INCLUDE-ED research, explores the outcomes of a community-based approach to empowering adults (parents of pupils) to make appropriate decisions about health issues and access services. This includes incorporating healthier habits into their daily lives (and those of their children) and regardless of their overall literacy status, becoming active in overcoming health inequalities and strengthening social cohesion. Transformation based on dialogue.
2.3. School in its virtual environment
J.F. Cérisier & A. Popuri, "Computers and School: Indian and French students' discourse". (EJE, 3/2011)
This article which describes a survey carried out in schools in India (621 usable responses) and France (339 usable responses) with pupils aged 11 - 18 years old. The aim was to explore students' perspectives and the impact that school computers have on their school and personal lives. Hence the survey was exploring the dynamics of students' relationships with computers at school and how they differ from their relationships with computers in general. The results show the extent to which their behaviours are determined by local cultures which in fact illustrate intersections between digital competence and social competence and expectations. For the French students there is a clear issue of conquering "margins of freedom" both in school and at home. Interestingly their results showed differences with comparable surveys some years ago when the use of the computer was very important for communicating with friends whereas in the current survey a lot of communication related activities appear to have migrated to mobile phones. On the other hand using computers to search for information showed an increase in this survey. The authors suggest that in school information searching which is assisted by staff gives way to networking and collaborative working at home that includes instant messaging. Their research shows the importance young people attribute to technology in the way they conduct their social lives, i.e. in the case of French students using technologies to shape private and community spaces partly because of the huge lack of public spaces that respond to their needs. Technology is also a form of emancipation from very organised and full timetables! This contrasts with more circumscribed and normative uses in school. The authors conclude: "policy makers would also benefit, both in India and in France, from a study of how young people make the best of the small gaps left to them by the educational system and which they use to create their own private culture and world that they share with their peers".
2.4. Individual students background
This section looks at three aspects of students' individual background:
a) The socio-economic aspects
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b) Supporting students to become learners
c) Whole child approaches
a) The socio-economic aspects:
Andy Green, "Lifelong Learning, Equality and Social Cohesion" (EJE 2/2011)
This article compares the evidence from the 2009 PISA survey on the distribution of skills amongst 15-year-olds in different regions and country groups and explores how education systems in these regions contribute to different levels of inequality. In the second part, it presents evidence from surveys on adult skills and attitudes on how skills inequality affects social attitudes and social cohesion. It argues that the evidence for developed countries is that education and training are major determinants of an individual's life chances, not only in employment, but also in terms of the broader social outcomes. The author refers evidence that more educated people tend to be healthier, to live longer, to commit fewer violent crimes (McMahon, 1999) and to experience a greater sense of well-being (Schuller et al., 2001), are more likely to be tolerant of others and other cultures, to trust other people and institutions, and to be active citizens, participating in their communities and engaged in politics (Putnam, 2000). Though these associations can be found in most developed countries in individual level analysis they do not necessarily hold in cross-country analysis because other contextual factors come into play (Green, Preston, & Janmaat, 2006). The article examines the importance of how education and skills are distributed, rather than the average levels of education noting that education and training systems shape social outcomes both in the way they socialise children and in the way they distribute skills. The author analyses data from PISA including the influence of socio-economic backgrounds and family background. The principal message from the research is about the importance of educational equality in particular in social cohesion. Green concludes that "Countries which achieve more equal education and which, on our evidence, benefit thereof in terms of social cohesion, are countries which believe in the virtues of equality and design their education systems to enhance it".
I. Nicaise,"A Smart Social Inclusion Policy for the EU: the role of education and training", (EJE, 2/2012)
This article by Nicaise focuses on education and the cycle of disadvantage and the strong correlation between the poverty risk of households and the level of education of their head in so far as families with a low-educated head face a poverty risk which is twice that of those where the head has completed secondary education (24% versus 13% on average for the EU27). The article examines how young people born and growing up in poverty have far fewer opportunities to benefit from education and have less access to good quality services in early childhood, participate less in kindergarten, start primary school with more arrears, accumulate larger deficits throughout their education careers, are more often referred to special education, are systematically orientated into lower-quality vocational tracks, and drop out more easily without any qualification. It refers to empirical research for the EU that has shown that children living in low income households, whose parents have low qualifications, are unemployed or are at risk of "in work poverty", who live in inadequate housing and disadvantaged neighbourhoods, and/or who come from a migrant or ethnic minority background are much less likely to obtain good qualifications at school (Machin, 2006).
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The author refers to the theoretical literature on equality in education thatdistinguishes between three sources of inequality at the individual level: unequal "talents" (mostly interpreted as genetic endowments); unequal effort or preferences, which are the responsibility of the individual; unequal opportunities relate to causes that are exogenous to the education system, such as the living conditions of disadvantaged groups. And adds a fourth which he suggests is probably the most important source of inequality: unequal treatment or discrimination. This includes the more frequent referral of children from immigrant or poor backgrounds to vocational tracks or in some countries the fact that Roma children may attend totally different and lower level school. Nicaise argues that "By the end of basic education, children from socially disadvantaged or minority backgrounds may have accumulated such deficits in verbal ability that it is almost impossible for them to enter general secondary education. If the curriculum in primary school had better valued instrumental forms of knowledge, it would have generated more equal success scores of lower-class pupils and led to more positive motivation and orientation. " He discusses the link between early school leaving and social disadvantage, including the need for crosssectoral policies involving education, social work, youth, employment and social protection as well as education system reforms that reduce stratification, disenchantment and exclusion of vulnerable groups as well as early targeting. He concludes on the need for massive investment in education, especially in basic skills and echoes existing research that suggests that both the fairness and the efficiency of education systems can be significantly improved by pulling down barriers in E&T systems and calls for a consensus "on the human rights perspective that the 'right to learn' is intimately linked with human dignity (and therefore rather absolute), irrespective of social background, ethnicity or indeed (perceived) talents."
b) Supporting students to become learners:
Farrington, C.A., Roderick, M., Allensworth, E., Nagaoka, J., Keyes, T.S., Johnson, D.W., & Beechum, N.O. (2012). Teaching adolescents to become learners. The role of noncognitive factors in shaping school performance: A critical literature review. Chicago: University of Chicago Consortium on Chicago School Research.
This literature review was produced by the University of Chicago Consortium on Chicago School Research in partnership with Lumina Foundation and Raikes Foundation. The aim was to develop a coherent and evidence-based framework for considering the role of noncognitive factors in academic performance and to identify critical gaps in the knowledge base and in the link between research and practice. The review sees this as a prerequisite for policymakers, practitioners, and education funders who would wish to assess the potential of non-cognitive factors as levers for increasing student educational attainment. The review found evidence to suggest that the best leverage points for improving student performance are in helping teachers understand the relationship between classroom context and student behaviours, providing teachers with clear strategies for creating classrooms that promote positive academic mindsets in students, and building teacher capacity to help students develop strategies that will enhance their learning and understanding of course material. The study acknowledges that while a growing body of knowledge suggests that non-cognitive factors play a key role in student success, it is unclear how all the different types of noncognitive factors interact to shape academic performance or what their implications are for educational practice. The report reviews the research on non-cognitive factors with a focus on students in the middle grades in high school in the USA and in the transition to college. They focus on identifying which non-cognitive factors matter for students' long-term success, clarifying why and how these factors matter, determining if these factors are malleable and
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responsive to context, determining if they play a role in persistent racial/ethnic or gender gaps in academic achievement, and illuminating how educators might best support the development of important non-cognitive factors within their schools and classrooms. They identify five general categories of non-cognitive factors related to academic performance:
ACADEMIC BEHAVIORS: Going to Class, Doing Homework, Organizing Materials, Participating, Studying
ACADEMIC PERSEVERANCE: Grit, Tenacity, Delayed Gratification, Self-Discipline, SelfControl
ACADEMIC MINDSETS: I belong in this academic community. My ability and competence grow with my effort. I can succeed at this. This work has value for me.
LEARNING STRATEGIES: Study Skills, Metacognitive Strategies, Self-Regulated Learning, Goal-Setting
SOCIAL SKILLS: Interpersonal Skills, Empathy, Cooperation, Assertion, and Responsibility
c) Whole child approaches:
This third example is taken from My transition Guide, which was developed by the children's services with the Scottish Highland Children's Forum in order to help young people plan their transition from school, to see the skills and achievements they already have and how to use them to make plans. The aim is that young people are engaged in their transition, they know who they are, where they want to go and what they need to get there. The guide presents a methodology for planning and as part of that lists of questions that echo the Curriculum for Excellence aims.
http://www.highland.gov.uk/NR/rdonlyres/C4A1C8E0-577A-431E-8C63-500DFDC4BF32/0/transitionguidepgp.pdf
What is interesting in this example is the use of the transversal aims for children and young people in the Curriculum of Excellence in the specific situation of transition from school to work which means broadening the items from classroom learning to their lives in general around three pillars: How I grow; What I need; and My wider world (see diagram below).
Am I Safe, Healthy and Nurtured? Do I
• have a place to live where I feel supported, happy and cared for?
• feel secure in my home and community and safe from people who could harm me?
• have confidence to be able to plan for myself and to take informed risks?
• maintain a healthy body and mind?
• know how I feel and have someone to talk to?
• feel confident to be able to get support to make informed and healthy choices?
Am I Included and Active? Do I
73
• have the support to be accepted and understood as an equal at home, work

and play?
• like and respect myself and am I willing to 'have a go'?
Am I Respected and Responsible. Am I a Responsible Citizen? Do I
• feel heard and involved in decisions that affect me and others?
• have opportunities and encouragement to be confident?
• understand the values and beliefs of others and look at the wider world?
Am I a Confident Individual? Am I an Effective Contributor? Do I
• feel comfortable with myself and others?
• set achievable goals for myself?
• understand the skills and needs of others and myself?
• understand the importance of being a team player?
Am I a Successful Learner? Do I
• have curiosity; do I want to learn?
• have enthusiasm; am I keen to learn?
• have determination; do I keep trying?
Am I achieving my wishes? Do I
• feel supported to achieve my goals?
The aims of the Transition Map are to
• help you to think about your own wishes.
• help you to think about the support you might need.
• give you a tool to help you to plan and be heard.
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They are illustrated in this star and the image below:
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3. Specific key competences
This section considers two out of the four transversal competences within the European Reference Framework. The competences 'learning to learn' and 'sense of initiative and entrepreneurship' will be considered in the next updated version of this literature review. The key competences that tend to be more linked to specific school disciplines are not included in this section, i.e.
☒ Communication in the mother tongue;
☒ Communication in foreign languages;
☒ Mathematical competence and basic competences in science and technology;
☒ Digital competence;
3.1. Social and civic competences
This section addresses on the one hand social and emotional literacy/learning and on the other hand civic competences.
3.1.1. Social and Emotional Learning/Literacy:
In recent years there has been a growth of programmes in schools to develop more personalised approaches to learning and different groups of competences for example social and emotional literacy, competences for citizenship and democracy, health promoting programmes, rights promoting programmes, etc. Researchers, such as Michel Fielding, maintain that: "education must be person-centred, democratic and aim at the flourishing of each individual as a human being". Other approaches emphasise process, for example the work of the iNet project, which focuses on system redesign as a path to educational transformation, defining the challenge as 'getting schools from mass production to mass customisation" with the aim that more of the educational needs of more of the students are met more fully than ever before. This includes moving from teaching to learning and to the "deep learning" that will equip students for the 21st century world of work. This approach identifies and develops "nine gateways" clustered into four areas through which a school may successfully explore personalising learning (deep learning (assessment for learning, student voice, learning to learn); deep support (mentoring & coaching; advice & guidance); deep experience (new technologies, curriculum); deep leadership (design and organisation; workforce reform). According to their definition, one of the outcomes for a learner experiencing "deep learning" through personalisation of their learning will be linked to the general well-being: "An articulate, autonomous but collaborative learner, with high metacognitive control and the generic skills of learning, gained through engaging educational experiences with enriched opportunities and challenges, and supported by various people, materials and ICT linked to general well-being but crucially focused on learning, in schools whose culture and structures sustain the continuous co-construction of education through shared leadership".
It is interesting to note that all of these approaches, whether focused on content on process, highlight some of the same capacities for children and young people to develop. They all emphasise albeit in different ways:
- Addressing the 'whole' child;
- The need for learning to be person-centred or human-centred;
- Capacities that will support the child's flourishing;
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- The transformative capacity of learning.
They are illustrated here by looking in more detail at social and emotional literacy, sometimes referred to as social and emotional education or learning, other times as a form of literacy (or sometimes termed skills). Whatever the title used, the intention is to stimulate well-rounded growth in young people and to enhance their academic achievement. It serves as a preventative strategy as well as also contributing to the improvement of their physical and mental health (Clouder 2008, p12). Furthermore proponents of this type of approach, such as the authors contributing to the two international publications on Social and Emotional Education (2008 & 2011) published by the Fundación Marcelino Botín, emphasise the importance of training adults (teachers, parents, professional, etc.) as a prerequisite in working towards developing the well-being of children and young people. There is an interesting assumption in this report that certain capabilities or competences are human rights to which all children should have access as it is vital for the sake of the health of future societies. They are:
- The ability to relate well to others
- To cooperate
- To manage and resolve conflict
- To act autonomously
- The ability to act within the larger context
- To form and conduct life plans and personal projects
- To defend and assert one's rights, interests, limits and needs
- To use language, symbols and texts
- The ability to use knowledge and information interactively and the ability to use technology interactivity.
The Introduction emphasises that "a child needs to understand her own feelings in order to recognize those in others" (p40) which echoes both the "learning to be" and the "learning to live together" pillars of the UNESCO report.
Depending on the context, approaches to emotional and social learning may involve introducing specific modules into the curriculum or ensuring that children and young people have the opportunities to develop the competences identified across the whole curriculum and through all the different types of classroom and extra-curricular activities proposed in a school.
Very substantial work in this field has been carried out in the USA by CASEL, the Collaborative for Academic, Social and Emotional Learning (http://casel.org/), which addresses five essential areas of social and emotional development:
- Self-awareness (recognising one's capacities, strengths, emotions and values)
- Self-management (managing emotions, and behaviours, persevering in overcoming obstacles)
- Social awareness (showing understanding and empathy for others)
- Relationship skills (forming positive relationships, teamwork, conflict resolutions)
- Responsible decision-making (making ethical, constructive choices about personal and social behaviour).
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The CASEL research suggests that benefits of paying attention to social and emotional learning include the following:
Better academic performance: Students who participated in SEL programs increased achievement scores an average of 11 percentile points versus students who did not participate.
Improved school attitudes and behaviours: SEL instils greater motivation to learn, deeper commitment to school, increased time devoted to schoolwork, better classroom behaviour, and improved attendance and graduation rates.
Fewer negative behaviours: Among students receiving SEL instruction, disruptive class behaviour, noncompliance, aggression, delinquent acts, and disciplinary referrals decrease significantly.
Reduced emotional distress: Reports of student depression, anxiety, stress, or social withdrawal significantly decrease among students receiving SEL instruction.
Kevin Haggerty, Jenna Elgin, Andrew Woolley, (2011) Social-Emotional Learning Assessment Measures for Middle School Youth, Social Development Research Group, University of Washington. Commissioned by the Raikes Foundation
This is a review carried out to identify solid assessment tools for social and emotional learning from among the tools that exist for use by US schools. For the purpose of the review, the researchers used the social and emotional competencies identified by the CASEL as a framework for identifying and reviewing various assessment measures. The five interrelated social and emotional competencies as described by CASEL include:
- Self-Awareness. Accurately assessing one's feelings, interests, values, and strengths; maintaining a well-grounded sense of self-confidence.
- Self-Management. Regulating one's emotions to handle stress, controlling impulses, and persevering in addressing challenges; expressing emotions appropriately; and setting and monitoring progress toward personal and academic goals.
- Social Awareness. Being able to take the perspective of and empathize with others; recognizing and appreciating individual and group similarities and differences; and recognizing and making the best use of family, school, and community resources.
- Relationship Skills. Establishing and maintaining healthy and rewarding relationships based on cooperation; resisting inappropriate social pressure; preventing, managing, and resolving interpersonal conflict, and seeking help when needed.
- Responsible Decision Making. Making decisions based on consideration of ethical standards, safety concerns, appropriate social norms, respect for others, and likely consequences of various actions; applying decision-making skills to academic and social situations; and contributing to the well-being of one's school and community.
The rational of the review is that children with strong Social-Emotional Learning (SEL) skills have been shown to excel in school, both socially and academically and it is considered that SEL programmes are imperative to address the social, emotional, and academic needs of students
The research conducted an extensive review of the literature on social and emotional learning in US middle school students and identified and evaluated 73 instruments. The criteria used were: what the constructs measure, target age group, strengths and
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weaknesses of the tool, reliability and validity. The aim was to test which tools have sound psychometric properties, are suited for programme evaluation, are readily available for schools to access and were not developed for use with a specific programme. 73 instruments were reviewed and 10 met their criteria for inclusion in this report.
In addition to improving students' social-emotional skills, the review reports that SEL programmes improve students' performance in the classroom (Payton et al., 2008) with an increase of 11% to 17% in test scores (Payton et al., 2008). The researchers conclude that SEL is 'absolutely crucial to children's success in school, both academically and socially'.
Hence reliable and valid assessment tools are necessary to conduct needs assessments and monitor the success of SEL programmes over time.
Immordino-Yang, M.H., Damasio, A. (2007) 'We Feel Therefore We Learn: The Relevance of Affective and Social Neuroscience to Education'. Mind Brain and Education.
In their article, We Feel Therefore We Learn: The Relevance of Affective and Social Neuroscience to Education, Mary Helen Immordino-Yang and Antonio Damasio explore how the connections between emotion, social functioning and decision-making have the potential to revolutionize our understanding of the role of affect in education. The authors emphasise that recent findings underscore "the critical role of emotion in bringing previously acquired knowledge to inform real-world decision-making in social contexts, they suggest the intriguing possibility that emotional processes are required for the skills and knowledge acquired in school to transfer to novel situations and real life". They conclude by saying that when "we educators fail to appreciate the importance of students" emotions, we fail to appreciate a critical force in students' learning.
C. Pert & N. Marriott, "The Science of Emotions and Consciousness" in Goleman D. et al, (2008) Measuring the Immeasurable; the scientific case for spirituality, Sounds True.
The research of Candace Pert, a neuroscientist and pharmacologist, explores the importance of emotions to learning based on a theory of emotions and on the notion of a vast network of communication coordinating the brain and body that she calls the "bodymind". Two important notions developed are that people acquire knowledge with their entire "bodymind" and that learning is an emotional event impacted by how one feels. She discusses how the experience of reality is filtered through memories, which give a specific interpretation and meaning to each person's experience in so far that everything that is learned is filtered through past experience: "what you experience as reality is your story of what happened". She links this to trauma, but also to the ordinary scars of growing up that continue to affect our lives. Her research and that of others suggests that recall is stored throughout the body in a psychosomatic network extending through all the systems of the organism and that much of memory is emotion-driven with two possible effects. Emotions can bring a recollection to the surface or bury a memory below awareness where it can affect perceptions, behaviours, health, etc.
3.1.2. Civic competences
W. Edelstein, "Education for Democracy: reasons and strategies", (EJE, 1.2011)
The author argues that major developments in Western industrial societies towards individualism, increasing social complexity and globalisation present serious threats to basic requirements of stable societies and expose democracy to the corrosion of its socio-moral resources such as social trust and civic commitment to the public good. In order to grow and
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to flourish, the competences and capabilities that constitute these resources require systematic cultivation through educational processes. HE argues that the school is the only institution that can provide the appropriate experience of a democratic life to all members of society. At present, however, schools are generally far from this goal. This article describes strategies for providing the experience of schools as democratic life-worlds from an early age: classroom councils as tools for democratic self-government and sites for socio-moral learning; projects of service learning in the community which can be successfully organized by these councils; and early experiences of civic engagement in community contexts as part of democratic classroom practice.
3.2. Cultural awareness and expression
The next contribution moves from the immediately "local" environment to partnerships in the case of Creative Partnerships:
McLellan, R., Galton, M., Steward, S. and Page, C. (2012). The Impact of Creative

Partnerships on the Wellbeing of Children and Young People. Newcastle: CCE
http://www.creativityculture education.org/the-impact-of-creative-initiatives-on-wellbeing-aliterature-review
This study aimed to explore how the Creative Partnerships programme had affected student wellbeing and the degree to which creative approaches had become embedded in areas of the curriculum other than those directly involving creative practitioners. The researchers discuss approaches to creativity and creative learning, including the Creative Partnerships programme, the notion of wellbeing and interests in measuring it, and motivation as the link between creative learning and wellbeing. The research draws on self-determination theory (SDT) where the achievement of learning goals associated with developing expertise and metacognitive wisdom, result in a sense of psychological wellbeing through satisfaction of core needs. This theory suggests that people's innate needs include competence, autonomy and relatedness, and the findings of this report suggest that these are the very qualities that underpin the central aim of the Creative Partnerships programme. Important themes are discussed in the report, including student voice, school ethos, the development of creative transferable skills, the effects of performativity culture, collaborative approaches, levels of student engagement, positive relationships in the classroom, and pastoral care.
Impact on primary schools
- Creative Partnerships' approach to fostering wellbeing was radically different from that in the other case study schools. In the latter wellbeing was a means to an end whereas in Creative Partnerships schools no distinction was made between creativity and wellbeing which meant that creative learning tended to permeate the whole curriculum.
- Student voice was crucial to promoting wellbeing and in helping students to function effectively both personally and socially. The extent to which students were able to have their views recognized and contribute to decision making had been taken further in Creative Partnerships schools.
- There was little evidence to suggest that there was a typology of creative practices. Where differences did exist this could be attributed to the fact that Creative Partnerships schools were at different stages of their learning journey, rather than because they adopted different approaches for developing their pupils' creativity.
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- There was little evidence that creative learning was promoted through specific 'arts based' approaches to learning. In all Creative Partnerships schools the emphasis was on generic pedagogies rather than pedagogic subject knowledge. Specialist knowledge and skills were only introduced when it helped students to develop their own ideas. The emphasis was mainly on helping students to think flexibly, strategically and creatively.
Impact on secondary schools
- In general, the survey indicated that there were no overall differences in wellbeing between students attending Creative Partnerships schools and other schools. Issues in the matching of schools at secondary level, together with previous research indicating that variation in wellbeing scores attributable to between school differences is small, might have accounted for this finding.
- The qualitative data can provide insight into how Creative Partnerships work impacts upon student wellbeing but as schools have many strategies in place to support wellbeing disaggregating the impact of any one strategy is difficult. The context within which secondary schools operate cannot be ignored, particularly the performativity culture with its focus on examination results. This has implications for student wellbeing in terms of lack of choice and exam pressures can be controlling and therefore thwart the need for autonomy, and can also lead to a de-valuing of creativity and wellbeing.
- The main motivation for schools joining the Creative Partnerships Programme was school improvement. Creative Partnerships work had generally focused on one departmental area or an identified group of students (e.g. disaffected Year 9 boys) in the first instance, although the number of departments and variety of projects undertaken increased as schools gained more experience of Creative Partnerships work. There was little evidence in the two case studies that where creative practitioners had worked with teachers to improve practice, this had impacted on pedagogy in other lessons beyond the Creative Partnerships projects.
- Self-determination theory was applied to identify elements of Creative Partnerships work that promoted wellbeing through the satisfaction of the core needs for autonomy, competence and relatedness. A number of projects were seen to meet these needs in different ways and particularly promoted interpersonal and perceived competence wellbeing. There was less evidence that Creative Partnerships work influenced life satisfaction. It also had the potential to have a positive impact on wellbeing through factors not captured by SDT, e.g. aspiration and teacher wellbeing. In a small number of cases, Creative Partnerships work also had the potential to reduce wellbeing in the way it had been implemented, i.e. depending on the quality of the work.
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Section 3: Key competences in France
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The framework of exchange and discussion regarding competences has, until recent years, been more of a theme in the French-speaking world as a whole than in each individual French-speaking country. Before the development of the European Reference Framework for Key Competences, Belgium (Missions decree - 1997) and Québec in Canada ('Pedagogical renewal' reform since the beginning of the 2000s) had indeed put the notion of competences at the heart of their reforms, and in doing so provoked scientific writing on the subject which has been widely circulated in the French-speaking community in Europe (Belgium, France, Luxembourg and Switzerland) and in North America (Québec) as well as in the African French-speaking countries.
Debates and experiences regarding competences have thus been included in the development of local educational culture and policy, rather than adhering to the frameworks of key competences, such as those which have been subsequently developed by the European Union.
1. The approach to competences in France and in the Frenchspeaking World
1.1. Professional origins of the notion
The notion of competence developed mainly in the professional world in the 70s, where it progressively replaced the notion of qualifications, to better consider the role of the person and the responsibility of the employee in unforeseeable circumstances (Legendre 2008). Secondly, the notion was encompassed in the context of adult education and vocational training, in order to identify the underlying "knowledge in action" of efficient activity and, if possible, ensure their application (Ropé and Tanguy, 1994). In the domain of adult education, still today the notion of competences is mentioned the most spontaneously, particularly with regards to life-long education (CAFOC, 2012).
It was only at a later stage in the 90s, that competence was encompassed in the context of school education, to characterise the unique competences of adaptation of an individual to new situations, and their evolutionary competences based on advances in technology and society (Rey, Carette, Defrance and Kahn, 2006).
Parallels were drawn between the separation of micro-objectives of the school curriculum (behavioural theory of pedagogy by objectives) and a Taylorist economy on one hand, and the approach by competences and a modern economy mainly based on people's autonomy and flexibility (Jonnaert, Barette et al). 2004).
One of the leading theorists on competences, Guy Le Boterf, is a consultant in human resources management, and has studied the notion of competences primarily in the context of training within professional organisations. He believes that the defining of competence as an extension of knowledge, skills and attitudes is too weak. His view is that competence is the combination of abilities or aptitudes for resolving a given problem. It consists in "knowing

how to act", which is the combination of - and not the addition of - various resources in a working environment:
- Knowing how to mobilize others;
- Knowing how to integrate;
- Knowing how to transfer.

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Competence is in mobilisation. For the individual, it is a question of going from knowing how to knowing how to act in a particular unforeseen situations (le Boterf, 2010)
1.2. The opening up of schools to the divisive outside world
The "professional" origins of the notion of competences have provoked opposition, in that the notion stifles educational ambition and leads to a utilitarian perspective (Crahay 2006, Boutin 2004).
For some researchers, education stakeholders or schools of thought, the notion of competence is a Trojan horse for employers - possibly backed up by international organisations and States, in order to adapt their education systems to their own needs - going against the transferral of heritage or critical thinking in education which schools should focus on, in their eyes (see for example Del Rey, 2010; Hirtt, 2010; Laval, Vergne, Clément and Dreux, 2011).
Equally, some intellectuals in France are very keen on the teaching of the "great works" in literature, history or art - subjects which are sometimes classified as Humanities or Humanistic Culture, a notion which was added to the French socle commun (core curriculum) of knowledge and competences in 2005 (Lelièvre, 2009).
This humanistic culture, which is essentially found in the school context, would, according to some, be threatened by the transformation of subjects into competences, in which the study of literature or civilisation would be, for example, diluted as "written or oral expression" or "communication".
These suspicions should be put in the context of the educational institution in France, particularly with regards to its political role in the fabric of the Republic and then national socialisation, something which one famous French sociologist calls the programme institutionnel (institutional programme) of schools (Dubet, 2002).
Conversely, for other researchers, it is precisely this better, desirable link between school and a society which pleads for recourse to competences.
Philippe Perrenoud (2004, 2011) has therefore worked for some years on the best way, for schools to prepare students for their personal, social, economic and civic lives.
His view is that getting through school is not an end in itself: the student should be capable of applying his educational achievements outside of school, in different unforeseen and complex situations. In doing so, this leads to an interest in building competences for the transfer of knowledge to other situations through their integration into wider competences, incorporating further concepts of reflection, decision making, action and adaptation.
He argues for educational content to be significantly more controlled with regards to social, cultural and professional life as "if school prepares for life, changes in the world and to people's lives should logically be the main factor determining evolution of the curriculum, at least as much as the transformation of knowledge is. Ultimately, it should only be the knowledge that affects people's lives and their understanding of the world that should be the force behind changes to educational programmes.

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" (Perrenoud, 2008).

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Please note that all quotations in this section of the literature review have been translated from the original French into English by a subcontracted translator.

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Other experts note that existing disciplines are often a creation of teaching itself, particularly in Secondary Education, however they do not really correspond to the creations of the worlds of science, culture or economics (Gauthier, 2006).
For sociologist Denis Meuret, schooling in France was marked by the heritage of conceptions made by the early 20th century sociologist Emile Durkheim, as a result of which school is seen as a front to protect individuals from unwholesome influences in the world - an idea which creates a negative view of everything external to schools, and furthermore makes structuring of teaching very difficult alongside what pupils learn and experiment with in their lives outside of school (Meuret, 2007).
He believes that this tradition has favoured the changes necessary to integrate advances of a scientific and technical nature that arose in the form of academic knowledge, but it is significantly less suited to "post-modern" strategic changes, in which case the quality and competences of the person should be improved first, giving more meaning to learning.
1.3. The dominant definitions of competence in the context of education in the French-speaking world
In the French language and in daily life, competence can assume very different meanings: legal authority (the legitimacy of an institution to act in a particular domain, for example), overall quality given to an individual, technical ability in a domain, etc.
In education, it can also define an overall competence (close to a key competence), such as the ability to express oneself in public, as well as elementary parts of knowledge or expertise: learning a spelling rule or ability in using a tool, for example.
Some researchers use the term competence without distinction in a number of ways (competence in mathematics, competence in spelling, competence in writing comments on texts) to describe all learning in school, at a level of detail which has little to do with the concept of key competences (Morlaix, 2009).
Other works on the other hand look to define competences with a holistic and integrated view, closer to the definition of key competences.
Competence-based approaches aim principally to break from the traditionally dominant method of teaching by objectives seen in schools during the 20th Century, which separates the transmission of knowledge within disciplines into objectives to be reached at each level of schooling.
Here however, the integration of learning into an overall logic is favoured, with the concern that the final aim of the competence to be acquired is not only at the end of this process, but is included from the outset and conditions the very way in which the different constituent parts of the competence are built by the student. In this sense, it is knowledge in action, built for action in problem-solving situations, series of tasks, etc.
The complex skills in the field of competences rest in putting into use multiple elements of knowledge and expertise to enable an integration of achievements into teaching. (Roegiers, 2000).
The knowledge put into use are transformed and re-contextualized. It is in carrying out actions that competences must be inferred, which is where the importance of evaluation in

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appropriate situations lies. This also raises the issue of "attitudes", otherwise difficult to evaluate with indirect evidence (Scallon, 2007). Being confronted with a concrete situation is essential, and competence is rarely demonstrated by a figure or a numbered result, but by means of an overall judgement. It is not an abstract ability separate from any context: competence is finalised and contextualised.
In Québec, a number of studies on competence have been made with regards to the Observatory of Educational Reforms, leading to the creation of a UNESCO chair on curricular development at the University of Québec in Montréal.
According to the researchers involved, at least three constant elements appear to make up the concept of competence in contemporary literature:
- Competence is based on the putting to use and coordination of a variety of resources by a person in a particular situation;
- A competence can develop only in that situation;
- A competence can only be attained if fully completed.
However, in their opinion, this is not a sufficient curricular definition for the creation of study programmes. In particular, the question of coordinating resources is often misunderstood, in most cases being reduced to juxtapose existing disciplinary elements. However, more than the sum of resources, competence is the result of the efficient coordination of such resources (Jonnaert, Barette et al., 2004). The "explicit and thought-out competence" seems to be largely absent in thoughts on contemporary teaching and learning when considering competence in education. Nevertheless, the researchers warn that "if situations are important, they are not enough".
When discussing competences as a new educational paradigm, we are therefore referring to the possibility for an individual to apply an integrated set of resources to oneself, to solve a series of problem situations, as defined by X. Roegiers (2004).
The taking into account of competences in the world of training denotes, in a general way, the gradual shift from a focus on knowledge, considered as a preamble to activity and so often decontextualized, to a consideration of activities in which this knowledge embodies itself (Legendre, 2008).
We are no longer in a framework of stable knowledge in its disciplinary configuration, but rather in one of knowledge in a state of dynamic construction, where knowledge is constantly reconstructed by the student. The aim is not so much to convey a science of academic knowledge passed down through heritage, as to be productive in the application of knowledge that is expressed by practical results and the demonstration of expertise. Therefore knowledge is called upon to become competences, through activity-centred teaching (Lemaître & Hatano, 2007).
For Bernard Rey's team at the Université Libre de Bruxelles, the main interest in competence is to make students work on sufficiently wide-ranging activities, which are meaningful and whose function is evident to them, even if it is often difficult to determine which competences are directly relevant to their social and professional lives.
Three degrees of competences are distinguished, of which only the last two can really deserve to be called "competences":

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- An elementary competence: knowing how to carry out a task in response to a signal (automated procedure, skill)
- A competence framework: to be able to interpret an unknown situation and choose the elementary competence most suited to it;
- A complex competence: to choose and combine multiple competences to respond to a new and complex situation (Rey, Carette, Defrance & Kahn, 2006).
2. The establishment of competences in France
The teaching of competences did not wait for the 2005 law to establish itself, although sometimes referred to by different names. For example, this is the case in many primary schools where the learning of certain competences coexists with the teaching of traditional school subjects, in particular in schools which experiment with different teaching techniques, such as Freinet's Pedagogy (Reuter, 2007).
It is also the case in certain secondary schools, which promote interdisciplinary work, contextualize grades, and insist on cooperation or creativity, in the framework of experimental lower secondary Schools (Cédelle, 2009) or regular lower secondary schools, where they use their ability to experiment as permitted by law (Di Martino and Sanchez, 2011; Duffez, 2012).
2.1. The law on compulsory schooling implemented in 2005
In the autumn of 2003, the French government established a commission headed up by an expert, Claude Thélot, to arrange a significant consultation on the future of the French education system. The conclusions from this consultation were submitted in a report published in 2004 entitled Pour la réussite de tous les élèves (For the success of all students) (Thelot, 2004), clearly evoking a base of "essential" knowledge that a school must allow students to acquire.
One result of this consultation would lead to the law on the French socle commun de connaissances et de compétences (core curriculum of knowledge and competences), passed on the 23rd April 2005, dealing with four areas: command of the French language, command of the principal elements of mathematics, a culture of humanities and science giving the ability to freely live as a citizen, and a command of common techniques in information and communication (Raulin, 2006, 2008).
A further decree in 2006 made this base clearer, and was composed of 7 major competences, each one broken down into knowledge, skills and attitude, largely inspired by the European Key Competences:
- command of the French language;
- use of a foreign language;
- basic competence in mathematics, science and technology;
- command of common techniques in ICT;
- humanistic culture;
- social and civic competence;
- Autonomy and initiative.

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It can be noted that the "French" version of the competences transforms "sense of initiative and entrepreneurship" into simply "sense of initiative", to prevent ideological clashes, and raises humanistic culture to the rank of the major competences, while the European framework speaks only of "cultural awareness and expression".
In many ways, this law is a milestone in the history of French education. This is one of the few times where legislation has intervened in the field of education, and is almost without precedent since the late 19th Century, when political power dictated the curricula with the introduction of the concepts of ''common'' and of ''competences''. This in an education system which was one of the last in the Western World to develop a common lower secondary teaching system (the "collège unique") in which disciplines are organised within a structured framework at an academic and institutional level (Gauthier and Le Gouvello, 2010; Rey, 2010).
However, some believe that the way in which the socle commun (core curriculum) is written in such a way that it "refers to the old teaching by objectives method in three fields of classification: knowledge, attitudes and skills" (Clerc, 2012) behind the apparent distribution between knowledge, attitudes and skills. Others are concerned that the description of the main areas, similar to the European Key Competences, do not mention the issue of specific skills which are useful in everyday life, nor, more importantly, do they describe specific situations or series of situations (Perrenoud, 2011). Is there not the risk then that we fall back on the closest possible meaning of what we commonly call subject "competences", which are more familiar and already used?
New concepts and rapid change, mixed with the frequent hesitancy of successive governments led to many reports highlighting that the introduction of measures relating to primary and secondary schooling was half-hearted, slow and incomplete (Grosperrin, 2010, HCEE 2011).
In some areas, the competence approach was more easily adopted due to the existing widespread use of the concept. This is the case in physical education and sport, Modern Foreign Languages (influenced by the European Framework for Languages) or in ICT (Cerisier, Devachelle, Rizza and Nguyen, 2008).
In other areas, competences were often heavily assimilated to related subject teaching (native language teaching, mathematics, history) or the lack of conformity was difficult to police.
This is particularly true in the case of competences "social and civic competences" and "sense of autonomy and initiative" which raised concerns among teachers as it seemed to

lead more to influencing attitudes rather than objective knowledge.
2.2. Subjects and Competences
The implementation of competences through a common framework is hampered by reluctance expressed variably by some teachers, trades unions, and associations who are attached to the subject framework, and fear that the competence-based approach means that subjects in the socle commun (core curriculum) are weakened or watered-down.
This reluctance is mainly found in secondary education, which is mainly structured around subjects, rather than in primary education.
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For certain actors in the education field, academic subjects represents a framework of protection for schools against the pressures of society and employers, and in a more general way against a utilitarian view of the school which the competence approach is suspected of causing.
Despite there seemingly being a prevailing consensus in France that there are no allencompassing competences which can be acquired outside of the learning of subjects, some of those involved in the sector and some researchers are continually concerned that "knowledge" becomes watered-down within transversal "competences".
With regards to research on the subject, these concerns are found in a number of articles.
The most well-known is that of Marcel Crahay (2006), where he discounts the idea that knowledge is secondary in the competence approach, and that the ability to react to a particular situation tends to lead to forgetting the necessary decontextualisation at the heart of all processes of cognitive construction.
Maggay Schneider-Gilliot (2006) also believes that by focussing on competences common to multiple disciplines, the myth of transferability leads to the elimination of the specifics of the theories relating to knowledge and thereby creating a phenomenon of "decategorisation" of the issues studied in school.
2.3. Upper Secondary School (Lycée)
Beyond compulsory education, the issue of competences is also raised in an indirect manner, notably in the context of upper secondary education, which corresponds to "Lycée" in France.
In both technological and general education, a recent reform (Moisan and Cook, 2011) notably provides for the creation of a "personalised teaching" system, giving attention margin of manoeuvre to the local initiative of teachers and schools. The general aim is to assist Lycée students to build on competences which have otherwise been poorly acquired during traditional lessons. This can be found in certain personalised teaching systems in activities which resemble either the key competence of "learning to learn", or parts of the key competences (such as oral communication) which are not explicitly developed within the framework of standard subjects.
2.4. Assessing Competences
The Theoretical Model
It can be noted that in all countries that have introduced a competence-based approach assessment is a crucial factor and therefore needs to be considered at the same time as curricular reform and not afterwards, as is so often the case (De Ketele, 2009).
Scallon's view (2007) is that it is fundamentally important to distinguish between the recall of knowledge and its use. In response to the use of standardised tests, performance should be considered and focussed on, taking into account many features:
1) challenging situations, to encourage giving a relatively advanced and more complex response, rather than simply tickling a box
2) authenticity - to come as close as possible to a situation where real-life questions are evaluated
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3) Interactivity
4) The multi-dimensional aspect of learning, to take good account of abilities, as well as self-motivation and self-confidence
5) Requirements, expectations and standards. A threshold of expected success or levels of competences should be defined, rather than fixing a cut-off point which separates students who fail from those who succeed; as is the case with the measurement model.
6) The importance of judgement to carry out a complete evaluation without merely confronting isolated work and one-offs.
7) Processes as elements to be assessed, to give importance to path monitoring (process) and the growth of the individual, not only the final result (product)
8) Integration and assessment of learning, considering the notion of interactive regulation of teaching
9) the participation of the student in assessment as well as self-assessment, intended as a constituent part of competence (i.e. having the ability to think about one's own progress)
Among the multiple possible definitions, there is common agreement that a competent student is one who is capable of resolving complex and unforeseen tasks that require an element of choice and a combination of learned procedures (Carette, 2009, 2008). The idea of previously unseen and complex tasks is clearly in contrast with current mainstream school assessments. In other words, in order to resolve the problem, the student must choose and combine procedures having identified those which are most relevant to the situation in hand, and this process of "framing" is an essential component of competence.
Nevertheless, there should be no confusion made between a "complex" situation and a "complicated" situation. This is not a matter of having students confront infeasible or overlycomplicated tasks for their level; it is more a question of encouraging an alternative application of solutions already known or used (Gérard, 2008).
For most French-speaking researchers, the competence approach requires input based on the situation, not based on knowledge (Ayott-Beaudet & Jonnaert, 2011; Perrenoud, 2011, Beckers and Voss, 2008). Every competence has a situation and a group of situations relating to it, in which the competence develops and grows. This leads to the provision, for public policy, of a bank of situations or a similar pooling device aiding the assessment of competences for teachers.
If the competence is demonstrated during assessment, the "perceived" competence should also be considered, i.e. the feeling of being more or less competent in relation to a proposed task (Van Der Maren & Loye, 2011). The question of being able to reflect and give a meaning to learning is just as, if not more important than what could be considered as a purely behavioural dimension.
Being competent does not mean adding skills and attitudes to knowledge, but knowing what one knows, how one knows it, and with what force to react to confront a new situation (Clerc, 2012).
Assessment of Competences and Traditional Assessment
Grade-based assessments usually appear to be poorly compatible with an assessment of competences. This is particularly the case where the result is a certification based on an
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average grade weighted over multiple disciplines (Dauvisis, 2007). In certain countries, as is the case in France or in some cantons of Switzerland, many experiments are trying and have tried to find alternatives to marking. At primary school level, if assessment notebooks, portfolios or coloured symbols ("traffic lights") are fairly common experiments designed to eliminate marking at the beginning of secondary education still frequently provoke debate and come across opposition, especially from parents. These are parents who fear the loss of a "simple" method of evaluating their child's performance at school, as well as the fear of falling behind the rest of the school system still measured by competition based on grades.
The system also sends discouraging signs concerning this type of experiment, by maintaining marking systems based on traditional programmes alongside competence-based programmes, as is the case in France where the "Brevet" exam at age 14-15 has been maintained alongside the socle commun (core curriculum) (Colsaët and Mevel, 2011).
Some researchers have devised a multi-level assessment to evaluate the ability to classify and mobilise resources when attempting to master a competence, above and beyond mastering procedures (Rey, Carette, Defrance & Kahn, 2006). Their aim is to first evaluate complex competences, in other words, the ability of students to select and combine multiple competences from those already acquired, in order to adequately resolve a previously unseen problem. To do this, they have conceived an assessment method made up of three successive tests from the most complex (the competence) to the most simple (the procedure).
However, as one of the researchers involved points out, this system is not completely infallible: individuals do not demonstrate the same behaviour when faced with situations which nevertheless require the same "mental reasoning" but in different forms (Kahn, 2012).
Various tools are used in the French-speaking world in support of the assessment of competences, among which portfolios, learning files, and concept maps (Tardif, 2006).
In France, the assessment of competences is carried out using a tool called the "livret personel de compétences" (personal competences book) (Houchot and Robine, 2007; Delatouche 2012), which has been strongly criticised by teaching staff for a number of reasons:
- the numerous items behind each competence (dozens of times each time) which makes it seem as a sort of "tick box factory" and can create an artificial certification of competences by adding micro-tasks;
- a lack of coherence between competences, the book and traditional programmes;
- a binary certification of achieved/not achieved which does not reflect the progressive nature of learning;
- unrealistic or misunderstood levels of expectation;
- Certain traditional tests such as the "Brevet des collèges" (at the end of lower secondary education) coexisting with the certification of competences in the book at the end of Year 10, both having a different logic behind them.
The chaotic implementation of the competence book in the French context is a demonstration of the fragile nature of the changes which began with the 2005 law. As is often the case, assessment methods concentrate and bring out all of the debates and problems which have not previously been solved, among which:
1) a major reform, the socle commun (core curriculum), which has not been directly challenged but the political leadership of it was only intermittently sustained, leaving many ambiguities pending with regards to issues concerning compulsory education;
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2) a fundamental and controversial question in teaching, grading and examinations, discussed in a time of confusion and mix-ups, when in fact what is at stake is a symbolic and practical representation for teachers and students, but also as a way of evaluating education for parents and other people involved in the system.
3) The assessment of competences is a difficult challenge, for which the solution is yet to be found for most of those who are involved, outside of mainland France, while the atmosphere around the education system is somewhat anxious meaning that the lack of certainty, guesswork and improvised teaching methods is difficult to accept (Rey, 2012).
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Annex 1: KeyCoNet Literature Review Matrix
In this review we are trying to gain an understanding of:
- What types of research have been undertaken for the topics and issues that interest us and what are the key results and outcomes that influence how KCs are being developed and implemented in different countries?
- What are the major contributions of international (including European) organisations and research centres?
- What are the main issues or messages that will influence our work?
- Are countries designing and implementing KC (or other terminology) frameworks based on research or on other criteria?
- Do we know if KCs make a difference to learning and if so how? What is their contribution?
- Is there evidence to suggest that the EU framework could usefully be fine-tuned?
- Etc.

Topics and issues	Scientific review/books	Reports international organisations/EU organisations	Key reports about initiatives (EU MS + other countries)
KC Theoretical framework: Research on the 8 KCs + key research on transversal skills and other foundational competences/capacities e.g. resilience, self-efficacy, etc. (UH, using ATC21s scientific reviews)			

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KC Implementation
Approaches to implementation at practice level (classroom/learning environment): (UH)
Themes:
- carefully designed & appropriately orchestrated implementation strategy,
- appropriate resources used effectively
- pedagogy of competence development,
- innovative learning environments,
- intensive use of ICT facilities,
- project work & teamwork,
- individualised techniques of organising learning

To be considered at:
- School level (including networks, learning organisations)
- Local/regional
- National
- Interaction of all levels
- non formal environment (local and/or virtual)

Approaches to implementation at system(ic) level:
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Curriculum design and implementation (frameworks, etc.) (EIEPS + EUN)
Teacher training: (EUN)
Consequences and implications for:
- Defining and acquiring their own KCs to facilitate learning of KCs
- Initial and continuous Professional Development (CPD)

Assessment of key competences
Issues: e.g. "measuring what matters"
New assessment methods
School organisation: (EUN)
- Opening to and partnership with the surrounding community
- Role of teachers (in the school) and team work at teacher level
- Etc.

Non formal learning environments: (UH + EIEPS contribution)
- Better suited/more efficient for some specific competences?
- Which complementarity with formal environment?
- Etc.

Socio-economic issues: (EIEPS)
- schools as learning communities to combat disadvantage
- school in its local environment
- school in its virtual environment (?)
- individual students background

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Research and evidence based policy implementation (EIEPS + EUN)
Research:
How countries use research or not and of what types? With what outcomes?
Policy formulation: what do we learn about different approaches to formulation in terms of outcomes?
Policy evaluation: is there any? who does it? how? What does it tell us?
International influences on policy and research:
E.g: DeSeCo, PISA, etc.
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Annex 2: Major categories of terminology about competences and skills used in the EU
The data in this table was collected through country "fiches" used for J. Gordon, G. Halasz, M. Krawczyk, T. Leney, A. Michel, D. Pepper, E. Putkiewicz, J. Wisniewski, Key Competences in Europe: Opening doors for lifelong learners across the school curriculum and teacher education, (Warsaw, CASE-Center for Social and Economic Research) undertaken for the Directorate General Education and Culture of the European Commission, (2009). The examples are not exhaustive, but are given as illustrations.

Major category of terminology	Sub-categories	Countries examples
Competences (HU): "kompetencia". A frequent synonym is "képesség" which can be translated as aptitude or skill. (see below also under key competences)		
(IT) Competenze: knowledge and abilities that each pupil will turn into personal competencies with the help of the school. This was first introduced in the Portfolio delle competenze (portfolio), through the reform of 2003.		
(PT): competência - knowledge in action or in use. It refers to the integrated development of knowledge, skills and attitudes for all stages of basic education. Thus, competence does not involve adding a set of skills and attitudes to a certain body of knowledge. Instead, it relates to the promotion or integrated development of skills and attitudes which are conducive to the use of this knowledge in different situations with which the pupil may or may not be familiar.		
(Sl): Kompetenza tends to be used. (traditionally Slovenian term was zmoznost.): Developing and demonstrated abilities of individuals which enable them to act creatively, effectively and ethically in complex, unforeseeable and changed circumstances in professional, social and private life.		
Competence has three components:		
- Acquisition of theoretical, conceptual and abstract knowledge;		

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- Developing skills, expertise and procedural knowledge to solve problems;		
- Developing an autonomous and ethical stance towards other people, community and environment – responsibility and autonomy.		
These are identified as cognitive, functional and socialisation perspectives.		
Key competences (AT) The terms Schlüsselqualifikationen (key qualifications), Schlüsselkompetenzen (key competencies) and, in particular, dynamische Fertigkeiten (dynamic skills) are used when talking about subject-independent, transversal competencies. The terms Grundfertigkeiten (basic skills) and Grundkompetenzen (basic competencies) are used when trying to convey the notion in its broadest sense.		
All subject-specific competence must therefore be supported and complemented by Sozialkompetenz (social competence, i.e. taking on responsibility, co-operation, initiative, active participation, team spirit) and Selbstkompetenz (personal competence, i.e. development of individual talents and capabilities, being aware of one's own strengths and weaknesses, self-reflection). Promotion of these two dynamic skills should prepare pupils for real-life situations that cannot be addressed through the simple recall of information. Coping with such situations requires self-esteem, initiative, acceptance of responsibility, and cooperation with others.		
(see below basic competences)		
(BG) *Ключови компетентности* no legal existence but used by teachers and decision-makers		
(CY) Key competences as core knowledge		
(CZ) *Klíčové* kompetence		
(DK) Nøglekompetence generally refers to the ability to apply knowledge and qualifications to differing situation but not part education terminology. The National Competence Account has identified competencies for professional life and personal development (environmental and natural competencies, physical competencies, social competencies and learning competencies) based on the DeSeCo Report. They are:		
- social competence,		

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- literacy competence,		
- learning competence,		
- communicative competence,		
- self-management competence,		
- democratic competence,		

- ecological competence,
- cultural competence,
- health, sports and physical competence,
- creative and innovative competence.
(FI) Avaintaito = key competence & avaintaidot = key competences (see below cross-curricula themes)
(HU) The strategy of the ministry contained a list of specific key competences to be developed, as follows:
- the capability of using various learning techniques (NB this is close to learning to learn)
- the capability of intelligent learning
- the capability to apply knowledge
- instrumental competences (like communication, mathematics or ICT-related competences)
- social competences
- value orientation (the capability to understand and use norms and values)
(LT): Esminė kompetencija or Bendrasis gebėjimas
(MT): Key competences- 'Competence' is the proven ability to use knowledge, skills and other abilities to perform a function against a given standard in work or study situations and in professional and/or personal development. In the EQF, "competence' is described in terms of responsibility and autonomy.
Educational objectives are defined and explained in a competence format, under the headings
knowledge/information, skills, and attitudes as follows:
- Self-awareness and the development of a System of Ethical and Moral Values;
- The Development of Citizens and a Democratic Environment;
- Developing a Sense of Identity through Creative Expression;
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- Religious Education;
- Strengthening of Gender Equality;
- Education on Human Sexuality;
- Preparing Education Consumers;
- Media Education;
- Effective and Productive Participation in the World of Work;
- Education for Leisure;
- Wise Choices in the Field of Health
- Greater Awareness of the Role of Science and Technology in Everyday Life;
- Competence in Communication;
- Preparation for Change.
Key Competences are listed as part of the National Qualifications Framework rather than as part of the curriculum (which predates the NQF). They are available for levels 1 to 3 of the NQF (up to the Secondary Education Certificate) and mirror the EU framework.
'The learning how to learn, self-directed learning and autonomous learning are the means for further personal development and keys to personal success in a career. There will hardly be any attempt at learning and training later on in life if the child is not equipped with literacy and numeracy, the ability to obtain and pass on information, the use [of] information and communication technology, the ability to analyse, to plan, to execute, the evaluate, the ability to work in a team - as a leader, as a team player, the need to be creative and entrepreneurial. This knowledge and these skills are essential elements of the National Framework curriculum."
(NL) Encompetenties (meaning, literally, core competencies). (See also below under general objectives)
(RO) Competenţă cheie
(SK) Kľúčové kompetencie
(SI) Key competences now described as: learning to learn; social skills; ICT; planning and developing one's career;
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entrepreneurship; environmental responsibility; safety at work.
Cross-curricular
key competences
(DE): general (subject-independent) competencies essential in order to operate effectively at personal and professional level. They are not limited to cognitive abilities and represent complex operational competencies:
- are required for and supported by different subjects and subject areas,
- help solve complex, holistic tasks in real-life contexts,
- can be transferred to new situations not covered by the curriculum,
- can be characterised as general abilities.
Basic
competences
(AT) Grundkompetenzen- used to convey the notion in its broadest sense.
(EL) ΒΑΣΙΚΕΣ ΔΕΞΙΟΤΗΤΕΣ - referring to IT skills
ΕΠΙΚΟΙΝΩΝΙΑΚΕΣ ΚΑΙ ΚΟΙΝΩΝΙΑΚΕΣ ΔΕΞΙΟΤΗΤΕΣ - communications and social competencies
(ES) Competencias Básicas. The Spanish Royal Academy gives the following definition for Competencia: Skill, aptitude, suitability to do something or to take part in a subject. They are part of the enseñanzas mínimas, the core curriculum defined by central government (objectives, key competences, contents & evaluation criteria) that serves as a reference for the Autonomous Communities. The key competences respect at least 3 criteria: within reach of the majority, relevant in a wide range of areas of life, contribute to lifelong learning. They are:
- Competence in linguistic communication
- Mathematical competence
- Competence n knowledge of and interaction with the physical world
- Information processing and digital competence
- Social and civic competence
- Cultural and artistic competence
- Learning to learn
- Autonomy and personal initiative
Tiana et al (2011)
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Foundation of
competences
(BE-fr) In compulsory education there is the Socles de compétences (foundations of competences) and the compétences terminales (final competences achieved). The socles de compétences are a "formal system of reference that sets out, in a structured way, which competencies must be exercised until the end of the first eight years of compulsory education, and those for which proficiency must be attained at the end of each stage, because they are considered necessary for social insertion and the pursuit of studies". As guarantors of school democratisation, learning guides, and assessment watchdogs, the socles de compétences mark out the difficult path that must lead not only to equal access to school, but also to equal results from education and to equal requirements for all children. They include both interdisciplinary and cross-curricula competencies, which, when gradually acquired, ensure the development of the pupil and his/her personality, and subject-related competencies.
(FR) Socles de connaissances et de compétences :
- Command of the French language
- Command of at least one modern foreign language
- Command of the main elements of mathematics and of science and technology
- Practical knowledge of information and communication
- Cultural education/awareness to enable participation in society/the exercise of citizenship
- Civic and social competences
- Autonomy and initiative
(LU) Socle de compétences. In German the term is: Kompetenzorientierte Bildungsstandards:
- Logical reasoning - mathematics
- Language, Letzeburgesch and opening to languages
- Discovery of the world by all the senses
- Psycho-motor skills, corporal expression and health
- Creativity, awakening to aesthetics and culture
- Living together and values.
There are transversal competences: citizenship, autonomy and personal aspirations
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Final competences
(to be achieved)
(BE-fr) Compétences terminales (see above)
General
competences
(EE) üldpädevus - general competence is used in the national curriculum and consists of four competences:
- learning competence - the ability to manage one's learning activities by using efficient learning strategies and suitable learning style; to motivate oneself to learning, to search for necessary information, have an overview of one's knowledge, to relate one's knowledge to that created by other people and create new knowledge, to monitor and assess one's thinking and learning activities.
- activity competence - ability to notice problems and to solve them, to plan one's activities, set goals and foresee expected results, to choose activity means, to assess the results on activities, ability to cooperate.
- value competence - ability to perceive one's relation to other people, to one's own and other cultures nature and things created by humans to assess relations between people and activities in relation to general moral norms.
- Self-definition competence - ability to understand and assess oneself, understand the meaning of one's activities and behaviour sin society, to shape one's personality.
They are specified depending on the age group. There are also subject competences which describe the content of each subject (see section 1 for the list of subjects). As a result of the integration of subject competences and general competences students will have domain-specific competences:
- valdkonnapädevus - domain-specific competence. There are seven domain-specific competences that are wider than those covered by individual subjects and should cover each aspect of the interaction a person has with the world surrounding him/her as well as her/himself. They are: nature competence; social competence; reflection & interaction competence; communication competence; technology competence; art competence; mathematics competence. There are expected outcomes per age group.
- kohustuslikud läbivad teemad - compulsory cross-curricular themes. There are four that are not taught as separate subjects (although they can be) but have to be covered while learning other subjects. They are: environment and sustainable development, career & career planning, media studies, ICT & security.
- ainealased teadmised ja oskused - skills and knowledge in specific subjects. The second part of the national curriculum that consists of subject curricula states the competences for each one to be attained by all pupils at a specific school stages.
New competences (LT) The Provisions of the National Education Strategy 2003-2012 stresses the need to develop citizenship, entrepreneurship, digital competences, learning to learn and cultural awareness competences. It refers to cross
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curricular competences: To ensure the quality of education development the contents of education shall be updated and related to new competencies of an individual: a more consistent transition to the new contents development policy oriented towards development of general abilities, values, provision of the necessary competencies based not so much on the transfer of knowledge, as on their analysis, critical assessment and practical application; such competencies shall relate the contents of education to actual life, actual problems and their solutions.
Essential
competences
(PT) Competências essenciais (essential competencies) constitute the body of general and subject specific knowledge which is considered indispensable for all citizens in today's society. In particular, it is essential to identify the knowledge which enables pupils to develop their understanding of the nature of each subject and its processes, as well as a positive attitude towards intellectual activity and the practical work it entails.
At the end of basic education, it is expected that pupils should have acquired the following general essential competencies, involving the ability to do the following:
- mobilise cultural, scientific and technological knowledge to understand reality and deal with everyday situations and problems;
- use languages from the different fields of cultural, scientific and technological knowledge to express themselves effectively;
- use the Portuguese language to communicate proficiently and structure their thoughts;
- use foreign languages to communicate proficiently in everyday situations and assimilate information;
- adopt personalised working and learning methodologies geared to achieving set objectives;
- investigate, select and organise information in order to transform it into knowledge that can be mobilised;
- adopt appropriate problem-solving and decision-making strategies;
- carry out activities independently, responsibly and creatively;
- cooperate with others in common tasks and projects.
Capacities (UK - Scot.) The purpose of the Curriculum for Excellence is encapsulated in the four capacities - to enable each child or young person to be a successful learner, a confident individual, a responsible citizen and an effective contributor.
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The curriculum aims to ensure that all children and young people in Scotland develop the knowledge, skills and attributes they will need if they are to flourish in life, learning and work, now and in the future. Each one is expressed as attributes and capacities to be achieved through learning experiences and outcomes which are a set of statements which describe the expectations for learning and progression for each curriculum area.
Competence
Account
(DK) Nationalt Kompetenceregnskab National Competence Account - key competences defined as acting as axes of rotation, activating professional competences and serving as a pre-requisite for the acquisition of professional competences.
Skills (UK- Eng.) Skills:
- personal learning and thinking skills
- key skills/functional skills
- cross-curricular themes/whole curriculum dimensions
The personal learning and thinking skills (PLTS) framework contains six groups of skills: independent enquirers; creative thinkers; reflective learners; team workers; self-managers; effective participators. There are also crosscurriculum dimensions which can provide a focus for work within and between subjects and across the curriculum. They include: identity and cultural diversity; healthy lifestyles; community participation; enterprise; global dimension and sustainable development; technology and the media; creativity and critical thinking.
UK (Nn. Ire.): At the heart of the curriculum lies an explicit emphasis on the development of skills and capabilities for life-long learning and for contributing effectively to society. These whole curriculum skills and capabilities consist of:
- Cross-Curricular Skills (Communication, Using Maths and Using ICT)
- Thinking Skills and Personal Capabilities - TSPCs - (including Managing Information, Thinking, Problem Solving and Decision-Making, Being Creative, Working with Others, and Self-Management)
They are embedded and infused throughout the revised Northern Ireland Curriculum at each key stage and pupils should have opportunities to acquire, develop and demonstrate these skills in all areas of the curriculum.
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UK (Wales): Skills Framework for 3 to 19 year olds:
- Developing thinking: Learners develop their thinking across the curriculum through the processes of planning, developing and reflecting.
- Developing communication: Learners develop their communication skills across the curriculum through the skills of oracy, reading, writing and wider communication.
- Developing ICT: Learners develop their ICT skills across the curriculum by finding, developing, creating and presenting information and ideas and by using a wide range of equipment and software.
- Developing number: Learners develop their number skills across the curriculum by using mathematical information, calculating, and interpreting and presenting findings.
Basic skills (AT) Grundfertigkeiten- used to convey the notion in its broadest sense. (see also above under key competences)
Dynamic skills (AT) Dynamische Fertigkeiten (dynamic skills) - subject-independent transversal competencies. . (see also above under key competences)
Skill / aptitude (HU) "kompetencia" - a frequent synonym is "képesség" which might be translated as aptitude or skill (see also above under competences)
Key skills (IE) Key Skills Framework:
- Primary: the ability to question, analyse, investigate, think critically, solve problems
- Junior secondary: interact effectively with others: communication and literacy, numeracy, manipulative skills, information technology, thinking and learning, problem solving, personal and interpersonal, social
- Senior cycle: information processing, critical and creative thinking, working with others, communicating,

being personally effective.
Cross-curricular
skills
(PL) A number of cross-curricular transversal skills are mentioned:
- planning, organising and assessing pupil's learning, taking responsibility for one's learning process,
- effective communication in various circumstances, presenting one's viewpoint and acknowledging the
views of others, proper use of language, preparation for public presentations,
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- effective cooperation within a group, building interpersonal relations, taking individual and collective
decisions,
- solving problems in a creative way,
- searching for, ranging and using information from various sources, effective usage of ICT,
- using the acquired knowledge in practice an creating of necessary experiences and patterns of behaviour,
- developing personal interest, acquiring methods of negotiative ways of solving conflicts and social
problems.
Goals and
objectives
Core objectives (NL) Core objectives (which relate to subjects) and general objectives (which are cross-curricular) are used.
Development
objectives and final
objectives
(BE-fl) Ontwikkelingsdoelen and Eindtermen
Final objectives are minimum objectives with regard to knowledge, insight, skills and attitudes, which the
educational authorities consider necessary and attainable for a specific pupil population. Developmental objectives
are minimum objectives which the educational authorities consider desirable for a specific pupil population (Decree
4 February 2003). The final objectives and developmental objectives define which knowledge, insight, skills and
attitudes society considers necessary or desirable for pupils at the various levels of education. Thus, final
objectives and developmental
Goals to be
achieved and
goals to strive for
(LV) The National Standard for General Secondary Education determines the goals common for all general
secondary education curricula:
- to provide pupils with knowledge and skills enabling them to prepare for further education;
- to encourage the development of their personality and of their physical and mental capacities, and to
develop their understanding of health as a condition for the quality of life;
- to encourage the development of positive, critical and socially active attitude, and to develop
understanding of rights and obligations of Latvian citizens; to develop the ability to study independently
and improve knowledge as well as create motivation for lifelong learning and a purposeful career.
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(SE) The term "key competences" does not figure in the documents determining the Swedish education system.
Instead the concept of goals is used: goals to strive for (quality of education process) and goals to be attained
(learning outcomes). Amongst the goals are the knowledge, skills and attitudes corresponding to particular key
competences. The goals are not ranked in order of importance. There is a nation-wide curriculum for pre-school,
compulsory school and upper secondary school. (See annex 1 for list)
General Objectives (NL) General objectives are regarded as important for all learning areas in the primary curriculum are:
• A good working attitude
• Use of learning strategies
• Reflection on one's own actions and learning
• Expression of one's own thoughts and feelings
• Respectful listening to and criticising of others' opinions
• Acquisition and processing of information
• Development of self-confidence
• Respectful and responsible dealing with others
• Care and appreciation for the living environment
Secondary level: the six general objectives are:
- Cross-disciplinary themes: Within the context of broad and balanced consideration of people and society,
pupils should obtain a certain level of insight into their position within their immediate personal
environment, as well as the wider social environment
- Learning to do: In situations as close as possible to real life, pupils should learn how to develop further
certain skills acquired at school, making use of ICT where appropriate.
- Learning to learn: Pupils should learn to acquire knowledge and skills, making use of ICT where
appropriate. To this end, they should learn (amongst other things) certain strategies for improving the
learning process.
- Learning to communicate: Pupils should learn how to develop further certain social and communicative
skills, on an interactive basis where appropriate. T
- Learning to reflect upon the learning process: Pupils should learn to be analytical about and to control the
learning process, by reflecting upon their own performance.
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- Learning to reflect upon the future: Pupils should learn to be analytical about their options for the
future/prospects and interests, by reflecting upon their own performance.
Training objectives (CY) TVET Curriculum includes training objectives (the content, the teaching materials, the students' profile and
suggested approaches and methods in order to achieve the training objectives.)
Goals in the
curriculum
(HU) The NCC contains the following parts: (1) the description of the role of the NCC in public education, (2) the
common values of education, (3) the basic goals (including the development of key competencies and the
development of a number of other areas), (4) the principles of differentiation, (5) equity principles, (6) the principles
of the elaboration of school level curricula with references to other national regulatory documents, (7) the
standards themselves.
In the NCC the standards are defined as learning outcomes, i.e. knowledge, skills, competencies, attitudes etc. To
be possessed by the pupils by the end of the specific cycle.
The NCC 2003 also contained a thesaurus which defined the notion of "competence based" and the relation
between "competence" and "knowledge". According to this. "The 'competence based approach means a
commitment that determines the taxonomy of the curriculum. In the background of this approach there is a theory
of personality which considers the competences (personal, cognitive, social and special competences) as the main
components of the personality... [This approach] links the competences to specific activities and tasks that are
achieved by the human being: someone is competent in relation with an activity if he/she is capable to achieve the
tasks related with this activity." This was the basis for restructuring the NCC so that instead of the element of
national culture ("knowledge") it was focusing on specific developmental tasks that are supposed to develop
specific competences.
Cross-curricular
themes
(FI) The national core curriculum defines the goals, main content of the various subjects & cross-curricular themes.
They are intended as an explicit part of the curriculum to be taught across the whole curriculum, and anchored in
particular areas of the subject curriculum. They are educational challenges with social significance. At the same
time, they are current statements on values. The objectives behind them are
- observe and analyse contemporary phenomena and operating environments;
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- express justified ideas of a desirable future;
- assess their own lifestyle and prevailing trends from a future perspective; and
- make choices and take action for the future that they consider as being desirable.
For the upper grades of basic education, the cross-curricular themes are:
- Growth of the person
- Cultural identity and internationalism
- Media skills and communication
- Participatory citizenship and entrepreneurship
- Responsibility for the environment, well-being and a sustainable future
- Safety and traffic competences
- Technology and the individual
For all upper secondary pupils in both the general and vocational strands the cross-curricular themes are:
- Active citizenship and entrepreneurship
- Welfare and safety
- Sustainable development
- Cultural identity and knowledge of cultures
- Technology and society
- Communication and media competence
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Annex 3: Sweden: goals to strive towards and goals to attain
The 'goals to strive towards' detailed in the curriculum documents include that schools should ensure that all pupils:
- Develop the ability to form and express ethical viewpoints based on knowledge and personal experiences.
- Respect the intrinsic value of other people.
- Reject the oppression and abusive treatment of other people and assist in supporting them.
- Can empathise with and understand the situations other people are in and can develop the will to act with their best interests at heart.
- Show respect for their immediate environment as well as for the environment in its wider perspective.
- Develop a sense of curiosity and the desire to learn.
- Develop their own individual way of learning.
- Develop confidence in their own ability.
- Feel a sense of security and learn to consider and show respect in their dealings with others.
- Learn to carry out research, and to learn and work independently and together with others.
- Acquire good knowledge in school subjects and subject areas to develop themselves and prepare for the future.
- Develop a rich and varied language and understand the importance of cultivating this.
- Learn to communicate in foreign languages.
- Learn to listen, discuss, reason and use their knowledge as a tool to formulate and test assumptions as well as to solve problems.
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- Reflect on their experiences and critically examine and value statements and relationships.
- Acquire sufficient knowledge and experience to be able to make well considered choices over further education and vocational orientation.
- Take personal responsibility for their studies and working environment.
- Gradually exercise increasingly greater influence over their education.
- Have an understanding of democratic principles and develop their ability to work democratically.
- Acquire sufficient knowledge and experience to be able to examine different options and make decisions concerning their own future.
- Develop the ability to assess their results themselves.
'Goals to attain' in the compulsory school include to:
- Have mastered Swedish and to be able to listen and read as well as to express ideas and thoughts in the spoken and written language.
- Have mastered basic mathematical principles and be able to use these in everyday life.
- Know and understand basic concepts and contexts within the natural sciences as well as within technical, social and human areas of
knowledge.
- Have developed the ability to express themselves creatively and be interested in participating in the range of cultural activities that society has
to offer.
- Be familiar with central parts of the Swedish, Nordic and Western cultural heritages.
- Be aware of the culture, language, religions and history of national minorities.
- Be able to develop and use their knowledge and experience in as many different forms of expression as possible covering language, images,
music, drama and dance.
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- Have developed their understanding of other cultures.
- Be able to communicate in speech and writing in English.
- Know the basis of society's laws and norms as well as their own rights and obligations in school and society.
- Be aware of the interdependence of countries and different parts of the world.
- Be aware of the requirements for a good environment and understand basic ecological contexts.
- Have a basic knowledge of the requirements to maintain good health and to understand the importance of lifestyle for health and the
environment.
- Have some knowledge of the media and of their role in relation to the media.
- Be able to use information technology as a tool in their search for knowledge.
- Develop their learning and to acquire deeper knowledge in a number of individually selected subject areas.
Sources: Eurydice note and Skolverket http://www.skolverket.se/sb/d/354/a/959
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Annex 4: Matrix of example behaviours by domain - Developing a Universal Toolkit for Measuring the Impact of Provision on the
Outcomes Achieved by Young People for Extending Entitlement and Learning Pathways 14-19 (13 October 2006). Prepared by
AlphaPlus Consultancy Ltd, Netton Old Farmhouse, Bishopstone, Salisbury, Wiltshire, UK -SP5 4DD.
GC Domain Context Min competence
Level 1
Level 2 Level 3 Mid competence
Level 4
Level 5 Level 6 Max competence
Level 7
Knowing myself Reflection Has a tendency to
brood about past
failures. Can become
depressed.
Personal reflection
often leads to a
somewhat negative
outlook on life.
Spends more time
thinking about the
past than the future.
Uses personal
reflection to adopt a
more positive
approach to life.
Reflects on past
experience in order to
adopt a more positive
view of life.
Is mostly forward
looking and knows
how to learn from
personal experience.
Is very forwardlooking. Learns well
from personal
experience.
Knowing myself awareness of
own qualities
and options for
change
Has no significant
awareness of own
qualities and options
for change.
Has little awareness
of own qualities and
options for change.

Has a limited
awareness of own
qualities and options
for change.
Is aware of own
qualities and options
for change.
Shows a broad
awareness of own
qualities and options
for change.
Is well aware of own
qualities and options
for change.
Is fully aware of own
qualities and options
for change.
Knowing myself planning and
implement
changes
Does not plan ahead.
Is not interested in
implementing
changes in personal
development.
Rarely plans ahead.
Has little interest in
implementing
changes in personal
development.
Occasionally plans
ahead. Makes an
effort to implement
changes in personal
development.
Makes some effort to
plan ahead but lacks
organisation. Is
interested in
implemented
changes in personal
development.
Usually plans ahead.
Is interested in
implementing
changes in some
aspects of personal
development.
Almost always plans
ahead. Is keen to
implement changes
in personal
development.
Always plans well
ahead. Knows how
to implement
changes in personal
development and
does so.
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GC Domain Context Min competence
Level 1
Level 2 Level 3 Mid competence
Level 4
Level 5 Level 6 Max competence
Level 7
Coping with
Feelings
understanding
feelings and
how they
influence
thinking and
behavior
Always displays
continuous outbursts
of temper. Reacts
negatively to any
criticism. Unable to
identify and name
own feelings
Nearly always
displays frequent
outbursts of temper,
unable to take
criticism, unable to
name or identify own
feelings and the
affect on their
reactions
Frequently displays
outbursts of temper,
reacts defensively to
criticism. Able to
name and describe
feelings. Often able to
identify and name
own feelings, but
displays lack of
understanding of the
impact on their
reactions
Displays occasional
outbursts of temper
or extreme mood
swings due to factors
beyond their control.
Can accept criticism
but does not take
positive action. Can
identify their own
feelings but do not
always reflect on how
it affects their
reactions
Occasionally loses
their temper with
others or in stressful
situations. Usually
accepts criticism and
sometimes takes
positive action as a
result. Can identify
their feelings and
sometimes show
understanding of how
it influences their
reactions
Rarely loses their
temper with others or
in stressful situations.
Accepts most
criticism and often
takes positive action
as a result. Can
identify their feelings
and understand how
it influences their
reactions
Never loses their
temper with others or
in stressful
situations. Accepts
criticism and takes
positive action as a
result. Can identify
their feelings and
understand how it
influences their
reactions and the
reactions of others.
Coping with
Feelings
appropriate
expression in
situations and
linking feelings
to actions
Finds considerable
difficulty expressing
personal feelings. Is
unable to link feelings
to actions.
Is largely unable to
express personal
feelings. Has little
ability to link feelings
to actions.
Can sometimes
express personal
feelings. Has a
limited ability to link
feelings to actions.
Displays some ability
to express personal
feelings. Can
sometimes link
feelings to actions.
Is usually able to
express personal
feelings. Is mostly
able to link feelings to
actions.
Is good at expressing
personal feelings. Is
able to link feelings to
actions in most
situations.
Expressed personal
feelings with
complete ease. Is
always able to link
feelings to actions.
Coping with
Feelings
recognise
feelings
produced by
prejudice and
discrimination
Shows no awareness
of the feelings
produced by
prejudice and
discrimination.
Shows little
awareness of the
feelings produced by
prejudice and
discrimination.
Shows a limited
awareness of the

feelings produced by prejudice and discrimination.
Shows some awareness of the feelings produced by prejudice and discrimination.
Usually shows awareness of the feelings produced by prejudice and discrimination.
Almost always shows awareness of the feelings produced by prejudice and discrimination.
Is always sensitively aware of the feelings produced by prejudice and discrimination.
Coping with feelings
understand and respond to feelings of others
Shows no awareness or understanding of the feelings of others.
Shows little awareness or understanding of the feelings of others.
Shows some awareness and understanding of the feelings of others but makes little attempt to respond to them.
Shows some awareness and understanding of the feelings of others and responds to a limited degree.
Usually shows awareness and understanding of the feelings of others and is able to respond in a suitable way.
Almost always shows awareness and understanding of the feelings of others and makes an effort to respond appropriately.
Shows a sensitive awareness and understanding of the feelings of others and always responds sympathetically.
122
GC Domain Context Min competence Level 1
Level 2 Level 3 Mid competence Level 4
Level 5 Level 6 Max competence Level 7
Holding Beliefs put forward your opinion, appreciate others and recognise their influence
Never puts forward an opinion. Does not appreciate others or recognise their influence.
Rarely puts forward an opinion. Has a very limited appreciation of others and usually fails to recognise their influence.
Will put forward an opinion but with little confidence. Has some appreciation of others and can sometimes appreciate their influence.
Is wiling to put forward an opinion on occasions and makes a real effort to appreciate others and recognise their influence.
Is able to put forward an opinion with some conviction. Shows some appreciation of others and can recognise their influence.
Usually puts forward an opinion confidently. Mostly appreciates others and recognises their influence.
Puts forward an opinion with confidence. Always appreciates others and recognises their influence.
Holding Beliefs understand what beliefs are and how they affect attitudes and behavior
Shows no understanding of the beliefs of others. Is insular in outlook. Prefers not to mix with people of different beliefs.
Shows little understanding of the beliefs of others. Has little interest in how beliefs affect attitudes and behaviour.
Makes some effort to understand the beliefs of others. Limited understanding of how they affect attitudes and behaviour.
Shows some understanding of what beliefs are and how they affect attitudes and behaviour. Can be suspicious of people with different beliefs.
Usually shows understanding of the beliefs of others. Realises how they affect attitudes and behaviour.
Almost always shows understanding of the beliefs of others and how they affect attitudes and behaviour.
Shows a deep understanding of the beliefs of others. Is very open-minded and receptive to the beliefs of others.
Handling Relationships
recognise and use skills to make and keep different types of relationship
Is unable to recognise and use skills to make and keep different kinds of relationships.
Has difficulty in recognising and using skills to make and keep different kinds of relationships.
Has a limited ability to recognise and use skills to make and keep different kinds of relationships.
Shows some ability to recognise and use skills to make and keep different kinds of relationships.
Is usually able to recognise and use skills to make and keep different kinds of relationships.

Is almost always able
to recognise and use
skills to make and
keep different kinds
of relationships.
Is always able to
recognise and use
skills to make and
keep different kinds
of relationships.
Handling
Relationships
understand
and use
approaches to
difficult
relationships
Has no
understanding of how
to use approaches to
difficult relationships.
Has little
understanding of how
to use approaches to
difficult relationships.
Has a limited
understanding of how
to use approaches to
difficult relationships.
Shows some
understanding of how
to use approaches to
difficult relationships.
Is usually able to
understand how to
use approaches to
difficult relationships.
Is almost always able
to understand how to
use approaches to
difficult relationships.
Is highly skilled at
how to use
approaches to
difficult relationships.
123
GC Domain Context Min competence
Level 1
Level 2 Level 3 Mid competence
Level 4
Level 5 Level 6 Max competence
Level 7
Handling
Relationships
recognise
proper and
improper use
of power and
control
Is unable to
recognise proper and
improper use of
power and control.
Has little ability to
recognise proper and
improper use of
power and control.
Has a limited ability to
recognise proper and
improper use of
power and control.
Shows some ability to
recognise proper and
improper use of
power and control.
Shows a broad
awareness of how to
recognise proper and
improper use of
power and control.
Is almost always able
to recognise proper
and improper use of
power and control.
Is highly skilled at
recognising proper
and improper use of
power and control.
Handling
Relationships
recognise
difference
between
aggressive and
assertive
behaviour
Is unable to
recognise the
difference between
aggressive and
assertive behaviour.
Is rarely able to
recognise the
difference between
aggressive and
assertive behaviour.
Is only occasionally
able to recognise the
difference between
aggressive and
assertive behaviour.
Is sometimes able to
recognise the
difference between
aggressive and
assertive behaviour.
Is usually able to
recognise the
difference between
aggressive and
assertive behaviour.
Is almost always able
to recognise the
difference between
aggressive and
assertive behaviour.
Is always able to
recognise the
difference between
aggressive and
assertive behaviour.
Handling
Relationships
use different
approaches to
giving and
receiving
feedback
Has no
understanding of how
to use different
approaches to giving
and receiving
feedback.
Has only a minimal
understanding of how
to use different
approaches to giving
and receiving
feedback.
Has a limited
understanding of how
to use different
approaches to giving
and receiving
feedback.
Shows some
understanding of how
to use different
approaches to giving
and receiving
feedback.
Is usually able to
understand how to
use different
approaches to giving
and receiving
feedback.
Is almost always to
understand how to
use different
approaches to giving
and receiving
feedback.
Is highly skilled at
understanding how
to use different
approaches to giving
and receiving
feedback.
Getting and
giving support
knowledge and
understanding
of available
support
Has no knowledge or
understanding that
help or support is
available.
Has only minimal
knowledge and
understanding of the
help and support that
is available.
Has a limited
knowledge and
understanding of the
help and support that
is available.
Is aware of some
knowledge and
understanding of the
help and support that
is available.
Shows broad
awareness,
knowledge and
understanding of the
range of help and
support that is
available.

Is well aware of and
understands well the
range of help and
support that is
available.
Is fully aware of the
full range of help and
support that is
available.
Getting and
giving support
understands
how to get
support
Never knows where
or to whom to turn for
appropriate help and
support.
Rarely knows where
or to whom to turn for
appropriate help and
support and only
rarely follows this
through.
Occasionally seems
to know where or to
whom to turn for
appropriate help and
support but often fails
to follow this up.
Is likely to know
where and who to
contact for
appropriate help and
support and quite
often follows this up -
but not always.
Usually knows where
to get help and
support and, more
often than not, follows
things through.
Often turns to the
appropriate source of
help and support,
and mostly follows
things through.
Always turns to the
appropriate source of
help and support,
and always follows
things through.
124
GC Domain Context Min competence
Level 1
Level 2 Level 3 Mid competence
Level 4
Level 5 Level 6 Max competence
Level 7
Getting and
giving support
understands
how to give
support to
others
Has a very poor
grasp of how to help
others and is almost
never prepared to
give the help that
others may need.
Has only a very
limited grasp of the
needs of others in
terms of the support
they may require or
how to provide it.
Has only a limited
grasp of the needs of
others in terms of the
support they may
require or how to
provide it.
Has some grasp of
the needs of others in
terms of the support
they may require and
an understanding of
how to provide it.
Usually appreciates
that others may need
help at times and is
usually willing to
provide the support
they need.
Almost always
appreciates when
others may need help
and is almost always
willing to provide it.
Always appreciates
that others may need
help at times and is
always willing to
provide it.
Exploring risks personal
motivation and
recognising
own reaction to
taking risks
Shows almost no
motivation, fails to
see opportunities and
is unable to recognise
when a risk is worth
taking or not.
Shows a little selfmotivation and is very
occasionally aware of
the possible
repercussions and
reactions of risk
taking to self.
Shows some selfmotivation and is
occasionally aware of
the possible
repercussions and
reactions of risk
taking to self.
Shows reasonable
self-motivation and is
aware of some of the
possible
repercussions and
reactions of risk
taking to self.
Shows self-motivation
and is aware of many
of the possible
repercussions and
reactions of risk
taking to self.
Is well motivated and
is well aware of many
of the possible
repercussions and
reactions of risk
taking to self.
Is very well
motivated and is fully
aware of all the
possible
repercussions and
reactions of risk
taking to self.
Exploring risks shows
awareness of
situations that
may present
risk to self
and/or others
Shows no awareness
of situations and
occasions that may
present risks to self
and/or others.
Shows minimal
awareness of
situations and
occasions that may
present risks to self
and/or others.
Occasionally shows
some awareness of
situations and
occasions that may
present risks to self
and/or others.
Displays some
awareness of
situations and
occasions that may
present risks to self
and/or others.
Frequently shows
awareness of
situations and
occasions that may
present risks to self
and/or others.
Nearly always shows
awareness of
situations and
occasions that may
present risks to self
and/or others.
Always shows
awareness of
situations and
occasions that may
present risks to self
and/or others.
Exploring risks understands
how to practise
safe sex
Displays no
awareness, interest
or understanding of
how to practise safe
sex.
Displays little
awareness and
understanding of how

to practise safe sex.
Displays limited
awareness and
understanding of how
to practise safe sex.
Displays an
awareness and
understanding of how
to practise safe sex
Displays good
awareness and a
sound understanding
of how to practise
safe sex.
Displays almost
complete awareness
and understanding of
how to practise safe
sex.
Displays complete
awareness and
understanding of
how to practise safe
sex.
125
GC Domain Context Min competence
Level 1
Level 2 Level 3 Mid competence
Level 4
Level 5 Level 6 Max competence
Level 7
Exploring risks recognise risk
of drug use
and how to
reduce it
Shows no recognition
of the risks involved
in drug use or how to
reduce them.
Can only recognise a
very small number of
risks involved with
drug use and very
few ways of reducing
them.
Is able to recognise a
number of the risks of
drug use and knows
a few ways of
reducing them.
Is able to recognise
some of the risks of
drug use and knows
a number of
appropriate ways of
reducing them.
Has a sound
knowledge and
understanding of
many of the risks
involved in drug use
and appropriate ways
of reducing them.
Has a very good
knowledge and
understanding of
most of the major
risks involved in drug
use and most
appropriate ways of
reducing them.
Shows complete
knowledge and
understanding of all
the major risks
involved in drug use
and all appropriate
ways of reducing
them.
Managing Myself appreciate
resources
required to
achieve
personal goals
Is unable to
appreciate resources
required to achieve
personal goals.
Is largely unable to
appreciate resources
required to achieve
personal goals.
Has only a limited
ability to appreciate
resources required to
achieve personal
goals.
Has some
appreciation of
resources required to
achieve personal
goals.
Has a sound
appreciation of
resources required to
achieve personal
goals.
Has a good
appreciation of
resources required to
achieve personal
goals.
If fully appreciative of
resources required to
achieve personal
goals.
Managing Myself ability to
manage time,
money and
lifestyle
Shows no ability to
manage time, money
and lifestyle.
Is largely unable to
manage time, money
and lifestyle.
Shows only a limited
ability to manage
time, money and
lifestyle.
Shows some ability to
manage time, money
and lifestyle.
Is usually able to
manage time, money
and lifestyle.
Is mostly able to
manage time, money
and lifestyle.
Is fully able to
manage time, money
and lifestyle.
Managing Myself understand
importance of
diet and
exercise in a
healthy
lifestyle
Shows no
understanding of the
importance of diet
and exercise in a
healthy lifestyle.
Shows little
understanding of the
importance of diet
and exercise in a
healthy lifestyle.
Shows only a limited
understanding of the
importance of diet
and exercise in a
healthy lifestyle.
Shows some
understanding of the
importance of diet
and exercise in a
healthy lifestyle.
Shows a sound
understanding of the
importance of diet
and exercise in a
healthy lifestyle.
Has a very good
understanding and
appreciation of the
importance of diet
and exercise in a
healthy lifestyle.
Has a full
understanding and
appreciation of the
importance of diet
and exercise in a
healthy lifestyle.
Using Information understand
how to get
information
Does not understand
how to get
information.
Has a little
understanding of how
to get information.
Occasionally
understands how to
get information.
Has a reasonable
understanding of how
to get information.
Has a good
understanding of how
to get information.
Has a very good
understanding of how
to get information.
Fully understands
how to get
information.
Using Information understand
how to supply
information

Does not understand
how to supply
information.
Has a little
understanding of how
to supply information.
Occasionally
understands how to
supply information.
Has a reasonable
understanding of how
to supply information.
Has a good
understanding of how
to supply information.
Has a very good
understanding of how
to supply information.
Fully understands
how to supply
information.
126
GC Domain Context Min competence
Level 1
Level 2 Level 3 Mid competence
Level 4
Level 5 Level 6 Max competence
Level 7
Rights and
responsibilities
understand
personal rights
and
responsibilities
Never displays any
understanding of
personal rights and
responsibilities
Only rarely displays
an understanding of
personal rights and
responsibilities.
Occasionally displays
an understanding of
personal rights and
responsibilities.
Usually displays
some understanding
of personal rights and
responsibilities.
Displays a good
understanding of
personal rights and
responsibilities on
most occasions.
Displays a very
sound understanding
of personal rights and
responsibilities on
most occasions.
Fully understands
his/her personal
rights and
responsibilities.
Rights and
responsibilities
understand
how your
attitudes to
R&R affect you
and others
Has a virtually no
understanding of how
personal attitudes
towards rights and
responsibilities are
likely to affect oneself
and others.
Has a very limited
understanding of how
personal attitudes
towards rights and
responsibilities are
likely to affect oneself
and others.
Has a limited
understanding of how
personal attitudes
towards rights and
responsibilities are
likely to affect oneself
and others.
Has some
understanding of how
personal attitudes
towards rights and
responsibilities are
likely to affect oneself
and others.
Has a sound
understanding of how
personal attitudes
towards rights and
responsibilities are
likely to affect oneself
and others.
Has a very good
understanding of how
personal attitudes
towards rights and
responsibilities are
likely to affect oneself
and others.
Fully understands
how personal
attitudes towards
rights and
responsibilities are
likely to affect
oneself and others.
Rights and
responsibilities
appreciate and
act on your
personal rights
and
responsibilities
Displays no
appreciation of
personal rights and
responsibilities and
never knows how to
act on them.
Displays very limited
appreciation of
personal rights and
responsibilities but
occasionally knows
how to act on them.
Displays a limited
appreciation of
personal rights and
responsibilities but
occasionally knows
how to act on them.
Displays an
appreciation of
personal rights and
responsibilities and
usually knows how to
act on them.
Displays a sound
appreciation of
personal rights and
responsibilities and
often knows how to
act on them.
Displays a very good
appreciation of
personal rights and
responsibilities and
knows how to act on
them in most
instances.
Fully appreciates
his/her personal
rights and
responsibilities and
knows exactly how to
act on them.

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29 quotations

Codes (15)###HLT, #Areal life, #RME, $$Approaches to teaching Mathematics, $$mathematics education, $$meaningful learning of Maths, $$teaching methods and strategies, $factors that influence teaching and learning, $learner aspects, $mathematics, %role players, &pedagogy, @@value of using ICT, @Devices, @Tools

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Aiming for 21
st
. Century Skills
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Introduction
There is a saying, "In education everything happens 50 years later". If
true, this may have been not too problematic in older times, when changes
were slow, but in times like ours where changes are extremely fast, it will be
disastrous. Instead of being behind, education should be ahead. Primaryschool, for instance, will have to prepare today's students for their entrance
in society, which will be about twenty years from now. However, this is not
the case. Instead, Tony Wagner (2008) speaks of an "achievement gap" between what schools (in the USA) are teaching and what students will need
to succeed in today's global knowledge economy. He argues that, "students
are simply not learning the skills that matter most for the twenty-first century" (ibid, 8-9). And he goes on to say that, "Our system of public education—our curricula, teaching methods, and the tests we require students to
take—were created in a different century for the needs of another era. They
are hopelessly outdated." (ibid, 8-9). If we focus on employability, we may
observe, that future employees in countries such as the USA will have to
compete with colleagues in other countries with similar skills who work for
lower wages. But, more importantly, the skills that the current and future
jobs require anywhere, differ significantly from what current education offers. Wagner interviewed numerous CEO's of large companies and he found
a strong communality in what they look for in new employees; as one of
them phrases it: "First and foremost, I look for someone who asks good
questions." (ibid, 2). The underlying rationale is that these employees will
have to function in dynamic organizations. Employees today continuously

have to learn new things. From this perspective, asking the right questions,
is an important skill. In this respect, there is, however, a huge gap between
what society demands and what schools offer. Prevailing language educa-Aiming for 21
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tion, for instance, aims at technical reading skills, but does not ask students
to think about what they read. Mathematics instruction also often focuses on
skills, instead of understanding. Schools, Wagner (ibid) goes on to say, are
not designed to support students in learning to think. The reason for that, he
argues, is that we as a society never asked schools to teach students to think.
This is reflected in national tests that are not designed to assess the ability to
reason and analyze. The point is, the world has changed, schools have not.
Next to the dynamics of the modern workplace, there is also a shift in
the type of work people do. Empirical research by the economists Levy and
Murnane (2006) shows that employment involving cognitive and manual
routine tasks in the USA dropped between 1960 and 2000, while employment involving analytical and inter-active non-routine tasks has grown in
the same period. This change especially concerned industries that rapidly
automatized their production. Parallel to the development in industry, similar changes occurred in other areas where a strong computerization took
place. This change happened on all levels of education. Jobs with a high
routine character are disappearing. Jobs that will be offering good prospects
for the future, are jobs which concern non-routine tasks, which are tasks that
require flexibility, creativity, problem solving skills, and complex communication skills. Autor, Levy, and Murnane (2003) refer to examples such as
reacting to irregularities, improving a production process, or managing people. The jobs of the future are the ones that ask for flexibility, creativity,
lifelong learning, and social skills. The latter are jobs that require communication skills, or face-to-face interaction—such as selling cars or managing
people. These changes do not only affect the decline or rise specific jobs;
existing jobs are changing as well. Secretaries, and bank employees for instance have got more complex tasks since word processors and ATM's have
taken over the more simple tasks.
Earlier, Friedman (2005) pointed to the fact that the effects of computerization and globalization overlap and reinforce each other. Routine tasks
can easily be outsourced, since information technology enables a quick and
easy worldwide exchange of information. The latter also makes it possible
to outsource business services, such as call-centers, or the work of accountants and computer programmers. Another effect of globalization is that it
forces companies to work as efficient as possible. This requires companies
to immediately implement computerization and outsourcing when it is economically profitable and strengthens the market position of the company. It
also demands of the company to be on the lookout for opportunities to im-32 Koeno Gravemeijer
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prove efficiency. As a consequence, working processes will have to be
adapted continuously. This in turn, puts high demands on the workers, who
have to have a certain level of general and mathematical literacy to be able
to keep up.
In summary we may conclude that schools will have to change to comply with the requirements imposed by our rapidly changing society. In this
paper we will especially look into the consequences for mathematics education. We will start by taking a closer look at the 21
st
century skills. This will
reveal the need for changes in the nature of the instructional process, and we
will discuss what this means for mathematics education by reviewing the
main characteristics of, annex requirements for, problem-centered, interactive, mathematics education. Next we will turn to the content of mathematics education, which also has to change. Will we start with the 21
st
century
skills.
21
st
century skills.
Voogt & Pareja Roblin (2010), who reviewed the literature around five
theoretical frameworks on 21
st
century skills, observe that the need for 21
st
century skills is mostly addressed by private or business initiatives, while
educational leaders, practitioners and the educational community do not actively participate in the debate. The frameworks they reviewed concern: the
'Partnership for 21st century skills' (Partnership for 21
st
century skills,
2008), 'EnGauge' (North Central Regional Educational Laboratory and the
Metiri Group, 2003), 'Assessment and Teaching of 21st Century Skills'
(ATCS) (Binkley, Erstad, Herman, Raizen, Ripley & Rumble (2010), 'National Educational Technology Standards' (NETS) (Roblyer, 2000), and
'Technological Literacy for the 2012 National Assessment of Educational
Progress' (NAEP). Dede (2009) compares similar frameworks in his review
of 21
st
century skills. Both review studies show that the frameworks appear
to strongly agree on the need for skills in the areas of communication, collaboration, ICT literacy, and social/cultural awareness. We may take Wagner's (2008) list of what he calls, 'the new survival skills', as exemplary:
1. critical thinking and problem solving
2. collaborating and leading by influence
3. agility and adaptability
4. initiative and entrepreneurism
5. effective oral and written communication
6. accessing and analyzing information
7. curiosity and imagination. Aiming for 21
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. Century Skills 33
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We may also follow Wagner (ibid) in arguing that the 21
st
century skills
are not just about employability
1
. They will also have to include broader
goals, such as becoming a responsible, active and well-informed citizen.
Wagner (ibid) typifies this broader goal by asking if the students would
eventually be well-equipped for acting as a member of a jury within the US
legal system: "Would they know how to distinguish fact from opinion,
weigh evidence, listen with both head and heart, wrestle with the sometimes
conflicting principles of justice and mercy, and work to seek the truth with
their fellow jurors?" (Wagner, 2008, xvi-xvii). This characterization resonates with the theme of this conference: "Democracy in mathematics curriculum", and the corresponding issue of contributing to critical thinking
and decision-making in the society. This begs the question of how to translate these lofty goals into instructional practice.
Mathematics education for 21
st
. century skills
Most publications on 21
st
century skills do not discuss how these goals
might be achieved in education. It seems fair to say, however, that adopting
21
st
. century skills as educational goals will primarily affect the way instructional practice is shaped. Mathematics education seems to be a good place to
incorporate these goals. Dam, & Volman (2004), for instance, conclude on
basis of a review study that critical thinking is best developed in conjunction
with science and mathematics education. But also in a more general sense, it
may be argued that interactive, problem-centered, mathematics education
will contribute to most 21
st
. century skills. However, if we want to integrate
the objective of fostering 21
st
. century skills with attaining the regular goals
of mathematics education, we will have to find a way by which problem
solving, collaborating, communicating etc. serves the mathematics goals. In
theory this is not too problematic, since the literature on reform mathematics
points to this very type of activities—problem solving, collaborating, communicating etc.—as necessary conditions to enable students to construct
meaningful mathematics. "In theory", because we know that creating
mathematics classrooms which can be characterized as interactive and problem-centered is not easy. We will discuss this issue by reviewing the main
characteristics of, annex requirements for, problem-centered, interactive,
mathematics education.
Local instruction theories
By reform mathematics we mean an approach to mathematics education
that assumes that students have to construct or reconstruct mathematics by 34 Koeno Gravemeijer
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themselves with help of the teacher and the textbook. Instead of reconstructing, one may also speak of reinventing—following the tradition of realistic
mathematics education (RME) (Gravemeijer, 2008). One of the prerequisites for fostering reinvention—which has been a focus of RME research—
is the availability of local instruction theories that serve the need for guidelines about the routes along which students might reinvent pieces of mathematics. A local instruction theory consists of a theory about a possible learning process for a given topic, and the means of supporting that process.
Mark that a local instruction theory should not be confused with a scripted
textbook. The local instruction theory is meant to function as a framework
of reference for teachers.
Hypothetical learning trajectories
On the basis of such a framework of reference a teacher may design socalled 'hypothetical learning trajectories' (Simon, 1995) for the actual instructional activities in his or her classroom at a given moment in time. With
the term hypothetical learning trajectory, Simon (1995) refers to the notion
that a teacher has to deliberate on what mental activities the students might
engage in as they participate in the instructional activities, he or she is considering. A decisive criterion of choice then will be how those mental activities relate to the chosen learning goals. He emphasizes the hypothetical
character of those learning trajectories; the teachers are to analyze the reactions of the students in light of the stipulated learning trajectory to find out
in how far the actual learning trajectory corresponds with what was envisioned. Based on this information the teacher has to construe new or adapted
instructional activities in connection with a revised learning trajectory.
Classroom social norms
In addition to a sound planning of the instructional activities, another
prerequisite for inquiry mathematics concerns the participation of the students. For the point is that students do not easily engage in problem solving
and reasoning in regular classrooms (Desforges & Cockburn, 1987). Cobb
& Yackel (1996) argue that this is not surprising, because students are often
familiar with a classroom culture in which the classroom social norms
(Cobb & Yackel, 1996) are that the teacher has the right answers, that the
students are expected to follow given procedures, and that correct answers
are more important than one's own reasoning. In this type of classrooms
teachers usually ask questions of which they already know the answer.
Apart from being used to this situation, students have learned what to expect Aiming for 21
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and what is expected from them. In relation to this, Brousseau (1988) speaks
of an implicit 'didactical contract'. Significant, however, is that the students
have learned this by experience, not because the teacher told them so.
In contrast to what is common in school mathematics, the classroom has
to work as a research, annex learning, community. To make this happen, the
students have to adopt classroom social norms that fit an inquiry-oriented
classroom culture. These encompass, the obligation to explain and justify
one's solutions, to try and understand other students' reasoning, and to ask
questions if one does not understand, and challenge arguments one does not
agree with. In addition to those social norms, Cobb & Yackel (1996) observe that the teacher also has to establish socio-mathematical norms, which
relate to what mathematics is. This is expressed in norms about issues such
as, what counts as a mathematical problem, what counts as a mathematical
solution, and what counts as a more sophisticated solution. In regard to the
latter, we may argue that socio-mathematical norms lay the basis for the intellectual autonomy of the students, as it enables them to decide for themselves on mathematical progress.

Task orientation
In addition to appropriating inquiry-based norms, students also have to
be inclined to invest effort in solving mathematical problems, discussing solutions, and discussing the underlying ideas. Students may engage in learning activities for different reasons. The attitude of students in a mathematics
classroom can be broadly divided in two categories, ego orientation and
task orientation (Jagacinski & Nicholls, 1984). Ego orientation implies that
the student is very conscious of the way he or she might be perceived by
others. Ego-oriented students are afraid to fail, or to look stupid in the eyes
of their fellow students, or the teacher. As a consequence, they may choose
not to even try to solve a given problem, in order to avoid embarrassment.
Task orientation on the other hand implies that the student's concern is with
the task itself, and on finding ways of solving that task. Research shows that
task orientation and ego orientation can be influenced by teachers.
Cobb, Yackel & Wood (1989) report on a study on a socio-constructivist
classroom, where task orientation was fostered. Part of their approach was
to change the classroom culture from one of competition, where students
compare themselves with each other, and with the criteria set by the teacher,
into a classroom culture, where students would measure success by comparing their results with their own results earlier. We may think of the latter
perspective as one similar to that of an amateur painter or amateur musician. 36 Koeno Gravemeijer
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An amateur musician would not think of comparing him- or herself with
others; there would always be many people performing better. Instead an
amateur musician would be pleased if he or she would master a piece, which
he/she could not play some time ago. A similar situation is possible in a
mathematics classroom, where the goal for the students would be personal
growth. Here, experiencing an 'Aha-Erlebnis', for instance, may function as
an incentive. In such a classroom, students might even protest to be given
'the solution', for that would deprive them from the satisfaction of figuring
out things for themselves. The aforementioned research of Cobb et al.
(1989) shows that a classroom culture that emphases the exchange of ideas,
and the development of mathematical understanding as a collaborative endeavor, may foster the task orientation of the students.
Mathematical interest
In connection with student motivation, we may add that next to the obvious "pragmatic interest", which realistic problems appeal to, students will
also have to develop "mathematical interest" (Gravemeijer & Van Eerde,
2009). This concerns the preparedness of the students to the investigate solution procedures, concepts and so on, from a pure mathematical perspective. This is a necessary condition for construing more sophisticated mathematics. Mathematical interest will rarely come naturally, but has to be cultivated by the teacher by
asking questions such as: What is the general principle here? Why does this work? Does it always work? Can we describe it in a
more precise manner? We may assume that the teacher can foster the students' mathematical interest by making mathematical questions a topic of
conversation, and by showing a genuine interest in the students' mathematical reasoning.
Framing topics for discussion
Another essential role of the teacher concerns the orchestration of productive whole-class discussions. For this the teacher has to identify the differences in mathematical understanding which underlie the variation in student responses (Cobb, 1997). Next, he or she has to frame these underlying
mathematical issues as topics for whole-class discussions. And finally, he or
she has to orchestrate a productive whole-class discussion on those topic in

order to foster higher levels of understanding.
In a similar feign, Stein, Engle, Smith, & Hughes (2008), espouse five
practices for promoting productive disciplinary engagement: anticipating
students' mathematical responses, monitoring student responses, purpose-Aiming for 21
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fully selecting student responses for public display, purposefully sequencing
student responses, and connecting students responses. The first two overlap
to some extent with Simon's hypothetical learning trajectory, whereas the
last three are related to identifying, framing, and discussing mathematical
issues. Although a difference may be that Stein et al. (2008) seem to try to
address both mathematical ideas and the solution strategies of the students,
while Cobb (1997) emphasizes the mathematical issues that underlie student
solutions, and tries to steer away from a focus on solution strategies as such.
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Century mathematical skills
The publications on 21
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century skills mostly list general skills. Elaborations of what changes are needed in the content of the various topics that are
taught, are very scarce. With some exceptions, there is also little attention is
given to the content of the mathematics curriculum. One of the reasons may
be that one of the effects of computerization is that mathematics becomes
invisible. Mathematics is hidden in integrated systems, such as spreadsheets,
automatics cashiers, and automated production lines. However, a consequence of this is that many people become "mathematics consumers" , as
Levy en Murnane (2006, 19) put it. They argue that people who use computerized systems are expected to make decisions on the basis of the output
of hidden mathematical calculations. The computerized equipment does the
actual calculation. However, they go on to say, if the decision maker does
not understand the underlying mathematics, he or she is very vulnerable to
serious errors of judgment. If we follow this line of reasoning, we may discern two seemingly conflicting tendencies. On the one hand, we appear to
need less and less mathematics, since various apparatus take over a growing
number of mathematical tasks. On the other hand, we will need more
mathematics as we develop into 'mathematics consumers', who become increasingly dependent of the quantitative information and mathematical
models—which we ought to understand.
The aforementioned hidden character of mathematics in the information
society makes it difficult to establish, what mathematics students will have
to learn to become sensible mathematics consumers. To answer this question, we will follow two tracks. The first concerns research of mathematics
at the workplace, the second concerns an analysis of the role of computers
as interface between the physical world and the computer user. 38 Koeno Gravemeijer
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Techno-mathematical literacy's
When trying to relate the goals of mathematics education to the requirements of the workplace, a complication is that the mathematics that is
used at the workplace is strikingly different from conventional mathematics
(Hoyles & Noss, 2003; Roth, 2005). To describe this specific kind of
mathematics, Hoyles and Noss (2003) coined the term "technomathematical literacy's"—or TmL's for short. TmL's are defined as idiosyncratic forms of mathematics that are shaped by work-place practices,
tasks, and tools. Acting successfully at the workplace is dependent on a
combination of mathematical knowledge and contextual knowledge.
In a recapitulatory report, Hoyles, Noss, Kent & Bakker (2010) mention
as one of their key findings, that artefacts comprising symbolic information
in the form of numbers, tables and graphs are often understood as "pseudo
mathematics". This means that these representations are conceived as labels
or pictures, without an understanding of the underlying mathematics. They
argue that, "Symbolic information thus failed to fulfill its intended role in
facilitating communication across 'boundaries' between communities within
and beyond the workplace" (ibid, 196). Important mathematical skills that
come to the fore in their research are: reading tables and graphs, identifying
and measuring key variables, reasoning about models in terms of the key relationships between variables, and representing and interpreting data.
We may further note that the role of TmL's does not simply boil down to
applying mathematical knowledge. Students will have to be able to flexibly
adapt their existing mathematical knowledge or adopt new knowledge. This
asks for experience with a variety of non-canonical forms of mathematics.
Quantitative models of reality
Apart from studying mathematics at the workplace, we may trace
mathematical skills that are needed in a computerized environment, by analyzing the role computers and computerized appliances play as interfaces
between the users and the concrete reality. In relation to this, we may discern the following processes:
- reality is quantified to make it accessible for computers—as computers only work with numbers;
- these numbers are processed by the computer on basis of models that
describe interdependencies between variables;
- the output, which often has some mathematical form, is interpreted.
Thus in order to understand how a computer deals with reality, one must,
- have some idea of what quantifying (or measuring) entails; Aiming for 21
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- understand at some level, what a variable is, how we can reason
about interdependencies between variables, and how computer models represent reality;
- has to be able to interpret the output of computers.
An important aspect of a measuring is that there will always be some inaccuracy and uncertainty involved. In general students do not realize that
there will always be some measurement error, or that a repeated measurement may result in a different outcome. Nor do they realize that variance is
part of industrial production, even though they will be aware of the phenomenon of natural variance in nature. We would argue that it is important
for students to develop this kind of understanding, for many of the numbers
they will have to work with will be the result of sampling. We may further
follow Jones (1971) in his observation that measuring an object comprises
assigning a value to a variable; what is measured is not the object but a
property of that object. The significance of this observation comes to the
fore when we look at co-variation. In such cases we do not compare the individual lengths and weights of certain persons, but we study how "length"
varies with "weight".
Fortunately information technology offers eminent possibilities for creating educational tools for helping students to come to grips with these
ideas. Hoyles et al. (2010), for instance, report on the use of interactive
computer tools that present symbolic information such as graphs, models
expressed in algebraic symbols, or numerical measures. Experiments with
such computer tools (which they call, technology-enhanced boundary objects) show that they can be employed successfully in enhancing the meanings of the symbolic information by making them more visible and manipulable, and therefore more accessible. We may note that this use of computer tools is in line with Kaput's, faith in the educational potential of information technology that allows for the use of dynamic representations (Kaput &
Schorr, 2007). Computers can show the numerical results or graphical representations of measuring activities real time on a screen, but also offer representations of simulations that can be manipulated at will.
We may mention two examples of the educational use of dynamic representations in service of the mathematical objectives mentioned above. One
concerns a simulation of a train on a rail track, which is accompanied by a
graphical representation of the speed of the train, which can be used to foster an understanding of co-variation and informal calculus (Galen &
Gravemeijer, 2010). A second example concerns an introduction to explora-40 Koeno Gravemeijer
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tory data analyses with the help of so-called computer minitools (Gravemeijer & Cobb, 2006), that allow for a variety of representations and ways of
structuring of data sets. While analyzing data with the help of these minitools, students may develop a qualitative notion of a distribution of data as
an object with certain characteristics, such as shape and spread.
Conclusion
We started by observing that there is a significant gap between what
students learn at school and what modern society demands. Computerization
and globalization result in a change in job requirements that involves a shift
from routine tasks towards non-routine tasks. The jobs of the future require
flexibility, creativity, problem solving, life-long learning and complex
communication skills. Our fast changing information society not only put
new demands on workers, it also affects our roles as learners and citizens.
In relation to this, we referred to Wagner's (2008) benchmark that the students eventually would have to become well-equipped for acting as a member of a jury in the US legal system.
Similar to Wagner, many publications in this area use the term, "21
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century skills"; skills that may be summarized as flexibility, critical thinking, problem solving, collaborating, and communicating. We argued that
these skills fit very well with problem-centered, interactive, mathematics
education, and we reviewed the most important characteristics of this type
of mathematics education.
We further discussed what changes in the content of mathematics education would be needed. Both research on mathematics in the workplace, and
an analysis of the role of computers as interface between the physical reality
and the user, point to topics such as, measuring, tables, graphs, variables,
models of relationships between variables, and elementary statistics.
Finally we indicated how computer tools may help students in coming to
grips with the aforementioned mathematics, by offering dynamic representations. In conclusion, we want to stress the urgency of a reconsideration of
the goals of mathematics education in light of the demands of the 21
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century, and the need to start experimenting with educational practices that may
foster these goals.
Notes Aiming for 21
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. Century Skills 41
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1. Mark that most publications on 21
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century skills stem from Western
countries, which implies a certain bias. We may argue, however, that
the 21
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century skills that are identified will be or become significant
for other countries as well.
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49 quotations

Codes (20)###HLT, ###design of activities, ###emergent modelling, ###guided reinvention, ##characteristics of RME, ##principles of RME, ##real life, #RME, $$approaches to teaching Mathematics, $$$modelling, $$learner reasoning, $$mathematics education, $$meaningful learning of Maths, $factors that influence teaching and learning, $mathematics, $mathematics content, &motivation, &pedagogy, &social aspects, @@ICT systems: virtual learning environment (VLE)

Memos (0)

Document body:

KOENO GRAVEMEIJER and MICHIEL DOORMAN
CONTEXT PROBLEMS IN REALISTIC MATHEMATICS
EDUCATION: A CALCULUS COURSE AS AN EXAMPLE
ABSTRACT. This article discusses the role of context problems, as they are used in the Dutch approach that is known as realistic mathematics education (RME). In RME, context problems are intended for supporting a reinvention process that enables students to come to grips with formal mathematics. This approach is primarily described from an instructional-design perspective. The instructional designer tries to construe a route by which the conventional mathematics can be reinvented. Such a reinvention route will be paved with context problems that offer the students opportunities for progressive mathematizing. Context problems are defined as problems of which the problem situation is experientially real to the student. An RME design for a calculus course is taken as an example, to illustrate that the theory based on the design heuristic using context problems and modeling, which was developed forprimary school mathematics, also fits an advanced topic such as calculus. Special attention is given to the RME heuristic that refer to the role models can play in a shift from a model ofsituated activity to a modelfor mathematical reasoning. In light ofthis model-of/model-for shift, it is argued that discrete functions and their graphs play a key role as an intermediary between the context problems that have to be solved and the formal calculus that is developed.
1. INTRODUCTION
The role of context problems used to be limited to the applications that would be addressed at the end ofa learning sequence – as a kind ofadd on. Nowadays, context problems have a more central role. They are endorsed because of today's emphasis on the usefulness of what is learned, and because of their presumed motivational power. Context problems play a more encompassing role in the Dutch approach that is known as realistic mathematics education (RME). In RME context problems play a role from the start onwards. Here they are defined as problems ofwhich the problem situation is experientially real to the student. Under this definition, a pure mathematical problem can be a context problem too. Provided that the mathematics involved offers a context, that is to say, is experientially real for the student.
In RME, the point of departure is that context problems can function as anchoring points for the reinvention of mathematics by the students
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themselves. *Moreover*, guided reinvention offers a way out ofthe generally perceived dilemma ofhow to bridge the gap between informal knowledge and formal mathematics. This issue is at the heart ofthis article: How can we help students to come to grips withformal mathematics?
We will take a calculus course as an example, and show that in the reinvention approach, the role ofcontext problems and ofsymbolizing and modeling are tightly interwoven. Actually, we build upon the work that has been done on symbolizing and modeling in primary-school mathematics (Streefland, 1985; Treffers, 1991; Gravemeijer, 1994, 1999). We try to show that the framework that has been developed for primary school can also be used for such an advanced topic as calculus.
We start by following Tall's critique on a formal approach to calculus teaching, by explicating the problems of instruction based on formal logical analysis. Next we discuss some alternatives before moving on to an elaboration ofthe RME approach.
The RME calculus sequence is inspired by the history ofmathematics. We will describe some elements ofthe history ofcalculus from the findings at the Merton College in the 14th century until Galileo that are interesting from an instructional design point of view. We argue that discrete functions and their graphs played a key role as an intermediary between the context problems that had to be solved and the formal calculus that was being developed. We will finish with a discussion of the RME approach of creating the opportunity to let formal mathematics emerge, instead of trying to bridge a gap between formal and informal knowledge.
2. TRADITIONAL CALCULUS INSTRUCTION AND SOME ALTERNATIVES
Traditional set up
According to Tall (1991), mathematicians tend to make a typical error when they design an instructional sequence for calculus. The general approach of a mathematician is to try to simplify a complex mathematical topic, by breaking it up in smaller parts, that can be ordered in a sequence that is logical from a mathematical point ofview. 'From the expert's viewpoint the components may be seen as part ofa whole. But the student may see the pieces as they are presented, in isolation, like separate pieces of a jigsaw puzzle for which no total picture is available.' Tall (1991, p. 17). It may be even worse, ifthe student does not realize that there is a big picture. The student may imagine every piece as an isolated picture, which will severely hinder a synthesis. The result may be that the student constructs113 an image of each individual piece, without ever succeeding in bringing all pieces together in one whole.
As an example, Tall describes a possible sequence for differentiation. He presents the following line of reasoning. To be able to understand the derivative f
0
.x/, one has to have the concept of a limit at one's disposal.
For, one has to take the limit ofthe difference quotient .f.xCh/–f.x//=h where h tends to zero. Thus the concept of a limit has to precede the derivative. Furthermore, one might decide that it is easier to take the limit in the case where x is fixed. The next step then would be to let x vary, to introduce the idea of a derivative in this manner.
For the student, however, the introduction ofthe limit concept suddenly appears for no reason, with all the cognitive problems this may bring. The next big problem is in the shift from a limit with a fixed x to a varying x, since taking a limit in one point is substantially different from perceiving f
0
.x/ as a function ofwhich the values describe the gradient of a graph of f.x/.
Alternative approaches
A sensible alternative, according to Tall, would be to look for situations that can function as informal starting points, from which cognitive growth is possible. In this context, Tall (cited in Bishop et al., 1996) argues for more emphasis on visualizing mathematical concepts and more enactive experiences in mathematics education. The students should first experience a qualitative, global, introduction ofa mathematical concept. This qualitative introduction then should create the need for a more formal description ofthe concepts involved. 'Graphic calculus' is such an alternative approach developed by Tall (1985, 1986).
What is characteristic in Tall's approach is the dynamical aspect. The dominant perspective is that of a running variable. Here the graph plays a central role: the graph is used to see how the dependent variable (e.g. y, or f.x/) changes when the independent value (x) changes at a constant rate. What is looked at primarily, are the changes in the dependent variable and the rate ofthese changes, which can be described in terms ofincrease, decrease, and gradient. This helps the students develop an intuitive feel for the derivative. However, the notion of the derivative as a measure for the rate of change stays rather implicit, since the primary focus is on the gradient ofthe graph.
Tall (cited in Bishop et al., 1996, p. 314) observes that the problem for a calculus course is in the transition from meaningful discussions based on visual imagery to formal mathematical reasoning. Students interpret a definition that is based on visual imagery as a description, as a model of114 the picture, instead ofa mathematical definition that can be used for formal reasoning.
Maybe the visual exploration stands too much on its own. In Tall's proposal, the relationship between graphs and real-life phenomena is restricted to a short introductory phase; for the main part, the functions and graphs are contained in a mathematical world.
Linking authentic experience and a mathematical symbol system
In contrast to Tall, Kaput (1994a), emphasizes the relationship between mathematical symbol systems like graphs and everyday reality.
The problem, in his view, is the gap between the island offormal mathematics and the mainland of real human experience. He elucidates this gap with the difference between mathematical functions that are defined by algebraic formulas, and empirical functions that describe everyday-life phenomena. To underscore his point, he quotes Thomas Tucker's rhetorical question: 'Are all functions encountered in real life given by closed algebraic formulas? Are any?' (cited in Kaput, 1994a, 384). He observes that most educational software available does not address this problem. This, he argues, also holds for computer programs that link functions and graphs. For, in terms of the island metaphor, both algebraic functions and graphs of algebraic functions belong to the island of formal mathematics. To attack the island problem, Kaput seeks situations where the students can maximally exploit their own authentic experience to investigate, and to come to grips with these formal representations. He tries to create such a situation with software he is developing under the name of Math Cars. The power ofthe device is in the internal linkage between the various display systems. In this way, the everyday experience of motion in a vehicle can be linked to the formal graph representations. This linkage then offers the students the opportunity to test the conjectures they develop about the graphic representations.
We may note that Kaput takes the ready-made symbol system as point ofdeparture, which is consistent with his concept ofa mathematical system that is distinct from our everyday-life experience. Others, however, try to help students develop or reinvent this symbol system themselves.
Inventing graphing
DiSessa, Hammer, Shern, and Kolpakowski (1991), who describe an instruction period where the students invented graphing, for instance, report such an approach. Albeit, not as a result of ample instructional planning. As a matter of fact, the invention process was more or less incidental. The students had been programming simulations of real-life motions with a115 Logo-like turtle that left a trail of dots across the screen. Next, the students were asked to come up with a paper-and-pencil way to represent the motion story of one of the simulations they worked on. The solutions of the students, that were to some extent inspired by the dot-track of the computer simulation, formed the starting point for a series of discussions and activities, in which a graph-type representation of this motion story emerged.
While acknowledging that the guidance by the teacher may have influenced the invention process more than the notion of invention suggests (DiSessa et al., 1991), we can argue that, if it would be possible to have the students invent distance-time and speed-time graphs by themselves, the dichotomy between formal mathematics and authentic experience Kaput presupposes, would not arise. For the mathematical ways of symbolizing would emerge in a natural way in the students' activities, and the accompanying formal mathematics would be experienced as an extension oftheir own authentic experience.
Following Meira (1995), we can take this line of thinking one step further, by acknowledging a dialectical relation between notations-in-use and mathematical sense making. According to this dynamic point ofview, it is in the process of symbolizing that symbolizations emerge and develop their meaning. In this process, notational systems shape the very activities from which they emerge, while at the same time, the activities shape the meanings that emerge.
Discussion
Looking at the alternatives to traditional calculus instruction presented above, we may discern the following characteristics. The overall goal is to design an insightful instructional sequence. Furthermore, the students should be given the opportunity to ground their understanding in their own informal knowledge. Finally, the instruction should be based on the students' own contributions to the teaching-learning process. We may discern three different orientations within these alternatives:
- Helping the students to develop qualitative (generic) notions that can function as a basis for their understanding;
- Creating a learning environment where the students can come to grips with the basic ideas by developing and testing hypotheses;
- Fostering the (re)invention of calculus.
A difference between the first two and the third is that the first two try to help the students bridge the gap between their informal knowledge and the formal mathematics, where the third tries to transcend this dichotomy by116 aiming at a process in which the formal mathematics emerges from the mathematical activity of the students. This is also the objective of RME, where instructional design is aimed at creating optimal opportunities for the emergence of formal mathematical knowledge.
3. GUIDED REINVENTION AND PROGRESSIVE MATHEMATIZATION
The underlying thread of all the alternative approaches is the belief that learning mathematics should have the characteristics of cognitive growth, and not of a process of stacking pieces of knowledge. This perspective is consistent with a more general view that the way in which mankind developed mathematical knowledge, is also the way in which individuals should acquire mathematical knowledge. A view that is, for instance, advanced by Polya (1963), and Freudenthal (1973, 1991).
Freudenthal's point of departure is in his critique of traditional mathematics education. He fiercely opposes what he calls an anti-didactical inversion (Freudenthal, 1973), where the end results of the work ofmathematicians are taken as the starting points for mathematics education. As an alternative he advocates that mathematics education should take its point ofdeparture primarily in mathematics as an activity, and not in mathematics as a ready-made-system (Freudenthal, 1971, 1973, 1991). With this adage, he has laid the foundation for RME. For him the core mathematical activity is 'mathematizing', which stands for organizing from a mathematical perspective. Freudenthal sees this activity ofthe students as a way to reinvent mathematics.
Note that the students are not expected to reinvent everything by themselves. In relation to this, Freudenthal (1991) speaks ofguided reinvention; for him, the emphasis is on the character of the learning process rather than on invention as such. The idea is to allow learners to come to regard the knowledge they acquire as their own private knowledge, knowledge for which they themselves are responsible. The latter implies that certain social norms (Yackel and Cobb, 1996) have to be in place. For instance, norms like: you do not learn mathematics by guessing what the teacher has in mind, but by figuring things out for yourself.
According to Freudenthal, mathematizing may involve both mathematizing everyday-life subject matter and mathematizing mathematical subject matter (Freudenthal, 1971). He does not see a fundamental difference between the two activities. Therefore, education might start with mathematizing everyday-life subject matter. Reinvention, however, demands that the students mathematize their own mathematical activity as well. In

relation to this, Treffers (1987) discerns horizontal and vertical mathem-117
atization. Horizontal mathematization refers to the process ofdescribing a
context problem in mathematical terms – to be able to solve it with mathematical means. Vertical mathematization refers to mathematizing one's
own mathematical activity. Through vertical mathematization, the student
reaches a higher level of mathematics. It is in the process of progressive mathematization – which comprises both the horizontal and vertical
component – that the students construct (new) mathematics.
Freudenthal (1971, p. 417) expresses this as 'the operational matter on
one level becomes a subject matter on the next level'. Although Freudenthal
has micro levels in mind, a connection can be made with Sfard's (1991)
more macroscopic account ofmathematical development based on historical analyses. She observes that the history ofmathematics can be characterized as an ongoing process ofreification in which processes are reinterpreted as objects. One ofthe examples she gives is that of functions. They
firstly appear as calculational prescriptions. This process takes so much
attention that the character of the operations does not get much attention.
Gradually, however, distinctions are made, and various sorts of functions
are discerned. More and more, functions are treated as objects, with certain characteristics. Sfard's analysis suggests that students will have to go
through the same process; students will only be in the position to grasp
the notion of a function as an object, if they have ample experience with
functions as procedures.
An instructional design perspective
In a reinvention approach, context problems play a key role. Well-chosen
context problems offer opportunities for the students to develop informal,
highly context-specific solution strategies. These informal solution procedures then, may function as foothold inventions, or as catalysts for curtailment, formalization or generalization. In short, in RME, context problems
are the basis for progressive mathematization. The instructional designer
tries to construe a set of context problems that can lead to a series of
processes of horizontal and vertical mathematization that together result
in the reinvention ofthe mathematics that one is aiming for. Basically, the
guiding question for the designers is: How couldIhave invented this? Here
the designer will take into account his/her own knowledge and learning
experience. Moreover, the designer can look at the history ofmathematics
as a source of inspiration, and at informal solution strategies of students
who are solving applied problems for which they do not know the standard
solution procedures yet (see Streefland, 1990; and Gravemeijer, 1994 for
examples).118
Figure 1. Solving 35 C : : : D 64 on the empty number line
Figure 2. Modeling 95 – 19 D 95 – 20 C 1.
Research on the design ofprimary-school RME sequences has shown,
that the concept of emergent models can function as a powerful design
heuristic (Gravemeijer, 1999). Here, the point of departure is in situationspecific solution methods, which are then modeled. First context problems are selected that offer the students the opportunity to develop these
situation-specific methods. Then, if they do, these methods are modeled.
In this sense, the models emerge from the activity ofthe students. Even if
the models are not actually invented by the students, great care is taken to
approximate student invention as closely as possible by choosing models
that link up with the learning history ofthe students. Another criterion is in
the potential of the models to support vertical mathematizing. The idea is
to look for models that can be generalized and formalized to develop into
entities of their own, which as such can become models for mathematical
reasoning.
As an example we may take an instructional sequence in which a ruler
comes to the fore as a model of iterating measurement units, and develops into a modelfor reasoning about mental computation strategies with
numbers up to 100 (Stephan, 1998; Gravemeijer, 1999). In this sequence,
the students measure various lengths by iterating some basic unit ofmeasurement, and a larger measurement unit consisting often basic units. This
measuring with 'tens' and 'ones' is modeled with a ruler of 100 units,
made ofunits often and one. Then the activity ofmeasuring is extended to
incrementing, decrementing and comparing lengths. These situations give
rise to counting strategies that are represented by arcs on a schematized
ruler or as an 'empty number line' (Whitney, 1988; Treffers, 1991) (see
Figure 1).
Eventually, the symbolizations on the empty number line representation will be used to explain and justify strategies like solving 95-19 by119
subtracting 20 and adding one (see Figure 2). In this manner, the number
line functions as a model for mathematical reasoning.
The shift from model of to model/for concurs with a shift in the way
the student thinks about the model, from models that derive their meaning
from the modeled context situation, to thinking about mathematical relations. In the latter phase, thinking about number relations will dominate the
use ofthe number line. In relation to this, we can discern different types of
activity (Gravemeijer, 1994; Gravemeijer, Cobb, Bowers and Whitenack,
in press):
(1) activity in the task setting (measuring with units often and one)
(2) referential activity (interpreting positions on the ruler as signifying
results ofiterating a measurement unit)
(3) general activity (using the ruler/number line to reason about computation methods)
(4) formal mathematical reasoning (reasoning with number relations within the mathematical reality of a framework ofnumber relations).
Note that, the term 'model' must be understood in a holistic sense. It is not
just the inscription
1
, but everything that comes with it that constitutes the
model in RME. Furthermore, the same model may encompass a cascade of
inscriptions (Lehrer and Romberg, 1996); e.g. from regular ruler to empty
number line. The label 'emergent' emphasizes the continuity in this process. This label is also used to refer to the fact that the model emerges from
the activity ofthe students. In addition, the mathematics that one is aiming
for (e.g. the development of a framework ofnumber relations), emerges in
the subsequent process.
We may note that the goal is not only to help the students elaborate their
informal understanding and informal solution strategies in such a manner
that they can develop more formal mathematical insights and strategies.
The objective is also to preserve the connection between the mathematical
concepts and that which these concepts describe. The students' final understanding of the formal mathematics should stay connected with, or as
Freudenthal would say, should be 'rooted in', their understanding ofthese
experientially real, everyday-life phenomena.
In relation to this, we want to emphasize that we see modeling and
symbolizing as an integral part of an organizing activity that aims at coming to grips with a problem situation. In other words, the development
of the model and of its constituting symbolizations/inscriptions go hand
in hand with the development of the mathematical conceptualization of
the problem situation. On one hand, the symbolization derives its meaning from the situation that it describes. At the other hand, the way the
problem situation is perceived is highly influenced by the symbolization120
'through which' the situation is seen. In this sense, we agree with Meira
(1995, p. 270) who advocates a dynamic activity-oriented view, according
to which symbolizations and meaning co-evolve in a dialectic process.
In the following we want to try to cast the design of the RME calculus course in terms of this emergent-models design heuristics. We will
start out by sketching a few important moments in the history of calculus.
Since, although reinvention does not necessarily imply that the history of
mathematics has to be the point of departure, this RME calculus course
happens to be an example of how inspiring the history of mathematics
can be for mathematics educators. Moreover, the history ofcalculus offers
some insight into what might be constituted as the emergent model in
calculus.
4. A HISTORICAL SKETCH
The emergence ofkinematics at Merton College
In the first half of the 14th century logicians and mathematicians associated with Merton College, Oxford, investigated velocity as a measure of
motion. They tried to find a description ofthe distance traversed by a body
moving with a uniform accelerated motion (Clagett, 1959). This problem
was not easy because the velocity of the body constantly changes and
the concept of motion and change of motion was defined very generally
according to Aristotle. Change and motion referred to temperature, size
and place (Lindberg, 1992). All these qualities of a body could change.
The interpretation of motion as change of place became the central issue
studied at Merton College; Clagett describes this as 'the emergence of
kinematics at Merton College'.
In those days the scientists at Merton College used the notion of instantaneous velocity and descriptions of the velocity of a moving object.
But there was no clear definition ofvelocity such as: the distance covered
divided by the traversal time (1s=1t). And surely no definition ofinstantaneous velocity as a limit ofthis division (ds=dt), because they could only
work, in the tradition ofEuclid, with proportions of equivalent units.
The most important result that was achieved is 'Merton's rule': When
the velocity of an object increases with equal parts in equal time intervals
from zero to a velocity v in a time interval t, then the distance traveled is
equal to halfthe distance traveled by an object that moves with a constant
velocity v in that time interval t. In modern symbols s.t/ D 1=2 v t.121
Figure 3. Velocity represented by the height ofthe rectangles.
Figure 4. Oresme's proof(pictures from a 15th century copy ofOresme's 'De configurationibus qualitum' reprinted from Claget, 1959).
They added the conjecture that the distance covered in the first half
of such a motion is one third of the distance covered in the second half.
However, this appeared to be hard to prove.
The graph as a model ofvelocity-time problems
It was Nichole Oresme (ca. 1360) who started to draw graphs of situations
in order to visualize the problem. He described how geometrical elements
like lines or rectangles can be used to represent the value of a variable:
the length of a line or the area of the rectangle represents the value of the
variable. These elements can be put along a horizontal line that represents
time.
When investigating the distance traveled by an object that moves with
constantly changing velocity Oresme knew that the velocity increases with
equivalent quantities in equivalent time intervals
2
. A graph where the velocity is represented by the height ofa rectangle, and the chosen time interval
by the width ofit, is shown in Figure 3.
Oresme could approximate the distance traveled in each time interval
by taking into account the length of the interval and the 'constant' velocity during the interval. The area ofthe rectangle represents this distance.
Oresme called the diagonal top-line of the graph, which results when you
choose the time-intervals very small, 'the line ofintensity'. He noticed that
it is a straight line and that the sum of the areas of all the rectangles is an
approximation ofthe area ofthe triangle between the line of intensity and122
the horizontal axis. As a result the distance traveled equals the area ofthe
triangle: 1=2 v t (Figure 4).
The power of this graphical representation comes to the fore when
you try to prove the hypothesis: the distance covered in the first half of
a uniformly accelerated motion is one third of the distance covered in the
second half. This can be deduced easily with the areas in the graph.
Galileo investigatingfreefall
The interesting fact of these results is that they dealt with abstract phenomena. Kinematics was an argumentative science, not an experimental
science. It was Galileo in the seventeenth century that made the shift to
experimental physics. He conjectured that an object in free fall moved with
a constantly increasing velocity. He designed experiments with objects
moving with a uniform acceleration and applied the Merton rule to a real
motion.
Furthermore, Galileo gave an explanation for the quadratic relation
between time and distance, using sums and differences. He argued that
the ratio between the distance traveled in two equal time intervals is 1:3;
if you divide time into four equal intervals it is 1:3:5:7; and it continues
as such a ratio determined by a sequence of odd numbers. Galileo knew
that these sequences added up to a square and with this property he tested
his conjecture. He designed a slide with nails on one side. The distances
between the nails were proportional to the odd numbers above. It appeared
that a rolling ball needed the same time too pass each consecutive nail.
Galileo's experiments confirmed that the falling distance increased linearly during each time-interval and from this result he concluded that the
relation between distance and time must be quadratic. The integral of the
linear function was thus found; an actual start with calculus was made. We
leave history here, to consider what we can learn from this.
Looking through the lens ofemergent models
Looking at the history of calculus from a modeling perspective, we see a
development ofcalculus that starts with modeling problems about velocity
and distance. Initially these problems are tackled with discrete approximations, inscribed by discrete graphs. Later, similar graphs – initially discrete
and later continuous – form the basis for more formal calculus. From
a modeling perspective, we could say that graphs of discrete functions
come to the fore as models of situations, in which velocity and distance
vary, while these graphs later develop into modelsfor formal mathematical
reasoning about calculus. In relation to this we can speak of a process of
reification. However, it is not the graph that is reified, it is the activity that is123
Figure 5. Sums and increments.
reified. E.g. the act ofsumming (differentiating) is reified, and becomes the
mathematical object 'integral' ('derivative'). The inscription – the graph –
visually supports both the activity and its reification.
We want to add that we do not take this notion ofreification too literally
here. For many students, the end result will be something in between a
process and an object – especially in the case of calculus. To emphasize
that the underlying process will be an integral part of the mathematical
object that is developed in this manner, Tall (cited in Bishop et al., 1996)
uses the term 'procept'. Moreover, it should be acknowledged that the development will not be as linear as our description suggests. Students may
shift back and forth between process and object conceptions, depending on
the problems they have to solve.
In looking at the history of calculus, Kaput's (1994b) characterization
ofcalculus as 'the mathematics ofchange' comes to mind. In the process of
trying to get a handle on change, the method of approximating a constant
changing velocity with the help of discrete functions plays a key role. In
a sense, the sum- and difference-series can be seen as a crude predecessor
of calculus. This idea can be exploited in instructional design by starting
the sequence with investigating discrete patterns. Moreover the mathematics of sum- and difference-series offers the opportunity of reinventing a
discrete variant ofthe main theorem of calculus in an early stage.
Another aspect ofthe history that seems relevant for instructional design
is the embedding in the context of speed and distance. This seems to be
a context that is suited for high school students. For, in general, speed
is a concept they still grapple with. Moreover the idea of instantaneous
velocity seems to be more assessable than a seemingly simple concept
such as average speed.124
5. RME CALCULUS COURSE

These deliberations are elaborated in a calculus sequence that is developed
for students of 16–17 years, in science-oriented high school classes
3
. The
unit starts with activities on series. Properties of, and the relations between
series, their sums and their increments are investigated by the students (see
Figure 5). In this process, the students develop (algebraic) tools that can
be used for problems later on. Moreover, it is a first introduction to the
relationship between sums, summation symbols, increments and difference
symbols. The notion of limit is only used in an intuitive way until the end
ofthe unit.
The relationship between sums and differences is represented with the
introduced symbols by: 1n
2
D 2k C 1 and
P
.2k C 1/ D n
2
. These
properties are also visualized with graphs. In general the discrete case of
the main theorem is posed: 1S.n/ D D.k/ and
P
D.k/ D S.n/ − S.0/.
Students use graphic calculators when dealing with these investigations.
From graphs and tables they can deduce that 13
n
D 2 3
k
and, with the
theorem they can prove that
P
3
k
D 1=2.3
n
− 3
0
/.
After this discrete analysis the shift towards modeling oftime/distance/
velocity is made. Note that a key element ofthe notion ofemergent models
is that the models first come to the fore as models of situations that are
experientially real for the students. It is in line with this notion that discrete
graphs are not introduced as an arbitrary Symbol System, but as models
of discrete approximations of a motion. Since the students are already
familiar with continuous graphs, an approach to graphing speed that is
not familiar to the students, is chosen. The point of departure is in the
medieval notion ofinstantaneous speed. The issue of symbolizing speed is
introduced in the context ofa narrative about Galileo's work. The students
are informed about the definition of instantaneous velocity in Galileo's
time in terms ofthe distance that is covered ifthe moving object maintains
its instantaneous velocity for a given period of time. In this context, the
problem is posed of how to visualize the motion of an object that moves
with varying speed. While struggling with this problem, the students may
come up with the idea ofsymbolizing instantaneous velocities with bars. If
not, this option may be presented to them, after they have been struggling
with this problem for a while.
As an aside, it may be noted that there will always be a tension between
a bottom-up approach that capitalizes on the inventions ofthe students and
the need, (a) to reach certain given educational goals, and (b) to plan instructional activities in advance. As a consequence, a top-down element is
inevitable in instruction. The key consideration for us, however, is that the125
Figure 6. A discrete approximation ofa constant changing velocity.
students experience these top-down elements as bottom-up: as solutions
they could have invented for themselves. For the instructional designer,
this boils down to striving to keep the gap between 'where the students
are' and what is being introduced as small as possible. Furthermore, the
teachers will be able to reduce the gap in interaction with their students.
Returning to the content, we want to stress, that although the idea of
using bars (i.e. small rectangles) may be handed to the students, this does
not mean that there are no problems left to solve. A central problem is that
of coordinating the height and the width of the bars when using them to
visualize a discrete approximation of a movement.
Next, the topic of investigation will be the relation between the 'area
of the graph', and the total distance covered over a longer period of time.
Figuring this out demands ofthe students that they come to grips with the
relationship between the motion, the representation, and the approximation. The whole process, in which the way of modeling motion, and the
conceptualization of the motion that is being modeled, co-evolve, can be
seen as a form of guided reinvention that can be contrasted with learning
some rule which magically equates area with distance.
The students are told the story ofGalileo, who presumed that the motion
of a free-falling object was with constantly increasing velocity. They are
asked to graph the discrete approximation of such a motion (see Figure 6).
Subsequently Galileo's problem is posed: What distance is covered by
the object? Here, after solving the discrete approximations, the students are
expected to make the connection between the area of the discrete graph,
and the area ofthe triangle that is created by the continuous graph: s.t / D
.1 10t/=2 D 5t
2
. This resulting formula reveals the quadratic relation
between time and distance that Galileo used to test his hypotheses empirically. With this calculation, the first step is made in a process in which the
attention shifts from describing motion, to calculating primitives.
After this introduction to integral calculus, students work on problems
that deal with differential calculus. That is, they have to determine velocities from distance-time graphs and formulas. In subsequent activities,126
the model will begin to function as a model for reasoning about integrating/differentiating arbitrary functions on the one hand, and of standard
algebraic functions on the other hand. At the same time, a shift is made
from problems cast in terms of everyday-life contexts to a focus on the
mathematical concepts and relations. To make such a shift possible for the
students, they will have to develop a mathematical framework ofreference
that enables them to look at these types of problems mathematically (see
Gravemeijer, 1999). It is exactly the emergence of such a framework that
this sequence tries to foster.
In the envisioned process ofprogressive mathematization, symbolizing
and modeling play a key role. The central model is that of a discrete function, in combination with the inscription of a discrete graph. This model
is the basis for both integration and differentiation through sums and differences. Although the discrete function can be seen as the backbone of
the model, the visual representation also plays an important role. In line
with Latour (cited in Meira, 1995) we would argue that the use of visual
representations helps one to focus on the mathematics.

6. DISCUSSION, CREATING MATHEMATICAL REALITY

The question we asked at the beginning of this article was: How can we
help students to come to grips with formal mathematics? Central in our
exposition was the RME approach. This approach distinguishes itselffrom
many other approaches, in that it tries to transcend the dichotomy between
informal and formal knowledge, by designing a hypothetical learning trajectory along which the students can reinvent formal mathematics. Ideally,
the actual learning trajectory unfolds in such a manner, that the formal
mathematics emerges in the mathematical activity of the students. This
ideal is connected to Freudenthal's (1991) contention that 'mathematics
should start and stay within common sense'. Freudenthal intended this
adage to be interpreted dynamically and argued that common sense is not
static. He noted, for example, that what is common sense for a mathematician differs significantly from what is common sense for a lay person. In
addition, he emphasized that common sense evolves in the course oflearning. Thus, in the first phase ofthe sequence, describing momentary speed
in terms of the distance that would be covered in case of constant speed,
is a common-sense activity. In a similar manner, a discrete approximation
of varying velocity can be seen as a common sense activity. By the end
ofthe sequence, acting in an environment structured in terms of areas and
gradients ofgraphs will have become common sense for the students.127
This development can also be taken to exemplify what is meant by
context problems in RME. As we indicated earlier, context problems are
defined as problem situations that are experientially real to the student.
The above example shows that this experiential reality grows with the
mathematical development ofthe student. Freudenthal explicates: 'I prefer
to apply the term "reality" to that which at a certain stage common sense experiences as real' (Freudenthal, 1991, p. 17). This use ofthe term reality in
RME is highly compatible with Greeno's (1991) environmental metaphor.
From this we conclude that the overall goal of instructional design is to
support the gradual emergence of a taken-as-shared mathematical reality.
If the students experience the process of reinventing mathematics as expanding common sense, then they will experience no dichotomy between
everyday life experience and mathematics. Both will be part of the same
reality.
We may note a reflexive relation between the use of context problems
and the development ofthe student's experiential reality. On the one hand,
the context problems are rooted in this reality, on the other hand, solving
these context problems helps the students to expand their reality. Notwithstanding this dynamic character of reality that defines context problems,
starting points for instructional sequences will often link up with everydaylife experience of students. It is precisely the connection with velocity
and distance that offers the students the means to reason and act in a
meaningful manner from the start.

NOTES

1
Following Lehrer and Romberg (1996) we use the term inscriptions to refer to
symbolizations that have physical form.
2
Some mathematicians argue that Oresme's proof of Merton's rule is not allowed.
First he should have defined velocity as a differential quotient and then deduced
the distance traversed by graphical integration. The mathematical historian Dijksterhuis discusses this and defends Oresme by stating: It is a situation which occurs
regularly in the history ofmathematics: mathematical concepts are often – maybe
even: usually – used intuitively for a long time before they can be described accurately, and fundamental theorems are understood intuitively before they are proven
(Dijksterhuis, 1980).
3
The sequence is called Sum & Difference, Distance & Speed (Som & Verschil,
Afstand & Snelheid – Kindt, 1997). One ofthe chapters ofthis sequence is based
on a text ofPolya on the history ofGalileo (Polya, 1963). Our description does not
correspond with the existing unit in every detail, but anticipates certain revisions.128

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Design research on how IT may support the development of symbols and meaning
in mathematics education
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Introduction
This paper sketches the methodology and some results of three studies that were part of a comprehensive research project called "IT and mathematics education". With the theory of Realistic Mathematics Education as a background, the project investigated the contribution of IT tools for the guided reinvention of mathematical concepts and the influence of IT use on the students' process of symbolising. Each of the studies focussed on a different mathematical topic and used different types of technological tools. The three studies that are discussed in this paper combine the methodology of design research with a prominent role for the Hypothetical Learning Trajectory as a research instrument in all phases of design research (design, teaching experiment, retrospective analysis).
In the first section we state the overarching research question. In section 2 we describe the research methodology. Each of the phases of the research cycles is addressed: the preliminary phase in which a Hypothetical Learning Trajectory and instructional activities are designed, the teaching experiment phase and the phase of the retrospective analysis. In section 3 we address each of the three studies subsequently and present some exemplary results. Section 4 contains a discussion, and section 5 states common conclusions from the three studies.
1. Research Question
One of the salient characteristics of mathematics is the use of symbols. This is not merely an external characteristic, as mathematical symbols are an integral part of mathematics. It is hard to think about algebra without the use of literal expressions, or to understand calculus without pointing to tangents and areas beneath graphs, or to grasp the notion of a normal distribution in statistics without referring to an image such as a bell shape. This intertwinement of meaning and visual representations poses a problem for mathematics education. Experts - like teachers and instructional designers - tend to see these symbols as carriers of meaning. For them, those graphs and literal expressions are transparent; they can "see the mathematics through it", so to speak. The students, however, do not have the necessary mathematical background to interpret those symbolic representations in that manner. As a consequence, teachers will have to explain to the students what there is to see, and how to reason with those symbolic representations. This, Cobb et al. (1992) point out, leads to proceduralising and algorithmetising and the loss of meaning - or to, as van Oers (2000) calls it, "pseudo mathematics". To find a way out of this dilemma, one may consider the history of mathematics to investigate how meaning and symbols emerged. It turns out that mathematical symbols did not arrive ready-made, with full-fletched meaning. Instead, one can discern a reflexive process in which symbolising and the development of meaning co-evolve (Meira, 1995). Symbolising, here, refers to inventing and using a series of symbols. In relation to this, Latour (1990) and others (e.g. Roth & McGinn, 1998) speak of a "cascade of inscriptions". This notion of a cascade of inscriptions has its counterpart in semiotic concepts as "chain of signification" (Walkerdine, 1988; Whitson, 1997).
The current challenge for mathematics educators is to develop mathematics education that is in line with these dynamic conceptions of symbolising and development of meaning. The task of researchers is to shed light on the key elements of this type of mathematics education.
In 1998 a group of projects was started around the premises that IT could be used to support the aforementioned type of mathematics education. Symbolic representations could be embedded in various IT tools such as computers and hand-held calculators. Furthermore, this type of learning environments might create opportunities for exploration and reflection. This group of projects was Design research on IT ORD-2003 2
funded by the PROO (a branch of the Netherlands Organisation for Scientific Research, NWO) as a comprehensive research project entitled "IT and mathematics education" (grant number 575-36-003). This research encompassed five projects, three of which we will discus in this paper. These three projects focussed on the research question:

How can the use of IT in mathematics education help students reinvent mathematical concepts and representations?
This overarching question is elaborated on for three topics, the notion of distribution in statistics, calculus or, as Kaput calls it, "the mathematics of change and variation", and the concept of parameter in algebra. For each research project, this implies a dual goal:
- On the one hand, answering the question on how to shape the teaching and learning process on this specific topic while integrating IT in it, and
- on the other hand, investigating the role of IT, and, more specifically, the reflexive relations between IT-embedded symbol use and the development of meaning.
Given these goals, we chose what is called design research, or developmental research (Gravemeijer, 1994b, 1998), as our research method. Following Brown (1992), Cobb, Confrey, diSessa, Lehrer and Schauble (2003) refer to this type of research as design experiments, which they elucidate in the following manner:
Prototypically, design experiments entail both "engineering" particular forms of learning and systematically studying those forms of learning within the context defined by the means of supporting them. This designed context is subject to test and revision, and successive iterations that result play a role similar to that of systematic variation in experiment.
(Cobb, Confrey, diSessa, Lehrer & Schauble, 2003, p. 9)
In this description, the two aspects of our research goal come to the fore in (a) the design of means of support for particular forms of learning, and (b) the study of those forms of learning. In each of the research projects under discussion, the backbone of the design is formed by the design, development and revision of a hypothetical learning trajectory (a term that we will elaborate on later). The enactment of the instructional activities that are part of the hypothetical learning trajectory in teaching experiments creates the opportunity to study IT-embedded symbol use and the development of meaning in the context of the actual learning trajectory.
2. Design Research Methodology
The design research approach we follow in the three projects has a cyclic character: each design research study consists of research cycles in which thought experiments and teaching experiments alternate. We distinguish macro cycles that concern the global level of the teaching experiments, and micro cycles that concern the level of subsequent lessons. Gravemeijer argued that the micro cycles lead to a cumulative effect of small steps, in which teaching experiments provide "feed-forward" for the next thought experiments and teaching experiments (Gravemeijer, 1994a). A macro cycle of design research consists of three phases: the preliminary design phase, the teaching experiment phase, and the phase of retrospective analysis.
A second characteristic of design research is the importance of the development of a learning trajectory that is made tangible in instructional activities (Gravemeijer, 1994a). The design of instructional activities is more than a necessity for carrying out teaching experiments. The design process forces the researcher to make explicit choices, hypotheses and expectations that otherwise might have remained implicit. The development of the design also indicates how the emphasis within Design research on IT ORD-2003 3
the theoretical development may shift and how the researcher's insights and hypotheses develop. As Edelson argues, design is a meaningful part of the research methodology:
(...) design research explicitly exploits the design process as an opportunity to advance the researchers understanding of teaching, learning, and educational systems. Design research may still incorporate the same types of outcome-based evaluation that characterise traditional theory testing, however, it recognizes design as an important approach to research in its own right.
(Edelson, 2002, p.107)
This is particularly the case when the theoretical framework involved is under construction.
One approach to design research is to immediately adapt the instructional activities after every lesson according to the experiences. This leads to many short micro cycles and can be an effective method, as shown by the following quotation that describes a situation where adaptations "on the fly" led to a micro cycle method:
The time pressure was considerable, but it had one advantage: experiences in the classroom could be incorporated immediately. Consequently, practical experience assumed great significance as a 'feed forward'; the components of the curriculum, which were still to be developed, could be adjusted directly on the basis of classroom experiences. (Gravemeijer, 1994a, p. 36)
In these studies, however, we started the teaching experiments with a complete set of instructional activities. Due to practical circumstances in two of the projects, adaptations were limited to decisions on skipping tasks or stressing other assignments. Therefore, the cyclic character of the research primarily concerns the macro cycles.
2.1. Hypothetical learning trajectory
Within each macro level research cycle, we distinguish three phases: the preliminary design phase, the teaching experiment phase, and the phase of retrospective analysis. The first phase of preliminary design includes two related parts, the development of a Hypothetical Learning Trajectory (HLT) and the design of instructional activities. In the next four sections we elaborate on each of these (partial) phases. Of course, each phase was slightly different in the three projects; here we describe common aspects of each phase within the different projects.
The first phase of each research cycle includes the development of a "Hypothetical learning trajectory" - a term that is taken from Simon (1995). Originally, Simon used the HLT for designing and planning short teaching cycles of one or two lessons. In our study, however, we developed HLTs for teaching experiments that lasted for sequences varying from 8 to 20 lessons. As a consequence, the HLT comes close to the concept of a local instruction theory (Gravemeijer, 1994a). Also, this interpretation of the HLT is close to the scenario concept as it is used by Klaassen (1995). A second difference with Simon's approach is that Simon took a teacher's perspective, whereas we take a researcher's perspective.
The development of an HLT involves the choice or design of instructional activities in relation to the assessment of the starting level of understanding, the formulation of the end goal and the conjectured mental activities of the students. Essential in Simon's notion of a HLT is that it is hypothetical; when the instructional activities are acted out, the teacher - or researcher in our case - will be looking for evidence of whether these conjectures can be verified, or should be rejected.
For the design of the student activities, their motivation and the estimation of their mental effects, the designer makes full use of his domain specific knowledge, his repertoire of activities and representations, his teaching experience, and his view on the teaching and learning of the topic. After Design research on IT ORD-2003 4
a field test by means of a teaching experiment, the HLT will usually be adapted and changed. These changes, based on the experiences in the classroom, start a new round through the mathematical teaching cycle, and, in terms of the design research method, the next research cycle.
The concept of the HLT may seem to suggest that all students follow the same learning trajectory at the same speed. This is not how the HLT should be understood. Rather than a rigid structure, the HLT represents a learning route that is broader than one single track and has a particular bandwidth.
Because of its stress on the mental activities of the students and on the motivation of the expected results by the designer, the HLT concept is an adequate research instrument for monitoring the development of the designed instructional activities and the accompanying hypotheses within design research. It provides a means of capturing the researcher's thinking and its development throughout the research and helps in getting from problem analysis to design solution.
2.2. Design of instructional activities
The preliminary design phase of the design research cycles includes the development of the HLT and the instructional activities. Of course, the development of an HLT and the design of instructional activities are closely related: the HLT guides the design of the instructional activities, but choices made in the design process may lead to reconsideration of the HLT. The expectations of the students' mental activities established in the HLT are elaborated into specific key activities in the instructional materials. In this section, we describe our design method and principles.
The design of instructional activities in these studies included the development of student text booklets, teacher guides, solutions to the assignments, tests and software. While designing these materials, choices and intentions were captured and motivated, to inform the teacher and to keep track of the development of the designer's insights. When the materials were about to be finalised, these aims and expectations were described at the task level. Key items, that embodied the main phases in the HLT, were identified. These items reflected the relevant aspects of the intended learning process and were based on the conceptual analysis of the topic. The identification of key items guided observations and prepared for the retrospective data analysis. Finally, teacher guides as well as observation instructions were written, to make intentions and expectations clear to teachers and observers. During the design phase, products were presented to colleagues, teachers and observers. This led to feedback that forced the researcher to be explicit about goals and aims, and that provided opportunities for improving all the materials.
While designing instructional activities, the key question is what meaningful problems may foster students' cognitive development according to the goals of the HLT. Three design principles guided the design process: guided reinvention, didactical phenomenology and mediating models.
The design principle of guided reinvention involves reconstructing the natural way of developing a mathematical concept from a given problem situation. A method for this can be to try to think how you would approach a problem situation if it were new to you, or, as Gravemeijer phrases it, "think how you might have figured it out yourself" (Gravemeijer 1994, p. 179). In practice, this is not always easy to do, because as a domain expert it is hard to think as if you were a freshman. The history of the domain can be informative on specific difficulties concerning concept development (e.g. Bakker, 2003; Gravemeijer & Doorman, 1999).
The second design principle, didactical phenomenology, was developed by Freudenthal (Freudenthal, 1983; Gravemeijer, 1994a; Gravemeijer & Cobb, 2001). Gravemeijer explains it as follows:
Didactical phenomenology points to applications as a possible source. Following on the idea that mathematics developed as increasing mathematisation of what where originally solutions to practical problems, it may be concluded that the starting points for the re-invention process can

be found in current applications. (Gravemeijer 1994, p. 179) Design research on IT ORD-2003 5
Didactical phenomenology aims at confronting the students with phenomena that "beg to be organised" by means of mathematical structures. In that way, students are invited to build up mathematical concepts. Meaningful contexts, from real life or "experientially real" in another way, are sources for generating such phenomena (de Lange, Burrill, Romberg, & van Reeuwijk, 1993; Treffers, 1987). The question, therefore, is to find meaningful problem contexts that may foster the development of the targeted mathematical objects. The context should be perceived as natural and meaningful, and offer an orientation basis for mathematisation.
The last remark leads to the third design principle, the use of mediating models (Gravemeijer, 1994a). In the design phase we try to find problem situations that lead to models that initially represent the concrete problem situation, but in the meantime have the potential to develop into general models for an abstract world of mathematical objects and relations.
2.3. Teaching experiments
The second phase of the design research cycle is the phase of the teaching experiment, in which the prior expectations embedded in the HLT and the instructional activities are confronted with classroom reality. The term "teaching experiment" is borrowed from Steffe (Steffe & Thompson, 2000). The word "experiment" is not referring to an experimental group - control group design. In this section we explain how the teaching experiments were carried out; in particular, we pay attention to the data sampling techniques used during the teaching experiments.
The research questions share a process character: they concern the development of understanding of mathematical concepts. Therefore, we focussed on data that reflected the learning process and provided insight into the thinking of the students. The main sources of data, therefore, were observations of student behaviour and interviews with students. The observations took place on three levels: classroom level, group level and individual level. Observations at classroom level concerned classroom discussions, explanations and demonstrations that were audio and video taped. These plenary observations were completed by written data from students, such as handed in tasks and notebooks.
Observations at group level took place while the students were working on the instructional activities in pairs or small groups. Short interviews were held with pairs of students. In addition to this, the observers made field notes.
The lessons were evaluated with the teachers. These discussions led to consequences for the following lessons, as is characteristic with design research micro cycles. In particular, the organisation of the next lesson and the content of the plenary parts were discussed. Also, decisions were taken about skipping (parts of) tasks because of time pressure. Such decisions were written down in the teaching experiment logbook.
2.4. Retrospective analysis
The third phase of a design research cycle is the phase of retrospective analysis. It includes data analysis, reflection on the findings and the formulation of the feed-forward for the next research cycle. In this section we focus on the data analysis method.
The first step of the retrospective analysis concerned elaborating on the data. A selection from video and audiotapes was made by event sampling. Criteria for the selection were the relevance of the fragment for the research questions and for the HLT of the teaching experiment in particular. Data concerning key items was always selected and these selections were transcribed verbatim. From the videotaped work of pairs of students, only a limited part of the registered time was transcribed verbatim, as students often proceeded slowly while working on their own, and did not always explain their method. In such cases, the tapes were summarised into descriptions. Long plenary explanations and classroom discussions were summarised as well, unless they contained interesting student contributions. Registrations where students were not working at all, or where organisational matters were discussed, were not elaborated on. The written work from the students was surveyed and Design research on IT ORD-2003 6
analysed, especially the work on key items, tests and hand-in tasks. Results were summarised in partial analyses.
The first phase of the analysis consisted of working through the protocols with an open approach that was inspired by the constant comparative method (Glaser & Strauss, 1967; Strauss & Corbin, 1988). Remarkable events or trends were noted as conjectures and were confronted with the expectations based on the HLT and the instructional activities. The second phase of analysis concerned looking for trends by means of sorting events and analysing patterns. The findings were summarised illustrated by prototypical observations. These conjectures were tested by surveying the data to find counterexamples or other interpretations, and by data triangulation: we analysed the other data sources, and in particular the written student material, to find instances that confirmed or rejected the conjectures. Analysis of the written materials often evoked a reconsideration of the protocols.
Analysis was continued in this way until saturation, which meant that no new elements were added to the analysis and no conclusions were subject to change.
The third phase in analysing the data was the interpretation of the findings and the comparison with the preliminary expectations of the HLT. Also, explanations for the differences between expectations and findings were developed. These conclusions and interpretations functioned as feed-forward for the formulation of new hypotheses for the next cycle in the research.
3. Exemplary cases from the three projects
3.1. A case from the statistics project
Background
The aim of this project, entitled "symbolising in statistics education", was to develop an instruction theory for early statistics education, with a focus on the notion of distribution of univariate data in relation to sampling. Students tend to perceive a data set as a collection of individual data points (Hancock, Kaput, & Goldsmith, 1992), whereas statistical data analysis is mainly about describing and predicting aggregate features of data set. The end goal of the instructional unit was that students should come to view a data set as a whole and develop a notion of distribution as an object that can have different features. The notion of distribution is a key concept of statistics and it is intimately tied to the idea of shape. A common representation of the normal distribution, for instance, is the bell shape. Such shapes are, of course, not just visual images, but signify how the data are distributed in terms of frequency or density. This means that students need to develop the meaning of such distributions before they can interpret conventional shapes such as the bell shape as symbols with meaning.
The teaching experiment that we report on here was in an eighth-grade class (second class havo-vwo) with 30 students, and lasted for ten lessons of 50 minutes. The students had learned about means and bar graphs, but no other statistical concepts or graphs, before the start of the teaching experiment.
Design of the HLT
The representational backbone of the Hypothetical Learning Trajectory (HLT) was the series of graphs in Figure 1. Students started with value-bar graphs of minitool 1; they could hide the bars, so that only the endpoints remained. If these endpoints were dropped (mentally), they got the dot plot of minitool 2. In minitool 2, students can organise data sets into equal intervals (precursor to histogram), two equal groups (median), four equal groups (box plots), and so on. While solving statistical problems with these minitools, students reasoned with different aspects of distribution, such as mean, spread, majority and outliers. We then wanted students to reason with continuous shapes, as opposed to discrete dot plots. Our conjecture was that this transition from a discrete plurality of data values to a continuous entity of a distribution is important for fostering a notion of distribution as an object. Once shape would be an object that students could reason about, the next step was to distinguish different shapes that are precursors to different distributions (uniform, symmetrical, skewed). Design research on IT ORD-2003 7
Figure 1. Minitool 1, with and without bars, and minitool 2 (the last two are the same data set).
Teaching experiment phase
One of the conjectures that emerged during an earlier cycle of design research was that experience with a so-called growing sample would help the students to reason about shapes of distributions, which is seen as a pre-stage to working with distribution as an object. The task of the students was to predict graphs of data of hypothetical situations with a particular sample size and compare their predictions with actual data. Then a larger sample had to be predicted and compared with actual data. In this way the shape of a distribution in relation to particular distribution aspects could become a topic of discussion.
In the first cycle of growing a sample, students had to predict weight data of ten eighth graders. We give examples of the work of three students that form a representative group of the class. Figure 2 shows the different graphs that were made: value-bar graphs, only the endpoints of the bars, and dot plots. These graphs correspond to the minitool graphs students had worked with in previous lessons (Figure 1). When comparing their own graphs with real data graphs, students often used informal attributes such as "together" and "spread out" to indicate range and spread.

Figure 2. Students' predictions of sample size 10, and three actual samples of 10.
Christel: The middle-most [graph with actual data] best resembles mine because the weights are closely together and that is also the case in my graph. It lies between 35 and 75 [kg]. Design research on IT ORD-2003 8
Susan: The other [real data] are more weights together and mine are further apart.
In the second cycle of growing a sample, students had to predict data for a whole class and for three classes after the feedback of the samples of ten data points in dot plots.
Figure 3. Susan's predictions for one and three classes and an actual sample of three classes.
Rikkert: My spread is different.
Christel: Mine resembles the sample, but I have more people around a certain weight and I do not really have outliers because I have 10 about the 70 and 80 and the real sample has only 6 around the 70 and 80.
Susan: With the 27 there are outliers and there is spread; with the 67 there are more together and more around the average.
In contrast to the first cycle, students used nouns instead of just attributes for comparing the graphs. This might seem trivial, but statistically it makes a difference whether we say, "the dots are spread out" or "the spread is large". Once spread is an object that can be talked about, it can be measured, for instance with range or interquartile range.
Figure 4. Students' predictions of shapes in the third cycle of growing a sample.
In the third cycle of growing samples students made their first remarks about shape. The continuous shapes that they proposed were pyramid, semicircle and bell shape. Using these shapes in the sixth lesson, students came to reason about different aspects of distributions, including mean, outliers and skewness, in relation to sampling.
Retrospective analysis
From the analysis we concluded that students' experience with the minitools supported their reasoning about growing samples and that this activity of growing samples in turn supported the development of Design research on IT ORD-2003 9
a notion of distribution in relation to sampling. In the lessons after this fourth lesson, students could work with growing samples in the second minitool. We claim that the activity that students worked on in the fourth lesson fitted well in the HLT. One indication for this is that a considerable part of the HLT was retrieved in one lesson. In fact, Treffers and colleagues use this as a criterion for the quality of an activity in a learning trajectory that aims at guided reinvention.
What becomes visible in the cross section of solutions of the complete group at a certain moment is the longitudinal section of the learning trajectory that has to be followed by each student. (Treffers, Streefland, & de Moor, 1994, p.147; translation from Dutch)
And indeed we encountered the whole representational backbone of the HLT in the fourth lesson:
1. Value-bar graph of minitool 1;
2. Only the endpoints of the bars;
3. Dot plot of minitool 2;
4. Dot plot with continuous shape on top of it;
5. Continuous shape.
From the analyses of students' reasoning with these graphs and shapes (symbols), we can infer that students did indeed start to develop meanings as intended. From a systematic analysis that was inspired by the constant comparative method of Glaser and Strauss (1967), we concluded that students in all seventh and eighth-grade classrooms tended to think of distributions as consisting of low, average, and high values. While working with diagrams both on paper and the computer screen, they started to develop a more subtle view on distribution.
The semiotic framework of diagrammatic reasoning (Peirce, 1976) helped in describing the process of meaning development in relation to statistical diagrams, in particular in the formation of mathematical objects. Not all of the students developed a notion of distribution as an object, but they did learn to reason about shapes in relation to frequency and density. Diagrammatic reasoning consists of constructing diagrams, experimenting with them, and reflecting on the results. It proved useful to let students invent their own diagrams and let them compare these with other students' diagrams. The minitools were used optimally during the experimentation phase, which formed the basis for reflection during class discussions.
The insights that have been developed in this project are of course domain-specific and situated in the Dutch context. In that sense, the theoretical contribution of such design research is "humble", as Cobb et al. phrase it (2003, p. 9). Yet, the fact that the HLT is theoretically and empirically grounded and is tailored to a specific domain makes it directly applicable to educational practice. Because the HLT has proved viable in different classrooms in different variants (also in the USA) we assume it can form the basis for a more general instruction theory on early statistics education.
3.2. A case from the calculus study
Background and design of HLT
The aim of this project, entitled "design research for the teaching and learning of calculus and kinematics with ICT", is to find out how students can learn the basic principles of calculus and kinematics by modelling motion. Many alternatives to the traditional transfer method stress the importance of modelling activities. In these alternatives two approaches are recognised. One uses simulations for exploratory modelling and the other uses (computer) tools for expressive modelling. Exploratory approaches stress the discovery of concepts while students work with formal representations in computer simulations. In the expressive approach, students' expressions during their modelling activities are the starting points for evolving concepts. This approach has many similarities with the importance of mediating models in realistic mathematics education. The expressive approach also fits the discussion about a dialectic relation between symbolising and development of understanding. With symbolising we refer to theories on how we create, use and adapt symbols and Design research on IT ORD-2003 10
how this relates to concept development as presented in the previous statistics study. In this project we focus on the role of graphical representations in the process of learning the principles of calculus and kinematics.
Nowadays, graphs are used in calculus and kinematics education as representations for describing change of velocity or distance travelled during a time interval. Students are expected to give meaning to the relation between distance travelled and velocity through characteristics of these graphs such as area and slope. The use of such instructional materials is based on a representational view (Cobb et al., 1992), which assumes that instructional materials can represent scientific knowledge, and that scientific concepts can be made accessible without fully taking into account the limitations of the knowledgebase of the students into which they have to be integrated. Elements of this view can still be found in exploratory approaches.
Cobb et al. oppose this view. In line with their reasoning, we claim that symbolisations and knowledge of motion can co-evolve in a learning process. Theories on symbolising give rise to heuristics for designing a learning route within which the mathematical and scientific knowledge emerges from the activity of the students (Gravemeijer et al. 2000). In this route, the creation, use and adaptation of various graphical representations are interwoven with learners' activities in a series of sciencepractices, from modelling discrete measurements to reasoning with continuous models of motion. Mathematical and physical aspects are integrated in these activities. Our focus is on students' contributions during these practices. Consequently, for understanding their reasoning we use the design research approach of planning and testing the envisioned trajectory in classroom situations, a research approach that aims at creating an educational setting for investigating how a trajectory works, instead of trying to decide whether it works.
The learning route - inspired by the domain history - is tried out and revised during teaching experiments in three tenth-grade classes. We collected data by video and audio taping whole class discussions and group work. The videotapes were used to analyse students' discourses and students' written materials with respect to the conjectured teaching and learning process.
Teaching experiment phase
We illustrate the change in how students think and talk about a model with the following episode. The trajectory starts with questions about a weather forecast. The teacher discusses the change of position of a hurricane with students: when will it reach land? This problem is posed as a leading question throughout the unit as a context for the need of grasping change. After the emergence of time series as

useful tools for describing change of position, students work with situations that are described by stroboscopic photographs. The idea is that students come up with measurements of displacements, and that it makes sense to display them graphically for finding and extrapolating patterns. Two types of discrete graphs are discussed: graphs of displacements (distances between successive positions) and graphs of the total distance travelled. Note that discrete graphs are not introduced as an arbitrary symbol system, but emerge as models of discrete approximations of a motion, that link up with prior activities and students' experiences. By using the computer program Flash students are able to investigate many situations. During these activities their attention shifts from describing specific situations to properties of these discrete graphs and the relation with kinematical concepts.
Our findings confirm such a change in reasoning. In the beginning students refer to distances between successive positions. After a while they reason using the global shapes and properties of graphs and motion. An example of such reasoning concerns an exercise about a zebra that is running at constant speed and a cheetah that starts hunting the zebra. The question is whether the cheetah will catch up with the zebra. In the graphs the successive measurements of the zebra and the cheetah are displayed.
The following discussion takes place between an observer and two students (Rob and Anna). Design research on IT ORD-2003 11
Figure 5. Distance-travelled and displacement graphs in Flash.
Observer: Oh yes. So why did you choose the one for the total distance [left graph in Fig. 5]?
Rob: Because it's the total distance that they cover and then you can-
Anna: Then you can see if they catch up with each other.
Observer: And can't you see that in the other [right graph]? There you can also see that the red [grey] catches up with blue [dark grey]?
Rob: Yes, but -
Anna: Yes, but that's at one moment. That only means that it's going faster at that moment but not that it'll catch up with the zebra.
Retrospective analysis
A difference between the displacements graph and the distance-travelled graph is the difference between the interpretations of the horizontal (time) axis. A value in the distance-travelled graph represents a distance from the start until the corresponding time, while a value in the displacements graph represents a distance in the corresponding time interval. Anna's last observation is an important step in the process of building the model of a velocity-time graph (and everything that comes with it). The qualitative analyses show that during the practices students re-invent and develop graphical symbolisations, as well as the language and the scientific concepts that come with them. However, these inventions only became explicit after interventions by an observer or by the teacher. Additionally, we found that the teacher had a crucial role during the classroom discussions. It was not always easy to organise the discussions in line with the intended process. Sometimes the teacher reacted to students' contributions in terms of the inscriptions or concepts aimed at. In those cases students awaited further explanation. The discussions appeared to be especially productive when the teacher organised classroom discussions about students' contributions in such a way that the students themselves posed the problems that had to be solved, and reflected on their answers. In a second teaching experiment we arranged a setting where the teacher had more information about the possible contributions of the students and the way in which they could be organised. Additionally, we designed activities for classroom discussions. In this experiment we found that this resulted in discussions where students anticipated on the intended concepts and reflected on their contributions.
In this approach the construction and interpretation of graphs and the scientific concepts are rooted in the activities of the students through emerging models. This ensures that the mathematical and physical concepts aimed at are firmly rooted in students' understanding of everyday phenomena. On the basis of our findings we conclude that classroom discussions where students discuss their solutions and pose new problems to be solved, are essential for a learning process during which symbolisations and knowledge of motion co-evolve. Design research on IT ORD-2003 12
3.3. A case from the algebra study
Background
The aim of the third project, entitled 'Learning algebra in a computer algebra environment', was to investigate the contribution of computer algebra use to the development of algebraic concepts and techniques. A computer algebra system (CAS) is a software package, available on both PCs and hand-held devices, which is able to carry out algebraic procedures such as solving equations, substituting expressions and simplifying formulas. The availability of such powerful tools for algebra raises the question of how this kind of technology can be integrated into algebra education and how this integration affects the role of by-hand algebra skills. What is the relation between the work in the computer algebra environment, the paperand-pencil work, and the students' mental understanding?
One perspective for investigating this question is the instrumental approach to using technology. By instrumentation we mean the process of developing techniques in the technological environment (Artigue, 2002). Such a technique is built up by the students for solving a specific type of task, and is condensed into so-called instrumentation schemes. An instrumentation scheme includes technical knowledge and skills for manipulating a technological tool, in this case the computer algebra environment, as well as conceptual knowledge that gives meaning to these technical manipulations. This intertwinement of technical aspects related to the tool, and mental aspects related to the students' conceptual understanding makes the instrumentation perspective relevant for our purpose.
Our claim here is that this instrumental approach allows for understanding the interaction between student and technological tool, and in particular the difficulties that are encountered (Drijvers, 2000). Furthermore, we argue that a carefully orchestrated instrumentation of the technological tool, both by the instructional activities and the teacher, may improve conceptual understanding, which also affects paper-and-pencil techniques.
The teaching experiments in which we found evidence for these claims took place in ninth- and tenth-grade classes of high achieving students (in the Dutch school system: vwo3 and vwo4) and lasted 15 - 20 lessons each. Before the experiments, the students had some experience with algebra, but little experience in using technology for learning mathematics. During the teaching experiments, they used the TI-89 symbolic calculator as computer algebra environment. We illustrate our claims with an example on substituting expressions.
Design of HLT
The pre-test of one of the teaching experiments revealed that hardly any of the students were able replace y in the equation x
2
+ y
2
= 10 by a - x , in other words to substitute y = a - x into the equation x
2
+ y
2
= 10.
During the teaching experiment the students carried out such substitutions using the computer algebra device. Fig. # shows the syntax of the substitution using the vertical substitution bar. Entering expression1 | variable = expression2 results in replacing all instances of variable in expression1 by expression2.
Figure 6. Substituting an expression in the CASDesign research on IT ORD-2003 13
While carrying out such substitutions, the students had to build up the substitute instrumentation scheme. This scheme involved:
1. Noticing that a substitution is required to obtain a result;
2. Remembering the way substitution is carried out in the computer algebra environment: the meaning of the vertical bar symbol and the syntax of the command;
3. Realising which expressions play the roles of expression1 and expression2, and in particular considering expression2 as an object instead of a process;
4. Being able to type in the command correctly;
5. Accepting the resulting expression as a satisfying answer in spite of its "lack-ofclosure" (see below).
Steps 2 and 4 of this scheme have a primarily technical character, whereas the other steps are mainly conceptual. One of the first examples of substitution concerned substituting the expression for the area of the bottom of a cylinder into the volume function:
V = A•H | A = π•R
2
. The fact that the problem situation was meaningful to the students helped them to understand the substitution. Meanwhile, the fact that the result, V = π•H•R
2
, still contained multiplication symbols was not satisfying to some of the students, as they wanted to be able to carry out the multiplications to get a numerical answer. This is the socalled lack-of-closure obstacle that was referred to in step 5 of the scheme (Tall & Thomas, 1991).
Teaching experiment phase
By means of practising substitution in the CAS, the students got used to expressions as answers and seemed to consider expressions as sensible results. However, their notion of substitution often was somewhat vague, as the following observation indicates:
Observer: And what happens now for example in V = A•H wherein A = π•R
2
?
Maria: Well that it just makes one formula out of it.
Observer: Yes, and how does it do that then?
Maria: By means of joining them or something?
Later in the teaching experiment, students tried to substitute non-isolated forms, such as for example x
2
+ y
2
= 10 | x + y = a. This error was not only a syntactical matter, but also revealed an imprecise concept of substitution in step 3 of the scheme. In the teaching, a proper view on substitution was supported by means of visualisation:
x
2
+ y
2
= 10 | x = z + 1
The instrumentation of substitution within the computer algebra environment led to an improved conceptual insight on the issues of steps 3 and 5 of the scheme. This was confirmed by data from the post-experiment interviews, during which the computer algebra device was no longer available to the students. Most students showed to be able to substitute y = a - x into the equation x
2
+ y
2
= 10. As fig. # shows, one of them used the vertical bar to indicate the substitution, which clearly was inspired the notation in the computer algebra environment. He wrote down x
2
+ (a - x)
2
= 10 and explained his result by saying: "Here is already a y and here it says what y is, so you just have to replace this with that, I was just thinking too difficultly".
Figure 7. Transfer of notation Design research on IT ORD-2003 14
Retrospective analysis
Looking back at the results of this study, we conclude that the perspective of instrumentation was helpful for understanding the interaction between student and technological tool and for studying the effect of computer algebra use on conceptual development. The question is how these findings link up with theories on symbolising, which stress the signification process in which the student develops his own symbolisations, and improves and uses them in relation to the meaning that he attaches to them (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Gravemeijer et al., 2000). Our study did not show much of this bottom-up process, because the computer algebra environment already offers a symbol system and is not very flexible concerning notation and syntax. However, the confrontation with a ready-made symbol system also involves the process of developing meaning while using the system. This links up with the technical and conceptual aspects of instrumentation. In that sense, theories on symbolisation provided a background for understanding the growing sense for the symbols within the computer algebra environment. Furthermore, the use of realistic problem situations such as the cylinder task provided a reference to a meaningful situation for the students. The use of the technological tool did not make such a reference redundant and in the meantime did not hinder generalisation and abstraction.
4. Discussion
In the statistics project, different instances of diagrammatic reasoning have been analysed. It proved useful to let students construct their own diagrams once they had some experience with specially designed software, and reflect on the diagrams they had made in relation to the problem that had to be solved. The alternation of providing carefully designed software and letting students reinvent their own graphs and strategies proved useful. Looking back we notice that the growing samples activity shows a subtle mix of providing and designing (cf. van Dijk et al., 2003). The students had already experience with minitool graphs there were provided, but in the fourth lesson they had to design their own graphs for new situations (very large samples). After the first cycle of growing samples, students received feedback with dot plots. In other words, there was opportunity for a reinvention process, but this was guided carefully. This means that the answer that our design research gives to such questions is subtle and situated, but not easy to generalise.
As intended in the HLT, students did indeed learn to reason about distributions and sampling in a meaningful way in relation to statistical diagrams. This learning process was analysed semiotically and, as expected, the relationship between symbolising and the development of meaning and mathematical objects could be characterised as reflexive.
Also, in the calculus project it appeared that symbols and reasoning could develop in a reflexive process. The IT-tools afforded students many situations with presented representations to analyse. During these analyses the character of the representations changed from situation-close to organising tools for modelling motion. In this way, through emerging models construction and interpretation of graphs and scientific concepts became rooted in the activities of the students. Moreover, we found that classroom discussions where students discuss their solutions and pose new problems to be solved are essential for a learning process during which symbolisations and knowledge of motion co-evolve.
The project on learning algebra in a computer algebra environment showed that the technical skills using the computer algebra device required, were closely related to a conceptual understanding of the mathematics behind the technique. Developing computer algebra techniques and instrumentation schemes proved to be difficult for the students. Overcoming these difficulties often involved a conceptual development that was relevant for the targeted insight. The joint development of technical skills and conceptual understanding is seen as support for the instrumental approach to using IT tools. Design research on IT ORD-2003 15
5. Conclusions
On the issue of the research question from paragraph 1, we notice an agreement on the tension between the bottom-up reinvention process and the top-down character of IT use. Moreover, we notice an agreement on the importance of the perspective of symbolising, though in a different way for each of the studies. On the issue of the methodology, we conclude that the notion of Hypothetical Learning Trajectory was helpful in the preliminary phase of each of the research cycles, and that gathering data

by means of dedicated mini-interviews with students was a fruitful strategy. During the teaching
experiments and retrospective analyses the HLT also served in guiding the observations and analysis.
We now briefly elaborate on each of these conclusions.
In each of the studies we noticed a tension between the targeted bottom-up reinvention process and the
top-down character of IT. The term guided reinvention is used for a process in which students
experience learning mathematics as if they could have invented it themselves, while being supported
by the teacher and instructional materials (including the IT-tools). The IT-tools intrinsically have a
top-down character because they offer a restricted repertoire of representations and. For computer
algebra this top-down character is the strongest, because the large variety of possibilities is combined
with a restricted flexibility in symbols, notation and syntax. Somehow this disadvantage of IT-tools
has to be compensated by instruction; reflection, interaction, carefully designed problems, and so on.
It is important that the teacher not just compares different diagrams and solutions with the students,
but also stimulates students to formulate which problems they solve, why certain solutions work and
in which direction future activities have to take place.
The focus on symbolising proved viable. In the projects on statistics and calculus, using semiotic and
perception theories turned out useful for analysing the relationship between symbolising and
development of meaning. We assume on the basis of these projects and prior research that a carefully
designed trajectory of symbol and meaning development is necessary to give students the
opportunities to learn mathematics. In that process, students need to get ample opportunities for their
own constructions and reflection on them. In the algebra project, this issue was different, because the
computer algebra systems are ready-made symbol systems that are hard to change. As a consequence,
the focus was not on the development of increasingly sophisticated symbols, but on the development
of meaning within an existing symbol system. Realistic contexts proved important in that. The
approach of instrumentation, which considers dealing with a ready-made symbol system in relation to
conceptual development, can be seen as a specification of the symbolising issue in the case of readymade systems.
With respect to the methodology of design research, we consider the Hypothetical Learning
Trajectory a useful instrument in all phases of design research. During the design phase it is the
theoretically grounded vision of the learning process, which is specified for concrete instructional
activities. During the teaching experiments, the HLT guides observations, data collections, which
questions have to be asked during mini-interviews, and offers a framework for educational decisions
during the teaching experiment. The mini-interviews proved useful for getting detailed information
about what symbols meant for students and how they reasoned with these symbols. In the
retrospective analysis phase, the HLT served as a guideline for data selection and offered conjectures
that could be tested during the analysis. Design research on IT ORD-2003 16
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Examples of activities

36 quotations

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How Emergent Models May Foster
the Constitution of Formal Mathematics
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This article deals with the role that so-called emergent models can play in the process
of constituting formal mathematics. The underlying philosophy is that formal mathematics is something that is, or should be, constituted by the students themselves. In the
instructional design theory for realistic mathematics education, models always have
been employed to foster a process in which formal mathematics is reinvented by the
students themselves. This article describes how the use of models became more and
more explicated over time and developed into the notion of emergent models. The design of an instructional sequence, which deals with flexible mental computation strategies for addition and subtraction up to 100, is taken as an instance for elaborating
what is meant by emergent models and what role they play in fostering the constitution of formal mathematics. The analysis shows that there are 3 interrelated processes.
First. at a more holistic level, there is aglobal transition in which "the model" initially
emerges as a model of informal mathematical activity and then gradually develops
into a model for more formal mathematical reasoning. Second, the transition from
"model of' to "model for" involves the constitution of anew mathematical reality that
can be denoted formal in relation to the original starting points of the students. Third,
in the series of instructional activities, there is not 1 model, but the model actually is
shaped as a series of signs, in which each new sign comes to signify activity with a
previous sign in a chain of signification.
The difficulty of learning mathematics often is attributed to what is perceived as the
gap between everyday life experience and what is colloquially called formal mathematics. In the approach discussed in this article, however, everyday-life experi-
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ence and formal mathematics are not seen as two disjunct entities. Instead, what is
aimed for is a process of gradual growth in which formal mathematics comes to the
fore as a natural extension of the student's experiential reality. The focus in this article will be on the role so-called emergent models can play in such a process. The
perspective taken here is that of instructional design. The design of an instructional
sequence that aims at linear measurement and mental arithmetic will be used as an

occasion to elaborate on the role of emergent models.
To provide the necessary background, I start by sketching the context of the instructional design approach, within which the use of emergent models can be depicted as a design heuristic. The instructional design under discussion is a design in the context of developmental or transformational research aiming at the development of a domain-specific instruction theory for realistic mathematics education (RME; Gravemeijer, 1994a; Streefland, 1991; Treffers, 1987). This RME instruction theory is not a fixed, ready-made theory but more a theory that iselaborated and refined continuously. Because of this, I think it is useful to describe the history of the way in which the role of models is conceived in RME. Further, before finally working out the intended exemplary instructional sequence, I sketch the history of that sequence.
I adopt a semihistorical approach in this article. As a consequence, I sometimes use descriptions that may seem unclear or questionable without elaboration, but I do so to do justice to the history.
DEVELOPMENTAL RESEARCH
The design heuristic, which is addressed here, is part of an instructional design approach, which, in turn, is part of a research approach that is known as developmnl research (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Freudenthal, 1991; Gravemeijer, 1994b, 1998) or as trar7sformntiotlnl research (National Council of Teachers of Mathematics Research Advisory Committee, 1988). Developmental research is best described as an integration of design and research. That is, the design of instructional sequences serves as research on an instruction theory. This type of research poses the question: What does mathematics education that fulfills certain prechosen points of departure look like?
In the first phase of a developmental research project, a preliminary design of an instructional sequence is constituted. In the second phase, this preliminary design is elaborated and adjusted in a series of cyclic processes of deliberations on and trials of instructional activities. These cycles are to a large extent similar to Simon's (1995) mathematical teaching cycles.
In the last phase of the developmental research project, the knowledge assembled during the experiments with the various instructional activities is used to (re)construct an optimal instructional sequence. A central goal of the research work is to explicate the local instruction theory that underlies this reconstructed in-
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structional sequence and to justify this theory in terms of theoretical deliberations and empirical assessments (see Gravemeijer, 1994a, 1994b, 1998).
This article is limited to the first part of such a developmental research activity. It focuses on the preliminary design of an instructional sequence. It should be noted that such a preliminary design does not necessarily have to be worked out in detailed descriptions of instructional activities because a core element of the developmental research approach is that instructional activities are designed and redesigned during the educational experiment. The experimental phase is not entered with a fixed sequence of instructional activities. Instead, the shape and form of the instructional activities that are actually used in the classroom are decided up on a day-to-day basis while taking into account what is learned from the preceding activities in the classroom.
The preliminary design may consist of a set of instructional activities that are supplemented with guidelines. The guidelines indicate both the order in which the instructional activities are to be enacted and the conjectured learning route in terms of the mental activities the students might engage in as they participate in the instructional activities. In relation to this, we speak of a local instruction theory. The concept of a conjectured local instruction theory is comparable with Simon's (1995) notion of a hypothetical learning trajectory. Such a notion encompasses the "consideration of the learning goal, the learning activities, and the thinking and learning in which the students might engage" (p. 133).
However, there are two differences: (a) The hypothetical learning trajectory deals with a small number of instructional activities, whereas the local instruction theory encompasses a whole instructional sequence; and (b) the hypothetical learning trajectory is tailored to the teacher's own classroom at a given moment in time, whereas the local instruction theory is more general. In my view, the connection between the two is that the local instruction theory entails a framework against which teachers can construe hypothetical learning trajectories that fit the actual situation in their classrooms.
If we return to the three phases of developmental research, we see that a conjectured local instruction theory is construed in the first phase. Subsequently, this conjectured local instruction theory is elaborated, refined, and adjusted in the second phase, the phase of the educational experiment. Then, in the third phase, the improved local instruction theory is reconstructed andjustified. The issue1 deal with in this article is the design of the conjectured local instruction theory in the first phase of adevelopmental research project. The focus is on the use of emergent models as a design heuristic, which can be seen as a constituting element of the RME theory.
The RME theory is not a fixed a priori theory but is a theory that is always "under construction." RME is being developed in an ongoing process of designing,
Downloaded by [North West University] at 07:05 27 May 2015 experimenting, analyzing, and reflecting. Central to this development is the role of developmental research. There is a reflexive relation between RME theory and developmental research. On one hand, RME theory guides the design and research; on the other hand, developmental research serves the elaboration and refinement of the theory. The goal of the RME research program is to determine how mathematics education can be presented to students to facilitate their reinvention of mathematics.
The point of departure is Freudenthal's (1973, 1991) adage of "mathematics as a human activity." For him, mathematics was primarily an activity. Mathematics as a system of organized knowledge was secondary for him. According to Freudenthal, this is how it ought to be for students as well. In his view, students should learn mathematics by mathematizing: both subject matter from reality and their own mathematical activity. Via a process of progressive mathematization, the students should be given the opportunity to reinvent mathematics. In this manner, Freudenthal transcended the apparent dichotomy between mathematics as an activity and mathematics as a body of knowledge.
In fact, the reinvention principle is the first instructional design heuristic in RME. According to the reinvention principle, a learning route has to be mapped out along which the students can find the intended mathematics for themselves (Freudenthal, 1973). To do so, the curriculum developer starts with a thought experiment by imagining a route the students could have invented themselves. Knowledge of the history of mathematics can be used as a heuristic device in this process. Note that the students are not expected to reinvent everything on their own. In relation to this, Freudenthal (1991) noted guided reinvention. For him, the emphasis is on the character of the learning process rather than on inventing as such. The idea is to allow learners to come to regard the knowledge they acquire as their own private knowledge, knowledge for which they themselves are responsible. On the teaching side, students should be given the opportunity to build their own mathematical knowledge store on the basis of such a learning process. In relation to this, it may be noted that the RME theory is primarily a theory about knowledge construction; the idea is not to motivate students with everyday-life contexts but to look for contexts that are experientially real for the students and can be used as starting points for progressive mathematization.
At a closer look, the history of the reinvention heuristic reveals the dynamics of the RME theory. Initially, the reinvention principle was interpreted rather literally, referring to the history of mathematics as a source of inspiration for the design of instructional sequences. Gradually, however, students' informal solution methods have been recognized as a source of inspiration. Sometimes, the informal ways by which students solve applied problems "anticipate" the more formal procedures we are aiming for (Streefland, 1991). For instance, Streefland found that, when solving problems such as dividing three pizzas among four children, students ar-
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rived at solution methods that contained the seeds of more formal procedures for adding and subtracting fractions with unequal denominators. For example, the students came up with solutions that corresponded with 3 divided by 4, 3 times one fourth, one half plus one fourth, or I minus one fourth. Written in a formal notation, one could conclude from this: ¾ = 3 x ¼ = ½ + +, and = 1 - -. In this sense, the informal solution methods of the students anticipated conventional standard procedures (e.g., ¾ + + + - + = x) , and one could think of ways to help students develop procedures similar to the conventional standard procedures.
RME started out as a vision, or as a philosophy of mathematics education that still had to be worked out. This is being done in developmental research projects, in which each time the research question is: What would mathematics education, which fulfills the initial points of departure, look like for a given topic? By reflecting on the yield of these projects, RME theory is developed, new theoretical insights in turn are used in subsequent research projects, and hence RME is developed in an iterative process of elaboration, adjustment, and refinement.
THE ROLE OF MODELS
In recent times, the RME community has expressed increasing interest in the role of models. From early on, it has been clear that models play a different role in RME than in mainstream information-processing approaches or in cultural-historical approaches, as elaborated by Gal'perin (1969).
In the latter approaches, models come to the fore as didactic models that embody the formal mathematics that is to be taught. That is, the formal abstract mathematics is "concretized" to be made accessible for the students. Actually, the students are to discover the mathematics that is concretized by the designer. In the RME approach, the models are not derived from the intended mathematics. Instead, the models are grounded in the contextual problems that are to be solved by the students. The models in RME are related to modeling; the starting point is in the contextual situation of the problem that has to be solved. This problem is modeled by the students who solve it with help of the construed model. The premise here is that students who work with these models will be encouraged to (re)invent the more formal mathematics.
Treffers (1987) discussed "bridging by vertical instruments" (p. 248). Here, vertical instruments also included schemes, diagrams, and symbols, although models were given a central position, and, although other roles of models were mentioned, it was emphasized that models primarily fulfill a bridging function between the phenomenological appearances of mathematics in reality on the one hand and the formal mathematics on the other. Still, models were treated as part of a broader category of instruments with a global function. In later publications, a more specific role of certain models was singled out:
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It is especially by means of strong models that children are given the opportunity to bridge the gap between informal, context-bound work and the formal, standardized manner of operation through the constructive contribution of the chi0ren themselves. They must be "models of' context problems that will serve as "models for" the pure subject restrictedand applied arithmetic in the pertaining area-multifaceted, in other words (Treffers, 199 1, p. 33).
Despite the reference to the constructive contribution of the students, this description may evoke an image of two independent worlds: that of the student and that of the formal mathematics, which have to be brought together. Such an image, however, would not do justice to the RME adage that students have to construct their own mathematics. To put it differently, formal mathematics is not something "out there" with which the student has to connect. Instead, formal mathematics is seen as something that grows out of the students' activity. The students are expected to develop formal mathematics by way of mathematizing their own informal mathematical activities. The notion of two disjunct worlds as it is taken from traditional mathematics education would not fit here.
The objective of the RME approach is that students experience formal mathematics no differently from informal mathematics. Freudenthal (1991) expressed this by speaking of mathematics as starting within common sense and staying within common sense. The premise is that common sense develops as the mathematics-that is, part of this common sense--develops. Within this perspective, we can distinguish between formal and it-form-nl by denoting formal mathematical reasorling as a form of reasoning that builds on arguments that are located in the newly formed mathematical reality. Seen this way, the distinction between informal and formal is a relative distinction-a distinction that is relative to a certain topic and that can be made especially from an observer perspective.
Streefland (1985) argued that models can support the process of mathematical growth: "Reality [is] attenuated into a model, a 'model of,' an after image" (p. 63). The model and the modeling facilitate reflection. Moreover, one may find that the model is also useful in other situations. The model can become exemplary, a "model for mathematical control of new realities" (p. 63). He elaborated: "In the mental shift from after-image to pre-image, realizing and level raising manifest themselves in the learning process. The aforementioned shift in perspective mirrors the new acquired insight in the wider applicability of the model (p. 63)."
In this sense, one can speak of a "model of' informal mathematical activity becoming a "model for" more formal mathematical reasoning. This shift can be exemplified with the fair-sharing situations that Streefland (1991) employed for instruction on fractions. In this setup, problems about sharing pizzas were modeled
'In the translation to English, the Dutch word voorbeeld, literally meaning"example," was translated as prr-image.
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by the students by drawing partitioning of circles that signify pizzas (model of). Later, the students used similar drawings to support their reasoning about relations between fractions (model for).
THE EMPTY NUMBER LINE
Thinking about the transition from model of to model for sparked a new impulse, with the research on the empty number line as a means of supporting flexible mental computation strategies for addition and subtraction up to 100.
Inspired by Whitney (1988), Treffers (1991) introduced the use of an empty number line in RME.
An empty number line can be used to describe the solution of (two digit) addition and subtraction problems by marking the numbers involved and drawing the "jumps" that correspond with the partial calculation. Solving 64 - 29 by subtracting 4, 1 0, 1 0, and 5, respectively, would result in the inscription shown in Figure I. Such a solution method, which could be seen as a form of curtailed counting

down, belongs to the most common informal solution methods that students use. Accordingly, we may discern between collection-type solution methods and linear, counting-type solution methods. In the first type, the student partitions each number into a "10s part" and a "1s part" and then adds or subtracts those separately. In the second type of method, the student increments or decrements directly from one number, as in the previous example. The empty number line sequence is especially thought to fit with counting-type rather than collection-type methods. Moreover, the use of the empty number line not only fits these methods but also supports them. The number line supports the execution of counting-type methods by offering a way of scaffolding both partial calculations and partial results. However, the number line has more advantages. One of these advantages is in the flexibility of the empty number line. The students can adapt the model to their thinking. We may contrast this with the common practice with base-10 blocks if we look at the way these blocks typically are used in instructional practices that aim at preparing students for the column algorithm (while acknowledging that these blocks can be used in a variety of other ways).
In this type of instruction, the students are expected to conform their thinking to the regulations that anticipate the written algorithms. When subtracting 37 from 84, the students are expected to start with the 1s to find that you cannot subtract 7 from 4, then exchange one 10 for ten 1s, which makes seven 10s and fourteen 1s.
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Next, the seven 1s can be taken away, followed by the three 10s, which leaves four 10s and seven 1s, equaling 47.
In the approach of Whitney (1988) and Treffers (1991), the model is not employed to steer the student's thinking; instead, the model is meant to be adapted to fit the student's thinking. The student decides what steps to make and marks the number line accordingly. It may be noted that something similar can be done with base-10 blocks, but Whitney and Treffers both chose the number line because of their preference for counting-type solution methods.
Another reason for choosing the empty number line is that it also can be used to depict more sophisticated strategies such as compensating. For example, 95 - 19 can be solved by first subtracting 20 from 95 and then adding 1. The empty number line can be used to explain and justify this strategy (see Figure 2).
Note that here the number line plays a different role than when used to model a context problem. This difference helps us to come to grips with the shift from model of to model for. At first, the focus is on the relation of thecontext problem to the number line; later, the numerical-mathematical relations become more important. At first, we encourage the student to try to mirror the problem situation by the number line representation; later, we encourage the student to use an incongruent representation if that would facilitate the calculation (see Figure 3).
Given the previous characteristics, the empty number line worked as a catalyst for the way the *RME* community was thinking about models. It helped the researchers realize that the shift from model of to model for concurs with a shift in the students' thinking, from thinking about the modeled context situation to a focus on mathematical relations. In the latter phase, number relations dominate the
75 76 95 FIGURE 2 Compensating
↑ FIGURE 3 Solving subtraction an
78 95 taking away or as a missing addend.
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FIGURE 4 Levels of activity
/ referential
/ task setting
/
use of the number line. Therefore, we can discern two different types of activity (Gravemeijer, 1994a; Gravemeijer, Cobb, Bowers, & Whitenack, in press):
1. Referential activity, in which "models of' refer to activity in the problem setting described in instructional activities.
2. General activity, in which "models for" make possible a focus on interpretations and solutions independently of situation-specific imagery.
These general types of activity can be seen as different levels of activity, which can be completed with (a) a level of activity in the task setting itself and (b) a level of activity in which the student no longer needs a model. Together, these levels can elaborate the distinction between model of and model for by identifying four general types of activity, as shown in Figure 4.
Activity in the task setting involves interpretations and solutions that depend on understanding how to act in the problem setting (often out-of-school settings). Referential activity involves models of that refer to activity in the setting described in instructional activities. General activity involves models for that facilitate a focus on interpretations and solutions independent of situation-specific imagery. The activity of formal matjematical reasoning is no longer dependent on the support of models for to achieve mathematical activity.
Although the four types of activity clearly involve a developmental progression, this does not imply that they involve a strictly ordered hierarchy. In practice, discussions on the level of general activity and of formal mathematical reasoning frequently fold back to referential activity or even to activity in the setting (McClain & Cobb, 1998).
The goal of describing the four levels of activity is to illustrate that models are initially tied to activity in specific settings and involve situation-specific imagery. At the referential level, models are grounded in students' understandings of paradigmatic, experientially real settings. In whole-class discussions, these models are integral to explanations in which students describe how they interpreted and solved tasks centering on the starting-point settings. General activity begins to
Downloaded by [North West University] at 07:05 27 May 2015 emerge as students' reasoning loses its dependency on situation-specific imagery, and the role of models gradually changes as they take on a life of their own. This transition can be seen as a process of reification, wherein the students begin to reflect collectively on their referential activity. In the process, the model becomes an entity in its own right and serves more as a means of mathematical reasoning than as a way to symbolize mathematical activity grounded in particular settings.
Here a connection can be made with the reification processes in mathematics as observed by Sfard (1991). Sfard noted that, in the history of mathematics, repeatedly processes develop into mathematical entities with an object-like character. Examples include the development from counting as a process into numbers as entities, as well as the shift from algebraic functions as recipes for processing an input to obtain an output, to functions as mathematical entities with certain characteristics (e.g., linear functions and quadratic functions).
The analogy with this process of objectivation, or reification, however, begs the question: What is it that is being reified? The analogy with the transition from process to object-like entity, observed by Sfard (1991), illuminates the point that it is not the inscription that is being reified but some activity. To identify the activity, it is necessary to look at the original instructional sequence in which the use of the empty number line is incorporated. In this instructional sequence, the empty number line is preceded by activities with a bead string (Treffers, 1991; Whitney, 1988). Initially, a bead string with 100 beads colored in groups of 1 0 is used for counting games. The beads are counted from left to right, and the number is indicated with a clothes peg that marks the total beads at its left. Students find out that the decimal structure can be used to count efficiently. Using the number line concept, the students solve tasks such as "Count 38 and add 24 more. Which number do you get?" In doing so, they might look for 38 on the bead string (possibly realizing that 38 is 2 less than 40); then, they might add 1 0 beads (knowing that the result will be 48) and another 1 0 beads (arriving at 58); and finally, they might add 4 (possibly as 2 and 2). In this manner, counting games on the bead string anticipate the use of the number line, which is introduced subsequently to model the activities on the bead string.
Initially, in the mode-of phase, the actions on the number line signify actions on the bead string (e.g., actions of counting beads, encompassing resultative counting, counting on, and counting down). It is this activity that is reified. In this manner, acting with the number line functions as an intermediary between counting beads and acting with numbers as elements of a network of number relations. In this sense, the empty number line changes from functioning as a model of acting with the bead string to functioning as a model for reasoning with number relations. Because the beads arecolored in groups of 10, working with the bead string offers the opportunity to develop decuples-referenced solution methods. As a consequence, it may be expected that the students develop a framework of number relations that revolve around the decimal structure of our number system. In conclusion, we could argue
Downloaded by [North West University] at 07:05 27 May 2015 that working with the number line helps students to understand this decimal structure in such a manner that the numbers up to 100 start to derive their meaning from a system of number relations that become meaningful in their own right. In this sense, the goal of the number line sequence corresponds with what Greeno (1991) referred to with his environmental metaphor of number sense.
PROBLEMS OF THE ORIGINAL EMPTY
NUMBER LINE SEQUENCE
Despite all its qualities, however, the empty number line sequence described previously also has some problematic aspects. One drawback concerns the artificial character of the bead string as a sort of tactile microworld. Another drawback is that there is no credible motive. There appears to be no reason for someone to carry out all sorts of counting activities with a bead string. Both points do not sit well with the RME approach, even though it could be argued that the bead string can be seen as a tactile representation of the number sequence that already is grasped by the students.
This critique was the catalyst for efforts to design an instructional sequence that capitalizes on linear measurement as a basis for the introduction of the empty number line (McClain, Cobb, Gravemeijer, & Estes, in press; Stephan, Cobb, Gravemeijer, & McClain, 1998).
To place this effort in the right context, however, I first elaborate on an earlier teaching experiment, in which we tried to circumvent the bead string. In that teaching experiment, we tried to build on activities known under the label "The Candy Shop" (see Cobb et al., 1997). To some extent, this choice was rather accidental. According to the preliminary design, the plan was to introduce the bead string following the candy shop activities. However, the state of affairs in the classroom seemed to justify the introduction of the empty number line as a means of representing the actions that were executed with the candies.
The instructional intent of the candy shop sequence was to support the students' initial coordination of units of 10 and 1. The story revolved around the production and sale of candies that were represented by unifix cubes. Within the scenario of the candy shop, the problem of how to pack the candies was made a topic for discussion. After having discussed the pros and cons of various options, the teacher revealed that the owner of the shop, Mrs. Wright, chose to make rolls of 10. When the students were familiar with packing candies into rolls of 10, other problems were posed within the candy shop scenario. Mathematically, this meant that the students were asked to generate different partitionings of a given collection of candies and to solve problems that involved incrementing and decrementing numbers of candies as required by the packing and unpacking process. The empty number line was introduced in a whole-class mental computation activity called "Target."
Downloaded by [North West University] at 07:05 27 May 2015 In this game-like activity, the task of the students was to determine how much one would have to change a given number to reach a fixed target number. During the Target activity, the ways in which the students solved these problems were symbolized by drawing number lines (see Figure 5).
The empty number line was subsequently extended to 100 during an activity in which it served as a record of simulated transactions in the candy shop. The hypothesis that the empty number line could be grounded in the candy shop activities initially appeared to be confirmed by the facility with which the children, working in pairs, created number lines to tell stories about what happened in the candy shop. However, later events in the classroom led us to revise our view. Problems arose when the teacher posed addition and subtraction tasks by drawing a horizontal empty number line and describing transactions in the candy shop but not acting them out. Here, it showed that the students interpreted the empty number line in a variety of different, yet personally meaningful, ways. The differences in interpretation are illustrated in Figure 6.
One of the tasks posed corresponded to 90 - 8 (i.e., "Mrs. Wright has 90 pieces of candy, and she sells 8 of them"). As part of a counting-back strategy for solving 90 - 8, some of the students interpreted this figure as showing that 3 candies had been subtracted: 89, 88, and 87. Other students argued that 2 candies had been taken away. It further became apparent that, for some students, the 9th decade when counting by 1 0 comprised 80,8 1 , 82, . . . , 89. They appeared to reason that taking away 8 from the 9th decade would leave the candies signified by 80 and 8 1, and so there would be 81 candies left.
In retrospect, we concluded that the students adopted the empty number line primarily as a way of notating and not as a way of modeling a (mental) activity. We inferred that this calculational interpretation is stimulated by the fact that the students conceptualized the candy shop activities from a collection perspective and not from a linear-counting perspective. In the candy shop setting, the students were routinely structuring quantities in 10s and 1s. This split in 10s and 1S then guided their solution methods, whereas the empty number line representation did not lend itself to this type of solution method.
FIGURE 5 Symbolizing a solution
for 6 + . . . = 13 in the context of the
Target activity.
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FIGURE 6 Counting down
The original approach aimed to encourage students to model situation-specific solution methods that were situated in the activity of manipulating the bead string. In that situation, we expected the students to develop solution methods of a linear type, such as counting on or counting down in bigger or smaller jumps while using the decuples as reference points. Next, that activity was modeled by the students. Their past experience of working with the bead string provided the necessary imagery behind the increments on the number line.
In the aforementioned teaching experiment, this imagery was lacking. As a consequence, the students had to create a semantic grounding for the empty number line individually and privately by interpreting it in terms of counting activity. They did this in a variety of ways, which made it impossible for them to communicate effectively about what had previously been an unproblematic activity.
MEASUREMENT AS AN ALTERNATIVE

FOR THE BEAD STRING
In the original instructional sequence, the activities with the bead string provided the necessary imagery. Note, however, that the way the bead string has to be interpreted is not self-evident. In this respect, we thought that measuring might offer a better basis for the imagery behind the number line. I explain this by comparing the bead string with measuring.
Just like the empty number line, the bead string can be interpreted in a cardinal or an ordinal manner. This may be illustrated with an incident during an informal empty number line teaching experiment that was carried out in West Lafayette, Indiana, in 1991. During this teaching experiment, we made a transition from working with a bead string to working with activity sheets on which bead strings were depicted. When the students were asked to mark a number (e.g., 53) they began to discuss the positioning of the mark. Should it go on the 53rd bead or next to it? (see Figure 7).
Eventually, this issue was resolved by the realization that the 53 refers to a cardinal number; thus, there were a total of 53 beads. Actually, this would be expressed best by circling all 53 beads or by putting one marker before the first bead and one behind the 53rd bead. Instead, the simpler convention was established that all beads to the left of the mark would count.
Downloaded by [North West University] at 07:05 27 May 2015 FIGURE 7 Different ways of marking 53 on the bead string.
1 1 1 1 1 1 1
FIGURE8 Different ways of mark-
5 3
ing on a ruler.
What is interesting is that the same issue arises in measurement. Applied to a measurement situation, the question might be: How do we mark the ruler? Do we mark the 53rd interval, or do we mark the boundary between the 53rd and the 54th interval? (see Figure 8).
Note that this relates to a conceptual understanding of measurement. Alternatively, from a mathematical point of view, it is not sufficient for the students to learn to measure accurately. Students will have to come to interpret the activity of measuring as the accumulation of distance (cf. Thompson & Thompson, 1994). The latter implies that each number word used in the activity of iterating signifies the total measure of the distance measured until that moment. Another aspect of a conceptual understanding is that the students conceive of the results of measuring as structured quantities. For example, a distance of 20 units can be thought of as being composed of 20 measurement units or as composed of two distances of 10 units or of distances of 5 units and 15 units.
DESIGN HEURISTIC
Given the similarities between the desired quantitative interpretation of the number line and measurement, it seemed sensible to investigate whether linear measurement might provide a sound basis for the introduction of the empty number line. This is where the emergent models design heuristic can show its usefulness. This heuristic can be used to think through a possible sequence. Starting with the hypothesis that such a sequence could be understood as a transition from model of to model for, guiding questions arise. For instance:
Downloaded by [North West University] at 07:05 27 May 2015 What is the model?
What activity is being modeled?
What is the model a model for?
What does the transition from model of to model for entail?
I now endeavor to deconstruct these questions. I start with the observation that there is a strong similarity between the empty number line and a ruler, and we can imagine a sequence in which activities with a number line build on activities with a ruler. However, from an instructional design perspective, the question arises: What do we take as the model? We could construe a conjectured local instruction theory, encompassing a sequence of two models that each pass through a model-of or model-for transition. Reasoning this way, we might find even more model-of or model-for shifts. One could discern various local shifts. The result of such an approach would be thattheemergent models notion loses muchof its meaning. Instead, I prefer to preserve the model-of or model-for language for the description of a more encompassing process that embraces a substantial instructional sequence. To do so, the idea of a model has to be interpreted rather broadly. This is done by interpreting the empty number line as a schematized ruler. In this manner, both the ruler and the empty number line can be seen as two manifestations of the same model.
If we take the ruler as the model, we must identify the activity in which the ruler is the model. Speculating on the genesis of the ruler in history, it makes sense to take the view that the ruler came about as a curtailment of iterating a measurement unit. So the ruler can be seen as a model of iterating some measurement unit. From our earlier reflections on the empty number line, we already know that the empty number line can be seen as a model for mathematical reasoning in the context of mental computation strategies for reasoning with numbers up to 100.
The connection between the two could be in the relation between measurement as accumulating distance and measurement as a cardinal interpretation of distance on the number line. In the context of measurement, distances on a ruler or number line would have the status of magnitudes. As the context of measuring moves becomes less dominant, the interpretation shifts toward numerical quantities in general. Finally, even the direct reference to specified countable objects becomes less prominent as the numbers on the empty number line start to derive their meaning from a framework of number relations.
In conclusion, we can envisage something like the following trajectory that may be embedded in a story line or in a series of stories:
1. The students start with measuring various lengths by iterating some measurement unit. This activity of iterating is modeled with a ruler.
2. The activity of measuring is extended to incrementing, decrementing, and comparing.
Downloaded by [North West University] at 07:05 27 May 2015 3. These activities are represented with the help of arcs on a schematized ruler.
4. This schematized representation is used as a way of scaffolding and as a way of communicating solution methods for all sorts of addition and subtraction problems. (Note that this trajectory is also in agreement with the reinvention heuristic.)
In such a sequence, special attention should be given to the decimal structure of our number system. In the context of measurement, this can be done by curtailing the iterating activity by means of a larger measurement unit, that is, a unit of 10. Measuring with 10s and 1s can form the basis for the development of a ruler that is structured by 10s and 1s. At the same time, it may support the development of strategies that use decuples as reference points.
A feature that may cause problems in such a setup is the rigid character of a ruler on one hand and the more global character of the empty number line on the other. Positions on a ruler have to correspond to exact distances. However, on the number line the distances between the numbers do not have to correspond exactly with their numerical values. The latter is essential because, otherwise, it would be impossible to represent unknown numbers or unknown differences. On reflection, it makes sense to differentiate between measuring and representing solution methods. Incrementing, decrementing, and comparing originate in measuring. However, when solving this type of problem, the opportunity arises to generate solutions by reasoning. For instance, consider the problem: "What is the result of incrementing 64 by 28?A possible line of reasoning could be as follows:
1. You first measure 64.
2. Then add six 1s, which would get you to 70.
3. Then measure 2 times 10, which would get you to 80 and 90, respectively, and finally, add two 1s, which gets you to 92.
When describing this method on an empty number line, it would be sufficient to show that you start at 64, add 6, arrive at 70, and then add 10, arriving at 80, add another 10, arriving at 90, and add 2, arriving at 92. To try to strive for an exact proportional representation of all the jumps would severely hamper a flexible use of the number line. Therefore, we will have to distinguish between the ruler as a measurement instrument and the empty number line as a means of expressing solution methods. This distinction should be drawn both at the preliminary design level and in the enactment in the classroom. In a global sense, we are speaking of one model; in practice, however, this model manifests itself in various symbolic representations.
In an earlier publication on the empty number line (Cobb et al., 1997), we already acknowledged that the model may take various shapes. We differentiated between the model as an overarching concept and the symbolic representations that are used in the instructional sequence. To describe this counterpart, we used
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FIGURE 9 A sign constituted of a
signifier and a signified. signified
] sm
signifier 2
] sign 1 = signified 2
signified 1
FIGURE 10 Sign 1 as a signified for Sign 2
empty number line = signifier 3
ruler = signifier 2
xlmits
sign 2 = signified 3
] sign 1 = sign1fied 2
FIGURE 11 Preliminary chain of signification
the notion of a chain of signification (cf. Walkerdine, 1988; Whitson, 1997). The concept of a chain of signification is grounded in the idea that a sign comprises a signified-signifier pair (Lacan, 1977; see Figure9). A chain of signification then is constituted when the activity with one sign comes to signify activity with a previous sign. As the previous sign starts to serve as a signified for a new signifier, a signified-signifier pair is formed. This pair constitutes a link in a more global chain of signification (see Figure 10).
In the measurement-mental-arithmetic sequence in question, the chain of signification basically would look like the one shown in Figure 11. Iterating individual units along the lengths of items that have to be measured would be the starting point. This would come to be curtailed to iterating tens and ones. In this way, sign 1 is formed, of which "iterating individual units" is the signified, and "iterating 10s and 1s" is the signifier. The whole sign 1 (consisting of signified 1 and signifier 1) then becomes the signified (signified 2) for the next signifier: a ruler (signifier 2). These two together shape sign 2, which in turn becomes the signified (signified 3) for the empty number line (signifier 3), and sign 3 is formed.
Based on this preliminary chain of signification, we designed an instructional sequence for a first grade classroom. In elaborating the sequence, the actual chain of signification became more complicated than the one sketched previously (see Stephan et al., 1998). This was because (a) we introduced two different basic measurement units and (b) we inserted the so-called measurement stick to facilitate the transition from the ruler (which we called the measurement strip) to the empty number line.
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The measurement sequence started with measuring by pacing "heel to toe." The idea was that measuring by pacing with your own feet offers a natural situation for iterating. Measuring with centimeters or inches would probably relate to familiarity with the centimeter as a sort of measurement strip and would therefore interfere with the intended buildup. Moreover, measuring with one's feet creates the opportunity to raise awareness for the need of a standardized measurement unit. The initial activities were situated in a scenario in which the king's foot was used as the standard measure.
The instructional intent was that the students came to interpret their pacing activity as an accumulation of distances. This imagery formed the basis for the next step in the sequence: measuring with a larger measurement unit. The scenario of measuring the king was employed to make the students think of more efficient ways of measuring. After a whole-class discussion of alternatives, the teacher told the students how this problem was solved in the kingdom. The solution was to make five footprints of the king's foot and to paste these together. This solution was discussed by the students before they began making their own footstrips. With the help of these footstrips, the students curtailed their counting of individual paces to counting by five paces. The intent was that the footstrips were seen as records of pacing, whereas the placement of one footstrip signified five paced feet.
Next, a shift was made to another measurement unit in the context of a story about Smurfs". The teacher explained that Smurfs are small, dwarf-like people and that one particular Smurf was called on to measure a wall. Because of a lack of measurement tools, he used the food cans he had with him. As a consequence, more Smurfs started measuring with food cans. According to the story, these cans happened to have the size of a unifix cube. This made it possible for the students to act out the measurement activities of the Smurfs. Measuring with unifix cubes was curtailed in the same way as the measurement with the footprints. That is, a "Smurf bar" of 1 0 unifix cubes was constructed to represent 10 cans. Again, the story provided a need for this curtailment, and the students were provided with ample opportunity to invent and discuss solutions before the teacher explained how the problem was solved by the Smurfs. This pattern was repeated in all subsequent steps of the sequence. Let me note, however, that during the teaching experiment, more often than not, at least some of the students came up with suggestions that were similar to the solutions chosen in the story.
Measuring with the cubes and the Smurf bar became a matter of coordinating units of 10 and 1. Measuring started by iterating units of 10, which were then adjusted by adding or subtracting 1s. In this manner, measuring with 10s and 1s helped the students in structuring the number sequence up to 100. At the same time, a basis was being laid for the construction of a measurement strip that comprised 1 0 units of 10, each subdivided into 1 0 units of 1 cube. The idea is that mea-
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suring with the measurement strip is grounded in the imagery of measuring units of 1 0 and I. Thus, for the students, measuring with the strip signified iterating with the Smurf bar and individual cubes. The ultimate aim here was that, in measuring the length of an item and reading off a number, the students would signify the space that extends from the beginning of the measurement strip to the line signifying the end of the item. The students were acting in a spatial environment in which space was already partitioned.
In the following activities, a shift was made from actually measuring items to

reasoning about lengths when solving tasks involving incrementing, decrementing, and comparing lengths of objects that were not physically present (e.g., comparing the measures of the heights of sunflowers in the context of a sunflower contest). These tasks offered opportunities for developing solution methods based on curtailed counting (using the decimal structure as a framework of reference).
The shift away from measuring (as an activity) to reasoning about magnitudes was carried on with the introduction of a measurement stick. The measurement stick was introduced as a yardstick that was attached to a quay wall. The measurement stick had to be used to read the height of the water in the canal. (The context of the canal gives rise to all sorts of problems involving the rise and fall of the water level, the drafts of boats, and shallows in the river.) After discussing the students' suggestions about what a measurement stick would have to look like to make it easy to read, the measurement stick (see Figure 12) was introduced as the solution of the person responsible.
The appearance of the measurement stick was to some extent similar to the original bead string. An important feature was that, although there were no numbers, values were easy to read by counting 10s and 1s. The conjecture was that this feature created opportunities for developing strategies that used the decimal structure. Note that the objective was not that students should acquire particular calculational methods. Instead, the focus was on the students' construction of numerical relations that might give rise to these methods. In this respect, attention was focused on activities that involved reading off numbers, indicating numbers, and analyzing number patterns. Numbers close to a decuple, for instance, could be identified by using that decuple as a referent (e.g., 53 = 50 + 3; 38 = 40 - 2). These relations could be exploited when analyzing patterns that correspond to jumps of 10.
Problems in the context of water heights could be solved in two ways: by counting on the measurement stick or by arithmetical reasoning. The empty number line was introduced as a way of symbolizing the measurement strip arithmetical solu-
FIGURE 12 Measurement stick. Note that here the stick is shown in a horizontal position; in the context of the water heights, the stick has a vertical position.
Downloaded by [North West University] at 07:05 27 May 2015 tion methods that were grounded in reasoning with "10s and 1s." (A jump on the number line described a move on the measurement stick that, in turn, was seen as signifying iterating Smurf bars, unifix cubes, or both.) The empty number line was introduced as a way of communicating solution methods but also functioned as a way of scaffolding. Finally, a generalization was made from magnitudes to numerical quantities in general; the students were asked to solve various addition and subtraction context problems while using the empty number line as a means to record their thinking.
On reflection, it can be argued that the new sequence based on measuring fits better with the RME philosophy than does the original bead-string-based sequence. Actually, the measuring-based sequence can be taken as an exemplary case to illustrate the contrast, mentioned previously, between the use of models that are construed as concretized formal mathematics and models that emerge in the process of modeling. Treffers (1987) elaborated on a distinction of enrhodiment versus embeddedness when he compared Dienes's (1963) teaching-learning theory with the RME instruction theory (which relates to the work of Van Hiele, 1973). Treffers wrote:
Whereas in the first phases of Dienes' learning cycles the children come in touch with concretized mathematical structures, with embodiments of games in artificially created environments, the Van Hiele basic levels entail an exploration of those aspects of reality for which the mathematical concepts and structures (can) serve as means of organization: "embeddedness" versus "embodiment"; the naturally organizable versus the artificially organized matter. (p. 275)
CONCLUSIONS
As our example of the measurement-mental-arithmetic sequence shows, the model in the model-of or model-for metaphor has to be understood as a global overarching concept. In our example, the ruler is the emerging model. The ruler emerges as a model of iterating a measurement unit (or measurement units). In this sense, the ruler is grounded in the activity of measuring. Gradually, however, the ruler changes character as the attention shifts from measuring to reasoning about the results of measuring. Finally, a schematized ruler becomes a model for reasoning about flexible mental-arithmetic strategies for numbers up to 100.
In the concrete elaboration in the sequence, the overarching model assumes various manifestations. The series of manifestations can be described in terms of a chain of signification. The chain of signification is in a sense the counterpart, on a more specific level, of what the model is on a more general level. A chain of signification is constituted as a sign that "slides under" a succeeding signifier (Walkerdine, 1988). That is, on one hand, the new signified encompasses the previous sign, whereas, on the other hand, the meaning of the previous sign changes.
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The latter relates to the constitution of a new sign, in which the signifier and the signified are reflexively related; the signified gives meaning to the signifier, whereas the signifier shapes the signified.
From a mathematics education perspective, the notion of a chain of signification is helpful in that it shows how formal mathematical signs can be rooted in concrete activities of the students. The dynamic character of the chain of signification as such justifies the term emergent models. However, the meaning of the label entergerzt is broader. It refers both to the process by which models emerge within RME and to the process by which these models support the emergence of formal mathematical knowledge.
A crucial aspect of the emergent models heuristic is that the shift from model of to model for is reflexively related to the creation of a new mathematical reality. This can be elucidated by the role of the ruler as a model. What was expected was that, in the course of the sequence, a shift would take place in the student's view of numbers as referents of distances to numbers as mathematical entities. This shift involves a transition from viewing numbers as tied to identifiable objects or units (i.e., numbers as constituents of magnitudes-"37 feet") to viewing numbers as entities on their own ("37"). For the student, a number viewed as a mathematical entity still has quantitative meaning, but this meaning is no longer dependent on its connection with identifiable distances or with specified countable objects. In the students' experienced world, numbers viewed as mathematical entities derive their meaning from their place in a network of number relations. Such a network might include relations such as 37 = 30 + 7, 37 = 3 x 10 + 7, 37 = 20 + 17, and 37 = 40 - 3. The critical aspect of this network is that the students' understanding of these relations transcend individual cases. That is, when students form notions of mathematical entities, they come to view relations like those just noted as holding for any quantity of 37 objects (including a magnitude of 37 units).
This shift from numbers as referents to numbers as mathematical entities is reflexively related to the transition from model of to model for. The students' actions with the model foster the constitution of a framework of number relations. However, through the students' development of this framework of number relations, the model can take its role as a model for mathematical reasoning.
In conclusion, this analysis shows that the emergent models heuristic refers to three interrelated processes. The first process consists of the global transition, in which the model first emerges as a model of informal mathematical activity and then gradually develops into a model for more formal mathematical reasoning. The second process involves the constitution of a new mathematical reality, which emerges as reflexively related to the model-of or model-for transition. The third process concerns the emergence of a chain of signification that can be seen as the concrete manifestation of the model in a series of signs, in which each new sign comes to signify an activity with a previous sign.
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Modelling in Cross-Disciplinary Authentic Student Research Projects
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In the Dutch secondary education system, students must carry out at the end of their school career a rather large research or design project to demonstrate their ability to apply acquired knowledge and skills while pursuing a research question or design goal in some depth. They are encouraged to choose the topic themselves and they are to some extent free in setting up their work. Ideally, the students do not only see it as a compulsory subject but also enjoy the stimulating aspects of doing their own research or design. Challenging and authentic projects, which are representative for actual research and design work done by professionals, seem effective in this respect. The focus of this paper is on how the use of ICT can contribute to the realisation of such projects in mathematics and science education and on how it can give students opportunities to take the nature and level of their work close to the characteristics of work of experts in the field. Three examples of students' inquiry work are analysed with regard to ICT usage, authenticity and resemblance with an expert's approach.
1 INTRODUCTION
In the Dutch examination programme of upper secondary education, students are obliged to build up a portfolio consisting of several small practical investigation tasks (4-10 hrs) and one rather large cross-disciplinary research or design project (80 hrs). The main instructional purpose of the small investigation tasks is to give students opportunities to (1) build up general competencies such as research skills, ICT skills, communication skills, and so on; (2) deepen or enlarge existing mathematical and scientific knowledge; and (3) become more proficient in applying knowledge and skills in practice. In the final project, usually done in the last year before the final central examination, students must demonstrate their achieved level of knowledge, skills, and attitude by the creation of a masterpiece of independent work on a topic of their own choice. For example, in a research project students must demonstrate their research abilities, ranging from choosing a manageable problem, formulating a good research question and structuring their work to drawing conclusions and presenting the research results. The final project is part of the examination programme and a sufficient mark is needed for obtaining a secondary school diploma. Notwithstanding that the final project is an assessment, it is seen by many curriculum developers and by many teachers as a chance to expose students to the real world of research and design, and as an opportunity to let students enjoy doing research and development on a subject of their own choice. 'Challenge' and 'authenticity' seem to be useful keywords for describing activities that really turn students on. Because contemporary R&D work makes heavy use of ICT and is often done in multidisciplinary teams, it is natural to raise the following educational research questions: How can the use of ICT contribute to the realisation of challenging cross-disciplinary authentic research projects in mathematics and science education, which are representative for actual research done by real scientists, and how can it give students opportunities to take the nature and level of their work close to the characteristics of work of experts in the field?
In order to be able to answer the above questions, the meaning of the adjectives 'cross-disciplinary' and 'authentic' is first explicated in section 2 in the context of student inquiry work and mathematical modelling. This is especially required because there exist competing views on authentic and realistic student work in research in mathematics and science education. The adjective 'challenging' depends very much on the level of expertise and motivation of the individual student doing his or her research project and is discussed only in an exemplary way. In sections 3 to 5, three illuminating examples of students' inquiry work are analysed with regard to ICT usage, authenticity and resemblance with an expert's approach. Two of these examples are secondary school students' final research projects, one about bungee jumping and another one about gait analysis. A third example, about the chaotic motion of a spring pendulum, comes from a CAS-supported investigation of a first-year university physics student, only three months after entering the university. The analysis of all three projects together leads to conclusions about the use of ICT for realising student research projects that resemble actual research practice.
2 AUTHENTIC INQUIRY BY STUDENTS:
WHAT DOES IT *MEAN*?
Researchers on mathematics and science education use terms like authentic science, authentic practice, authentic learning, authentic inquiry, authentic modelling, and so on (see, for example, Edelson & Reiser, 2006; Palm, 2002, 2006; Roth, 1995; Roth et al., 2008; Schwartz & Crawford, 2004; Vos, 2009; Weiss et al., 2009; Wellington, 1998; Woolnough, 2000). The meaning of the adjective 'authentic' is so diverse, even when connected with a single noun, that it is wise to first explicate the meaning of 'cross-disciplinary authentic student research project' that is used in this paper and to compare it with other uses of this term in inquirybased mathematics and science education.
I refer to an 'authentic student research project' as a students' investigation having the following characteristics:
o Students work on a self-selected, challenging, messy, illdefined, open-ended problem that is rooted in a real life situation instead of a more abstract or ideal situation; [2
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o Students do not follow some standard recipes, but they examine their problem from different perspectives, using a variety of resources and high-order skills.
o A broad range of competencies is required to make the project a success, and one of these abilities is making good use of ICT for information gathering, data acquisition, data processing and analysis, problem-solving, reporting, and communication with team members;
o Students' work is open-ended in the sense that there exist multiple methods or approaches to obtain many possible or even competing results and answers. The student researchers actually decide whether the investigation can be finished for whatever reason;
o It offers students the opportunity to be in contact with contemporary, cross-disciplinary research and to learn about the nature of mathematics and science;
o Students disclose their own understanding through a portfolio and/or a more or less polished product like a report, paper, or presentation.
A project is called 'cross-disciplinary' when more than one discipline contributes in an essential way to the process of coming to an understanding of the problem situation. The student research projects presented in this paper are mainly rooted in applied mathematics and physics. The term 'crossdisciplinary' is used, and not a term like 'interdisciplinary,' to emphasise that all disciplines are required to get satisfactory results: The whole is more than the sum of the parts.
In science education literature much has been written about authentic inquiry that gives students the opportunity to learn science by doing science like 'real' scientists, as opposed to carrying out practical work that follows cookbook recipes and has hardly any resemblance with experimental work done in real laboratories (Roth, 1995; Schwartz & Crawford, 2004; Woolnough, 2000). It is noted that an inquiry approach is effective in developing core skills in students, improves their attitudes towards science and technology, and allows students to get an honest picture of the nature of science (Schwartz & Crawford, 2004). Authentic scientific work is linked with the real practice of doing science rather than with the methods narratives of published research that keep up the myth of the 'scientific method.' Educational research literature (cf. Chinn & Malhotra, 2002) often emphasises the differences between the work of experts and novices, and it questions whether authentic science is feasible in educational practice because of its complexity and crossdisciplinary nature, the problems with assessing student work, and its requirements regarding content knowledge, research abilities, engagement, motivation, and time availability. However, according to Roth et al. (2008) the problem of how to organise authentic science lessons does not lie with the level of agreement between school science and laboratory science but rather with the levels of control, authority, mastery, and authorship that students are enabled to exercise.
Following this line of thinking, the emphasis in this paper is on the students' acts, that is, on what the students really did, which decisions they made, how they used ICT to put their ideas into action, and how their work can be characterised as authentic work, rather than on the details of the mathematical and scientific knowledge that the students applied.
In research literature on mathematics education, the adjective 'authentic' is mostly associated with applications of mathematics and with mathematical modelling. Common ground seems to be Lebow's (1993) description of an authentic activity as experiences of personal relevance that permit learners to practise skills in environments that have considerable resemblance to those in which the skills are used. It is derived from socio-constructivist and RME principles of locating learning in the context of reality and of looking upon mathematics as a human activity in which sense making by students through participation in the mathematical representation process is highly valued. For a review of the different meanings attached to a words like '(in)authentic' in mathematics education in general, and in the teaching and learning of mathematical modelling in particular, see for example (Palm, 2002, 2006) and (Vos, 2009), respectively.
3 PHYSICS OF BUNGEE JUMPING
The work of the student researchers who carried out the bungee jumping project resembles the work of 'real' scientists regarding affective and cognitive processes in authentic inquiry. To begin with, there were (1) the students' intrinsic motivation to select the subject of their investigation; and (2) the generation of their own research question(s). The students wrote in their report that they teamed up to investigate the physics of bungee jumping, triggered by their own interest and the article of Menz (1993) on www.bungee.com.
In particular, they were intrigued by the alleged 'greaterthan-g acceleration' of a bungee jumper. The students formulated the following research question: "How large is the acceleration at a bungee jump and to what extent is this acceleration influenced by the relative mass of the rope and the jumper?" They hypothesised that the acceleration would be greater than g and that this effect would be more dramatic in case the rope is relatively heavier compared to the jumper.
In order to find quantitative support for their hypothesis the students designed an experiment in which they recorded with a webcam the motion of dropped wooden blocks of various weight attached to ropes of various stiffness and collected position-time data through video measurements.
The students realised that working with a scale model of a bungee jumper or a wooden block is not the same as investigating real bungee jumping, but that it would provide them with enough information on what happens in reality and could lead to a good understanding. This indicates a research design in which procedures to address a problem are determined, variables to investigate are selected, control of variables is thought of, and measurements are planned. Note that not the measured position is the variable of interest, but a derived quantity, viz., acceleration. The students had learned in previous video analysis activities that adequate numerical derivates could be obtained from measured position data.
After some trials, the students realised that the stiffness of the rope plays only a minor role and that the mass ratio between rope and objects needs to be large enough to notice an outstanding result. Therefore they repeated the experiment with objects of larger mass ratio. This is another aspect of authentic inquiry: Researchers are responsible for 3]
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detecting flaws in their experimental set-up and must decide how to adjust their original plans. In this case, the students decided to change the mass ratio and to concentrate on the moment when the object has fallen a distance equal to the rest length of the elastic (because they had observed that the acceleration is greatest at this point of the motion). They collected more data and determined the graph of the acceleration at this moment as a function of the mass ratio of elastic and block. They compared their data plot with the graph of a theoretical relationship taken from (Kagan & Kott, 1996).
The students noted that these graphs were alike, with the theoretical values just a bit higher. They attributed the difference mainly to the development of heat during the motion. Again, this indicates a behaviour of the students that resembles the attitude of competent researchers: Comparing own results with work of others and trying to explain differences by scientific reasoning.
The student researchers not only investigated the acceleration of a bungee jumper by quantitative methods, but also tried to explain the greater-than-g acceleration both on qualitative, theory-driven grounds, as well as through a more detailed mathematical model of the phenomenon. The students followed the approach of Kagan and Kott (1996), in which it is noted that here one does not deal with a falling rigid body, but instead with a system consisting of the falling object and the rope. The interested reader is referred to (Heck et al., 2010) for more ICT-supported experimentation and a detailed description of a mathematical model of the system that can be implemented in a modelling and simulation tool to compare experimental results with theory. The main point to make here is that the students' work contains clear signs of their understanding that observation, experiment and theory are related tools for scientific thinking.
The students wrote an article about their work and got it published (Dubbelaar & Brantjes, 2003). This article triggered quite a number of reactions in the journal and for almost a year on Internet. There were complaints about the quality of physics teaching in the Netherlands, arguing that obviously acceleration of a bungee jumper could clearly not exceed the acceleration of gravity and that the students' work

proved that the level of physics education in the Netherlands
had decreased in the last decades. The editorial commentary
on this was subtle, but to the point: “The students who wrote
the paper may consider it a compliment that scepticism overcame professional physicists and physics teachers. That’s
how (or maybe it is just the point that) experienced intuition
can be wrong.” In other words, the students’ work revealed
that in practice, scientists are not as rationally, non-algorithmic thinking and having an open mind as commonly pictured
in literature about science and scientists.
Regarding the use of ICT in this particular student
research project, the following main roles can be identified:
o Internet gave the students access to many of the same resources that professionals use in their research. They
could gather information and get in contact with experts;
o The students used video analysis software, in particular
the video tool of the learning and authoring environment
named Coach (Heck et al., 2009), to collect, visualise,
process, and analyse data. Numerical differentiation,
computing the area under the curve, and function fitting
were the main mathematical tools used in the data processing and data analysis.
o The students used a standard text processor and presentation tool to report their results.
4 GAIT ANALYSIS
In this project, a high school student collected and
analysed gait data in much the same way movement scientists do, namely, via recording and measurement of motions
with a video tool and via electromyography, i.e., measurement of muscle activity. The authenticity of the student project was rooted (1) in the use of inexpensive tools (a webcam, a simple EMG/ECG set and Coach software), which are
on the one hand fit for educational practice and on the other
hand in essence close to the techniques used by biomechanists; and (2) in the student’s use of the theoretical framework,
nomenclature, and research methods of practitioners. That is
to say, the student conducted many aspects of motion analysis herself: She formulated research questions about a selfselected gait pattern; gathered information; designed and carried out experiments; processed, analysed and interpreted data; and finally published results (Heck & van Dongen, 2008).
The main research question in this project was: “How
goes human gait?” It was specialised as follows:
(1) “Which phases are distinguished in the gait cycle?”
(2) “What muscle activity happens during gait?”
(3) “How do bones and joints make gait possible?”
The first and the third sub question were addressed by
video measurement with a webcam of the planar motion of
the leg around the hip and knee joint during normal walk on
a motorised treadmill in a sports centre. The recorded data
were analysed in a knee angle vs. hip angle diagram and the
periodic leg motion was modelled as a force-driven harmonic
oscillator (Holt et al., 1990). The angular motion curve of the
knee joint is described in this model by a sum of two sine
functions, in which one frequency is nearly twice as large as
the other one. This can be verified via a curve stripping regression technique (i.e., repeated sine regression), or via
automated sinusoidal regression based on advanced, highresolution methods. Figure 1 shows the knee joint angle vs.
time point plot for normal walking at a speed of 5 km/h. The
diagram also shows the graph of the best function fit with a
sum of two sine functions. For a description of the experimental setting of the video recording of the normal walk, for
a picture of the measured knee angle vs. hip angle diagram,
and for the connection between points on this parametric
curve and the phases of the gait cycles, the reader is referred
to (Heck & van Dongen, 2008).
Muscle activity is typically studied using dynamic
electromyography (EMG). In case of measurement of activity of superficial muscles, surface electrodes are placed on the
skin surface to detect the electric activity responsible for contraction of muscles (Kamen & Gabriel, 2010). EMG recording is rather difficult because correct placement of electrodes
is critical. Processing and interpreting an EMG for a muscle
is also not easy. This research project offered the student the [4
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opportunity to personally experience such challenges faced
in gait analysis. She not only learned science content by
doing science, but also got a valid picture of how scientific
knowledge is obtained and what the nature and status of scientific knowledge is. These are in fact commonly identified
aims of practical work (‘knowing that,’ ‘knowing how,’ and
‘knowing about;’ cf. Wellington, 1998; Woolnough, 2000).
Although the student found it difficult to process and
interpret the recorded EMG data, she managed to read off
from the EMG signal when muscle activity was on and off in
normal walking and to interpret the EMG signal in terms of
phases and events in the gait cycle. Essential for successful
analysis is that the data acquisition system allows simultaneous measurement with sensors and video capture. In Coach,
the video clip and the measured data are synchronised: This
means that clicking with the mouse on a point on the graph
automatically shows the corresponding video frame and that
selecting a particular frame highlights the corresponding
points in diagrams, when scanning mode is on. Scrubbing,
i.e., advancing or reversing the video clip manually and
simultaneously watching the points in the data graphs that
correspond with the video frame in view, is then an effective
means to reveal that maxima and minima in the clearly periodic, raw and processed EMG signal are consistently linked to
certain gait events. The lower diagram in Figure 2 shows the
graph of the processed EMG signal during one gait cycle:
‘Processing’ means here that the absolute value of the raw
signal is taken with respect to the zero level of the signal (socalled full-wave rectification) and that the rectified signal is
smoothed via a moving average. These techniques exemplify
the use of basic mathematics in biomechanical practice.
Peaks in the diagram correspond with muscle activity and
corresponding gait events have been marked in the diagram.
The most general conclusion of the student, based on the
processed EMG diagram, was that the gastrocnemius is predominantly active during stance phase.
The role of ICT in the gait analysis project did not differ much from that in the bungee jumping project, except that
the hardware and software offered the student here the
opportunity to collect and work with high-quality, real-time
data about human gait in much the same way movement
scientists do. Regarding mathematical content, emphasis was
put in the project on data processing, regression, visualisation
and interpretation of graphs in a real world context.
Figure 1 Knee joint angle vs. time:
collected data and a regression curve.
Figure 2 Processed EMG signal of the gastrocnemius
muscle for normal walking at low speed
5 MOTION OF THE SPRING PENDULUM
A learning activity can also make use of authentic
software tools, i.e., computer environments that are used by
professional mathematicians and scientists on the workplace.
Computer algebra systems belong to this category of tools. A
popular system, Mathematica (www.wolfram.com), is advertised as “an environment for doing anything technical, that is,
not for computing only, but also for modelling, simulation,
visualisation, development, documentation, and deployment.” All of these purposes are relevant for doing research
and hardly anyone ignores the advantages of using ICT for
this type of work. At the Science Faculty of the University of
Amsterdam freshmen get the opportunity to experience this
in a short (10 hrs) CAS-supported investigation. Students can
choose a project out of a given list, work it out in groups of
two, and finally report on it through a Mathematica notebook. Other aims of the student projects are to motivate the
freshmen in their first steps in the discipline and to let them
experience that their present knowledge of mathematics and
science, empowered by a computer algebra system, already
suffices for investigating a phenomenon at a rather deep level. The authentic nature of the activity also lies in ample
choice of exploratory pathways: the students themselves select the research methodology and the level of complexity to
which they feel themselves challenged; a general research
question is already given to direct the research. The fact that
all projects concern well-known, solved problems does not
matter much: the students are unfamiliar with the problems,
they get the opportunity to explore and solve them in their
own way, or they can follow in the footsteps of researchers.
I discuss the work of a physics student who chose to
study the motion of a spring pendulum, for which the given
research question was: “Is the orbit of a given spring pendulum predictable and to what extent is the motion influenced by small changes in initial conditions?” The spring
pendulum is an understandable and yet intriguing example of 5]
Modelling in Cross-Disciplinary Authentic Student Research Projects
International Journal for Technology in Mathematics Education, Volume 17, No X
a nonlinear physics problem that shows chaotic behaviour
(van der Weele & de Kleine, 1996). Although its mathematical modelling is often used in mechanics courses to exemplify the Euler-Lagrange formalism, the model can also be
derived from first principles of Newton mechanics. Figure 3
shows the Cartesian and polar coordinate systems used in
studying the motion of the pendulum bob with mass m that is
attached to a massless spring of rest length l. The spring can
make on its turn a two-dimensional pendular motion around
a fixed point that is taken as origin of the coordinate system.
Figure 3 The spring pendulum configuration
There are two basic forces in the problem when dissipation is not taken into account: a constant gravity force
(where g denotes the acceleration of gravity), and a spring
force that is assumed to obey Hooke’s law, which means that
the spring force is proportional to the extension of the spring
(the proportionality constant is denoted by k). In Cartesian
coordinates, Newton’s second law of motion and decomposition of the forces acting on the mass in the coordinate directions leads to the following system of differential equations:
2 2 2 2
1 , 1
kx l ky l
x y g
m m x y x y
□ □ □ □
= – – = – – – □ □ □ □
+ +
□ □ □ □
JJ JJ
In order to increase the authenticity of the research project,
the student must derive the equations of motion. This is uncommon in secondary school practice: There, mathematical
equations are often given without much explanation.
Hereafter, the student determined the equilibrium positions: there are a stable equilibrium point below the origin
and an unstable point above the origin under reasonable conditions. The student implemented the equations of motion in
Mathematica and verified that the numerical solution of the
differential equations with the spring pendulum bob initially
at rest in one of the equilibrium points remains constant. He
chose for this and for further computations the following
rational parameter values (which we take as well):
2 2
m g k g = = = = 1 20kg, 981/100 m s , 1kg s , 981/1000 m
He motivated his choice of fractions instead of floating-point
number by the intention to work as much as possible in exact
arithmetic, trying to avoid in this way problems with numerical precision and reducing errors introduced by numerical
computation. The noticeable point here is that the student apparently realised that he should distinguish between phenomena that inevitably occur in reality and those that are caused
by inexact, numerical computations. This is a good sign of
reflection about the quality of research. The student’s choice
of parameter values actually implies that a remarkable behaviour of the spring pendulum can be found, ranging from periodic to chaotic motion (cf., van der Weele & de Kleine,
1996). Figure 4 shows rather stable oscillatory movements in
case the bob starts at position (1.45, -1.45) and (0.4, 0.4),
respectively, with zero speed. Figure 5 shows the periodic
orbit of the bob when it starts at (0.1, 0.281058) without
speed and at a point nearby, viz., (0.1, 0.281058). This figure
illustrates what is meant by chaos: A small change in the
initial condition can lead to a dramatic change in behaviour
of the solution. This is not a consequence of numerical computations in a given working precision, but a fundamental
property of the dynamical system.
Being confronted with regular and chaotic behaviour
of solutions of the motion of a spring pendulum, the student
set himself the task to systematically analyse in what regions
chaos occurs. For example, he looked at various starting
points on the x-axis. Figure 6 shows how close periodic and
chaotic motions can be: The starting points for the orbits are
(1.496, 0) and (1.495, 0), respectively.
Figure 4 Two oscillatory motions of a spring pendulum
Figure 5 Motions starting at nearby points close to origin
Figure 6 Motions starting at nearby points along the x-axis
The student explored the motion systematically and
discovered that small deviations from the stable equilibrium
point and large extensions of the spring pendulum do not
lead to chaotic behaviour. He also concluded that there are
regions in space where chaos occurs and where chaos hardly
occurs. Of course he did not have the background knowledge
to get advanced results. But this was not the aim of the
project and lab assistants better make not many suggestions
in such directions. Instead, they should give students ample
opportunity and encourage them to discover things on their
own, explore ideas, and enjoy doing investigations. [6
André Heck
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6 CONCLUSION
The educational issue in the described students’ investigative work is the ICT-supported interaction between experimental work and mathematical modelling, in which the obtainment and interpretation of results is based on methods
from mathematics and science. The fact that students must
apply their mathematical and scientific knowledge in a meaningful way in a concrete context leads at the same time to
consolidation and deepening of this knowledge. The examples in this paper show that ICT can contribute to the realisation of authentic inquiry and raise its level by allowing students to gather information about the subject and be in contact with experts; to collect real-time data of good quality; to
process, analyse and visualise data; to do computations that

are otherwise impossible; to build and use computer models
of dynamic systems in much the same way as professionals
do; to compare results from experiments, models, and theory;
and to report results. Besides, students can develop, practise
and demonstrate the following research abilities:
o formulate good research questions that guide the work;
o design and implement an experiment for collection of
relevant data;
o apply mathematical knowledge and techniques, and science concepts in new situations;
o effectively use a computer in research work;
o construct, test, evaluate, and improve computer models,
and have insight in their role in science;
o interpret, theoretically underpin and reflect on results;
o collaborate with others in an investigation task.
The student-driven experimental design, the modelling process, the underlying thinking processes, the discussions with
peers, the effective use of ICT, and the improvement of students' mathematical and scientific literacy in the research are
considered more important than the obtained results.
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BIOGRAPHICAL NOTES
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Codes (21)###emergent modelling, ###guided reinvention, ##real life, #RME, #RME theory, $$$approaches to teaching Mathematics, $$$modelling, $$meaningful learning of Maths, $factors that influence teaching and learning, $learner aspects, $mathematics, $mathematics content, %role players, &motivation, &pedagogy, @@challenges when working with ICT, @@ICT systems: virtual learning environment (VLE), @@value of using ICT, @Devices, @Tools, =methodology

Memos (0)

Document body:

Applets for learning school algebra and calculus
Experiences from secondary school practice with an
integrated learning environment for mathematics
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Abstract. In recent years, the use of ICT tools for learning of mathematics received a lot of attention. New computer programs include digital
chapters for existing schoolbooks, Java applets, dynamic software for geometry and statistics, computer algebra systems, computer aided testing
and assessment systems, and many more. In this paper we will focus on
the development and use of Java applets for learning and practising algebra and calculus at secondary school level. We have explored in what
ways applets can help student to develop algebraic understanding and
algebraic skills. Especially applets that provide feedback with scoring
facilities and that are fully integrated in a learning environment with
which students and teachers are familiar have great potential. Of course,
teachers should be able to create exercises and tests based on their own
ideas: the customizable SCORM compliant Java applet generator makes
this possible. We will discuss our experiences from development work
and classroom tests during the GALOIS project.
1 The GALOIS project
Since December 2004 the GALOIS project aims to realize an integrated learning
environment for learning and doing mathematics in which
- students can practise mathematics anytime and anywhere;
- mathematical assignments can be generated 'at random', so that the amount
of exercise material becomes unlimited;
- activities and answers of students are stored;
- "intelligent" feedback on students' work is given automatically.
We wish to reach these goals by using open source technologies and open standards. We try to translate and extend existing technologies into concepts that
work in the practice of school mathematics. We also make ICT-rich learning
materials and try them out in the classroom. In this paper we describe our
classroom experiences with exercise material at secondary school level. We discuss the following topics in the given order:
- the design and use of mathematical Java applets from the Freudenthal
Institute, called WisWeb applets [1],2
- the use of the 'Digitale Wiskunde Oefenomgeving' [Digital Mathematical
Exercise Environment] developed at the Freudenthal Institute [2];
- the use of the SCORM standard [3] to make teaching and learning with
WisWeb applets fully operational in an existing virtual learning environment,
in this case the open source system Moodle [4];
- intelligent feedback in the JavaScript-based testing and assignment environment DITwis [5], which is also integrated in an existing VLE, in this case
Moodle again, via the SCORM standard;
- the design and use of a customizable SCORM compliant applet generator
for learning school algebra and calculus.
By the way, GALOIS is a Dutch acronym for Ge¨integreerde Algebra¨ische LeerOmgeving In School [Integrated Algebraic Learning Environment in School].
The Dutch Ministry of Education, Culture and Science financially supports this
innovation project in the Vooruit! framework until 2007. For more information we
refer to the website www.stmichaelcollege.nl/galois/index.php?lang=en.
2 Design and use of WisWeb applets
In the framework of the projects WisWeb and Welp [1] a large number of Java
applets for learning mathematics have been developed by a team of researchers
and curriculum developers at the Freudenthal Institute in collaboration with
teachers at secondary schools. The last three authors of this paper were involved from early days, tried out prototypes of these little computer programs
in the classroom, and improved the software and accompanying instructional
materials. Knowledge on how to use the software to enrich the learning was
built up in this way. The use of applets for learning mathematics was expected
to contribute to visualization of mathematical concepts, exploration of context
situations (including purely mathematical contexts), and to bridging the gap between informal knowledge and formal mathematics. The Realistic Mathematics
Education (RME [6]) approach was the key theoretical framework for the design
of the software and of the instructional material, and for the flanking educational
research. Considerable eijort was put into the design and field testing of applets
for students to develop algebraic understanding and skills, because this topic is
for ages an important topic in mathematics education, but at the same time it
is an obstacle for many a student. This kind of applets were used and further
developed in the GALOIS project.
Two kinds of applets can be distinguished: model applets, which help to develop mathematical understanding (concepts), and exercise applets, which support the development of mathematical skills. In model applets visual representations are used as models for mathematical concepts. Within these representations
students can work on the basis of their own ideas and experiment freely. An example of a model applet is the 'Geometric Algebra 2D' applet shown in Figure 1.
It allows adding and multiplying numbers and variables by combining line segments to new line segments and rectangles. The lengths of the line segments and3
the areas of the rectangles are displayed as expressions. By splitting and joining
rectangles students can develop insight in equivalence of various expressions.
Fig. 1. The 'Geometric Algebra 2D' applet.
A model applet such as the 'Geometric Algebra 2D' applet has an open
nature: it allows students to work with it on the basis of their own constructions
and productions. The ICT tool is only expected to help students to mathematise
the situation on the basis of their activities. Activity sheets may guide students
in making progress from informal to more formal mathematical activity. We refer
to [7, 8] for further reading about the concept of the emergent-modeling design
heuristic, in which the objective is to help students model their own informal
mathematics activity and gradually develop this model into a model for more
formal mathematical reasoning.
In the second class of Java applets that have been developed, the exercise
applets, this open structure is replaced by a more closed structure. In exercise
applets, a number of tasks of certain type are given to the students, their work is
checked and feedback is provided. Figure 2 illustrates the 'Area Algebra' applet,
in which the area model is used to verify equivalence of algebraic expressions.
The main task is to find the correct expressions for areas and edges of rectangles.
The applet is an interactive version of the well-known geometrical representation
of the distributive law of algebra. A formula like 3(y+2) is treated as the area of
a rectangle with length 3 and width y+2. This results in a stepwise formula with
a part equal to 3y and a part equal to 6. Students intuitively learn and see that
3(y + 2) is equivalent to 3y + 6. Of course, this model has some limitations when
it comes to negative numbers, but then it can easily be replaced with a simple
table. Feedback in this applet is direct: green symbols indicate which part of
the answer is correct, and red, orange and green circles around the assignment4
numbers indicate wrong, incomplete and correct answers. This stimulates the
students to go on and try another one, and to get as many correct answers as
possible. Also very important, the stored feedback provides both the student and
the teacher detailed information about where and what kind of mistakes have
been made. The screen shot shows an example: maybe the student had simply
forgotten to fill out a text box in the rectangle; maybe he or she had worked in
a too formal manner and written 32 with 3 · 2 in mind, just as one writes 3y
when the product is actually meant.
Fig. 2. The 'Area Algebra' applet with immediate feedback.
The method of completing squares for solving quadratic equations links up
well with the above area model. It is supported in the exercise applet 'Quadratic
Equations' shown in Figure 3. The applet presents a quadratic equation that is
to be solved step by step. With the red buttons on top, the student can select an
operation that is to be performed on both sides of the equation. The goal is to
simplify the equation and eventually end with x = solutions. With this applet,
a student has to think and decide what operation to perform. Two versions of
the applets are available: one in which the computer performs the calculation
at each step, thus allowing the student to concentrate on the solution strategy,
and one in which the student has to do the calculation himself and the program
only verifies whether intermediate expressions are still equivalent. A 'gaming'
feature can be added to the second version by letting students earn points for
entering correct intermediate results. Using help or letting the computer perform
an intermediate step costs points. The idea is that changing a set of mathematical
exercises into a game with various levels motivates students to work hard.5
Fig. 3. The 'Quadratic Equations' applet, using the completed squares method and

the balance strategy for solving equations.
As was noted before, the Java applets have been published on the WisWebsite (www.wisweb.nl) and have been field tested extensively at secondary schools in collaboration with teachers and educational researchers [9-11]. We quote some findings on the use of school algebra applets [11]:
"It was found that the use of applets has add-on value: they are fun and motivate students; they allow students to work at their own level of thinking and thus better address individual dijerence between students; the visual, interactive and dynamic features of applets makes the mathematics more easy to understand; thanks to the calculation power of the applets one can focus on the mathematical concepts and models; students are more creative and get more self esteem; the applets form a model students can fall back on; the practice and feedback features are much more powerful than pencil-and-paper exercises. However, to take fully advantage of these opportunities of applets, applets should be integrated in the daily mathematics class routine. Using the computer should not be a voluntary thing that is extra beyond regular math class."
The last issue played an important role in the GALOIS project. The successes with mathematical applets and other ICT tools actually lead to higher expectations. For a teacher it would be nice if he or she could review after a computer aided lesson what the students actually did, what progress they made, which problems arose during learning and to which mathematical subjects attention must be paid in the next lessons. Also, if assignments were given outside of class, this mechanism of 'student tracking' would aid a teacher in determining student ability. An even more basic advantage would be the possibility of checking whether a student has actually done his or her homework. At secondary school level -- with students at age between 12 and 18 years -- this added bonus is often overlooked.
All this would enable teachers to use their actual 'contact time' more ejj-ciently. On the other hand, students also benefit from monitoring student performance: a digital archive of computer based activities gives these activities a6 firm place in the curriculum and improves the learning process while it enables students to review mistakes made and it stores feedback. This systematic approach within one learning environment enhances the already positive ejects of direct, instantaneous feedback.
Many Virtual Learning Environments (VLE) store student work, and it seems that the use of VLEs in secondary education in the Netherlands is on the rise. Nonetheless, with these environments, schools still have to add their own educational tools and lecture notes. Reuse ofICT-components and exchange of learning materials with colleagues without conversion problems promotes this. One could even argue that a VLE only deserves its name when one can actually learn within the system, which implies that an integrated learning environment actually has content. A logical next step in creating an integrated learning environment for mathematics is to see whether mathematics programs can communicate with standard VLEs to store user feedback and aid sharing of content. For this we used in the GALOIS project the SCORM standard.
3 The DWO math environment
The DWO math environment [2] is a web-based application in which a student logs in on a central server, works with an applet, and automatically receives feedback from this applet. Student results are stored so that students can stop working in the DWO and return later, without loosing any results. Teachers can monitor the progress of individual students, but also of entire classes (Figures 4 and 5 show examples).
At the secondary school St. Micha¨el College, the DWO math environment was used for example in the second grade (12 to 14 year olds) in combination with written material, replacing two chapters on algebra. This first learning unit gave students a first acquaintance with mathematical expressions and above all the important mathematical concept of 'variable'. It also aimed to instil some 'symbol sense' and the ability to rewrite mathematical formulas. One of the applets used was the 'Area Algebra', which we described in the previous section (Figure 2). Students were given the opportunity to go through the instructional materials at their own pace, not only at school but also at home. Because the DWO math environment stores the results and feedback of the exercises of all students, the teachers could not only see which assignments students answered incorrectly, but also watch the mistakes in detail.
As a matter of fact, just the knowledge that most pupils did not have any problems with certain subjects was valuable information in practice, too: the teachers could pay more attention to the things the students did not know or had not mastered well enough, instead of teaching about things the students already knew. Of course, in the classes there were dijerences in the progress that students made because they had dijerent levels of 'talent' for the subject. Individual results from students in one class (Figure 4) and in other classes (Figure 5) helped with tracking student performance individually as well as globally, and at a more detailed level than in traditional classroom instruction.7
To correctly interpret the results shown in Figures 4 and 5, it is important to know that in this case teachers had given the students homework to make the first two parts of the applet, resulting in the highest attainable score of 42%. In this case five out of seven classes score higher than the expected percentage. Upon even closer inspection 25% of all students correctly made more than two thirds of all assignments, and 5% of all students even scored 100%. This is a clear sign that the applets highly motivated the students, even beyond the requested activity level.
Fig. 4. The results of all students in one class using the 'Area Algebra' Applet.
Fig. 5. The results of all 2nd grade classes using the 'Area Algebra' Applet.8
In summary, use of the DWO helps the teacher to get good insight into the learning processes of pupils and to make grounded choices in the construction of his or her lessons. With the possibilities of such an interactive environment the polarity between classroom and individual teaching has to be reconsidered. Teaching can shift from a classroom approach, with instruction and discussion, to a more individualistic approach with custom feedback. To students and teachers alike, the DWO is seen as a clear and intuitive environment with not too many options and features. Administration of the DWO is in the hands of an external organization (in this case the Freudenthal Institute) and teachers do not have to concern themselves with the technical side of things.
It also is possible to construct new assignments within a couple of Java applets. This feature aids a teacher in making custom made assignments for every school level within a given subject. To evaluate the performance of students it is also possible to give an exam within the DWO environment. A teacher makes a new module that can be accessed during a limited time span. In this module buttons for feedback and marking have been removed, so now a student has to do everything himself. A teacher does have access to the marking button to facilitate the marking of these assignments.
There are some disadvantages, too. When a school already uses a VLE, with all kinds of communication facilities such as email and bulletin board available, it could be considered cumbersome that yet another environment is added to the already long list. Especially when more and more persons start learning and teaching with the aid of computers, the necessity of an integrated and uniform learning environment, becomes more and more apparent. In some cases it is preferable that administration of the system lies with the own organization (for example when one wants to indicate whether parts of a learning module belong to a basic or an advanced track). These pros and cons of the DWO environment were also mentioned during interviews with teachers and students. We refer to [12] for details about practical problems, user wishes and other remarks that came up on usage of the DWO math environment in classroom experiments.
4 Using SCORM to monitor student performance with applets in an existing VLE
A big advantage of using an existing virtual learning environment with which students and teachers are familiar is that many things run very smoothly: the users know how to log in on the system, how to send emails to groups of student or to individuals, how to work with other built-in communication tools, and how to inspect students' results and export them into the school administrative system. There are a couple of requirements when aiming to integrate applets into an existing learning environment:
- An applet and the VLE have to be able to communicate with each other;
- It should be possible to share the content in various VLEs, for it could very well be that existing programs and learning environments change over the years at school;9
- It should be possible to use this mechanism for other applets or computer programs.
The use of existing standards seems to be the best method to reach these goals, with a clear distinction between the role a VLE plays and the role the program plays. This distinction also means that suppliers of learning environments can operate independently from the suppliers of educational programs, but with the certainty that both work well together. A well-known and often used model is the Sharable Content Object Reference Model, abbreviated as SCORM [3]. It aims to integrate several specifications into one standard describing technical requirements for storing digital learning materials and the interaction between VLE and these materials. SCORM 2004 consists of seven dijerent standards, including requirements for
- metadata (What type of material? What is the target audience? Who has made this material? How can use this material? etc);
- constructing SCORM packages;
- for navigation through the package;
- for 'running' the package in a particular VLE.
Many system suppliers already support the SCORM model, e.g., Moodle [4] and Blackboard [13]. Within the GALOIS project we adapted applets and exercise programs so that they would be SCORM compliant. We intended to use SCORM compliant packages in Moodle, which is called MichelangElo at the St. Micha¨el College, but by using a standard we wanted to ensure that our results could be used in any SCORM compliant learning environment.
Fig. 6. The 'Herleiden' [Rewriting] applet in Moodle.10
One of the applets that we made SCORM compliant is the 'Herleiden' applet, shown in Figure 6. This applet was actually used in a fourth grade class to practise rewriting of given expressions and to reinforce symbol sense. Examples of given assignments concern expanding and rewriting mathematical expressions like 2·3(2a+2b)-2c+a·(a+2), (a+b)(b+c)-2ac, and (2x+y)
2 -(2x-y)
2
. Many
students had problems with this type of assignments, despite the attention given to formula manipulation in the years before and the successes at that time. A common mistake in teaching and learning: not only does one have to teach a certain mathematical skill or ability, it also has to be maintained. Practice makes perfect, but in this case it would be helpful if students could practise a lot without giving the teacher an unnecessary extra workload. A web-based exercise program makes this possible. In our school situation, students could practise rewriting formulas during computer workshops at school as well as at home. Students' scores and results were stored in the system, which enabled the teacher to see whether students have a good command of the algebraic skill of removing brackets and combining like term.
Within MichelangElo it is also possible to export results to text and spreadsheet format. With slight modifications they can even be exported into existing school administration systems. When giving assignments, there naturally are students who do not do their homework. Low scores easily reveal this. Moodle's internal messenger can be used to address these students. Problems are also discovered easily, especially because 'the road to an answer' is also stored in the VLE [14]. For example, some students still made the common mistake to rewrite
(2a + b)
2
as 4a
2 + b
2
. This misconception was addressed in the next lesson by using the area model to explain why this equivalence doesn't hold. We noticed in practice that the students -- at the dijicult age of 16 -- were highly motivated by this novel way of practising an otherwise dijicult and boring subject; they found the applets attractive and motivating. The short-term eject of using the applets in the VLE was that students seemed to get a better understanding of the structure of formulas and that the teachers could monitor student performances and address common mistakes more easily. All this happened within a VLE that was already in use at school and that both teachers and students were familiar with.
5 Intelligent feedback
A teacher must be all ears and have eyes in the back of one's head during classroom work when he or she puts ejort in interactive education that takes individual dijerences of students into account. Occasionally -- mostly when working with small groups or in a face-to-face situation -- one gets the feeling of having realized 'customized education' in which dijerentiation based on the needs of the students came into being. Normally, a teacher is forced to aim at the average student or the interaction is merely a random check. Issues like a large number of students in a class, a limited number of contact hours, time pressure, and so on, often make things more complicated.11
Whether ICT can make the dijerence here is not clear yet. Of course, a modern teacher benefits nowadays from all kinds of computer programs and Internet (for lesson material and communication), but the optimism that ICT can contribute to ejective teaching and learning is not shared by everybody.
It is tempting to state that an online exercise environment enables students to practise mathematics independently and as much as needed, without increasing the work load of a teacher. But often the interaction is rather primitive and limited: answers are marked correct or incorrect without any explanation or further information. The student is left with questions like "What's wrong? Should I type a comma or a decimal point (or the other way around)?", "Is my answer not accurate enough (or too accurate)?", and "Did I make a mistake and, if so, what kind of error: a careless mistake, a reading error, or a conceptual misunderstanding?" When they call in a teacher he or she may hope to find out quickly where the shoe pinches.
An interesting question is to what extent the expertise of a mathematics teacher, i.e., the knowledge about mathematics, teaching and learning, and about the students, can be built into an exercise environment in a such that student receive useful feedback when they make a mistake. We refer to this as 'intelligent feedback' and 'tailored feedback'.

In education, feedback is inextricably bound up with learning processes [15]:
It is usually meant to give students insight in their own learning process or
functioning, to confirm, strengthen or transform their knowledge, and to give
teachers insight in the competencies of their students so that they can make their
teaching more ejective. In this paper we restrict ourselves to 'local feedback', i.e.,
to feedback that is based on the typed answer or another interaction provoked
by a question or task. Dijerent forms of local feedback can be distinguished
and often more than one of these forms is present in a single reaction. Some
examples:
- conceptual: explanation or elucidation of a particular reaction (comments on
the accuracy ofthe given response, instructions of how to input mathematical
expressions, etc.);
- clarifying: supportive remarks concerning content (hints, references to learning materials, etc.);
- evaluative: information about total score, time spent on the task, and so on;
- ajective: for example, an encouragement or a indication of correctness of
intermediate results;
- advisory: think of suggestions for further practice.
We speak about intelligent feedback when it is based on an analysis of the
student's answer or action and when it goes beyond a simple correct/incorrect
sign or a mark. For example, an incorrect answer is annotated with a short
explanation of what might have gone wrong. Also, a hint might be given or a
student gets a second chance. Another example is the highlighting of a correct,
but mathematically inferior answer (e.g.,
2
4 and
2
50 are usually simplified to
2 and 5
2
2, respectively). Intelligent feedback is a detailed reaction based on the
following.12
1. expert knowledge about the mathematical content;
2. a model in which common mistakes and actions are rubricated;
3. knowledge about learning and instruction so that the most suitable feedback
can be chosen.
In the GALOIS project we have experimented with tailored feedback for
exercises in algebra and arithmetic. For an algebraic example we refer to [16].
Table 1 shows various forms of feedback on typical answers to an arithmetic
question.
Question: Your bicycle tyre has a puncture. Each minute 8% of air escapes. What
percentage has escaped after 11 minutes? Round your answer to the nearest integer.
Response Feedback Explanation
60 Excellent! correct answer
60.04 Fine, but too accurate too accurate answer
60.04 Your answer cannot be
interpreted; use the decimal point
not recognized by parser
59 or 61 Close to the correct answer. A
round-oj error?
probably a round-oj error
40 This amount of air is left well-known mistake
133 Confusing addition of 8% ? 1.08
11
= 2.33; well-known mistake
88 No, this is not the way it goes. Use
the growth factor.
notorious mistake (8 times 11)
other Wrong answer; have again a close
look at the section on exponential
decay
God knows what went wrong.
Table 1. Example of tailored feedback on students' responses.
This form of feedback is not in the future. Tailored feedback exists in the
computer algebra based testing and assignment environment STACK [17] and
in the custom-made JavaScript-based testing and assignment environment called
DITwis [5]. At first, DITwis was developed as an independent means to practice
and test certain concepts without teacher's intervention. However, we thought
it would be useful to see whether integration of DITwis into an existing VLE
would be beneficial. Again we made this program SCORM compliant. We used
the system in a third grade (trigonometry) and a fourth grade (combinatorics
and probability) class. The advantages of integration soon became apparent, as
well as (temporarily) some disadvantages. Many of these findings overlap with
the aforementioned ones, with a positive emphasis on: better insight in student
work and problems that students face; the possibility of practising anytime,
anywhere; the added bonus in the communication facilities of a standard VLE.
6 Design and use of a customizable SCORM compliant
applet generator
It would be even better if an applet could be configured within the learning
environment in such a way that custom assignments could be made. This pos-13
sibility has been successfully explored in the last phase of the GALOIS project
in 2006-2007. In this section we discuss a prototype of a customizable SCORM
compliant applet generator for learning school algebra and calculus.
The primary design goal was to enable teachers to create exercises and tests
for their students similar to the existing WisWeb applets, but set up according
to their own ideas. WisWeb applets dealing with the same mathematical topic
vary with respect to feedback, ranging from marking correct/incorrect only to
symbols indicating (in)correctness of intermediate results, and with respect to
marking of answers (e.g., dijerent marks for fully simplified answers and correct,
but not completely worked out responses). We wanted to leave the decision on
the level of feedback and the marking system to the teacher who creates algebraic exercises and tests. Furthermore, teachers should be able to create SCORM
compliant applets so that they can use them in any virtual learning environment
that supports this standard. In this way, teachers could exchange their materials
more easily. Since teachers are not computer programmers, the process of creating SCORM packages containing Java based exercises should be as simple as
possible. Teachers should be able to concentrate on the mathematical contents
and instructional design instead of spending time and ejort on technicalities. A
teacher should also be able to look at the end result through the eyes of students. Keeping in mind that most teachers have no programming experience, the
process of creating randomized questions should be kept simple, too.
Let us first look at a real example of an exercise on computing derivatives,
which was used in a test for 4th grade pre-university students. Figure 7 is a screen
shot of a prototype of the customizable SCORM compliant applet generator.
Fig. 7. Creating an exercise with the applet generator.14
When this exercise together with all the other exercises in the test is saved,
the program generates a SCORM package that can be uploaded into a VLE that
supports this standard. That's all! In such a learning environment, marking of
the students' work is automatically dealt with. Slight modification of the Moodle
environment [14] and other VLEs allowed us to save and reuse the state in which
a student leaves the applet. The advantage of this in school practice was that
a student could temporarily stop working at the computer and return later to
the tasks, that a student could go back to an earlier task and improve his or her
work, and that a teacher could not only see the marks of the students, but also
inspect their work and see how they had done the exercises. Figure 8 shows a
screen shot of how the applet generated looks like when used in Blackboard. The
example shows that the randomly generated exercise was not completed yet, but
was already marked with 6 points; combination of like terms would lead to the
full score of 10 points.
Fig. 8. Use of the generated applet in Blackboard.
Let us examine in more detail the implementation of the question shown in
Figure 8. It naturally includes more than just the question posed to the student.
The minimum information which needs to be entered to create a viable question
is the text of the question, the teacher's provided answer, and the marking. For
simple questions this is entirely straightforward. Having mastered this, a teacher
can seek to include randomly generated parameters, a more complicated expression generation, a tool menu, and more sophisticated marking of (intermediate)
expressions entered by a student. We detail the fields in the applet generator in
this section to illustrate the authoring of questions.
Question variables are of the form parameter = value, separated by a semicolon when they are specified in one line. The value can be a sequence of integers
separated by comma's or a range of values (denoted as a starting value and an15
end value, separated by the ellipsis operator ..). When a question is instantiated, a random choice from possible integer values will be made. It is possible to
use previously defined question variables when making a new one. For example,
i=1..5; j=i^2 will lead to a random choice of i and j such that j = i
2
and
i=1..5; j=1..i will lead to a random choice of i and j such that i ≥ j;
Initial formula is the expression or equation posed to the student in the
work area. You use this field for example, when you do not want the formula in
the question text. In the given example, the function whose derivative must be
calculated is specified in the question text.
Question text is the actual question or exercise posed to the student. It is
more or less a compulsory field which the teacher must enter for a valid question.
It may contain text and formulas. In Figure 7 you see that the formula (invoked
by pressing the F button) is a mathematical expression surrounded by hash
signs (#). When the question is instantiated, all parameter values are randomly
chosen, their values are substituted in the parts that are between hash signs,
and basic simplifications (e.g., combining like powers, adding numbers, removal
of superfluous zeros and ones in expressions — think of replacements such as
x + 0 → x, 1x → x, x
1 → x, and so on →) are automatically carried out.
Apparently, the following values were chosen for the parameter in the given
example: a = 4, b = 2, q = 2, and d = 4. More than one pair of hash signs may
be present in a formula to allow maximal flexibility in the formula generated. An
example says more than words. Assume that a, b and n are question variables
leading to nonzero natural numbers. In Table 2 are listed formulas with 1, 2,
and 4 pairs of hash signs leading to dijerent formula types (a concrete example
is given for a = 4, b = 6, and n = 12).
Formula Explanation Sample
#a#/#n#' + #b#/#n# The task 'as it is'
4
12 +
6
12
#a/n# + #b/n# Simplification of fraction, but no addition
1
3 +
1
2
#a/n + b/n# The final answer
5
6
Table 2. Example of the use of hash signs in formula generation.
Correct answer is another compulsory field that the teacher must enter; a
checkbox is used to indicate that the answer is an equation instead of an algebraic
expression. By using pairs of hash signs, the type of answer that is considered
as correct in a randomized question may be specified (as we have seen before).
But there exist other ways of specifying the level of correctness. When the box
Equivalence is checked, any student answer that is equivalent to the teacher's
answer is marked as correct and a number of points is scored. When the box
Rewriting is checked, additional points can be earned by the student when
he or she has simplified an intermediate result. In the given example, extra
points are given when the answer is a polynomial in expanded form, i.e., without
brackets. Special forms that have been implemented in the prototype of the
applet generator are expressions in which16
- no brackets are left in polynomial expressions (e.g., (x + 1)
2 → x
2 + 2x + 1);
- powers with the same base have been combined into one power (x
2
x
3 → x
5
);
- a formula has been rewritten such that no powers with fractional or negative
exponents occur (x
−2 → 1
x2 and x
5
2 → x
2/
x);
- a rational expression has been rewritten into a single fraction (
1
2 +
1
3 → 5
6
);
- logarithms have been combined as much as possible into a single logarithm
(2 ln 3 + ln 8 → ln 72);
- only simple logarithms appear in a formula (ln 72 → 2 ln 3 + 3 ln 2).
When the box Exact has been checked, a student can score extra points when
he or she enters exactly the answer that the teacher had in mind . When the box
Steps enabled has been checked, a student can enter intermediate expressions
in the work field, gets feedback on correctness, and may even score points for
correct intermediate expressions.

Levels of sets of exercises can be specified in the left bottom corner of the user
interface of the applet generator. In the example shown in Figure 7, the dijerent
levels have been named after the dijerentiation rule that must be applied. But
one can also divide the exercises in a test into sets of easy and dijcult exercises.
The number of levels and the number of questions at each level can be set by
pressing the up/down and left/right arrows, respectively.
Title of the exercise is optional. It can only contain plain text and no formulas. Like in the question field, you can only use alphabetic characters and digits.
Use of accents and HTML coding is not provided (yet).
Fig. 9. Creating an exercise about addition of two fractions.17
The dedicated use of hash signs in randomized questions and the sophisticated processing of (intermediate) student's answers are very convenient for
intelligent marking of responses. Such intelligent question generation and marking of answers is not easily implemented in computer algebra based systems like
STACK [17] or Maple T.A. [18]. Making an exercise in steps is not possible in
these systems either, unless it has been explicitly implemented by the author of
the exercise. The following example shows how easy it is to implement an addition of two randomly generated fractions and to give dijerent marking when a
student is able to
– write the fractions such that they have a common denominator (2 points);
- rewrite the sum correctly as a single fraction, but not with the smallest
possible denominator (5 points);
– find the exact teacher's answer (10 points).
Figure 9 shows the applet generator in action. Figure 10 shows a correct answer
to the question. The marking changed in the intermediate steps from 2 points
to 5 points, and finally to the full score of 10 points.
Fig. 10. Creating an exercise about addition of two fractions.
The main drawback of the generated Java applets is that there is no computer
algebra system behind it (yet). Thus, equivalence testing is done numerically.
Although this may lead to erroneous results – after all, inequivalent algebraic
expressions may evaluate to the same numerical result – in practice it works
quite satisfactory.18
7 Conclusion
Existing mathematical tools become even more powerful and useful when results
and performance of students can be tracked online. Use of a common standard,
in this case SCORM, enables the sharing of content, but also of storing student performance. Students know that they can be and are monitored by their
teacher. Together with other features in virtual learning environments, especially for communication, students are stimulated to do their homework, or even
more than that. Students do not seem to mind this 'big brother is watching
you' concept. On the contrary, they seem to appreciate the fact that a teacher
has good insight into common mistakes and can use this knowledge to address
certain problems more ejectively. They also seem to find that computer practice
prepares them better for written exams than making assignments from a book.
Computer aided learning is also seen as a more 'fun' way of practising otherwise
boring subjects. Direct feedback adds to this appreciation as well as the fact that
small mistakes can easily be corrected. Using direct feedback, giving students
hints and enabling a 'trial and error' approach makes it possible for students
to explore dijerent strategies for solving a given problem. But exercise applets
without these facilities have also been developed.
In our opinion, digital assignments and diagnostic testing add to the motivation and performance of students in mathematics education. By making use
of computers, the extra time involved for teachers is acceptable. Contact time
can ejectively be reorganized because students that need extra attention are
signalled earlier. One of the success factors of the use of ICT would be that
all teachers in an organization agree on the vision they have on mathematics
education. Ideally, less experienced teachers would be supported by experienced
teachers and experiences are shared alike to evaluate and improve the use of ICT
in the classroom. From technological point of view, SCORM contributes to the
realization of meaningful ICT-rich education. The customizable SCORM compliant Java applet generator allows teachers to create exercises and tests for their
students in an easy way and based on their own ideas. These tests can be exchanged with fellow teachers because of the use of a widely used standard. Field
tests at school confirm that all works well in practice, although many wishes for
improvement of the software arise at this stage of the development work. Linking
the applets to a backbone computer algebra system and having more question
types at one's disposal are two of the envisioned extensions. The developed ICT
tools have a surplus value for mathematics education at secondary school, and
probably as well for other educational levels: experiments at the University of
Amsterdam in remedial math for freshmen and in math intensive courses for
first year chemistry students take place at this moment.
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Empowered
Applets for
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Algebra and
Calculus
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We describe the development and use of Mathematica empowered
applets for learning and practising algebra and calculus at secondary
school level. We discuss how Mathematica can be used as back engine
to evaluate students' answers and provide appropriate feedback. These
applets provide scoring facilities, are fully integrated in a virtual learning environment in which answers are stored, and can be edited by
teachers to create their own exercises and tests. Students benefit from
the applets because the exercises help them to develop mathematical
understanding and skills more thoroughly than is possible with pencil
and paper. Using the applets, a teacher can review after a computer
aided lesson what the students actually did, what progress they made,
which problems arose during learning and to which mathematical
subjects attention must be paid in the next lessons. In other words, the
applets help the teacher to get insight into the learning process of
students and to make grounded choices in the construction of his or
her lessons.

Many students at secondary school level and beyond have difficulties in
mastering algebraic skills. Often the algebraic manipulations are carried out in
a superficial way, using rules that are not meaningfully connected with the
underlying concepts. To help secondary school students to develop real algebraic understanding (insights and concepts) and to learn and practice algebraic skills in sense-making ways, designers of the Freudenthal Institute have
developed several sequences of programs, Java applets, that (1) enable
students to connect the formal mathematical activities in a meaningful way
with the underlying concepts and (2) provide exercises with feedback and
have scoring facilities in a game-like setting. Later these applets were integrated in a virtual learning environment, in which information on the results
and the learning could be captured. At the moment the Freudenthal Institute
is extending this learning environment with authoring facilities, to create new
applets without the need for specialized programming knowledge. This
enables teachers to design their own interactive activities and tests on their
own ideas.
Many of the interactive exercises are using build-in computer algebra alike
facilities for evaluating students' answers and be for generating proper feedback. The evaluation of an answer can be the verification of the correctness
of the final answer to a question, but it may also be the verification of the
correctness of an intermediate computational step entered by a student. For
the more advanced mathematical topics, the need for a professional
computer algebra system like Mathematica to do this job is felt more and
more. As a proof of concept, some of the developed applets are redesigned
for using Mathematica as back engine to evaluate students' answers. The
screen shot in Figure 1 gives an impression of the user interface for the case
of exercising integration by parts.
Mathematica Empowered Applets 3
IMS08_paper_Heck_Boon_vanVelthoven.nb 4/15/08 The Mathematica Journal volume:issue © year Wolfram Media, Inc.Figure 1. The 'Integrating by Parts' applet with immediate feedback
In this paper we will discuss our experiences from development work and
classroom tests. We will pay special attention to the way Mathematica is used
as back engine. We will also exemplify this use of Mathematica in the context
of geometry education. The article is organized as follows. First we discuss
the theoretical framework for the design and use of the educational applets,
their use in the classroom via a virtual learning environment, the feedback
provided in the applets, and the authoring of the applets. Hereafter we
discuss in detail the use of Mathematica as back engine to evaluate students'
answers on the basis of the exercise applet for integrating by parts. Finally we
give a short example of the use of Mathematica in geometric theorem proving
when using a dynamic geometry applet. We conclude with our ideas for future
development resulting from our experience with the prototypical Mathematica
empowered Java applets that have been developed so far.

4 Heck, Boon, van Velthoven
The Mathematica Journal volume:issue © year Wolfram Media, Inc. IMS08_paper_Heck_Boon_vanVelthoven.nb 4/15/08â Design and Use of Applets for Learning Mathematics
â Theoretical Framework
In the last decade, a team of researchers, curriculum developers and software designers at the Freudenthal Institute have developed in collaboration
with secondary school teachers various applets for learning mathematics.
This development effort was made with the expectation that use of such mathematical applets would contribute to visualization of mathematical concepts,
exploration of context situations (including purely mathematical contexts), and
to bridging the gap between informal knowledge and formal mathematics.
The Realistic Mathematics Education (RME, [1]) approach was the key theoretical framework for the design of the computer programs and the accompanying lesson material, and for the flanking educational research. The key
element of the RME approach for the design and use of the applets is that the
problems that they present are real and meaningful to the students and worth
the effort of solving. Considerable work was put into the design and field
testing of applets for students to develop algebraic understanding and skills,
to introduce and strengthen the notion of function, and to develop geometrical
insight and spatial abilities.
Two kinds of applets can be distinguished:
è model applets, which help to develop mathematical understanding;
è exercise applets, which help to develop mathematical skills.
In model applets visual representations are used as models for mathematical
concepts. Within these representations students can work on the basis of
their own ideas and experiment freely. An example of a model applet is the
"Geometric Algebra 2D" applet shown in Figure 2. It allows adding and multiplying numbers and variables by combining line segments to new line
segments and rectangles. The lengths of the line segments and the areas of
the rectangles are displayed as expressions. By splitting and joining rectangles students can develop insight in equivalence of various algebraic
expressions.
Mathematica Empowered Applets 5
IMS08_paper_Heck_Boon_vanVelthoven.nb 4/15/08 The Mathematica Journal volume:issue © year Wolfram Media, Inc.Figure 2. The 'Geometric Algebra 2D' applet.
A model applet such as the 'Geometric Algebra 2D' applet has an open
nature: it allows students to work with it on the basis of their own constructions and productions. The ICT tool is only expected to help students to mathematise the situation on the basis of their activities. Activity sheets may guide
students in making progress from informal to more formal mathematical
activity.
Exercise applets have a more closed structure: a number of tasks of certain
type are given to the students, their work is checked and feedback is
provided. Figure 3 illustrates the 'Area Algebra' applet, in which the area
model is used to verify equivalence of algebraic expressions. The main task
is to find the correct expressions for areas and edges of rectangles. The
applet is an interactive version of the well-known geometrical representation
of the distributive law of algebra. A formula like 3 Hy + 2L is treated as the area
of a rectangle with length 3 and width Hy + 2L. This results in a stepwise
formula with a part equal to 3 y and a part equal to 6. Students intuitively learn
and see that 3 Hy + 2L is equivalent to H3 y + 6L.
Feedback in this applet is direct: green symbols in the work area indicate
which part of the answer is correct, and red, orange and green circles around
the assignment numbers indicate wrong, incomplete and correct answers.
This stimulates the students to go on and try another one, and to get as many
correct answers as possible. And what is also very important, the stored feedback provides both the student and the teacher detailed information about
where and what kind of mistakes have been made. The screen shot shows
an example: maybe the student had simplify forgotten to fill out a text box in
the rectangle; maybe (s)he had worked in a too formal manner and written 32
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The Mathematica Journal volume:issue © year Wolfram Media, Inc. IMS08_paper_Heck_Boon_vanVelthoven.nb 4/15/08the rectangle; maybe (s)he had worked in a too formal manner and written 32
with 3 y2 in mind, just as one writes 3 y when the product is meant.
Figure 3. The 'Area' applet with immediate feedback.
The method of completing squares for solving quadratic equations links up
well with the above area model. It is supported in the exercise applet 'Quadratic Equations' shown in Figure 4. The applet presents a quadratic equation
that is to be solved step by step. With the red buttons on top, the student can
select an operation that is to be performed on both sides of the equation. The
goal is to simplify the equation and eventually end with x = … solutions. With
this applet, a student has to think and decide what operation to perform. Two
versions of the applets are available:
è one in which the computer performs the calculation at each step,
thus allowing the student to concentrate on the solution strategy;
è one in which the student has to do the calculation himself and the
program only verifies whether intermediate expressions are still
equivalent.
Mathematica Empowered Applets 7
IMS08_paper_Heck_Boon_vanVelthoven.nb 4/15/08 The Mathematica Journal volume:issue © year Wolfram Media, Inc.Figure 4. The 'Quadratic Equations' applet, using the completed squares method and the
balance strategy for solving equations.
A gaming feature can be added to the second version by letting students earn
points for entering correct intermediate results. Using help or letting the
computer perform an intermediate step cost points. The idea, proven to be
true in field tests, is that changing a set of mathematical exercises into a
game with various levels motivates students to work hard.
One of the other exercise applets is the applet called 'Herleiden' (Dutch for
"rewriting") and shown in Figure 5. This applet presents mathematical expressions that are to be simplified step by step, where a student gets in each step
feedback in the from of a cross or checkmark whether he or she is still on the
right track or not. The purpose of this applet is to practise rewriting of given
algebraic expressions and to reinforce students' symbol sense.
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The Mathematica Journal volume:issue © year Wolfram Media, Inc. IMS08_paper_Heck_Boon_vanVelthoven.nb 4/15/08Figure 5. The 'Herleiden' applet with intermediate feedback.
While a student works with an applet (s) he receives instantaneous local feedback, i.e., feedback that is based on the typed answer or another interaction
provoked by a question or task. Different forms of local feedback can be distinguished and often more than one of these forms is present in a single reaction. In Figures 1, 3, 4, and 5 green marks and red crosses indicate which
part of the answer is correct and incorrect, respectively. In addition, red,
orange and green circles around the assignment numbers indicate wrong,
incomplete and correct answers. Another form of feedback in the applets are
messages informing the student that "the notion used is invalid" or that
"superfluous solutions are present in the answer". Via such intelligent, tailored
feedback that goes beyond a simple correct/incorrect sign or a mark a
student gets the impression that a teacher is looking over his/her shoulder to
make useful remarks, to warn for common mistakes, and to give hints if
needed.
Mathematica Empowered Applets 9
IMS08_paper_Heck_Boon_vanVelthoven.nb 4/15/08 The Mathematica Journal volume:issue © year Wolfram Media, Inc.â The Digital Mathematics Environment
The applets can be embedded in a simple electronic learning environment,
which is called the Digital Mathematics Environment (DME [2]; in Dutch called
the "Digitale Wiskunde Oefenomgeving", DWO). The DME is an answer to the
fleeting character of the work with applets, which both students and teachers
often experience. In the DME, a student logs in on a central server, works
with an applet, and automatically receives feedback from the applet.
Students' results are stored on a central server. This allows the student to
review his/her work, correct it, and continue it in any location with Internet
access, particularly at home. The teacher can check and grade the students'
work and (s)he can monitor the learning process of individual students and of
entire classes. An example of an overview of students' results is shown in
Figure 6.
Figure 6. The results of all students in one class using the "Area Algebra" applet.
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The Mathematica Journal volume:issue © year Wolfram Media, Inc. IMS08_paper_Heck_Boon_vanVelthoven.nb 4/15/08â User-Friendly Authoring Facilities for Teachers
A promising new feature of the DME is the authoring tool for designing new
exercise applets. A prototype of this tool was developed in the Galois project
[3]. Lately much effort is put in the further development. The editor of the
authoring tool enables teachers to create exercises and tests for their
students similar to the existing applets, but set up according to their own
ideas. The decision on the level of feedback and the marking system is left to
the teacher who creates exercises and tests. Since teachers are not
computer programmers, the process of creating the interactive exercises is
kept as simple as possible. Teachers should be able to concentrate on the
mathematical contents and instructional design instead of spending time and
effort on technicalities. A teacher or author is able to look at the final result of
the applet creation through the eyes of students. Keeping in mind that most
teachers have no programming experience, the process of creating randomized questions should be kept simple too. Although designed for teachers, at
the moment textbook authors use this authoring tool as well to design the
accompanying ICT-materials for students.
â Interoperability
The two functionalities of the DME (learning environment and authoring environment) can be separated. As an authoring environment, the DME can, on
the one hand, package the designed activities in so-called SCORM-compliant
learning objects [4] that can be used in other learning environments, for
example Moodle or other SCORM-compliant environments. On the other
hand, as a learning environment, the DME can also embed other (SCORMcompliant) learning objects that are not designed and produced with the DME.
This possibility, especially the adoption of the SCORM-standard, facilitates
the collaboration with other systems and designers.
â Description of the Authoring Tool
Figure 7 is a screen shot of the editor of the authoring tool.
Mathematica Empowered Applets 11
IMS08_paper_Heck_Boon_vanVelthoven.nb 4/15/08 The Mathematica Journal volume:issue © year Wolfram Media, Inc.Figure 7. Creation of an exercise with (the prototype of) the applet generator.
We describe the fields in the above editor to illustrate the authoring of questions; more details can be found in [5].
Question variables are of the form parameter = value, separated by a semicolon when they are specified in one line. The value can be a sequence of integers separated by comma's or a range of values (denoted as a starting value
and an end value, separated by the ellipsis (..). When a question is instantiated, a random choice from possible integer values will be made.
Initial formula is the expression or equation posed to the student in the work
area. You use this field for example, when you do not want the formula in the
question text. In the given example, the function whose derivative must be
calculated is specified in the question text.
Question text is the actual question or exercise posed to the student. It is
more or less a compulsory field that the teacher must enter for a valid question. It may contain text and formulas. In Figure 7 you see that the formula
(invoked by pressing the F button) is a mathematical expression surrounded
by hash signs (#). When the question is instantiated, all parameter values are
randomly chosen, their values are substituted in the parts that are between
hash signs, and basic simplifications (e.g., combining like terms, adding
numbers, removal of superfluous zeros and ones in expressions) are automatically carried out.
Correct answer is another compulsory field that the teacher must enter; a
checkbox is used to indicate that the answer is an equation instead of an algebraic expression. By using pairs of hash signs, the type of answer that is
considered as correct in a randomized question may be specified (as we have
seen before). But there exist other ways of specifying the level of correctness.
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The Mathematica Journal volume:issue © year Wolfram Media, Inc. IMS08_paper_Heck_Boon_vanVelthoven.nb 4/15/08seen before). But there exist other ways of specifying the level of correctness.
When the box 'Equivalence' is checked, any student answer that is equivalent
to the teacher's answer is marked as correct and a number of points is
scored. When the box 'Rewriting' is checked, additional points can be earned
by the student when he or she has simplified an intermediate result. In the
given example, extra points are given when the answer is a polynomial in
expanded form, i.e., without brackets. Other special forms have been implemented in the editor.
Levels of sets of exercises can be specified in the left bottom corner of the
user interface of the applet generator. In the example shown in Figure 7, the
different levels have been named after the differentiation rule that must be
applied. But one can also divide the exercises in a test into sets of easy and
difficult exercises. The number of levels and the number of questions at each
level can be set by pressing the up/down and left/right arrows, respectively.
Title of the exercise is optional. It can only contain plain text and no formulas.
Like in the question field, you can only use alphabetic characters and digits.
Use of accents and HTML coding is not provided (yet).
â Mathematica Empowerment of the Applets
The main drawback of the generated Java applets is that there is no real
computer algebra system behind it yet. Some limited computer algebra facilities are embedded, like expanding polynomials, simplifying a limited number
of expression types and performing substitutions. But most of the equivalence
testing is done numerically via a randomized process: random values are
repeatedly substituted for the variables in the expressions under consideration and only when the results are notably different from each other, nonequivalence of the expressions is proven. If the tests are passed for enough
substitutions it is assumed that equivalence holds. In a few cases, formal
equivalence is tested, e.g., in case of arithmetic tasks and when removal of
brackets in polynomial expression is applied. Although this numerical equivalence testing may lead to erroneous results — after all, nonequivalent algebraic expressions may evaluate to the same numerical result — in practice it
works quite satisfactory for simple algebraic expressions like polynomials.
However, when more complicated expressions or mathematical functions
come into play, the need for algebraic equivalence testing is felt more and
more. A modern computer algebra system can do the work in the background. In the implementation of the prototypical Mathematica empowered
Java applets we use the Simplify command to this end.
â A Problematic Example: Integration by Parts
Let us have a look again at the example of the Mathematica empowered Java
applet in Figure 1. The task is to compute the definite integral
∫ (1)
1
2
x
3
ln x „x .
Mathematica Empowered Applets 13
IMS08_paper_Heck_Boon_vanVelthoven.nb 4/15/08 The Mathematica Journal volume:issue © year Wolfram Media, Inc.Integrating by parts leads to the following equivalent expression:
Hln xL (2)
x
4
4
1
2
-∫
1
2 1
x
x
4
4
„x ,
which can be worked out into
4 ln 2 - ∫ (3)
1

2 x
3
4
„x
But how can we easily check equivalence of expressions (1), (2) and (3)?
Randomized equivalence testing is not possible and numerical integration
does not seem appropriate either! Only use of a computer algebra system
can really solve this equivalence problem, which is transformed into a zero
test problem (also referred to as the problem of normal form simplification)
that is in general more easily solved than the problem of bringing expressions
to their canonical form.
á Example of a Mathematica empowered Java Applet: Integration
by Parts
Verification of the final answer and the intermediate results in the 'Integration
by Parts' example shown in Figure 1 is carried out as follows. Whenever the
user enters an expression, the parser checks whether the computer algebra
system is really necessary for equivalence testing. For example, when the
mathematical notation for integration is encountered, the decision is made
that Mathematica is indeed needed and the whole expression that was
entered by the user is translated toward a Mathematica expression. This translated expression is send to a Mathematica server accessible to the user and
running in the background together with the previous expression (translated
as well according to the syntax of the Mathematica language). The computer
algebra system uses the Simplify command to verify whether the difference
between these two Mathematica expressions is equal to zero. If so, the two
expressions in the applet are equivalent and a checkmark indicating the
student that (s)he may be on the right track can be set. If a non-zero difference was obtained by the Mathematica zero test, the two expressions in the
Java applet are most probably not equivalent and appropriate feedback can
be provided. You may be worried that failure of the Mathematica zero test
does not guarantee that the expressions are non-equivalent – after all, you
may write zero in such a complicated way that even the Mathematica
command Simplify or FullSimplify cannot recognize it as zero – but it is
our experience that this does not happen for mathematical expressions at the
difficulty level of secondary school.
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The Mathematica Journal volume:issue © year Wolfram Media, Inc. IMS08_paper_Heck_Boon_vanVelthoven.nb 4/15/08á Implementation Details
Our approach is similar to WebMathematica: the student interacts via a
browser with a Java applet (entering expression, dragging objects, etc.) and
this interaction leads to one or more Mathematica expressions that are then
carried out by a Mathematica program run on a server that is accessible to
the user. The main difference is that we do not use the computer algebra
system for evaluation of each and every mathematical expression. Our
approach is simple: after the Mathematica command is carried out, the Mathematica session is 'cleaned' in that the user can no longer refer to the previous
command or result.
All communication tools are based on Java Technology: Servlet technology,
Java technology and special purpose Java technology developed for example
in the OpenMath project [6]. The Java tools govern the interaction with the
mathematical service offered by Mathematica. They make it possible to
contact the computational back engine, submit queries to the service, and
also take care of handling the responses obtained from this back engine. The
browser-webserver based architecture used for supporting the exercise
applets is shown in Figure 8.
Figure 8. The general scheme of the Mathematica empowered architecture
We explain in some detail the main components shown in this picture.
A web browser with Java support represents the client. It will present views of
the Java applets on an external server to the user, interact with the user, and
communicate user interaction (mainly mathematical expressions entered) to
the server. The responsibility for interaction is mostly on the server side.
The Java applet, which is a either a model applet or an exercise applet,
caters to presentation, communication, and context. It supports a wide range
of actions ranging from entry and handling mathematical expressions to
providing hints and offering graph tools to support the students in doing their
tasks.
The Mathematica web service serves as an interface to the computer algebra
system. We have made use of standard protocols, among which those developed by RIACA at the University of Eindhoven [7]. In short, it is sockets-based
technology that allows a Mathematica service to run the Mathematica
package across the web and that relies on a dedicated program to handle the
actual connection to/from a Mathematica engine. In order to run a Mathematica web service, a web server (e.g., Apache Tomcat) must have been set
up. A main difference with other approaches such as MathDox [8] is that in
our prototypical examples we have not yet used OpenMath technology [6] in
most applets: input to and from the Mathematica engine goes via strings in
Mathematica Empowered Applets 15
IMS08_paper_Heck_Boon_vanVelthoven.nb 4/15/08 The Mathematica Journal volume:issue © year Wolfram Media, Inc.most applets: input to and from the Mathematica engine goes via strings in
the syntax of the Mathematica language.
á Another Example of Mathematica Empowerment: Geometric
Reasoning with a Dynamic Geometry Applet
A dynamic geometry applet lets students build geometric constructions with a
digital ruler and compass. The students can test whether certain assertions
hold in their configuration. The applet is able to verify numerically these assertions for the displayed construction. The students can move the free points of
the construction to other positions and the program will automatically test the
assertions again. In popular packages like Cabri Geometry and Cinderella, a
probabilistic prover is used to verify assertions [9].
A very simple example is the midpoint theorem:
The midpoint theorem.
Let ABC be a triangle and let D and E be the midpoints of AC and BC, respectively.Then the line DE is parallel to the base AB.
Figure 9. Snapshot of the dynamic geometry applet about the midpoint theorem.
A snapshot of the dynamic geometry applet is shown in Figure 9. When
students make the construction, they may quickly conjecture the parallelism
of the two lines drawn. The system offers tools to check such facts and here
the text label in the middle of the screen informs the students that the system
agrees with the assertion of parallelism of the lines. Dragging one of the free
points A, B, or C, may convince the students about correctness of the statement.
The next step is that the students are able to verify their assertions in the
general case. This is where Mathematica comes in. The applet replaces coordinates of the free points with variables and redoes the construction symbolically. Below we discuss the approach that applies for our concrete example.
We may choose a coordinate system such that the coordinates of the five
points A, B, C, D, and E are as follows:
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The Mathematica Journal volume:issue © year Wolfram Media, Inc. IMS08_paper_Heck_Boon_vanVelthoven.nb 4/15/08AH0, 0L
è BH1, 0L
è CHx, yL
è DI
x
2
,
y
2
M
è EI
1+x
2
,
y
2
M
The statement that the line AB is parallel to DE means in algebraic form that
HxA - xBLHyD - yEL - HyA - yBLHxD - xEL = 0 . (4)
Expression (4) is fed to Mathematica.
H0 - 1L+ Iyë2 - yë2M - H0 - 0L +Ixë2 - H1 + xL ë2M 0
True
Because of our smart choice of the coordinate system, the formulas became
very simple and Mathematica could perform the above zero testing by simply
evaluating the expression entered. In general the mathematical formulas
won't be that simple and the use of a computer algebra system is needed.
Suppose that we do not make any assumption on the coordinate system, i.e.,
suppose that the coordinates of the five points are as follows:
è AHxA, yAL
è BHxB, yBL
è CHxC, yCL
è DI
xA+xC
2
,
yA+yC
2
M
è EI
xB+xC
2
,
yB+yC
2
M
In this case we must rely on the Simplify command of Mathematica.
HxA - xBL+IHyA + yCL ë2 - HyB + yCL ë2M -
HyA - yBL+IHxA + xCL ë2 - HxB + xCL ë2M 0
-
1
2
H-xB - xCL +
1
2
HxA + xCL HyA - yBL +
HxA - xBL
1
2
H-yB - yCL +
1
2
HyA + yCL 0
Simplify@%D
True
The boolean value proves that the assertion holds. Especially when intersection points or points that are constrained to a particular line or circle come into
play, one may easily get a full page formula, which Mathematica will happily
simplify to True.
Mathematica Empowered Applets 17
IMS08_paper_Heck_Boon_vanVelthoven.nb 4/15/08 The Mathematica Journal volume:issue © year Wolfram Media, Inc.Alternatively, the constraints for the points D and E can be formulated as polynomial equations.
2 xD - HxA + xCL = 0
2 yD - HyA + yCL = 0
2 xE - HxB + xCL = 0
2 yE - HyA + yCL = 0
Then the GroebnerBasis method can be applied to prove the geometric
theorem (see [10] for a short gentle introduction to the theory): Use the PolynomialReduce command on a GroebnerBasis of the constraints.
polys =
82 xD - HxA + xCL, 2 yD - HyA + yCL, 2 xE - HxB + xCL, 2 yE - HyB + yCL<;
vars = 8xA, xB, xC, xD, xE, yA, yB, yC, yD, yE<;
gb = GroebnerBasis@polys, varsD;
Last@PolynomialReduce@
HxA - xBL HyD - yEL - HyA - yBL HxD - xEL, gb, varsDD 0
True
We end this section with a few remarks on the implementation of Mathematica support in geometric theorem proving. The approach is similar to the
architecture in Figure 8. However this time the OpenMath [6] technology is
used. The Java applet will contact a servlet, sending it the configuration and
assertion in OpenMath style. This servlet then reformulates the assertion in
an algebraic expression also in OpenMath style. With this formula the servlet
will call a Mathematica service as developed by RIACA [7] after converting the
OpenMath expressions to the Mathematica syntax and vice versa.
á Conclusion
We conclude with a summary of our experiences with web-based mathematical tools for learning and practising algebra and with our ideas for future development resulting from our experience with the prototypical Mathematica
empowered Java applets that have been developed so far.
First we quote some findings on the use of school algebra applets [11]:
"It was found that the use of applets has add-on value: they are fun and motivate students; they allow students to work at their own level of thinking and
thus better address individual difference between students; the visual, interactive and dynamic features of applets makes the mathematics more easy to
understand; thanks to the calculation power of the applets one can focus on
the mathematical concepts and models; students are more creative and get
more self esteem; the applets form a model students can fall back on; the practice and feedback features are much more powerful than pencil-and-paper
exercises. However, to take fully advantage of these opportunities of applets,
applets should be integrated in the daily mathematics class routine. Using the
computed should not be a voluntary thing that is extra beyond regular math
class."
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The Mathematica Journal volume:issue © year Wolfram Media, Inc. IMS08_paper_Heck_Boon_vanVelthoven.nb 4/15/08The last issue played an important role in the GALOIS project that took place
in the period 2005-2007 [3]. This project aimed to realize an integrated
learning environment for learning and doing mathematics in which
è students can practise mathematics anytime and anywhere;
è mathematical assignments can be generated 'at random', so that the
amount of exercise material becomes unlimited; mathematical
assignments can be generated 'at random', so that the amount of
exercise material becomes unlimited;
è activities and answers of students are stored;
è intelligent feedback on students' work is given automatically.
This project and its one-year extension project SAGE led much to the work
presented in this article. We quote the conclusion drawn in [3]:
"Existing mathematical tools become even more powerful and useful when

results and performance of students can be tracked online in a virtual learning
environment with which teachers and students are familiar. Use of a common
standard, in this case SCORM, enables the sharing of content, but also of
storing student performance. Students know that they can be and are monitored by their teacher. Together with other features in virtual learning environments, especially for communication, students are stimulated to do their homework, or even more than that. Students do not seem to mind this 'big brother
is watching you' concept. On the contrary, they seem to appreciate the fact
that a teacher has good insight into common mistakes and can use this knowledge to address certain problems more effectively. They also seem to find
that computer practice prepares them better for written exams than making
assignments from a book. Computer aided learning is also seen as a more
'fun' way of practicing otherwise 'boring' subjects. Using direct feedback,
giving students hints and enabling a 'trial and error' approach makes it
possible for students to explore different strategies for solving a given
problem.
In our opinion, digital assignments and diagnostic testing add to the motivation and performance of students in mathematics education. By making use of
computers, the extra time involved for teachers is acceptable. Contact time
can effectively be reorganized because students that need extra attention are
signaled earlier."
In field tests at schools during the past projects it was found that all works
well in practice, although many wishes for improvement of the software arose
at these staged of the development work. Linking the applets to a backbone
computer algebra system and having more question types at one's disposal
were two of the envisioned extensions. The first steps to meet the first wish
have been made with the Mathematica empowered Java applets.
à Additional Material
Sample applets are provided on Internet: to get an idea login as guest on the
website www.fi.uu.nl/dwo/en/ (accessed April 2008).
Mathematica Empowered Applets 19
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Research on e-Learning and ICT in Education
Information and Communication Technologies (ICT) have had a huge impact on
contemporary society, fundamentally changing the way that we communicate,
work, and entertain. Education is one of the fi elds where ICT applications have
been used extensively over the years.
In the early days of computers, educators saw the potential of digital material to
improve education, at least from a management and economic point of view: digital
content can be easily managed and distributed to large groups of learners. With the
rapid evolution of computers (with increased processing power, screen displays, and
usability) digital learning materials offered additional affordances over "traditional"
print materials that can signifi cantly improve the quality of education such as
multimedia (i.e., multiple representations of the learning material), hypermedia
(i.e., non-serial access to the learning material), and interactivity (i.e., active engagement with the learning material). This gave birth to a whole new series of initiatives
around the world from ministries, educational organizations, companies, etc., who
developed digital learning resources and educational software.
With the advent of the Internet and the web, educators realized the tremendous
potential for distributing the digital learning materials and for supporting new forms
of web-based learning. This resulted in the development of e-Learning systems, which
mainly support the sharing of digital learning material and facilitate the communication between learners and educators. At the same time, lifelong learning emerged
as a vital necessity since all citizens need to be educated throughout their lives in
order to remain competitive in the knowledge-based economy. To meet these needs
of supporting access to education and training to anyone, anytime, anyplace, the
e-Learning industry has experienced rapid growth over the past decade becoming
the second largest industry evolving around the Internet and the web (second only
to the e-Health industry).
In this context, the fi eld of ICT in Education and e-Learning in particular has
attracted increasing research interest worldwide during the past decades. This
research is constantly evolving, especially since technology itself is also evolving at
rapid pace and new devices, solutions, and practices are becoming increasingly vi
available. This extensive research interest in the potential of technology to support
learning has led to a considerable body of knowledge. Still, many issues remain
open such as:
• How can ICT improve learning in preschool and primary education settings?
• How can it improve teaching of programming?
• How can the new Web 2.0 tools support learning in different educational settings?
• How can technology be used to support learning in museums and other cultural
institutions?
• How can teachers use ICT to improve teaching and learning practices?
This volume aims to contribute to the literature in ICT in Education and
e- Learning by addressing several core issues. The Volume includes 19 chapters
which cover a wide variety of topics.
• The fi rst part includes three chapters which attempt to situate ICT in the broader
educational context. Underwood questions why digital technology has penetrated our lives so much, but has failed to make an impact in the classroom.
Mikropoulos , Sampson , Nikopoulos , and Pintelas explore the evolution of educational
technology through a bibliometric study. Apostolopoulou , Panagiotakopoulos ,
and Karatrantou conclude this part through an investigation of the learning theories underlying the development of educational applications for supporting
teaching and learning of Mathematics, Physics, and Chemistry in Secondary
Education.
• The second part includes three chapters which focus on ICT use in preschool and
primary school settings. Nikolopoulou investigates how educational software is
used in kindergartens. Zaranis examines how ICT can facilitate fi rst graders'
Geometry concepts. Halki and Politis investigate how educational software
affects the learning outcomes of primary school students in high level skills of
critical thinking and programming.
• The third part includes two chapters which address the teaching of programming
concepts through ICT. Malliarakis , Satratzemi , and Xinogalos investigate teaching
of programming through educational games. Misirli and Komis also target programming through robotics in the context of early childhood education.
• The fourth part includes three chapters which investigate how Web 2.0 technologies can affect education. Eteokleous - Grigoriou and Photiou investigate how
blogs can be integrated in primary education. Altanopoulou , Katsanos , and
Tselios investigate the effectiveness of Wikis in undergraduate education.
Kazanidis , Valsamidis , Kontogiannis , and Karakos address the evaluation of
courseware at the exams, usage, and content level.
• The fi fth part includes two chapters which explore technology-based learning in
museums and cultural institutions. Yiannoutsou and Avouris propose the use of
digital games as a means to actively involve museum visitors to participate in the
process of culture creation. Nikonanou and Bounia discuss digital applications
created by museums and other cultural institutions.
• The sixth part includes three chapters which investigate how ICT affects Pre- and
In-service teachers and their practices. Karasavvidis and Kollias examine the
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ways in-service teachers integrate technology in their designs after an extensive
Professional Development Training program. Khaneboubi and Beauné investigate the effect of ICT in French middle schools involved in a national endowment program on digital textbooks. Vekiri investigates the challenges and needs
that should be addressed in teacher preparation for educational technology.
• Finally, the seventh part includes three chapters which address specialized topics
in ICT in Education. Tegos , Karakostas , and Demetriadis address conversational
pedagogical agents in individual and collaborative learning settings. Apostolidis
and Tsiatsos present a prototype device that measures the anxiety level of a person
by collecting bio-signals. Chatzara , Karagiannidis , Mavropoulou , and Stamatis
discuss the potential value of using Digital Storytelling for teaching social skills
to children with Autism Spectrum Disorders.
The following sections describe the parts and chapters in detail.
Part I: Situating ICT in Education
Underwood questions why digital technology is a ubiquitous tool outside of the
classroom but is less well received within the classroom. She addresses the questions

of what happens if we—the educators—fail to embrace technology, and if technology
could take education out of the classroom and the hands of educators themselves. The
chapter proposes that digital technologies can act as a change agent, a galvanizing
force in a way that other previous innovations have failed to do. In doing so, we may
need to "merge and evolve," i.e., educators allow ourselves to adapt and respond to
the possibilities afforded by the technology and embrace innovation.
Mikropoulos , Sampson , Nikopoulos , and Pintelas explore the evolution of educational technology through a bibliometric study. They analyze the 849 papers presented
in a specifi c series of educational technology conferences (HCICTE from 2000 to
2012), in order to study the e-Learning scientifi c community in Greece, to identify the
evolution of salient topics, and the emergence of the trends in the fi eld. Their analysis
reveals that there is a wide involvement of researchers in e-Learning innovations at
schools and less basic and applied technological research, indicating the need for a
more balanced research in both theoretical issues and technological aspects.
Apostolopoulou , Panagiotakopoulos , and Karatrantou investigate the learning
theories underlying the development of educational applications for supporting
teaching and learning of Mathematics, Physics, and Chemistry in Secondary
Education. They argue that it is important for teachers to be able to recognize the
theories behind these educational applications, in order to utilize them in the
most effective way. Their study involved 50 in-service teachers who answered a
questionnaire, and 3 teacher trainers who were interviewed. The results showed that
there is an increasing tendency towards the utilization of the Theories of Construction
of knowledge, and that the perceptions of the in-service teachers are in agreement
with the aims of the developers of the applications.
Research on e-Learning and ICT in Educationviii
Part II: ICT in Preschool and Primary Education
Nikolopoulou investigates educational software use in kindergartens. She interviews
25 teachers from 17 Greek kindergartens to identify the educational software commonly used by children in the classes, how this software is used, and the main diffi -
culties children face in the use of this educational software. Her analysis reveals that
a variety of educational software is used in these classes, most of them being
open-ended software aiming to advance language, reading, and writing skills. The
main diffi culties that children face with the use of this software are related to the
requisite motor skills and the language readiness required for their operation.
Zaranis investigates how ICT can improve fi rst grade students' basic geometry achievement, especially regarding circles and triangles. He employs the
Realistic Mathematics Education (RME) and the van Hiele models, and compares it against the "traditional" teaching methodology. He experiments with two
student groups involving 234 fi rst graders, and concludes that teaching and learning through ICT is an interactive process for students at the fi rst grade level and
has a positive effect for learning Geometric concepts when employing RME and
the van Hiele model.
Halki and Politis investigate the use of ICT for improving critical thinking and
programming skills of primary school children. They experiment with two student
groups, one using the Scratch educational programming language and the other
educational games and experiential learning. They conclude that the two groups
presented almost same results in most learning outcomes they examined. The differentiation referred only to debugging skills (which were higher for the group
using the Scratch), and to cooperativeness and teamwork skills (which were higher
for the group without ICT support).
Part III: ICT and Teaching Programming
Malliarakis , Satratzemi , and Xinogalos examine the teaching of programming
through educational games. They offer a critical review of the most well-known
educational games for teaching programming, which can guide the development of
future applications. Their analysis is based on a framework which summarizes the
functionality that should be supported by educational games for teaching programming. The proposed framework has been based on an analysis of 70 papers from the
related literature.
Misirli and Komis address the teaching of programming for early childhood education through robotics. They propose a conceptual framework for designing educational scenarios that integrate programmable toys as a guide to teaching programming
concepts. Their framework involves seven phases for designing educational scenarios, including identifi cation of the teaching subject, identifi cation of children's
prior knowledge, determination of scenario goals, selection of ICT teaching materials,
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design of scenario activities, scenario assessment, and scenario documentation.
Their framework has been tested and verifi ed by 46 educators on 864 children
between the ages of 4-6.
Part IV: Web 2.0 Tools and Learning
Eteokleous - Grigoriou and Photiou investigate the integration of Blogs as tools in
primary education, and describe the development of a Community of Inquiry (CoI)
within a blended learning environment developed through hybrid learning. They
analyze the social, cognitive, and teaching presence within a CoI in a specifi c class,
where students use a blog to achieve specifi c learning objectives. The results of the
study reveal the development of a CoI through a blended learning environment and
the potential of using blogs as tools to achieve specifi c learning objectives, and
highlight the important role of the educator and lesson's design.
Altanopoulou , Katsanos , and Tselios investigate the effectiveness of a framed
wiki-based learning activity. They report a study which involve 139 fi rst year undergraduates who used Wikis to learn about Web 2.0 and its applications in the
context of an introductory course. The study demonstrated signifi cant improvement
in learning outcomes, both for students with low and high initial performance.
In addition, it was found that students benefi ted from the activity regardless of their
role in the Wiki project. Furthermore, results showed that students with a higher
number of logged Wiki actions had a signifi cantly higher learning gain compared to
those that were less active.
Kazanidis , Valsamidis , Kontogiannis , and Karakos propose a method which
evaluates courseware at the exams, usage, and content level. It defi nes different
measures and metrics for each level, and it uses data mining techniques, such as
classifi cation, clustering, association rule mining, and regression analysis, in order
to discover possible dependencies of the e-learning data. The proposed method was
successfully tested to e-learning data from a Greek University, and the results con-
fi rmed the validity of the approach and showed a relationship among the components of the proposed three-level schema.
Part V: ICT for Learning in Museums
Yiannoutsou and Avouris analyze two different trends that have informed technology
for learning in cultural institutions during recent years. The fi rst one supports the
information consumption metaphor where museums produce "information" in digital
or other form for the visitor to consume, while the second one invites visitors to participate in the process of culture creation. They then discuss game design as an example of participatory activity and they identify its learning dimensions. They analyze
the role of technology in providing a scaffold that can help museum audience to
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construct games which can function as "public artifacts" and can be added to the
museum's assets, enhancing audience engagement and community building.
Nikonanou and Bounia present a qualitative evaluation study of digital applications created by Greek museums and other cultural institutions during the past
few years. The study is based on contemporary theoretical approaches in the fi eld
of museum education and aims to explore the extent to which these approaches
are taken into account when designing a digital application for museum education use. The research highlights that there is a growing interest in developing
ICT applications on the part of the museums, but also that there are many possibilities regarding interactivity, active involvement, collaboration, and creative
expression that are still open for consideration and exploitation by cultural
institutions.
Part VI: ICT and Pre- and In-service Teacher Practices
Karasavvidis and Kollias aimed to bridge the knowledge gap of how specifi c teacher
groups respond to in-service Professional Development Training (PDT) on ICT
pedagogy. They examined how teachers' backgrounds infl uence their responses to
an extensive PDT program. They report data from a multiple case study involving
three in-service teachers who had constructivist teaching philosophies and high academic qualifi cations. The study inquired whether the teachers integrated technology
in ways that supported or transformed their existing practices. The fi rst study fi nding is that the teachers integrated technology in their designs in ways that supported
their existing practices, closely following the dominant science education paradigm
in Greece. The second study fi nding is that the teacher participants were very reluctant
to consider other types of technology integration that would lead to more transformative teaching practices. The authors conclude that their results cast serious doubts
on the potential of contemporary PDT programs to transform teaching practices
through technology and discuss the implications of their work for future conceptualizations of PDT.
Khaneboubi and Beauné investigate the effect of ICT in four French middle
schools involved in a national endowment program on digital textbooks. The analysis of two datasets collected in 2010 and 2012, involving 89 teachers, provides useful insights into the school dynamics in this context. The use of technologies by
teachers in the classrooms seems to be determined by the quality of infrastructure
and equipment reliability. Gender and seniority appear as discriminating factors in
the perception of technologies' availability. And teachers in charge of pedagogical
uses of ICT in each school are almost exclusively sought for technical support.
Vekiri investigates the challenges and needs that should be addressed in teacher
preparation for educational technology. She presents a study which analyzed 30 pre-
service elementary school teachers' lesson plans, representing their fi rst attempts to
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design a web-based lesson. The analysis focused on the types of activities they had
designed, the characteristics of the web resources they had selected, and the scaffolding techniques they had planned to use to support their students. Study fi ndings
highlight that, in order to use the Internet productively and creatively, teachers need
to develop complex forms of knowledge that require the integration of knowledge
about technology, pedagogy, and content.
Part VII: ICT for Specialized Uses
Tegos , Karakostas , and Demetriadis address conversational pedagogical agents
which can guide and scaffold student dialogue using natural language in both individual and collaborative learning settings. They present the results of an experimental collaborative learning activity exploring whether the different agent roles (peer
or tutor) may affect the students' perceptions of the agent or their conversational
style in their responses to it. The study fi ndings provide valuable insights into how
the different agent' appearance and communication styles can have an impact on the
degree of formality in students' utterances.
Apostolidis and Tsiatsos present a prototype device that measures the anxiety
level of a person by collecting bio-signals, namely Galvanic Skin Response and
Photoplethysmography. They pilot-tested their device with 13 students, who suggested that the anxiety measurement may be useful to learning activities such as
examinations and workshops, that they are willing to reuse these sensors in other
academic activities, and that stress level awareness can motivate students to
regulate themselves.
Chatzara , Karagiannidis , Mavropoulou , and Stamatis discuss the potential value
of using Digital Storytelling for teaching social skills to children with Autism
Spectrum Disorders. They present the DiSSA (Digital Structured Storytelling for
Autism) tool, a software application for creating digital stories following a structured approach. The system is aimed to cater for the needs of students in the autistic
spectrum, taking advantage of structured teaching in the design of the application.
It accommodates user's performance data towards the learning outcomes and
thereby provides feedback to other users (such as, teachers or parents) about children's
progress. DiSSA has been pilot-tested with four pupils (7-11 years) with a formal
diagnosis of mild autism, and has demonstrated to be successful in terms of user
satisfaction and user task accomplishment.
We would like to express our deepest appreciation to a large number of people
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Naturally Resistant

Although many in education would argue that we are in a constant state of fl ux the changes, irritating though they may be, are often superfi cial. As Papert (1993) has argued the education system is highly resistant to change. The proposition explored in this chapter is that digital technologies can act as a change agent, a galvanising force in a way that other previous innovations have failed to do so. This places high expectations on the use of the technology but can it deliver? There are still many who doubt the usefulness of such technologies in the classroom, yet we are very accepting of digital tools in our daily life. Why is this the case? While plotting a timeline of technological development, and even projecting into the future, is a relatively simple task, assessing the impact of those technological developments on teaching and learning is far more problematic. The underlying ambivalence to the educational use of technology makes future projections diffi cult, and at fi rst sight the prospects are not encouraging.

A Brief History of Digital Technology

Learning technologies are not a recent idea although it is no easy matter to identify their inception; they are as old as written language itself (Westera, 2010). For a long time the slate and the chalkboard were the dominant educational technologies, but the twentieth century proved to be a century of rapid technological advance inside and outside of the classroom. First technologies such as fi lm and radio were added

Digital Technologies: An Effective Educational
Change Agent?

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to the teacher's toolbox, and then, towards the end of the last century, digital technologies fi rst crept and then thrust themselves onto the educational scene. Digital technologies are different from previous technologies in a number of ways not least because they are in a constant state of rapid evolution. This is clearly illustrated by Apple's i-Phone. Just a few years ago Apple released this touchscreen 2G (second generation, that is, digital) mobile phone, which I purchased with pride. It was not just a phone of course; it came with Internet access, a camera, note pad and much more. It was a communication system and mini-offi ce all rolled into one. Four years on and my i-Phone became a poor relation of its 3G cousin the i-Phone 3. As is the way of each new generation of such technologies, the i-Phone 3 was of course faster and had greater memory capacity and enhanced functionalities such video capture and streaming alongside the still image camera. However it too has been superceded by new generations of i-Phones. My fi rst-generation i-Phone is not able to play the cult game "Angry Birds" nor does it allow me to access You-Tube clips or have face-to-face real-time conversations.

That such an iconic object as the original i-Phone became effectively obsolete in less than 4 years, and its successor in even less time than that, graphically illustrates the speed and nature of technology development, which, on the whole, have improved what already exists and also added new functionalities. These new functionalities have resulted in a rapid move towards the personalisation of content, where every user has his or her own custom distribution channel. However, although the technology has become more powerful, that is, it has more on-board memory, a more powerful central processing units (CPU), and most signifi cantly wireless network access, we fi nd activities such as playing "Angry Birds" or sending short video clips absorb all the additional power. This has resulted in a new industry supplying digital storage space in the "clouds". Large clusters of networked servers supply vast processing power and storage capacity at low cost removing the need for large personal data stores for your personal images or music fi les (Johnson, Levine, & Smith, 2009).

So the technology is faster, more powerful and has greater functionality at a reduced cost, and yet it still disappoints because we want more.

And the Impact of That Technology Is?

As technology has spread through our society new behaviours and new ways of working have emerged; for instance mobile phones have become indispensable to the operation of small independent businesses. This is how we now contact our local plumber or electrician (Crabtree & Roberts, 2003). Further it has become a ubiquitous tool of the young. In the UK over 90 % of all 11-21-year-olds had access to a mobile phone (Haste, 2005), and by 2006 49 % of 8-11-year-olds and 82 % of 12-15-yearolds had their own phones (OfCom, 2006). Moreover, 82 % of 8-11-year-olds and 93 % of 12-15-year-olds spent time texting. Texting has become such a widespread and valued activity among many young learners (c.f., Plester & Wood, 2009).

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Typically 16-24-year-olds spend more time on their mobile phones and social networking sites than watching television (OfCom, 2010).

Other changes have been seismic shifts rather than modifi cations of behaviour. For example, few would have predicted the impact of technology on news reporting, an impact that has led to the rise of the citizen reporter. Armed with a camera phone anyone can be both reporter and editor of current events. What was once hidden is now exposed across the world on You-Tube, often before the offi cial news networks. Such changes necessarily affect the structures of a society. The result is that institutions fi nd themselves unable to handle key changes in the rhythms and patterns of emerging human behaviours, and so new or transformed institutions emerge. Thus the Iranian Government made a huge effort to block images and news reports of the violence following the 2009 elections, but there was still leakage of news (Palser, 2009). However in the USA political structure was transformed post the 2008 presidential elections, which were won and lost in cyberspace. The now US President Obama had three times as many supporters signed up on Facebook than his republican rival John McCain, and 500 million blog posts mentioned Obama compared to 150 mentioning McCain (Aronson, 2012). Obama managed to do what many politicians in the West have failed to, that is, engage a new generation in politics by using their preferred communication tools.

And the Impact on Education Is?

Learners of all ages are exhibiting new behaviours as a result of these ubiquitous high-functioning technologies. Changes may be relatively mundane, such as replacing the school satchel with a memory stick, or profound, as when learners seek out expertise beyond the traditional classroom or move from text to more visual modes of representation. For example we reported on two primary schools, both of which were linked to the same theatre group in order to write a play as part of the web play

project (Underwood et al., 2005). While signifi cant, these changes are not necessarily transformational, but it could be argued that given the formal framework of education such transformation may not be possible. Lowendahl (2009) argues that the education system has a history of resistance to change, a resistance born out of disappointment when the "hype cycle" of technology in education fails to deliver. This cycle which starts with a techno-romantic phase often leads to disillusionment when the technology fails to deliver nirvana. There is then a need to pass through to a slope of enlightenment, that is, to make a realistic assessment of what technology can and cannot do, before reaching a plateau of productivity when the technology actually delivers to realistic goals.
Both students and tutors have been shown to underuse many high-level functions such as the communication tools (Sclater, 2010), and teachers often fi nd the more advanced functionalities diffi cult to customise (Severance, Hardin & Whyte, 2008). Our own research has shown that while there have been signifi cant advances in educational technology they have not always brought about measurable shifts in

user behaviour. A study of learning platform use in eight technically savvy English secondary schools (Underwood & Stiller, 2014) showed that while teachers' intentions to use the range of functions available to them on the learning platform varied from extensive to as little as possible, those expressing a desire to use the system creatively found themselves held back by mundane barriers associated with time, personal skills and curriculum demands. Innovative uses of blogs, wikis, and other tools remained aspirational at best even in these schools where technology innovation, at a surface level at least, was actively encouraged.
So while Lowendahl calls for more realism and less hype in technology innovation and acceptance, Westera (2010) argues that the main barrier to effective embedding of technology is a little too much realism. The computer has been used as a sensible teaching aid, quite useful for a specifi c subset of learning activities, but it has never challenged the educational system as a whole. Resnick (2006) picks up this argument by raising the following question: "'Which is the odd one out: the television, the computer or the paintbrush?'" He argues that the potential of digital technologies will not be realised until we think of them as modern equivalents of the paintbrush and not as televisions. That is, we need to start seeing computers not simply as information machines but also as a new medium for creative design and expression.
Although those of us who teach see ourselves as innovators, the truth is that change in education is very slow. Seymour Papert (1993) graphically depicted the immovability of education in his story about surgeons and teachers. He argued that a surgeon from a century ago would not recognise a modern operating room but while a teacher might be puzzled by some of the resources in the new classroom, he or she would nevertheless feel at home. Papert posed the question as to why, when so much has changed over the last century, there has not been a comparable change in the way we educate our children. It is not because new tools, the equivalent of the surgeon's heart monitor machine, do not exist. So is it because teachers are inherently resistant to change? While we are cautious professionals we are not Luddites,
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entrenched
opponents of change. The slow pace of change is more to do with need. Medical practice needed to change because people were dying, but traditional methods of teaching do result in children learning, so why are we in such a hurry to change?
So Why Change?
From Aviram and Talmi's (2005) point of view, the centrality of digital technologies in education is both assured and inevitable. That perceived inevitability is built on the assumption of the omnipresence of ICT in our everyday lives and the rise of the generation of digital natives (Prensky, 2001). Indeed Prensky argues that today's students are no longer the people our educational system was designed to teach.
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Luddites: Workers who violently resisted the introduction of new machinery into the textile industry in nineteenth-century England.

Teachers are told that they have to attract students' interest and attention (Simplicio, 2000), recognising that learners are growing up surrounded by video games, mobile phones and other digital media, all of which are leading to new learner expectations of an acceptable educational environment (Pedró, 2006). However, Watson (2001) queries the automatic link made between the everyday and educational uses of technology, as it does not take into account the fact that technology often fi ts uncomfortably with teachers' professional judgments. In support of Watson's analysis, the evidence shows that technologies that move teachers outside their comfort zone tend to have a slower take-up and high rejection rates. The conclusion from our own work is that positive impacts are more likely when linked to a teacher's existing pedagogical philosophy, hence the rapid acceptance of interactive whiteboards (IWB) compared to virtual learning (Underwood et al., 2010). Indeed half of the teachers we interviewed identifi ed the IWB as their "must-have" technology. However, some teachers expressed unease with this position, for example stating "IWB and PowerPoint, sadly ...", acknowledging that there were other more exciting ways of using technology although they were not exploiting these opportunities themselves. The discontinuity between teachers and technology may be more deep-seated than a clash with professional practice though. It may lie in the nature of those who choose to teach. For example, it is not age or sex but membership of the teaching profession that is the defi ning characteristic of low involvement with video games (Sandford, Ulksak, Facer & Rudd, 2006). Books are the preferred tool of this group as a whole. If teachers, as a group, are inherently low technology users compared to the general population, does this mean that there is a natural resistance to the embedding of technology into the educational processes and practices? While this rather negative portrayal of the teaching profession may be valid in some cases, our evidence of a decade of national research projects presents a more positive picture of a profession that is cautious but constructive in its approach to innovation.
Three Possible Ways Forward
It has long been argued that any good teaching system aligns the teaching method and assessment to the stated learning objectives (Lebrun, 2007). How do we achieve this alignment? There are broadly three strategic responses to the demands to go digital (Underwood & Dillon, 2011).
1. Minimise the use of technology : This approach minimises the demands on teachers and maintains the status quo. However, such a strategy raises very real issues of equality. Those learners with access to technology outside of the school will be advantaged, leaving a digital underclass of learners who lack either the economic or the cultural support that would make these technology tools available to them.
2. Use technology to support current practice : Accept technology where it fi ts current educational structures and practices. This approach recognises technology as a useful tool in the right place but removes the role of catalyst for change. So we fi nd that some innovations are more readily assimilated into the classroom than

others; for example digital whiteboards is a case in point. However other ubiquitous technologies such as 3G phones have minimal impact (Wang, Shen, Novak, & Pan, 2009) and are resisted by teacher unions (Robinson, 2010). This way forward feels comfortable and of course is in widespread use. However, there is an inherent risk that such use will lead to learner disaffection and rejection of the educational process, particularly in the case of the digitally savvy learner.
3. Merge and evolve : Here, we educators allow ourselves to adapt and respond to the possibilities afforded by the technology and embrace innovation. The approach recognises that digital technologies necessarily require us to reassess how learners learn and teachers teach. From this perspective we need to think about how schools or learning ecologies are organised, including the role of technology, to support meaningful student achievement. Schools will move to be more open educational institutions that "dramatically change their views on knowledge, assessment and the teacher, student and information relationships" (Hernandez & Goodison, 2004 , p. xvi). One such example is the emergence of the personal web that will allow learners and teachers to customise the web to their own needs and interests using a range of data management and tracking tools. A second example would be the simulated contexts provided by virtual environments in which participants interact with digital objects and tools, such as historical photographs or virtual microscopes (Clarke-Midura & Dede, 2010).
But Even If We Decide to Merge and Evolve?
The Perils of Joining the Net Generation
In 2009 a comparison of faculty and student responses indicates that students were much more likely than faculty to use Facebook and were signifi cantly more open to the possibility of using Facebook and similar technologies to support classroom work. Faculty members were predisposed to use more "traditional" technologies such as email (Roblyer et al., 2010). However, there are tutors who have joined the Facebook generation, and in doing so they have taken the decision to merge and evolve. However, this brings its own perils. Just as we, the tutors, begin to feel in charge of the technology we are reminded that the potential risks of the digital world are not just for learners but for tutors as well.
A study by Sleigh, Smith, and Laboe (2013) of students' responses to tutors' Facebook pages is illuminating. They examined whether the specifi c type of self-disclosure on a tutor's profi le would affect students' perceptions of the tutor including their expectations of the tutor's classroom practice. Students reported being most interested in professional information on a tutor's Facebook profi le, yet they reported being least infl uenced by that professional profi le. They found that tutors who were seen as social individuals had high popularity ratings but the sting in the tail was that, although their profi les were viewed as entertaining, they were judged as inappropriate for a professional. This perception resulted in such tutors

being assessed as less skilled professionally. It would appear that students form perceptions about the classroom environment and about their tutors based on the specifi c details disclosed in tutors' Facebook profi les.
So while tapping into students' interest SNSs may be an effective way for tutors and institutions to communicate and stimulate their students (Junco, 2011), the technology can be revealing and the students' developing perceptions and expectations may not always be what we intended or desired.
And Then There Are MOOCs
Massive open online courses (MOOCs) may seem somewhat tangential to what has gone before in this short piece, but they raise considerable issues for educators and possibly learners too. They are the dream scenario:
Nothing has more potential to unlock a billion more brains to solve the world's biggest problems. And nothing has more potential to enable us to reimagine higher education than the massive open online course, or MOOC, platforms that are being developed by the likes of Stanford and the Massachusetts Institute of Technology and companies like Coursera and Udacity. (Friedman, 2013 , page SR1)
So what are MOOCs, and why are they arousing such interest? MOOCs have been developed to support large-scale, open-access participation that includes varying levels of interaction. While such systems can appear prosaic at fi rst sight, delivering traditional course materials such as videos, readings and exercises through the web, it is the use of interactive user forums designed to build a community of learners and tutors that elevates the MOOC from a delivery system to a new approach to knowledge and learning. Such knowledge, according to Downes (2005), is created by interaction and is not simply a relationship or a distributive pattern between one fact or idea and another. In essence connective knowledge is knowledge of the connection.
For a connectivist dynamic system to exist, Downes (2005) argues that there must be learner autonomy, group diversity, openness and interactivity and connectedness defi ned as follows:
1. Autonomy—The level of learner autonomy must be high, and the learners must be more than simply managed participants who receive rather than create of knowledge.
2. Diversity—The community will have a diverse membership and not be a self-perpetuating in-group maintaining the status quo and stifl ing new ideas and connections.
3. Openness—This will support free-fl owing communication at various levels of activities with easy access for participants with no clear boundaries between membership and non-membership.
4. Interactivity and connectedness will produce knowledge that is unique within the community. Such knowledge will very likely be complex, representing not simple statements of fact or principle, but rather refl ecting a community response to complex phenomena.

However, this utopian dream often falls short even in networks that are open and allow autonomy as only certain perspectives of those of participants occupying the highly connected nodes tend to be circulated to the network as a whole for consideration.
The University of Manitoba delivered an early interpretation of the MOOC principles in the autumn term of 2008. Predicated on this new perception of knowledge and way of learning "connectionism" (Downes, 2005 ; Siemens, 2004), CCK08 was designed to enable participants to engage with the theory and practice of connectivism. Over 2,000 participants signed up for the course, including Mackness, Mak and Williams (2010) who were keen to explore the Downes's model of learning in practice. They found that all four characteristics of a MOOC, autonomy, diversity, openness and connectedness/interactivity, were present in the MOOC, but that they did not necessarily lead to an effective learning experience. The more autonomous, diverse and open the course, and the more connected the learners, the more the potential for their learning to be limited by the lack of structure, support and moderation normally associated with an online course. These students fell back engaging in traditional groups as opposed to the open network. The fi nding that there can be too much autonomy has already established in other learning situations (Underwood et al., 2010). These responses constrain the possibility of having the positive experiences of autonomy, diversity, openness and connectedness/interactivity normally expected of an online network. The research suggests that the question of whether a large open online network can be fused with a course has yet to be resolved. Further research studies with larger samples are needed, as is an investigation into the ethical considerations that may need to be taken into account when testing new theory and practice on course participants.
However, the nightmare scenario would see MOOCs as leading to the demise of universities, colleges and even upper secondary or high school education
I believe that online education will be an important building block of teaching in the future…I see great potential in having the opportunity to learn no matter where you are. You don't need to be in school or in a lecture hall of a university anymore, and therefore I believe that this will dramatically change our lives.
(Dr. Angela Merkel, Chancellor of the Federal Republic of Germany (https://moocfellowship.org/ 2013, no page)
One cannot deny the size of the MOOC impact on higher education, for example:
Mitch Duneier, a Princeton sociology professor, wrote an essay in The Chronicle of Higher Education in the fall about his experience teaching a class through Coursera: "A few months ago, just as the campus of Princeton University had grown nearly silent after commencement, 40,000 students from 113 countries arrived here via the Internet to take a free course in introductory sociology. (Friedman, 2013 , no page)
Further the fi rst UK-based MOOC platform, FutureLearn, is intended to go live in autumn 2013. It will be populated by MOOC courses designed by its 21 member institutions. It is estimated that it will cost about £30,000 (35,000 euros) to develop a MOOC for this platform (Parr, 2013a). While MOOCs can be linked to other tools such as Facebook such linkage can result in negative consequences. For example,

Kop, Fournier and Mak (2011) found that students had privacy and security concerns about Facebook. It would appear that the level of trust, feelings of confi dence and the sense of presence and community are crucial to students engaging with the system, and Facebook leaves them uneasy.
But There Is a Twist in the Tail
Completion rates for MOOC courses, defi ned by the number of students being awarded some form of certifi cate, are alarmingly low. The Times Higher Education quotes an average fi gure of 7 %, that is, 93 % of students failing to complete such courses (Parr, 2013b). There is signifi cant variation in these rates as is shown by the following three courses, all mounted on Coursera (Jordan, 2013). A History of the World course at Princeton University which ran 2012-2013 is recorded as having the poorest completion rate at 0.7 % of students enrolled; that is, 581 of the 83,00

who enrolled were certifi cated. However the University of Edinburgh had a completion rate of 2.3 % for its Artifi cial Intelligence Planning in 2013, that is, 660 students out of 28,689 students enrolled on the course. The most successful course in terms of completion rates according to Jordan was a course in Functional Programming Principles from the Ecole Polytechnique Fédérale de Lausanne from 2012-09 to 2012-11. Here 19.2 % completed the course, that is, 9,600 students out of the 50,000 who enrolled.
The headline fi ndings from Jordan's (2013) data are encouraging for those who would hold back the march of online teaching and learning. However, look deeper and you will see that the World History course made assignment completion optional. This suggests that the course was designed with more than one audience in mind. Yes there were students of history seeking qualifi cation, but there were also those who were simply interested in the topic, many of whom I would surmise are learners of the third age, that is, retirees coming back to a subject that interested them in the past but was not seen as providing job skills when younger. How many students are enrolling on these courses as top-up, tasters or as a hobby is yet to be established. The fi ndings from the Edinburgh and Lausanne courses might be of greater concern, as these are not "hobby" subjects for most people. However, even here the numbers passing the course are not insignifi cant; 9,600 completions in the case of the Lausanne course should not be viewed as an inconsequential with the potential to impact on more traditionally provided courses. That really is cost- effective education.
Where Do You Stand?
Attempts to bed in new technologies necessarily involve some level of disturbance to the educational system. The degree to which these perturbations are tolerated will affect technology acceptance. This raises the question of whether the educational
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system will allow itself to be transformed or not although the rise of MOOCs may take these decisions out of the hands of educational establishment at least at the university level even if schools are more future proof.
Of course there is always that cynical old "truism". In the UK education circles it has long been the belief that universities will survive all educational revolutions because in the end the middle classes have to send their children somewhere to fi nd suitable husbands and wives. With the advent of social networking and online dating even this role is now under threat.
The equally cynical view of schools as state-provided babysitting services has yet to be questioned although one can see the rise of plugged-in children. In the end if we do not adapt then for some learners the educational system will become increasingly irrelevant and they will carve out a learning environment for themselves, dipping into the formal system only when they see the need. As Prensky (2001) points out we ignore the fundamental fact that in a digital world the students themselves have changed. Will we as educators change with them or will the majority of us become increasingly less relevant?
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Introduction
During the past years, the increased interest for applying technology to improve learning and teaching (Spector, 2012) has led to the evolution of this discipline from educational technology (ET) defi ned as "the study and ethical practice of facilitating learning and improving performance by creating, using, and managing appropriate technological processes and resources" (Januszewski & Molenda, 2008) to technology-enhanced learning (TeL) referring to a transformative movement in learning and teaching that exploits technological advances for offering learning experiences not possible to be organized in current formal educational settings (Haythornthwaite & Andrews, 2011). Educational technology and TeL are now mature interdisciplinary research areas, and there are already literature studies aiming at the study of scientifi c communities, the identifi cation and evolution of salient topics, and the emergence of the trends in the fi eld (Cho, Park, Jo, & Suh, 2013 ; Kinshuk, Huang, Sampson, & Chen, 2013 ; Masood, 2004 ; Pham, Derntl, & Klamma, 2012). Such studies are important since they can provide insights into the
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Based on a Bibliometric Study
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development of a particular research area. Typically, social network analysis and bibliometric approaches using co-word, performance analysis, and science mapping have been exploited for this purpose (Cobo, López-Herrera, Herrera-Viedma, & Herrera, 2011 ; Masood, 2004 ; Muñoz-Leiva, Sánchez-Fernández, Liébana-Cabanillas, & Martínez-Fiestas, 2013).
In educational technology and TeL, there are some efforts to analyze research trends and scientifi c communities using as sources articles published in journals (Cho et al., 2013 ; Klein, 1997 ; LeBaron & McDonough, 2009 ; Masood, 2004). On the other hand, Randolph, Julnes, Bednarik, and Sutinen have indicated that "there were no practically or statistically signifi cant differences between the articles published in journals and those published in conference proceedings on any of the indicators of methodological quality" (2007). As a result, there have been similar studies where the sources were articles published in international conference such as the IEEE International Conference on Advanced Learning Technologies (ICALT) (Pham et al., 2012 ; Randolph et al., 2005) and the Artifi cial Intelligence in Education conference organized by the International Artifi cial Intelligence in Education (AIED) Society (Rourke, Anerson, Garrison, & Archer, 2001).
In this work, we aim to investigate the evolution of the educational technology fi eld in a European Union member state, namely, Greece, through the analysis of the category of targeted research outcomes and the key topics of interest that have been emerged in all papers presented at the main national scientifi c conference of this fi eld between 2000 and 2012. More specifi cally, we analyzed papers presented in the biannual conference of the Hellenic Scientifi c Association for ICT in Education, a nonprofi t scientifi c association founded in 2000 with the aim to promote research and development in the fi eld of educational technology. Through this analysis, we aim to identify trends between three different categories of targeted research outcomes (theory, technology, practice) and the evolution of different key topics of interest over a period of 12 years. We then discuss these trends in relation to educational technology policies and programs that have stimulated both research and development activities as well as mainstreaming activities aiming to widespread adoption of educational technologies in school education.
Method
Sample
The sample of this study consists of the population, namely, all the articles published in the proceedings of all seven biannual national conferences organized by the Hellenic Scientifi c Association for ICT in Education between 2000 and 2012.
The association is the only scientifi c, on behalf of membership criteria, body in the fi eld in Greece and represents almost the entire research community in Greece, and its conferences are the biggest in Greece. Thus, it can be considered that a

bibliometric analysis based on the association's proceedings covers the Greek community of ICT in education in Greece. The number of articles presented and published in both printed and electronic form is 849 which constitutes a signifi cant number of articles from which we can extract meaningful conclusions. Figure 1 shows the distribution of all articles in the seven biannual conferences from 2000 to 2012. It can be seen that there is a reasonably balanced distribution with peaks during 2002-2006 due to the implementation of a large-scale public-funded program for introducing ICT in school education which stimulated interest from both educational practitioners and researchers.
Bibliometric Approach
For the purposes of the present study, the co-word analysis, as a powerful quantitative content analysis technique "in mapping the strength of association between information items in textual data" (Cobo et al., 2011), is applied. Moreover, co- word analysis is used for temporal analysis and "develops a performance analysis of specifi c themes using a series of basic bibliometric indicators" (Muñoz-Leiva et al., 2013).
The key topics of interest are formed by studying the common keywords from

the corpus of the articles. The keywords used are retrieved from the conference
index, author's keywords, and their combinations. Moreover, two different coders
studied a specifi c number of articles and found common keywords at a level of 80 %
that is usually characterized as standard (Rourke et al., 2001). Based on the common
keywords, clusters of keywords are formed, thus creating the key topics of interest.
75
150
163
144
104
122
91
-20
20
60
100
140
180
2000 2002 2004 2006 2008 2010 2012
Total number of articles
Year of Conference
Fig. 1 Population of articles per biannual conference (2000-2012)

Results
First, we have identifi ed three different categories of targeted research outcomes,
namely, theory, technology, and practice. The fi rst category concerns with theoretical issues of educational technology such as "pedagogy in TeL" and "learning
design: theoretical aspects." Thus, we refer to this category as "theory." The second
category concerns with a variety of learning technologies, from "authoring tools for
educational content and learning designs" and "course management systems" to
"wireless, mobile, and ubiquitous technologies" and "Web 2.0 and social computing technologies," and involves articles that emphasize on learning systems based
on these technologies. We refer to this category as "technology." The third category
concerns with those topics of interest that relate to the practical pedagogical exploitation of ICT in formal and informal learning settings, including ICT in teaching
various subject domains, and other implementation issues of ICT in education.
Thus, we refer to this category as "practice."
Table 1 presents the categories of targeted research outcomes (level 1) and their
corresponding key topics of interest per category of research that have arisen from
the co-word analysis (level 2), together with their frequencies (%) and absolute
frequencies for the entire 12-year period. It can be noticed that the balance between
the three different categories of targeted research (theory—5.18 %, technology—10.95 %, and practice—83.86 %) indicates that the vast majority of conducted
research concerns with the practical use of existing theories and technologies in
school education.
Figure 2 illustrates how these trends are evolved over time during the 12-year
period. It can be noticed that basic research ("theory") tends to be reduced and
applied technological research ("technology") has a constant increase (mainly due
to the continuing development of new technologies), but still the core bulk of
research concerns "practice."
This is consistent with our hypothesis that the community considers that the fi eld
of educational technology has matured enough to mainly concern mainstreaming
(that is, large-scale implementation of technology-supported educational innovations in formal setting, like schools), rather than basic research or even applied
technological research. This is consistent with current European Union policies and
trends in TeL.
Figures 3 , 4 , and 5 illustrate the time evolution of key topics of interest in each
category of targeted research outcomes during the 12-year period.
From Fig. 3 it can be noticed that in category "theory" the main topics of interest
have been "learning design: theoretical aspects" and "pedagogy in TeL," whereas
there are some sporadic contribution in "learning theories in TeL" and "e- assessment:
theory and methods." Even so, the topic with a consistent interest over time has been
the "pedagogy in TeL," which is reasonable since the implementation of large-scale
ICT in school education programs stimulates interest in basic research related with
the rethinking of pedagogy in TeL.

Table 1 Key topics of interest in national conferences

Level 1: Category of targeted research outcomes	Level 2: Key topics of interest	Frequencies (%) of level 1 categories over 2000-2012	Absolute frequencies of level 1 categories over 2000-2012 N (= 849)	Frequencies (%) of level 2 topics over 2000-2012	Absolute frequencies of level 2 topics over 2000-2012 (N = 849)
Theory	Pedagogy in technology- enhanced learning	5.18	44	2.00	17
	Learning theories in technology - enhanced learning	0.35	3		
	Learning design: Theoretical aspects	2.71	23		
	e-Assessment: Theory and methods	0.12	1		
Technology	Authoring tools (educational content, learning design)	10.95	93	2.47	21
	Course management systems	0.59	5		
	Virtual environments and worlds	2.00	17		
	Digital games	0.24	2		
	Educational robotics	0.24	2		
	Wireless, mobile, and ubiquitous technologies for learning	0.35	3		
	Collaborative technologies	2.00	17		
	e-Assessment tools	1.77	15		
	Tools for education of people with disabilities	0.47	4		
	Web 2.0 and social computing in education	0.24	2		
	Hypermedia and multimedia	0.59	5		

(continued)

Table 1 (continued)

Level 1: Category of targeted research outcomes	Level 2: Key topics of interest	Frequencies (%) of level 1 categories over 2000-2012	Absolute frequencies of level 1 categories over 2000-2012 N (= 849)	Frequencies (%) of level 2 topics over 2000-2012	Absolute frequencies of level 2 topics over 2000-2012 (N = 849)
Practice	ICT in science education	83.86	712	13.19	112
	ICT in mathematics education	7.54	64		
	ICT in language and literature education	7.89	67		
	ICT in history education	6.01	51		
	ICT in foreign language education	3.42	29		
	ICT in computer science education	10.13	86		
	e-Assessment: Implementation issues	2.71	23		
	Distance learning: Implementation issues	11.90	101		
	Digital literacy	8.13	69		
	Continuing professional development programs—teachers' training	8.01	68		
	Educational management	2.00	17		
	Educational policy	1.18	10		
	ICT in education of people with disabilities	1.77	15		

From Fig. 4 it can be noticed that in category "technology" the main topics of
interest have been "authoring tools for educational content and learning designs,"
"virtual environment and worlds," "collaborative technologies," and "e-assessment
tools." It is also interesting to notice that there is a consistent interest on these topics
over time. This is reasonable since authoring tools and e-assessment tools are
0
4
8
2000 2002 2004 2006 2008 2010 2012
NUmber of articles
Year of Conference
Articles' distribution for theme "theory"
Pedagogy in Technologyenhanced Learning
Learning Theories in Technology-enhanced Learning
Learning Design: Theoretical Aspects
e-Assessment: Theory and Methods
Fig. 3 Time evolution of key topics of interest in "theory"
0
40
80
120
2000 2002 2004 2006 2008 2010 2012
THEORY
TECHNOLOGY
PRACTICE
Fig. 2 Categories of targeted research outcomes over time

directly useful in the implementation of large-scale ICT in school education programs,
whereas collaborative technologies are attracting practical interest since collaborative learning aspects are considered a major educational innovation that can be
supported by technology. On the other hand, applied technological research on topics
such as digital games and educational robotics, as well as wireless, mobile, and
ubiquitous technologies, appears not to be supported by the Greek research
community, mainly due to the lack of national industry that develops such
technologies.
From Fig. 5 it can be noticed that in category "practice" the main topics of interest have been "ICT in science education," "ICT in computer science education," and
"distance learning: implementation issues." This is reasonable because science,
0
5
10
2000 2002 2004 2006 2008 2010 2012
Articles' distribution for theme "technology"
Authoring Tools (Educational Content, Learning Design)
Course Management Systems
Virtual Environments and Worlds
Digital Games
Educational Robotics
Wireless, Mobile and Ubiquitous Technologies for Learning
Collaborative Technologies
e-Assessment Tools
Tools for education of people with disabilities
Web 2.0 and social computing in education
Hypermedia and multimedia
Fig. 4 Time evolution of key topics of interest in "technology"
30
25
20

15
10
5
0
2000 2002 2004 2006 2008 2010
ICT in science education
Articles' distribution for theme "practice"
ICT in mathematics education
ICT in language and literature education
ICT in history education
ICT in foreign language education
ICT in computer science education
e-Assessment: Implementation Issues
Distance learning: Implementation Issues
Digital literacy
Educational management
Educational policy
ICT in education of people with disabilities
Continuing Professional Development Programs–
Teachers' Training
2012
Fig. 5 Time evolution of key topics of interest in "practice"
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technology, engineering, and mathematics (STEM) are the key school curriculum
subject domain topics that can benefi t the most from innovative technology-
supported teaching and learning strategies (Rocard et al., 2007). This is consistent
with fi nding in other member states of the European Union and all over the world
(Rocard et al., 2007).
Conclusions
Educational technology and TeL are now mature interdisciplinary research areas,
and there are already literature studies aiming at the study of scientifi c communities,
the identifi cation and evolution of salient topics, and the emergence of the trends in
the fi eld. Such studies are important since they can provide insights into the development of a particular research area.
In this work, we investigated the evolution of the educational technology fi eld in
a European Union member state, namely, Greece, through the analysis of trends
between three different categories of targeted research outcomes (theory, technology, practice) and the corresponding key topics of interest that have been emerged
in all 849 papers presented at the main national scientifi c conference of this fi eld
between 2000 and 2012.
Some interesting conclusions can be drawn for the discussion of the obtained
results. It appears that the educational technology community, at least in Greece,
considers that the fi eld has matured enough to mainly concern mainstreaming (that
is, large-scale implementation of technology-supported educational innovations in
formal setting, like schools), rather than basic research or even applied technological research. This can be a misleading route which can result to disappointments,
since using existing technologies, mainly developed out of the context of learning
and education, simply used as a facilitator for implementing incremental innovations in school education has potential risks of failure. In our view, the truly transformative value of educational technologies in formal and informal learning requires
disruptive innovations that question the why-what-how-where of learning and
teaching in the digital era. To this end, basic and applied technological and pedagogical interdisciplinary research is needed more than any other time in the past.
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Introduction
In the history of pedagogical thought and psychology, every defi nition of learning
is supported by an entire world view, a sequence of ontological assumptions concerning nature, man, and the relation between them. In parallel, the phenomenon
of learning is approached by distinct disciplines in a different way (Bigge &
Shermis, 1999). The variety of Learning Theories has led us to the conclusion that
the phenomenon of learning constitutes a complex and composite fact, of various
dimensions, since no theory on its own is able to describe and interpret the whole
range of learning and its dimensions. The ideological currents and the theories of
learning, as well as the methods via which the latter contribute to the shaping of
teaching, are various. Among the most widespread schools of thought are those
which consider learning as (Ertmer & Newby, 1993 ; Nagowah & Nagowah, 2009):
a change in the observable behavior (Theories of Behavior – Behaviorism), an
internal mental process of the learner (Cognitive Theories – Cognitivism), or an
adjustment of mental models to accommodate new experiences (Theories of
Construction of Knowledge– Constructivism). Teaching with or without the
implementation of ICT tools is every time based on certain acceptances regarding
what learners have to learn and which way of learning is the best. These arguments
are substantially connected directly to a certain learning theory. Thus, the educational software, as well, as a means of learning, is always based on a certain or
certain theories of learning, incorporated in it by its creators (Mcleod, 2003 ;
Roblyer, 2006).
Theories of Learning in Math and Science
Educational Software
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Today it is commonly accepted that ICT integration in the educational process is
able to support educational activities in a rich educational environment in which
learners can have multiple representations of complex phenomena and even be individually supported in the construction of knowledge in an authentic learning environment (Cooper & Brna, 2002). Worldwide research, though, has shown that
teachers take advantage of capabilities of this kind in a restricted manner, using
ICTs mainly for supporting traditional teaching practices and not for designing and
implementing educational activities which involve learners actively in the learning
process (Sang, Valcke, van Braak, & Tondeur, 2010 ; Tondeur, van Keer, van Braak,
& Valcke, 2008).
In the present research paper an attempt is being made to study teachers' perceptions about applications of educational software which deal with the syllabus of
Mathematics, Physics, and Chemistry of Secondary Education regarding the embedded learning theories. The main research question was to identify which groups of
theories of learning are embedded in certain applications , and to what extent ,
according to in - service teachers ' perceptions . These perceptions were correlated
with the ones of experienced teachers in the use of ICT in the classroom and the
statements of the developers of the applications under study as they described in the
supporting material of each application.
Theories of Learning and Educational Software
Each one of the three groups of learning theories can be used in the educational
software (Mcleod, 2003 ; Roblyer, 2006). Which one precisely and at which point of
the software it will be applied depends on the users' background knowledge and
their characteristics, as well as the discipline presented via the educational software.
A suffi cient condition for this to happen is that the design and production team be
aware of both the positive and negative elements of each theory, in order for the team
to utilize them appropriately (Panagiotakopoulos, Pierrakeas, & Pintelas, 2003).
Theories of Behavior
Learning is defi ned as a change in learner's behavior arising from experiences, as
well as the tasks set by the teacher. Learning occurs through the reinforcement of
the desired behavior either through its reward (positive reinforcement) or through
punishment (negative reinforcement) (Nagowah & Nagowah, 2009). The teacher
plays a central role as a transmitter of knowledge to learners and as a basic factor in
the educational process which reinforces the desired behavior .
A learning environment designed in the context of the Theories of Behavior must
have specifi c characteristics concerning the structure of the curriculum in short
units, gradual progress of the syllabus according to the learner's learning pace,

exposition of the syllabus in increasing order of diffi culty, and direct verifi cation of
the learner's response (Ertmer & Newby, 1993).
Theories of Behavior constitute the fi rst Learning Theories, which were utilized
for the theoretical support of technology application in education. Educational
applications integrating the aforementioned theories are mostly of the type of tutorials and drill and practice, and are judged effi cient at providing supervisory teaching
as well as at consolidating a low level of knowledge and skills (Nagowah &
Nagowah, 2009 ; Panagiotakopoulos, Pierrakeas, & Pintelas, 2003).
Cognitive Theories
Cognitive Learning Theories have ensued from a reaction to Theories of Behavior,
since researchers considered that the emphasis laid by the latter theories on the connection of stimulus-response was not effi cient enough to interpret human activity
during the learning process. They focus on the mental processes put into motion on
the basis of the stimuli presented in the human perceptual and cognitive systems.
That is, they probe into the functions of the cerebral mechanisms which are related
to learning and describe their relations so as for them to be better understood. To the
researchers of Cognitive Theories learning is a change in behavior, in thought, in
understanding, and in feelings (McLeod, 2003 ; Porpodas, 2000).
These new models focused on the processes of codifi cation, knowledge representation, on information storage, and on the retrieval and integration of new knowledge into the preceding pieces of information. The aim is still knowledge transfer in
the most effective and effi cient way. Cognitive Theories have imposed on education
metaphors, the analysis of complex concepts into simpler ones and the scrupulous
organization of the educational material from the simple to the complicated
(McLeod, 2003 ; Panagiotakopoulos, Pierrakeas, & Pintelas, 2003).
As far as the interaction between the learner and the software is concerned, there
is no strictly predetermined sequence in which the learner has to "learn by heart" the
subtopics of the syllabus, but he can study any of its subsections and test his knowledge wherever he considers (Anderson, Redder, & Simon, 1998).
Theories of Construction of Knowledge
Learning is a subjective and internal process of concept construction and is deemed
to be the result of organizing and adjusting new pieces of information to already
existing knowledge. That is, learning demands the rearrangement and reconstruction of the individual's mental structures, so that they adjust to new knowledge, and
also "adapt" new knowledge to the subordinate mental structures (Shunk, 2004).
A central role is played by the learner, who assumes an active role in the construction of his knowledge, by the preexisting knowledge of the learner, which has to be

modifi ed and expanded as a result of learning and by the teacher, who assumes a
supportive-consultative role in learners' activity (McLeod, 2003).
The researchers have adopted in their studies several approaches of constructivism, from the view of the theory, which supports that learning is simply the process
of adjusting our mental models to accommodate new experiences and involves constructing one's own knowledge from one's own experiences, to sociocultural
approach, which emphasizes learning from experience and discourse in authentic
learning environments, through collaboration and social interactions (Jonassen,
1999 ; Mahoney & Granvold, 2005).
A learning environment which is designed to comply with the Theories of
Construction of Knowledge, according to Boyle (1997), must have specifi c features
which aim at constructing knowledge, such as providing the learner with experiences, integrating knowledge in a realistic environment directly related to the real
world, encouraging the affi rmation and expression of perceptions in the learning
process, consolidating knowledge via social experience, encouraging the use of
multiple forms of knowledge representation, and encouraging self-awareness in the
process of constructing knowledge. Educational applications, which are designed to
take the aforementioned theories into consideration, should encourage a series of
procedures and support the creation of such teaching contexts which would advocate the idea of Knowledge Construction by the learner himself/herself while trying
to solve problems and interacting with the environment in this attempt. They should
also encourage learners' self-expression and provide authentic learning activities
(Edgar, 2012 ; Jonassen, 1999).
Educational Software for Maths and Science
Until today, there have been many studies about the potential of the educational
software to contribute positively to the learning process. These studies have concluded that the use of educational software as an educational tool under certain
conditions can be extended to all educational sectors (Bowman, Hodges, Allison, &
Wineman, 1999 ; Vanucci & Colla, 2010).
In the fi eld of mathematics education student's active involvement in their learning has the main role to their engaging with mathematical ideas as it is very important to them to participate in forming and testing hypotheses, trying out models, and
developing reasoned solutions to authentic problems. The use of software applications such as dynamic geometry, graphing and data-modeling packages, databases,
interactive games, simulations, and programming tools enables students to reach
accurate feedback and gain positive motivation (Barzel, Drijvers, Maschietto, &
Trouche, 2005). The use of such applications also allows students to focus on strategies and interpretations and not on complex computational calculations. In this way
the use of ICT in mathematics education supports constructivist pedagogy and supports students to explore mathematical concepts, relations and procedures, promoting higher order thinking and problem-solving strategies which are in line with the

new trends in mathematics education (Ruthven and Hennessy, 2002).
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In the fi eld of Science, tools such as the ones used for collecting, processing, and
interpreting data, databases and spreadsheets, graphic tools, modeling environments, multimedia software intended for simulating processes and phenomena and
for conducting "virtual" experiments, publication and presentation tools and, fi nally,
computer-controlled apparatuses can offer prospects for interactive educational
activities (Osborne & Hennessy, 2003).
By utilizing such tools appropriately learners are given the opportunity of working in an environment adapted to specialized educational needs, of comprehending
natural phenomena and natural laws by using hypotheses and trial and error methods, and of experimenting in a structured way by exploring a model and discovering
the elements of the simulation environment. Furthermore, learners are enabled to
use a variety of representations (images, graphs, data tables etc.), which are useful
in comprehending concepts and their interrelation. Finally, learners are given the
opportunity of expressing concept representation and cognitive models of the
surrounding natural world (Osborne & Hennessy, 2003 ; Velázquez-Marcano,
Williamson, Ashkenazi, Tasker, & Williamson, 2004).
Today innovative educational activities, which contribute to the interconnection
of concepts and, by extension, highlight the homogeneity of Sciences, are of special
interest in the educational process. The contribution of new technologies towards
that direction seems to be quite signifi cant, due to the fact that they provide the
opportunity to highlight the modern way in which Science and Maths are taught.
These technologies also create new prospects in the context of knowledge exploration and collaborative learning as the Theories of Construction of Knowledge
prescribe.
Studies have shown that teaching which takes advantage of ICT applications
proves to be much more effective, regarding the comprehension of Science-related
concepts, processes, and phenomena, than activities designed in a traditional way
(Redish, Saul, & Steinberg, 1997), since learners are given the possibility of a direct
visualization of natural or/and chemical phenomena (Bowman et al., 1999).
Methodology
For the needs of our research a sample was used which is, to a great extent, a
"convenience sample" (Bryman, 1995). To achieve greater reliability and control of
internal validity, the collection of data was done by using two research tools: a
questionnaire—as the main research tool—and a semi-structured interview (Cohen,
Manion, & Morisson, 2007). The questionnaire was addressed to teachers of Science
and Maths in Secondary Education, who participated in a national in-service training course. The objective of this course was to educate 28,100 teachers of primary
and secondary education, concerning the utilization and application of ICTs in the
teaching practice (http://b-epipedo2.cti.gr/en). Participants, after the training course,
expressed their perceptions on the educational software applications under investigation, by answering to specifi c questions related to the Learning Theory or Theories
which the applications utilize.
Theories of Learning in Math and Science Educational Software30
The questionnaire consisted of 50 questions which were divided into two parts,
one for collecting demographic data and the other for investigating the main research
question. The fi rst part included questions about the teachers' personal profi le with
special emphasis on their dealing with applications of educational software and on
their getting accustomed to using it. The second part (main part) of the questionnaire consisted of 40 questions related to the features of Learning Theories. Fifteen
out of the questions posed concerned the Theories of Behavior, seven of them the
Cognitive Theories, and eighteen of them the Theories of Construction of
Knowledge. The questions lent themselves to two types of answer, the Yes-No type
and the 5-grade Likert-type Scale (1 = Never, 2 = Rarely, 3 = Sometimes, 4 = Often,
5 = Always). According to the existing research on the features of learning theories
when apply to learning environments, a set of questions was formulated, which fall
into the following axes (Anderson, Reder, & Simon, 1998 ; Atkins, 1993 ; Edgar,
2012 ; Ertmer & Newby, 1993 ; Nagowah & Nagowah, 2009):
• The transparency of the aim and the learning objectives to be achieved
• The existence of a logical distinction and cohesion of the instructional units
• The way in which the syllabus is presented in the software
• The relation between the learner's prior knowledge and the content to be studied
• The cultivating of student's ability to interpret facts and phenomena
• The way a learner can be evaluated/self-evaluated
• Whether the educational activities are able to connect real-life situations based
on learners' experiences with representations of the real world and daily life
• Whether they cultivate the use of critical thinking methods allowing for the
learner's autonomous course in constructing knowledge
• Whether learners' active participation is promoted and in what way
• Whether the collaborative approach to knowledge and problem-solving is
encouraged by cultivating learners' creativity and imagination at the same time
The software applications chosen to be studied are the ones suggested by the
Hellenic Ministry of Education and the Pedagogical Institute to be used at schools
and which are used at the abovementioned in-service training course (3 closed type
and 2 open type ones, 4 regarding Science and 1 regarding Mathematics) out of
those most implemented in class.
Thirty-two (32) teachers of Science completed the questionnaire for each one of
the four software applications for science, and eighteen (18) teachers of Mathematics
completed the questionnaire for the application of mathematics. In order for the data
to be processed, the answers to the questionnaires were input to statistical data analysis software. At fi rst, the reliability coeffi cient α (Chronbach's alpha) of the questionnaire responses, regarding every group of the participants (Science teachers,
Mathematics teachers), was estimated. In order for the teachers' descriptive answers
to questions to be classifi ed, a content analysis of the responses was conducted
(Bogdan & Bilken, 1982).
The interviews conducted with participants who were in key positions in the
abovementioned in-service training course (Cohen et al., 2007). Interviewees were
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three teacher trainers specializing on the utilization and application of ICTs in the
teaching practice, who had great training and research experience on issues regarding the utilization of ICTs in the teaching practice. These trainers were educators in
the aforementioned in-service teachers' training course. Interviews aimed to provide qualitative data from the view of the trainer-expert, complementary or explanatory to the quantitative data provided by the questionnaires. The interview consisted
of 23 questions which were divided into two parts. In the fi rst part demographic data
were recorded while in the second part an attempt was made to collect data via
open-ended questions related to the research issues. A content analysis to the interviewees' responses was conducted.
Findings and Analysis
The values for the coeffi cient of internal consistency (Chronbach's alpha) of the
questionnaire responses were 0.842 for Science teachers and 0.853 for Mathematics
teachers. This fact shows strong reliability of the answers. The sample's characteristics are presented in Table 1 .
Twenty-seven (27) of the Science teachers answered that they use general purpose software for the needs of their lesson very often, whereas fi ve (5) of them use
it rarely. Twelve (12) of the Mathematics teachers also use general purpose software
often, while six (6) of them use it rarely.
The main criteria for the selection of the 3 teacher trainers who participated in
the interview were their involvement in the teacher's training courses and their
experience in using educational software applications in the classroom. They all had
a teaching experience of more than 15 years, great experience in training and
research, as well as in using educational software in class.
The following paragraphs present the results of the quantitative (Table 2) as well
as the qualitative research of the fi ve (5) characteristic software applications studied. The presentation of the quantitative results is made in relation to the number of
questions (related to every Learning Theory), which the teachers estimated that the
software fulfi lls to a "Full" (grade 4 and 5 of the scale) or "Partly" (grade 3 and 2 of
the scale) extent.
Table 1 Characteristics of the sample
Science teachers Mathematics teachers
Age range: 30-60 32-57
Teaching experience range: 3-31 7-31
Gender (M | F): 22 | 10 11 | 7
Total: 32
a
18
a
18 Physics, 6 Chemistry, 5 Biology, and 3 Geology teachers
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More specifi cally:
A Wonderful Journey in the World of Physics
The "A Wonderful Journey in the World of Physics" software fully satisfi es six (6)
questions about the Theories of Behavior, three (3) questions about the Cognitive
Theories and eleven (11) questions about the Theories of Construction of Knowledge.
It also partly satisfi es nine (9) questions about the Theories of Behavior, two (2)
questions about Cognitive Theories, and fi ve (5) questions about the Theories of
Construction of Knowledge.
The same fi ndings also emerged from the teacher trainers' interviews, who
responded that although the specifi c software is suitable for being utilized in class
through activities aiming at Knowledge Construction by learners, it also includes
elements of the Theories of Behavior. More specifi cally, there seemed to be a distinction between instructional units and a representation of the syllabus in various
ways. The examples, simulations and videos provided allow and promote the connection between the learner's prior knowledge and the content to be studied. Thus,
the learner's skill to interpret phenomena is cultivated and, simultaneously, a connection is being realized between the learner's experience and the representation of
his/her daily life.
Interactive Physics
Regarding the "Interactive Physics" software, due to the fact that it is a type of open
software, the teachers who participated in the research chose not to answer to all of
the questions in the Likert Scale or in the nominal Yes/No scale, but to answer periphrastically to some of them by making a comment. Based on the questions answered,
in the scales, this software fully satisfi es seven (7) questions about the Theories
of Construction of Knowledge and one (1) question about Cognitive Theories.
Table 2 Frequencies of answers per learning theories and software
Software/learning theory
Behaviorism
(15 criteria)
Cognitivism
(7 criteria)
Constructivism
(18 criteria)
Full Partly Full Partly Full Partly
A wonderful journey in the world of Physics 6 9 3 2 11 5
Interactive Physics 0 0 1 0 7 0
Composite Laboratory Environment—C.L.E. 0 3 2 1 11 4
The Wonderful World of Chemistry 7 8 4 3 9 7
The Geometer's Sketchpad 0 0 1 0 7 0
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In the rest of the questions, it becomes evident from the teachers' comments that this
software provides the user-teacher with the ability to design and support educational
activities aiming at his/her learners' Construction of Knowledge by planning and
utilizing appropriate educational scenarios and worksheets in the classroom.
On the basis of the aforementioned data, the descriptive answers of the participants to the rest of the questions and the teacher trainers' answers in the interviews
it could be concluded that this software distinctly applies the Theories of Construction
of Knowledge. Teacher trainers agree that, during the utilization of the software, the
teacher plays an especially active role. The teacher should, every time, design and
create a simulation which provides his/her learners the opportunity to work on powerfully exploratory learning activities via appropriately designed educational scenarios and specifi c worksheets.
Composite Laboratory Environment (C.L.E.)
The "Composite Laboratory Environment—C.L.E." software fully satisfi es eleven
(11) questions about the Theories of Construction of Knowledge and two (2) questions about Cognitive Theory. It also partly fulfi lls three (3) questions about the
Theories of Behavior, one (1) question about the Cognitive Theories and four (4)
questions about the Theories of Construction of Knowledge. Based on the aforementioned results, it would be concluded that the software orientates towards those
which mainly integrate the Theories of Construction of Knowledge.
The interviews lead to the same conclusions, as well. Due to the virtual experiments it provides, this type of software encourages the learner's ability to interpret
facts and phenomena. The virtual experiments and the videos included in the software are realistic to a great extent, a fact which facilitates the connection between
real-life situations from the learner's experiences and what is being studied through
the representation of the real world and everyday life.
The Wonderful World of Chemistry
The "The Wonderful World of Chemistry" software fully satisfi es seven (7) questions about the Theories of Behavior, four (4) questions about Cognitive Theories,
and nine (9) questions about the Theories of Construction of Knowledge. It also
partly satisfi es eight (8) questions about the Theories of Behavior, three (3) about
Cognitive Theories, and seven (7) questions about the Theories of Construction of
Knowledge.
The teacher trainers agree that this type of software is mainly utilized in activities
based on the Theories of Construction of Knowledge. However, it includes elements
from the Theories of Behavior, as well. The software includes distinct instructional
units. The syllabus presentation is carried out through defi nitions, examples, rules,
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and simulation examples. The possibility of learner evaluation is provided. The
learner is informed on his/her answer correctness through positive or negative feedback. Moreover, via examples, simulations, and videos, the learner's skill to interpret facts and phenomena is cultivated.
The Geometer's Sketchpad
As far as "The Geometer's Sketchpad" is concerned, since it constitutes open software, the teachers participating in the research chose not to answer to all the questions in the Likert Scale or in the nominal Yes/No scale. They also answered
periphrastically to some of them by providing a comment. Based on the questions
answered in the scales, this type of software fully satisfi es seven (7) questions about
the Theories of Construction of Knowledge and very much one (1) question about
the Cognitive Learning Theories. Thus, this software could be classifi ed as one of
the types integrating the Theories of Construction of Knowledge.
On the basis of the aforementioned data, the descriptive answers of the participants to the rest of the questions and the teacher trainer's answers in the interviews
it could be concluded that the specifi c software application is placed among those
applications which clearly serves the purpose of the Theories of Construction of
Knowledge, since the learner is evidently involved in the construction and structuring of his/her knowledge. It constitutes interactive software, which enables the
learner to be actively involved in geometrical design and construction activities,
which strongly promote learner autonomy.
All fi ndings are in agreement with the objectives of the developers of the fi ve
applications studied, as reported in the supporting material of each application.
Discussion and Conclusions
In the present research paper the dominating Learning Theories were studied in
terms of their existence in applications of Educational Software for Science and
Mathematics, according to the perceptions of a sample of in-service teachers. These
perceptions resulted from the participants' responses on questions concerning the
characteristics of fi ve specifi c educational applications.
According to the sample's perceptions, the results of the fi ve applications presented allow us to conclude that open type applications (Interactive Physics, The
Geometer's Sketchpad) serve the Theories of Construction of Knowledge, since
through them the teacher is able not only to create new interactional applications
according to his/her learners' educational needs, but also to encourage learners in
creating by themselves their own simulations and models of interpreting processes
and phenomena. These fi ndings are in agreement also with the three expert teachers'
responses.
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The fi ndings from both the answers of the sample and the three expert teachers

interviewed about the close type applications showed that these applications incorporate characteristics of both Theories of Behavior and Theories of Construction of
Knowledge. All these fi ndings fall in with the statements of the developers of the
fi ve applications studied, as reported in the supporting material of each application.
Taking under consideration that the dominating theories of learning are the Theories
of Construction of Knowledge, as they can broaden the students' skills better than
others (Malabar & Pountney, 2002), it is obvious that the fi ve applications which
were studied and are proposed for use in the Greek schools incorporate these theories in whole or in part.
Moreover, it is important for a teacher to be able to recognize specifi c characteristics of an educational application concerning the embedded learning theories in
order to utilize the application in the most proper way in the educational process.
Teachers should be supported via suitable training courses towards this direction.
However, studies have shown that many factors affect, eventually, the way of integrating ICTs in the educational process. Such factors may be related, except for
teacher training, to the technological infrastructure of school units, to both technological and pedagogical support in applying educational activities in class, and to
teachers' beliefs about teaching and learning, their attitudes towards ICTs in the
fi eld of education, and towards life in general, skills of using ICTs, etc. (Sang et al.,
2010). All the aforementioned factors should be studied, taken into account and
dealt with in order for ICTs to be fully integrated into school class and practice.
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Introduction
Although Information and Communication Technology (ICT) use and particularly
computer use in early childhood educational settings is under-examined in comparison to other educational levels (e.g., primary and secondary education), the debate
is no longer one of should we or should not we use computers in early childhood
settings. Electronic culture is already an integral part of early childhood experience
for many youngsters (Parette & Blum, 2013). Research studies (Clements & Sarama,
2003 ; Haugland, 2005 ; McCarrick & Li, 2007 ; McKenney & Voogt, 2012 ; Siraj-
Blatchford & Siraj-Blatchford, 2006 ; Stephen & Plowman, 2003 , 2008 ; Yelland,
2005) reported that computer can be used as a tool to support learning, and assist
communication, collaboration, creativity, and language development in young children. Judge, Puckett, and Cabuk (2004) reported that it is increasingly important for
early childhood educators to introduce and use computers in their settings, particularly for those children who do not have access at home. They assert that offering
access to computers in early childhood settings helps to reduce the digital divide.
The essential role of early childhood teachers in the improvement of children's
computer/ICT related experiences has also been reported (Stephen & Plowman,
2008). Kindergarten teachers are, for example, responsible for selecting educational
software and deciding on the ways of its use. The aim of this research study was to
investigate educational software use by young children in kindergarten classes.
For the purpose of this paper, specifi c terms used are briefl y explained. Initially,
the term computers is used as synonymous and as more preferable to the terms ICT
(Information and Communication Technology) and technology . Apart from computer
software, a number of products that incorporate some aspects of ICT are available to
Educational Software Use in Kindergarten
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young children such as electronic-musical keyboards, programmable interactive
toys, and digital cameras. However, practitioners defi ne ICT more narrowly as computers and printers and this view is very infl uential (Stephen & Plowman, 2008) till
now. Moreover, this study focuses on computers and software that runs on computers as, for the present, this is the predominant technological device for those kindergarten classrooms in Greece that have access to technology. The term educational
software use refers to children's computer related activities, as well as the observation of such activities. The term early childhood settings is used as synonymous to
the terms kindergartens and preschools . This term refers to kindergarten classrooms
(formal educational settings) that attend children above 3 years old.
Theoretical Background
Educational Software Usage in Early Childhood Settings
Regarding educational software usage in kindergarten classes, different software
programs/applications have been used such as interactive multimedia environments,
games, painting/drawing programs, Word processing, and Logo programming
(Chera & Wood, 2003 ; Clements, 2000 ; Labbo, Sprague, Montero, & Font, 2000 ;
Sung, Chang, & Lee, 2008). The boundaries between different software applications produced for young children are not necessarily fi xed, as different applications
are presented in a form of play designed to attract and sustain children's attention.
For example, both commercial and educational CD-ROMs incorporate the play
component, which should not be seen solely for recreation or fun purposes
(Verenikina, Herrington, Peterson, & Mantei, 2010). Early childhood educational
software contains activities which quite frequently focus on early development
skills such as sorting-classifying, matching, following instructions, and spatial reasoning (Stephen & Plowman, 2003). Regarding its usage in early childhood educational settings, McKenney's and Voogt's study (2010) showed that children 4-7
year old reported playing games, practicing words/maths, and drawing as the most
frequently activities done in kindergarten classrooms. Marsh et al. (2005) reported
frequent use of painting/drawing packages and rare use of the Internet.
Although there are a variety of educational software programs used by children
in kindergarten classes, there is limited empirical evidence on how educational software is used in classrooms. For example, Ljung-Djärf, Åberg-Bengtsson, and
Ottosson (2005) reported that computer can be used, among others, as an available
option or as an essential activity. Computer use is often something that may be
allowed between planned or adult-led activities, which means that it is typically
used during the time that is organized as free play (Howard, Miles, & Rees-Davies,
2012 ; Ljung-Djärf, 2008). In America, the National Association for the Education
of Young Children and Fred Rogers Center for early learning and children's media
(NAEYC - FRC 2012), in their position statement regarding the role of technology
in preschool classrooms, state that (a) technology and interactive media tools must
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be used appropriately, and (b) technology integration is effective when integrated
into the environment, curriculum, and daily routines. When computer use takes
place within kindergarten classes, the role of teacher/adult intervention in supporting and extending children's experiences has been emphasized (Nir-Gal & Klein,
2004 ; Stephen & Plowman, 2003). For example, McKenney and Voogt (2009)
found that teacher intervention elicited young children's engagement with literacy
concepts and that children were able to work independently with the computer program after a few instruction sessions. Other studies have shown that children are
largely left alone at the computer with the teachers seldom to participate in what
goes on there (Plowman & Stephen, 2005) and, in particular, when children played
computer games there was lack of teacher intervention (Ljung-Djärf et al., 2005 ;
Vangsnes, Økland, & Krumsvik, 2012).
In Greece there are a few empirical studies regarding educational software or
Internet usage in kindergarten classes (Chronaki & Stergiou, 2005 ; Fesakis,
Sofroniou, & Mavroudi, 2011). For example, Chronaki and Stergiou (2005) compared computer use in two kindergartens, carrying out interviews with teachers and
children. This paper presents research fi ndings regarding educational software use
in kindergarten classes in Athens, Greece. Fesakis et al. (2011) carried out an experimental case study of a learning activity meant for teaching preschoolers geometric
concepts, via the use of communication tools from the Internet. The results of this
study constitute part of a research project regarding ICT use-integration in kindergarten classes in Greece. Thus, some information about ICT in early childhood
education in Greece is provided below.
ICT in Early Childhood Education in Greece
The Greek educational system is centrally organized and the main bodies of educational policy and planning are the Ministry of Education (YPEPTH) and the
Pedagogical Institute (PI). Until recently, there was a lack of a central plan for
the introduction of ICT. The Pedagogical Institute has published a framework for the
introduction of ICT in teaching and learning, the so-called "Cross-Thematic
Curriculum Framework for ICTs." For early childhood education, it sets directions
for programs regarding planning and development of activities in the context of the
following subjects: language, mathematics, environmental studies, creation/expression, and computer science (YPEPTH - PI 2003). These programs are not considered as independent subjects, but it is suggested to be taken into account when
planning and implementing meaningful and purposeful activities for the children.
Among the essential prerequisites for computer integration in early childhood
education are the placement of the computer/ICT in class (i.e., the so called "computer corner") and its inclusion in kindergarten's daily teaching practice (Komis,
2004). In order to successfully implement the curriculum, it is essential for teachers
to be provided with the appropriate training and for early childhood classrooms with
the appropriate resources. Regarding resources, many kindergartens have lately
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acquired computers for use by the children. However, those kindergartens with a
computer in their classrooms have, more or less, similar technology facilities (i.e.,
predominantly one or two computers). There are no computer labs in kindergartens.
Regarding teachers, they are responsible for translating into practice the expectations/visions of curricula planners. The "Teachers' training on ICT in Education"
program (YPEPTH - PI 2009), which is the most widespread in Greece, included
the training of early childhood teachers as well. The fi rst phase of the program
included training in technical skills and has been attended by the majority of early
childhood teachers. The second phase of the program which is dedicated to providing teachers with the pedagogical skills for ICT integration in the classrooms has
been attended and is currently being attended by a number of teachers (the actual
number is not known yet, as it forms part of internal evaluation of the program which
still takes place). This large scale inservice training aims, among others, to familiarize teachers with appropriate educational software and the skills to adopt-integrate
ICT in their everyday teaching practices. Within the Greek context, there is currently
poor uptake of computers in early childhood settings (Nikolopoulou, 2009).
Method
Objectives of the Study
As stated earlier, the aim of this research study was to investigate the use of educational software in kindergarten classes. Three basic parameters of the study were as
follows: (1) the presence of at least one computer in the kindergarten classroom and
its use by preschool children, (2) the voluntary participation of kindergarten teachers, and (3) the study to be carried out without artifi cial intervention by the researcher
(i.e., into the "natural" classroom environment). For the purpose of this research,
offi cial permission was provided by the research department of the Greek
Pedagogical Institute and all kindergarten schools and their participants were treated
anonymously. The objectives of this study were:
1. To identify the educational software commonly used by children in kindergarten
classes.
2. To investigate how educational software is used in kindergarten classes.
3. To fi nd out possible diffi culties children face in the use of educational software.
Sample
Seventeen state kindergartens from Athens, in Greece, participated in this study and
their teachers agreed to voluntarily participate in the research project. All kindergarten teachers were women (age range: 26-55), most of whom had attended the fi rst
phase of the "Teachers' training on ICT in Education" program (i.e., introductory
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training in ICT skills). However, none of them had attended the second phase of this
program which regards pedagogical training in the use of ICT in classes (this phase
has started only recently, see YPEPTH 2012). All schools were committed to the
same National Curriculum guidelines. The ages of the children ranged from 4 to
6.5 years. However, over 88 % of the children were aged 5-6.5 years old, as kindergarten attendance in Greece is now obligatory for this age group (i.e., the age group
that will attend primary school during the next academic year). Table 1 shows the characteristics of kindergartens, as derived from interviews/discussions with the teachers

and observations made in classes. In order to maintain the anonymity, the codes
N1-N17 (N1: kindergarten 1, N2: kindergarten 2, etc.) were used. Each kindergarten participated in the research with one or more classes (full-day and/or classic
class, as shown in the second column of Table 1). Classes held between 13 and 26
children, except those serving children with special needs. Table 1 also shows the
number of teachers interviewed, the total number and ages of children, the number
of computers, as well as the frequency and the duration of computer use (per child).
One or two computers were available in 15 (out of the 17) kindergartens and this
refl ects the typical situation in Greek kindergartens. The computer(s) were located
in the classroom, they were set up on a table at the computer corner, while each
computer had two kindergarten-sized chairs placed before it: to facilitate collaboration and to provide a place for children to sit and watch their peers if they are waiting to use the computer on their own (according to teachers' explanations). In only
one school (N9), the teacher had her desk in the classroom and as a result it was the
same computer (located on teacher's desk) being used by both the children and the
class teacher (for administrative purposes etc.). Computer use appeared to be an
established practice during the last 3 years.
Almost all schools (except N17) were located in central semi-urban areas of the
Greek capital, in medium (neither high nor low) socioeconomic areas, without variation in ethnic background. Only the school N17 was located in a working class
socioeconomic area, around 17 km away from the city center. Around 90 % of the
children of the sample spoke Greek as a fi rst language.
Data Collection Process and Research Instruments
The data were collected between January 2009 and June 2010 and consisted of interviews and informal discussions with the kindergarten teachers, observations and
fi eld notes of class activities. All data were collected by the researcher and author of
this paper. Qualitative approaches seem practical and valuable for early childhood
settings, although they include small samples and are not easily generalizable
(Nikolopoulou, 2010). The interviews with the teachers were conducted before the
observation sessions, they took place in the classroom (after the preschool day
ended) and they were recorded digitally and transcribed. Approximately 8 h of interview recordings was collected and transcribed. The semi-structured interviews were
based on specifi c axes related to the research objectives. The following questions
were used: What educational software programs are used by children in the class?

Table 1 The characteristics of kindergartens
Kindergarten Type of class(es)
No. of
teachers
No. of
children
Ages of
children
No. of
computers
Frequency of
computer use (per child)
Duration of
computer use (min)
N1 Full-day
a
1 23 4-6 2 2-3 Times/week 10
N2 Full-day 1 25 5-6 1 Once/week 10-15
N3 Classic
b
2 26 4-6 3 Once/week 30 (max)
N4 Full-day 1 13 4,5-6 1 Once/week 5-15
N5 Classic 1 15 4,5-6,5 2 Every day 10
N6 Full-day 2 21 5-6 1 Once/week 8-15
N7 Classic +, full- day 2 28 5-6 2 Once/week 10
N8 Full-day 2 22 5,5-6 1 Once/month 8
N9 Classic +, full- day 2 43 4,5-6,5 2 Once/week 10
N10 Full-day 1 25 5-6 2 Once/week 10-15
N11 Full-day +, 2 classic 2 57 4-6 1 Every 3 weeks 10-15
N12 Classic +, full- day 2 41 4,5-6,5 2 Weekly –once/month 10-15
N13 2 Full- day 2 49 5-6 3 Once/week 10-20
N14 Classic 1 23 5-6 1 Once/month 10-15
N15 Classic 1 22 5-6 1 Once/week 10-20
N16 Special needs 1 4 4-6 1 Once/week 10-20
N17 Special needs 1 4 4-6 2 Weekly/bi-monthly 10-20
a
Full-day class: children attend kindergarten from 8:30 a.m. to 4 p.m.
b
Classic class: children attend kindergarten from 8:30 a.m.to 12:30 p.m.

How are these programs being used? Do children face diffi culties when using
specifi c software programs? All teachers were confronted with the same set of core
questions, while follow-up questions were formulated in order to offer teachers the
opportunity to introduce unexpected ideas and thoughts. The interviews also
addressed areas such as demographics and computer availability/access.
In parallel with the interviews, in each kindergarten were conducted two or three
observations of computer use (i.e., two or three sessions, each one in a different
day), of total duration of 1.6-3 h. All observations took place from January (2009 or
2010) onwards, after negotiation with class teachers. This time period, being
3 months after the commencement of the academic year, was proposed by the teachers because they wanted their children to be initially acquainted to the new kindergarten environment before any research took place. The observations were
naturalistic observations of the children's and teachers' activity in the classroom. As
the researcher was also an observer, she sat close to the computer corner and she
needed to be mindful of her position in the classroom (e.g., as a researcher observing children using computers there was a challenge of supporting their engagement). During the observations, a small number of children (two or three) sat at the
computer corner, while the rest of the class was engaged in other free play activities.
Children did engage with the computer activities in ways that suggested that they
were comfortable with the researcher's presence. The observations involved the
writing of detailed fi eld notes (what took place around the computer etc.) during and
immediately after the observations. As fi eld notes were reviewed memos were
recorded to document emerging thoughts, feelings and questions regarding the
observations. Informal discussions with the teachers took place during and mainly
immediately after the observations (e.g., for clarifi cations on various actions).
In order to investigate the fi rst research objective, the data were collected via
interviews with the teachers. For the investigation of the second and third objectives, interviews and observations/fi eld notes were used. As a main parameter of the
study was to be carried out without an imposed intervention by the researcher (i.e.,
in the natural environment of the everyday practice), it was the teachers who decided
on the timing of the observations, as well as on the software used. There was also
negotiation with the teachers with regard to the number of sessions observed (i.e.,
two or three sessions).
Data Analysis
The qualitative data gathered (from interviews, observations, fi eld notes) were each
analyzed using classical content analysis. All the data sources were analyzed to fi nd
codes that could be organized into categories. The codes were produced deductively
and then included as descriptive information about the data (Leech & Onwuegbuzie,
2007). Results derived from one source of data were supported by other data
sources. Some of the interview statements most strongly supporting the categories
are presented.

Ethical Considerations
Regarding ethical considerations, kindergarten teachers were informed about the
nature, duration, and the aim of the research study. Ethical issues arise when investigating dependent, vulnerable members of society such as young children in early
childhood education settings (Morgan, 2010). Gaining informed consent from
research participants is widely regarded as central to ethical research practice and in
institutional settings such as schools, access tends to be mediated by teachers, managers, etc. (Heath, Charles, Crow, & Wiles, 2007). Issues of anonymity and confi -
dentiality are also included in ethical considerations, thus the above parameters
were assured in this study (i.e., all kindergartens and participants were replaced
with codes). Additionally, initial and ongoing consent with teachers was negotiated
and participants were informed about research outcomes.
Results and Discussion
This section describes and discusses the results in three subsections, according to
the research objectives, with parallel presentation of indicative excerpts from the
interviews. This is done because direct quotes from the interviews can help readers
to acquire a more complete view of the events and situations (Forman & Hall, 2005).
The fi ndings are presented for all schools, descriptively.
Educational Software Commonly Used by Children
in Kindergarten Classes
Table 2 shows the educational software/programs most commonly used by young
children in classes, as reported by kindergarten teachers: the MS Paint, the commercial series Ram Kid/Kide Pedia (which include many games), followed by the use of
Table 2 Educational programs commonly used in kindergarten classes by children
No. of times
mentioned
MS Paint (and painting/drawing programs such as TuxPaint) 13
Ram Kid, Kide Pedia (commercial series that include many games) 11
Educational CD-ROMs (e.g., "My class") 9
Word processing (MS Word) 8
"Explorer of the computer–Electronic Postman" (educational software distributed
to most all-day classes (by the Greek Ministry of Education)
5
Internet 5
Digital games (commercial) 3
MS PowerPoint 2

educational CD-ROMs (e.g., "My class," "Salto and Zelia," "On the road safely")
and the MS Word. The programs "MS Paint" and "MS Word" are included in every
computer as parts of the Microsoft Offi ce, and thus, there is no extra cost for the
kindergartens. Taken into account the limited school budgets, such open ended software is suggested to be embraced in kindergarten classrooms (and as a consequence,
it is interesting to investigate examples of good practice and what actually children
learn). Additionally, the educational software "Explorer of the computer–Electronic
Postman" (contains language and mathematics learning activities) has been distributed recently, during the academic year 2008-2009, by the Greek Ministry of
Education in most full-day kindergarten classes without any cost. Some kindergarten teachers use this software in their class and characterized it as particularly interesting. For example, in N4 (full-day class) it is used as a supporting tool for language
and mathematics, while the teachers in N7 (classic and full- day classes) use it systematically during the hour of free play activities. However, other teachers (in N1,
N2) who have this software in the kindergarten, do not use it in their class because
they do not consider it as appropriate for the children. For example, "I do not consider it (the software) appropriate as to its design-placement of specifi c buttons on
screen–because the children can be easily lost" (teacher in N1), or "It has a long
introduction and the kids get bored…" (teacher in N2).
Some programs (especially the commercial series Ram Kid/Kide Pedia and the
educational CD-ROMs) are frequently brought in classes either by the teachers (e.g.,
in kindergartens N2, N10, N11, N12, N14) or by the parents (e.g., in N3, N9, N10,
N11, N12). Table 2 reveals a limited range of programs used in classes and this may
be attributed to the tight school fi nances. The fi ndings are in some agreement with
earlier research. For example, Marsh et al. (2005) reported frequent use of painting/
drawing programs and rare use of the Internet, while others (Lee & O'Rourke, 2006)
found frequent use of CD-ROMs (e.g., games, talking books). In Greece, Chronaki
and Stergiou (2005) found that children carried out educational activities (writing
words and numbers, painting), aiming at children's familiarization with the computer. The reasons for using computers with children in kindergarten classes (discussed in another paper) were the development of language and fi ne motor skills,
and the computer's contribution as an incentive in the learning process. In particular,
the MS Word was mainly used for the development of language, reading, and writing skills, for emergent literacy skills and in order to support curriculum learning
objectives in language–as well as for providing practice in the recognition of the
alphabet letters. The Internet (in those kindergartens which had a connection) was
reported to be mainly used for downloading/playing games, for carrying out project
work and for downloading educational material and photographs.
Independently of the programs used and the tight budgets, the role of the
kindergarten teachers was essential because they decided on the program(s) used by
their children. An example of an interview excerpt was: "We have some of the Ram
Kid series, we intend to buy them with our money. A mother has brought the program 'On the road safely' and when we talk about traffi c education we use this very
much… We fi rst look at the software, i.e., we see whether the activities included are
appropriate for our children. We have found some (programs) that were very nice

but they did not correspond to our children's level and interests" (teacher in N11).
Different decisions made by the teachers are mentioned in the following sections of
the paper. The crucial role of the kindergarten teachers in the computer environment
has been reported in literature (Keengwe & Onchwari, 2009 ; Nir-Gal & Klein,
2004). The essential role of the teachers in this study is also highlighted in the subsequent sections of this paper.
How Educational Software Is Used in Kindergarten Classes
This section discusses how educational software was used in the classes, and it also
includes fi ndings regarding when the software was used. Computer use took place,
mainly, during the hour of free activities/play at the computer corner. As most teachers stated, one hour every morning or every afternoon is devoted to computer use.
These fi ndings are in agreement with the relevant literature which refers to computer
use in kindergarten's daily practice so that children can understand its contribution
towards teaching and learning (Plowman & Stephen, 2003), and to that computer use
took place during the free play activities (Ljung-Djärf, 2008 ; Ljung- Djärf et al.,
2005). In this study, the frequency of computer use, per child, was found to vary
between 2 and 3 times per week and once per month (see Table 1). In most kindergartens, teachers reported that every child was entitled to work on the computer (if
s/he wanted) once per week with an average time of 10-20 min. In about half of the
kindergartens there was a sort of a systematic way for the children to have access to
the computer, such as a reference plan. In cases of a lack of a reference plan, children's use of the computer was not recorded, but some teachers said they did remember which children passed from the computer corner. Educational software was often
reported to be used in the context of a project in combination with other non-computer based activities (e.g., in N1, N2, N3, N5), or for producing the class' newspaper (in N1). Computer use is suggested to be combined with off- computer classroom
activities, as relevant literature has shown that this way can lead to better learning
effects (Haugland, 2000 ; McKenney & Voogt, 2009). Parette and Blum (2013)
discussed elements of technology integration and identifi ed key activities in which
technologies can be used across preschool settings. Such activities may embrace the
use of educational software in order to support and enhance children's leaning.
Class organization in computer environment was found to, mainly, take place in
small groups because as two teachers explained: "It works very well as a group
because the other child sits next to it and says 'you haven't seen this!' or 'no, we must
not do this', i.e., they discuss, they help each other, the child waits his/her turn"
(teachers in N11). An exception to group work constituted the kindergarten N17,
which serves children with special needs. In this case, each child works with the
teacher's help in order to achieve specifi c objectives: "The autistic child, who can not
interact simultaneously with the screen and the teacher, focuses on the screen… It is
important for children with special needs to acquire fi ne motor skills and to understand the cause-effect relationship. There are diffi culties depending on the child, for

example with the mouse. I do help them… Two objectives can be pursued in parallel,
for example, development of fi ne motor and cognitive skills" (teacher in N17).

Class observations revealed that, in most cases, when children were using different programs there was some type of teacher intervention and guidance. For example, when they were using the program MS Word there was an initial guidance from the teachers and afterwards an intervention-assistance when children did not know specifi c functions (transition to the next line, error erasing, etc.). The class observations (in kindergartens N1, N3, N4, N6, N7, N9, N10) revealed that children, with the help of their teachers, used successfully different functions of the MS Word such as "delete an error" (button "Backspace"), "transfer to the next line" (button "Enter"), "select a color," "drag and drop," and "save a fi le." Children were happy to write their name, to select-change a color and to change the size of the letters of their name (as well as of other words). It was also observed that children cooperated and helped each other for the successful outcome of a learning activity. In any case, it was the teacher who knew her children in the class and she decided on the type of intervention needed. For example, an interview excerpt was: "Children sit (on the computer) several times on their own and seek help from other children—they believe they know—rather than from myself... I consider it important to sit next to very young children because when they cannot fi nd something they get bored and leave the computer" (teacher in N7). Teacher's guidance/mediation could be broadly divided into three categories, which were also observed to take place in combination: (a) step-by-step guidance for teaching new concepts/skills, (b) initial explanations (e.g., function of specifi c buttons) provided by the teachers and then independent work by the children, and (c) teacher guidance/assistance whenever children asked for it. These types of teacher guidance are discussed among the recommendations made by researchers (Plowman & Stephen, 2005 ; Stephen & Plowman, 2008) for more effective computer integration in early childhood settings. Regarding teacher guidance, the use of computer games (the series Ram Kid, Kide Pedia, etc.) constituted an exception, as children played on their own (without asking for help), they often chose activities they had played before and knew in advance the correct answers (they did not even wait to listen to the instructions), and they never asked for teacher's help. Regarding this point, there is an agreement with other research studies (Ljung-Djärf et al., 2005 ; Vangsnes et al., 2012) which reported lack of teacher intervention when children played computer games. As evidence-based guidelines for computer use in preschool education are limited (Siraj-Blatchford & Siraj-Blatchford, 2006), it is rather uncertain for teachers on how to achieve the visions-claims reported in literature. The issue of teacher guidance when children use different educational software programs in kindergarten classes is suggested for further investigation.

Teachers reported that children's computer work is often displayed in the classroom. For example, children's drawings made with the program MS Paint alternate as a background on the class computer (in N2), children's printed work is displayed at the end of the year (in N9), while all children's computer work is copied onto a CD for the parents (in N10). For example, "There is a computer folder which is given to parents at the end of the year...it includes all work the child has done on

the computer (printouts with names, pictures on Paint etc)" (teacher in N10). In such a way, computer use is considered as an activity carried out in kindergarten's daily practice, an activity like other school activities that may foster the links between school and parents. Additionally, several teachers noted examples of children's evident pleasure during their engagement with ICT (i.e., an aspect of children's preschool experiences that is valued, as promoting a positive disposition towards learning): "After help as to where to click, she was happy to print, a short phrase she wrote using MS Word, independently" (teacher in N4).

The interviews with the teachers also revealed the time period (timing) during which some of the programs are being used in classes. It was shown that computer games are mainly used at the beginning of the academic year, while the word processor is used by the end of the year. Indicative quotes from interviews were: "Computer games are mainly used at the beginning of the year, so that children become acquainted with the use of the mouse, while the word processor is used by the end of the year" (teacher in N3), "Children, especially by the end of the year, recognize the letters of the alphabet, they can make the transfer between lowercase and uppercase letters... thus the word processor is used by the end of the year" (in N7), and "The letters on keyboard are uppercase, while the children in kindergarten learn initially the lowercase letter... the (program) Word is used in spring" (in N9). In another kindergarten where the uppercase letters are being taught fi rst, "Every child who enters the class in the morning is encouraged to sit on the computer to write her /his name on a separate line. They have been taught the uppercase letters and now (month May) we are in the process of teaching the lowercase letters. After March they started to write their name in the Word every morning, as a kind of register -till then they wrote it on a paper" (teacher in N6). It appears that some teachers decide on the learning goals to be achieved in their class before they decide on the type of software, and this decision/sequence is in agreement with earlier research (Haugland, 2005 ; Lin, 2012). For example, Lin (2012) suggests that teachers can choose the learning objectives and design a series of activities (software use could be among them) in order to accomplish learning goals.

Diffi culties Children Face in the Use of Educational Software

The interviews with the teachers in combination with class observations revealed some diffi culties children encountered when using the programs MS Paint and MS Word. Interestingly, although the use of MS Paint was most commonly reported (as shown in Table 2), in almost one third of the kindergartens (in N7, N9, N10, N11, and N16) children faced some sort of diffi culties. Examples of excerpts from the interviews were: "The children initially preferred games but now they are also using MS Paint with my help, because they have diffi culties" (teacher in N10) and "The MS Paint is not used because i fi nd it diffi cult... in order to get a quality result - painting—it needs time for training" (teacher in N7). One factor that makes the program MS Paint diffi cult is that some children have not developed the necessary fi ne motor skills: for example, they have got used to draw a house much easier by

hand using traditional materials, rather than by using the Paint program. Drawing with pencils and other traditional materials is an essential traditional activity and the use of the software does not come to replace it but to support and extend children's experiences. Guided interaction (e.g., demonstrating how to use the eraser/paintbrush, or providing feedback so as to encourage the child's efforts) is a way for helping the children overcome some diffi culties. Overall, the role of learning in the use of software programs is important and (as a teacher mentioned) time is needed for children's training and practice.

Regarding the use of MS Word, almost in every class where its use was observed, children faced diffi culties in identifying specifi c alphabet letters on the keyboard—because the Greek and the English characters were both present on some buttons. This diffi culty could be overcome by placing on the keyboard, stickers with the Greek uppercase letters, a procedure which was successfully followed in kindergarten N6. Furthermore, another diffi culty was that children pressed continually one button, with the consequent need to delete several characters (i.e., by using the Backspace button). In order for the MS Word to be used in class, it is suggested for children to have an initial acquaintance with the alphabet letters: this helps children to identify/recognize the relevant letters on the keyboard. Thus, the time period the Word is introduced in class is very important. It has been mentioned that teachers chose to introduce the program in spring, towards the end of the academic year. The role of learning in the use of MS Word is important. For example, as two teachers reported informally, children are sometimes given text to copy and as their skill and accuracy develop, they are expected to type and print short sentences.

Aspects of observations together with the interviews revealed that some children haven't developed the necessary fi ne motor skills for the use of the software. At the beginning of the academic year some problems appear mainly with those children who do not have a computer at home, but afterwards all children seem to become accustomed to computer use. Diffi culties with the mouse seem to be easily overcome, through practice. Examples from interview excerpts were: "Some children need physical guidance to support their development of mouse control. However, after few attempts they get mouse control and become more confi dent" (teacher in N8), "Children are more familiar with the mouse—because they play games—rather than with the keyboard. They often press continually the keyboard buttons... I've noticed differences in children's fi ne motor skills due to computer use at home" (teacher in N7), "We found out that some children do not have computer at home and at the beginning they face diffi culties. The computer helps in the development of fi ne motor skills -the mouse helps a lot...In general, when computer games are carefully selected they help a lot" (teacher in N11) and "Gender differences are not intense, the differences appear due to different access to computer at home and due to the frequency of choosing the computer at school...sometimes, children 4 years old handle the mouse much better than older children who rarely choose to use the computer at school" (teacher in N6). Additionally, in a class with children with special needs there were present some initial diffi culties, which were then overcome: "We started with the painting program where children struggled a bit because it did not come out—what they wanted—i.e., what comes out with the hand. They faced diffi culties with the use of the mouse...Later on, we put children's names on

the computer and they were encouraged to write them with different characters" (teacher in N16). The interview excerpts reveal the role of learning/practice as well as the crucial role of the kindergarten teachers in the study. Computers have the potential to support young children's classroom experiences, but for this to happen "it is necessary for teachers to carefully plan for, and articulate to children, suitable classroom tasks" (Kervin & Mantei, 2009 , p. 30).

Some of the interview excerpts presented in this paper, as well as others not shown here, revealed issues related (a) to the use of computer by children at home and (b) to children's gender. The already developed children's fi ne motor skills were often attributed to the use of computer at home. Recent research (McKenney & Voogt, 2010 ; Plowman, McPake, & Stephen, 2008) reported that young children have access to and use the computer/ICT at home and many children had acquired the basic computer skills already from home (before they entered kindergarten). As there are differences regarding the skills children bring to classroom use of computers, it is suggested for teachers to be aware of these skills (in order, for example, to provide support for those children without previous technological experience).

Regarding children's gender, the results appeared contradictory. In some kindergartens (in N9, N10, N12), children's gender appears to be related to the frequency of computer use (is higher for boys) or to children's preferences to different types of software. Some teachers reported that boys were more interested in using the computer because they chose it more frequently during the hours of free activities, while the girls preferred to be engaged in other activities. Two interview excerpts regarding children's preferences were: "Girls prefer activities with puzzles, painting" (in N9) and "Boys prefer action games, games that provide scores" (in N9, N12).

However, in other kindergartens (e.g., in N6 and N7) no gender differences were reported, as "When it comes their turn to work on the computer all children want – it did not happen a child not to want (the computer) and to choose another activity" (teacher in N7). In literature, gender differences regarded children's preferences towards interface design (Passig & Levin, 2000) and software's characters (Littleton, Light, Joiner, Messer, & Barnes, 1992). In particular, young boys seem to be attracted by movement, "male" characters and action games, while young girls seem to prefer the colorful buttons of a program and "female" characters. However, other studies did not report on signifi cant gender differences when children use computers in class (Hatzigianni & Margetts, 2012 ; Shawareb, 2011). As a consequence, the issue of gender can be further investigated.

Concluding Comments and Recommendations for Future Research

The research described in this paper provides some useful insight for researchers and teachers about educational software usage in kindergarten classes. The class observations and the interviews with the teachers revealed some issues diffi cult to be explored through large-scale quantitative surveys. The data derived from the 17

kindergartens cannot be generalized due to the small sample and its origin from one region. However, taken into account that in Greece (as well as in other countries) there is limited empirical research, this study contributes to our understanding of computer use and classroom practices with young children. Concluding comments are presented below.

The most commonly used programs were painting programs (especially the MS Paint), the commercial series Ram Kid/Kide Pedia (which include many games), educational CD-ROMs and the MS Word, while the use of the Internet was rare. For early childhood settings with tight budgets, the use of software programs that are free, low cost or downloadable seems a good choice. In parallel, researchers have emphasized the issue of using developmentally appropriate software, in appropriate ways (Parette & Blum, 2013). Classroom activities designed using appropriate technologies and NAEYC principles (e.g., the technologies should align with the curriculum, the choice of technology should be based on how it serves classroom learning, see NAEYC – FRC 2012) support the developmental learning needs of young learners. It is the case that not all commercial computer games are developmentally appropriate for use by young children. As a starting point, software/games need to be carefully evaluated by teachers (a process facilitated when teachers have attended appropriate training) before any usage in the classroom.

This study found that computer use took place, mainly, during the hour of free activities at the computer corner. "Playing with the computer" designates a series of qualitatively different activities in which children can, for example, engage in drawing, writing or play (Plowman & Stephen, 2005), and computer use in the hour of free play activities is linked to literacy and pedagogy. For example, the manipulation of symbols and images on computer screen represents a new form of symbolic play, and there is potential in the development of children's higher order thinking (Verenikina et al., 2010 ; Yelland, 2005). Wohlwend (2009) demonstrated that 5-7-year-old children were accessing new literacies through pretend play—to explore iPods and video games— while Morgan (2010) indicated that teachers value and promote "playful" and interactive technology experiences as vehicles for 3-7-year-old children's learning. Appropriate ways for computer/technology integration could include technology-supported learning experiences during the hour of free activities: for example, instances where children are provided with technology support that help them complete tasks in classroom activities. With regard to the third objective, children encountered some diffi culties in using the programs MS Paint and MS Word. Some diffi culties with the MS Paint were mainly attributed to children's underdeveloped fi ne motor skills. The role of learning in the use of such programs is important, because these programs support and extend children's experiences.

The crucial role of early childhood teachers, in the whole process of computer use in kindergarten classes, has emerged. Initially, teachers' participation in this project was voluntary and they suggested the timing for the observations, while later they decided on the program(s) used in their classes and on class organization. The essential role of kindergarten teachers in ICT environments, for supporting and extending children's experiences, has been extensively discussed in relevant literature (Kervin & Mantei, 2009 ; Stephen & Plowman, 2008). Selwyn (2011) proposed the cultivation

of a critical digital literacy approach: for example, the development of creative spaces utilizing digital technologies that bridge formal/informal divide. As young children are entering early childhood settings with dispositions that may not have been part of their repertoire of skills in past decades, early childhood educators need to be aware of this and to develop new learning experiences for young children (Zevenbergen & Logan, 2008). The teachers are the designers and implementers of learning activities for children. The role of the teachers is related to and has implications for teacher training. Teachers' skills, views, and classroom practices have all implications for inservice training in the pedagogical uses of ICT. According to Parette, Quesenberry, and Blum (2010), both preservice education and in-service professional development are important so that early childhood teachers develop and apply skills in integrating and using ICT in classroom settings.

The limitations of this study included the limited amount of time and the limited fi nancial resources. For example, if video recordings were used during class observations, this could have highlighted the interactions between young children and software, as well as the interactions among children (their gestures, dialogues, etc.). It is noted that this study did not investigate the acquisition/development of skills and knowledge through the use of educational software, and this will constitute the focus of another paper.

As there is little empirical evidence on the use of computers in early childhood

settings, the following issues are suggested for future research. Investigation of the
activities children do on the class computer and the development of specifi c skills.
The research has now moved on from questions about whether ICT can help children learn (Stephen & Plowman, 2008), with the need to investigate further how and
whether it makes a difference in young children's learning and development. For
example, Haugland (2000) discussed for developmentally appropriate activities,
while Yelland and Kilderry (2010) reported that ICT as a resource is being
underutilized and children are being denied access to learning opportunities that
promote open-ended applications, problem solving situations/tasks that result in
varied learning outcomes. The link between early childhood pedagogy and ICT
could also be further explored. It is a fact that software and hardware are constantly
changing. Whilst this does not necessarily mean that the pedagogical issues of using
ICT in early childhood settings automatically change as well, it cannot be assumed
that they remain constant either. There is a need, for example, to identify quality
practices and appropriate types of teacher intervention. As Parette and Blum (2013)
reported, the key challenge for today's teachers is how to use technologies effectively and effi ciently in order to support learning experiences for young children in
the classroom. Future research could also investigate to what extent the factors
"children's use of computer/ICT at home" and "children's gender" impact on children's use of ICT in early childhood settings. Finally, some other broader fi elds that
might be of interest to educators/researchers include teachers' attitudes and motivations to use ICT in kindergartens. Provided that today's children are surrounded by
different technological tools, it is worth investigating how teachers embrace and
utilize a broader range of technologies, apart from computers, to support young
children's learning in classrooms.
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Introduction
Nowadays, it is becoming increasingly accepted that various ICT applications are
developmentally appropriate technological resources for children of primary and
secondary school age (Druin & Fast, 2002 ; Passey, 2006 ; Plowman & Stephen,
2003 ; Zaranis, 2011). It is obvious now that the original "aphorism" and the ignoring of ICT are no longer effective strategies and are gradually replaced by more
realistic solutions as infl uenced by related research (Remtulla, 2010).
Technology, in this enlarged view, acts as a catalyst in their social interaction and
moreover provides additional opportunities for a rich learning environment that is
consistent with the modern era. International research confi rms that the implementation of computers in education can have positive results in different subjects (Chou
& Liu, 2005 ; Finegan & Austin, 2002 ; Livingstone, 2012). These technologies can
therefore play an essential role in achieving the objectives of the primary curriculum
in all sectors and subjects if supported by developmentally appropriate software
applications embedded in appropriate educational scenarios (Dwyer, 2007 ; Fisher,
Denning, Higgins, & Loveless, 2012 ; Lee, 2009 ; Zaranis & Oikonomidis 2009).
ICT and Mathematics
In the most ideal setting, information communication technologies are treated as a
tool for teaching and learning (Burnett, 2009 ; Sife, Lwoga, & Sanga, 2007 ;
Sutherland et al., 2004). They are used as a tool for the students to become more
Geometry Teaching Through ICT
in Primary School
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familiar with new technology and to integrate investigation, communication, and
understanding across the full range of the curriculum. Particularly, in the cognitive
fi eld of mathematics an evaluation of learning outcomes regarding computer based
mathematical teaching in students showed that computer-assisted learning can signifi cantly help in developing proper mathematical skills and the cultivation of deeper
conceptual thinking in comparison to the traditional mathematical teaching method
(Dimakos & Zaranis 2010 ; Hardman, 2005 ; Keong, Horani, & Daniel, 2005).
Researchers suggest that the mathematical diffi culties students encounter later
are correlated with insuffi cient development of mathematical thinking in their early
years (Bobis et al., 2005 ; Gersten, Jordan, & Flojo, 2005). Various researches'
results relate the appropriate use of computers with the ability of students to more
effi ciently understand the different mathematical notions (Howie & Blignaut, 2009 ;
Trouche & Drijvers, 2010). Thus, it becomes obvious that in the primary school
level a very attractive environment of investigating the computer use in mathematics
education emerges. Indeed, a vast number of studies show a positive interrelation
between the use of computers and the development of mathematical thinking in
school (Clements, 2002 ; Vale & Leder, 2004). Nonetheless, computer based activities should refl ect the theoretical ideas behind them (Clements & Sarama, 2004 ;
Disanayake, Karunananda, & Lekamge, 2007).
Following this principle, the software designed and the students' activities developed and examined for the purposes of the current study were inspired by the framework of Realistic Mathematics Education (RME) and the van Hiele model. RME
was based and developed principally upon Freudenthal's (1968) view of mathematics
as a "human activity." According to his perspective, in order for mathematics to be
of human value, it has to be taught so as to be useful, it has to be closely related to
reality, close to children and relevant to the society (Freudenthal, 1968 ; Van den
Heuvel-Panhuizen & Wijers, 2005). RME is an active and constantly evolving theory of teaching and learning mathematics (Van den Heuvel-Panhuizen, 2001).
Indicative of this, the learning and teaching trajectories with intermediate attainment targets were fi rst conducted for the subject of mathematical calculation at the
primary school level and extended to the subject of geometry (Van den Heuvel-
Panhuizen & Buys, 2008).
In the whole trajectory of the RME teaching theory, fi ve main characteristics of
understanding geometry concepts are involved: (a) introducing a problem using a
realistic context; (b) identifying the main objects of the problem; (c) using appropriate social interaction and teacher intervention to refi ne the models of the problem;
(d) encouraging the process of reinvention with the development of the problem;
and (e) focusing on the connections and aspects of mathematics in general (Van den
Heuvel-Panhuizen, 2001). These should be the main focuses of the learning and
teaching procedure concerning geometry in primary school.
Moreover, the theory of the van Hiele model deals specifi cally with geometric
thought as it develops through several levels of sophistication under the infl uence of
a school curriculum (Clements & Battista, 1992). The van Hiele model uses fi ve
levels, however, for the fi rst grade students only the fi rst two levels were used. In the
Visual Level, students were able to identify fi gures such as circles and triangles as
visual gestalts. Conceptualization at this level includes being able to name, reproduce
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and group similar geometric objects by visual recognition. For instance, they might
say that a fi gure is a rectangle because it looks like a door. In the Descriptive/
Analytic Level, students were able to identify shapes from their properties.
Conceptualization at this level includes identifying geometric objects according to
their properties. For example, a student sees a rhombus as a fi gure with four equal
sides (Zaranis, 2012). Following the theoretical framework that blends together
Realistic Mathematics Education (RME), the van Hiele model of geometric thinking and the use of ICT in primary school, we designed a new model referred to as

the First Class Primary Shape Model (FCPSM) which consisted of fi ve levels.
The majority of previous studies aggregately examined the effects of various
teaching on the geometric shapes. However, a small number of studies have found
that in the fi rst levels of primary school, various shapes are understood differently
by students (Walcott et al., 2009 ; Wong et al., 2007). These studies examined the
impact of various teaching interventions on each shape distinctively. Specifi cally, in
the case of circles, only the thickness and size may be varied. Rectangles and triangles may also share differences in characteristics of thickness and size with the
addition of other variable characteristics such as length of sides and orientation.
Moreover, triangles may vary in the degree of their angles, resulting in a greater
variety of shapes. Thus, from the perspective of characteristic features, circles are
the simplest. On the contrary, triangles are the most complex, and rectangles are of
intermediate diffi culty (Satlow & Newcombe, 1998 ; Wong et al., 2011).
Our study was based on the above mentioned international literature; we set out
to investigate the following research question: Is the concept of triangles more dif-
fi cult to understand than circles for the fi rst grade students?
In addition, based on the previous studies, we set out to investigate the following
hypotheses:
1. The students who will be taught circles with educational intervention based on
FCPSM will have a signifi cant improvement in comparison to those taught using
the traditional teaching method according to the fi rst grade curriculum.
2. The students who will be taught triangles with educational intervention based on
FCPSM will have a signifi cant improvement in comparison to those taught using
the traditional teaching method according to the fi rst grade curriculum.
The present study makes an important contribution to the literature; we examined and compared the effects of a new model which combines computer and non-
computer activities for teaching circles and triangles separately.
Methodology
Subjects
The study was carried out during the 2011-2012 school year in ten public primary
schools located in the cities of Rethymno in Crete (two classes) and Athens (eight
classes). It was an experimental research which compared the FCPSM teaching

process to traditional teaching based on fi rst grade curriculum. The sample included
234 fi rst graders consisting of 123 girls and 111 boys age 6-7 years old. There were
two groups in the study, one control (n = 121) and one experimental (n = 113). In the
control group there was not a computer available for the students' use. The classes
in the experimental group, in order to participate, were required to have a laptop
computer and a video projector available for use by children as part of the teaching
procedure. For the uniformity of the survey, instructions were given to the teachers
who taught in the experimental or control groups. Teachers who participated in the
study had university degrees in education. The teachers in the experimental group
ranged from 40 to 50-year-olds with in-service teaching experience ranging from 18
to 28 years. The teachers in the control group were of ages 38-48 years with in-
service teaching experience ranging from 15 to 25 years.
Instructional Intervention
In the second phase the control group taught with traditional teaching according to
kindergarten curriculum. Group and individual activities were given to children
every day. The experimental group covered the same material at roughly the same
time according to the FCPSM teaching. The content of the 4-week syllabus of the
FCPSM divided in fi ve levels. It comprised of shapes including a variety of topics
concerning fundamental geometry concepts and focus on circles and triangles.
The fi rst level of the teaching intervention was according to the fi rst characteristic of the teaching theory of RME which introduces a problem using a realistic
context. In this level a story called "The Family of Shapes" was presented to the
students. Various geometrical shapes including circles and triangles are presented in
this story in a fi ctional family setting where each member of the family represents a
particular shape. This story was designed using Flash CS3 Professional Edition and
presented with the video projector in the classroom. In this story Mrs. Square and
Mr. Triangle (Fig. 1 -left) had a daughter Miss Circle. Later in the story, Miss Circle
met Mr. Rectangle, got married, and had many children that looked like their parents and grandparents.
The second level started with an activity which identifi es the main objects of the
problem as underlined by the RME theory. According to the fi rst level of the van
Hiele model and the second characteristics of the RME theory, the students (a) identifi ed various geometrical shapes by examining objects in their classrooms, (b) drew
certain geometrical shapes (Fig. 1 -right), and (c) named the shape of a set of various
objects presented to them by the teacher.
The third level of the teaching procedure included activities using appropriate
social interaction to refi ne the models of the problem. Firstly, a child had to draw a
shape from a bag and to fi nd out the name of the shape without seeing it. Then, the
students split into groups and each group had to construct a dog or a man using a set
of shapes. The fi nal part of this level involved a computer-based group activity in
which the student had to recognize and choose the correct shape among other shapes
(Fig. 2 -left) as implied by the fi rst van Hiele level.

In the fourth level of the teaching procedure, according to the second level of the
van Hiele model and the RME theory, we implemented a process of reinvention
with the development of the shapes in a higher cognitive level where the properties
of an object were presented in a software activity (Fig. 2 -right). Next, the children
were separated into groups and cooperated with one another to make a shape with
their bodies on the fl oor (Fig. 3 -left). Then, they divided into groups and each group
was assigned a specifi c shape. Each of these groups was instructed to make fake
cookies from plasticine in the form of the groups' assigned shape. At the end of this
activity, the students in each group shared their cookies with the other groups.
In the last level of the teaching process, as the second level of van Hiele theory
implied, the students played card games with the properties of shapes. The goal
of this card game is to focus on the connections and aspects of the properties of
circles and triangles as implied by the fi fth characteristic of the RME theory.
There were two kinds of card games: the white cards with shapes and their properties of sides and the yellow cards with shapes and their properties of angles. The
students are separated into groups and each student in a group starts with a number
of cards in their hand. Each student takes his/her turn throwing a card on the table.
Fig. 1 Students watched the story "The Family of Shapes" (left —fi rst level) and did a drawing
activity (right —second level)
Fig. 2 The student had to recognize shapes (left —third level) and students viewed the properties
of a triangle on the computer (right —fourth level)

The purpose of the game is to match cards with identical properties and to collect
those cards. At the end of the game the child who had the majority of cards was the
winner. Afterwards, there were computer activities where the children had to recognize the shapes from using only the properties of the shapes. The computer presents
a picture as a puzzle. The students then must drag and drop the corresponding shape,
based the geometrical property displayed on the screen, into the box (Fig. 3 -right).
In the software represented above, once an activity is selected, a problem is
announced verbally and directions are given to the user through a recorded message.
The feedback users get after following these directions is represented by two different screens, one positive and one negative. This feedback is accompanied with the
corresponding audio messages. In the fi rst case, the user receives a "well-done"
message and in the second case the user gets a "try it again" message. In both cases,
though, an effort was made to keep these messages to be as little emphatic as possible. This way, the children's' interest is focused more on the mathematical procedure of the application rather than their result or the competition.
Educational Measures
The present research was conducted in three phases. In the fi rst phase, the pretest
was given to the classes of the experimental and control groups during the beginning of December 2011 to isolate the effects of the treatment by looking for inherent
inequities in the geometry achievement potential of the two groups. The pretest was
referred to as First Class Primary Shape Test (FCPST) and it contained thirty tasks
in total. The fi rst twenty tasks of the FCPST were based on the fi rst level of the Van
Hiele model and the last ten were from the test that was used in the research by
Clements, Swaminathan, Zeitler-Hannibal, and Sarama (1999) based on the second
van Hiele level. Due to the young age of the students, the pretests were administrated in the class with explicit and detailed instructions of the teacher. These were
pencil-and-paper tasks in which the children were asked to select shapes including
Fig. 3 Students made a shape with their bodies on the fl oor (left —fourth level) and completed a
software activity (right —fi fth level)

rectangles, squares, triangles, and circles on a page with various geometric fi gures
(Fig. 4). Each task had a weighted grade that was computed from the student's
answers. Particularly, for each correct answer the student was given one grade and
lost one grade for each incorrect. Scores were computed for each of the individual
geometrical tasks of the FCPST. Since the numbers of problems varied across tasks,
a mean proportion of correct responses for each of the thirty tasks were produced by
dividing the number of correct responses by the total number of problems on that
task (possible range of standardized scores, 0-1.00). Moreover, for the purposes of
the present study, we measured the score of circles and the score of the triangles
separately. The scale used to measure the unstandardized scores for circles is based
on a scale from 0 to 69. The scale used to measure the unstandardized scores for
triangles is based on a scale from 0 to 40.
Similarly, during the third and fi nal phase of the study, after the teaching intervention, the same test (FCPST) was given to all students in both the experimental
and control groups as a posttest at the beginning of March 2012 to measure their
improvement on circles and triangles separately.
Research Design
The present study was a quasi-experimental design with one experimental and one
control groups. Eight fi rst grade classes from Athens and two from Rethymno participated in this study. From the eight classes located in Athens, we randomly
assigned four classes to the control group and the remaining four classes were
Fig. 4 Student marks circles

assigned to the experimental group. We then randomly assigned one of the two
classes from Rethymno to the control group and the other class to the experimental
group. Therefore, fi ve classes were in the control group (n = 121) and the remaining
fi ve classes were in the experimental group (n = 113).
Results
A set of analyses was conducted to determine the effects of the mathematics intervention on fi rst grade students' geometry knowledge for circles and triangles. The
pretest and posttest were taken by 234 students. Analysis of the data was carried out
using the SPSS (ver. 19) statistical analysis computer program. The independent
variables were the group (experimental group and control group) and the shape (circles or triangles). The dependent variable was the students' FCPST posttest score.
Comparing the Diffi culty Between Understanding
Circles and Triangles
The fi rst analysis was a paired t -test among the students' FCPST pretest scores of
circles and triangles in order to examine whether the concept of triangles is more
diffi cult to understand than circles. There was a signifi cant difference in the students' FCPST pretest scores for circles (M = 0.94, SD = 0.04) and triangles (M = 0.83,
SD = 0.07); t (233) = 25.93, p <0.001.
The descriptive statistics for students' scores for circles and triangles are presented in Table 1 . Observing the standardized values, students scored higher for
circles and triangles after receiving the intervention than before receiving it.
Moreover, the score of the students in triangles was lower than those in circles after
the teaching intervention (Table 1).
Evaluate the Effectiveness of FCPSM for Circles
Before conducting the analysis of ANCOVA on the students' FCPST posttest scores
for circles to evaluate the effectiveness of the intervention, checks were performed
to confi rm that there were no violations of the assumptions of homogeneity of
Table 1 Descriptive statistics for students' standardized scores of experimental and control groups
Circles Triangles
Pretest Posttest Pretest Posttest
Group M SD M SD M SD M SD
Control 0.94 0.04 0.95 0.03 0.82 0.05 0.85 0.06
Experimental 0.95 0.04 0.97 0.03 0.83 0.08 0.91 0.06

variances and homogeneity of regression slopes (Pallant, 2001). The result of
Levene's test when pretest for circles was included in the model as a covariate was
not signifi cant, indicating that the group variances were equal, F (1, 232) = 0.476,
p = 0.491; hence, the assumption of homogeneity of variance was not been violated.
The value of the covariance by dependent variable interaction (group × score for
circles) was not signifi cant, F (1, 230) = 2.042, p = 0.154, η^2 = 0.009; therefore, the
assumption homogeneity of regression slopes was tenable. After adjusting for
FCPST scores for circles in the pretest (covariate), the following results were
obtained from the analysis of covariance (ANCOVA). A statistically signifi cant main
effect was found for type of intervention on the FCPST posttest scores for circles,
F (1, 231) = 46.513, p < 0.001, η^2 = 0.168 (Table 2); thus, the experimental group performed signifi cantly higher in the FCPST posttest for circles than the control group.
Evaluate the Effectiveness of FCPSM for Triangles
Then, the analysis of ANCOVA on the students' FCPST posttest scores for triangles
was performed to evaluate the effectiveness of the intervention. The result of Levene's
test when pretest for triangles was included in the model as a covariate was not signifi cant, indicating that the group variances were equal, F (1, 232) = 0.044, p = 0.834;
hence, the assumption of homogeneity of variance was not been violated.
The value of the covariance by dependent variable interaction (group × score for
triangles) was not signifi cant, F (1, 230) = 0.214, p = 0.644, η^2 = 0.001; therefore, the
assumption homogeneity of regression slopes was tenable. After adjusting for
FCPST scores for triangles in the pretest (covariate), the following results were
obtained from the analysis of covariance (ANCOVA). A statistically signifi cant
main effect was found for type of intervention on the FCPST posttest scores for
triangles, F (1, 231) = 69.715, p < 0.001, η^2 = 0.232 (Table 3); thus, the experimental
group performed signifi cantly higher in the FCPST posttest for triangles than the
control group.
Table 2 Comparison of student scores for circles in posttest: ANCOVA analysis
Sources Type III sum of squares d f Mean squares F Sig. Partial eta squared
Pretest 0.105 1 0.105 155.990 0.000 0.403
Group 0.031 1 0.031 46.513 0.000 0.168
Error 0.156 231 0.001
Table 3 Comparison of student scores for triangles in posttest: ANCOVA analysis
Sources Type III sum of squares d f Mean squares F Sig. Partial eta squared
Pretest 0.197 1 0.197 69.645 0.000 0.232
Group 0.197 1 0.197 69.715 0.000 0.232
Error 0.654 231 0.003

Results of this study expand the research on the effects of appropriate software

embedded in a computerized environment as a tool for visualization and mathematical reasoning used alongside with specially designed activities (Bobis et al., 2005 ; Dimakos & Zaranis, 2010 ; Dissanayake et al., 2007 ; Gersten et al., 2005 ; Howie & Blignaut, 2009 ; Starkey, Klein, & Wakeley, 2004 ; Trouche & Drijvers, 2010 ; Wong et al., 2007 , 2011 ; Zaranis, 2012). Also, the outcomes of the present study create a new teaching model with computer and non-computer activities based on the theoretical framework that blends together Realistic Mathematics Education and the van Hiele model of geometric thinking in primary school.

Discussion

The overall purpose of the study was to investigate the impact of instructional intervention using the First Class Primary Shape Model (specially designed mathematics activities and software based on Realistic Mathematics Education and the van Hiele model) for the purpose of teaching the basic geometrical concepts of circles and triangles in regard to the geometry competence of the fi rst grade students of primary school. In this research, we found that it is more diffi cult for fi rst grade students to understand the concept of triangles than circles. These students carried a relatively great amount of geometric knowledge from kindergarten and as a result, it is possible that the students understand circles and their properties better than triangles because triangles are more complex shapes than circles (Bobis et al., 2005 ; Clements & Sarama, 2004). Our fi ndings agree with similar researches (Satlow & Newcombe, 1998 ; Walcott et al., 2009 ; Wong et al., 2007 , 2011) which implied that circles are simpler and triangles are most complex. As a result, the research question answered positively.

Moreover, we found that the students that were taught circles with educational intervention based on FCPSM had a signifi cant improvement compared to those taught using the traditional teaching method according to the fi rst grade curriculum (hypothesis 1). Our results overlap with the results of other analogous studies which indicate the positive effects of a computer based-model of teaching geometry (Bobis et al., 2005 ; Dissanayake et al., 2007 ; Wong et al., 2007 ; Zaranis, 2012). Therefore, the fi rst hypothesis was confi rmed.

Also, our fi ndings suggest that the students were taught triangles with educational intervention based on FCPSM had a signifi cant improvement compared to those taught using the traditional teaching method (hypothesis 2). Our outcomes overlie the results of other similar studies which indicate the positive effects of a computer based teaching model for geometric shapes (Dimakos & Zaranis, 2010 ; Howie & Blignaut, 2009 ; Starkey et al., 2004 ; Trouche & Drijvers, 2010 ; Wong et al., 2011). Thus, the second hypothesis was answered positively. In addition, as it mentioned in the results section, students in both the experimental and the control groups scored higher for shapes of circles or triangles after receiving the teaching intervention than before the teaching process.

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Moreover, an important statistical outcome of the present study was that the partial eta squared for triangles (η
2
= 0.232, Table 3) is higher than it was for circles
(η
2
= 0.168, Table 2). This outcomes supports that, our teaching intervention had a somewhat greater impact in learning triangles than of learning circles for fi rst grade students. Also, the present study demonstrates that our fi rst grade teaching process, the non-computer based mathematics activities and the computer based mathematics activities, may support the learning of both diffi cult and simple mathematics in the primary school level. Our fi ndings agree with other similar researches supporting the effective role of ICT in education and more specifi cally in mathematical reasoning (Clements & Sarama, 2004 ; Dimakos and Zaranis 2010 ; Furner & Marinas, 2007 ; Hardman, 2005 ; Howie & Blignaut, 2009 ; Keong et al., 2005 ; Vale & Leder, 2004).

The above discussion should be referenced in light of some of the limitations of this study. The fi rst limitation of the study is that the data collected was from the participants residing the cities of Athens and Rethymno. The second limitation was the generalizability of this study which was limited to participants attending public schools. Therefore, the results from this research can be generalized only to similar groups of students. The results may not adequately describe students from other regions of Greece. However, as the study was of small scale and context specifi c, any application of the fi ndings should be done with caution.

Furthermore, the undertaken computer assisted educational procedure revealed an extended interest for the tasks involved from the part of the students. It is an ongoing challenge for the refl ective teacher to decide how this technology can be best utilized in education; especially in light of the current researches on the effects of such an implementation. This study is one small piece in the puzzle of geometry education in primary schools.

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Introduction

A series of studies suggest that learning programming languages affects the development of critical and structured thinking and the acquiring of higher thinking skills, while at the same time programming languages support learning in other areas (Mikropoulos, 2004). From 1970 and for almost 2 decades Informatics was taught mainly in secondary education and focused mainly on learning some programming language (Komis, 2004). At the same time there has been research on the didactics of programming. It focused on charting the relevant students' initial misconceptions so as to develop standards, both for the teaching materials and for the teaching methods that were in agreement with constructivism (Jimoyiannis, 2005).

The need for constructivist and exploratory approaches in teaching Informatics become even more relevant with the introduction of Programming, as part of the module "Programming the computer" of the Cross-thematic Curriculum Framework (CCF), in the 5th and 6th grades of Primary education in Greece a few years ago, specifi cally in 2011 (CCF, 2011).

Teaching Informatics in Primary School
With and Without Educational Software
Support

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Research Questions

In order to address the research question, the following points were taking into consideration.

• There are no textbooks available for teaching Programming in the Primary School (Pedagogical Institute, 2013) and the teaching material presently offered is limited and often has not been previously implemented and assessed (Ministry of Education, 2013).

• Results of a series of studies focused on the experiential teaching of computer science without any use of computers show that the experiential learning of programming promotes collaboration and teamwork (Bell, 2000 ; Bell, Alexander, Freema, & Grimley, 2009 ; Curzon & McOwan, 2008 ; Taub, Ben-Ari, & Armoni, 2009).

• In the CCF for ICT teaching in primary education, the initial students' misconceptions have not been taken into account. However, available research points out that students assume that the computer has "anthropomorphic characteristics" and that it has a "hidden intelligence" (Pea, 1986 ; Taylor, 1990),

Through these three points the parameters of the research question were determined.

The purpose of this study is to assess the learning outcomes of teaching scenarios, designed by our team, for the module "Programming the computer" of the CCF for teaching Informatics in the 5th Grade. These teaching scenarios promoted problem solving and the development of critical thinking and included exploratory, constructive and collaborative activities (CCF, 2011). Moreover, the initial students' misconceptions and students' specifi c cognitive diffi culties related to learning computer programming were taken into account.

The following research questions were addressed:

1. What is the improvement in critical thinking skills (Matsagouras, 2000 , 2005 ,

2007) that is achieved through the teaching of the designed module "Programming the computer" in a programming environment?
2. What is the improvement in critical thinking skills (Matsagouras, 2000 , 2005 , 2007) that is achieved through the teaching of the designed module "Programming the computer" without computer usage, namely experientially?
3. Does the teaching of the designed module whether in a programming environment or without the use of computer (experientially) promote collaboration, inquiry, critical thinking and development of problem solving skills, in accordance with the general purpose of ICT integration in the Primary School (CCF, 2011)?
Theoretical Framework
Since the design of the module followed the constructivist approach, it took into account the students' initial misconceptions and the cognitive diffi culties that they face (Skoumios, 2011). Summarising the studies on the cognitive
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diffi culties and the students' initial misconceptions on Programming leads to the following conclusions:
• Students need to not only understand the syntax and the specifi c semantics of a programming language, but also to use properly and in a structurally coherent way these programming language tools, which are not related to their prior practical- experiential knowledge (Jimoyiannis, 2005).
• The concept of some terms, already known to the students, such as "equal", "data", found in other cognitive areas (mathematics, science) are not identical to that in the fi eld of Programming. This differentiated concept creates diffi culties for students in understanding and writing in a programming environment (Jimoyiannis, 2005).
• In any programming environment, students face diffi culties in writing commands following the correct syntax, since this is the fi rst time they come in contact with the syntax and semantics of the specifi c programming language commands. With no exception, programming languages require a rigorous design in the syntax and structure of their own commands (Jimoyiannis, 2005).
• Students cannot understand the role of the computational machine during the execution of program. Students have diffi culty in understanding the data fl ow in a computer system (Jimoyiannis, 2005). They also have diffi culty to develop effective models for the computer and its operation during the execution of a program (Bonar & Soloway, 1985 ; Rogalski & Vergnaud, 1987). Students perceive hardware as a "black box" and attribute to it several anthropomorphic characteristics as if it is a "mental giant" or has "hidden intelligence" (Pea, 1986 ; Taylor, 1990). Moreover, students create their own model based on their initial perceptions about computer operation.
• Students face diffi culties to learn how to use the available tools of a programming environment, prior to begin learning the main programming concepts of the programming language itself (Du Boulay, 1989).
The main objectives set by the CCF for teaching the module "Programming the computer" are introducing to and familiarising students with:
• The concept of algorithms.
• The use of strict syntax and interpretation of basic commands solving a concrete problem, through their implementation in a visual programming environment.
• The breaking up of problems into simpler ones.
• The composition of a task from individual components.
• The use of debugging techniques (CCF, 2011).
The computational fl ow of data directly related to a program's execution is not among the objectives that students have to achieve, nor is the interpretation of a program written in any programming language into the language "understood" by the computer.
As mentioned above, the main purpose of this study is to assess learning outcomes of teaching scenarios that were created on the basis of the CCF guidelines taking into account the students' misconceptions regarding the computational fl ow
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of data. However, the teaching scenarios had objectives that promoted the development of critical thinking and followed the learning levels of critical thinking delineated by Matsagouras. Matsagouras (2000 , 2005 , 2007) schema of learning levels of critical thinking was affected by Bloom's taxonomy of learning domains' and was further developed in a more constructivist point of view (Table 1).
Data Collection
This is the recall and reproduction level of information, where students were asked to retrieve knowledge referring to terminology, specifi c clues, events, etc. (Matsagouras, 2007 ; Oosterhof, 2010 ; Sofos, 2012).
Data Management
This is the skills and concepts level, where the main focus is on procedural knowledge specifi ed by differentiation, concepts and rules. With respect to differentiation what matters is that students differentiate or fi nd similarities among individual stimuli received from the environment. With respect to concepts, what matters is that students perceive the characteristics that can be leveraged so that materials, phenomena, situations, etc. can be classifi ed (Sofos, 2012).
Table 1 Learning levels of critical thinking (Matsagouras 2000 , 2005 , 2007)

Learning levels	Skills	Learning outcomes
First level		
Informational learning (best learning outcome is the completeness of information)		
Data collection		
	1. Observation	
	2. Identifi cation	
	3. Recall	
Information		
Second level		
Organisational learning (best learning outcome is the ability on building concepts)		
Data management		
	1. Comparison	
	2. Categorisation	
	3. Regularisation	
	4. Classifi cation	
Concepts, simple correlations		
Third level		
Analytical learning (best learning outcome is the ability on formulating generalisations)		
Data analysis		
	1. Structural elements analysis	
	2. Relations differentiation	
	3. Pattern differentiation	
	4. Distinguishes facts from opinions/estimations	
	5. Clarifi cation	
Generalisations—principles- schemes		
Fourth level		
Productive learning (best learning outcome is the ability on problem solving)		
Going beyond the data		
	1. Explanation	
	2. Prediction—Hypothesis	
	3. Confi rmation—Inference	
	4. Organisation	
Problem solving		
Theory/prototype generation of cognitive/cultural products		

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Data Analysis
This is the level of generalisation and strategic thinking. This level's main characteristics are the representation of the problem, the classifi cation of the problem, the selection of strategy, the solving method (Sofos, 2012).
Going Beyond the Data
This is the level of extended operation, thought and research, which refers to going beyond the data. The main characteristic of this level is the scientifi c approach adopted using various points of view, leading to hypotheses formulation, disciplined inquiry, stating of explanations and decisions, extracting conclusions, etc. These skills are based on the types of learning that are relevant to forming composite rules and to problem solving (Sofos, 2012).
Methodology
The sample consisted of 35 5th grade students, of a primary school located in the capital of one of the largest Greek islands. The students were divided into two groups. The fi rst, the experimental group (EG) consisting of 16 people, used the educational programming environment Scratch to elaborate the teaching scenarios. The second, the control group (CG) consisting of 19 people, used no technological support, elaborated the didactic scenarios through playing games and so achieving experiential learning.
Scratch is an educational programming language and multimedia authoring tool allowing users to use event driven programming. Scratch can be used by students for a range of educational constructivist purposes from math and science projects, including simulations and visualisations of experiments.
Description of the Educational Factors
Four teaching scenarios were developed for the research needs of this study. Each scenario comprised some educational activities and each student group elaborated them for approximately 2 teaching hours. The EG accomplished the activities by using Scratch. The necessary feedback provided to the students was coming mainly from the educational programming environment and the teacher acted as an auxiliary partner. In contrast, the CG accomplished the activities in question through the use of play, where experiential learning was taking place, whilst the feedback was provided publicly in the classroom mainly by the teacher, who acted also as an organiser. Titles, objectives and duration of teaching scenarios (Table 2) were identical for both groups of students.
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The phases of the teaching scenarios as well as the types of teaching techniques according Matsagouras (Matsagouras, 2005) used overall in all units are presented in Table 3 .
The 1st Section, "What language does the computer 'speak'?" (based on the scenario developed in the context of the web site Computer Science Unplugged: csunplugged.org) dealt with the computer's binary numbering system, i.e. how the data registered by the man in a PC can be converted by the PC in a language that the computer can process (Fig. 1).
In the 2nd Section, teaching focused on the strict syntax of commands. The students took the roles of the computer and the programmer, of robots and their operators so as to conclude that a computer, unlike men, understands only certain and strictly indexed commands (Fig. 2).
Table 2 Titles and duration of didactic scenarios

Section/issue	Teaching hours (approximately)
1. What language does the computer "speak"?	2
2. I articulate the commands and move the robot	2
3. I build scenarios by using orders	2
4. I execute and correct fi nal scenarios	2

Table 3 Phases and teaching styles/techniques on teaching scenarios

S/n	Teaching phases	Teaching styles/techniques
1	Presentation geared to problematic situations	Discussion among students and between students and teacher Questions-answers, thought disclosure
2	Highlighting and clarifying initial perceptions	Discussion among students and between students and teacher Plan design for the emergence of students' initial perceptions
3	Constructive processing	Practice, discussion among students and between students and teacher Discussion on everyday life similar cases, trial implementation of proposed decisions
4	Application to new situations, feedback	Discussion among students and between students and teacher, role play, practice, challenge, personal support (experimental group), feedback at the class level (control group)
5	Refl ection	Discussion among students and between students and teacher
6	Evaluation	Written tests

Fig. 1 Worksheet (on the left) that CG students were asked to fi ll with the help of their classmates (on the centre) and the EG students to fi ll with the help of Scratch (on the right)
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In the 3rd Section, teaching concerned the creation of sequences of a set of commands in order to create a desired image. Students were asked to create sequences that were syntactically and semantically correct (Fig. 3).
The 4th Section dealt with the execution of a sequence of commands that was provided to them and they were asked to analyse its structural elements in order to predict the result of the scenario and debug possible errors inside the code (Fig. 4).
Fig. 2 On the left shows the CG students who are trying to fi nd the right commands so as to move their classmate-robot to the desired position. On the right shows the corresponding setting in Scratch for EG students
Fig. 3 Tasks: (1) send the cat to eat the fi sh, (2) make minnow build the cage to protect the starfi sh from the shark
Fig. 4 Task (on the left): a kitty must save her friend whose house is burned and a solution by EG student (on the centre) and CG student (on the right)
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Data Collection and Analysis
The data collection and analysis of the study was based on:
1. Students' answers in Worksheets and Evaluation Sheets that they were given during teaching. Each question in the Worksheets and Evaluation Sheets related to a specifi c skill according to the levels of critical thinking presented in the theory part of this paper and the analysis was based on counting the frequency of correct answers, indicating the mastering (or not) of the relevant skills (e.g. confi rmation, error fi nding, extract conclusions, etc.).
An example of evaluation questions is presented below:
Given the two scenarios of commands below please answer the questions: (1) Write the numbers of the commands that cause movement in the kitty; (skill:

recognition). (2) Are commands 7 and 9 the same? Justify your answer; (skill: distinction). (3) What is the difference between the two scenarios shown below? What different would happen if the kitty was performing scenario 3 instead of scenario 2; (skill: comparison). (4) Which set of commands is repeated in Scenario 1; (skill: pattern recognition).
2. The observation sheet. The observation sheet was containing general observations about students' behaviour and attitudes, namely: children's participation in the learning process, their learning interoperability, their tensions, their assistance from other peers or the teacher, etc. Four sheets were complemented in each teaching module, while the fi nal observation sheet was the result of the averaging responses from each individual sheet (low—1, moderate—2, or high—3 level).
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Results
Results are classifi ed into four categories according to levels of critical thinking: data collection, organisation, analysis and going beyond. Each fi gure compares the learning outcomes as produced by the EG and the CG. The experimental group presented slightly better results with regard to data collection and organisation (Figs. 5 and 6). In data analysis, in contrast, the CG has showed slightly better results (Fig. 7) while in going beyond the data the EG displays higher rates in the
Fig. 5 Learning outcomes and data collection skills
Fig. 6 Learning outcomes and data organisation skills
Fig. 7 Learning outcomes and data analysis skills
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responses with fewer errors (Fig. 8). This difference in rates, is mainly due to the error correction skills developed more by the students in the EG because of the immediate and individualised feedback offered by the Scratch, unlike the CG were feedback was neither direct nor individualised, as it was offered at the class level.
This is confi rmed by the application of the statistical criterion t which shows a statistically signifi cant difference between the two research groups as far as the debugging is concerned. Most likely, there is no such statistically signifi cant difference as far as the problem solution is concerned (Fig. 9).
Fig. 8 Learning outcomes and going beyond the data skills
Independent Samples Test
Levene's Test for Equality of Variances t-test for Equality of Means
F Sig. t df Sig. (2-tailed) Mean Differ ence Std. Error Differen ce 95% Confidence Interval of the Difference Lower Upper
Problem 1 solution Equal variances assumed ,001 ,979 -1,143 28 ,263 -,393 ,344 -1,097 ,311
Equal variances not assumed -1,141 27,319 ,264 -,393 ,344 -1,099 ,313
Problem 1 debugging Equal variances assumed 1,747 ,200 3,412 21 ,003 1,545 ,453 ,603 2,487
Equal variances not assumed 3,451 20,467 ,002 1,545 ,448 ,613 2,478
Problem 2 solution Equal variances assumed ,675 ,418 -,693 29 ,494 -,238 ,343 -,938 ,463
Equal variances not assumed -,690 27,951 ,496 -,238 ,344 -,942 ,467
Problem 2 debugging Equal variances assumed 2,986 ,108 4,798 13 ,000 1,833 ,382 1,008 2,659
Equal variances not assumed 5,500 12,075 ,000 1,833 ,333 1,108 2,559
Problem 3 solution Equal variances assumed ,079 ,781 -,446 32 ,659 -,147 ,331 -,821 ,526
Equal variances not assumed -,447 30,597 ,658 -,147 ,330 -,820 ,525
Problem 3 debugging Equal variances assumed 43,022 ,000 3,818 16 ,002 1,700 ,445 ,756 2,644
Equal variances not assumed 4,295 9,000 ,002 1,700 ,396 ,805 2,595
Code debugging Equal variances assumed ,458 ,503 -,389 32 ,700 -,126 ,325 -,788 ,536
Equal variances not assumed -,391 30,919 ,698 -,126 ,323 -,785 ,532
Fig. 9 T -Test statistics
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From the results taken from the observation sheet, students both in the EG and the CG collaborated highly with each other, except that in the CG there was both intra and inter group cooperation. In addition, in the CG students were more enthusiastic in expressing their views. This was in agreement with the teaching method, which in the case of the CG was always collective (solving the exercise, experiential activities, etc.) and was perceived as a play by the students.
Conclusions
Both problem-solving and critical thinking, characteristics that were directly related to the skills that correspond to higher levels of critical thinking (see Table 1), showed equal level in both groups. The two groups were different in that the students without PC usage had a greater diffi culty to correct errors than the students with PC usage due to the lack of immediate and individualised feedback. On the other hand, with regard to cooperativeness, the results of our study agree with those of respective investigations which support that teaching without PC enhances the interest and teamwork of students (Bell, 2000 ; Bell et al., 2009 ; Curzon & McOwan, 2008 ; Taub et al., 2009).
The survey results taken from the observation sheet showed that by implementing those specifi c teaching scenarios, the cooperativity was promoted to a higher degree in CG students. Although the role of the software in developing the "going beyond the data skills" dimension should not be underestimated, the positive results offered by experiential learning relative to the promotion of collaboration and teamwork amongst students are also important.
We are therefore led to the suggestion of devising didactic scenarios that could combine both software and experiential learning process. Such a didactic proposal could be developed with virtual reality software, whose results would be of particular interest.
Moreover, the didactic scenarios that were designed for the section "What language does the computer 'speak'" dealt with the computer's binary numbering system and turned out to be effective in both conditions. The learning results show that students at this age are able to understand this particular section. By applying the same methodology there could be developed didactic scenarios for other modules (e.g. the development of a microworld simulating the computational data fl ow, the formation of an image on a computer screen etc.) and assessed in the process of curricula teaching transformation for ICT teaching in primary education (Sofos & Kron, 2010).
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Introduction
Students of the twenty-fi rst century have been growing up in a highly digital world, where they learn and react very differently during the learning process in comparison to students not familiarised with technology. This is also due to the fact that computer games invade students' lives from a very early age, making them signifi - cantly accustomed to many of the computer's functionalities.
At the same time, students continue to face diffi culties in computer science courses such as computer programming, even though their familiarisation with computers would suggest their learning of programming would become easier nowadays. The extended research studies carried out the last two decades state that these diffi culties are still present and students seem to be even less interested in programming (Lahtinen, Ala-Mutka, & Jarvinen, 2005).
The hindering of these diffi culties is important, as successful teaching and learning of computer programming can be extremely benefi cial for twenty-fi rst century generation students. It enables the development of various competences such as critical thinking, by allowing students to create their own programs, and of concept analysis and problem solving, as they are usually required to decipher a given scenario and interpret it into lines of code. Moreover, students learn to work in groups and collaborate with each other in their effort to develop executable programs while exercising in exchanging expertise and communicating ideas. Thus,
Educational Games for Teaching Computer Programming
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overall computer programming empowers students to become lifelong learners, a very important benefi t for this ever-growing world of knowledge since they can

transfer their skills to a number of future work domains (engineering, computer
science, etc.) (Law, Lee, & Yu, 2010).
An interesting proposal for alleviating the problems faced is the incorporation of
educational games or serious games within computer programming courses. The
term educational games is commonly used to describe computer games that are used
as educational tools and provide interactive and appealing activities that attract students' interest for learning (Gunter, Kenny, & Vick, 2008). These games reinforce
students' intrinsic motivation through the sense of challenge; they pique their curiosity, enforce a sense of security as well as stimulate their imagination (Ho, Chung,
& Tsai, 2006). Also, students are able to achieve specifi c goals and view the success
results immediately, not only when they complete the game but also when they pass
the game's stages, a process that increases their self-confi dence and helps them trust
their decision making skills. Hence, there is a need to readjust currently followed
learning techniques according to these newly emerged technological trends that can
more easily and effi ciently pull the new generation of students towards computer
programming education.
The main purpose of this paper is to review existing educational games that aim
to teach computer programming and correspondingly review how effectively they
support students in achieving the educational goals that teachers set in each case.
The rest of the paper is organised as follows: the next section provides information
regarding the requirements that have been identifi ed in the literature regarding the
development of educational games. The following section presents a review and
comparative analysis of educational games for teaching computer programming
elements, giving emphasis in features that correspond to the requirements identifi ed
in the previous section. Finally, the paper concludes with an overview of the work
done in the fi eld in terms of how well and to what extend the studied educational
games cover the identifi ed requirements and suggests future work.
Educational Games Requirements Specifi cation
Our work was carried out following a rigorous review methodology, where we
searched a variety of academic databases (e.g. Web of Science, Scopus, CiteSeer
and Google Scholar) for identifying relevant papers. During this search, we used
keywords such as: educational games, computer programming, requirements, facilitating tools, etc., in various combinations. Over 70 papers were originally identifi ed
and in the end 20 were used for our review after thorough fi ltering as the most relevant to our scope of interest.
We have studied thoroughly case studies that have proposed and developed frameworks for educational games (Becker, 2010 ; De Freitas, & Jarvis, 2006 ; Salen &
Zimmerman, 2004 ; Yusoff, Crowder, Gilbert, & Wills, 2009 ; Zualkernan, 2006).
According to these works, the development of an educational game should be carried
out after the examination of a number of aspects, where each aspect determines the

features that should be supported in an educational game. All frameworks include
similar requirements as concepts that are considered important. In this work, we
choose to follow the one suggested by Becker (2010), because its concepts encompass
all the ones included in previous frameworks. Thus, we consider it to be a more
abstract superset of features that should be supported by all educational games.
Initially, it is suggested that the educational goals should be investigated across
two axes by using a different viewpoint, so that their specifi cation will be complete.
These axes are:
• Cognitive axis , relating to mental competencies (Knowledge). Educational goals
should make sure that the information received by the students begin from the
fi rst category in the Bloom's taxonomy (Knowledge) and end in the fi nal and
most complex category (Evaluation) successfully.
• Emotional axis , relating to emotions or emotional areas (Attitude). Educational
goals should enable students to handle given situations through the enticement of
their emotions. For example, the need and thus the desire to free a prisoner during a game motivates students to solve the assigned task faster and correctly so
that they can experience the emotions followed by accomplishing the goal.
Additionally, it is important to select a proper framework that will guide the
learning experience with the incorporation of educational games. The construction
of a development framework requires the determination of various elements that
together structure the framework's components. For example, the processes of the
real world that will need to be simulated into the game (which movements will be
allowed, how will the virtual world be constructed, how will the players be represented, etc.) need to be clearly identifi ed. This is a very important step as it determines the educational scenarios that can be supported by the game's environment
and thus affects the entire learning process.
Following the framework's specifi cation, an architecture will be constructed
based on the components identifi ed that will have to be available in an authentic
game environment. These components include:
• The scenario ' s space . Students are introduced to the game's storyline once they
logon to the environment with a short description of the plot as well as a brief
overview of the basic activities they will have to execute.
• Relevant cases . Students are provided with a set of pre-solved similar cases from
which they can get a better insight of the game's knowledge and skills requirements and thus be better prepared for when it is their turn to solve assigned tasks.
• Information resources . Students can access information relevant to the task at
hand whenever they are in need of assistance.
• Facilitating tools . A set of tools is included that students can use when they are trying to execute a task and that help them build new knowledge. Additionally, the
game provides tools that underpin student communication as well as discussion with
the teachers regarding any questions, thoughts or refl ection on the virtual world.
Also, it is essential to distinguish information regarding the student, such as
learning goals, learning style (holistic, analytical, etc.) as well cognitive limitations
(e.g. behavioural competences that can affect their learning).

Teachers should also be able to set educational goals that will have to be accomplished by the students during the game by assigning specifi c activities for them to
participate in. This way, students achieve interim goals by successfully completing
tasks that will lead them to absorbing the fi nal learning outcomes set by the teachers.
The above features will underpin the selection of an authentic scenario that will
provide an attractive story to go along with the game's virtual world. To this end,
students should be presented with an interesting and motivating problem that needs
solving and it is best if the plot is similar to the ones available in existing computer
games. This way, students will be already familiar with the overall concept and the
activities they will engage in will seem more like games than teaching exercises.
Similarly, the individual problems that students will be called to solve during the
game should be consistent with the educational goals as well as with any cognitive
limitations that may be apparent to the teachers. This is one more reason why both
these features are required to be supported in an educational game.
Another important feature is the constant and explanatory feedback provided to
the students during their navigation through the game's levels. This feedback should
be represented in a form of messages that guide students towards understanding
what they did right, what they did wrong and how they can achieve their goals, even
through the mistakes they have made. This scaffolding technique ensures that students realise why their actions did not lead to successful task execution and thus
they will be able to perform better in their next endeavour.
Finally, a number of generic conditions need to be taken into consideration while
designing and developing an educational game. Such is the location and type of the
education to take place (e.g. offl ine, online, blended learning). For example, if the
learning process will be realised entirely online, then the game requires all the educational material to be uploaded within the environment and communication tools
should be very well supported and plentiful. Moreover, the course's duration will
determine how many hours students will spend on the game and how many scenarios need to be constructed according to the elements that need to be taught in this
duration. All of the above require adequate preparation from the teacher and proper
confi guration of the environment so as to exploit all of the game's benefi ts and foster knowledge and skills development by the students.
Educational Games for Computer Programming Education
This section presents a series of games that have been developed specifi cally for
computer programming courses. The review of these games was carried out based on
the specifi cations identifi ed and described in the previous section, in the cases where
they were explicitly identifi ed by the relevant literature. Moreover, all chosen games
have addressed both the cognitive axis and the emotional axis during their development. Thus, in each game students start out by receiving pieces of information, and
through their engagement with the game, they move on up the Bloom's taxonomy
to the evaluation step by refl ecting on their progress and fi nalising assignments.
Also, the emotional axis is addressed via the game scenarios. All scenarios stimulate
emotions that motivate students to go through all tasks in order to win.

Two major categories could be distinguished during the research, which sort
educational games based on the educational goals they aim to support. The fi rst
category includes games that focus on teaching specifi c computer programming units
while the second category represents games that cover multiple educational goals
and thus computer programming material. A review for each category is presented
in the next two subsections, followed by a comparative analysis presented in Table 1 .
Educational Games Focused on Teaching
a Specifi c Unit of Learning
Catacombs . It is a three-dimensional multiplayer game that aims to teach students
how to declare variables and use simple and nested if statements and loops. According
to the game's scenario, each player represents a wizard that has to rescue two children trapped within catacombs. Towards this goal, the wizards have to answer multiple choice questions trying to solve a given programming code that will help them
complete their quests. The answers to the given questions automatically create executable lines of code in a micro-language. If the answers are correct, the wizards
progress through the game's levels; otherwise they are given corresponding feedback
as to what they answered wrong and are prompted to try again. The game records
experience scores for each student and provides explanatory messages as a scaffolding mechanism (Barnes, Chaffi n, Powell, & Lipford, 2008 ; Barnes et al., 2007).
Saving Princess Sera . It is a two-dimensional game that enables students' scaffolding through explanatory messages directed to the player. Each player has to try and
save a princess named Sera who has been abducted by a monster named Gargamel,
on her sixteenth birthday. Students are required to complete a number of quests in
order to progress in the plot of the game. Towards this goal, they complete lines of
code that will result to an executable program or they have to correctly map existing
lines of code to their proper position or order within a program employing a drag
and drop functionality. This way, students learn the quick-sort algorithm along with
simple and nested loops with the usage of a micro-language (Barnes et al., 2007 ;
Barnes et al., 2008).
EleMental : The Recurrence . It is a three-dimensional game that aims to teach students how to execute recursion and depth-fi rst search transversal using the C# programming language. The player has to navigate across a virtual binary tree by
employing the depth-fi rst transversal and complete three quests by applying
recursion. Two avatars named Ele and Cera help students during the game in various
ways. For example, once the code is written, Ele crosses the binary tree according
to how the written code is deployed, while Cera explains exactly what the code is
producing at a specifi c moment (Chaffi n, Doran, Hicks, & Barnes, 2009).
Wu ' s Castle . It is a two-dimensional role playing game that aims to teach students
loops and arrays through interactive activities. Each player is a wizard that can control an army of snowmen. Players recognise logical errors at lines of code written in
the C++ programming language. The game allows arrays management through

changing the parameters inside the loops and movement of the characters through
the execution of nested loops (Eagle & Barnes, 2009).
Robozzle . It is an online puzzle game that provides a series of predefi ned commands
ready for use and does not show any actual code. According to the game's scenario,
users have to build functions that will help them achieve each given task in a grid
and tiles virtual world. Users can run their functions and see how their hero will
move across the world and can therefore easily detect what mistakes they have made
and reprogram accordingly (Li & Watson, 2011).
LightBot . LightBot (Piteira & Haddad, 2011) is an online puzzle game similar to
Robozzle. It includes a series of predefi ned commands and no actual code or programming language. Additionally, users have to complete given tasks by building
their own functions and moving the hero across a grid and tiles environment and
light all the blue tiles. Once a task is completed, the user can move to the next level,
which requires the construction of more complex functions.
TALENT . TALENT (Maragos & Grigoriadou, 2011) focuses on teaching if statements and loops in the forms of algorithms by using a micro-language. Each player
is an archaeologist that has to navigate across the virtual environment by completing
a series of tasks and collect objects that are available at specifi c locations for their
future exhibition at a museum. Towards this goal, students can drag and drop lines
of code as well as write them in an editor whenever requested. As a scaffolding
mechanism, TALENT provides an agent that acts as a mentor and helps students
when needed as well as suggest what their next mission should be.
Educational Games Focused on Teaching Multiple Units
of Learning
Robocode . It is a two-dimensional environment that aims to teach computer programming using the Java language. The game comprises of a programming editor,
robots and a virtual arena, and students are required to program a robot that will
compete against one another in the arena. Students familiarise themselves with the
basic commands of structured computer programming and object-oriented programming (e.g. inheritance, polymorphism) while they try to build a robot ready for
combat. During its construction, the robot inherits basic methods that can later be
extended by students according to the behaviour they want their robots to have
inside the arena (O'Kelly & Gibson, 2006).
M.U.P.P.E.T.S . It is a three-dimensional, Web-based and collaborative game that aims
to teach object building and in general to familiarise students with the basic concepts
of object-oriented programming using the Java language. Students create a robot that
has to fi ght another robot inside a virtual arena, interact with the objects they build
and write and compile their lines of code within the embedded development environment that includes a commands console (Phelps, Bierre, & Parks, 2003).

Table 1 Overview of the educational games for computer programming courses
Game Programming elements Programming language Programming activities Special characteristics
Catacombs Variables; simple and nested if
statements; loops
Micro- language Multiple choice questions; fi lling
out lines of code
Three dimensional; multiplayer;
success scores; scaffolding
with explanatory messages
from the hero
Saving Sera If statements; recursion Micro- language Filling out lines of code; mapping
parts of code in corresponding
locations; multiple choice questions
Two dimensional; scaffolding
EleMental Recursion; depth-fi rst search (DFS)
algorithm
C# Depth-fi rst search algorithm; moving
the hero on a fantastical binary tree
Three dimensional; scaffolding
ProgBPlay Structured computer programming,
if-statements, loops
Ada, C, Java, OCaml,
Scratch, Compalgo
Completing eight missions Multiplayer; infi nite scenarios;
available
Wu's Castle Loops; arrays C++ Arrays management; movement
of the hero; identifi cation
of logical mistakes in code
Interaction; role playing

Robozzle Functions No code used Creating functions through the hero's movement
Interactive; available
LightBot Functions No code used Creating functions through the hero's movement
Web based; available
TALENT If statements; loops Micro- language drag and drop lines of code; writing lines of code in an editor
Explanatory messages
Robocode Structured computer programming; object- oriented programming
Java Writing lines of code Every robot stores data of past activities; available
M.U.P.P.E.T.S. Three- dimensional objects; object- oriented programming
Java Interaction with the developed objects; writing lines of code; compile
Collaborative; three dimensional; commands panel; embedded development environment
PlayLogo 3D Basic concepts and commands of structured programming
Logo Creation of heroes; navigation across the "galaxy" by writing commands
Interaction; multiplayer; three dimensional; available
Gidget Analysis and design of basic algorithms
Micro- language Fixing problems and programs; interaction with a personalised robot
Web based; explanatory messages

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ProgßPlay . It is a library currently integrated in the Web-based, real-time multi-player strategy game Kernel Panic, which enables constant interaction amongst users. Students can program their own heroes and form alliances with each other aiming to prevail in the game. ProgßPlay allows students to choose the language in which they prefer to code their programs amongst programming languages such as Ada, C, Java, OCaml, Scratch and Compalgo (Muratet, Torguet, Viallet, & Jessel, 2011).

PlayLogo 3D . It is a three-dimensional, role playing game that allows interaction amongst multiple users and aims to teach basic concepts of structured computer programming. Users are required to program their heroes by writing the corresponding lines of code in the LOGO language, and navigate them across the environment. More specifi cally, the virtual world consists of the spaceship X-15 located on a constellation of the Andromeda galaxy, where a contest is held each year amongst pilot-robots (Paliokas, Arapidis, & Mpimpitsos, 2011).

Gidget . It is a Web-based game where students can program using a simplifi ed programming language created specifi cally for the game in order to learn how to design and analyse basic algorithms. A robot named Gidget has problems with a part of his software and thus cannot complete its tasks. Therefore, students are called in to help Gidget by either fi xing wrong lines of code, or by completing missing code within given programs. During these processes, students receive constant feedback of their progress (Lee & Ko, 2011).

The above table presents the study with a structured representation of features supported by the most commonly known educational games for computer programming. The programming elements, characteristics as well programming activities identifi ed can be considered as concepts that describe the fi eld. Thus, future researchers and game developers should take them into consideration when designing a new and advanced educational game for computer programming.

Discussion

In this section we provide an overview of the development of existing educational games for computer programming, and their limitations. The results are categorised based on the features identifi ed that should be taken into consideration during the design and development of an educational game for computer programming.

Educational goals . The educational goals seem to cover both the cognitive and emotional axes . Within the games, these goals are clearly focused in the computer programming concepts that each game aims to teach. This is especially the case in the educational games that cover specifi c units of learning, and thus the desired learning outcomes are more clearly identifi ed. The emotional goals seem to be accomplished through the numerous attractive scenarios available in each game.

The problems students are required to solve are consistent with the set educational goals and their cognitive limitations. In the educational games focused in specifi c units of learning, students execute and complete quests that teach them knowledge that is relevant to the programming concepts set in the goals. As an

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example, the simple and nested loops in Catacombs are taught through the completion of lines of code, and their correct syntax allows students to pass to the next level, while the same concepts are taught in Wu's Castle when students move their characters across the world and recognise logical errors. On the other hand, the second category of educational games (e.g. Robocode, M.U.P.P.E.T.S, PlayLogo 3D) employs problems that allow students to interact with each other and execute multiple tasks that will teach them all the basic concepts of computer programming.

Framework . Educational games that focus on specifi c units of learning seem to have properly defi ned a framework for their employment in educational contexts. However, games that teach multiple and more complex units of learning, and thus cover multiple educational goals usually set several frameworks. It should be noted that the games Lightbot and Robozzle do not defi ne any framework.

Scenario ' s space . All educational games present and work based on a scenario in order to attract and motivate students. In some cases introductory information is provided to the players in regard to the virtual world (e.g. PlayLogo 3D, Wu's Castle).

Information resources . Most educational games provide explanatory messages . The games where this feature is more fully supported are Catacombs, Saving Sera, EleMental: The Recurrence and Wu's Castle. Moreover, scaffolding techniques are provided through these explanatory messages that appear while students are trying to solve their quests (e.g. Catacombs, Saving Sera, EleMental).

Facilitating tools . Tools where students can write requested lines of code exist in the Catacombs, Saving Sera, EleMental, Robocode and M.U.P.P.E.T.S games. In addition, multiplayer games (e.g. PlayLogo 3D, M.U.P.P.E.T.S., ProgßPlay, Catacombs) include features where students can communicate and interact with one another.

Generic conditions . The generic conditions have been taken into consideration. This has been carried out more effi ciently in the educational games that cover specifi c units of learning rather than in the ones that teach multiple and complex computer programming concepts. On the other hand, they have not been considered at all during the design of the Robozzle and Lightbot games.

It should be noted that none of the studied games provides relevant cases that can prepare students for the activities they will be required to execute. The existence of this feature would signifi cantly increase the quality of the games, since it would provide useful tutorials and guidelines for learners.

We also have to mention that many of the aforementioned information regarding these games derive exclusively from the relevant literature, since they are not available for access. This fact also results in our inability to exploit them in the learning process and actually test them against set educational goals in computer programming courses. Summing up, it seems that all studied games include scenarios that motivate learners, clearly indicate the educational goals that need to be reached and include problems that are set up as specifi ed above. Other features, such as facilitating tools, information resources, one or several frameworks and taking into consideration generic conditions are supported by the majority of the studied games. However, none of the games appear to support relevant cases to prepare learners before engaging with the environments or to act as manuals for when learners require guidance.

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Conclusions

The main implications derived from the analysis include that most games have been developed to cover programming concepts (such as variables, simple and nested if statements, loops, arrays, functions) with the exception of M.U.P.P.E.T.S. and Robocode that cover more complex concepts such as object-oriented programming. We do not consider the fact that the games do not tackle all programming concepts as a disadvantage, since they seem to successfully fulfi l the educational goals they set regarding the group of concepts they aim to teach.

Also, the study elaborated on the added values of using educational games in computer programming. Our research revealed a number of interesting principles that can help us understand why educational games can improve teaching and learning of computer programming. For example, games seem to have a facilitating role in the learning process during the teaching of specifi ed concepts and could play a small or a big part in the entire course's implementation process, depending on the generic conditions. More specifi cally, depending on the nature of the course (online, offl ine, blended), materials, communications, exams, etc. could be supported on different levels by the games' environments. To this end, educational games can provide a number of characteristics, such as storytelling, scaffolding and interactivity, which increase motivation for participation in class as well as attract students to complete their tasks through interesting scenarios.

We examined the educational games in terms of the educational value that they bring, and we derived that they can provide students with:

• Clear educational goals and learning outputs, ensuring that they know what they have to do to achieve the required knowledge and skills.
• A familiar and immersive environment that attracts students' attention facilitates their active participation and increases their motivation.
• Interesting scenarios with comprehensive problems they have to solve, which enable them to learn in a contextual manner (learning specifi c units of learning periodically).
• Tools that help them communicate and collaborate with their classmates, improving their group work skills and guiding them through the learning process by explaining the possible mistakes they make. These tools can be applied either with chat functionalities or with different types of interactions between the learners and the game while trying to achieve and fulfi l common goals.

Furthermore, the study's fi ndings have implications regarding the design of future educational games focused on computer programming, listing and elaborating on the requirements educational game designers and developers should strive to support and thus setting the foundations for future holistic environments.

Educational games can also assist teachers teach programming in their courses by designing the game and setting up its parameters. For example, teachers can use educational games to plan their courses and monitor students' interactions, progress and evaluation through their activities in the game. The establishment of the educational goals, learning outcomes and the setting up of a scenario that will

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delineate the curriculum into units of learning also enables teachers to be better prepared and have a deeper knowledge of the materials they teach and get more skilled in course planning.

On the other hand, a signifi cant limitation identifi ed in the existing educational games focused on computer programming courses is the ability of the teacher to confi gure the environment according to the pedagogical goals related to the respective unit of learning. Additionally, the collaboration concept could be reinforced and better supported within multiplayer educational games so that they can teach more complex programming concepts that will be more effi ciently understood through team-based learning activities.

The evaluations carried out during the pilots studies, showed that the majority of learners expressed positive attitudes towards the examined environments. Thus, this enables the initial implications of our research to exploit the features considered important by the literature and presented throughout the paper during future design and development of educational games.

Such games will fully support all identifi ed specifi cations and features and will aim to teach in-depth more complex concepts, such as object-oriented programming.

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Introduction
The term "Educational Robotics" refers to the teaching practice during which the
students use the robots to construct knowledge with the help of or for the robots
themselves. The term appeared in the 1960s through the educational approach of the
Logo programming language. Within this context educational robotics consists of
an educational approach which recruits programmable devices to improve the learning process through project-based learning. It is defi ned by the use of Information
and Communication Technologies (ICT) in its own affordances for observation,
analysis, modelling and control of various physical procedures (Depover, Karsenti,
& Komis, 2007). It concerns an approach which allows the learner to familiarise
himself with the Information and Communication Technologies and use them to
defi ne a plan, to organise and fi nd a specifi c solution to the given problem exchanging his opinion with those of others (Denis & Baron, 1993 ; Leroux, Nonnon, &
Ginestié, 2005). The cognitive abilities that develop in early childhood with the use
of robotics have been studied since the introduction of the Logo educational
approach. A distinct category of educational robotics is the Logo-like programmable toys which are appropriate in early childhood and primary education. These
programmable toys are programmable robots which are controlled by the user for
the respective movement or path they are ordered to execute. In some cases the connection with the computing environment may be used. The child conceives and
defi nes the commands which are introduced into the robot following the principles
of the Logo programming language. This robotics subcategory is inscribed in the
Robotics and Programming Concepts
in Early Childhood Education: A Conceptual
Framework for Designing Educational
Scenarios
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psychopedagogical approach of the Logo language, supporting the development of
the metacognitive ability, with which the children refl ect on the cognitive process
adopted, improving the ability of problem-solving and promoting the ability of spatial orientation (Clements & Nastasi, 1999 ; Clements & Sarama, 2002).
An Overview of Programmable Toys in Early Childhood
The Logo programming language developed in the mid-1960s at the M.I.T.
(Massachusetts Institute of Technology). At the same time various types of fl oor-
robots which are programmed with the Logo language make their appearance to aid
the implementation of the new programming language in educational contexts after
further research activity. Since then a series of robots featuring a common Logo
programming language implementation have been used for educational purposes.
A literature review of the integration of robotics in education allows us to distinguish two approaches: (a) the use of educational kits for the construction and function of the robotic system using the appropriate programming language and (b) the
use of educational kits with pre-constructed robotic systems using the appropriate
programming language.
The fi rst category includes kits like LEGO ®
-WeDo™ and LEGO ®
-Mindstorms™.
The second category includes the robotic programmable toys such as the Roamer,
the Bee-Bot, the Pro-Bot and the Constructa-Bot. The latter require the user to
design and compose a program and execute it in order to achieve a goal, thereby
solving a problem. Their characteristic is that they are programmed by novice users
either through the use of a computer or through the device itself. Most of the programmable toys use the ideas of the initial Logo fl oor robot—the Logo turtle—and
have similar functional and operational characteristics.
Table 1 presents an overview of early childhood and primary education robotic
devices (Table 1). It is clear that the majority of the robotic systems refers mainly to
pre-constructed robotic devices the so-called "programmable toys" (Hirst, Johnson,
Petre, Price, & Richards, 2003). Some have a more attractive appearance and function due to the production of sensory stimuli one of which is the Bee-Bot programmable toy while some others are less complex and are more of a machine such as
the PIXIE. In any case, the child designs and defi nes the total amount of commands
which are introduced into the robot following the principles of the Logo programming language. The added value of this specifi c programming language is the fact
that it is appropriate for early childhood development and support abilities such as
problem-solving, metacognitive thought as well as skills such as counting, spatial
orientation and measurement (Bers, 2008 ; Highfi eld, 2010 ; Highfi eld & Mulligan,
2008 ; Highfi eld, Mulligan, & Hedberg, 2008 ; Papert, 1980).
Two interesting conclusions can be drawn from this table. First, programmable
toys have been under development for the last 40 years within the context of Logo
language paradigm. These toys have a Logo-like robust interface and are technological devices using this specifi c language.

Table 1 Robotic devices for early childhood and primary education
Year of
development Name of Kit Language Category Firmware
Novice users/
Age group
Learning
Area Image
Late 1970s BIGTRAK (1979) Logo Programmable toy MB N/A Maths
Literacy
Arts
During 1980s Tortue Jeulin T2 Logo Programmable toy, Computer,
Pedagogical material
Evreux N/A Maths
Literacy
Arts
Tortue Jeulin T3 Logo Programmable toy, Computer,
Pedagogical material
N/A Maths
Literacy
Arts
Turtle (1985) Logo Programmable toy,
Programming by computer
Valiant Technology N/A Maths
Literacy
Arts
During 1990s Roamer (1999) Logo Programmable toy 4-9 Maths
Literacy
Arts
Roamer-Too (2008) Logo Programmable toy 4-7 Maths
Literacy
Arts
PIXIE Logo Programmable toy, Software Swallow 5-8 Maths
Literacy
Arts
PIP Logo Programmable toy, Software 7-12 Maths
Literacy
Arts
PIPPIN Logo Programmable toy, Software 7-12 Maths
Literacy
Arts
(continued)

Table 1 (continued)
Year of
development Name of Kit Language Category Firmware
Novice users/
Age group
Learning
Area Image
During 2000s Lego Mindstorms
RCX Brick (1998)
Logo Construction kit, Software Lego >7 STEM
Lego Mindstorms
NXT (2006)
Logo Construction kit, Software >7 STEM
Lego WeDo (2009) Logo Construction kit Software >7 STEM
Bee-Bot (2005) Logo Programmable toy Software,
Pedagogical material
TTS 4-7 Maths
Literacy
Arts
Constructa-Bot Logo Programmable toy 4-7 Maths
Literacy
Arts
Pro-Bot (2006) Logo Programmable toy, Software to
transfer programs from the
concrete object
7-12 Maths
Literacy
Arts

Second they are supported by appropriate educational material and are, apart
from the LEGO ®
robotics kits (LEGO ®
Mindstorms™ & LEGO ®
WeDo™), pre-
constructed systems requiring no construction. Their manipulation focuses on
developing the user programming abilities rather than engineering/technology ones.
Their educational applications focus on STEM whereas those which are addressed
to younger children such as the Bee-Bot may cover other cognitive areas as well
(Language, Arts). Most of the recent toys (Bee-Bot and Pro-Bot) have more user
friendly (robust) interface and a more playful appearance.
Educational Robotics in Early Childhood Education
The implementation of educational robotics in early childhood education is seen as
a way of introducing various concepts and developing different abilities.
Robotics is an interesting cognitive domain because it is a tool through which
children have the opportunity to approach mathematical concepts, applying strategies such as problem-solving, inquiry and experimentation (Rogers & Portsmore,
2004). It is worth mentioning that robotics is an educational approach with a variable dimension, which it can be easily integrated in various educational settings
(Bers & Horn, 2010). Furthermore, teaching about and through computer programming and robotics using developmentally appropriate approaches increases children's sequencing abilities (Kazakoff, Sullivan, & Bers, 2013). During the planning
and constructing procedure of a robotic model children of early childhood put into
action cognitive abilities which are under development (Papert, 1980). Programming
concepts which may be developed within computing environments are not always
developed for children of this age range. There are usually environments which
require users to develop the ability of abstract thought.
Hirst et al. (2003) dealt with the review, the description and the presentation of
robotic environments which are based on the technology of the LEGO ®
Mindstorms™ robotics systems. They propose that these specifi c systems bridge
the gap between how the user acts concerning more abstract computing systems and
specifi c conventional tools. They also propose that the creation of a more individualised system of a graphic microworld for novice users must be created so that they
are progressively drawn towards a more advanced programming environment.
Therefore, educational robotics in early childhood education uses appropriate
cognitive tools, emphasising on tangible use. The use of such tools which is the case
of programmable toys is a factor of motivation infusing the interest in children and
their actions towards learning. Those tools are developmentally appropriate as they
are based on playing and consisting of meaningful action and reaction (Highfi eld,
2010 ; Highfi eld et al., 2008 ; Highfi eld & Mulligan, 2008).
Some researchers suggest that learning through interaction with a programmable
toy, the construction of more abstract cognitive structures and the development of
social skills is reinforced (Bers & Horn, 2010 ; Yelland, 2007).

However researchers such as Greff (1998 , 2001) attempted to reinforce the learning context and developed appropriate teaching materials based on a language of
graphic representation of the commands for the approach of algorithmic concepts.
A positive aspect of this language was that it offered the user directness because of the
appropriateness of its structure and planning during the creation of a program. Thus,
algorithms were planned in order to direct the fl oor-robot to complete a specifi c path.
Other researchers used the programmable toy fl oor-robot Roamer in their study.
João-Monteiro, Cristóvão-Morgado, Bulas-Cruz, and Morgado (2003) report the
results of their teaching intervention within the context of the ICEI programme in
preschool settings in Portugal. The fl oor-robot Roamer was used to support teachers
in using the ICT as a cognitive tool. The added value of this particular robot lies in
its potential to develop mathematical concepts in children in early childhood.
A look at the studies published in the last decade shows that the use of Logo-like
programmable toy Bee-Bot lies in the centre of the scientifi c interest for this specifi c
age group.
Beraza, Pina, and Demo (2010) in their study presented teacher-orientated robotics activities in order to support teachers in their practice. They claim that the programmable toy Bee-Bot is suitable for early childhood and primary education but
provides limited programming opportunities. For this reason and for encouraging
teachers in designing suitable educational scenarios with programmable toys robots
they proposed the use of the Arduino platform.
In the study by Highfi led (2010) 33 children of early childhood and their teachers
chose the Bee-Bots and Pro-Bots from a range of robotic toys. Through a learning
process of a combination of robotic toys and engaging tasks mathematical thinking
and sustained engagement was promoted.
Pekarova (2008) studied the development of effective teaching practices and
attractive activities for children through digital technologies in early childhood education. Her results show the need of an organised context for teaching programming
concepts with the use of the Bee-Bot. However, since this procedure is not suffi cient
enough for activating children's inner motives, the formation and the organisation
of appropriate problem-solving tasks as well as the development of teaching materials are required.
Highfi eld and Mulligan (2008) describe various instances where early childhood
and primary school children interacted with the programmable Bee-Bot toys. The

desired outcome came through the experimentation with the programmable toy as
children applied different strategies in order to discover its functions and features.
Especially the experimentation gave children the opportunity to discover the feature
of programmable toy to rotate rather than move aside. Similar outcomes have also
been stated by Highfi eld et al. (2008) for the development of mathematical concepts
in children of early childhood when using the Bee-Bot programmable toy. This toy
facilitates children's learning and in particular the way to approach topics such as
measurement and geometrical transformations as opposed to traditional teacher-
orientated teaching.
Similar conclusions are drawn by De Michele, Demo, and Siega (2008) where
children from primary education used the programmable toy Bee-Bot to program
through mathematical concepts of multiplication and addition in typical classroom
teaching practices.
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Overall, research has shown that there is no systematic and principle-based framework for teaching educational robotics concepts in early childhood settings. What
appears to be missing is a developmentally appropriate educational context for developing programming abilities and reinforcing inner motives of children in early childhood. This is important because integration and use of a programmable toy within an
appropriate teaching and learning context may infuse cognitive development (Depover
et al., 2007) such as mathematical skills and problem-solving abilities in children.
Furthermore, Csink and Farkas (2010) gave emphasis to the use of fl oor-robot
Roamer to teach programming concepts integrated within the curriculum and the
additional methodology. They also propose teaching programming that should be in
a context with role-playing and 3D games instead of using computer software. The
same point of view for integration of programming concepts within the offi cial curriculum is shared by De Michele et al. (2008). Scientifi c activity in the fi eld of
educational robotics and especially in the fi eld of programmable toys seems to infl uence the curriculum of various countries such as those of England, Australia,
Croatia, Estonia and Hungary (Csink & Farkas, 2010) as well as that of Greece in
which a clearly distinct thematic approach has been integrated for the teaching of
programming. In some cases like England, Australia, Scotland and Greece there is
an explicit reference to the use of programmable toys. In England specifi cally programmable toys have been integrated in the mathematical learning area and children
learn to program by designing paths, a procedure through which they develop abilities such as spatial orientation. In Australia the Ministry of Education introduced a
teaching guide concerning the integration and use of the Bee-Bot programmable toy
in early childhood education (Kopelke, 2007). In Greece a new curriculum proposal
announced in 2011 (The Greek Institute of Educational Policy, 2011) is currently
being piloted and includes the integration and use of programmable toys in early
childhood and primary education. In addition, in Malta since 2011 and Scotland
since 2013 educational programmes targeting the use of the Bee-Bot programmable
toy have been launched. In these countries, the Bee-Bot programmable toy is being
used as a teaching and learning tool in several academic subjects (ICT, Mathematics,
Language, Social Studies, and Physical Education).
A Methodology Framework for Educational Scenarios
in Programming and Robotics in Early Childhood
As the preceding literature review of educational robotics in early childhood education clearly shows, the relative research is limited in references of implemented
teaching activities without systematic educational design or organization. In some
cases the researchers have tried to gather students' representations concerning the
content of the robot (through the use of a single question) as well as try to record and
present students' programming strategies during free experimentation of the children
with the Bee-Bot (Pekarova, 2008). In other cases an attempt to trace children's initial
representations and their manipulation of the Bee-Bot is recorded. However, there is
still lack of systematic observation and recording of children's learning processes
(Highfi eld, 2010 ; Highfi eld et al., 2008 ; Highfi eld & Mulligan, 2008).
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In the above studies we can additionally observe that there is no specifi c or organised meaningful context. In other words, they lack in presenting a context suitable for
teaching programming as well as for problem-solving situations. Especially the more
recent studies use the researcher's demonstration of a particular path and the aim is
that children reproduce the same path (Highfi eld et al., 2008 ; Highfi eld & Mulligan,
2008). On the contrary it is Pekarova (2008) who claims that the programmable toy
has attractive features and functions for children of this age but these elements are
not effi cient enough. There is a need for a clear and appropriate planning program
construction orientated for the reinforcement and the function of teamwork.
Greff (1998) shows that the creation of a pseudo-language through the graphic
representation of the commands of the Roamer programmable toy is an appropriate
teaching strategy for the visualization of the programming procedure. This procedure provides user with the opportunity to not only visualize a program but also
refl ect on and correct its content. This visualisation technique was evaluated by
other researchers using a different methodology. In the study of João-Monteiro et al.
(2003), the users were asked to programme the toy to reach a desired position or
goal as it is represented on a coloured patterned command on a card.
It is worth mentioning the fact that most studies which used the Bee-Bot programmable toy were not integrated in typical classroom teaching practices. They
appear to be more focused on free experimentation with the tool and orientated
towards the implementation form the scientists themselves who act either as facilitators of the research or are supported by the classroom teacher (Highfi eld et al.,
2008 ; Highfi eld & Mulligan, 2008 ; Pekarova, 2008). On the other hand, Greff
(1996) reports the implementation of a teaching intervention towards the development of a programming ability while placing the children either in the position of
the robot or in the position of the user under real classroom conditions. Moreover,
Greff (2001) and João-Monteiro et al. (2003) describe cases of use of the Roamer
robot in typical classroom conditions for the development of programming and
mathematical concepts.
As far as the cognitive context is concerned, concepts referring to either premature programming structures (the sequence of commands) or belonging to mathematical learning areas (counting, shapes) were approached. João-Monteiro et al.
(2003) in their study implemented the cross-curricular approach in order for children to construct programming concepts.
The primary aim of this study is to propose an appropriate educational framework for organising the teaching process, one that emphasises the understanding,
design and implementation of robotics and programming concepts. The study is
situated in the wider scientifi c research context of robotics use in early childhood
education. It follows the design based research mixed research model and uses
the method of multiple case studies for collecting qualitative and quantitative developmental data (Kelly, Lesh, & Baek, 2008). This kind of research deals with the
design, development and evaluation of educational programmes or constructions
as part of the teaching practice in such a way that the most effi cient conditions
for teaching and learning are assured for the benefi t of everyone involved
(Depover et al., 2007).
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Since there is no existing framework for teaching programming in early childhood the approach proposed below is an innovative, new tool for educators. This
framework is based on a conceptualization where methodological and pedagogical
issues are suitably integrated to facilitate the teaching and learning process, thereby
addressing the defi ciencies of former studies identifi ed above. Such defi ciencies
have resulted in unsystematic and unstructured educational interventions, which
failed to address issues related to the teaching of programming, educational practices and pedagogical principles. The teaching of programming includes concepts
such as algorithms, programs, memory and debugging. Moreover, a structured and
systematic educational intervention should give emphasis on and carefully select
the appropriate teaching approaches. Such issues refer to defi ning an objective and
the goals which meet early childhood children's needs, to using additional teaching
strategies, to comprising didactic transposition on programming concepts and to
developing appropriate teaching material. These teaching practices are based on
pedagogical principles such as project-based learning, child-centred learning, collaborative learning, and well-organised learning environments.
The proposed educational and methodological framework includes seven (7) distinct phases (Fig. 1) for designing an educational scenario. The importance of designing an educational scenario is to include issues of programming concepts, educational
practices and pedagogical principles to lead the teaching process. Therefore the
selection of a specifi c subject orients teaching to set an objective, as well as the goals
for delivering its content through seven (7) distinct phases, which in fact present different instances of the planning and the implementation process. These phases are
found in the core part of the whole design are closely interrelated and interact at the
same time (Komis, 2010 ; Komis, Tzavara, Kansenti, Collin, & Simard, 2013).
From an educator's point of view designing an educational scenario is very
important as it addresses issues such as the integration of ICT at least in some of the
phases using a computer or robotic device during its implementation and application. The structure of current methodology describes the method with which the
participants (in our case early childhood educators) are asked to use ICT in a suitable planned and well-organised context. Thus, the educational scenarios used
within this study adopt concepts and themes of computer science and mathematics
curricula expecting children to construct programming concepts by using the Bee-
Bot programmable toy.
Based on the above conceptualization, a teaching model was developed for educational robotics which was structured after appropriate adaptations as those which
emerged by analysing former studies. So a conceptual model was created of an
educational scenario appropriately structured and adapted to the cognitive early
childhood children's needs. Anticipations and adaptations that took place aspired to
minimise methodological faults with the aim of maximising the validity of research
fi ndings. Pekarova (2008) mentioned that emphasis on teaching programming in
early childhood children should be given to plan a well-organised and systematic
teaching intervention. Particularly, all the adaptations that were taken into account
were divided into two categories concerning different approaches such as (a) methodological and (b) pedagogical.
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Methodological Approaches
The methodological approaches include: (a) the organisation of the educational scenario, (b) the introduction and the integration of didactic transposition of programming and informatics concepts, (c) the development of instructional design, (d) the
integration of inherent teaching strategies in pedagogical and informatics design, (e)
the use of explicitly stated teaching contracts and (f) the development of research
protocols (instruments gathering data) for each individual.
Educational Scenario
The organisation of the educational scenario (structure and content) followed a spiral
developmental procedure, comprising the six (6) following steps: (1) Design, (2)
Implementation, (3) Evaluation, (4) Modifi cation, (5) Re-implementation and (6)
Re-evaluation with the aim of potentially adapting it in every different educational
context or group of children. The interventions which used the programmable toy
Bee-Bot do not explicitly mention a systematic implementation to gather data for
evaluation and feedback for modifi cation (De Michele et al., 2008 ; Highfi eld, 2010 ;
Fig. 1 The phases in designing educational scenarios for robotics
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Highfi eld et al., 2008 ; Highfi eld & Mulligan, 2008 ; João-Monteiro et al., 2003 ;
Pekarova, 2008). Every educational scenario we designed was pilot tested in a typical
classroom setting, before being formally implemented by the in-service teachers.
Didactic Transposition of Programming and Informatics Concept
Moreover, in the published literature there is no illustrated integration of didactic
transposition of programming and informatics concepts. All of them are giving
more emphasis on the development of abilities concerning mathematical concepts,
rather than programming and informatics ones. In our case, the integration of programming and informatics concepts created a more explicit context not only for the
children but also for the teachers. For children it is self-evident to deliver the epistemological knowledge under appropriate transposition, while for the teachers it
was necessary since they had no previous experience on robotics.
Instructional Design
A development of instructional design underlined the teaching of programming and
informatics concepts by increasing the level of cognitive diffi culty. This meant that
each educational scenario had been designed by increasing the diffi culty level in its
goals and activities. Researchers, using interventions focused on curricular subjects
may have used similar adaptations, however this is not clearly stated (De Michele
et al., 2008 ; João-Monteiro et al., 2003).
Teaching Strategies
Being a more structured and theory-based educational scenario, this has integrated
inherent teaching strategies such as problem-solving, cognitive confl ict and inquiring. Especially in pedagogical and informatics educational design these strategies
provide an ascending engagement and learning process.
Teaching Contract
Furthermore, the use of an explicitly stated teaching contract, customised for each
activity was clearly defi ned, in order to arrange the class' settlement and to motivate
the self-regulation of group members, as well as that of the whole class.
Research Protocols
To trace pre and evaluate post-intervention children's ideas, conceptions and representations using Bee-Bot and its functions, research protocols were developed and
fi lled for each individual. Additionally, this process includes children's personal
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graphic representations (pre and post drawings) showing their ideas about the
Bee- Bot. These two different protocols provide the educator with the opportunity to
verify early childhood children's verbal representations, since at this age different
approaches are required to facilitate verbal communication.
Pedagogical Approaches
The pedagogical approaches include: (a) the development of a pseudo-language, (b)
the development of additional teaching materials, (c) the initiation/organisation of
an appropriate learning context and (d) the appropriate adaptation for implementation by in-service teachers in typical classrooms settings, taking the role of facilitators and co-researchers.
Pseudo-Language
The development of a pseudo-language, through a series of graphical representation
of commands on cards—based on Greff's study (1998), who had argued that developing such language is developmentally and pedagogical appropriate for the visualization of programming procedures for children in early childhood. This procedure
provides the user with the opportunity to not only visualize a program but also
refl ect on and correct its content. In particular, every command of Bee-Bot's interface has been represented on a card containing three different semiotic systems.
Those are: (1) the image or representation of the real object/button, (2) the image of
Bee-Bot's interface with the additional button highlighted and (3) the word corresponding to the additional command (Fig. 2).
Every synthesis of card-commands constitutes a full program, which could be
executed by the Bee-Bot (Fig. 3). It is also structured according to the program's
syntax process (CLEAR-COMMAND OF DIRECTION/ORIENTATION-GO) and
placed vertically following of the principles of the Logo language. Each card is
15 cm wide, the same as the Bee-Bot's step length, facilitating children to develop
learning strategies for the interrelation between cards, the Bee-Bot's step and the
squares of gridded mats.
Fig. 2 Graphical representation of Bee-Bot's commands on cards
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Teaching Materials
Alongside the previous pseudo-language teaching material was developed to support and facilitate the teaching process. Gridded mats of 15 cm each square (the
Bee-Bot's step length) on A3 laminated papers were created. This kind of material
enabled children to consider the sequence of commands of an algorithm. The child/
user is supported and reinforced to fi nd the appropriate commands/cards of orientation and direction, visualizing the program, matching each command/card to a
square. Moreover, a series of additional pictures or 3D objects (animals, toys, sticks/
ribbons for measuring, adhesive bookmarks), were customised for each educational
scenario depending on the learning context they were integrated to.
Learning Context
For each educational scenario an adequate learning context was designed, so as to
introduce problem-solving situations as well as to support the educator's teaching
practices concerning programming concepts. The so-called learning context initiated an inquiry/problem-solving situation with a developmentally appropriate way,
taking into account the children's prior knowledge and experience on programmable toys, as well as programming and mathematical concepts. Through this context
open questions (questions stimulating productive activity) were set up for each goal
to lead inquiry and problem-solving activities. The learning context takes advantage
Fig. 3 Examples of programs developed by a team while working on a teaching activity on orientation concepts
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of the Bee-Bot's animated and playful appearance, enabling children to inquire.
In that way the programmable toy is integrated as a member of the team, enabling
children to relate to its action and see themselves as co-researchers.
The previous adaptations are based on Papert's ideas about creating a developmentally appropriate constructivist programming environment where concrete material facilitates the construction of abstract ideas and refl ection on them (Papert, 1980).
Role of Teachers

Last but not least, an appropriate adaptation for implementation by in-service teachers in typical classrooms, taking the role of facilitators and co-researchers was taken into account. This approach is implied by the broader context of educational robotics thus is the constructionism (Papert, 1980) and the social-constructivism (Vygotsky, 1978). Each teaching activity provided teachers with the knowledge of the learning process they should have, in order for children to construct knowledge. Each activity is comprised of: the learning goal, the question to be explored (inquiry question), one or more teaching strategies, the additional teaching material, the class organisation, the description of the process and fi nally the conclusion--reference to the prospected learning outcome. Moreover, the scenario documentation (phase 7) comprised of useful references (scientifi c knowledge) on the children's cognitive development on each teaching subject, additional appropriate vocabulary on programming concepts and guidelines on how to assess and implement the interviews with each child.
Therefore the idea was to divide each educational scenario in two distinct sections. The fi rst section (shared across all scenarios) introduces programming concepts and provides children with basic skills/knowledge to understand the functions and the use of the programmable toy. Within that section a transition from exploration and investigation to more structured activities is applied. Respectively those skills were delivered and extended in the second section where children were engaged in more open-ended activities focused on inquiry and experimentation of mathematical and programming concepts. Mathematical concepts such as spatial awareness are aside to programming concepts due to the Logo-language structure.
For example in one educational scenario children applied the robot step as a unit of measure to extend their understanding on length concepts.
The conception and design of educational scenarios of robotics and programming in early childhood education adopt concepts from the cognitive fi elds of computer science and mathematics. The scientifi c fi eld of educational robotics concerning the use of the programmable toys is part of the learning theories of constructionism and social constructivism (Bers, 2008 ; Depover et al., 2007 ; Papert, 1980 ; Resnick, 2006). Both these theoretical models lead to the development of abilities of a cognitive style such as inquiry, experimentation, observation, and recording as well as of a social context such as cooperation and discussion with others, refl ection on fi nal conclusions and share the results.
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An Empirical Investigation of the Framework
Subjects and Setting
The framework outlined above has been implemented for 3 years (2010-2013) through the scientifi c European project Fibonacci by 46 educators and 864 children between the ages of 4-6. Eventually valuable data was gathered from 38 educators and 674 children. It follows the design based research mixed research model and uses the method of multiple case studies for collecting qualitative and quantitative developmental data (Kelly et al., 2008). A concurrent triangulation approach was applied which "provides quantitative statistical results followed by qualitative quotes that support or disconfi rm the quantitative results" (Creswell, 2009). The quantitative analysis of the data has not yet been completed; however initial results provide interesting details about the proposed framework.
Measures and Data Collection
Concerning the techniques gathering the quantitative data; different research protocols (instruments) were developed and introduced to the distinct phases of an educational scenario (Fig. 1). All these instruments were tools for educators for recording and evaluating the learning process. A structured interview was introduced at the phase where children's prior representations and knowledge about the programmable toy Bee-Bot were to be identifi ed. It comprised of eleven (11) open-ended questions and was used to assess pre and to evaluate post-intervention children's ideas. These questions formed the categorical variables. The results from the post-intervention interview are embedded in the phase where the scenario activities are designed and particular to the sub-category of subject evaluation process. In the same sub-category three more instruments have been applied. After the fi rst section of teaching activities (function and use of programmable toy), an instrument for assessing every child's prior knowledge on the mathematical concepts was introduced. Another instrument was used to record each child's evaluation on the programming and maths concepts. Its structure provided us with more categorical variables. Moreover, every child was accessed for his refl ection on the programmable toy. All these instruments shared the same technique of a structured interview.
Data Analysis
As far as the qualitative analysis is concerned, data from records and videos has been collected but not yet completed. In particular, records are written notes kept by teachers on a daily basis during the implementation of an educational scenario along with notes regarding each child's evaluation on programming and maths.
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The data gathered from several case studies (92 children) on the fi rst year of the implementation, indicated that children were having diffi culties understanding orientation concepts (Komis & Misirli, 2011). This educational scenario was focused on measurement, enabling children to use the robot step as a unit of measure. The fi ndings indicated the need to engage children with spatial concepts and especially directionality. Therefore an appropriate educational scenario was implemented. In the third year, one more scenario was developed. It was aimed to develop concepts of iteration and sequencing of patterns as those could be initiated through the concept of a program and its structure.
The analysis clearly shows the development of algorithmic thought and programmable abilities as well as the evolution of abilities in diverse mathematical structures. The development and the evolution of these specifi c cognitive abilities is facilitated through designing and implementing developmentally appropriate educational scenarios which in turn lead to the creation of relative cognitive representations on the function and the use of the Bee-Bot programmable toy (Komis & Misirli, 2011 , 2012 ; Misirli & Komis, 2012).
Data gathered through structured interviews and drawings from ninety two (92) children show a signifi cant shift to more qualitative representations in the post tests. This became apparent by the fact that more explanations was provided (by the children) about the programmable toy's functions and controls (Misirli & Komis, 2012).
The interpretation of these preliminary fi ndings attributed to the conception of appropriate teaching activities illustrating and highlighting programmable toy's functions and controls (Greff, 2005).
Concerning the development of algorithmic thought, this was formed through the cognitive process of problem-solving which in our case demanded from each child to plan a spatial path. Thus, a spatial path is suffi ciently abstract for this age group, the pseudo-language--a series of card commands--alongside the developed teaching material engaged children in tasks were algorithms were visualised (Komis & Misirli, 2011). The same study showed that a small group of children tried to plan a path mentally without visualizing it. Moreover, gender differences were not signifi cant but differences among different age groups were. The pre and post assessment showed that children aged fi ve and six understand programming concepts more easily.
A very interesting fi nding is the design and the programming of algorithms by early childhood children (Komis & Misirli, 2013 ; Misirli & Komis, 2013). During this cognitive construction of an algorithm the children follow two (2) steps: (a) initially they verbalise the abstract conception of the algorithm and (b) they proceed to the syntax of the algorithm by implementing the programming stages respectively. While the syntax procedure takes place, a variance of the amount of commands which are used and rate between three (3) cards-commands and twelve (12) cards-commands for direction and orientation, is observed. On the whole the design of scenario activities (phase 5), were structured and organised to initially familiarise children with the abstract process of conceiving an algorithm. According to Vygotsky (1978) thought is considered as "inner speech" and is the result of language. The "inner speech" was in educational scenario the time when children verbalised (thinking aloud) an algorithm and thus planned their program and
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consequently organised a strategy to process. This cognitive construction leads children to modelling the solution of a problem as they form it verbally and consequently indicate it by their fi nger or the programmable toy itself. Moreover, all the activities of a scenario (phase 5) had a teaching contract placed from the very beginning when children were introduced to the activities of cognitive and psychological preparation. It is worth mentioning that all children for the 3 years intervention proceeded to verbalizing an algorithm having the programmable toy as system of reference and not their body as it was proposed from the studies of Greff (1998) and De Michele et al. (2008).
Discussion
The preliminary fi ndings drawn from the present conceptualization and its implementation show that although the programmable toy Bee-Bot has a limited command set (Beraza et al., 2010 ; Kazakoff et al., 2013) it may be a cognitive tool for children if a systematic, structured, principled and theory-based framework is used.
The application of the educational scenarios in typical classrooms in early childhood education, showed that preliminary concepts of programming could be developed through the use of programmable toys. Thus the cognitive potential (Depover et al., 2007) applies not only to the development of mathematical abilities but also to the development of programming abilities and problem-solving situations.
Our proposed framework is validated by quantitative and qualitative data gathered from cases where in-service teachers integrated the educational scenarios in their teaching, without having prior knowledge neither on robotics nor on educational scenarios. This implementation in typical classroom contexts also distinguishes our approach from other researches on programmable toys, which were conducted mostly by researchers and in some cases they have demonstrated to children how to input a program (Beraza et al., 2010 ; Highfi eld, 2010 ; Highfi eld et al., 2008 ; Highfi eld & Mulligan, 2008 ; Pekarova, 2008). Furthermore, our work demonstrates that the methodological and pedagogical approaches we used to underline this conceptualization, were effi ciently addressing the defi ciencies of previous studies. Our fi ndings show that although children had no prior experience with programmable toys they fi nally achieved the cognitive objectives set through inquiry-based activities. They were able to build sequential programs based on graphical representations and transfer them to the programmable toy's tangible interface in a learning context, underlined by appropriate methodological and pedagogical approaches.
It seems that the development of programming skills (algorithmic thinking, concept of memory, debugging, structure of sequence, and inputting strategies) requires an appropriate conceptualization to effi ciently motivate young children. Based on these preliminary fi ndings, we reach the same conclusion as Highfi eld et al. (2008): "the Bee-Bot has the potential to enhance children's development of mathematical concepts, particularly … measurement processes much earlier that traditionally expected." On the top of that we also found spatial orientation (Misirli & Komis, 2013) and sequencing skills.
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The present study has drawn conclusions from a specifi c educational context, but the fi ndings need to be validated in other educational contexts. Several questions such as the role of the teacher in the learning process, the potential differences if the educational scenario is applied it a digital rather than a physical environment, the transfer of programming concepts to other learning areas and metacognitive processes remain open. Our study could be extended and expanded in the future allowing for the above questions to be answered.
In conclusion, the conceptual framework proposed above consists a tool with cognitive potential for early childhood children to develop initial programming concepts via a developmentally appropriate and supportive learning environment. It allows each child to create his/her personal learning "trajectory," one that fulfi ls his/her own learning needs. There is evidence to suggest that this conceptual model can be integrated into everyday teaching practices and into the everyday processes. In addition, it does not necessarily require educators to be specialized in educational robotics and programming.
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Introduction
With the rapid diffusion of the Internet; new approaches to learning were created
(Crosta, 2004). As a result, the interest in the development and use of online learning has been steadily increasing (Dabbagh & Kitsantas, 2004) providing "anytime,
anywhere learning." More specifi cally, the technological advancement in information technology and telecommunications resulted in the development of the Web 2.0
and created the appropriate framework for user participation. The traditional one-
way communication is transformed into a two-way communication, and process of
information. In Web 2.0 users are Contributing, Collaborating, Creating—the 3C's
(Ala-Mutka, Punie, & Ferrari, 2009). Various online tools have emerged such as
blogs, wikis, discussion forums, online collaborative documents, online sharing of
documents, pictures and videos, podcasts, RSS feed, etc. Millions of people use
various social and professional networks, such as Facebook, MySpace, Twitter,
Delicious, Flickr, LinkedIn, and Live Journal. Nowadays, with the advent of Web
2.0, the Internet has become truly interactive. The aforementioned tools and networks are excellent examples of how defi nitions, ideas, photographs, videos, and
voice can be shared over a powerful Web 2.0 Internet. Technology provides a realistic, visually compelling, and motivating interactive environment for developing
the life skills and knowledge needed for today's globalized, hi-tech environment
(Goddard, 2002). The Web 2.0 technologies became an essential tool of daily life
and a crucial part of students' personal knowledge tools (Lee, Miller, & Newnham,
2008). Consequently, the Web 2.0 tools can be educationally exploited for teaching
and learning purposes towards achieving educational objectives, thus transforming
social to educational networking.
Integrating Blogs in Primary Education
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Blogs are one of the most popular Web 2.0 tools. In 2009, 133 million blogs were
online and two new blogs are created every second! It is supported that blogs have
educational value given their characteristics and the opportunities provided to users
(Davis, 2005 ; Eid Neurolearning, 2005 ; Richardson, 2010). Even though they were
around for years, they recently emerged as a popular means of communication,
discussion, collaboration, and information sharing. Blogs are Web publishing tools
which provide teachers and students an interactive platform where text, images, and
links to other blogs, Web pages, are posted mostly focusing on a particular subject.
It is supported that blogs have educational value given their characteristics and the
opportunities provided to users (Richardson, 2010). Blogs are most popular among
students since they are virtual and can be worked at any time and place (Richardson,
2010). It is extremely benefi cial to integrate Web 2.0 tools; in this case blogs, as
learning-cognitive tools, to add educational value and enhance the teaching and
learning process and promote the development of higher-order thinking skills, such
as application, synthesis, evaluation, creation (Anderson & Krathwohl, 2001). It can
be also suggested that by default blogs facilitate and promote the development of a
community. The members of the community are the bloggers, since bloggers share
a common interest (a specifi c subject under "investigation"); connecting to each
other by posting comments and discussing. Students extensive use of technology is
possible to facilitate the integration of blogs (and technology overall) as tools in the
teaching and learning process. Being part of our students' digital world might be
more possible to raise their interest, motivate them, transform the classroom environment, and properly prepare them for the rapidly changing information society's
needs and demands.
Main Aim of the Study
The current study evaluates blog integration as an educational tool within the teaching and learning process and specifi cally within the Language and Linguistics
course in fi fth grade (Subject examined: The time machine) and its role in developing a Community of Inquiry (CoI). The research objectives that guided the current
study are the following:
- To investigate the effectiveness and role of blog in achieving specifi c learning
objectives when it is integrated as an educational tool within the teaching and
learning process,
- To examine the development of a blended learning environment with the use of
blog in relation to in-classroom activity,
- To identify how the role of educator and students differentiate within a blended
learning environment,
- To investigate the development of a Community of Inquiry (CoI), by identifying
the existence of the three parameters that characterizes a CoI: cognitive, social,
and teaching presence.
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Theoretical Background
Blogs
Blog is considered an important tool of the Web 2.0 toolbox (Richardson, 2009) and
blogging has become one of the most popular Web 2.0 activities. The origins of the
blog emerged from the short term of "web log" (Bauer, 2011); an online chronological collection of personal commentary and links that was fi rst used by Barger (1997).
A blog is a Web site that is maintained by an individual or a group where readers can
comment on blog posts to supply more information and discuss various issues (Allen,
2011). A person that owns a blog and/or posts messages on blogs is called a blogger
and the actions within its environment are known as blogging (Hill, 2004). Blogs
contain text, graphics, images, videos, and hyperlinks to other Web sites. Bloggers
comment on the posts, discuss, argue, and provide their opinions. Blogs create conditions for interaction, refl ection, ideas' synthesis, exchange and discussion,
self-evaluation, and feedback (Petko, 2011 ; Sim & Hew, 2010 ; Zawilinski, 2009).
Blogs and Education
Given the great educational blogging potential, numerous educators have already
started using blogs in the classroom. Blogs can be integrated as educational tools
across the curriculum, from primary to higher education, achieving collaboration
among students and educators even in different schools and countries. Many studies
explored the features and educational benefi ts that blogging offers to students, and
discussed the major blog uses in education (Churchill, 2009 , Downes, 2004 ,
Richardson, 2009 ; Richardson, 2010 ; Siemens, 2005). Specifi cally, studies integrated blogs into their teaching exploring the learning value of blogging (Chen,
Cannon, Gabrio, & Leifer, 2005 , Makri & Kynigos, 2007), i.e., for group teaching,
collaborative learning, and Web-based collaboration (Grassley & Bartoletti, 2009).
Educational blogs provide a practical online platform for discussions that hastens
the acquisition of knowledge and learning (Liu & Chang, 2010). Additionally,
blogs provide space for the students to refl ect and publish their thoughts and understandings, and opportunities for feedback, scaffolding of new ideas, as well as collaborative learning. Blogs also promote the development of higher-order thinking
skills such as application, synthesis, evaluation, creation (Anderson & Krathwohl,
2001), and improve fl exibility in teaching and learning. Finally, blogs feature
hyperlinks, which help students understand the relational and contextual basis of
knowledge, knowledge construction, meaning making and experience connective
writing (Liu & Chang, 2010 ; Penrod, 2007 ; Richardson, 2009). According to
Oravec (2002) blogs encourage self-expression and collaboration, which in turn are
refl ected in enhanced critical thinking skills. A blog offers extended interactivity,
increasing students' involvement and motivation (Eteokleous & Pavlou, 2010).
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Students can receive instant feedback from the instructor, peers, and other visitors,
which enhances learning effi ciency (Kaplan, Piskin, & Bol, 2010). Blog participants
perform connective writing since they need to read carefully and critically, and
develop context that is clear, organized, and convincing. There is a synthesis of
ideas, self- evaluation, and refl ection. Experiencing blog writing promotes critical,
analytical, relational, and creative thinking. It also combines collegiality and social
interaction, developing working and social relationships among teachers, educators, and professionals (Davis, 2005 ; Eid Neurolearning, 2005 ; Richardson, 2010).
Eteokleous (2011) explains the development of student-centered environments and
how students become educational content creators, having an increased role in the
teaching and learning process when blogs are used as educational tools. Instructor's
role in a blog learning environment is extremely important. Glogoff (2005) emphasizes that the instructors should be aware of the delicate balance between the synchronicity of time and place on the one hand and the need to keep discussions
focused on the topic. Additionally, Kim (2008) states that the success of the system
relies on teachers' capability in providing the appropriate resources. So, it can be
suggested that although teaching and refl ecting through blogs constitutes an effective medium for teaching, it must be applied in a proper way with the guidance of
the instructor in order to foster the best teaching practices (Eteokleous & Nisiforou,
2013a , 2013b ; Karaman, 2011).
Blog's pedagogical affordances were examined and reported by various researchers. First of all, it is supported that blogs motivate students to engage positively in
the writing process (Barrios, 2003 ; Cottle, 2009 ; Shiffl et, 2008 ; Trammel & Ferdig,
2004). Additionally, blogs enhance participation and interactive communication
opportunities (Angelaina & Jimoyiannis, 2012) promoting both individualized
(Cottle, 2009 ; Shiffl et, 2008) and group refl ection on learning experiences. The
blog's pedagogical affordances also include the support of authentic learning tasks
through peer assessment and formative evaluation of student work (Angelaina &
Jimoyiannis, 2012) as well as the promotion of critical thinking and increases learner
autonomy (Richardson, 2010). Finally, blog facilitates student collaboration within
a community of learners (Nelson & Fernheimer, 2003) and encourages and support
blended learning activities by effectively changing formal and informal learning.
Blog as Cognitive-Learning Tool
Blogs serve numerous purposes such as personal, professional, and educational.
Focusing on the educational use of blog, four extra categories can be identifi ed:
Blog as online course tool, blog as a discussion forum, blog as a research tool, and
blog as cognitive-learning tool (Eteokleous & Nisiforou, 2013a , 2013b). For the
purposes of the current paper, it is important to better explain the cognitive-learning
tool category. Eteokleous and Nisiforou (2013a , 2013b) attempted to defi ne blogs as
cognitive-learning tools and examine their role in the teaching and learning process
as well their effectiveness in achieving specifi c learning objectives.
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The current paper suggests that blog integration in the teaching and learning
practice is defi ned as the use of blogs by students as a cognitive-learning tool that
enhances their learning experience and supports the achievement of specifi c learning goals. It can be integrated in all educational levels (i.e., pre-primary, primary,
and secondary, as well as higher education in numerous departments) and fi elds and
in various subjects (i.e., Mathematics, Literature, Science, etc.). This approach is
related to the learning with computers or computers as mindtools (Jonassen, 1999b),
where computers and overall technology is introduced as students' partners within
the teaching and learning process. Learning with technology and effects of technology use characterizes this trend. Learning with requires integrating computers and
overall technology as mindtools in the classrooms to support constructive learning.
Educators embed or apply technology capacity in the context of ongoing teaching
and learning in different school subjects. Based on the above, students learn how to
use various technology applications not as an end in themselves, but as tools that
help them execute their tasks and promote the balanced development of their mental
abilities. As a result they do not learn from technology, but technologies support
meaning generated by students (Bielaczyc & Collins, 1999 ; CTGV, 2003 ; Jonassen,
1999a , 1999b ; Jonassen, 2000).
Blog is integrated within the teaching and learning practice for numerous years;
however, the development of a pedagogical framework is extremely important. It
will highlight a set of key criteria and parameters for blogs to be integrated as
cognitive- learning tools within the teaching and learning process in order to achieve
real blogging. Additionally, educator's and students' role should be further examined and clarifi ed. Finally, research on the following needs to be conducted: blog's
design, format, content used and uploaded, tools and gadgets employed.
Community of Learning
Communities of learning in schools have been examined by various researchers
providing defi nitions, characteristics, the role as well as the importance of
communities of learning. According to Wenger, McDermott, and Snyder (2002)
communities of learning "…are groups of people who share a concern, a set of
problems, a passion about a topic, and who deepen their knowledge and expertise
in this area by interacting on an ongoing basis" (p. 4). Few years later, Wenger
(2004) defi ned communities of learning as "…groups of people who share a passion for something that they know how to do, and who interact regularly in order
to learn how to do it better" (p. 2). In the learning environments, the role of a community is to support and facilitate socially constructed knowledge (Job-Sluder &
Barab, 2004 ; Palloff & Pratt, 2005). Finally, Loving, Schroeder, Kang, Shimek,
and Herbert (2007) supported that "A learning community is a group of autonomous, independent individuals who are drawn together by shared values, goals,
and interests and committed to knowledge construction through intensive dialogues, interaction, and collaboration" (p. 179).
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For a learning community to be developed; face-to-face interactions are not
necessary. Nevertheless, various online tools can be employed for the development
of a learning community. As Loving et al. (2007) suggest, "these virtual learning
communities can be built in two forms, synchronous and asynchronous. Blogs are
considered one of those tools that can support and promote the development of a
learning community, either a fully immersed learning community or either a blended
learning community (Oravec, 2003). Consequently, blogs have an important role to
play in building an online community of learning.
Community of Inquiry
The current study examines the development of a Community of Inquiry (CoI).

C. S. Peirce and John Dewey were the fi rst philosophers to introduce the concept of
the community of inquiry. The introductory concept described the nature of knowledge, formation and the process of scientifi c inquiry. An educational community of
inquiry is a group of individuals involved in a process of empirical or conceptual
inquiry into problematic situations. Those individuals collaboratively engage in
purposeful critical discourse and reff ection to construct personal meaning and con-
fi rm mutual understanding. The CoI requires intersubjective agreement among
those involved in the process of inquiry for legitimacy since it emphasizes that
knowledge is necessarily embedded within a social context (Seixas, 1993 ; Sharp,
2007).
The theoretical framework of the current study focuses on the Community of
Inquiry (CoI) model as has been suggested by Shea and Bidjerano (2010) where
social, cognitive, and teaching presence are related. The model is based on the work
of Garrison, Anderson, and Archer (2000) which introduced the original model of
CoI. The CoI model assumes that effective online learning requires the development
of a community that supports meaningful inquiry and learning (Shea, 2006).
Garrison et al. (2000) developed this model which assumes that deep and meaningful learning results when there are suffi cient levels of three components: teaching,
social, and cognitive presence. The model outlines the theoretical elements essential
to successful knowledge construction in collaborative online environments. The
social presence relates to the establishment of a supportive environment such that
students feel socially and emotionally connected to each other and to the instructor
in a computer-mediated environment. The elements of the social presence are demonstrated through emotional expression, open communication, and group cohesion.
The teaching presence involves the design, facilitation, and direction of cognitive
and social processes leading to personally meaningful and educationally worthwhile
learning outcomes. Elements of the teaching presence include setting curriculum
and activities, shaping constructive discourse, and focusing and resolving issues.
The cognitive presence is defi ned as the extent to which learners are able to construct and confi rm meaning through continuous suggestion and discussion in a critical community of inquiry. The elements of the cognitive presence include triggering

event (sense of puzzlement), exploration (sharing information and ideas), integration (connecting ideas), and resolution (synthesizing and applying new ideas)
(Garrison & Arbaugh, 2007 ; Swan et al., 2008).
Research Methodology
To address the above, a case study approach was employed where qualitative
(through in-classroom and blog's observations) and quantitative (through questionnaires) data was collected (Creswell, 2003). The classroom intervention took place
within the context of the Language and Linguistic and Art courses during October-
December 2011. The unit delivered was "Time Machine" and 20 fi fth graders participated at the study. For the purposes of teaching the lessons a blog was developed
using Blogger . The blog was integrated as an educational tool in the teaching and
learning process within six, 40 min lessons (fi ve Language and Linguistics lessons
and one Art lesson). A blended learning environment was developed through in-
classroom and online activities as well as homework activities. An introductory
lesson took place at the computer lab in order to familiarize students with the blog
(uses and tools) since it was the very fi rst time used by the educators and the students. Students were given notes regarding the blog use. Regarding the rest of the
lessons, both the classroom and the computer lab were used. Additionally, in some
cases students were asked to use the blog at home.
Various questions were posted on the blog throughout the fi ve Language and
Linguistics lessons. The open questions posted at the blog were related to the subject of the theme delivered ("Time Machine") and they aimed to contribute in
achieving the lesson's objectives. The educator constantly reminded students to
comment on their classmates' blog—posts, report if they agree/disagree, provide
arguments, suggestions, etc. The goal was for a discussion to be conducted, where
students' opinions, views, thinking would be revealed at the blog wall. The teaching
intervention was designed and developed by the authors in collaboration with two
teachers which then delivered the lessons.
The quantitative data collection method conducted using a questionnaire given to
students by the completion of the three lessons in the presence of the teacher. Thus,
the students had the opportunity to make any clarifying questions about the questionnaire, which was created based on the Community of Inquiry questionnaire
(Swan et al., 2008). The questionnaire consisted of two parts: (1) Demographic
Data (e.g., gender, country of origin, use and frequency of computer use, use and
frequency of Internet use) and (2) Communities of inquiry statements which consists of three main parameters: teaching, social, and cognitive presence. Each
parameter consists of several sub-parameters; as a result the questionnaire includes
a total of 34 statements. Students were asked to rank the 34 statements using a 5
point Likert scale, where 1 = Strongly Disagree, 2 = Disagree, 3 = Neutral, 4 = Agree,
and 5 = Strongly Agree. The statistical package SPSS (Version 19) was employed in
order for the quantitative data analysis to be performed. It includes descriptive

statistics, namely frequencies, percentages, averages, and standard deviations for all
variables of the questionnaire, as well as Cronbach's alpha (α) for internal
consistency.
The qualitative method of data collection was conducted by observing in-
classroom and blog activity—teacher and students' blog postings. Blog postings
were analyzed using an open coding system, attaching labels to blog postings (words
or lines of data) and then describing the data at a concrete level, before moving to a
more conceptual level (Anfara, Brown, & Mangione, 2002). Firstly, this process was
conducted within each blog questions (students' responses) and then across all blog
questions (all students' responses). The iterative coding process employed leaded to
the identifi cation of themes. It is important to clarify that a blended learning environment was employed where a combination of in-classroom and online activities
were performed. Consequently, the students evaluated the development of a CoI
based on their experience within the blended learning environment. The data collection process was conducted during October–November 2011.
Results
Demographic Characteristics
Regarding the gender of the students, 55 % was boys and 45 % was girls, while the
country of students' origin varies. Specifi cally, the majority of the students were
coming from Cyprus (25 %), followed by Georgia and Ukraine (10 %). The vast
majority of the students (85 %) use the computer, of which 25 % use it once per
week and 5 % use it daily (mean = 2.8; SD = 1.11). The majority of the students
(70 %) replied that using the Internet, of which 25 % of students answered that
rarely uses it, while 15 % use it once per week (mean = 2.5; SD = 1.19). A 5-point
Likert scale was used, where 1 = no use and 5 = daily.
Communities of Inquiry: Blended Learning Environment
The analysis of the questionnaires revealed some interesting results regarding the
development of a Community of Inquiry (CoI) through the use of blog as the main
educational tool. Specifi cally, it can be supported that a CoI was developed through
the Lesson Time Machine, where the blog was integrated as an educational tool
within a blended learning environment (combination of in-classroom and online
blog-based activities).
Generally, it can be suggested that the teacher presence has a vital impact since
the mean was 4.80 (SD = 0.242; α = 0.765). The important role of the educator is
highlighted. Specifi cally, the results suggest that the teacher was properly organized,
gave immediate and appropriate instructions, and successfully played the role of the

facilitator, so that the students felt comfortable to engage in productive discussions
and reff ections. The aforementioned mainly took place within the in- classroom
activities. On the other hand, the educator did not participate in the blog discussions
besides providing questions and clear instructions on students' responsibilities. The
educator wanted to grant students the freedom and fl exibility needed to express
themselves within the blog, without interfering or providing any guidance, help,
and/or encouragement, thus not having an active role in the blog. Nevertheless, the
educator was observing and following blog activity. The teaching presence within
in-classroom and blog activity revealed to be really infl uential and it played an
important role within the teaching and learning process. The educator facilitated
and helped students within the teaching and learning process, providing them with
clear, direct instructions. Specifi cally, direct instruction was highly apparent mainly
within in-classroom activities than blog activities, since the educator managed to
focus discussion on relevant issues, helping students to gain understanding and better realize and comprehend the subjects under investigation. In addition, within inclassroom activities the educator provided constructive feedback to the students.
Through the blended learning environment developed the topics and subjects under
investigation were clear enough, and the educator provided freedom and fl exibility
to the students, as well as increased responsibility for their own learning based on
their needs, demands, and interests.
The same picture is observed regarding the social presence, though being graded
relatively lower scores than the teaching presence. The data supports that it was
signifi cantly noticeable (M = 4.42; SD = 0.51; α = 0.709). Students felt to great extent
that they belonged to a group in which they could freely express their views, opinions, and thoughts. The interaction developed among students within in-classroom
as well as through blog's activities seemed to have strengthened the sense of collaboration. The educator managed to develop a community where the educator and
students felt closed and connected to each other, felt part of the course (developed
sense of belonging in the course—group cohesion), not only communicating for
educational purposes, but interacting socially as well (Affective Expression).
Additionally, the participants freely expressed their ideas, views, and opinions
through not only the online tools but within in-classroom, vividly participating in
the course discussion and frequently interacting with the rest of the participants
(Open Communication). Students shared and discussed with their peer views and
opinions and in some cases they felt instant connection to other peers. An initial
introduction, a welcoming note, and the online ice-breaking activities from the educator helped the students to open up and develop a sense of belonging. Channels of
communication were developed among students and the educator through the in-
classroom and blog activity showing that the open communication parameter was
highly present. Consequently, it can be supported that the social presence has been
developed through the use of blogs as educational tools.
Finally, the cognitive presence, had also a really "strong" appearance (M = 4.57;
S.D. = 0.38; α = 0.801) within the blended learning environment developed (blog
and in-classroom activity), however not as strong as the teaching presence.
Specifi cally, it emerged that the topic of the lesson stimulated in a great degree

students' interest, and online postings enabled students to understand the basic concepts of the course. The educator gave various interesting activities, motivated
students through questions; however, she did not provide adequate extra academic
and scientifi c information and resources to study. More specifi cally, the triggering
events were greatly apparent attempting to attract and motivate students. The activities developed and performed piqued students curiosity and felt motivated to explore
content related questions. It seems that the questions/activities designed and performed increased students' interest regarding the curriculum concepts under investigation and that their interest, motivation, and curiosity were enhanced. Exploration
was one of the elements of the cognitive presence that did not get high scores since
the educator did not provide adequate academic and scientifi c information and
resources to the students. Additionally, brainstorming, as a process was promoted
and implemented through in-classroom and blog discussions, which it helped students come up with answers and solutions. Finally, the total mean for the three presences was 4.59 (SD = 0.38). Revealing that the development of Community of
Inquiry was achieved in a great degree.
Blog Activity
The blog activity reveals blog role in developing a blended-learning environment
where in-classroom and blog-based activities are integrated in achieving the lesson's objectives. Overall nine posts were uploaded by the educator. Specifi cally,
seven of the posts were questions related to the subject under investigation. The
very fi rst question posted at the blog was for testing purposes (" Try to comment on
a blog post "). This question was posted throughout the introductory course conducted in the computer lab in order to give the students the opportunity to experience blogging. Their responses were used by the teacher in order to begin the
in-classroom lesson and discussions regarding "Time machine."
The second question posted at the blog was related to a video watched and discussed as an introduction to the lesson "How do you imagine a time machine? Can
you describe it? How does it look like externally and internally? What can it do?"
Students' replies focused on describing the time machine internally and externally,
and what one can do when using a time machine. During the Language and
Linguistics course the following question was posted: "Imagine that you are in the
time machine ready to travel. Where would you like to travel, into the past or in the
future and why? What would you change there?" The current question aimed to
trigger their imagination. Analyzing students' responses it is revealed that most of
them had special preference to travel into the past, and specifi cally to fi nd out where
their ancestors lived and how they behaved. Additionally, students reported that
they would like to travel to the past in order to correct their mistakes, to travel to
ancient history eras such as Ancient Greece and the Dinosaurs' Era. Finally, some
others chose to travel to the future in order to fi nd out how life would be and to make

sure that they will make their dreams come true. The students were requested to
begin addressing the questions in class and continue at home.
The next step was to discuss an article included in the Language and Linguistics
book. The educator used the students' responses in the previous blog question as an
introduction to a book article. In particular, the article was taken from the Focus
magazine, entitled "Travelling in time" and uploaded at the blog in order for the
discussion to be continued online. After studying and discussing the article in the
class, two questions were posted on the blog. The fourth and fi fth questions were as
follows: "Changing the events in the past and in the future: What would you like to
change if you went to the past or the future? What event would you like to change
and/or infl uence?" and "Persons from the past: Which person from the past would
you like to represent? What decisions would you change if you were that person?"
The students started answering the question at school and continued at home.
By the completion of the Language and Linguistics course the students were
requested to visit the blog once more in order to vote in which era they would like
to travel using the time machine. It was a multiple question voting system. The
choices were the following: Dinosaurs Era, Ice Age, Ancient Greece, their childhood, the future, the era in which their ancestors lived (parents/grandparents), and
other. The students chose to travel to the future and childhood, to the Dinosaurs Era
and their ancestors' era. Once more, the educator requested the students to use the
blog from home.
Finally, during the Art Lesson, the students were asked to imagine and draw a
time machine. Their drawings were scanned and uploaded by the educator to the
blog. A blog post accompanied the drawings delineating the following: "Fifth graders draw a time machine! Below you can fi nd your classmates drawings. Each student is requested to write a few words about his/her drawing. You can also comment
on your classmates' drawings. We are all waiting to hear your opinions." For this
post students showed minimum interest, mainly due to the lack of time.
Discussion
Blog Use
Blog was integrated as a tool at school and at home. There was no extensive use of
the blog from home for educational purposes, even though the students reported
using the Internet relatively often at home. Even if students did not highly use the
blog; its educational potential was revealed, since blog exploitation facilitated
the achievement of the lesson's learning objectives. It can be also supported that the
blog was integrated as an educational tool on a satisfactory level/degree since it
provided a platform for students to interact and discuss issues related to the subject
under investigation, activated their imagination and fi nally helped to signifi cantly
achieve the learning goals set by the educator. Overall, the blog was well organized
and designed. The effective and successful exploitation use of various blog's tools,

functions, and settings by the educator was observed and facilitated the learning
process. For example, the posts uploaded by the educator were of different content,
text, picture (student drawings) and video. The educator also employed various gadgets such as: voting (in order to perform the eighth post/exercises), calendar (for
reminding students when and where the lessons will take place as well as when the
exercises were due), blog archives.

Students
Students freely expressed their personal views, thoughts, and opinions; however,
they did not comment on their classmates' posts. Unfortunately, the blog was not
employed as a tool to promote collaboration and refl ection. This result was kind of
expected given the novice educator and student experience in blogging and the inexistence of educator's blog role and appearance. Additionally, it is supported that the
design of the activities was problematic, since they did not clearly provide instructions and guidelines to students. There was limited discussion, interaction and dialogue among the students at the blog, something that did not happen through
in-classroom activities (where the students were lively participated). Students'
responses showed that they were able to freely express their personal opinions; however, they did not comment on their classmates' responses. Thus, the blog was not
employed as a tool for promoting cooperation and refl ective learning. Specifi cally,
the teacher did not create the appropriate environment for students' interaction, dialogue and discussion through the blog. Moreover, not all students had access to computers and Internet at home. Finally, it was the very fi rst time that such as tool (the
blog) was used by both students and the teacher for educational purposes.
Educator
Given the above, it is suggested that educator's role is extremely important. Even
though educator's overall presence was really infl uential, managing to develop a
blended learning environment; it is important for a number of elements to be taken
into consideration when designing and implementing blog-based activities integrated within a blended learning environment. The students needed more guidance,
monitoring, assistance (promote interaction, debate, and dialogue) while working in
an online environment. In order to effectively and successfully integrate blogs as
educational tools within the teaching and learning process a number of factors need
to be in place. First of all, the educator should have a greater presence and involvement within blog activity. The educator needs to directly lead and guide the blog
activity. Specifi cally, the educator's participation at the blog for scaffolding, stimulation and motivation purposes considers being extremely important.
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Having the same importance; specifi c and understandable instructions and guidelines regarding blog use and students' responsibilities and expectations is expected
to be given to the students. For example, the educators need to make sure that when
designing a blog and its posts (exercises), the following parameters are addressed:
frequency and consistency of use, initiative to begin discussion, initiative to continue discussion, quality of responses, minimum number of posts and responses to
classmates, deadlines, combination of blog and classroom activities; and provide
relevant information to the subject under investigation through links and extra readings. The educator's role is important and crucial. His engagement and involvement
in blog should be a continuously apparent while playing the role of the facilitator.
The educator needs to review students' posts, comment on students' responses, ask
questions, make observations, prompt and remind students to respond (Garrison &
Arbaugh, 2007). Also, it is important to motivate students, even providing grading
related motives.
One more important element is for the educator to highlight the importance of
dialogue, discussion, interaction and interactivity, coexistence, and collaboration
within a team, and fi nally the collaborative group result. The educator is responsible
in organizing lesson plans and developing activities to achieve the above learning
environments. It is also important that the educator organizes and develops well-
planned activities which use effectively blog's features, functions, and settings. For
example, blog activities are designed to be posted on the blog must have features
that promote critical thinking, refl ection, collaboration, dialogue, debate, expression
of opinions, and interaction (Zawilinski, 2009). Finally, to effectively design
blended learning environments; suffi cient time for students to use the blog should
be given, taking into account other parameters such as possession and use of computers and Internet access at home.
Conclusion
The possibility of incorporating the blog as an educational tool to a greater extent
within the primary education is highly evident given the results of the current study.
Additionally, the results of the study highlighted the employment of a blog as an
educational tool in order to design and develop blended learning environments
where Community of Inquiry is achieved. Specifi cally, the results revealed that both
cognitive and social presence as well as teaching presence contributed to knowledge
construction (Garrison & Vaughan, 2005). Nevertheless, the use of blog can be
characterized as satisfactory given the limited time it was active and its partial use
by the students. The current paper highlights the possibility of extensive blog integration as an educational tool in primary education taking into account various factors and suggests the development of a model that incorporates/explains the
requirements of effective integration blog in educational practice based on the following parameters: teacher, student, blogs, and content/activities. Through this
model, the role and interaction of the four parameters will be refl ected and explained.
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Finally, it is argued that the model of the blended community of inquiry is considered more suitable for primary education. Future research should focus on adapting
the current model of community of inquiry, creating and weighting of a model to the
characteristics of primary education.
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Introduction
Web 2.0 technologies can be used in education, especially for building projectbased learning activities (Duffy & Kirkley, 2004). Among these technologies, wikis
seem to offer rich collaboration possibilities (West & West, 2009). A wiki offers the
ability to edit a website by adding, modifying, and deleting pages as well as integrating hypermedia.
The open nature of the wiki technology provides opportunities for learning
(Mindel & Verma, 2006; Raman, Ryan, & Olfman, 2005; Wheeler & Wheeler,
2009), since all participants should work collaboratively in order to edit and improve
the content. Wikis as a collaboration tool can help students to write better (Mak &
Coniam, 2008) and can support collaborative knowledge creation (Raman et al.,
2005; Wagner, 2004). In addition, wikis can facilitate group learning (Carpenter &

Roberts, 2007), foster contribution to peers (West & West, 2009), and improve students' engagement (Molyneaux & Brumley, 2007).
Various skills, such as writing, IT, collaboration, and organizational skills (Lai &
Ng, 2011; Wheeler & Wheeler, 2009), can be improved with students' involvement
in wiki-based activities. Thus, students' participation in wiki projects can add value
Effectiveness of Wiki-Based Learning
in Higher Education
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to learning and to their professional success (Frydenberg, 2008). It seems that in
well-designed wiki-based activities students are positive towards this technology
(Tétard, Patokorpi, & Packalén, 2009). They believe that wikis are useful for sharing
knowledge (Elgort, Smith, & Toland, 2008) and a great tool for collaboration
(Deters, Cuthrell, & Stapleton, 2010). Unsurprisingly, wikis have been used in various contexts such as developing an online textbook (Ravid, Kalman, & Rafaeli,
2008) and supporting knowledge sharing (Raman et al., 2005). Wikis can also help
teachers to manage and mark their students' work (Deters et al., 2010). However,
the wiki openness may also be a disadvantage if the context and objectives of the
activity are not well determined (Parker & Chao, 2007). As a result, there are examples in which students did not actively participate in creating or editing context
(Cole, 2009; Ebner, Kickmeier-Rust, & Holzinger, 2008).
In order to avoid these situations and to provide a rich context and simultaneously a support structure in a wiki environment, West and West (2009) proposed an
instructional design process. This process includes the following steps: establish a
purpose for the wiki project, define and classify the wiki project's learning goals,
design a rich context and problem that support the achievement of the purpose and
goals, prepare students for work in the new environment, and promote a collaborative process through which active, social learning can take place (West & West,
2009, p. 22). This approach emphasizes scaffolding. In addition, West and West
(2009) used the Bloom's Taxonomy of Learning (Bloom, 1956) to classify the main
wiki project's learning domains. From this grouping three wiki project categories
emerged: (a) knowledge construction, (b) critical thinking, and (c) contextual application. The wiki activity design was based on the framework proposed by West and
West (2009), and its learning domain belongs to the category of knowledge construction. The goal of the designed activity was to learn general information about
Web 2.0 and its applications in the frame of a first-year academic course entitled
"Introduction to ICT."
The aim of the study presented in this chapter was to investigate the effect of a
framed, rigorously designed, wiki-based activity on the learning outcome. In specific, this chapter investigates:
• Students' learning gain after the wiki-based activity
• Whether the students with lower pretest score benefited from the activity at least
to the same extent as students with higher pretest score
• Whether students' learning performance was affected by their role while carrying out the activity
• Whether students with more logged wiki edits benefited more than students with
less wiki edits
This chapter is organized as follows: Initially, the research methodology, the
profile of the participants, and the design of the activity are described. Subsequently,
the research results are presented, focusing on learning outcome as assessed by an
appropriately designed pre- and posttest questionnaire.
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Methodology
Research Method and Materials
A single-group pretest-posttest design was adopted (Cohen, Manion, & Morrison,
2000). A questionnaire with closed questions was the data collection instrument.
The pretest questionnaire comprised both demographic (13 questions) and factual
knowledge questions (36 questions, each with four possible answers of which only
one was correct). The factual knowledge questions were primarily related to general
information about Web 2.0 and its applications, whereas the demographic questions
were related to personal information regarding ICT, Internet, and wiki usage and
adoption. The posttest comprised the same factual knowledge questions. The students were not informed that they would be asked to complete the questionnaire at
the beginning or the end of the activity.
The wikispaces service (www.wikispaces.com) was used both for the activity
announcement and as the platform provided to the students to construct their wiki.
The online questionnaire service SurveyMonkey (www.surveymonkey.com) was
used to create and distribute the questionnaires of the study. The obtained data were
organized and analyzed using Excel 2007 and SPSS v17.0. The activity presentation, students' presentation of their wikis, and completion of questionnaires took
place in the computer lab of the Department.
Procedure and Participants
All in all, 220 first-year university students participated in the study. The students
were divided freely into 44 groups comprising 5 members each. In the beginning of
the procedure an instruction on the wiki's basic functionality was given to the students. Subsequently, a compulsory assignment was presented to them in the form of
a wiki, realized by the researchers. Each team member had a specific role in the
group such as collector, organizer, editor, and verifier (West & West, 2009). The
responsibilities of each role are delineated in the following.
Eighty-one (81) of the students did not respond to either the pre- or posttest
assessment questionnaire and were excluded from the dataset. Analysis was conducted for the data collected by 139 students, 2 male and 137 female, aged 17–37
(mean=19.3, sd=3.5). The majority of the participants (127/139) were 17–22 years
old. They were attending a compulsory academic course entitled "Introduction to
ICT," offered in the first semester in the Department of Education and Early
Childhood Education at the University of Patras. Participation in the activity was
compulsory and was one of the five required mini-projects given to the students in
the context of the laboratory part of the lesson.
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Description ofthe Activity
The design of the activity was based on the framework proposed by West and West
(2009). Learning was expected to be achieved by engaging the students into four
processes: information seeking and retrieval, argumentation development and
refinement to support their thesis, cooperation among members, and their involvement with the wiki-editing process. The assignment was presented to the students
through an exemplary wiki, which was constructed by the researchers (available at
http://labtpewiki.wikispaces.com).
The exemplary wiki included the purpose and the objectives of the assignment,
detailed implementation instructions, expected learning outcome, evaluation criteria, and representative support material. In addition, the topics that students had to
cover were outlined and organized into subsections with a short description for each
one. Afterwards, the students of each group had to create their own wiki, in which
they would develop the topics of the assignment.
The topic of the designed activity was to learn general information about Web
2.0 and its applications. It was selected due to the following reasons: First, the students should be able to understand the impact of Web 2.0 on society in general and
on education in particular. In addition, the topic is suitable for covering a variety of
educational and technological aspects of Web 2.0, thus giving a fertile ground for
argumentation. Finally, it is a notable session of the course's overall outline. The
exemplary wiki included nine segment topics: (a) Web 2.0 definitions, (b) characteristics of Web 2.0, (c) YouTube and Slideshare, (d) Twitter, (e) Blogs, (f) Wikis, (g)
Skype, (h) description of an educational activity using these technologies, and (i)
potential risks of Web 2.0 use.
For this activity wikis were used to promote collaboration with peers and improve
their knowledge related to the aforementioned topics (West & West, 2009). The students had to search for information on all of these topics and seek additional material. Furthermore, it was stressed that usage of other's work should follow specific
rules since the open nature of Web 2.0 tools could lead to inappropriate use of content
from other sources, as reported in Huijser (2008) and West and West (2009). Students
were instructed on how to use and cite sources and were also informed that they
could only use freely available media or media under a creative commons license.
Each team member was assigned a specific role by the researchers. These roles
are delineated by West and West (2009). The first role was that of "collector" who
had the responsibility to obtain appropriate material relevant to each subtopic. Two
members of each group were "collectors." The second role was that of the "organizer" who was responsible to organize the collected material and to check its consistency and relatedness with the objectives of the project. The "editor" was
responsible to check grammar and syntax errors in the content and its compliance
with the provided format. The "verifier" was responsible to check the content for its
completeness, structure, and compliance with the objectives of the project. However,
all students were allowed to participate and contribute in every aspect of the
collaboration process.
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Finally, the students had to present their work briefly during the laboratory session of the course. Each project was graded by the researchers on a 1–100 scale. The
score was multiplied by the number of the group members and was given to the
students. Subsequently, the students in each group were asked to discuss and distribute these points fairly according to each member's contribution. As far as the score
distribution is concerned, a notable differentiation was observed in only 9 out of the
44 groups. Such grade distribution differentiations possibly indicate a lack of balanced collaboration.
Results
All in all, we analyzed data from 139 first-year university students involved in a
wiki-based activity in the context of an introductory ICT course. Table 1 presents
participants' demographic-related information in our dataset.
First, a reliability analysis of the provided 36-item knowledge assessment questionnaire was conducted. Reliability refers to the extent to which an instrument,
such as a questionnaire, yields the same results under consistent conditions
(Nunnally & Bernstein, 1994). It is most commonly measured using Cronbach's
alpha, which is a measure of internal consistency. Results showed that the initial
36-item questionnaire used in the study did not have sufficient reliability
(alpha=0.69) to meet the typical minimum standard of 0.70 (Nunnally & Bernstein,
1994). Two questions increased the alpha to 0.70 if they were deleted and thus were
excluded from subsequent test score computations.
Next, students' pretest and posttest correct answers in the knowledge assessment
questionnaire were converted to a composite test score on a 0–100 scale. In addition, a normalized learning gain score was produced for each participant by using
the formula proposed in Nelson et al. (2009) and defined as
G =
–
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score score
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score score
x
Table 1 Participants'
demographic-related
information in our dataset
Sample size N 139
Age Mean 19.3
SD 3.5
Range 17–37
Gender Male 2
Female 137
School stream Theoretical 125
Technological/scientific 14
Web usage frequency [1-5] Mean 4.2
SD 1.0
Range 2–5
Prior wiki usage Yes 60
No 79
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This score has the advantage of "normalizing the observed gain (the numerator)
against the amount of possible learning that could be achieved (the denominator)"
(Nelson et al., 2009, p. 1797). Table 2 presents descriptive statistics of the dependent variables measured in the study.
In all subsequent statistical analyses, we use the correlation coefficient r as an
effect size, which is calculated according to the formulas reported in Field (2009).
Did the Wiki-Based Activity Improve Students' Performance?
A dependent t-test was applied to compare students' pretest (M=43.6, SD=11.4)
and posttest (M=63.8, SD=13.6) performance, as measured by the provided knowledge assessment questionnaire. The differences between the test scores did not violate the assumption of normality (D(139)=0.98, p=0.059), and thus a parametric
test was selected. Results indicated that students achieved significantly higher
(t(138)=17.74, p<0.001, r=0.83) test scores after participating in the wikimediated learning activity. According to Cohen (1992), this is a very large effect
size, which demonstrates the learning effectiveness of a properly designed wikimediated learning activity.
Were Students with Lower Pretest Score Benefited at Least
to the Same Extent as Students with Higher Pretest Scores?
We recoded our dataset to create two between-subject groups based on students'
initial performance: (a) low initial performance (N=83), which included students
with pretest score below or equal to the median score of all students, and (b) high
initial performance (N=56), in which students with pretest score above the median
score of all students were assigned. Table 3 presents descriptive statistics of students'
pretest score, posttest score, and normalized learning gain in relation to these groups.
A two-tailed Mann–Whitney U test investigated the effect of students' initial
performance on their normalized learning gain. A nonparametric test was selected
because the assumption of normality was violated for the high initial performance
group (D(56)=0.95, p<0.05), and homogeneity of variance was also violated
(Levene's test, F(1,137)=9.39, p<0.01). Results indicated that although students
Table 2 Students' pre- and posttest scores in the knowledge assessment questionnaire and their
normalized learning gain
N
Pretest score [0-100] Posttest score [0-100] Normalized learning gain
a
[%]
Mean±95 % C.I. Mean±95 % C.I. Mean±95 % C.I.
139 43.6±1.9 63.8±2.3 35.1±3.9
a
Normalized learning gain is measured as (post–pre)/(max score–pre) (Nelson et al., 2009)
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with low initial performance showed a slightly higher (3.9 %) normalized learning
gain compared to those with high initial performance, this difference was not significant (z=0.14, p=0.889). Also, an insignificant correlation (rs=−0.04, p=0.668)
between students' pretest score and normalized learning gain was found.
Additional analyses after Bonferroni correction investigated whether the wikimediated activity improved students' score for both the low and high initial performance groups. A nonparametric test was selected for the high initial performance
group because the distribution of the differences in the dependent variable (test
score) between the two related conditions deviated significantly (D(56)=0.94,
p<0.01) from a normal distribution. Results showed that students' test scores were
significantly improved in both the low and high initial performance groups:
t(82)=17.16, p<0.001, r=0.88 and z=5.56, p<0.001, r=0.53, respectively.
All in all, the above results provide evidence that the wiki-mediated learning
activity was beneficial to students with lower initial performance, at least to the
same extent as those with higher initial performance.
Did Students' Role in the Wiki-Mediated Activity Affect Their

Learning Gain?
Table 4 presents students' performance grouped by their role in the wiki-based
activity: collector (N=53), organizer (N=28), editor (N=29), and verifier (N=29).
Table 4 shows that the lowest average learning gain (30.0 %) was observed for
students with the verifier role, whereas students with the editor role had the highest
learning gain (44.4 %) on average. However, a one-way ANOVA did not unveil any
Table 3 Students' pre- and posttest scores in the knowledge assessment questionnaire and their
normalized learning gain grouped by their initial performance
Initial performance group N
Pretest score [0-100] Posttest score [0-100]
Normalized
learning gain
a
[%]
Mean±95 % C.I. Mean±95 % C.I. Mean±95 % C.I.
Low 83 35.8±1.4 59.7±2.8 36.7±4.1
High 56 55.1±1.6 70.0±3.4 32.8±7.8
a
Normalized learning gain is measured as (post−pre)/(max score−pre) (Nelson et al., 2009)
Table 4 Students' pre- and posttest scores in the knowledge assessment questionnaire and their
normalized learning gain grouped by their role in the activity
Role in the wiki activity N
Pretest score [0-100] Posttest score [0-100]
Normalized
learning gain
a
[%]
Mean±95 % C.I. Mean±95 % C.I. Mean±95 % C.I.
Collector 53 42.9±3.0 63.3±4.1 35.4±6.6
Organizer 28 44.0±4.6 61.6±4.3 30.1±7.5
Editor 29 43.4±4.9 69.3±4.5 44.4±7.4
Verifier 29 44.6±3.9 61.6±5.4 30.0±10.6
a
Normalized learning gain is measured as (post−pre)/(max score−pre) (Nelson et al., 2009)
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significant learning gain differences between the four different students' roles
(F(3,135)=2.48, p=0.064).
Thus, results show that the wiki-mediated learning activity was beneficial to all
students, regardless of their specific role in the project.
Were Students with More Wiki Edits Benefited More Than
Students with Less Wiki Edits?
We recoded our dataset to create two between-subject groups based on students'
logged number of edits in the wiki: (a) low number of wiki edits (N=71), which
included students with a number of wiki edits below or equal to the median number
of wiki edits of all students, and (b) high number of wiki edits (N=68), in which
students with a number of wiki edits above the median number of wiki edits of all
students were assigned. Table 5 presents descriptive statistics of the measured
dependent variables in relation to these two groups.
The assumption of normality was violated for the high number of edits group
(D(68)=0.93, p<0.001); thus a nonparametric test was applied to investigate differences between the two groups. A two-tailed Mann–Whitney U test showed that
students with more wiki edits had a significantly higher learning gain (z=-4.32,
p<0.001, r=0.37) compared to students with less wiki edits. In addition, a significant correlation (rs=0.41, p<0.01) was found between students' number of logged
wiki edits and normalized learning gain: the more active the students were, the more
they improved their performance.
Additional analyses after Bonferroni correction investigated whether the wikimediated activity improved students' score for both the students with lower and
higher number of wiki edits. Results showed that students' test scores were significantly improved in both groups: t(70)=10.69, p<0.001, r=0.79 and t(67)=15.45,
p<0.001, r=0.88, respectively. As the effect sizes show, this improvement was of
higher magnitude for the students who were more active contributors in the wiki.
In sum, it was found that both students with low and high number of logged edits
in the wiki improved significantly their performance, but the latter had a significantly higher learning gain.
Table 5 Students' pre- and posttest scores in the knowledge assessment questionnaire and their
normalized learning gain grouped by their number of wiki edits
Number of logged wiki edits N
Pretest score [0-100] Posttest score [0-100]
Normalized
learning gain
a
[%]
Mean±95 % C.I. Mean±95 % C.I. Mean±95 % C.I.
Low 71 42.4±2.7 59.0±3.1 27.8±5.0
High 68 44.8±2.8 68.9±3.0 42.8±5.7
a
Normalized learning gain is measured as (post−pre)/(max score−pre) (Nelson et al., 2009)
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Conclusions
A study investigating the effectiveness of a wiki-mediated learning activity for ICT
education was presented. The evaluation was carried out using a one-group pretest-
posttest design. The results showed significant improvement in learning outcomes;
the average students' test score improved from 43.6/100 to 63.8/100. In addition, it
was found that the wiki-mediated learning activity was equally beneficial to students with lower and higher initial performance. These results suggest that a properly designed, framed wiki-based activity could substantially facilitate students to
learn by building content. In a similar vein, there are surveys which indicate that
wiki technology can be beneficial to students in various learning domains
(Mohammed, 2010), given that a carefully designed activity is introduced to them.
Similar findings are reported in Tselios, Altanopoulou, and Katsanos (2011) and
Tselios, Altanopoulou, and Komis (2011).
No significant learning gain differences between the four different student's roles
(i.e., collector, organizer, editor, verifier) in the wiki activity were identified. This
finding is in line with previous research (Strijbos, Martens, Jochems, & Broers,
2004; Tselios, Altanopoulou, & Katsanos, 2011; Tselios, Altanopoulou, & Komis,
2011) which indicates that roles do not affect group members' performance.
However, roles can help students who work collaboratively to build knowledge in
comparison to students with no distinct roles while collaborating (Schellens, Van
Keer, De Wever, & Valcke, 2007).
Furthermore, results showed that students who were more active contributors in
the wiki, as measured by their logged number of edits, had a significantly higher
learning gain.
However, the reported study is not without limitations. First, it should be noted
that our data is gender and age skewed; thus the findings might not be generalizable
to male students or older students involved in wiki-mediated learning activities.
Furthermore, the results obtained do not explain how the students have benefited
from their involvement in the activity. Future research goals constitute the design of
additional wiki-based activities in a variety of educational settings as well as investigation of the learners' behavioral intention to use wiki technology using technology acceptance models (Tselios, Daskalakis, & Papadopoulou, 2011). Moreover,
the relation between the observed students' activity and the learning outcome will
also be examined (Katsanos, Tselios, & Avouris, 2010).
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Introduction
E-learning is a modern learning method, based on information and communication
technologies (ICT). Its main characteristics are that it overcomes time and spatial
restrictions, since learners can attend the course wherever they are, assuming they
have adequate equipment, such as a computer connected to the Internet.
Courseware is a term that combines the words "course" with "software", and was
used originally to describe additional educational material for authors and learners
apart from the e-learning platform itself. The meaning of the term and usage has
expanded and can refer to the entire course and any additional material when used
in reference to an online or "computer formatted" classroom. The main objective of
constructing courseware is to help authors and developers to carry out the construction/development process of courseware and learning contents automatically. The
secondary objective is to promote the reuse/exchange of existing learning resources
among different users and systems (Romero & Ventura, 2010).
Courseware Evaluation Through Content,
Usage and Marking Assessment
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In any courseware, each course consists of the content, the organisation and the
management of the course, the teaching/learning process, the log data, social computing tools, etc. Although the quality of a course does not depend only on its content but also on its structure, organisation, support, delivery, etc., content plays a
crucial role for the success of e-learning. In order to ensure this fact, it is necessary
to apply processes of continuous evaluation and optimization of the educational

material (Kazanidis, Theodosiou, Petasakis, & Valsamidis, 2013). Consequently, it
is necessary to provide feedback to a course author, in order to show the means to
improve its courseware (Romero, Ventura, & De Bra, 2004). The evaluation of educational material can be made either directly by taking feedback from the learners
or through automated data mining techniques applied to the courses log fi les data
(Vialardi, Bravo, & Ortigosa, 2008).
Statistical analysis methods were applied in many cases in order to obtain relevant information from e-learning data. The higher education learner-evaluation data
were analysed in Jin, Wu, Liu, and Yan (2009). The number of visits and duration
per quarter, top search terms and number of downloads of e-learning resources were
described in Grob, Bensberg, and Kaderali (2004). The number of different pages
browsed and total time spent browsing different pages was also presented in Hwang,
Tsai, Tsai, and Tseng (2008).
Data mining (DM) in education uses computational approaches to analyse educational data in order to analyse upcoming educational issues. According to Romero
and Ventura (2010) "the term DM is used in a larger sense than the original/traditional DM defi nition". Although there is a great deal of research in the fi eld of DM
in e-learning that uses typical techniques, such as classifi cation, clustering, association rule mining, sequential mining, etc., there are also a signifi cant number of studies that use techniques belonging to the broader fi eld of DM, such as regression,
correlation, etc. The application of data mining (DM) in education is a rapidly
growing interdisciplinary area which combines a variety of areas, including individual learning from educational software, computer supported collaborative learning, computer-adaptive testing (and testing more broadly), and the factors that are
associated with student failure or non-retention on courses (Baker, 2010). Although
data mining (DM) in education uses computational approaches to analyse educational data in order to study educational questions, the methods are often different
from standard data mining methods (Baker & Yacef, 2010).
The approach in this paper is twofold. On the one hand, it goes backward to
examine whether the usage of the courses by the learners is affected by the educational content exposed by the authors. One the other hand, it examines whether the
usage of the courses by the learners on a course is related to the mean performance
of the learners on this course. It proposes some new metrics and measures, taking
into account several statistics concerning the courses. These include the number of
fi les and their sizes, the number of pages that each course has on the e-learning
platform, and statistics concerning the usage of the platform for each course by the
learners, such as the number of sessions, the number of visits, the duration of each
visit. These measures and metrics aim to help course authors and/or platform administrators review course usage, and fi nd online course weaknesses. Regression analysis is also used for the identifi cation of possible dependencies.

Method
The proposed method adopts a three-level schema for an e-learning platform. It uses
six measures and three metrics for both content and usage measurement. Finally,
classifi cation, clustering, association rule mining, and regression analysis are
applied to the e-learning data. More specifi cally, the values of the measures and
metrics and the mean marks from the corresponding courses are investigated for
possible dependencies.
The Three-Level Dependencies
A view of the proposed schema of the approach is depicted in Fig. 1 .
In Fig. 1 , the Content Level (CL) includes the educational material that is exposed
to the learners by the authors. It can be assessed with the use of measures. The Usage
Level (UL) includes the usage of the educational material by the learners. Other
measures and new metrics are also used to assess the usage. The Exams Level (EL)
includes mean marks, the learners performed for each course. The mapping between
CL and UL is one to one. Although the values of the measures may change each time
(affected by authors' actions and learners' usage), the assessment usually takes place
at the end of the semester. The mapping between UL and EL is one to many, since
there are n (n> 1) opportunities for every learner to participate in the exams.
Measures and Metrics
Some measures are used in the CL and some others in the UL of the courses. With
the measures of the Table 1 , we quantify the offered educational material to the
learners by the authors in terms of input variables on a course.
Exams performance1 Exams performance n
Course Usage
Educational Content of the Course
EXAMS
LEVEL
mapping
mappings
CONTENT
LEVEL
USAGE
LEVEL
Fig. 1 The 3-level approach

This category of measures is related to the courses' online educational content of
the courses. More specifi cally, the number of pages, the number of fi les and their
corresponding sizes give an estimation of the content quantity, which is a crucial
factor of online educational content. In some e-learning systems instead of the term
"page" the term "module" is used. If the number of fi les and their size are small, this
might be due to the weakness of the author to upload enough educational content
onto the online platform. If the course has a lot of fi les with large sizes this could lead
learners to face the cognitive overload problem and not study the course effectively.
With the measures of Table 2 (Valsamidis, Kontogiannis, Kazanidis, Theodosiou,
& Karakos, 2012), we quantify the usage of the available educational material by
the learners in terms of usage variables on a course.
The second category of measures helps researchers to discover learner activity
and follow up a course. The number of sessions show how many times learners have
logged in. The number of sessions and the number of visits viewed by all users are
counted for the calculation of course activity. Each session refl ects when a user logs
in to the platform, and after some activity, logs out from the platform. If there is no
activity, there is a timeout of 30 s. The number of visits refl ects how many pages
were viewed by all users. There are some pages of the course which were viewed by
many users, but there were also some other pages not so popular. If learners of a
specifi c course visit more pages for a long time, this means that course content is
interesting and useful for the learners. This could refl ect the course quality.
Consequently, a good course in terms of quality may help learners in their study.
The number of sessions could be compared with number of visits and duration. The
two later variables show if learners fi nd the course useful and like to visit its pages.
The next step is to defi ne some metrics, in order to qualify these quantity metrics.
In Table 3 , we calculate the quality of the available educational material on a course,
in terms of usage by the learners of output variables.
VPS Visits Sessions □ / ( ) V E/

(1)
Table 1 Content measures
Measure Description of the measure
Pages (P) The total number of pages (modules) existing on the course
Files (F) The total number of fi les on the course
Size (S) The total size of the existing fi les on the course
Table 2 Usage measures
Measure Description of the measure
Sessions (E) The total number of sessions per course viewed by all users
Visits (V) The total number of visits (page hits) per course by all users
Duration (D) The duration of (total) visits per course by all users

The VPS has a range of between (0, 1]. A low number of VPS outcomes indicates
that users stay more time in a page of a course. Since the VPS metric measures the
number of visited pages divided by the number of sessions of a course, the higher
the number of sessions in the course, the lower is the fraction. On the other hand, the
author may update the educational material with more visited pages. So, the fraction
has comparable values.
VPD Visits Duration □ / ( ) V D/

(2)
The VPD expresses the number of visits per duration. A high number of VPD
means that users change pages quickly into the course. This could be an indication
that users are experiencing diffi culties discovering the desirable content or they
already have assimilated most of the course content. Both metrics refl ect users'
behaviour related to the educational material.
We also defi ne a course quality metric called smooth or distinct Course Utilisation
and user Perception metric (CUP). This metric expresses how smooth, or selective,
or even randomly visit time of users per course is distributed over the academic
semester. That is, from Eq. (3), for n online courses a constant histogram break is
used equal to 1/2 n (break = 10 and n expresses weeks (n = 1…10)). Then the CUP
metric value is calculated as follows:
CUP □ □n h x h x max min □ □ □ □ □

(3)
where h (x) the histogram density estimate:
h x k n w □ □ □ / / 1

(4)
x = cell centred at x with width w that contains k data points and n × h (x) = (2/ n) k
if CUP → 0 then we have smooth course utilisation over time, while if CUP → 1
we have only distinct weeks of course utilisation. In cases of CUP > 1 we assume
that such courses maintain either a one-time utilisation or abnormal utilisation of
very low and very high. We consider such courses as fl apping or abandoned courses.
Data Mining
Data mining techniques have been applied to e-learning systems data by many
researchers. Apart from the analytical review by Romero and Ventura (2010), there
is some more domain specifi c. Castro, Vellido, Nebot, and Mugica (2007), among
Table 3 Quality metrics
Metric Description of the metric
VPS Visits per session
VPD Visits per duration
CUP Course utilisation and user perception

others, deal with the assessment of the students' learning performance, provide
course adaptation and learning recommendations based on the students' learning
behaviour, deal with the evaluation of learning material and educational Web-based
courses, provide feedback to both teachers and students of e-learning courses, and
detect a typical student's learning behaviour. A survey by Koutri, Avouris, and
Daskalaki (2005) provides an overview of the state of the art in research of Web
usage mining, while discussing the most relevant criteria for deciding on the suitability of these techniques for building an adaptive web site. One relevant study
(Kotsiantis, Pierrakeas, & Pintelas, 2004) predicts the students' performance, as
well as to assess the relevance of the attributes involved.
Students are assessed in the fi nal exams of the courses, and they are assigned a
mark according to their performance on the course. Having measured the students'
activity in the e-learning system according to the measures and metrics, it is possible to investigate whether there is a relationship between student activity in the
platform of the e-learning system and the marks of the students in the fi nal exams.
A well prepared and implemented online course may help learners in their study and
therefore allow them to achieve a better mark. Therefore, it would be useful to check
if the previous metrics and measures are related to a learner's mark.
In the classifi cation step, the algorithm 1R (Witten & Frank, 2005) may be
applied. It uses the minimum-error attribute for prediction, discretizing numeric
attributes (Holte, 1993). The attribute Mark has to be used as class, since it describes
the education outcome. In this step the attribute/s which best describe the classifi cation will be discovered.
The clustering step contains course clustering with the use of the SimpleKMeans
algorithm (Kaufmann & Rousseeuw, 1990) for unsupervised learning. SimpleKMeans
algorithm automatically handles a mixture of categorical and numerical attributes.
Furthermore, the algorithm automatically normalises numerical attributes when
doing distance computations.
According to Baker (2010) relationship mining is a technique which discovers
relationships between variables, in a data set with a large number of variables.
There are four types of relationship mining: association rule mining, correlation
mining, sequential pattern mining, and causal data mining. In this paper we focus
on association rule mining. Association rule mining is one of the most well studied
data mining tasks. It discovers relationships among attributes in databases, producing if-then statements concerning attribute-values . An association rule X → Y
expresses a close correlation among items in a database, in which transactions in
the database where X occurs, there is a high probability of having Y as well. In an
association rule X and Y are called respectively the antecedent and consequent of
the rule. The strength of such a rule is measured by values of its support and confi -
dence. The confi dence of the rule is the percentage of transactions with antecedent
X in the database that also contain the consequent Y. The support of the rule is the
percentage of transactions in the database that contains both the antecedent X and
the consequent Y in all transactions in the database. The Weka system has several

association rule-discovering algorithms available. The Apriori algorithm will be
used for fi nding association rules over discretized e-learning platform data.
Regression Analysis
Several regression techniques have been used to predict a student's academic performance (using stepwise linear regression) (Golding & Donalson, 2006), to identify variables that could predict success in college courses (using multiple
regression), to predict university students' satisfaction (using regression and decision tree analysis), to predict high school students' probabilities of success in university (Mcdonald, 2004), to predict a student's test score (using stepwise regression)
(Feng, Heffernan, & Koedinger, 2005) and to predict the probability a student has
of giving the correct answer to a problem in an ITS (using a robust ridge regression
algorithm) (Cetintas, Si, Xin, & Hord, 2009). Yu, Jannasch-Pennell, Digangi, and
Wasson (1999) use a multivariable regression model to predict a learner's performance from log and test scores in Web-based instruction. Multiple linear regression
is used for predicting the time to be spent on a learning page (Arnold, Scheines,
Beck, & Jerome, 2005). Thomas and Galambos (2004) use regression analysis for
predicting university learners' satisfaction.
Linear regression analysis is applied on the CL. The classifi er builds linear logistic regression models. In our case, the way the mean mark is affected by the metrics
VPS, CUP, and VPD of each course is examined. The regression coeffi cients show
the marginal value of input (VPS, CUP and VPD) required measuring Mark.
Results
Study Population and Context
The recording of specifi c data from the e-learning platform is the fi rst step. The
dataset was collected from the Open eClass e-learning platform (GUNet, 2012)
which is used at the Kavala Technological Education Institute (TEI). The data are
from the Spring semester of 2011 and involve 1,534 students and 34 different
courses and are obtained from the server log fi les.
Case Study
A view of the collected data is shown in Table 4 . The values of the measures of
Tables 1 and 2 , which express measures of UL and CL, are presented. The aforementioned measures contribute to the evaluation of courses content and usage.

First Mapping Dependencies
All the DM techniques were performed using the open source Weka. The data mining methods are applied to the measures of Table 4 . The results of the classifi cation based on the OneR algorithm show that the measure Pages is better classifi ed (described) by the measure Sessions as depicted in Fig. 2 . For the three bins of the instances (low, mid and high) are described by the low bin of grades. It is easily interpretable, since only one mean mark is greater than 7.66667.
Table 4 Tracked data, measures, metrics and marks
CID Sessions Visits Duration Pages Files Size VPS VPD CUP Mark
AD5104 480 1,567 2,891 6 785 785 3.26 0.542 0.213 5.78
AD5103 477 1,853 3,635 6 3,165 3,165 3.89 0.510 0.16 6.02
AD2104 61 93 116 5 2,065 2,065 1.51 0.802 2.844 4.42
AD2103 3,237 9,756 10,864 6 5,135 5,135 3.01 0.898 0.0462 7.26
AD6102 3,734 6,585 6,571 6 5,696 5,696 1.76 1.002 0.0977 6.2
AD6114 337 938 1,688 6 0 0 2.79 0.556 0.2844 6.48
AD6106 144 271 709 6 0 0 1.89 0.382 0.888 4.49
AD6105 910 2,340 3,586 6 7,198 7,198 2.57 0.653 0.1244 5.88
AD7107 1,115 4,334 6,778 6 42,277 42,277 3.89 0.639 0.0799 6.3
AD4108 378 1,627 3,185 6 100 100 4.30 0.511 0.1777 5.56
AD7105 728 2,184 3,760 6 2,471 2,471 3.00 0.581 0.1244 5.89
AD4101 709 1,501 3,284 6 250 250 2.12 0.457 0.1955 5.65
AD5102 539 921 1,912 6 0 0 1.71 0.482 0.355 6.23
AD5101 414 746 1,731 6 0 0 1.80 0.431 0.444 6.11
AD6112 256 593 1,321 6 410 410 2.31 0.449 1.4222 6.08
AD6111 383 1,352 2,719 6 1,355 1,355 3.53 0.497 0.071 6.34
AD2100 2,063 8,430 15,915 6 358 358 4.09 0.530 0.0319 7.67
AD3107 632 1,713 3,662 6 6,899 6,899 2.71 0.468 0.1777 6.66
AD5106 308 984 1,994 6 121 121 3.20 0.493 0.3555 6.31
AD5105 269 799 1,750 6 0 0 2.97 0.457 0.2844 6.09
AD7101 791 3,206 5,706 6 27 27 4.06 0.562 0.1155 6.05
AD7100 415 1,677 2,724 6 0 0 4.04 0.616 0.2488 6.28
AD6108 2,209 4,565 7,633 9 12,461 12,461 2.07 0.598 0.0888 6.71
AD6107 970 2,088 3,538 10 11,525 11,525 2.15 0.590 0.106 6.53
AD2106 4,793 10,091 14,551 12 5,943 5,943 2.11 0.693 0.0355 7.23
AD2105 5,538 11,832 16,780 13 52,318 52,318 2.14 0.705 0.0266 7.81
AD2107 3,726 10,113 13,824 6 2,206 2,206 2.71 0.732 0.0248 7.19
AD6100 2,697 5,271 8,199 6 29,290 29,290 1.95 0.643 0.053 6.67
AD7102 3,721 7,780 8,846 6 61,213 61,213 2.09 0.879 0.044 6.82
AD3106 706 2,533 5,115 6 108 108 3.59 0.495 0.106 5.98
AD3108 2,759 4,139 5,330 6 4,175 4,175 1.50 0.777 0.106 6.02
AD4100 616 1,401 2,564 6 0 0 2.27 0.546 0.213 6.22
AD3102 490 621 1,515 6 0 0 1.27 0.410 0.3556 4.45
AD4104 252 390 1,147 6 0 0 1.55 0.340 0.444 4.210
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The results of the clustering based on the SimpleKMeans algorithm (Kaufmann & Rousseeuw, 1990) show that there are two clusters which correspond to high Mark (cluster 0) and low Mark (cluster 1), as depicted in Fig. 3 . For most attributes of the two clusters there is a high degree of similarity. However, the values of Sessions, Visits, Duration, VPD and Mark are not equal.
The results of the association rule mining based on the Apriori algorithm (Agrawal & Srikant, 1994) show ten rules, as depicted in Table 5 .
Table 5 shows how a large number of association rules can be discovered. There are some uninteresting rules, such as rules 8 and 9, since there is obvious dependency between Duration and Visits. Specifi cally, there is dependency between the low range of Duration (-inf-5670.666667] and the low range of Visits (-inf-4006]. There are also redundant rules, rules with a generalisation of relationships of several rules, like rule 2 with rules 1, and 7, 3 with rules 5 and 9, and 4 with rules 6 and 10. There are some similar rules, rules with the same element in antecedent and consequent, but interchanged, such as rules 1, 2, 8 and rules 5, 3, 9 respectively. But there are also rules that show relevant information for educational purposes, like those that show conforming relationships, such as rules 1, 2, 3, 5 and 7. And there are also rules that show interesting relationships, such as rules 4, 6 and 10, which can be very useful for the author in decision making about the activities of their courses. For example, in rule 10, there is a dependency between low range of Visits (-inf- 4006] and the low range of Pages (-inf-7.666667]. Similar conclusions can be drawn from the other rules.
Fig. 2 Classifi cation results
Fig. 3 Clustering results
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Second Mapping Dependencies
Simple linear regression analysis was applied to the metrics and marks of Table 4 . The courses were classifi ed to three classes according to their mean marks. The fi rst class which corresponds to courses with low marks is described by the equation
Mark VPS CUP □ □ □ □ □ □ □ 1 46 0 9 2 73 . . .

(5)
The second class which corresponds to courses with mid marks is described by the equation
Mark VPS □ □ □ 1 21 0 55 . .

(6)
The third class which corresponds to courses with high marks is described by the equation
Mark VPS VPD □ □ □ □ □ □ 0 27 0 61 0 981 21 . . . .

(7)
An increase in the VPS metric leads to higher mean marks for each course. From the formulas (5), (6) and (7), the conclusion is drawn that an increase of one unit to the value of the metric VPS will increase the mean value of the Mark 0.9, 0.55 and 0.61 for the three classes respectively.
Table 5 Apriori algorithm based on confi dence metric
Best rules found
1. Visits = '(-inf-4006]' 23 → Sessions = '(-inf-1886.666667]' 23 conf:(1)
2. Visits = '(-inf-4006]' Duration = '(-inf- 5670.666667]' 22 → Sessions = '(-inf-1886.666667]' 22 conf:(1)
3. Sessions = '(-inf-1886.666667]' Duration = '(-inf-5670.666667]' 22 → Visits = '(-inf-4006]' 22 conf:(1)
4. Visits = '(-inf-4006]' Pages = '(-inf-7.666667]' 22 → Sessions = '(-inf-1886.666667]' 22 conf:(1)
5. Sessions = '(-inf-1886.666667]' 24 → Visits = '(-inf-4006]' 23 conf:(0.96)
6. Sessions = '(-inf-1886.666667]' 24 → Pages = '(-inf-7.666667]' 23 conf:(0.96)
7. Duration = '(-inf-5670.666667]' 23 → Sessions = '(-inf-1886.666667]' 22 conf:(0.96)
8. Duration = '(-inf-5670.666667]' 23 → Visits = '(-inf-4006]' 22 conf:(0.96)
9. Visits = '(-inf-4006]' 23 → Duration = '(-inf-5670.666667]' 22 conf:(0.96)
10. Visits = '(-inf-4006]' 23 → Pages = '(-inf-7.666667]' 22 conf:(0.96)
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Discussion and Conclusions
This study proposes an approach for discovering dependencies based on histories from e-learning data. It tackles the problem of analysing these data at three levels. Initially it examines whether the usage of the courses by the learners is affected by the educational content exposed by the authors. It is proved by the classifi cation results and the dependency between Sessions and Pages (Modules). Then, it examines whether the usage of the courses by the learners on a course is related to the mean performance of the learners on this course. This is proved by regression results and the dependency between VPS and Mark. It proposes some measures, such as the number of fi les and their sizes, the number of pages that each course has on the e-learning platform, the number of sessions, the number of visits and the duration of each visit. New metrics are also proposed to assess course usage. Three data mining techniques, classifi cation, clustering and association rule mining, were applied to the e-learning data at the fi rst two levels. Furthermore, regression analysis was applied to the same data at the last two levels.
The originality of the study lies in the different use of existing techniques. The study builds on existing work, but also extends it in a different way, encompassing the e-learning fi eld. It has the following advantages: (1) It is independent of a specifi c e-learning platform, since it is based on the Apache log fi les and not the e-learning platform itself. Thus, it can be easily implemented for every e-learning platform. (2) It uses measures and metrics, in order to facilitate the evaluation of each course in the e-learning platform and the authors to make proper adjustments to their course educational material. (3) It uses classifi cation, clustering, association rule mining and regression analysis in order discover possible dependencies of the e-learning data.
The results reveal dependencies between content of the course and its usage by the learners. There is a dependency between the number of modules (Pages) of the platform with the number of sessions and the number of visits. The results also confi rm the assertion that there is dependency between the students' usage on an e-learning platform with their corresponding performance in the exams.
Feedback about the approach was received by the authors. The authors were informed about the indexing results along with abstract directions on how to improve their courses. Most of authors increased the quality and the quantity of their educational material. They increased the quality by reorganising the educational material in a uniform, hierarchical and structured way. They also improved the quantity by embedding additional educational material. By updating educational material, both quality and quantity were increased. A major outcome through the process of informing the authors about the results is that the ranking of the courses constitutes an important motivation for the authors to try to improve their educational material. Because of their mutual competition, they want their courses to be highly ranked.
A few authors complained that their courses organisation does not assist them to have high fi nal scores in the ranking list. They argued that, for example, the measure Pages (Modules) is heavily infl uenced by the number of web pages used to organise the educational material. Thus, courses that have all their educational material organised
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in a few pages have a low Pages (Modules) score. They were asked again to reorganise the material for each course in the e-learning platform according to the order taught, in order to facilitate easier use by the students.
The fact that only 34 courses in one platform were investigated is a limitation to the study. Especially for the data mining techniques which demand large datasets. However, this was ineluctable, since the case study department implements this number of online courses. But the proposed approach seems to be quite reliable if the inspection takes place over a long time period.
There are also other limitations to the proposed approach. Although ten parameters were investigated, which are not few, it could be asserted that many more could also be investigated. In addition, the infl uence of collaboration among the learners could also be investigated, since this plays a crucial role in the educational process. This omission is due to a lack of suffi cient enough related logs of the corresponding modules (wikis, chats, forums etc.) of the e-learning platform. Moreover, there are a number of students who try to read the materials only just before the exams. No one denies that a good site with good material, which is updated frequently, exists, but rarely visited. On the other hand, a bad web site may have frequent visits because student visits are related to their expected mark. So, the frequency and pattern of an individual student accessing an instructor's materials will be an indicator to show such a student's "Laziness" or "Diligence".
The results of this research are remarkable from a pedagogical point of view. On the one hand, this approach contributes to the improvement of courseware content quality, since the proposed measures and metrics, and their correlations to students' marks provide the authors a feedback about their effi ciency of their courses. Improvement of course quality provides students the opportunity of asynchronous study of courses with actualized and optimal educational material. On the other hand, since students usage results are correlated with the students' grade, an online platform may provide authors with notifi cations about their students' online actions. According to our experiment, results usage is closely related to the students' marks. An increased usage leads to better student marks and therefore to an improved educational outcome. For example, a learning platform could record student actions and after the application of specifi c algorithms classify them into predefi ned groups according to their estimated performance. Authors could study these groups of students, and try to help and motivate weak students. They could provide the advanced students with more educational content in order to achieve a more in-depth learning programme. Thus, the learning performance of all students could be signifi cantly improved.
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Introduction
This chapter builds on previous work about the use of technologies for learning in
cultural institutions (Yiannoutsou, Bounia, Roussou, & Avouris, 2011) where an
analysis of selected cases revealed that technology mainly functioned as a medium
for information delivery. This use of technology treats culture as something that can
be "transferred" from the "knowledge holding" museum to the visitor. In this context museum experience is structured around the consumption metaphor: the
museum produces "information" in digital or other form, for the visitor to consume.
Studies evaluating this type of cultural experience used the term "museum fatigue"
to highlight visitor limited ability to remember, digest and utilise the information
offered (Bitgood, 2009). Another line of research, reports decrease in the audience
of museums and cultural institutions (Simon, 2010). Taking into account the above
observations we could argue that technology has been employed in various ways by
museums to support their reconnection with the public where we identify two main
trends with respect to the learning experience pursued. The fi rst focuses on refi ning
the information and the way it is delivered to the visitor. The second redefi nes the
role of the visitor and his/her relationship with the museum in the process of culture
creation. In this paper we will briefl y refer to technologies supporting the information delivery metaphor and we will further expand on how technologies can support
a learning experience based on visitor participation in the process of culture
creation.
Game Design as a context for Learning in
Cultural Institutions
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Personalization and Games: Same Content
in New Clothing?
In this section we will discuss the learning implications of two examples which
we consider that involve information transfer: personalization and technologysupported games. These two examples represent the current trend in technology
development to support and enhance museum experience (see for example the
funded EC projects in FP7 and the recent call for digital culture, Cordis, 2012)
applied in order to attract more visitors to the museums. Here we will only discuss
the rationale underlying in general the learning experience that seems to be supported by these two examples.
Personalization aims at adjusting and transforming cultural experience so as to
meet the experiences, interests and knowledge at the level of the individual visitor
or the group. To this end, technology suggests the most appropriate content to the
user based (i) on visitor profi les, (ii) on the cumulative visitors' history (Bohnert,
Zukerman, Berkovsky, Baldwin, & Sonenberg, 2008 ; de Gemmis, Lops, Semeraro,
& Basile, 2008) (iii) on user-generated content (e.g. tags and comments) and on
combinations of these methods. The rationale for applying such techniques is that
cultural heritage sites have a huge amount of information to present, which must be
fi ltered and personalised in order to enable the individual user to easily access it.
Although this approach supports visits adjusted to visitor interests and previous
experience, it is underlined by the concept of visitor as consumer of information;
more fi ne grained this time. Thus, personalised interaction eager to bring visitors
closer to the museum employs tools and methodologies to accommodate the visitors' interests but at the same time reproduces the initial distance between the visitor
and the authority of the museum who owns culture and now also knows what is best
for each visitor.
The infusion of new technologies in cultural institutions and more specifi cally,
of mobile technologies have resulted in new ways of experiencing games in cultural institutions (spatial awareness) and have helped in reaching a larger audience
(not only children but also adult visitors). We focus on mobile games because they
support the cultural experience during the visit of the cultural institutions as
opposed to games appearing for example in museum web sites aiming to prepare
for the visit or enhance the experience after the visit. Many mobile games however,
created for supporting cultural experience with the aim to engage users fail to bring
at the centre of the activity the cultural content in playful ways. So there are game
and story instances (see for example Akkerman, Admiraal, & Huizenga, 2009 ;
Psaty et al., 2008) where players seem to be enjoying the cultural experience but the
question remains as to how cultural content is integrated in these games/stories and
what game characteristics invoke visitor engagement. Our observation is that quite
often visitors engage with cultural content (e.g. explore the historical centre of a
city) in the context of following the plot of a mystery story or being engaged in a
role playing game. Yet in many cases engagement with cultural content remains at
a superfi cial level as it is not smoothly integrated in the story or the game

(Yiannoutsou & Avouris, 2012). As a result, engagement with the cultural content
(e.g. responding to questions, taking pictures) becomes the "price the visitors have
to pay" in order for the interesting things and the fun to carry on (e.g. to see what
happens next in the mystery story, or who is the murderer, or to continue playing
the game). Furthermore, the output of this process is factual information (e.g. visitors end up knowing that there is a convent in the city) leaving outside other aspects
of the cultural experience (creativity, fi nding connections with own experience,
gaining ownership over the cultural experience, etc.).
To sum up, in this section we focused on the learning involved when using personalization techniques and games to support cultural experiences. We discussed
these two examples due to their wide adoption in cultural institutions during the last
years and because they are claimed to result in active visitor engagement with cultural content. Our analysis showed that the mainstream use of these approaches ends
up often in consuming or collecting factual information by the visitor where the
museum is the entity that holds the knowledge for the visitor to collect it or consume
it. This is not to imply that game and personalization technologies can only support
the information consumption metaphor. Instead, as we will show in the next section,
personalised learning and games can offer rich learning opportunities if they are
integrated in a different rationale with respect to the role of the visitor, his/her relationship with the museum and the goal/nature of learning in cultural institutions.
Participation as a Context for Rethinking Technologies
that Support Learning in Museums
Participation-based cultural experience is based on the assumption that culture is
generated dynamically through the dialectic relationship between the museum and
the visitor (Simon, 2010). Proctor (2009) used the metaphor "From Parthenon to
Agora" to illustrate the shift from the perception of cultural experience as something that the museum holds and the visitors see but don't touch, to something that
can be discussed, shared and negotiated. Apparently, the role of the visitor in this
context changes to collaborator and partner (Simon, 2010). Furthermore participatory cultural experiences imply a new relationship between the visitor and the
museum which is not restricted to one off or fi rst time visits. Instead, participation
aims also at building an enduring relationship with existing audiences and communities (museum friends, volunteers, etc.) related to the museum (Black, 2005).
Building an enduring relationship between the visitor and the museum through visitor active participation enhances the cultural experience and enriches the content
and the impact of the museum on fi rst time visitors or one off visitors too (ibid).
In the wide spectrum of participatory activities (for a detailed presentation see
Simon, 2010) we identifi ed two types of activities relevant to our analysis. The fi rst
type of activity reserves for the visitor a role similar to the documentation process
performed by the museum. The proliferation of mobile technologies and social

media has supported the creation of user generated content using various crowd-
sourcing practices. Ridge (2011) has offered a grouping of these practices:
• Stating preferences , voting on interesting objects, comments, etc.
• Tagging : unstructured text associated with objects, see for example the Steve
project http://tagger.steve.museum/ which addresses social tagging as a process
that encourages visitor engagement and provides new ways of describing and
accessing culture . Twenty-one Institutions participate in the project with 97,041
Objects. In the site appears that 8,346 users have produced 552,105 terms for the
above objects.
• Debunking, criticising : arguing against other peoples' ideas, tags, etc. (see for
example the Freeze Tag project in Brooklyn museum http://www.brooklynmuseum.
org/opencollection/freeze_tag/start.php).
• Recording personal stories : personal memories associated to museum objects or
memorabilia made available to digital collections. See for example Europeana
1914-1918 project in http://www.europeana1914-1918.eu/en .
• Linking objects or categorising : grouping of objects or associating them with
themes.
The second type of activity aims at resuming or approaching cultural experience
through engaging visitors in the creations of "meta-artefacts"—i.e. games or stories
based on compositions of elements of cultural content—which are supposed to have
a public status. The idea of involving visitors in creating computer-based public
artefacts that make use of cultural content is new. It builds on a theoretical background that acknowledges the gap in the communication between the museum and
the visitor and calls for active participation of visitors in the dialogue with the museums (Hein, 2006 ; Simon, 2010).
Three examples are known and presented here: One comes from British Museum
which included in the museum activities family workshops on game design.
Participants were invited to build their own games inspired by the collections of the
British Museum (after visit experience). The new games could be uploaded on the
Web to be played at home or shared with friends. The second example comes from
Tate Gallery where young visitors (6-12 years) create drawings through game play
(Jackson, 2011) and fi lms for pieces of art. The third example is the idea of remixing
museum content for the creation of a visitor generated narrative (Fisher & TwissGarrity, 2007). This example is grounded on the observation that visitor centric exhibition narratives should not be the objective of the cultural experience, but instead
the focus should be cultural activities promoting the construction of narratives by the
visitors. Visitor generated narratives build on the transformative connection between
the visitor and the exhibit, asserted by Hein (2006). Transformative experiences can
occur when the visitor is encountered with challenges to discover connections with
the exhibits and is provided with the tools to analyse and manipulate the exhibits in
order to transform them into something new, related to his/her experience.
Although both activity types ("crowdsourcing" and "meta-artefacts") reserve an
active role for the visitors they have a drawback: visitor generated "products"—content
or artefacts—are almost never integrated in the museum's assets because of their low

quality (Simon, 2010). This problem is related to the open ended and unstructured
participatory activities:
When it comes to participatory activities, many educators feel that they should deliberately
remove scaffolding to allow participants to fully control their creative experience. This creates an open-ended environment that can feel daunting to would-be participants. … What if
I walked up to you on the street and asked you to make a video about your ideas of justice
in the next three minutes? Does that sound like a fun and rewarding casual activity to you?
(ibid, Chap. 1, p. 13)
What Simon described above draws upon an approach which asserts that learning in museums should focus in triggering visitor creativity and subjective interpretation of cultural content leaving aside the "knowledge of the museum" which
prevails in the information consumption metaphor. Simon showed that in participatory activities this perspective has its weaknesses. In the same line comes the idea
of "objectifi ed cultural capital" (Bourdieu, 1986) which explains that cultural experience is not just an issue of access (i.e. being able to visit a museum) but it is also
an issue of background knowledge that supports the person to appreciate and understand the value of a piece of art. Museums and cultural institutions offer in the
process of culture creation not only the objects-exhibits but also the background
knowledge necessary to value the exhibits. In our view the key in this process is how
background knowledge will become the means to an end (i.e. a tool for the visitor
to generate cultural experience) and not the end itself. This means that cultural
experience is not diminished into comprehending institutional knowledge instead,
the latter needs to come into the visitor's attention as material to be negotiated, discussed, shared and used for the construction of something new. In this context we
argue that technology can play a crucial role in supporting the cultural learning
experience and we further illustrate this by focusing on the example of game design
and its potential in supporting learning in cultural institutions.
Game Design as Learning Activity in Cultural Institutions
Game play is not a new practice for museums. The introduction of digital technologies resulted in revisiting the idea of game play and storytelling in museums.
Technologies today play a key role in interaction, interpretation, learning, content
creation through crowd-sourcing, outreach, marketing, etc. (for a detailed presentation and overview see Beale, 2011). Whereas there is an extensive analysis on game
play, research in cultural heritage sites have not addressed yet the idea of game
design as an end user activity.
Interestingly, research in the fi eld of technology enhanced learning has already
highlighted the learning potential not only of game play but also of game design and
development (Kafai, 2006). Game design activities were identifi ed as having the
potential of helping learners to build a new relationship with knowledge, as learners
feel ownership over the knowledge and experience deep interaction with the learning
concepts to be integrated in the game with a functional role. As Kafai, Franke, Ching,

and Shih (1998) observed, learners negotiated with learning concepts in this context,
in order for the game to be playable.
Research in the fi eld of game design by non-technical end users—such as students—has focused on the learning activity which has been analysed from two perspectives. The fi rst one focuses on studying learning related to programming or
specifi c skills and concepts (see for example Hoyles, Noss, Adamson, & Lowe
2001) for a discussion on children's causal reasoning and rule understanding during
game construction). The second and most recent perspective acknowledges game

design as a learning goal in itself (Hayes & Games, 2008). In this trend, researchers
have coined the term "Design thinking" to encapsulate the set of learning and meta-
cognitive skills involved, such as system based thinking, self-regulation, social,
technical, technological, artistic and linguistic skills (Robertson & Howells, 2008 ;
Salen, 2007). The analysis of learning through game design shows that the learning
experience is shaped and amplifi ed by the feeling of ownership over the games created by the learners, the participation in communities that share and exchange ideas
and the motivation inherent in game play and game design (Rieber, 1996 ; Robertson
& Howells, 2008 ; Salen, 2007).
Game creation in cultural institutions as participatory learning activity should be
integrated in activities that will give the chance to visitors to interact with museum staff
and discuss, negotiate and integrate in their games different aspects of cultural content.
Game creation can be supported by technological scaffolds (such as game templates)
and personalization techniques that present the museum view in order for the audience
constructions to meet their standards and become a public artefact that can be used by
other visitors, can be shared, revisited, discussed, changed and expanded.
Game Design Platforms and Their Relationship to Learning
When it comes to technology based scaffolds for game design there is a question we
need to address: Do we need to design game-creation platforms to support learning
in museums or we can use existing solutions such as KODU, storybricks, Game Star
Mechanic, Game maker, the Games Factory and many more (for a critical review of
technologies for game design see Hayes & Games, 2008). The answer here is that
the technologies used for game design are confi gured to support not only the creation of games but also to facilitate the other objectives related and integrated in
game design (e.g. the different types of learning, or in our case the cultural experience). Thus when game design is employed for purposes other than game creation
then the design tools consist of elements and support practices related to the purpose
for which game design is employed. To make this rationale more explicit we analyse next two examples of games that support end user design of the salient features,
rules and content. The fi rst one is a prototype game that is designed to support the
learning of spatial concepts and more specifi cally, issues related to orientation, map
alignment, use of systems of reference, etc. The example we selected is designed to
facilitate learning in school but it is appropriate for our analysis here because it
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reveals how the characteristics and affordances of the tool can facilitate specifi c
types of learning.
The "Treasure Hunt" Template
Unlike game design platforms (like KODU) the "Treasure hunt" tool does not lead
in designing all sorts of games. Instead, it integrates the idea of game design in what
we call game templates. Game templates represent a game genre (in our example
treasure hunt) and they can produce different instances of the same genre by manipulating and specifying basic game elements (For a detailed presentation see
Yiannoutsou, Sintoris, & Avouris, 2011) like the rules and the content (treasures,
hints, etc.). It is thus apparent that game templates are more limited with respect to
the spectrum of the produced games but they allow for designing of a learning experience that is oriented towards specifi c learning objectives (as we will show next).
The other advantage of game templates is that while they allow confi guration of
certain game elements according to the learning objectives, they also provide some
ready-made non confi gurable components which are integrated in the template to
guarantee high quality of the games without requiring from the user specialised
knowledge and lots of effort (like a 3D terrain for example). In a nutshell, game
templates allow for focused learning design and produce high quality games that
can meet the quality standards of the users.
In the example we present here, the non confi gurable game elements of the treasure hunt template are the representations of the terrain. These consist of a 2D and
a 3D representation of a neighbourhood which are dynamically linked together.
This means that motion of an agent in the 3D space results in a trail on the 2D representation; a treasure and a landmark as soon as they are placed on 2D space they
also appear on 3D space (see Fig. 1 below)
Fig. 1 The terrain of the treasure hunt template
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The dynamic link between the two representations is an important characteristic
of the template because game design takes place on the map (2D representation)
whereas game play takes place on the 3D space with the option for game designers
to make the map available or not (so as to make the game easier).
The confi gurable elements of the template are the content and the rules. The
content involves the number of the treasures and the hints that lead from one treasure to the other. The basic rule of the game remains the same, i.e. in treasure hunt
games the player has to collect a set of hints in order to be able to fi nd the treasure.
The template however provides some variations around this rule: i.e. instead of having one treasure and a set of hints, there are numbered treasures that lead from
the fi rst treasure to the second and so on. Furthermore, the designers can decide if the
game will consist of one player who moves in the neighbourhood searching for
the treasures or of one player and one thief also searching for treasures. In the latter
case the designer has to defi ne what will happen if the player encounters the thief by
selecting one of the three options available (see Fig. 2)
Another confi gurable element of the template is the defi nition of dangerous
streets (see Fig. 1). The designers can designate some streets, normally around treasures, as dangerous areas by clicking on them with the mouse. Next they need to
attribute to each area what will happen if the player and the thief step on this street.
There are three options here again, losing speed for a specifi c time period, losing all
the treasures collected or losing half of the treasures collected. The next step for the
designer is to defi ne the possibilities for the selected events to be activated by moving the cursor on a slider with values from 0 to 100 % (e.g. 60 % possibility for an
event to be activated if the player enters the street). To put it briefl y, with a set of
simple functionalities the treasure hunt template can support non technical designers to construct a rather sophisticated game.
Our concern however in this section was to show the relationship between the
characteristics (representations and functionalities) of the game template and the
learning pursued. As we mentioned earlier the treasure hunt template was designed
to promote spatial thinking and orientation skills. An analysis of data drawn from
experimentation of two groups of 14-year-old students fi rst showed that the treasure
hunt template did not put any cognitive load to students with respect to technical
issues (Yiannoutsou, Sintoris, et al., 2011). Second, student interaction with the
template (i.e. what was confi gurable and what wasn't, functionalities and representations) required that spatial concepts were used as instruments for game construction (ibid). This means that students in order to construct the game negotiated spatial
Fig. 2 Refi ning the rules of the game: defi ning the player-thief encounter
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concepts like the difference in the properties of the map and the 3D representation
of space—i.e. which one is more appropriate for orientation, what is visible and
what is not in each representation—the accuracy of directions, the use of absolute
frames of reference, etc. (see the next extract)
S2: Now the quiz. What should we write for the fi rst treasure?
S1: Go south
S2: Yeah, but it is not enough. All these are south (shows the streets at the bottom
of the map). She said [the Facilitator] that the directions should be precise
otherwise the game is not good. They will never fi nd it.
S1: Ok then, there is a rectangular here. It is the only south rectangular [he refers
to the shapes of the street on the map. (see Fig. 1)]
S2: That's good! Let's say "go to the southmost corner of the southmost
rectangular"
S1: They will fi nd it immediately if you are so precise.
S2: Yeah, but they will be on 3D they will not be seeing the map .
Extract 1 : Negotiation of spatial concepts (taken from Yiannoutsou, Sintoris,
et al., 2011)
The second confi gurable game to be analysed is from the fi eld of learning in cultural institutions described by Di Loretto, Divitini, Trimailovas, and Mander (2013)
The museum confi gurable game MagMar is a tool that allows students to create
question-answer games. The tool was used in, a study with 18 year old students
worked in groups to create multiple choice questions (with an indication for the right
answer) for pieces of art of a contemporary museum. The paintings were selected by
the teacher and were common for all students. Question construction was created
inside the museum and students had at their disposal a small picture of the painting
presented in MagMar, access to internet and a guide of the museum. Students were
also allowed to go to the room where the painting was located in order to fi nd material for the question construction. Correct answer to a question awarded the group
with a raise on their score. From the reported results with respect to the learning
experience we highlighted and site here two remarks:
(a) For the second item students walked around trying to fi nd content for possible
answers also in the information exposed in the exhibit (in the specifi c case, the
name of painters contemporary to the one of the focal painting). Though this is
clearly not suffi cient to draw any general conclusion, it was still interesting to
see how the game could actually motivate an exploration of the museum…
(b) When they were asked what they remembered from the museum visit, the students answered with the content of the questions they created (Di Loretto et al.,
2013 , pp. 527).
Although MagMar doesn't have the characteristics of game templates we
described earlier, it shows a rather successful implementation of the idea of visitor
participation through the construction of meta-artefacts that can be used during the
museum visit. It appears that the learning experience with MagMar proved to be
engaging, prompted for exploration and connection fi nding with other exhibits
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(remark a) and showed that the process of question construction proved to be more
effective—from a learning perspective—than the process of question answering
(remark b). Without ignoring that there might be criticism on MagMar related to the
emphasis on factual knowledge for example, and to questions about its playful
character, it nevertheless shows the learning potential of a participatory activity like
the ones we describe here. Apart from that, the idea of question constructin which
is central to MagMar is also central to the process of culture generation as it is
expressed by J. Landy:
"Artworks are not just fancy ways of delivering messages. One of their most
interesting and useful functions is not to provide answers but to offer questions we
have to answer ourselves. Answering the questions does not involve guessing the
artist's intention. Instead, it involves injecting something of ourselves". (J. Landy
extract from the video of his course at Stanford that comes under the title: "The art
of Living" http://humanexperience.stanford.edu/artofl ivingiVideos). A similar view
comes from Simon (2010) who stresses that interesting questions with respect to
learning are not those that aim at comprehending the institutional knowledge but
those that draw on the visitor's knowledge. These two comments can offer useful
insights in refi ning the design of MagMar so as to move from factual information to
more interesting learning objectives.
It becomes apparent then that if we want to employ game design in the cultural
experience we need to create a platform that engages users with what is considered
crucial for the cultural experience. We attempted to show this with identifying in
MagMar an underlying principle which is also important in the process of culture
generation. Another suggestion is that game design tools could focus on the connections the visitor can make between the different cultural artefacts and with overarching concepts, beliefs and narratives (Falk & Dierking, 2000 ; Hein, 2006).
Integration of Participatory Activities in Museum
and Visitor Practice
The visitor doesn't have all day to engage in game design. When and how do you expect this
to happen? He/she might have a couple of hours to visit the museum and that's all
The above remark—made by one of our colleagues when we fi rst presented the
idea of game design—is added here to show the contrast between the technologies
that promote a participatory cultural experience and the average visitor practice in a
museum which involves one off or fi rst time visits. It also stresses that a new practice—like participation—related to the use of a new technology cannot occur just as
a result of the introduction of this technology (Tallon & Walker, 2008). Instead, in
order to foster this practice we need to design activities integrating this technology
by making the most out of its potential and by describing the roles of the stakeholders (i.e. museum, curators, visitors). As we mentioned at the beginning of this paper,
participatory experience assumes a long term relationship between the visitor and
the museum and requires structuring in order to result in an output which meets the
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quality standards of its creator and can become part of the museum assets (Hein,
2006 ; Simon, 2010). Taking all these into account we envisage learning activities
that integrate game design technologies which can have the form of onsite or online
workshops taking place regularly and focusing on specifi c parts of the museum collection each time. In this context we expect that the institutional knowledge about a
specifi c set of exhibits will be discussed, analysed and negotiated between the audience and museum curators in order to be transformed and integrated into visitor
constructed games. Social media and platforms that will allow a visitor to present
and share his/her game with his/her friends and being able to play it when they visit
the museum is an important element of the game construction process.
Concluding Remarks
In this paper we discussed the use of technology as a medium for learning in cultural
institutions. Starting with technologies to support learning as information consumption, we moved to more recent approaches that involve visitor participation in the
process of culture creation. Our analysis highlighted that participatory activities
such as game design when scaffolded by technology and integrated in museum
activities can offer rich learning experiences which reserve for the visitor the role of
collaborator and partner and entail the creation of an enduring relationship with the
museum. We further analysed the idea of cultural experience as participation
through the presentation of the idea of game templates and how their characteristics
can be related to learning.
Our analysis of two confi gurable games showed that the design of game templates
in order to support a rich and engaging learning experience should be structured
around what is considered important to learn when visiting a museum. A fi nal observation involved the design of activities integrating and at the same time demonstrating how the new technology should be used in order to pursue the envisaged
participatory learning experience. This rather new approach needs to be further investigated and supported through specifi c game-design tools and empirical studies.
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Introduction

The use of ICT applications for the purpose of museum communication and learning tends to become a main feature of contemporary museum policy. In order to serve their educational goals, different types of museums produce ICT applications that follow the technological developments in the form and shape they take: they range from distribution material in the form of CD and DVD-ROMs to online applications or systems that can be used in terminals inside the museum. The extensive use of these applications in current museum practice has cast the spotlight on a new area of museum education research. The central issue is not anymore whether we should use technology (as it was in the past), but how we should use it, in what ways users could become involved in activities and what experiences they could gain from using technology. In this context, an extensive research project took place between 2008 and 2011 in Greece. In the fi rst stage of the project, we conducted a questionnaire-based research in order to record the use of ICT technologies for educational purposes by Greek museums (Bounia, Economou & Pitsiava, 2010). Based on the results of the fi rst phase, in 2010-2011 a systematic qualitative study was carried out on the educational applications that were at the time offered by Greek museums, in order to explore the possibilities inherent in each type of media used, in conjunction with the venues these media are associated with, their target groups, objectives and educational philosophy. In addition we aimed to explore

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whether and to what extent theoretical perspectives on museum learning, as these have been developed in the relevant literature during the last couple of decades, have been taken into consideration for the creation of such applications.

Theoretical Framework

The last 20 years or so have seen an evolution in understanding what kind of learning happens, or should happen, in museums (Perry, 2012). While in the past theories of learning focused on a "transference model", i.e. one that placed emphasis on information passing from the (museum) educator to the visitor/audience, current museum education theory has embraced different, inclusive and participatory models of learning. Instead of accepting their role as simply passing along information to visitors, museum professionals are involved in attempts to enable visitors to make meaning and actively create knowledge for themselves based on their identities and previous experiences.

The role of constructivism has been crucial in this discussion. According to Hein (1998 , 34), "learning requires active participation of the learner in both the way that the mind is employed and in the product of the activity, the knowledge that is required". Learners, in other words, are not just passive recipients of knowledge but active meaning makers, since "knowledge does not exist independently of the learner … It is constructed by the learner" (Hein, 1999 , 75). Furthermore, Hein and Alexander (1998) go further to argue that museums are, and should be, constructivist environments, where visitors can construct their own meanings by actively engaging with exhibits, programs, objects and phenomena (Perry, 2012 , 13).

According to the interactive model of museum experience, personal context is very important for museum learning, as "learning is a very personal experience, self motivated, emotionally satisfying and very personally rewarding and always constructed from a base of prior knowledge" (Falk & Dierking, 2000 , 33). In addition, sociocultural context is equally or sometimes even more important. Museum visitors' research has highlighted the importance of social agenda for museum visiting (Perry, 2012 , 11). The social context of a museum visit infl uences the perspectives and the experiences of every visitor (Falk & Dierking, 1992 , 3). In the words of Falk and Dierking (2000 , 37) "learning is both an individual and a group experience".

Museum educators have extensively used both these theoretical models in order to argue for a people-oriented (as opposed to object-centred) model of museum learning. Paris and Mercer (2002), summarising museum education research, suggested three models for the relationship between visitors and museum objects. The fi rst is the model of passive reception, i.e. the one where a linear, one-way communication is expected. In this traditional approach, museums are expected to provide information, in a creative and effective way, which is easily understood by the visitors. Despite its shortcomings, this model is still active in many institutions around the world. The second model refers to active construction, and is based on the theories of constructivism, as developed before. The third model is called "transactions" and

refers to a highly personalised understanding of learning, where the encounter of the visitor/group of visitors with the exhibits serves to reinforce or develop personal identity and beliefs in a rather idiosyncratic manner.

What all these theories have in common is the understanding that museums are spaces where a special kind of interaction takes place: visitors bearing their personal and/or social agendas encounter museum objects and use this encounter to create meaning, i.e. to learn. Learning within the museum environment is thus understood as a broad process, incorporating many different notions and strategies, ranging from passive reception to active construction and transaction (Perry, 2012 , 15).

Within the context of museums, J. Dewey's notion of "experience" and its value in education is also highly relevant. Dewey emphasises active practice, but also the fact that experience in order to be educative has to be organised. In other words, experiences should not be just "hands-on" but also "minds on" in order to be effective (Hein, 1998 , 2). Museum education has been historically related to Dewey's "learning by doing" philosophy (König 2002 , 34-35), since museums are places where experience is built through the encounter with material objects, but also by the very fact that museum visit is an activity in space and it involves human beings as material entities. It provides multiple opportunities for visitors to experience the world, and consequently it can encourage creative individual thinking and acting—the basis of a democratic society (see also König 2002 , 34-35; Sauter, 1994 , 171-187).

This experiential approach also relates to the possibilities offered by museums to integrate in the learning process senses other than the sight. Activities during which visitors design, draw, construct and change their environment and themselves are integral parts of museum education programmes (Rottmann, 1998), because "active learning occurs when people … interact with information and experiences at hand" (Tishman, 2009 , 2).

An important part of this interaction is the presence of museum exhibits. Museum education has always been concerned with the creation of suitable conditions for visitors' encounters with objects. Learning in museums is different than learning in schools or other similar environments, since it takes place "not in the shadow of objects" (Kerschensteiner 1925, quoted in König 2012 , 24) as in the case of schools, where books and words hold the primary position, but through objects. Museum exhibits have a tangible quality and usually aesthetic value. They are "real things", and they can allow for a large number of different narratives to be developed around and through them. Roberts (1997) has highlighted the idea of "narrative" as an educational tool along with the ability of museums in particular to become places par excellence for the construction of narrative.

In order to work with museum objects, museum education places special emphasis on how we "ask" objects (Durbin, Morris, & Wilkinson, 1990 ; Hennigar Shuh, 1999 ; Hooper-Greenhill, 1994). Museum objects are characterised by a set of values (or qualities) that make them suitable for communication and learning. They are tangible, and they have survived from the past to the present, thus being a unique gateway to that past. They challenge the senses, inspire observation and discussion and provoke creativity. They can support educational processes for the understanding of reality and the past through the senses. They are open to different interpretations,

providing opportunities for multiple narratives to be constructed around them and for open-ended learning experiences, meaning making and transaction. Finally, museum objects may serve aesthetic purposes. According to Weschenfelder and Zacharias (1992), the starting point of all museum education design and planning is (and should be) the pedagogical use of the values/qualities that museum objects have.

In the case of ICT applications for museums, the role of museum objects comes at the centre of attention, this time for a different reason. How are possibilities offered by authentic, "real" museum objects altered when they take digital form (see also Doonan & Boyd, 2008)? In this case, issues regarding the quality of digital representation, the advantages of digital copies for enhancing visitors' experience and the enlargement of experiences through new forms of digital mediation are brought to the forefront (also Bayne, Ross & Williamson, 2009 , Frost, 2010). Nowadays the borders between real, virtual and illusion, symbolic and abstract and tangible reality and its simulation are more and more fl uid (Zacharias, 1995 , 72), resulting in a growing loss of the experience of reality through the senses. This fact points out the importance of providing opportunities for the connection of the digital with the real world, for expanding the digital experience into the real world.

On the other hand, information and communication technologies provide increased educational opportunities for interaction, active involvement and personalisation. They provide, or can do so, museum visitors with opportunities to select content and create their own individual routes in their approach to knowledge, as well as opportunities to contribute content and exchange ideas and information with others, thus enhancing the social dimension of their experience. In addition, interactivity and the stimulation of users on an intellectual, emotional and social level are facilitated via the introduction of ICT technologies into museum education (Roussou, 2010 ; Witcomb, 2006).

The theoretical framework presented above poses a series of issues regarding museum education and the use of ICT applications in museums. The main concern has been whether digital learning environments can support experience and meaning making by the museum visitors, taking into account the values/qualities of museum objects, the knowledge and experiences that users bring with them, the social dimension of the museum experience and the emphasis of museums on creative and experimental learning.

As a result, a series of research questions were developed, which informed the process and the data analysis. We therefore investigated the extent to which ICT applications created and employed by Greek cultural institutions:

- Take into account the previous knowledge and identity of the user and promote meaning-making processes according to the constructivist approach to museum learning
- Take into account the decisive role played by the social dimension of the educational process in the formation of the visitor's experience
- Promote processes that encourage a constructive encounter between visitors and museum exhibits
- Use the educational values/qualities of museum objects

- Promote creative activity on the part of the users
- Offer possibilities of expanding the digital experience into the real world
- Explore the possibilities of interactivity to encourage a more active interpretation of museum material

Research Methodology

For the purposes of this research we collected a total of 25 applications that Greek museums of varying type, size and scope (Table 1) provided to the researchers and explicitly identifi ed as "educational". These were either in the form of a portable medium (CD-ROM/DVD-ROM) or web based; it was important that the institutions themselves considered these applications as "educational" and defi ned them as such, because this allows us to identify indirectly how institutions feel about education in general. We decided not to include in the research media or applications aiming mainly to museum communication, such as museum websites and social media, since their aims and objectives are not primarily educational (Bounia et al., 2010).

In order to analyse the applications we used qualitative content analysis (Krippendorff, 2004). The tool for this analysis was a form (Table 2) developed by the researchers based on the theoretical approaches and research questions presented above. The aim of the analysis was:

- To investigate the extent to which these applications utilise the aforementioned theories
- To study the opportunities for learning that each of the applications offers to their users

The aim of the study was neither to test the usability of each application nor to evaluate their use by the intended audience. This has been a research performed in order to evaluate the use of museological perspectives in the design of applications

aimed to complement museum learning.
After recording the identifi cation parameters of each application (namely institution, title, application form, funding, target group, languages) (see Table 1) the analysis form consisted of the following fi elds (Table 2):
• Presupposed knowledge/experience of the user
• Signifi cance of collections/objects (authenticity, materiality, aesthetic value, narrative)
• User involvement:
- Creativity-experimental learning
- Experience expansion in the real world
- Social context
Each of these parameters corresponds to a museum learning concern discussed previously.
Digital Applications in Museums: An Analysis from a Museum Education Perspective Table 1 The educational applications studied in this research
Museums/cultural
institutions
Educational
applications: Title Funding Application form Target group Languages
Benaki Museum "Play with a Painting
on Your Screen"
EU Operational Programme
"Information Society" and National
(Greek) Funds (General Secretariat
Research and Technology)
Online: http://www.benaki.
gr/electronic_game_gr.asp
Children Greek
Benaki Museum "The Hidden Treasure
of the Benaki
Museum"
Benaki Museum and General
Secretariat Research
and Technology
CD-ROM School children
(6-18 years)
Greek
Benaki Museum "Greece at the Benaki
Museum"
Benaki Museum DVD ROM Unspecifi ed Greek,
English
Benaki Museum "Dedicated to
Trelantonis"
Benaki Museum DVD ROM/multimedia
application
Children Greek
Museum of Traditional
Pottery/Centre for the
Study of Traditional
Pottery
"The Ceramics
of Crete in recent
years"
EU INTERREG II CD-ROM Unspecifi ed Greek,
English
The Network of Mani
Museums, Hellenic
Ministry of Culture
"Mani: A virtual tour" Hellenic Ministry of Culture and
European Regional Development
Fund of the European Union
DVD ROM Unspecifi ed Greek,
English
Museum of the
Macedonian Struggle
"The Struggle for
Macedonia:
Adventures at
the lake"
European Regional Development
Fund and national funds
Multimedia game Unspecifi ed Greek,
English,
German
State Museum of
Contemporary Art
"Educational CD.
Costakis Collection"
EU Operational Programme "Information
Society" and national funds
CD-ROM School children
6-12 years
Greek,
English,
Russian
Modern Greek Art
Museum, Municipality
of Rhodes
"Click on Art.
Educational Program
and Multimedia Game"
CD-ROM Children Greek Museums/cultural
institutions
Educational
applications: Title Funding Application form Target group Languages
Numismatic Museum "Children's website" Alexandros Onasis Foundation Online: http://www.nma.
gr/kids.coins/
Children Greek
Mystras
(archaeological site)
"The Fortifi ed City
of Mystras"
European Regional Development
Fund—Regional Operational Programme
of the Peloponnese
On-line: http://www.culture.gr/
culture/mystras-edu/
Children Greek
Museum of Cycladic Art "Games" Nikolaos Goulandris Foundation Online: http://www.cycladic.
gr/frontoffi ce/portal.asp?
cpage=NODE&cnode=135
&clang=0
Children Greek,
English
Acropolis Restoration
Service, Hellenic
Ministry of Culture,
rst Ephorate fi
of prehistoric and
classical Antiquities,
National Documentation
Centre-National Hellenic
Research Foundation
"The Parthenon
Frieze"
Operational Programme
"Information Society"
Online: http://www.
parthenonfrieze.gr/#/home
General public Greek,
English
Acropolis Museum "Athena, Goodness
of the Acropolis"
Acropolis Museum; Bodossakis Foundation;
Hellenic Ministry
of Culture
Online: http://www.acropolisathena.gr
Children Greek,
English
Folklife and Ethnological
Museum of
Macedonia-Thrace
"Mylos on the Road
to the Mills"
EU and national funds DVD ROM Children 10+ and
general public
Greek,
English
Historical Museum
of Crete
"Nikos Kazantzakis:
A traveller to places
and ideas"
Historical Museum of Crete and
The Society of Cretan
Historical Studies
Online: http://www.historicalmuseum.gr/educational/
Primary school groups
10-13 years
and secondary
school groups,
14-18 years
Greek
(continued)Table 1 (continued)
Museums/cultural
institutions
Educational
applications: Title Funding Application form Target group Languages
The Athens University
Museum
"Collectors of Stories" EU Operational Programme
"Information Society" and
National (Greek) Funds
CD-ROM Young audience Greek
Museum of the Olive and
Greek Olive Oil
in Sparta
"Bring out the Oil
for Us"
European Regional Development
Fund—Regional Operational Programme
of the Peloponnese
CD-ROM Children 7+ Greek
Natural History Museum
in Axioupolis, Kilkis
"Electronic games
about plants and
animals"
EU INTERREG III and national funds On-site applications Primary School
groups: fi rst
to second/third to
fourth/fi fth
to sixth grades
Greek
Museum of Byzantine
Culture
"Games" Archaeological Receipts Fund Online: http://www.mbp.gr/
edu/index.php?
option=com_content&
view=article&id=62
&Itemid=30&lang=el
Children Greek
Aiani Archaeological
Museum
"Educational
Games"
Unspecifi ed Online: http://www.
mouseioaianis.gr/CAME/
GAMEFORSITE/
WebGame.swf
Children Greek Museums/cultural
institutions
Educational
applications: Title Funding Application form Target group Languages

The Jewish Museum of Greece	"World War II and the Holocaust of Greek Jews, 1941-1944"	Jewish Museum of Greece	DVD ROM
General public, Schools	Greek, English	Historical Archive of the Aegean—Ergani	"Entrepreneurship and Innovation in Northeast Aegean at the end of the nineteenth and the beginning of the twentieth century"
EU Operational Programme "Information Society" and National (Greek) Funds	CD-ROM	General public	Greek, English, French
Herakleidon Museum	"Educational program: Art and Mathematics"	Museum funds	CD-ROM
School groups	Greek		
G. Iakovidis Digital Museum	"In the studio of the painter Georgios Iakovidis"	Educational and Cultural Foundation (N.G. Papademetriou)	CD-ROM
Children and School groups	Greek	188	

Results

General Characteristics: Target Groups

A wide range of bodies of varying size, legal status and fi nancial strength are actively involved in the production of educational multimedia applications in Greece. The number is growing as well as the interest of museums to acquire a presence in the digital sphere. These bodies seek opportunities to produce such applications either through funding by European programmes or through other kinds of funding, often by private foundations. Of course, there are also a number of self- funded initiatives, which refl ect the strong interest and even anxiety of some bodies to acquire a digital "educational voice". While the older applications take the conventional form of a CD/DVD-ROM, the latest ones are Internet based as the importance of easy user access is being increasingly recognised, while at the same time the technological developments providing such possibilities are constantly expanding. In most cases (23 out of 25) the applications are meant to be used outside the museum; in just two cases they are used on museum premises. In other words, the applications are meant to "accompany" visitors home after their museum visit, so as to consolidate the content of the visit and to reinforce and possibly extend the museum experience. In the case of applications accessible on the Internet, once again these are used at the user's personal space without an actual visit to the relevant museum being necessary. At the same time, it is also possible to see how these applications can motivate users to actually visit the museums that provide them, and so, in this respect, they can claim to have the role of introduction to the respective institutions.

As far as the user groups are concerned, three (3) applications are aimed at the "general public", while seven (7) are targeted specifi cally at school groups and the educational community. In three cases the school groups are further defi ned according to their educational level (primary or secondary) such as in the web application of the Historical Museum of Crete. "Children" in general also feature as a target group; more specifi cally, in these cases it is mainly the lower age limit of the users that is defi ned (usually ages 7-10). It is evident, therefore, that the applications treat the public, even the children/teenage public, as a unifi ed whole in contrast to museum education activities, which are usually aimed at highly specifi c age groups.

Table 2 Analysis form

1. General information Institution, title, application form, founding, target group, languages
2. Contents of applications Presupposed knowledge/experience of the user Signifi cance of collections/objects (authenticity, materiality, aesthetic value, narrative)
3. User involvement Creativity-experimental learning Experience expansion in real world Social context

An exception to this can be found at the Natural History Museum at Axioupolis, where the applications form part of the museum education activities and provide different activities for three different age groups: infants and the fi rst to second grades of primary school, the third to fourth grades of primary school and the fi fth to sixth grades of primary school (Economou, Nikonanou, Kasvikis, Economou, & Samaroudi, 2011). It is also worth mentioning that there are no applications in our sample which are intended for special needs groups, despite the fact that digital technologies can provide for different special needs (e.g. enlargements for visually impaired users).

Almost half of the applications (14 out of 25) are offered just in the Greek language. In the case of digital discs (CD-ROMs) there is usually a choice of other languages than Greek, such as in the application for the Costakis Collection, which is also offered in English and Russian, and also in the educational game produced by the Museum of the Macedonian Struggle, which is also offered in English and German, and the application of the Historical Archive of the Aegean—Ergani offered in English and French. Greek and English are offered in eight (8) applications (CD-ROMs and web applications), and the rest is only in Greek (see Table 1). This choice of languages is also relevant to the target audience of the educational provisions of the museums: most of the institutions want to use their applications to cater for tourists as well or for the audience of countries which have a special interest in their collections (as for instance in the case of the Costakis Collection which consists of paintings by Russian artists).

Contents of Applications: The Role of the Object

As far as the contents of the applications are concerned, the analysis examined to what extent the applications presuppose knowledge on the subject by the user. The applications were not found to presuppose any special knowledge on the part of the user, at least no greater level of knowledge than that provided by the applications themselves. In other words, these applications are self-contained packages in which both the provision of knowledge and the suggested activities function as a complete educational unit. They do not refer to the museum environment, and they do not correspond to activities or knowledge that users have had from previous visits or other experiences.

The second parameter that was examined in this section concerns the place that museum objects occupy in each application. Most of the applications (22 out of 25) contain museum exhibits at the centre of their content; in some cases (e.g. "The Parthenon Frieze"), the exhibits are the central feature of the content, while in some others (e.g. "Bring out the Oil for Us") museum exhibits play a minor role (even though they are still there). Applications of the fi rst type can also prepare users for a visit to the museum housing the relevant object, where they will be able to recognise the object in their real encounter with it on the museum premises. In most of the cases where a digital museum object occupies a central place in the application,

museum education processes are used in approaching it, such as observation, examination and interpretation (e.g. the Benaki Museum's "Play with a Painting on Your Screen"). More generally, it seems that the role museum objects occupy in the applications depends on their uniqueness or general appreciation, as, for instance, is the case with the Frieze from the Athenian Acropolis. In the case objects are considered more "common" or interesting for qualities other than aesthetics and uniqueness, as for instance in the case of the olive pressing machinery, their role in the applications is not that central and the focus goes more to their use or construction technique rather than the objects themselves.

When it comes to the values/qualities of objects, it is clear that those of authenticity, materiality and aesthetics can be used only to a limited extent in a digital environment since they are inseparably bound up with approaching the material substance of real museum objects (Bounia & Nikonanou, 2008). Similarly, despite the fact that in traditional museum educational programmes museum objects are at the centre of the educational process, in the digital applications they are not used in the same way, especially in the narrative.

In order to examine perceptions of materiality and aesthetics, we examined the quality of the digitisation in conjunction with the possibilities for making a visual examination of the object as well as the provision of three-dimensional images (Frost, 2010). The latter has not been used extensively in most of the cases—probably because of the cost of production—except for one case: the application produced by the State Museum of Contemporary Art; in this application technology is used to provide detailed views of art objects.

In order to focus on aesthetics in particular, we also examined the representation of digital spaces. The approach followed in this process is largely based on the conventional aesthetics of children's books, comics or animations or on aesthetic choices oriented towards the aesthetics of the museum objects in each collection (e.g. Costakis Collection). Exception to this is the application of the Numismatic Museum that offers a "screen" with an active "fairytale-like" image and numerous stimuli and prompts encouraging the user to explore and connect to organisations and events "outside the screen". It seems that museums present their objects in similar manners whether in the real or the digital environment: art museums tend to place more emphasis on the aesthetics, uniqueness and materiality of their objects, whereas museums of technology or natural history tend to place more emphasis on the processes and the circumstances connected to their collections and not necessarily on the objects as unique material testimonies.

As for another important value/quality of museum objects, i.e. the possibility to encourage multiple narratives, we realised during the process of analysis that this is very little exploited in the digital applications, although it is not subject to the same limitations of digital space as the three previously mentioned qualities/values. In terms of their narrative content, most of the museum objects presented in the applications have "closed" narratives, based on elements of a factual nature, while users are not encouraged to develop their own individual interpretations of them. Of particular interest in this respect is the application by the Numismatic Museum, which, using the museum exhibits as a starting point, provides users with the opportunity

to access further information via web links and make more imaginative connections (such as with not-for-profi t environmental groups when discussing coins bearing marine animal symbols).

User Involvement

Learning is not just about content; it is also about processes which concern what the user does and in what kind of context the interaction between learner and content happens (see also Dimaraki, 2008). As a result, it was important to examine whether and how users of the applications were encouraged to interact with museum material, whether and how creativity was promoted and fi nally whether, and how, the social context of the museum visit was exploited to encourage learning within the museum. Most applications present the user with a variety of activities, which provide different opportunities for interaction.

Most applications draw a distinction between "serious" and "play-based" learning, either providing extensive texts and information as more or less a prerequisite of "play" or offering activities that serve as a "gateway" to knowledge. In the fi rst category belong applications, which require the user to read the information and then to test his/her newly acquired knowledge by fi lling in crosswords and word search puzzles. Examples of such an approach are those of the "Games" section of the Alani Archaeological Museum's application or the multiple-choice questions provided in the application of the Historical Museum of Crete.

In the second category belong activities that serve as a "gateway" to knowledge; in these cases, the user receives information concerning the object that he/she is asked to "discover" as a result of the activity and not as a prerequisite for it. This is the case of the puzzles and memory games created by the Museum of Byzantine Culture or the Museum of Cycladic Art. In some other cases of the same category, such as the application of the Benaki Museum entitled "Hidden Treasure" or the games of the Numismatic Museum, the user gets access to information about museum objects, their provenance and so on after choosing them in the context of the activity.

The question regarding these cases is whether, and to what extent, users are motivated to explore the new knowledge provided to them after they have fi nished with the activity, since this new knowledge is not necessary in order to complete the activity.

Very few applications examined in this research place an emphasis on personal interpretations and the construction of individual meaning along the line of constructivist learning, as this has been discussed above. An exception to this is the creative activities suggested to users in some cases, usually related to art collections and art museums. For instance, the State Museum of Contemporary Art encourages users to employ symbols and elements from the museum artworks in order to create their own works of art and the Iakovidis Digital Museum encourages users to import their own texts. However, this allows for limited personal interpretation and does not take advantage of many other possibilities available through museum objects (Rakhochkine, 2003 ; Weschenfelder & Zacharias, 1992 , 174-88).

Interestingly no attempt is made to link digital to non-digital activities, either creative ones or experimental ones, despite the fact that such an attempt would help towards a more comprehensive use of the materiality of the objects and the involvement of the senses. In the few cases where users can print out their own digital creations, these serve more as a souvenir of their use of the application and less as a stimulus for creative off-screen activities (e.g. "The Fortifi ed City of Mystras"). In other words, there is no attempt to extend the scope of the educational activity beyond the confi nes of the screen in order to support educational processes that exploit the involvement of the senses.

Finally, as far as the social context of the experience is concerned, we came to realise that all of the applications examined are targeted at the individual user; that is, they do not involve multiple users or collaboration between users, even when they are targeted to school groups. It is evident, therefore, that the social dimension of the users' experience does not lie at the heart of the applications' educational design, despite its importance in the overall experience of museum visitors.

Conclusions

The aim of the research presented in this chapter was to explore whether and to what extent ICT applications created and used by cultural institutions in Greece incorporate the principles of museum education and take advantage of the possibilities offered by new technologies to maximise their effectiveness and appeal. We performed a qualitative content analysis, which highlighted some very interesting and thought-provoking issues.

Greek institutions invest increasingly more in the production of digital applications, thus aiming to enrich their educational provisions and encourage more visitors to take advantage of the museum collections and the knowledge they provide. These applications tend to be web based, so as to be as widely accessible as possible. This is also evident by the fact that the applications aim to address the needs of a wide variety of users and they do not target specifi c groups or needs.

Unlike real museum experiences though, where encounter with the museum object is at the centre of the educational process, museum digital applications do not take full advantage of the possibilities offered by them. Multiple interpretations, individual meaning making, encouragement of alternative or additional ideas produced by the user/visitor, social interaction, interactivity, creativity and engagement are not encouraged or supported to the extent that they could be. The applications seem to reinforce the model of passive reception, where the museum provides cultural content and the user is expected to accept and internalise the knowledge

provided. It is therefore necessary to reconsider the creation of museum educational
applications and to design them taking into account museum education theory and
the special qualities/values museums and their collections hold. It is important for
museum educational applications to encourage a more active involvement of users
with museum objects and to provide opportunities for more creative, engaging and
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enabling activities. At the same time, greater emphasis should be placed on strengthening the social dimension of the overall experience by utilising the possibilities
that information and communication technologies offer for user involvement and
collaboration and for the exchange of ideas and content.
This research aims to contribute to the discussion regarding digital media use in
museum education. We argue that it is time to reconsider the efforts made so far and
to promote an interdisciplinary approach that will offer a more personal, rewarding
and multi-layered learning experience for all.
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ICT and Pre- and In-service
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Introduction: Technology Uptake
The belief that technology in its various instantiations will transform educational
practice is very prevalent and dates back for at least a century. The assumption
behind the introduction of technology into educational systems was that it will
eventually make them more meaningful, interesting, and relevant for students,
thereby drastically improving the quality of learning. However, if there is one consistent fi nding from the past three decades of research on ICT use in education, it is
that technology has failed to transform teaching and learning practices.
There are two interrelated problems with technology use. First , research indicates that the extent of technology use in classrooms is rather low: teachers do not
appear to use technology in their practices to any considerable extent (Hinostroza,
Labbé, Brun, & Matamala, 2011 ; Norris, Sullivan, Poirot, & Soloway, 2003 ; Ward
& Parr, 2010 ; Webb & Cox, 2004 ; Wikan & Molster, 2011). Second , even when
teachers do embrace technology, it gets integrated in ways which sustain rather than
transform existing practices (Condie, Munro, Seagraves, & Kenesson, 2007 ; Cuban,
2001 ; Cuban, Kirkpatrick, & Peck, 2001 ; Donnelly, McGarr, & O'Reilly, 2011 ;
Eteokleous, 2008 ; Hayes, 2007 ; Hermans, Tondeur, van Braak, & Valcke, 2008 ; Li,
2007 ; Norton, McRobbie, & Cooper, 2000 ; OFSTED, 2004 ; Player-Koro, 2013 ;
Prestridge, 2012). On an international level, the SITES 2006 study indicated that
ICT adoption does not necessarily mean that traditional practices are abolished
Technology Integration in the Most Favorable
Conditions: Findings from a Professional
Development Training Program
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(Law, 2008). Similar evidence is reported on a national level, e.g., the UK (Selwyn,
2008 ; Smith, Rudd, & Coghlan, 2008 ; Yang, 2012) Ireland (McGarr, 2009) and
Greece (Vosniadou & Kollias, 2001). The low rate of classroom technology use and
the way technology is used to support existing practices are the primary reasons why
the vision of transforming education through technology has yet to be realized.
Why has it proven so diffi cult for teachers to use technologies in their practices?
Researchers have sought to determine the reasons behind this technology resistance.
More than a decade ago, Becker (2000a) identifi ed four enabling conditions for
technology adoption: technology access, training, curriculum compatibility, and
constructivist beliefs. Ertmer (1999 , 2005) attempted to further systematize technology resistance into obstacles that can be distinguished into fi rst-order and second-
order barriers. Typically, fi rst-order barriers are extrinsic to teachers while
second-order barriers are teacher related.
First-order barriers are beyond the direct control of the teacher and have to do with
what is provided by the local and state authorities in terms of technology infrastructure and support structures such as equipment, training, and support. First, technology access is one of the main conditions upon which technology integration
depends. Several studies report that one of the strongest predictors of technology
use is technology access (Becker, 2000a ; Eteokleous, 2008 ; Granger, Morbey,
Lotherington, Owston, & Wideman, 2002 ; Norris et al., 2003). Second, a certain
level of technological competence is required if teachers are to use technology.
A possible lack of technical skills might potentially undermine technology integration. Several studies report that the greater the personal ICT competence the more
likely the teachers were to use ICT in their classrooms (Eteokleous, 2008 ; Prestridge,
2012). Moreover, classroom integration of technology has been predicted by computer experience (Mueller, Wood, Willoughby, Ross, & Specht, 2008 ; Wood,
Mueller, Willoughby, Specht, & Deyoung, 2005). Third, technical support can also
be a hindrance to technology adoption. Several studies report that access to technical support can be a facilitator of technology use (Hayes, 2007 ; Penuel, Fishman,
Yamaguchi, & Gallagher, 2007). Finally, the issue of leadership is often stressed as
teachers need not only technical but also administrative support. Some studies
report that principals and school administrators can play a facilitatory role in terms
of technology adoption (Hayes, 2007 ; Law, 2008).
Technology adoption is clearly contingent on eliminating these fi rst-order barriers .
Addressing fi rst-order barriers required lavish funding so as to ensure the availability
of resources and training, both technical and pedagogical. Additionally, educational
authorities have restructured curricula so as to accommodate technology use and
foster technology integration. Progress on all fronts related to fi rst- or der barriers has
been steadily made over the years (Ertmer, 2005). The underlying assumption that
guided much of the thinking was that providing resources and support would somehow naturally lead to greater technology adoption (Ertmer, 1999). It turned out,
however, that resources and support were a necessary but not a suffi cient condition
for technology integration: second-order barriers played a critical role.
Second-order barriers involve teacher beliefs about teaching and learning (Ertmer,
1999). Teacher beliefs about teaching and learning might shape whether and how
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teachers eventually integrate technology in their classrooms. Therefore, teacher
beliefs have been the focus of much attention in the literature (Hermans et al., 2008 ;
van Braak, Tondeur, & Valcke, 2004 ; see also Baggott la Velle, McFarlane, John, &
Brawn, 2004 ; Ward & Parr, 2010). While addressing fi rst-order barriers was relatively straightforward, addressing second-order barriers proved considerably more
challenging (Ertmer, 2005). Generally speaking, second-order barriers have been
addressed mainly via professional development training (PDT) programs and activities of many forms.
Professional Development Training on ICT Pedagogy
Ertmer (2005) argued that teachers are likely to think about technology in the same
way they think about other educational innovations. Consequently, examining how
teachers approach innovations and what makes PDT programs effective might help
understand teachers' response to PDT on ICT integration in the classroom.
According to the literature on PDT, three properties have been singled out as being
critical for its success: form, length, and content. As far as form is concerned, many
forms of PDT have been found to be effective: workshops (Ertmer, Ottenbreit-
Leftwich, & York, 2007 ; Shriner, Schlee, Hamil, & Libler, 2009), seminars and
conferences (Ertmer et al., 2007), independent learning (Gray, Thomas, & Lewis,
2010), school-based professional development by staff (Gray et al., 2010), and personal coaching (Miller & Glover, 2007). When it comes to length PDT should be
both continuing (Miller & Glover, 2007) and sustained (Garet, Porter, Desimone,
Birman, & Yoon, 2001). Finally, with regard to content , research suggests that PDT
is more likely to be effective if it has a pedagogical rather than a technical orientation (Law, 2008 ; Law & Chow, 2008b). It is also likely to have an impact if the
primary focus is on the academic subject (Garet et al., 2001).
While some essential features of successful PDT have been identifi ed, there are
still areas of critical importance which are largely unexplored. More specifi cally, in
addition to form, length, and content, it has been argued that teachers themselves
are one of the most critical determinants of PDT success because their previous
experiences might infl uence the outcomes of any in-service training regardless of its
form, length, and content . The argument is that we need to consider what teachers
themselves bring to PDT sessions in terms of former experiences and practices
(Penuel et al., 2007). For example, Coburn (2004) has convincingly demonstrated
that teachers' responses to innovation appear to be mediated by their preexisting
world views and practices. Additionally, teachers' local contexts should also be
carefully considered when determining the effectiveness of a PDT program, as the
demands posed by the contexts of practice make teachers set specifi c priorities
(Penuel et al., 2007). PDT is bound to be interpreted in terms of the existing policies,
schedules, budgets, curricula, hardware, software, technical, and administrative support of teachers' local contexts. For instance, Zhao and Frank (2003) found that the
more strongly teachers believed that computers were compatible with their teaching
styles, the more often teachers reported using computers in their practices.
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Many teacher background variables have been systematically explored as predictors of ICT classroom use (e.g., Hermans et al., 2008 ; Law & Chow, 2008b ; Tondeur,
Hermans, van Braak, & Valcke, 2008 ; van Braak et al., 2004 ; Ward & Parr, 2010).
However, teacher background variables have not been systematically investigated as
predictors of PDT success even though their signifi cance has been recognized in the
aforementioned literature (Coburn, 2004 ; Penuel et al., 2007). In particular, when it
comes to PDT that is related to ICT integration in the classroom, researchers have
rarely focused on how teachers with specifi c backgrounds respond to PDT .
But in what ways can teachers belonging to specifi c groups be important for understanding the effectiveness of PDT for technology integration? As we argue in this
work, this is because examining teachers with specifi c—and more particularly
favorable—background properties is one way of determining the possible upper
range of technology integration that we can reasonably expect from PDT programs.
Technology integration can vary greatly along the sustain-transform continuum. At
one extreme, teachers might make no or limited use of technology. In this case, the

impact of technology will range from negligible to small. At the other extreme,
teachers might use technology a great deal. In this case, depending on the ways
technology gets used, its impact might be far-reaching, ultimately leading to the
transformation of teaching and learning practices. As the preceding literature review
shows, the majority of teachers do not use technology in their practices and those
who actually do tend to domesticate it rather than use it to change their practices.
Examining how the most committed, skilled, qualifi ed, or experienced teachers
respond to PDT in ICT use is a possible test of success for current in-service PDT
programs since it can be a measure of their maximal effectiveness along the sustain-
transform continuum of technology use. In other words, if PDT stands any chance
of achieving our highest aspirations relevant to transforming current educational
practices, then teachers with such qualities are the best possible candidates for
proving the case for PDTs.
To the best of our knowledge, there are no studies on how teacher background
properties such as skills, expertise, or qualifi cations might infl uence the effectiveness of a PDT . Consequently, we draw mainly on studies indicating certain teacher
background properties as being either highly conducive to technology adoption or
closely related to it. It seems reasonable to assume that the more properties facilitating technology integration teachers have before attending a PDT program, the less
ground these teachers would have to cover in terms of learning while attending the
PDT . Our assumption is that teachers with such properties will show the best and
most favorable response to PDT as they would have to make less progress compared
to other teachers.
ICT Use as a Function of Teacher Background
Only a handful of studies have closely examined specifi c teacher groups with respect
to technology adoption and use . One group of studies focused on exemplary technology-
using teachers to extract those background properties that make them distinct .
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Exemplary technology-using teachers use technology in their practices in innovative,
non conventional ways. In such studies the typical focus is on determining what makes
these teachers exemplary technology users, documenting their practices, investigating
their beliefs and pedagogical philosophies, and determining factors that either facilitate
or hinder their efforts to use technology (Angers & MacInmes, 2005 ; Becker, 2000b ;
Becker & Riel, 2000 ; Ertmer et al., 2007 ; Hadley & Sheingold, 1990 , 1993 ; Leftwich,
2007 ; Riel & Becker, 2008). This body of research shows that exemplary technologyusing teachers are different from other technology-using teachers and other teachers in
general in a number of ways. More specifi cally, exemplary technology-using teachers
actively seek more professional development activities than ordinary teachers, take
release time to follow such activities, are more willing to take risks and experiment
with technology, and overall have a high level of commitment to improving their students' learning through technology (Angers & MacInmes, 2005 ; Becker & Riel, 2000 ;
Hadley & Sheingold, 1990 ; Leftwich, 2007 ; Riel & Becker, 2008). While the contribution of such studies to our understanding of technology integration is critically important, this line of research has not focused on the processes through which these teachers
became exemplary. As a consequence, the personal learning trajectories of exemplary
technology- using teachers are unknown, especially in relation to PDT on ICT pedagogy . However the aforementioned characteristics of exemplary technology-using
teachers can work as rough guidelines in an attempt to locate groups of teachers with
background properties that maximize the potential of in-service PDT.
One group of teachers with special background properties which might be important for technology integration are teachers with constructivist beliefs. Several studies have indicated that exemplary technology-using teachers are also highly likely
to employ a constructivist, student-centered approach to teaching (Becker & Riel,
2000 ; Dexter, Anderson, & Becker, 1999 ; Hermans et al., 2008 ; Matzen & Edmunds,
2007 ; van Braak et al., 2004). Overall, a systematic relationship between constructivist approaches to learning and technology use has been reported in the literature:
constructivist beliefs are correlated with a higher rate of technology adoption. While
the relationship between constructivist teaching philosophies and technology use
has been well established in the literature, how exactly teachers who are very familiar with constructivist teaching and learning in a given subject area or grade level
respond to in-service PDT on pedagogical uses of ICT has not been explored.
Another group of teachers with specifi c background characteristics that might be
important for technology integration are teachers of high academic qualifi cations.
Compared to ordinary teachers, teachers who hold postgraduate degrees have by
defi nition a higher degree of specialization. Riel and Becker (2008) found that a
particular area in which professionally engaged teachers are differentiated from
other teachers is that they have invested more in their own education and master's
degrees were considered to be an indication of such an investment. As Riel and
Becker (2008) report, professionally engaged teachers were more likely to (a) have
a constructivist teaching philosophy and (b) use ICT more frequently and differently than other teachers (e.g., more tool applications, wider variety of applications). Although specialization might infl uence how teachers respond to PDT, how
teachers with a high degree of specialization, such as master's or Ph.D. degrees,
respond to PDT has not been investigated.
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Focus of the Study
Overall, there is a knowledge gap in terms of how specifi c teacher groups respond
to in-service PDT on ICT pedagogy . The present multiple case study aimed to
examine how one such group of teachers responded to a PDT program on ICT pedagogy. More specifi cally, our target was a group of three primary school teachers
who participated in an in-service PDT program offered by a University Training
Center (hereafter UTC) in Greece. These teachers were selected among the other
participants in the PDT program because they deviated maximally from the average
teacher in several ways. First, they had a high degree of expertise in the fi eld of science education as they all held relevant Ph.D. degrees. Second, they had a record of
academic publications in refereed journals, having authored or coauthored scholarly
papers in the area of science education. Third, they were all very experienced, as
their teaching experience ranged from 10 to 20 years of service. Fourth, none of
them were ICT novices as they all had previously used ICT in their teaching practices. Finally, two of them had participated in national funded research projects
which aimed to support science teaching with ICT while the third earned her Ph.D.
in a Teacher Education Department in Greece with a reputation for targeting ICT in
the teaching of science. For these reasons, the three teachers had backgrounds
which clearly set them apart from the general teacher population .
Given that these teachers participated in an in-service PDT program, their backgrounds were highly relevant for two main reasons. On the one hand, their specialization in science education ensured that they were, by defi nitio n , among the most
theoretically sophisticated teachers in terms of constructivist teaching philosophies
and pedagogies . Based on the literature reviewed above, they were the most likely
to respond favorably to technology integration given that constructivist beliefs are
related to classroom technology use (Becker & Riel, 2000 ; Dexter et al., 1999 ;
Hermans et al., 2008 ; Matzen & Edmunds, 2007 ; van Braak et al., 2004). On the
other hand, the fact that the three teachers held not only master's but also Ph.D.
degrees indicates a very high level of specialization . Thus, based on the fi ndings of
Riel and Becker (2008), this specialization would greatly facilitate in-service PDT
training on ICT pedagogy. Therefore, we assumed that from the whole teacher population these three participants were the most likely to respond favorably to PDT
not just on a superfi cial but also on a substantial level . In fact, we would go as far
as to argue that teachers of such backgrounds represent the ideal audience for seeding technology innovation concepts.
Given that the three teachers who participated in the PDT held constructivist
teaching philosophies and had high academic qualifi cations, this multiple case study
examined how they integrated technology in their practices along the sustain-
transform continuum.
Given the design challenge of creating instructional scenarios, implementing them
in their classrooms, refl ecting on them in the context of the PDT, and then revising
their initial instructional scenarios the following research questions were addressed:
1. How did the teachers integrate technology in their designs?
2. Where is technology integration situated on the sustain-transform continuum?
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3. What were teachers' refl ections on their designs?
4. How did the teachers revise their initial designs?
The fi rst question aims to provide an account of technology integration in the
context of their practicum so as to map out how the different technologies were
prescribed to be used. The second question explored whether technology integration
supported established practices or transformed them into new directions. The fi nal
two questions mapped out the teachers' responses by way of refl ection or redesigning to the design challenge, its implementation, and the feedback they received in
the UTC.
Method
Participants and Setting
Following the general European Union (EU) policy guidelines, the Greek authorities have adopted a two-level PDT program for primary and secondary teachers. In
2000 the Greek Ministry of Education (MoE) initiated a large EU-funded PDT program of teacher in ICT (see Demetriadis et al., 2003 ; Jimoyiannis & Komis, 2007 ,
for a comprehensive account of this program). The program had an explicit technological literacy orientation and aimed to develop teachers' ICT skills and competences. It had a total duration of 50 h and was conducted at special school-training
centers (STC). Thousands of teachers participated in this ICT training that continued through most of the decade.
In 2007 the MoE established EU-funded UTCs in academic institutions around
the country (Jimoyiannis, 2010 , provides a detailed account of this program). The
objective of these UTCs was to provide high-quality in-service PDT in the area of
pedagogical technology integration across the curriculum. The PDT curriculum
involved pedagogical issues regarding technology integration in all academic subjects and grade levels. Each PDT program lasted for 350 h and spanned a period of
6 months. All primary and secondary teachers who had successfully completed the
former training program were eligible for participation and could apply for a position. After completing the UTC in-service training programs, the participants could
take a centralized exam and, if successful, become offi cial ICT mentors in their
respective academic subjects. Following the cascade model which was adopted for
this PDT program, these teacher mentors would then provide pedagogical ICT
training for their fellow teachers in local STCs (see Fig. 1).
Starting in late 2007, three main in-service training programs were offered at the
UTC of the University of Thessaly, the authors' host institution. The present work
draws on data collected from the third in-service training program (2011-2012).
This program followed the general guidelines for successful PDT in terms of form
(lectures, seminars and workshops, independent learning, and personal coaching
through mentors), length (it was extensive covering 350 h and spanned a period of
6 months), and curriculum (clear pedagogical rather than technical orientation).
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A total of eight primary teachers signed up for the third in-service training program.
In this work we focus on three of these eight teachers because they naturally formed
a group of teachers with very special backgrounds.
PDT Curriculum
The in-service PDT program offered at the UTC in the University of Thessaly comprised a general part and a subject-and-grade-level specifi c part. The former had a
broad, introductory goal and addressed issues related to educational policy in the EU
and Greece, history of educational technology, learning theories, and how they relate
to educational software, taxonomies of educational software, technical and administration issues related to the school ICT laboratories, and adult education. This general part lasted for 160 h and provided the foundation upon which the second,
subject-specifi c part could build. The second part which lasted for 190 h focused on
how to specifi cally integrate technology in the teaching of various academic subjects
and grade levels. Both subject-specifi c and general-purpose software tools were
introduced. Particular emphasis was given to technology integration according to the
research literature for each academic subject. To this end, a number of experts specializing in the teaching of academic subjects were contracted as teachers. Following
the MoE mandates to ensure the highest possible quality of training, only university
staff or Ph.D. holders of various specializations were eligible to teach at the UTCs.
In addition to the theory (i.e., general and subject- specifi c part), the training program
also included a short 30-h practicum section. As part of the requirements of the
practicum section, the participants had to implement two of their instructional scenarios (a) in their own classrooms and (b) in collaborating STCs. Four teacher ICT
mentors who had already successfully completed previous versions of the University
of Thessaly UTC in-service PDT program were also contracted to mentor the
planning and refl ection components of the practicum section.
Fig. 1 A cascade model of PDT
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Design, Procedures, and Data Collection
Following the rationale of qualitative methodology (Lincoln & Guba, 2000), the
present study was conducted as a multiple case study (Yin, 2009). The study was
designed as a case study in an attempt to understand how teachers of constructivist
philosophies and high academic qualifi cations responded to an in-service PDT program on ICT pedagogy. In this multiple case study design, each teacher was treated
as a separate case in order to determine common underlying patterns through
replication.
The overall procedure followed is depicted in Fig. 2 . The teachers attended the
350-h in-service PDT program which involved both theory and practical applications of ICT across the curriculum. The theory section was concluded with the
design of ICT-based instructional scenarios. These instructional scenarios were put
to practice in the practicum section . Each teacher selected two of the instructional
scenarios designed in the course of the training and implemented them in their
classrooms. The practicum section was followed by a feedback session where the
teachers shared their experiences with the group and received feedback and
suggestions from their fellow teachers, the teacher ICT mentors, and the authors. In
the refl ection session which followed, the teachers were asked to revise their instructional scenarios in light of their experiences and the feedback received.
Due to the nature and focus of the study, many different types of data were collected in the course of the PDT. For the purposes of the work reported in this chapter, we draw on the following data sources:
(a) Instructional scenarios. As artifacts, instructional scenarios were of primary
interest as they embodied a teaching plan. The participating teachers developed
several instructional scenarios, following a detailed template that was provided
as part of the requirements of the training program. The teachers had the freedom to create any instructional scenario, in any subject, using any of the ICT
tools available, in any way they saw fi t. Following the theory and practice
guidelines of the PDT, the main requirement was that the integration of technology in their designs would have to have high added value.
(b) Group discussions. Whole-group discussions were also of primary interest as it
is during these that the teachers provided explicit accounts of their instructional
scenarios, thereby disclosing the rationale behind their designs. Group discussions were held during the feedback session and took place at the UTC with the
authors and the ICT-mentor teachers. These group discussions were tape-
recorded, and large portions were transcribed verbatim for further analysis.
Fig. 2 Overview of the procedure
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(c) Revised instructional scenarios . Revised instructional scenarios were meant to
embody teachers' refl ections following the implementation of instructional scenarios in their classrooms and the feedback session that followed. By refl ecting
on the elements the teachers perceived as problematic, revised instructional scenarios helped pinpoint the "corrective" measures needed. The revised instructional scenarios were also formally required for evaluation purposes.
(d) Participant observations and fi eld notes . In the course of the practicum and
refl ection sections, the authors took various notes of informal communications
with the participants (e.g., personal e-mails, informal discussions), questions
posed in the practicum section, and problems which surfaced in planning and
teaching. All such observations and notes were then combined with the rest of
the data to facilitate the analysis.
Data Analysis
Instructional scenarios and revised instructional scenarios were the data sources
through which we addressed the fi rst two research questions. They were analyzed
following established qualitative data analysis procedures. For each teacher case,
this involved data reduction, data display, conclusion drawing, and verifi cations
(Miles & Huberman, 1994). Each instructional scenario comprised several activities, and for the purposes of this work we used the instructional activity as our main
unit of analysis. Following a qualitative content analysis approach, we initially used
rough descriptive categories to classify technology use in each instructional activity,
arriving at general profi les of ICT use per instructional scenario. Subsequent passes
led to successive generalizations and mutual agreement between the researchers on
the main categories of technology use in the teacher designs. The categories used
are described next. These categories were used as indicators of constructivist theoretical underpinnings for the teachers and as a means of assessing which constructivist principles found their way to the instructional scenarios.
1. T echnology tools . This category included the various types of software tools
used such as stand-alone software or network applications (e.g., web browser).
This category assessed the presence of a constructivists' preference for a multitude of information sources so as to address students with different proclivities
and intensify the social embeddedness of the information provided. Although a
stand-alone software may indeed be specially designed in addressing particular

disciplinary needs, the current availability of easily accessible learning resources
through the Web makes them natural candidates for lessons addressed to digital
natives.
2. Information modality . This category addressed the types of content that the technology made available and included text, images, video, and audio. This category assessed the importance of providing information of different modalities so
as to supply multiple different representations of information.
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3. Information context . This category described the nature of the information
sources used, distinguishing between educational and authentic sources. It
assessed the extent to which authentic information resources and real-time data
possibly of local and personal interest were employed in the instructional scenarios. Although including information in the context of school necessitates distancing from the offi cial sources by learned communities, the bulk of information
available on the web by varying sources of expertise makes it practically feasible
to assess authentic information sources.
4. Students' role in technology use . This category referred to who used technology
(student vs. teacher), whether technology was used as a tool to process information (yes/no), the locus of choice of technology tools and sources (student vs.
teacher), the locus of choice relative to how technology tools were used (critical
decisions regarding technology use were made by the students vs. the teacher),
and the mode of technology use (individual vs. group use). This category
assessed various indicators of constructivist concerns for promoting student
agency in the learning process. From a constructivist learning viewpoint, (a)
students rather than teachers are expected to be the main users of technology, (b)
students are expected to use technology as a tool to process information rather
than simply consume information, and (c) students are supported in making the
choices regarding technology use.
5. Technology function . This category examined the specifi c role technology played
in terms of learning for every instructional activity. Technology was used for
providing information, providing representations (without manipulation by the
students), and providing opportunity for limited simulation (manipulation
demands were minimal). This category assessed the constructivist tendency to
harness the potential created by the access to rich information sources and to
strong tools for data exploration (e.g., to assess rich information sources, to synthesize information from various—often divergent—sources, to use real-time
data to draw conclusions) and to use the visualization affordances of the technology (e.g., to conceptually facilitate the transition from abstract to the concrete, to
use multiple representations perhaps in parallel to student manipulation). Finally,
since all instructional scenarios were related to science education, it also assessed
the presence of technology uses that are in sync with current constructivist learning environments in science education which capitalize (a) explorations of a
physical phenomenon in ways impossible in real life, (b) experimentation
(hypothesis formation and testing), and (c) developing science process skills.
Each instructional scenario activity was assessed with respect to the categories
mentioned above. All six instructional scenarios we analyzed were related to science education, four belonged to earth science, one to physics, and one to environmental studies. The analysis of the transcriptions of the group discussions and the
participant observations and fi eld notes focused on themes pertinent to the third
research question, i.e., how the teachers refl ected on the design challenge and its
implementation. On the one hand we examined if teachers thought that their designs
refl ected signifi cant departures from their current practices. On the other hand, we
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looked at whether they experienced the design of instructional scenarios as a challenging activity. These data were triangulated with the teachers' assessment comments about the implementation of their instructional scenarios that were included
in their instructional scenario reports.
Results
Technology Integration
The fi rst question focuses on how the teachers integrated technology in their
designs. The instructional scenarios were the main data sources used to answer in
this part of the analysis. In order to identify patterns, each instructional scenario for
every teacher was treated as a separate case. Despite differentiations, the analysis of
the instructional scenarios revealed similar patterns of technology integration. Due
to space limitations, one instructional scenario per teacher was randomly selected
and is presented here.
Tables 1 , 2 , and 3 present the results of the analysis of one lesson that each teacher
planned and carried out in the practicum section with respect to the categories of
technology tools, information modality, information context, and technology func-
Table 1 Teacher A: Grade: 6; academic subject: science; unit: physics, analysis, and synthesis of light
Technology tool Information modality Information context Technology function
Web browser Text, image Educational Information provider
Web browser Video Educational Information provider
Representation provider
Web browser Text, image Authentic Information provider
Web browser Animation Educational Limited simulation
Web browser Text, image, video, animation Educational Information provider
Representation provider
Limited simulation
Web browser Video Authentic Information provider
Table 2 Teacher B: Grade: 4, academic subject: environmental studies, unit: the weather
Technology tool Information modality Information context Technology function
Web browser Text, image, tables, charts Authentic Information provider
Web browser Text, images, tables, charts Authentic Information provider
Web browser Text, image Educational Information provider
Stand-alone software Text, image, animation Educational Information provider
Representation provider
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tion. Each lesson was actualized within two periods, which is approximately 90 min.
Each table row represents the different instructional activities in each lesson.
As can be seen in Tables 1 , 2 , and 3 , technology use involved (a) digital learning
resources and (b) interactive software applications. More specifi cally, taking into
consideration all of the designs, technology use was characterized by the use of both
general-purpose tools (such as a web browser) and the use of special-purpose educational software. The browser was primarily used for accessing information on the
World Wide Web and, to a lesser extent, for running web-based simulations and
animations. Although Web 2.0 sources (blog, wiki) were used in two cases by
teacher A, they were employed as information sources and the students were not
involved with the more constructive functionalities of these tools. The other type of
technology use involved stand-alone educational software. It should be noted however that the stand-alone software that was used was the result of research work
aiming towards addressing student misconceptions in the relevant natural science
domains. Overall, the browser reigned as the main software tool as it was dominant
in designs of two out of the three teachers.
In terms of modality, Tables 1 , 2 , and 3 show the presence of not only the textual
mode but also visual modes. Visual modes included realistic videos, simulations,
realistic videos that were annotated, realistic images, and charts. This variety is in
accordance to the professed constructivism of the teachers. On the other hand auditory modes have attracted less attention.
In terms of authenticity, only some of the information sources used were authentic ones. The majority of the sources were educational, i.e., were tailor-made for
educational purposes. Moreover even the authentic sources that were used were
authoritative in nature, thus coming as close as possible to univocal educational
sources. However the sources were appropriately selected so as to suit the targeted
students' age range.
Finally, with respect to the category of students' role in technology use, the
results do not show much variation. In all of the designs, technology was exclusively used by the students who collaborated in small groups to complete the assignments. However, technology tools and sources were explicitly selected by the
teachers. Moreover, the ways both the technology tools and the information sources
were to be used by the students were highly prescribed by the teachers through
worksheets. The worksheet was the main tool through which the teachers tried to
balance some freedom of choice for the students with a detailed specifi cation of the
technology use. Finally, technology tools were mainly used as a gateway to information, not as tools to process data and information or otherwise transform it.
Table 3 Teacher C: Grade: 6, academic subject: geography, unit: day-night cycle
Technology Information modality Information context Technology function
Stand-alone software Animation Educational Limited simulation
Stand-alone software Image Educational Representation provider
Stand-alone software Animation Educational Information provider
Limited simulation
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On the whole, the designs of the instructional scenarios were infl uenced by constructivist principles. The students were the main users of various information
sources and simulations in ways that facilitated the expression of their alternative
conceptions and a reality check of these conceptions. Nonetheless, two constraining
factors characterize most of the categories:
(a) Limitations of openness: Capitalizing on Web 2.0 functionalities, accessing
confl icting information sources, and accessing people outside the walls of
classroom are all nonexistent in the teacher designs.
(b) Limitations of students' agency: How and for what purpose technology is used
are prescribed by the teacher, and any source of challenge (like confl ictual
information) is avoided.
Technology Integration Along the Sustain-Transform
Continuum
The second research question focused on the technology leverage for implementing
science education instructional scenarios that were clearly going past current practices in Greece. The initial instructional scenarios and the revised ones were the
main data sources for this analysis. Teachers' refl ections were also used as a secondary data source but are reported in detail in part c of this section. The main functions
of technology can be seen in the rightmost column of Tables 1 , 2 , and 3 . More specifi cally, when situating the function that technology performs in the context of the
instructional scenarios we arrived at two main categories that express the leverage
of technology: accessibility and visualization . The fi rst refers to making accessible
content which would be inaccessible without technology. The second refers to the
visualization of physical phenomena and models in the context of providing 2D/3D
static or dynamic representations as well as other forms of representation.
The fi rst main function that technology played in the designs involved making
inaccessible information easily tangible. For example teacher B used a meteorological site run by a state agency to make accessible to students real-time data on the
current weather in different sites in Greece and Europe. Undoubtedly, using technology to access information which would be inaccessible through other means utilizes
the potential of technology to add currency, relevance, authenticity, multimodality,
and interest to one's teaching. Overall, the teachers used the information resources
to enrich the curriculum content which had to be delivered . On the other hand technology was not used to support engagement with students' own concerns and questions. Moreover, once the information was accessed no further demands of creative
craftsmanship either in processing the information or in interpreting the information
were put on the students. The absence of confl ictual or diffi cult-to- interpret information was further minimizing opportunities for this craftsmanship to be needed.
The second main technology function involved visualization. Instructional scenarios, worksheets, presentations of the instructional scenarios, and refl ections on
the instructional scenarios all centered on some form of presentation to the students.
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As the analysis of the instructional scenarios suggests, although technology was
used by the students themselves, technology was largely used for demonstration
purposes in order to "show" something as clearly as possible, so that (a) student
misconceptions are eradicated through cognitive confl ict and (b) students are provided with crucial external representations that facilitate the understanding of the
intended concept or process. For example, in the course of an instructional activity
teacher C asked the students to stop a simulation showing coordinated representations of the Earth and its position relative to the Sun at specifi c time intervals. After
each simulation freeze, the students had to answer specifi c questions which were
given in the worksheet. Undoubtedly, supporting visualization is one of the main
strengths of technology , and it is understandable why the teachers made such an
extensive use of technology-enabled visualizations. On the whole, the teachers did
take into consideration students' alternative conceptions and constructed sequences
of predefi ned experiences alternating raw production of students' ideas with the
"corrective" experience of superb visualization afforded by technology .
When considering the whole corpus of the instructional scenarios the following
common patterns emerged. First, simulations appeared in the teachers' designs, but
their use was extremely limited. For example, in the case of the coordinated representations time was the only variable that could change. Moreover the directions in
the student worksheets specifi ed the specifi c values of the time variable where the
students were instructed to freeze the simulation. Consequently, while on the surface the students appear to be actively controlling the simulation, from a learning
point of view nothing much would have been different had the teachers used a video
projector for a whole-class display of the simulation and had they posed similar
questions to the whole class.
Second, technology use by students for constructing hypotheses or transforming
and representing knowledge or managing the tasks was extremely sparse. There is
only one exception to this pattern, teacher C, who on one occasion used GoogleEarth
to create limited opportunities of manipulation and provided students with a genuine
inquiry question. However the conditions were unfavorable (time allowed, place of the
activity in the overall design) and rendered such an inquiry practically impossible.
Third, technology-enabled visualizations seemed to compete with physical
artifact- enabled visualizations as if the two were struggling to occupy the same slot
in the script of the didactical sequence. There are several manifestations of this. On
the one hand, simulations were used sequentially and not in parallel with more traditional "experiments." For instance, teacher A introduced a simulation quite some
time after a relevant experiment. On the other hand, teachers underplayed the visualizing and representational affordances of hands-on artifacts (such as the globe, construction, and manipulation of 3D artifacts). The teachers did not use the opportunity
to combine digital simulations with the use of hands-on artifacts; instead, they
showed an extreme faith on the effi ciency of digital visualization as a learning tool.
Finally, technology was often used (especially by teachers B and C) to stage a guided
presentation of the features of the visualized physical model and to compare these
features with selected and heavily transformed pseudo-authentic digital materials.
In this fi nal case digital reality took the place of physical reality both in terms of the
experimental means and of the observations.
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Teachers' Refl ection on the Design Challenge
and its Implementation
The third research question centered on teachers' refl ections on the whole PDT
experience and specifi cally looked at the teachers' perceptions of the design challenge and its implementation.
Technology integration as a challenge
Overall, the teachers did not experience the designs in the practicum section of the
PDT as a real challenge. This is not surprising given their high level of expertise. As
teacher A noted in the discussion of the refl ection session:
I have been using ICT in my teaching before this [PDT] program. I was certainly using ICT
in Science Education which is a subject I know really well. So, it's not that I learned something new that I've just started using in my teaching…That doesn't mean I did not profi t
somehow from attending the PDT. It [ICT] was more useful for other [academic] subjects.
But here [science education], since I taught in a domain that I know well, I feel that I would
have still delivered even if I had not attended the PDT (Teacher-A)
Here the teacher clearly delineates what she thought of the PDT, stressing that
she did not fi nd it informative enough in her domain of expertise. Overall, the teachers who participated in this study were very confi dent with their theoretical underpinnings in science education and often cited relevant sources in their refl ections.
For example, in the following excerpt, teacher A explains the theoretical guidelines
that guided their designs:
From the point of view of current approaches to Instruction in the Natural science, learning
is not just acquiring information but a continuous process of resolving of inner cognitive
confl icts. Those confl icts are created and resolved through active participation, communication and interaction between the student and the learning and social environment in the
classroom (Teacher-A)
This statement clearly refl ects the constructivist convictions of the teacher,
refl ecting both the nature of science learning and an instructional approach to science teaching.
In another occasion teacher C articulated his own stance about when the use of
ICT may be productive, largely corroborating our conclusions (section on
"Technology Integration Along the Sustain-Transform Continuum" above) about
the added value that teachers attributed to technology

… my conclusion regarding the use of ICT or what we call “digital resources” etc. is that
you aim to use ICT whenever you have no particular or no other ways of representation,
alternative ways of representation, presentation of a new concept or phenomenon and the
second way [of ICT use] is to use ICT in conjunction with the experiment etc. what we call
multiple representations, that is as an complementary medium, as a supplementary tool to
promote better understanding (Teacher-8)
Teachers' openness to change
Given that the main functions of technology involved information access and
visualization, in the group discussion session the authors suggested other ways of
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technology integration, linking back to the theory and practices of the PDT program.
It was pointed out to the teachers that technology might have been well integrated
in their designs, but this integration was limited with respect to the potential of
technology to support new forms of teaching and learning. Teachers' responses to
our proposals were completely unexpected. Not only did they defend their designs
but also claimed that both their designs and the ways technology was integrated in
these designs were nothing short of exceptional. From our point of view, it was puzzling that the teachers did not seem to be open to suggestions and refused to even
consider other proposals for contemplating new ways of technology integration
which would have resulted in a more substantial level of technology use, a level that
would have entailed a change in the teaching practices. In an effort to ground the
discussion in a concrete way and since visualization was a pivotal point in all of
their designs, each participant was asked to explicitly describe the function of visualization in terms of learning for his/her design. While the UTC training program
provided a broad conceptual framework for understanding technology use across
the curriculum, the teachers approached visualization ad hoc in their designs; that
is, they neither examined visualization in terms of a learning theory or a specifi c
conceptualization of learning nor did they consider the special mediating role visualization was to play in their students' learning . It was as if visualizations themselves would somehow provide most of the support needed by the students leaving
teachers with the task of selecting and pacing the appropriate technological tools to
supply the visualizations in a “just-in-time” fashion. Therefore, there appeared to
be dissociation between the concepts presented in the training curriculum and the
concepts the teachers invoked to explain why exactly they chose to use technology in
the ways they did . Essentially, they conceived technology as a gateway to information, fi tting a slot in the science education teaching script that they had mastered as
opposed to addressing the technology's learning functions and the role of technology
in mediating the learning of science content.
Teachers' Revised Instructional Scenarios
In addition to designing instructional scenarios, the PDT also involved implementing these scenarios in real-world settings and the teachers tried them out in their
classes in the practicum section of the program. Due to the PDT design, the teachers
were asked to refl ect on their experiences and to describe in detail how they would
change their designs based on their experiences with (a) the actual technology use
in their classes and (b) the feedback they received in the refl ection session. More
specifi cally, they were asked to revise their instructional scenarios as they see fi t so
as to achieve the maximum level of technology added value for the same learning
objectives. The resulting accounts of technology use would be idealized, free from
any sorts of constraints (time, curricular, infrastructure, student background knowledge, etc.). The analysis of these “idealized” instructional scenarios indicated that
the teachers stood by their original designs. The only changes made were minor
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ones and were unrelated to technology use or function per se . Consequently, information access and visualization remained the main technology functions in the
revised instructional scenarios.
Discussion
The last century has been characterized by recurrent visions of transforming education through various technologies. The high hopes that technology integration into
teaching practices would lead to their transformation have not been validated
(Condie et al., 2007 ; Cuban, 2001 ; Cuban et al., 2001 ; Donnelly et al., 2011 ;
Eteokleous, 2008 ; Hayes, 2007 ; Hermans et al., 2008 ; Li, 2007 ; Norton et al., 2000 ;
OFSTED, 2004 ; Player-Koro, 2013 ; Prestridge, 2012). Teachers either resist using
technology or use technology to sustain rather than transform their practices
(Donnelly et al., 2011 ; Law & Chow, 2008a ; Player-Koro, 2013). This failure to
transform education through technology has been attributed to fi rst- and second-
order barriers (Ertmer, 1999 , 2005). As research shows, fi rst-order barriers are a
necessary but not a suffi cient condition for technology integration. Therefore
second- order barriers need to be addressed, and one of the main tools to address
them has been teacher training, both preservice and in-service. While there is a
substantial body of research on what makes professional development effective, the
importance of factors related to teachers' backgrounds has not been thoroughly
explored yet (Coburn, 2004 ; Penuel et al., 2007). The present work contributes to
this knowledge gap by examining how a group of teachers who had constructivist
teaching philosophies and high academic qualifi cations responded to an extensive
in-service PDT program on ICT pedagogy. The special characteristics of the teacher
participants provide a measure of the limits of PDTs as a means to promote technology integration in educational practices in transformative ways.
Due to both the teachers' characteristics and the design of the PDT they participated in, there were no fi rst-order barriers hampering technology integration. With
regard to second-order barriers, these teachers were science education experts and
science education is a fi eld where constructivism is championed more than any
other educational fi eld (Duit & Treagust, 1998). It should also be noted that in
Greece most Ph.D. dissertations in science education adopt some version of the
constructivist paradigm. As the literature shows, teachers who have constructivist
beliefs are more likely than other teachers to use technology and also tend to use it
in more student-centered ways (Becker & Riel, 2000 ; Dexter et al., 1999 ; Hermans
et al., 2008 ; Watzen & Edmunds, 2007 ; van Braak et al., 2004). On the other hand,
one of the potential barriers to technology integration is the time and effort required
by teachers to adopt an innovation (Hayes, 2007 ; Penuel et al., 2007 ; Sandholtz &
Reilly, 2004 ; Tyack & Tobin, 1994). Teachers are often reluctant to embrace an
innovation because there is a lot of work involved in adopting it. In our case, however, the teachers were already accomplished, i.e., had a sound theoretical foundation which in principle should require minimal work and effort on their part
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regarding technology integration. Overall, because the teachers did not have much
theoretical ground to cover, we expected that their responses to PDT would be very
positive in two principal ways. First, in terms of technology integration, we expected
that technology would be instrumental in the success of a lesson. Second, we
expected that technology would not be a mere add-on to current teaching practices
but it would leverage them leading to transformations .
The former was fully corroborated by our fi ndings as the teachers integrated
technology in their lessons in a fi tting way, closely following the general principles
of constructivist learning . Firstly, the students themselves were the main users of
technology. That did not mean the use of technology for drill and practice purposes
as is common for novice teachers to do. A wide assortment of digital learning
resources was used in the instructional scenarios, giving them currency and relevance. These means were effective in promoting student engagement and facilitated
the students' recall of relevant prior knowledge. Secondly, collaborative work and
learning were promoted as the students worked in small groups to complete the
assignments. Students were indeed prompted to discuss the information and visualizations provided by technology, and certainly some questions could be solved
through the joint effort of the students. Thirdly, technology was instrumental for the
actualization of these designs and served the teachers' goal of achieving conceptual
change in the science topics targeted in each lesson. The teachers themselves
reported positive results through assessments they had embedded in the instructional scenarios and realized during the implementation of their designs. Based on
teachers' backgrounds and expertise, such high levels of technology integration
were hardly surprising and, as corroborated by their own comments, were to a certain degree mastered before following the current PDT program.
The latter, however, was not supported by our fi ndings. The analysis of the
instructional scenarios and in particular the specifi c technology functions the
teachers used indicate that technology was assimilated into their current practices.
To illustrate the nature of this assimilation, we will consider in some detail the
dominant instructional paradigm of current science education practices in Greece.
More specifi cally, this paradigm is an adaptation of the model of the “inquiry-
scaffolding teaching method” (Schmidkunz & Lindemann, 1992 , as reported in
Apostolakis et al., 2006). The science education teacher books for grades 5 and 6
elaborate on this didactical model and provide the general guidelines for its use.
According to the rationale covered in the teacher books, each lesson follows a specifi c sequence because
students' participation in inquiry is not unguided, but follows specifi c stages and is guided
through specifi c actions, so as to be practically realizable. At every point the teacher can
follow how students learn (Apostolakis et al., 2006 , p. 32).
This sequence includes a fi rst stage where the teacher transforms the subject he/
she has to teach into an initial question or problem. Relevant prior knowledge is
brought forth, and students are supported in proposing their ideas (“hypotheses”)
about the solution to the problem. Student misconceptions surface at this stage.
Then the students perform one or more experiments that the teacher has selected for
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them. As the teacher book suggests, during this phase the teacher should not be too
intrusive, allowing students to “really engage in inquiry.” However, the time constraints of the implementation indicate that the authors of the teacher guide consider
the experiments so well chosen that the students are either expected to arrive themselves at the intended conclusions or that they will be easily convinced by the arguments provided by the teacher. In the next stage, the teacher book proposes to hold
a discussion through which the class arrives at the intended interpretation of the
experiment and gradually towards answering the initial problem. There is no providence for the cases where students might propose new ideas that could be tested
through alterations to the experiments or through new experiments. While the
experiments do address students' misconceptions, they do not leave much space for
student initiative and creativity in the unfolding of the inquiry. During the closing
part of the lesson the teacher guide recommends that students compare their fi nal
answers with the ones they gave initially. In the fi nal stage proposed by the teacher
book, students should work on teacher-assigned exercises that are expected to lead
to a deeper understanding of the science material covered.
As outlined in the teacher guide above, the dominant science education paradigm
in Greece is strongly concerned about the pacing of the instruction, trying to balance
its constructivist theoretical underpinnings and the appropriation of conceptual change
literature with constraints that are inherent in the Greek educational system. Therefore
it does not take into account students' own needs and the scaffolding demands placed
on the teachers should they choose to support these needs. Out of the four main ICT
affordances that Webb (2005) outlined in her review of science learning with ICT-rich
environments, this dominant paradigm is compatible only with the following two:
(a) promoting cognitive development and (b) relating science to students' own experiences and data in the broader real world. The other two affordances namely
increasing students' self-management and enabling them to track their progress so that
teachers' time is freed to focus on supporting and enabling students learning; and facilitating data collection and presentation of data that helps students to understand and interpret
the data, and additionally frees students' time so that they have more time to focus on
developing conceptual understanding (Webb, 2005)
are not compatible with the concern about teacher control expressed in the above
model. This view is in line with other literature proposals for using ICT in science
education. For example, Chang (2013) argued that addressing student needs is probably a main factor in successful scaffolding of science learning through simulations.
On the other hand, Osborne and Hennessy (2003) noted the critical role ICT can
play for introducing students to scientifi c inquiry, for developing hypothesis formation and testing, for advancing science process skills, and for solving open-ended
problems through various technological tools.
The examination of instructional scenarios against this backdrop leads to the
conclusion that the teachers incorporated technology in their existing practices .
There are two main indications of this. First, the way that information was used in
the instructional scenarios expressed a strong concern for effi ciency in time management: all the information aimed to direct students towards the intended “correct”
interpretation . Even in the cases where authentic sources were used, they were as
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“school like” as possible: they were used in ways that would not demand judgment
and evaluation, only the selection of bits and pieces of relevant information (for an
alternative way of using information sources see Bell, 2000). Real-time, detailed,
complex, and authentic data sources were utilized in ways that looked more like a
guided tour. Finally, Web 2.0 resources were only used to access authoritative information indicating an entrenched practice that avoids introducing real- life confl ict in
the classroom. Second, technology was exclusively used in order to provide authoritative information . There is a striking similarity here between this technology role
and the role “experiments” play in the “inquiry-scaffolding teaching method” (as
adapted in the teacher guide).
Overall, the ways teachers integrated technology in their instructional scenarios
do not show any signifi cant departure from established science education practices
in Greece . Considering teachers' backgrounds this was not expected as the
conditions for transformation were very favorable. Zhao, Pugh, Sheldon, and Byers
(2002) reported that one of the factors that might affect classroom technology innovations is the distance from existing practice. In our case this distance was relatively
short given the teachers' starting point. Consequently, the teachers did not have
much ground to cover in order to integrate technology in a transformative manner,
i.e., along the lines proposed by Osborne and Hennessy (2003).
Interestingly enough, not only was technology merely assimilated into existing
practices, but the teachers also refused to question their teaching practices and were
not open to suggestions along this direction, that is, despite the persistent efforts
from the authors to explicitly point out the limitations in the ways they had integrated
technology in their teaching in the practicum section. In retrospect, there are several
possible explanations for this type of resistance. First , as the participants in our study
were already accomplished teachers and researchers, they probably did not come to
the PDT thinking that they would need to radically transform their teaching practices , much less of course in science education which was their domain of expertise.
In all likelihood, they considered that such radical transformations of their teaching
conceptions and practices had already taken place in the course of their professional
histories. Second , resistance might be due to the fact that the teachers felt that the
level of ICT integration they had already achieved was part of the roadmap that other
teachers (that they soon would be mentoring) would have to pass through in order to
achieve more highbrow goals. In this sense, they were probably excusing themselves
from putting cognitive resources in the direction of further pedagogical experimentation. Third , it could well be that we have experienced a ceiling effect as the teachers
were already accomplished and there was no room for progress . Unlike other PDT
studies (e.g., Dwyer, Ringstaff, & Sandholtz, 1990 ; Levin & Wadmany, 2005 ;
Prestridge, 2012) in which the entry level of teachers who participated was that of an
“average,” “traditional” teacher, in our study the three teacher participants could not
be considered “average” or “traditional” by any measure. Finally , it could be that the
constructivist practices teachers had adopted might have been producing better
results in validated tests than the practices of the average Greek teacher who still
strives to meet the guidelines of current teacher guides. This means that they did not
have many reasons to feel “pedagogical discontentment” (Southerland, Sowell,
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Blanchard, & Granger, 2011) with the teaching model they were following. It is
likely that neither their former practices nor the PDT program succeeded in generating the “pedagogical discontent” needed for energizing the teachers' search for ICT
integration of higher quality.
In terms of the sustain-transform technology integration continuum, we argue
that the responses of the teachers to the in-service PDT marks the upper limit of the
possible range of technology integration , at least in the context of Greece. Our fi ndings suggest that the teachers in our study who were very advanced theory-wise and
already using fully compatible teaching practices could not go past a certain degree
of technology use, that of sustaining existing practices . Although there were no fi rst-
or second-order barriers, the highest level of integration reached was to use technology as a gateway to information and supplying visual representations . When
considering this against the dominant science educational paradigm in Greek education we see that neither of these uses suggests a transformation of teaching practices. Interestingly enough, the teachers did not signifi cantly modify their initial
designs, even after (a) trying them out in their classroom and (b) receiving feedback
from the authors which highlighted several limitations and missed opportunities for
adopting new technology-based practices. This suggests that their vision of technology integration did not go past the ways they were integrating technology in their

practices before attending the PDT. As our fi ndings indicate, a more substantial
level of technology integration, namely one that would go past information accessibility and visualization and move towards new teaching practices, is probably not
very likely even in the most favorable conditions, i.e., with teachers who have constructivist teaching philosophies and very high qualifi cations.
Implications
The fi ndings of the present study have important implications for in-service PDT on
ICT pedagogy. If accomplished teachers can only go so far after attending an extensive in-service PDT program such as the one described in this study, then one can
only wonder how far average teachers might go in terms of technology integration
so as to achieve the much desired transformation. Not only did the participants not
move past a given level of integration, but they also refused to consider other types
of technology use. If this is the upper limit obtainable by teachers who hold constructivist philosophies and are highly qualifi ed, how realistic is it to expect any
further transformation of teaching practices through technology? That is, if technology integration does not lead to teaching practice transformation in the most favorable conditions, as the ones described in this study, then perhaps the time has come
to rethink PDT programs in ICT integration .
PDT has come to the spotlight because of the importance of second-order barriers for technology integration. As second-order barriers are considered to be intrinsic to teachers, the focus that much of PDT literature puts on teachers is
understandable. In the end, the teacher is broadly acknowledged as the most critical
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mediating factor for classroom technology use (Ertmer, 2005). The present study
clearly indicates potential limitations of current PDT programs if our objective is
the transformation of educational practices through ICT. We argue that to address
these limitations we will need to reconceptualize the way we approach in-service
PDT programs on ICT pedagogy. As we see it, there are three issues pertinent to this
reconceptualization.
First, w e need to redefi ne what technology integration really means . We need to
be very explicit about the range of integration as well as its nature. For example, if the
objective is simply to integrate technology so as to enrich the curriculum, then the
ways the teachers in our study integrated technology in their lessons are exemplary.
From this point of view, current implementations of PDT programs can be very effective. However, if the objective is to integrate technology so as to change current
teaching practices in specifi c directions (such as to foster student-centered learning,
meaningful learning, problem-based learning), then the ways our participants integrated technology in their teaching practices are quite limited. It is imperative to
defi ne clearly what this direction actually is. In this sense, technology integration
would have to be explicitly described not only in terms of teaching practices but also
in terms of student learning and the crucial mediating role technology can play in
order to achieve this learning. Recently, other researchers have also called for a
reconceptualization of what it means to teach with technology and stressed the
importance of sketching out such a vision (e.g., Ertmer & Ottenbreit- Leftwich, 2010).
Second, we need to address other possible background variables that could infl uence the effectiveness of PDTs . For example, the presence of pedagogical discontent
(Southerland et al., 2011) that was mentioned above is such a variable that might
make a difference. Such variables need not be strictly personal. They may be constructs that are strongly determined by the context of teachers' practices. For example
even teachers with constructivist beliefs may not have the opportunity in terms of
time or available assessment instruments to test the limits of their current designs and
thus to experience pedagogical discontent. Or they may not feel psychologically safe
to try innovations because they do not have the administrative support to try out very
innovative designs in order to conceptualize and desire new goals for their students.
Finally, while a focus on the individual teacher is indispensable, we need to
broaden this focus to take into consideration not just the teachers themselves but
also the contexts in which they function . Ultimately, the "grammar of schooling"
(Tyack & Tobin, 1994) is very important as it is these contexts that shape teacher
beliefs and attitudes. Take for example the ACOT report conclusion, in which
Dwyer et al. (1990) argue:
Although the direction of change in ACOT classrooms is promising, the pace of change is
slow, for even when innovative teachers alter their practices and beliefs, the cultural norms
continue to support lecture-based instruction, subject-centered curriculum, and measurement-
driven accountability. (p. 2).
This clearly delineates the power current norms have in shaping teacher thinking
and consequently to teacher responses to ICT integration—even for innovative
teachers. The importance of the context of an innovation has been stressed (Penuel
et al., 2007 ; Starkey, 2010). Therefore, regardless of technology familiarity and
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constructivist beliefs about learning, the material conditions of actual practice (i.e.,
curriculum, legislation, high stakes testing, working conditions, resources) exert
signifi cant pressures on how teachers eventually come to view innovations in general and technology in particular. As it has been demonstrated, all these infl uences
might eventually shape an object of activity for teachers that is markedly different
to the one envisaged by educators, reformers, researchers, parents, politicians, and
other stakeholders (Karasavvidis, 2009). For the most part, PDT research has failed
to employ theoretical frameworks that take into consideration not only the teacher—
as the alleviation of second-order barriers clearly demands—but also other contextual factors that have the power to shape teacher thoughts and practices. Future
studies need to draw on theoretical frameworks that help conceptualize PDT programs in systemic terms so that the individual teacher no longer remains the focal
point of attention and the sole unit of analysis.
Conclusion
The current research addressed a gap in the current literature with respect to the way
teacher background properties such as expertise and qualifi cations might infl uence
the effectiveness of PDT programs. The fi rst main study fi nding is that even after
attending an extensive in-service PDT program, three teachers with constructivist
teaching philosophies and high academic qualifi cations integrated technology in
ways that sustained rather than transformed their existing practices. The second
study fi nding is that the teacher participants found it very challenging to consider
other types of technology integration that would be more on the transform end of the
sustain-transform continuum. As teachers who hold constructivist beliefs and have
high levels of qualifi cation are expected to exhibit the most favorable response to
PDT programs, this work raises serious concerns with respect to how far contemporary PDT programs can go in the direction of transforming teaching practices
through technology. Despite its main limitation, namely the small number of teachers who participated, we think that the present study contributes to delineating the
upper limit of technology integration that could be realistically expected from mainstream PDT programs. Further research in this direction should take the "grammar
of schooling" into consideration and carefully examine the shaping infl uences of
context on teacher beliefs and, consequently, on their responses to PDT.
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Introduction
Among various programs of technological equipment in schools, one-to-one programs (1:1) are those that neutralize the access factors. Massive computer endowments are considered by administrators, educators, families, and politics as a part of any major change in a school. However, the relative failure of One Laptop Per Child (OLPC) program (Warschauer & Ames, 2010) shows that personal computers have no pedagogical or didactical added value without other basic plans such as teacher training, reasonable number of students per teacher, resources, and electricity. In a conservative competitive system, like in Western countries, computer uses in classrooms are not raising learning outcomes more than ordinary technology. For the Maine (USA) project, Silvernail (2005) argues that students' results are not affected by computers because the ability to search and fi nd knowledge is not part of the institutional assessment which is based on learning by heart.
However, the way teachers rule the classroom is affected by technology. Windschitl and Sahl (2002) claim that some teachers tend to create a more constructivist pedagogy which is close to the evidence reported by Bebell and Kay (2010). If the teaching modifi cation mechanism is not clearly identifi ed, we suggest that ICT implementations may be a way of affi rming a pedagogy overhaul. For example, Karsenti and Collin (2011) report that in Canada, at the Eastern Township School Board, technologies were part of an institutional renewal strategy in a similar way to
ICT Use in Secondary Education: Schooling
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the Grimes and Warschauer (2008) studies on a one-to-one laptop program in California. An example of this phenomenon is the emblematic adjudicate youth program realized with Seymour Papert (Cavallo, Papert, & Stager, 2004) or the Apple Classrooms of Tomorrow project (Dwyer, Ringstaff, Haymore, & Sandholtz, 1994).
In this perspective, achievement of ICT, as reported by Zucker and Light (2009), can be seen not only by direct effects of ICT on cognitive activities but also through a general pedagogical and didactical improvement, where computers constitute the visible part of the iceberg. Beyond schools saturated with technology, teachers' testimonies in regular lower secondary schools are central for understanding what are the key features of a sustainable and an effi cient use. In the Norwegian context, Wikan and Molster (2011) report that almost two-thirds of their sample were using ICT once a month, similar to France (Khaneboubi, 2007). In Sweden, on the basis of a focus group methodology, Erixon (2010) claims that insuffi cient access to computers is one of the most important reasons for not using ICT during classroom. The most frequent uses of ICT by teachers in secondary schools are for sustaining motivation and search, as pointed by Wikan and Molster (2011). Drent and Meelissen (2008) in the Netherlands claim that early adopters are the "anchor point for stimulating the innovative use of ICT in education." In particular, those teachers are well socialized and technically skilled in regard to their pedagogical needs.
Jimoyiannis and Komis (2007), in a quantitative study with Greek teachers, report that personal factors such as subject matter, teaching experience, or gender are associated with their perceptions about ICT in education. In England, Hennessy, Ruthven, and Brindley (2005) report that commitment in ICT activities is tempered by the infl uence of external constraints.
In France, after active national policies supporting the training of teachers for integrating technologies into their school practices, an approach of new artifacts has been developed, conceiving them as tools (Baron & Bruillard, 1996). Since the 1980s, this approach still dominates. Technology endowment programs in the French schools continue to be accumulated without necessarily being accompanied by national plans for teacher education. Furthermore, mandatory uses of ICT in the curricula are very sparse and decisions about curricula tend to reserve computing into specialized courses. The French one-to-one programs (Jaillet, 2004 ; Khaneboubi, 2010 ; Rinaudo, Turban, Delalande, & Dhana, 2008) show a weak state support in regard to the local authorities' implication.
In this chapter, we report a study conducted in 2012 following a fi rst work made in 2010, in four lower secondary schools where teachers and students have a relatively better access to technology and digital resources than in other French schools. Three of them were involved in a national state program on digital textbooks; they had been endowed in 2010 by a local state authority called "académie." Consequently, they were also involved in a series of programs that aim at identifying the conditions and limitations of ICT implementation. The disciplines mostly concerned were mathematics, social studies, literature, and technology and, occasionally, foreign languages and biology. We seek to understand dynamic and evolutions by taking "pictures" at two different times and, according to the actors, trying to retrace what happened.

The purpose of our survey, partially funded by the local state authority, was to get a better understanding of the evolution of ICT uses in the everyday teachers' professional running pattern in order to identify propitious and obstacle factors. To what extent computers are a signifi cant tool in the classrooms to achieve a teaching task? Do teachers have an equal access to equipment? What is the infl uence of organizational factors on teachers' utilizations of technologies?
Data Collection Methods
The investigation method has been guided by the framework made by Fulton, Glenn, and Valdez (2003 , 2004), who draw a list of necessary elements propitious for teaching with ICT. This framework has been adapted to the French context; seven factors were defi ned as follows:
- The role of the leadership in the school management: Is there a team for piloting ICT projects in the school? What is the importance of ICT in the general school project?
- The roles assumed by teacher(s) responsible for ICT in the school: In each secondary school one or several teachers are discharged for a part of their teaching duty to help and conduct pedagogical use of ICT among his or her colleagues. Very often, this function is allocated to early adopter as described by Drent and Meelissen (2008).
- The manner in which the support and maintenance are performed: In the French context, the national state is responsible for teaching contents and human resources, while authorities on buildings, equipment, and maintenance are devolved to local administrations such as territorial councils and city councils.
- Financing from local communities, state local authority, ministry ....
- State policy on ICT as well as local policies.
- Different support settings for teachers of different disciplines in their use of ICT: Training, guidance, meeting ....
- School position in the environment and links with parents and with local associations.
In the next section, we describe a preliminary study that has nurtured and served the construction of the study conducted in 2012. They are both based on Kathleen Fulton's framework.
The 2010 Pre-study
In 2010, we sought to get a better understanding according to the methodological elements explained in the previous section by performing 30 interviews with innovative teachers and management teams (Bruillard et al., 2011), following an

ethnographic perspective as documented by Woods (1996). For each middle school, we have conducted interviews with:
- The management team
- Teachers responsible for ICT
- Teachers-librarians
- Teachers who use digital textbooks or any kind of ICT
The interviews focused on the school characteristics, teaching experience of the teachers responsible for ICT and how they managed these responsibilities, organization of the endowment program in digital textbooks, and changes in teachers' practices related to these new means as well as those occurring due to other technologies available. We did not meet all the teachers, and only ICT users. Interviews showed disparities in the access to the computer room, and the Internet weaknesses were not perceived in the same way by all teachers. Furthermore, it was often suggested that all teachers were not equal in front of technical and educational assistance.
Literature teachers in particular (often women) seemed unaware of the existence of some equipment. Teachers responsible for ICT said that they did not have time to perform maintenance on all computers. For this reason, there was a high probability that computers used by teachers less close to the management team or in conflict with it did not get the same comfort of use as other teachers. Therefore, it seemed fruitful to test the hypothesis that all teachers do not have the same possibilities in front of infrastructure and equipment deficiencies and that technical assistance is not distributed equiprobably: patterns according to gender, age, or subject taught were potentially observable.
The 2012 Study
Since the 2010 study focused on innovative teachers in a qualitative way, in 2012 we sought to reach a greater number of teachers with a questionnaire in order to get a clearer outline of the dynamics at work in the schools and the possible new (declared) practices in the classroom and for preparation. Therefore, in 2012, interviews were conducted in three middle schools with eight teachers and managers and a questionnaire was filled by 89 teachers in four schools (Beauné, Khaneboubi, Tort, & Bruillard, 2013). This amount of exploitable questionnaires represents approximately 45 % of the teacher target population. Among the five schools participating in the first survey, one school was left out because of the difficulties encountered during the initial investigation.
Schools A, B, C, and E are located in suburban areas near Paris. School D is part of a rural area. Establishments A, B, and C are located in high-poverty area. However, these schools have a relatively large equipment and are known for their significant use of ICT in teaching and learning activities. Each one hosts approximately 500 students and 50 teachers and has more than 100 computers distributed in computer labs, libraries, technology classrooms, etc. There are at least one

computer lab and two sets of laptop per school. All the classrooms of the sixth and seventh grades are equipped with a computer and a projector in most cases. In three schools, all rooms are equipped with a computer. However, Internet and network access are not reliable in most places. The stability of the teaching staff such as managers' departures, teachers responsible for ICTs' departures, or major changes in the teams of teachers is not accomplished. In two institutions, an important and frequent turnover has been reported.
In short, in 2012 a quantitative and a qualitative study was made to complete a 2010 work. The first round of interviews made in 2010 was the foundation of our quantitative survey of 2012. Interviews were conducted with administrators and teachers responsible for ICT in three schools (A, D, and C); observations and informal interviews were conducted in the staff room for two schools (A and E). The interviews focused on the changes that occurred between 2010 and 2012. A particular issue was to establish a balance after 2 years of using digital textbooks. We seek to document the changes in the practices that have been observed between the two periods of data collection. When it was possible, interviews were conducted with managers and teachers responsible for ICT as in 2010. We have managed to conduct interviews with the same people or with those who occupy the same function in our school sample.
In those schools which get significantly more equipment than average schools, new equipment is not so frequent. No major changes in the equipment have been reported. Between 2010 and 2012, the main technology that entered the French school was interactive tablet. Less than 20 were endowed in one college of our sample. The weakness of the Internet access has not evolved. Analysis of speeches from managers and teachers responsible for ICT permits a comparison of evolution, a description of the inertia in the teachers' team involvement, and the dynamics at work in the schools.
Teachers' Questionnaire
The questionnaire fulfilled in 2012 is a part of a longer term project. We have sought

to build a database during several years in order to make comparative studies and
report experiences of teachers while there are technological developments. We also
aim at sharing our raw database. For now it is possible to present how the various
factors that can be found in the literature are combined in our context.
Based on the interview analysis of the 2010 study, a fi rst draft of the questionnaire
was tested in one middle school and one high school in another region. An anonymous four-page questionnaire (shown below in Appendix) was stabilized, with
about 60 questions grouped into six themes linked to ICT practices:
· Personal and professional information (status, discipline, experience background …)
· Mode of the course preparation (frequency uses of software, Internet, digital
textbook, etc.)
· Classroom activities (frequency uses of equipment in and out of the classroom,
student activities with ICT …)
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· Student assessment activities (where, how, what …)
· Classroom preparation activities (communication with administrators, colleagues,
parents …)
· Usage at home (personal equipment, social network uses…)
A total of 89 questionnaires were collected. We stayed in the teacher room in
four schools (A, C, D, and E) for 2 days on average, and we proposed to every
teacher we met to fi ll the questionnaire. In the French context, the participation of
volunteers is unavoidable to sample our target population, and other sampling
method is not possible in lower secondary schools. The sample is not as important
as we expected, but the collection rate is similar to other studies using the same
method (Khaneboubi, 2007 , 2010). We collected between 19 and 25 questionnaires
in each school among teachers who agreed to answer. Three half-days were required
for school A, two full days in school C, and one half-day in school E. In school D,
questionnaires were given to the teacher responsible for ICT who assured the distribution and the collection of questionnaires. In the four schools considered, a total of
89 questionnaires were collected which represents approximately 45 % of our target
population.
How to estimate the characteristics of individuals who have not responded to the
questionnaire? First, the sampling of volunteers excluded teachers who were not
favorable to ICT. Frequently, surveys on technologies tend to "attract" users or
teachers favorable to ICT and to "scare" the teachers less comfortable with technology. Then, for three schools, only teachers who attended the staff room were likely
to respond to the questionnaire. Teachers who have a disciplinary room where they
spend break times are excluded. The disappearance of smoking rooms has also
excluded smokers from the sample.
Survey analysis has been made by performing univariate summaries and chi square
test. Main ideas expressed during the interviews were crossed with questionnaire
data. A regression logistics has been computed to describe three variable interactions.
Open questions have been recoded in categories. On this basis, occurrences of the
category have been graphically represented in barplot with a package implemented in
the software named R (R Development Core Team, 2011). Translation has been made
for Figs. 3 and 4 by searching for the best analogy between French and English.
Sampling Presentation
The questionnaire required 20 min to be completed. This is a relatively long time
that demands concentration and, like all surveys of this type, a form of intrusion into
the professional privacy of respondents. Non-responses are more frequent toward
the end of the questionnaire than at the beginning. They are also more frequent for
open questions and tables. Finally, the format of the open questions includes impassable ambiguities. For example, if teachers distinguish Word and Excel , they do not
distinguish Writer from Calc and only declare " Open Offi ce " or " Libre Offi ce ."
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Those 89 usable questionnaires were fi lled by 57 % of women and 43 % of men.
Three-quarters of the sample had a class of sixth grade and 80 % of ninth grade.
As shown in Fig. 1 , teachers aged over 40 years are signifi cantly fewer than the
age group 25-40 years. More than half of the sample is made up of teachers between
25 and 40 years. Most of the staffs of school C are less than 40 years old, while
school D shows greater dispersion. Distribution in age groups is consistent with the
population of teachers in the direct environment because those schools are located
in an area which welcomes fresh tenured teachers.
As shown in Fig. 2 , we found in the sample 17 % of people who teach mathematics,
13.5 % literature, 11 % social studies, and 18 % foreign languages. Proportions of the
disciplines represented in the sample are consistent with the students' schedule discipline. There are more women who teach literature and foreign languages than mathematics and gym. As for age groups, the gender distribution is consistent with the trends
of the population from which the sample is coming from.
Respondents appear to be quite familiar with the Web: 46 % have a personal
account on Facebook, and 63 % report frequently using technologies; only seven
of them have a personal web page, four have a blog, and three have a Twitter
account. Only 15 % consider themselves advanced, while 79 % feel that they
have a beginner or an intermediate level. Men outnumber women to report an
advanced level.
25
0
5
10
20 30 40 50 60
count
Age groups
Schools
A
D
E
C
15
20
Fig. 1 Age repartition among middle schools
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Results
ICT in the Classrooms
As shown in Fig. 3 , teachers' main activities in classrooms are search for information , exercises , dynamic geometry , and screening activities . Seventy-eight percent
of the sample report that they ask their students to do information search. Among
them, 57 % state that they require students to do it at home, 44 % at the school
library, and 37 % in the computer lab.
However, these activities have only a relative importance for students with only
34 % who declare assessing activities with ICT, including 16 % for literature and
3 % for geometry. Usage in lesson by students is modest: only 46 % report having
made one or several students use the computer. Those elements reveal that the
importance of ICT in the classroom is humble in regard to teaching progressions.
This does not mean that technologies are not part of the teaching system.
Seventy-two percent of teachers claim to use computers in their classroom almost
every day, but the observations made in the staff room tend to show that it is probably mainly for administrative purposes. Only 34 % use the computer lab once a
month or a little more and 11 % once a week. As shown in Fig. 4 , in the classroom
the Internet is the most widely cited and then comes offi ce suites and dynamic
geometry application.
0
math
literature
english
gym
soc-stud
biology
other
physic
spanish
techno
music
arts
german
5
10
count
Disciplines
men
women
15
gender
Fig. 2 Discipline distribution in regard to gender
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Fig. 3 Classroom activity citation occurrences
Fig. 4 Software and applications used during the classroom citation occurrences
Classroom Preparation with ICT
On the other hand, usage for classroom preparation appears large and stable.
In preparing the course, technologies are particularly sought. Professional
web sources are preferred when researching information for setting the course.
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However, software packages are more important than disciplinary software, except
for dynamic geometry application.
Eighty-three percent of the sample report seeking information frequently to
prepare their classes. Most viewed websites were fi rst institutional websites, then
teachers' personal websites, and, third, professional association websites. YouTube ,
Wikipedia , and Google are much less cited. There is a signifi cant link between the
use of digital textbooks to prepare lessons and equipment reliability and availability.
In sum, when the material's availability is medium or low reliable, teachers are
signifi cantly less likely to report using the digital textbooks to prepare their classes.
The preferred instrument for the preparation of courses is the word processor:
87 % reported using it more than once a week and 62 % almost every day. Thus, 43 %
reported using an image-processing software more than once a month. Disciplinary
software mostly used in the preparation of lessons are free dynamic geometry
software: Geogebra , Geospace , Geoplan , Cabrigeometre , and Sesamath .
1
In general,
there is a predominance of proprietary software. For example, MSWord is the most
cited word processor with 24 occurrences whereas Libre Offi ce and Open Offi ce
come with 14 occurrences. Overall, free and open-source software are present but in
a secondary place.
Technologies appear mostly used professionally by teachers to communicate
with each other: 76 % claim that they use ICT to write to their colleagues and 58 %
with management and administrators. Seventy-four percent use the online homework notebooks. We fi nd the same proportions for the application counting the
missing students as 74 % reported. In addition, 95 % reported using the online
application for entering assessment results.
Infrastructure Issues
It is essential to enlighten some contrasts between the discourses of the administrative teams and those of the teachers, noted in the staff rooms. Administrators tend to
report that things are going well or that practices with ICT are more frequent,
whereas in the teachers' breakroom, a lot of claims are expressed: about delays in
digital textbooks' delivery, classrooms that are not equipped with interactive whiteboards, network dysfunctions, problems related to software for managing the school
life, equipment maintenance, etc. In two schools, the replacement of teachers
responsible for ICT had important implications: several remarks evoke responsibilities that are going beyond the scope of the teachers responsible for ICT in terms of
time, remuneration, and sometimes also skills.
1
Sesamath (www.sesamath.net) is an association aiming to spread under a free and open-source
license documents and educational software for mathematics.
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We noted that changes had occurred in the management teams between
the endowment and the 2012 study. A new head teacher had taken a position opposite to that of his predecessor and did not provide specifi c information about the
continuity of the endowment program. Similarly, two teachers newly responsible
for ICT said that everything was going well overall, thanks to a continuous handover.
In contact with the ancient responsibles, they also evoked a solid foundation but
they declared that it is diffi cult to make as well as them. They also said that they
refuse certain tasks or that they are not qualifi ed for it, such as setting up computers
to access and deploy new textbooks.
The lack of technical support given by institutions appears as a source of discontent among teachers responsible for ICT. Some data from the questionnaire show
this phenomenon. First, teachers responsible for ICT are sought for technical support almost exclusively: as 34 % stated, and only between 7 and 12 % indicated
requesting them for a pedagogical dimension which is their main assignment. This
result is probably linked to the departures of experimented teacher responsible for
ICT in two schools. Secondly, to the question "Do you give help to someone using
ICT in education?" 24 % responded "yes," including 21 % who stated that it is
for their colleagues. Thirty-four percent of respondents indicated asking their colleagues for help and educational aspects on the technical aspects.
During interviews, infrastructure access has been highlighted as a determinant
factor on practices. In these quite well-endowed schools with technological equipment, organizational and practical diffi culties in the management of computer fl eet
appear. These problems hinder the development of ICT usage: few signifi cant evolutions in terms of classroom practices are declared despite the tenacity of teams
and the almost intact motivation to seize the means in helping their students
succeed.
Access to Equipment
Nine out of ten teachers reported having a computer in their room, a projector,
and a computer room in the school. Fifty-two percent indicated that the material
is highly available and 39 % medium or low. Forty-fi ve percent indicated that the
equipment is very reliable, while 38 % described it as slightly or moderately
reliable: it is mainly Internet connections that are not robust and computers
obsolete.
Evidence shows that the equipment is not perceived as available in the same way
for all teachers: gender and seniority appear as discriminating factors in the perception of this availability. By performing a chi-square test in Table 1 , we found that
female teachers are signifi cantly more likely to report that the material is medium to
low available whereas men teachers are signifi cantly more likely to respond that the
material is available (chi-squared = 4.2286, p -value = 0.03975).
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By performing a logistic regression as described by Dalgaard (2004) we can
claim that in our sample, with an equal seniority, men have 3.2 greater chance to
declare equipment "very available" than women.
Seniority in the profession is also an issue to access equipment: teachers recently
tenured are less likely to agree that the material is available. In three schools, interviews show a difference of treatment between old and new teachers; older teachers
often have their own classrooms and signifi cantly more equipment than new teachers. The difference among the old and the new teachers can be a source of tension
within the teams.
Discussion
The purpose of our survey was to get a better understanding of the dynamics of ICT

uses in the everyday teachers' professional running pattern in order to identify propitious and obstacle factors. We considered, in this research, an initial situation and
the evolution of this situation of four schools that were involved in a national endowment program on digital textbook. These schools are better endowed with technology and digital resources than the other French schools. Interviews were conducted
in three middle schools with eight teachers and managers, and a questionnaire was
fi lled by 89 teachers in four schools.
According to our results, we can underline that in those relatively favorable
contexts for technology uses, ICT is not a major concern for teachers. Above all,
leading pupils to the exam success is the main goal, and technologies are not perceived as a reliable medium to do so. The departures of teachers locally responsible
for ICT impacted the coordination and the development of ICT's uses in teachers'
practices, even if this could appear as a local and punctual need. Teachers who
responded to our questionnaire used technology as a vehicle for their cultural practices in consistence with their age groups (Donnat, 2009) and professional affi liation (Bourdieu, 1984).
For investigators, in order to engage in these schools, create trust, and approach
a global vision, it is necessary to implement a long qualitative research based on
ethnographic methods and to combine it with a questionnaire survey. Two years
after the fi rst endowment in these schools, the managers' interest was not the same
as in the fi rst investigation and, thus, access to the schools was made more diffi cult
for collecting data. The complexity of the phenomenon we have sought to characterize only allows modest interpretations. It would be necessary to test further our
Table 1 Equipment
availability based on gender
Medium or low availability High availability
Women 24 21
Men 11 25
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results in other contexts and to investigate the fi eld with a much more qualitative
method. This study attempts to outline elements that are prominent in our methodological context, but it deserves a more ambitious continuity.
For teaching purposes, offi ce software are cited more frequently than disciplinary software. The main activity with the technologies is information search, but it is
mainly requested to be done outside the classroom. This result concurs with Wikan
and Molster (2011), who report that the use of ICT is mainly for sustaining motivation and for resources. Uses of a computer with a projector in the classroom remain
as one of the most important elements in the practice. We can't identify an important use of technology for assesment. The use of technologies by teachers in the
classroom seems to be determined by the infrastructure's quality and the equipment
reliability. Gender and seniority are discriminating factors in the availability perception of technologies. This element is convergent with what Jimoyiannis and Komis
(2007) have reported on gender.
On the other hand, usage for classroom preparation is important. Training and
educational support are essentially made by peers, and institutional support is
almost exclusively sought for technical reasons. As Erixon (2009) shows, insuffi -
cient access to computers is one of the most important reasons for teachers not using
ICT in their teaching. We can presume that teachers did not have an equal access to
computers. In our context, it is highly possible that technology is one of the "apanage" of individuals with a good legitimacy and a signifi cant weight in their establishments. Thus, a male teacher with seniority in his school will probably have a
better access to the equipment than a new young female teacher. To confi rm this
trend, it would be necessary to examine the possible interaction between the taught
discipline and the availability of equipment which is not permitted at this time with
our number (89) of respondents.
Leader postures on ICT and animation are a key feature pointed by Drent and
Meelissen (2008). As in a one-to-one project, teachers' use of ICT is revealing a
major issue of the general professional context. In all professional fi elds, seniority
is an indicator of legitimacy (Bourdieu, 1999). Moreover, disciplines do not have
the same prestige or importance. Other factors such as gender, diploma, and institutional status are a source of disparity. Depending on the school history and traditions, the ambience may be created in several ways. In other words, institutions can
promote achievement against archaism such as gender equity vigilance and
generational diversity program. From those impulses, a general organization of the
day-to- day work is collectively made and supported by the chief administrator. This
could be an extension of this work to characterize these elements in a long temporality and to link those evolutions with changes in the use of ICT.
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Appendix: Teacher's Questionnaire
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Introduction
The Internet provides quick and easy access to a large variety of information
resources and tools and therefore has the potential to support instructional
approaches that are consistent with current learning theories and educational reform
efforts, such as inquiry-oriented learning (Schofi eld, 2006). A typical characteristic
of these approaches is that learning activities evolve around open-ended questions
and ill-structured problems that are relevant to the real world. Activities also involve
student interaction and collaboration and require the use of, as well as promote the
development of, high-level cognitive and self-regulatory learning strategies.
Research has shown, however, that web-based learning, and online inquiry in particular, is very demanding and challenging both for the students and for the teachers.
Typically, students do not have suffi cient prior knowledge and fully developed cognitive skills to grapple with new subject matter while carrying out online inquiry
tasks (Kupier, Volman, & Terwel, 2005 ; Quintana, Zhang, & Krajcik, 2005 ; Zhang
& Quintana, 2012). Also, teachers need to be able to design learning activities that
engage students in meaningful learning, to select and provide appropriate resources,
and to possess a large repertoire of pedagogical techniques so as to effectively support their students (Wallace, 2004).
What do teachers need to know in order to design and implement web-based
instruction that utilizes the wealth of information resources available on the web in
ways that enhance student learning? The present study addresses this question
through an analysis of the lesson plans submitted by preservice teachers after attending an introductory course on the educational applications of ICTs. The study aims
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at examining preservice teachers' understandings of Internet use in the classroom,
so as to identify needs that should be addressed in teacher preparation for ICT
educational integration.
Literature Review
The Internet provides teachers with access to a wealth of information and a variety
of representation tools which they can use to improve teaching and learning. When
using it as an information source learners can fi nd content represented in multiple
visual formats, including photos, diagrams, graphs, videos, and animations, having
thus more opportunities to connect ideas and to construct rich mental models of the
concepts and phenomena they study (Moreno & Mayer, 2007 ; Osborne & Hennessy,
2003). In addition, the Internet can enable learning experiences that otherwise
would not be feasible inside the classroom. Interactive representations of content
such as interactive models and simulations enable students to test hypotheses,
observe patterns and relationships in data, and draw their own conclusions (Osborne
& Hennessy, 2003). Students can explore questions of personal interest and importance by accessing authentic sources (e.g., data archives of governmental organizations, museum collections, news archives of TV and radio stations) and real-time
data (e.g., satellite maps, reports on recent earthquakes) (Schofi eld, 2006).
The above make the Internet an ideal information resource for inquiry-oriented
approaches to learning (Schofi eld, 2006). Inquiry learning is hard to defi ne because
there are many conceptualizations and approaches to inquiry instruction, even
within the literature of the same fi eld such as science education. However, all of
these approaches share three critical elements: they evolve around authentic, complex questions, require student active engagement, and include at least some part of
the investigation cycle (question generation, study design, data collection, drawing
conclusions, and communicating fi ndings) (Anderson, 2002 ; Minner, Levy, &
Century, 2010). The Internet can support student inquiry in multiple ways. Students
can utilize the wealth of internet resources to ask open-ended questions about real-
world issues that are more interesting and meaningful to them. They can collect
real-time and authentic data as well as design and carry out virtual experiments to
test theories and predictions. They can engage actively in knowledge construction
because they have to interpret, organize, compare, and integrate all this information
to draw their own conclusions from their investigations.
While the Internet is an incredibly valuable resource that can support teaching and
learning in diverse ways, its use yields many challenges for both teachers and students. Information on the Internet is unstructured and unfi ltered. Inexperienced users
can get lost or distracted while navigating their way through hypertext and may come
across content that is not credible, relevant, or age and task appropriate. Users need
to know how to perform productive searches, to identify and select the right sources,
to evaluate and make sense of content that is presented in multimedia formats, and to
coordinate different information sources. Research shows that fi nding relevant and
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credible information online is a time-consuming and diffi cult task for young students,
who have several misconceptions about the nature of the Internet, do not understand
fully how search engines work, and lack suffi cient prior knowledge to come up with
good keywords and to select relevant websites from the list of their search results
(Julien & Barker, 2009 ; Kupier et al., 2005 ; Wallace, Kupperman, Krajcik, &
Soloway, 2000). As a result, in online inquiry-oriented activities students tend to
spend much of their time searching for information than engaging deeply with content (Wallace et al., 2000 ; Walraven, Brand-Gruwel, & Boshuizen, 2009 ; Zhang &
Quintana, 2012). Also, not only elementary students but also adolescents and many
adults are not able to discriminate between reliable and unreliable information, and
they often use insuffi cient and superfi cial criteria to judge the trustworthiness of
information, such as the appearance of a website, the amount of content, and the presence of pictures (Kupier et al., 2005 ; Walraven et al., 2009 ; Wiley et al., 2009).
In addition to the challenges regarding information search and evaluation, students face several diffi culties in their efforts to process the information they fi nd,
especially when they have to coordinate facts and data from multiple sources (Rouet,
2006). Due to the hypertext structure of web documents, comprehending Internet
text is a cognitively demanding activity that requires prior knowledge on website
structures and a range of self-regulation strategies, because readers in a sense compose their own texts through the choices they make on what to read and which links
to follow (Coiro & Dobler, 2007). Another challenge is the multimodal nature of
most online texts. Students often lack "visual literacy" skills to interpret and extract
important information from graphs, diagrams, and other complex visual displays as
well as to synthesize information from verbal and pictorial presentations of content
(Hannus & Hyönä, 1999 ; McTigue, 2009). Finally, integrating information from
multiple sources requires readers to evaluate the trustworthiness and the nature of
each source (e.g., fi rst-hand data vs. a report), corroborate information across
sources, and combine information from all sources to create a coherent whole
(Rouet, 2006). Young students tend to follow the easy way to get answers to open-
ended questions. They hope to fi nd a website that provides "the right answer" and

do not spontaneously engage in evaluation, selection, and synthesis of information
from multiple sources to draw their own conclusion (Wallace et al., 2000 ; Walraven
et al., 2009). In general, students are good at well-structured tasks, requiring the
retrieval of specifi c pieces of information from given websites, but not as successful
at ill-structured tasks that involve navigation, collection, and synthesis of information across multiple sites (OECD, 2011).
Web-based instruction is very demanding for the teachers because most websites
are not designed for teaching and learning purposes. Leaving aside the textbook,
which provides some structure for instruction, and working with the Internet require
extensive planning on teachers' part. Teachers need to search and select online
resources that meet the needs of their students, to transform them into instructional
materials, to design learning activities that are aligned with the curriculum, and to
build curricular coherence so that students "experience a reasonable progression of
ideas" from one activity or website to another (Wallace, 2004). Giving students
open-ended research assignments may require less preparation but places excessive
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demands on teachers during lesson enactment in the classroom. They need to
monitor the work of their students, whose research activities typically develop into
completely different directions as each group or individual comes across different
content, and to provide them with extensive scaffolding due to the diffi culties they
face with the search, evaluation, and critical synthesis of the information they fi nd
(Wallace, 2004 ; Zhang & Quintana, 2012).
Although school access to the Internet has become widespread in many countries, research shows that the Internet is underused in the classroom or it is used in
ways that do not utilize its full potential to enrich student learning (Schofi eld, 2006).
This is in line with research on ICT educational integration in general, showing that,
not only in Greece but also in other developed countries with higher ICT educational adoption rates, many teachers have not embraced ICTs, while most of those
who have integrated technology into their teaching have not employed it in ways
that support critical and active learning (Jimoyiannis & Komis, 2007 ; Osborne &
Hennessy, 2003 ; Webb & Cox, 2004). According to relevant empirical studies and
literature reviews, the problem is not so much teachers' lack of ICT skills but mostly
their lack of knowledge about how to teach with ICTs (Jimoyiannis & Komis, 2007 ;
Osborne & Hennessy, 2003 ; Webb & Cox, 2004).
Scholars have drawn on Schulman's idea of pedagogical content knowledge
(PCK) and proposed the notion of PCK of educational technology (Margerum-Leys
& Marx, 2002) or technological pedagogical content knowledge (TPCK) (Angeli &
Valanides, 2009 ; Mishra & Koehler, 2006) to describe the knowledge sources that
teachers need in order to successfully integrate technology into their teaching.
According to the various conceptualizations of TPCK (for a review see Voogt,
Fisser, Pareja Roblin, Tondeur, & van Braak, 2012), teaching with ICTs requires not
only technological knowledge, content knowledge, and pedagogical knowledge, but
also new, complex forms of knowledge that develop from the integration of knowledge about technology, pedagogy, and content. These other forms involve knowledge of the role of particular technologies in teaching and learning (technological
pedagogical knowledge), effective pedagogies for teaching specifi c content (PCK),
affordances of certain technologies for representing specifi c content (technological
content knowledge), and how to teach specifi c content with technology in a specifi c
context (TPCK). TPCK is a unique body of situated knowledge that develops when
teachers have to transform content, pedagogical, and technological knowledge so as
to solve problems regarding the use of technology as a teaching and learning tool in
various contexts (Angeli & Valanides, 2009 ; Benson & Ward, 2013 ; Mouza &
Karchmer-Klein, 2013).
Studies on the instructional use of the Internet as an information resource show
that both in-service and preservice teachers feel ill prepared to teach with the
Internet and express the need for web-related pedagogy, particularly for selecting
appropriate resources and tools, for designing activities in which these resources are
used creatively by the students to promote deep learning, and for techniques to scaffold the students (Childs, Twidle, Sorensen, & Godwin, 2007 ; Lee & Tsai, 2010 ;
Wallace, 2004) However, there is little research about what it takes for teachers to
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teach effectively with the Internet and what they need to be taught to inform teacher
education practice. The present study looked at preservice elementary school teachers' fi rst attempts to design a web-based lesson, in which the Internet was used as an
information resource, in order to identify challenges, misunderstandings, and learning needs that should be addressed in teacher preparation for educational technology integration. Lesson plan analysis addressed the following questions:
1. What learning activities did preservice teachers include in their lesson plans?
2. In case when they selected and provided specifi c Internet resources to their students, what were the characteristics of these resources?
3. What guidance and support did they consider providing to the students, through
both the materials they designed (e.g., student worksheets) and the specifi c techniques they were planning to employ in the classroom during instruction?
Method
Participants and Context
Participants were 30 preservice teachers who were enrolled in a 4-year undergraduate teacher education program in Greece during the spring term of 2009. At the time
of the study all students were attending an introductory course in educational technology that was offered in the third year of the program. The purpose of the course
was to familiarize students with educational software programs that were approved
by the Ministry of Education and were available in public schools for the teaching
of elementary school subjects. More specifi cally, students were expected to develop
knowledge and skills for evaluating and selecting educational software as well as
for designing learning activities that involved the use of educational software.
All course sections were led by the author. Instruction took place in a computer
lab and involved lectures, group work, and discussions. Students explored a variety
of software programs that differed in their pedagogical design, ranging from multimedia encyclopedias, tutorials, and behaviorist drill and practice programs to constructivist software designed to support open problem solving and inquiry-oriented
learning in various subject areas. Typically students collaborated in small groups to
examine an educational scenario involving the use of a particular software program
and were guided with worksheets to complete specifi c learning activities by undertaking the role of elementary school students. These scenarios and worksheets were
either part of the software packages or were developed by the author. Educational
scenarios employed a variety of student-centered, constructivist approaches to
learning, such as problem solving of open geometry problems, a science teaching
sequence designed to promote conceptual change through hypotheses testing and
experimentation with a simulation, and inquiry-oriented learning in history and
geography where students had to investigate open-ended questions by organizing,
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interpreting, and synthesizing information presented in verbal and various visual
formats. The worksheets that were used in the context of these scenarios modeled a
wide range of student scaffolding techniques that have been discussed in the literature (Puntambekar, Stylianou, & Goldstein, 2007 ; Quintana et al., 2005 ; van de Pol,
Volman, & Beishuizen, 2010), such as simple illustrated instructions on how to use
particular software features; prompts that recommended specifi c steps or requested
students to provide hypotheses, predictions, observations, and conclusions; as well
as prompts and tables for organizing information and drawing conclusions. Lectures
and discussions focused on the pedagogical design of each software program, its
affordances and the instructional approaches it could support, the potential diffi culties that elementary students might encounter while learning with it, and the scaffolding they would need to successfully meet the learning goals of the lesson. Also,
through discussions focusing on the analysis of each educational scenario, students
were encouraged to make connections between curriculum learning goals, pedagogy, and characteristics of the software.
One of the 3-h course sessions, which was structured in a similar manner, focused
on web learning. After an introductory lecture that addressed the challenges of web
learning for elementary students as well as the areas in which they need pedagogical
support, preservice teachers examined two educational scenarios (one designed by
the author and one found online) that introduced them to a variety of online information resources and some of the ways these resources could be utilized in instruction. In these scenarios students explored specifi c research questions using
pre-selected web resources, guided with worksheets. However, the scenarios differed in their pedagogical approach. The fi rst modeled an inquiry-oriented approach,
as it requested students to draw their own conclusions by organizing and comparing
information from multiple sources, including authentic data, while the other scenario requested students to answer a series of factual questions using a variety of
sites that provided predominantly encyclopedic knowledge. Preservice teachers
then compared the two scenarios focusing on the types of learning they promoted,
the affordances of the information resources, potential challenges for elementary
school students, and the types of support that was provided through the student
worksheets.
One of the course requirements, to be fulfi lled at the end of the course, was the
design of a 2-h lesson involving the use of ICTs. Students were provided with fi ve
options (i.e., using a specifi c software program to teach a specifi c curriculum topic),
one of which involved teaching an earth science topic using information resources
from the Internet. The topic, which was included in the sixth-grade geography curriculum, was entitled "Natural disasters and their impact in human lives" and
addressed the causes and the impact of earthquakes and volcanoes as well as the
phenomena of erosion, weathering, and deposition. Preservice teachers were provided with detailed instructions on what to consider in designing their lesson plans
(see section "Data" below) and were expected to use pedagogical approaches and
techniques that were modeled through the scenarios that they had explored in class
during the semester. Of the 116 undergraduate students who completed this requirement 30 designed a lesson that involved teaching earth science with the Internet.
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Data
The data of the present study consist of the lesson plans submitted by the 30 preservice teachers who chose to design a hypothetical lesson in which sixth-grade students were going to use learning and informational materials on the web. Following
the instructions that were provided, in these lesson plans preservice teachers had to
present the learning goals of the lesson; discuss the knowledge and skills as well as
the misunderstandings and diffi culties that sixth-grade students might have relative
to the content and the use of online materials based on the literature (i.e., relevant
articles and lecture slides that were included in course materials); explain in detail
how they would carry out the lesson and which teaching methods and techniques
they would use to support their students so as to attain the goals of the lesson;
describe how they would assess their students; and, fi nally, provide all the materials
they would use (e.g., worksheets, slides, tests, pre-selected websites) and explain
the rationale for their selection or design.
Analysis of instructional practice artifacts, such as lesson plans and learning materials designed by prospective or in-service teachers, has been used successfully in teacher
education research for the study of teacher thought processes and practices (e.g.,
Mouza & Karchmer-Klein, 2013 ; Silver, Mesa, Morris, Star, & Benken, 2009), either
as an alternative to or in combination with classroom observations and survey methods.
Although artifact analysis is a less direct method of teaching effectiveness research
compared to observations of actual teaching, relevant studies show that it enables
researchers to accurately characterize classroom practice and to draw valid inferences
about teacher knowledge and beliefs (Borko, Stecher, & Kuffner, 2007 ; Matsumura,
Garnier, Pascal, & Valdés, 2002). In the present study preservice teachers' lesson plans
were analyzed so as to identify knowledge gaps and possible misunderstandings
regarding the use of the Internet as an information resource in teaching and learning.
Data Analysis
Guided by the research questions of the study, data analysis attended to three issues:
the activities that preservice teachers had designed, the characteristics of the web
materials they had selected, and the guidance and support they had considered providing to their students.
The unit of analysis was the activity. Activities, which involved tasks assigned to
the students and instructional sequences carried out by the teacher, were identifi ed
based on preservice teachers' lesson plan descriptions and student worksheets.
Specifi cally, transition from one activity to the other was recognized through
changes in at least one of the following: the learning goal, the content, and the
teaching-learning approach. Activities that did not involve or were not directly relevant to the use of Internet resources (e.g., teacher-led question answering in the
beginning of the lesson to elicit student knowledge and experiences relevant to the
topic) were not included in the analysis.
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Activities . Instructional activities were classifi ed based on (a) whether they were
teacher directed or student centered and (b) the complexity of the online research
task that was assigned to the students (e.g., well-defi ned query to answer a factual
question vs. ill-structured research).
Websites . Websites that had been selected by preservice teachers, to be used in
teacher-led instruction or student-centered activities, were analyzed in terms of their
level of multimodality and interactivity . Multimodality refers to the range of modes
(verbal and nonverbal) that were used to present information (Moreno & Mayer,
2007). Each selected webpage was examined for the types of nonverbal representations (e.g., diagram, map, photo, animation) that were present besides printed words
or narration. Then a multimodality score was calculated to characterize the range of
pictorial representations that were utilized in the entire lesson plan. Specifi cally,
each lesson received 1 point for each type of visual representation that was utilized.
Two levels were used to code interactivity based on the classifi cation system
proposed by Moreno and Mayer (2007). A website provided low interactivity if it
allowed the user to perform simple navigation and search actions, such as click on
a link or button to move, select an option, or enter a simple query. High interactivity
was enabled when users could manipulate a simulation, control the pace and order
of a presentation, and receive feedback to their input.
Support . The coding scheme for analyzing teacher support, as refl ected in preservice
teachers' activity descriptions and student worksheets, was developed through multiple readings of the materials. I looked for scaffolding techniques that have been
recommended in the literature (e.g., Kupier et al., 2005 ; Li & Lim, 2008 ; Quintana
et al., 2005 ; Segers & Verhoeven, 2009 ; Wallace, 2004 ; Zhang & Quintana, 2012),
which were modeled or explained to preservice teachers during the course, as effective or needed to guide students with three aspects of web learning: searching for
information, evaluating information, and processing of information. Examples of
relevant techniques involve explaining how particular search engines work, modeling or recommending specifi c steps and actions, providing keyword lists, navigation
instructions and criteria for evaluating the trustworthiness of information on websites, and asking questions or providing prompts that help students retrieve, compare, organize, and interpret information. The fi nal coding scheme for the techniques
that were identifi ed during this iterative process is presented in Table 1 .
When the development of the fi nal coding scheme for activity type and teacher support was completed, one-third of the lesson plans were coded by an independent examiner. Inter-rater agreement was 94 % for activity type and 89.5 % for teacher support.
Results
Analysis of the teaching and learning activities that preservice teachers designed
shows that they utilized the Internet in four ways: (a) they used visual web resources
to enrich teacher-directed instruction , that is, teacher presentations and class
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discussions; (b) they included search activities requiring students to perform an
open- search web query; (c) they designed information retrieval activities in which
students would explore the content of one particular website to obtain factual information; and (d) they designed open-ended research activities requiring students to
search for, retrieve, and synthesize information about a particular topic from multiple web sources. Web-search activities involved searching for images and/or videos
relevant to lesson topics. Information retrieval activities included viewing images,
animations, and videos or reading a webpage with the purpose of obtaining factual
information, running a simulation (e.g., build a volcano and watch it explode), and
locating geographical information using Google Earth. Although the last two activities involved learning through interaction with a system, they were categorized as
information retrieval activities because students were asked to either make a simple
observation or answer a factual question (i.e., use Google Earth to fi nd the location
of particular volcanos). Often in information retrieval activities students were
guided with questions, to be provided orally by the teacher or through worksheets,
that aimed at helping them focus on and retrieve specifi c pieces of information.
Finally, in open-ended research activities students were asked to gather and present
information about a topic (e.g., the volcano of the Greek island Santorini, the most
disastrous earthquakes, or the most important volcanos on earth). These activities
were open ended in some sense because they required gathering information from
Table 1 Types of support for information search, evaluation, and processing by activity type
Types of support
Student activities
Information
search
(n = 8)
Information

retrieval
(n = 37)
Information
research
(n = 17)
Information search
Shows how to search with a search engine 1 1 4
Provides URL/keywords and site selection guidance 5 29 9
Shows how to navigate a site and what to select 13
Total 6 43 13
Evaluation of information
Shows how URLs can be used in source evaluation 1
Uses author as a criterion to evaluate content 1
Total 0 0 2
Processing of information
Asks questions to help focus on important ideas 2 23 2
Asks refl ection questions 1
Asks students to make a hypothesis 3
Asks students to interpret information 1
Provides a table to organize information 1
Asks summarization questions 1
Provides background information 6
Provides concept map to organize information 1
Total 2 34 5
Note : Numbers in columns show the presence of each type of support in each learning activity
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multiple sources (although it was likely that someone might locate a single webpage
containing enough information to cover the topic). However, none of these activities
involved the investigation of a complex, authentic question, requiring students to
come up with their own answers by interpreting, comparing, and integrating information, which is an essential element of inquiry-oriented learning.
None of the participants designed a "traditional" lesson where ICTs would be
used exclusively by the teacher to explain content or stimulate discussions. Most
preservice teachers (n = 17) used a combination of teacher-directed instruction and
activities to be carried out by the students, while the remaining 13 based their lesson
solely on information search, retrieval, and/or research activities. Overall, 15 participants included information retrieval activities, 15 designed open research activities, and 8 used web-search activities. Typically, students did not combine
information retrieval with open research activities. It seems that they either adopted
a "structured" approach, involving information retrieval activities, which in some
cases were combined with teacher-directed instruction and/or search activities, or
an "open" approach involving open research activities that were often combined
with teacher-directed instruction and/or search activities.
Twenty-seven preservice teachers selected webpages or materials found online,
such as photos and videos, to be used either in teacher-directed instruction or in
student activities. These websites or materials were analyzed for their level of multimodality, and a score was calculated for each lesson. Participants used fi ve types
of representations: photos, maps, diagrams, animations, and videos. The mean multimodality score for their group was relatively low (M = 2.55), indicating that on
average preservice teachers had utilized a small range of visual resources in their
lesson plans. A closer inspection of the data showed that while half of the participants (n = 14) had selected at least three types of representations, many participants
(n = 10) had used only photos. Typically, teacher presentations and discussions were
enriched with static visual displays, which were primarily photos and to a lesser
extent maps and explanatory diagrams, although some participants also included
videos and animations. In addition, preservice teachers tended to select sites that
enabled little interactivity. Only fi ve participants selected interactive representations
of content (e.g., simulations, Google Earth) to be utilized in the context of information retrieval activities.
An incidental fi nding from the analysis of the websites was that about half of the
participants had selected at least one resource that was inappropriate relevant to the
activity for which it was used. Specifi cally, some preservice teachers (n = 7) had
included content from unreliable sources such as commercial sites and personal
blogs, treating them uncritically as authoritative sources, although some of them provided scientifi cally inaccurate information. In addition, about half of the participants
used texts from sites that were designed for adults to provide students with background encyclopedic information on lesson topics. However, these texts could not be
easily accessible to elementary students due to text length and excessive use of scientifi c terms and sophisticated explanations. Also, some of the selected resources provided affordances not clearly tight to lesson learning goals, such as a simulation
that enabled students to create different types of volcanos by manipulating gas and
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viscosity levels. Finally, many preservice teachers selected images and videos that did
not communicate information clearly and effectively due to poor quality (e.g., low
resolution) or the provision of many unnecessary and irrelevant details.
An examination of the techniques that preservice teachers had considered using
so as to support their students, either orally or in the form of prompts and questions
in student worksheets, showed several interesting patterns. First of all, as Table 1
shows, most participants included a small range of techniques which focused mostly
on the search process and to a lesser extent on information processing. Regarding
the latter, the majority of the participants who planned to provide support on information processing used questions to help students retrieve specifi c pieces of information. Also, only one preservice teacher thought to provide students with support
on how to evaluate the trustworthiness of web content.
Another interesting pattern is that students were more likely to provide support
for the information retrieval activities than for the research activities. Typically, in
information retrieval activities preservice teachers guided students to locate specifi c
pages, by providing them with keywords or specifi c URLs, and gave navigation
instructions for websites that offered many options. This guidance was expected
because elementary students were going to use pre-selected sites. On the other hand,
participants who designed open research activities provided minimal or no guidance
on how students should search for, evaluate, and choose appropriate sites for their
projects. In addition, in information retrieval activities most preservice teachers provided students with questions to help them focus on important information when
exploring the content of websites. On the other hand, the vast majority of participants who included open research activities in their lesson plans did not consider
providing support to help their students retrieve, comprehend, organize, compare,
and integrate information that was relevant to their topics. Finally, students who
included web-search activities were likely to provide some support for the search
process, but only two of them thought to give students some instructions on what to
do with the results of their query.
Discussion
The data of this study represent preservice teachers' fi rst attempts to design a lesson
that involved teaching using Internet information resources. This particular group of
undergraduate students had one more year of study in their teacher preparation program, involving more coursework on teaching methods in the subject areas as well
as fi eld experience in local public schools and, therefore, more opportunities to
further develop their knowledge and skills about teaching and learning. So, it is
likely that their instructional materials would have been more sophisticated if they
had been asked to design a similar lesson near the end of their fourth year in the
program. Also, in the present study preservice teachers designed hypothetical lessons which were not going to be implemented in a real classroom with real students
and real technologies. This suggests that their ideas were not fully developed
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because preservice teachers did not have to think through all the details of an actual
lesson enactment, and their lesson plans were based on a hypothetical, abstract
classroom context.
Despite the above limitations, the patterns that emerged from the analysis of the
data provide useful insights into the possible misunderstandings and knowledge
gaps that preservice teachers may have about teaching with the Internet and about
online inquiry. Therefore, the conclusions of the present study can guide the design
of future teacher education and professional development courses and seminars
focusing on teaching with ICTs and, more specifi cally, on teaching with the use of
Internet information resources.
It is encouraging that all participants included in their lesson plans instructional
activities that involved the use of ICTs by their students, as opposed to the use of
technology exclusively as a demonstration tool in teacher-directed instruction. Two
distinct patterns emerged in their approaches. About half of preservice teachers
used a "structured" approach that involved information retrieval activities or some
combination of information retrieval activities, teacher-directed instruction, and
search activities, while the other half used an "open" approach, involving open
research activities or some combination of open research activities, teacher-directed
instruction, and search activities. Participants adopting a structured approach
appeared to be aware of the difficulties that their students might have with the information search process, with website evaluation and navigation, and with the extraction and comprehension of relevant content. So they provided guidance by directing
students to pre-selected sites and by helping them locate and focus on important
information through navigation instructions and factual questions. However, this
approach was over-structured because it limited independent learning and focused
on the acquisition of fragmented factual knowledge. Students were not encouraged
to engage actively with content by exploring hypotheses, organizing and comparing
information, and making new connections among ideas. On the other hand, preservice teachers who adopted an open approach seemed to have unrealistic expectations about elementary students' abilities to learn via open online research with
minimal guidance (Kirschner, Sweller, & Clark, 2006 ; Quintana et al., 2005). These
participants provided some support for the search process but no guidance on how
students were going to evaluate, select, and utilize the resources they would fi nd
online in order to achieve the learning goals of the lesson. Also, none of the preservice teachers designed an activity addressing an authentic inquiry question.
Elementary students were given some general topics or questions that required the
collection of factual information (e.g., "The volcano of Santorini," "What were
some of the most disastrous volcanic eruptions?"), as opposed to questions requiring them to draw conclusions through the corroboration and integration of information from multiple sources. It would be quite hard even for experienced teachers to
design an inquiry project that could be realized within two periods. However, it was
expected that preservice teachers would have included in the design of their lesson
plans more opportunities for deep and meaningful learning, utilizing some of the
pedagogical techniques that were modeled throughout their educational technology
course. It was also interesting that a large number of preservice teachers designed
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open research activities, although this approach was not modeled during the educational technology course. On the contrary, both through the lecture and the discussion of the educational scenarios that focused on web learning, it was stressed that
open research learning is an ineffective approach when implemented without extensive teacher scaffolding due to the many challenges that young students face with
the search, selection, evaluation, and processing of information.
The analysis of the types of support that preservice teachers had considered for
elementary students showed that most of the techniques they described concerned
the information search process. This indicates that preservice teachers were generally aware of some of the difficulties that elementary students might have with
online search and could also come up with specific techniques to make this process
more manageable. However, preservice teachers demonstrated a limited repertoire
of techniques to scaffold elementary students with web learning beyond the level of
factual information extraction. Very few participants considered assisting students
with the evaluation, selection, organization, critical analysis, and integration of
information or with the preparation of their project presentations. And none of the
participants considered it necessary to provide specific suggestions to help students
comprehend and think critically about the visual resources.
Overall, preservice teachers made extensive use of videos and static visual displays (i.e., photos, diagrams, and maps), but about half of them did not utilize the
full range of content representations that were available online. Specifi cally, ten
preservice teachers used only photos and only fi ve participants selected interactive
representations of content, which have the potential to engage students more actively
in learning. However, even these fi ve participants did not take full advantage of the
representations' affordances because their proposed use by the students was limited
to making simple observations and to retrieving factual information as opposed to
asking questions, testing predictions, or observing patterns. Also, those who chose
the "build-a-volcano" simulation were possibly attracted by the interactivity it provided, but they used it in a way that was not clearly relevant to their lesson goals. In
addition, according to the analysis of preservice teachers' website selections, many
of the participants did not know how to select appropriate sites for young students
in terms of the quality, trustworthiness, and age appropriateness of their content.
The fi ndings of the present study point out several issues that should be addressed
in teacher preparation programs regarding web-based instruction. In line with the
TPCK framework (Angeli & Valanides, 2009 ; Benson & Ward, 2013 ; Mouza &
Karchmer-Klein, 2013), they highlight that, in order to use the Internet productively
and creatively as an information resource, teachers need to develop complex forms
of knowledge that require the integration of knowledge about technology, pedagogy, and content.
First of all, as previous studies have shown (e.g., Childs et al., 2007 ; Wallace,
2004) it is clear that preservice teachers are in great need for instruction on how to
select appropriate web content. Selecting Internet resources involves several complex criteria that require the consideration of the technological characteristics of the
resources, their affordances for learning and their credibility, the subject matter, the
learning goals and the teaching approach, as well as students' prior knowledge and
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digital skills. Based on the fi ndings of this study, preservice teachers need to learn
how to evaluate websites for:
(a) Their general quality as learning resources using established, research-based
text-design and multimedia-design principles (Hartley, 2004 ; Mayer, 2005).
(b) The nature and credibility of their source.
(c) Their appropriateness for students of particular age relative to their prior knowledge and navigation abilities.
(d) Their relevance relative to lesson content and learning goals.
(e) Their affordances regarding the types of learning they can support.
In addition, preservice teachers need to develop technological pedagogical
knowledge about inquiry-oriented approaches to web-based instruction as well as
about students' potential difficulties with web-based learning, which concern not
only the process of information search and evaluation but also the comprehension of
multimodal documents and the critical analysis and synthesis of information from
multiple texts and data sources. Teachers need to also develop TPCK about techniques for scaffolding students to engage successfully and productively both with
the processes and with the content of online inquiry.
All the above are complex forms of knowledge which develop progressively and
require extensive teacher involvement in critical reasoning and problem solving
regarding authentic instructional situations (Angeli & Valanides, 2009 ; MargerumLeys & Marx, 2002 ; Mouza & Karchmer-Klein, 2013). In order to help preservice
teachers integrate knowledge of content, pedagogy, and technology, teacher preparation programs should provide student teachers with opportunities to analyze, plan,
enact, and refl ect on web-based instruction in the content areas. This in turn may
require that, at some level in the teacher preparation program, educational technology courses are integrated with fi eld practicum and teaching method courses, so that
preservice teachers are supported to transform and integrate theoretical knowledge
obtained through separate courses into practical forms of technological pedagogical
and technological pedagogical content knowledge.
In agreement with what research has shown about experienced in-service teachers
(Osborne & Hennessy, 2003 ; Schofi eld, 2006 ; Wallace, 2004), the participants of
this study could not find an alternative or a balance between two extreme approaches
to Internet use in the classroom: the open research and the over- structured approach,
both of which, for different reasons, do not enable learners to engage actively with
content. In the open research approach students are likely to encounter many inappropriate sites and to spend most of their time and effort searching for relevant
information, which severely reduces the cognitive resources they can allocate to the
comprehension, evaluation, and synthesis of the information (Segers & Verhoeven,
2009 ; Zhang & Quintana, 2012). Without enough guidance and scaffolding, students typically gain only superfi cial understandings of their research topics. Poor
learning outcomes may also be observed with the over- structured approach. The
latter offers security and direction, but, by focusing on factual learning and by constraining students' independence, it limits their opportunities to develop deep content understandings and critical skills. The fact that these two extreme approaches
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have been observed not only in the preservice teachers of this study but also among
experienced in-service teachers is related to the nature of the Internet itself and

reflects teachers' confusion regarding how to both capitalize on its potential to support independent, student-centered learning and set boundaries so as to ensure that students learn something while using it (Schofield, 2006 ; Wallace, 2004). Although several scholars (e.g., Li & Lim, 2008 ; Segers & Verhoeven, 2009 ; Zhang & Quintana, 2012) have been exploring the effectiveness of particular approaches (e.g., web quests), scaffolding techniques, and technological tools that aim at maximizing the benefits of web-based instruction and online inquiry for students, there is clearly a need for more research in this area to provide productive suggestions for teachers regarding alternatives to these two typical extremes.
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Introducing Conversational Agents
In the field of artificial intelligence, the terms "agent" or "intelligent system" refer to any entity that perceives its environment through sensors and acts upon it using effectors (Franklin & Graesser, 1997). However, through the prism of e-learning, a "pedagogical agent" refers to a computer-generated character typically employed to fulfil a series of pedagogical aims in an educational system (Gulz, Haake, Silvervarg, Sjödén, & Veletsianos, 2011).
In our work, we focus on "conversational agents," a subgroup of pedagogical agents that engage in a conversation with the learners using natural language. The type of communication occurring between a conversational agent and a learner can be text based, oral, or even nonverbal, including body language movements and facial expressions (Kerly, Ellis, & Bull, 2009).
Moreover, the graphical representation of conversational agents may also vary, ranging from a two-dimensional cartoonish appearance to a three-dimensional photo-realistic character (Veletsianos, 2010). Conversational agents that have a visual representation are frequently mentioned as "embodied" conversational agents (Cassell, Sullivan, Prevost, & Churchill, 2000). Research has repeatedly identified the agent visual appearance as an important design element, which affects learners' stereotypes or expectations of the agent intelligence (e.g., Haake & Gulz, 2008 ; Veletsianos, 2010). Indeed, the agent embodiment had a major impact on the evolution of conversational agents from the impersonal characters found in the intelligent tutoring systems (ITSs) of the past to the tangible personalized pedagogical agents of today (Gulz et al., 2011).
Conversational Agents for Learning:
How the Agent Role Affects Student Communication
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Another important factor regarding the agent design is the role of the agent in the learning environment. Conversational agents have been developed to serve multiple pedagogical roles including—but not limited to—coaches, tutors, motivators, or learning partners (Haake & Gulz, 2009). Many studies have been conducted to explore the various roles and uses of conversational agents in individual learning settings, where the agent has engaged in one-to-one interactions with the learner (Kerly et al., 2009). Agents acting as peer learners have been shown to lower students' anxiety and promote students' empathy (Chase, Chin, Oppezzo, & Schwartz, 2009). Additionally, it was reported that such agents tend to be less intrusive than agents acting as instructors (Baylor & Richards, 2010).
More recently, taking into account the pedagogical benefits of computer-supported collaborative learning (CSCL) (Dillenbourg, 1999), researchers have expressed their interest in assessing the use of conversational agents in providing dynamic collaborative learning support (e.g., Chaudhuri, Kumar, Howley, & Rosé, 2009 ; Walker, Rummel, & Koedinger, 2011). A study revealed that conversational agents can efficiently utilize both social and task-oriented intervention strategies to support students' collaboration (Kumar, Ai, Beuth, & Rosé, 2010). Furthermore, other studies explored the positive impact of conversational agents on collaborative learning settings by emphasizing on discourse scaffolding (Stahl, Rosé, O'Hara, & Powell, 2010), reflective prompting (Walker et al., 2011), or reasoning elicitation (Kumar, Rosé, Wang, Joshi, & Robinson, 2007).
Encouraging as such findings may be, several key questions have emerged. For instance, what types of collaborative problems are best suited for such conversational agent systems? (Harrer, McLaren, Walker, Bollen, & Sewall, 2006) Should the supportive prompts provided by the agent be solicited or unsolicited? (Chaudhuri et al., 2009) How can the different roles of the agent (e.g., tutor, peer, motivator) affect peer dialogue? What is the impact of the agent presence ("persona effect") on the behavior of students working together? (Veletsianos & Russell, 2014)
Following this potentially promising research direction, we have argued that conversational agents for collaborative learning can be designed by focusing on the role of the teacher as well as the peers' interactions occurring while students work together (Tegos, Demetriadis, & Tsiatsos, 2012). Based on this rationale, we have developed a prototype conversational agent system, named MentorChat (Tegos, Demetriadis, & Tsiatsos, 2014). In the following sections, we present an overview of the MentorChat system and an evaluation study exploring how the students' perceptions of the agent and their conversational behavior may be affected by the different roles (peer or tutor) of a conversational agent.
MentorChat System Overview
MentorChat is a cloud-based multimodal dialogue system that utilizes an embodied conversational agent to scaffold learners' discussions (Tegos et al., 2014). We have developed MentorChat as a domain-independent dialogue system that (a) promotes
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constructive peer interactions using facilitative agent interventions (prompts) and (b) enables the teacher to configure the support provided by the conversational agent.
MentorChat can support discussions in English or Greek and was implemented using modern web technologies, such as HTML5, CSS3, and AJAX. The system infrastructure is based on a client-server model, which allocates workloads between the server and the clients. Its architecture comprises three main modules: the student, the teacher, and the conversational agent module (Fig. 1).
The conversational agent of MentorChat is based upon the following three models:
• The peer interaction model, which records and stores the students' interactions in a computational format
• The domain model, which utilizes the teacher's domain knowledge representation in conjunction with the pattern-matching algorithms to determine whether an agent intervention would be appropriate
• The intervention model, which examines a series of various micro-parameters (e.g., the time passed since the last agent intervention) to determine if an intervention will eventually be displayed
A teacher can use MentorChat to design, deploy, and monitor an online dialogue- based learning activity. These can be accomplished using the MentorChat administration panels, which are available in the teacher's interface. More specifically, the teacher may set up the discussion topics/phases of the collaborative
Fig. 1 MentorChat system architecture
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activity (activity structure panel), manage the participating users and groups (user management panel), monitor groups' discussions (monitoring panel), or configure the agent domain model for the activity by inserting a set of rules or creating a concept map (domain modeling panel).
Students entering MentorChat are asked to collaborate with their partner(s) on a given task (Fig. 2, A) using text-based synchronous communication. During the students' dialogue, an animated humanlike conversational agent (Fig. 2, B) analyzes their discussion providing supportive interventions that trigger fruitful peer interactions on key domain concepts. Each agent intervention is dynamically displayed in a pop-up frame, next to the peers' chat frame (Fig. 2, C), allowing learners to complete their ongoing conversational interaction before answering the agent question.
Method
MentorChat was used in an experimental activity, which was conducted in the context of a computer science course offered by the Second Chance School of Thessaloniki in Greece. The aim of the study was to explore the impact of two different agent roles (peer vs. tutor) on students' perceptions and behavior. The total participants of the study were 24 Second Chance School students (13 males and 11 females), who were not able to attend mainstream secondary education for various socioeconomic reasons. Students were adults whose age ranged from 19 to 67 years (N = 24, M = 37.4, SD = 13.36). Although their nationality also varied (e.g., Albanians, Bulgarians, Greeks), all of them spoke Greek in class.
Fig. 2 A (translated) screenshot of the MentorChat student interface
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Procedure
In the course lectures, the students were introduced to the concepts of the Internet
and the web-based applications. The classroom sessions involved many discussions
around synchronous and asynchronous communication tools, micro-blogging, blogging, and social networking. The MentorChat environment was also presented to
the students as an example of an online collaborative learning environment.
After a 3-week period, an experimental activity was carried out in the computer
lab of the Second Chance School for 2 teaching hours (90 min). The participating
students were asked to use MentorChat to discuss the web applications they had
learned and used in class. They were also informed that during their conversation a
virtual peer or tutor would raise some questions, which they should discuss within
their group in order to provide a joint answer.
The agent was confi gured by the two classroom teachers to raise issues regarding
social networking, search engines, and modern communication tools. In particular,
the teachers used the MentorChat domain authoring panel to form the agent domain
model by entering a set of rules. Each rule consisted of a domain concept (a key
word or phrase—e.g., "Mozilla Firefox") along with a particular intervention. The
interventions were refl ective questions that asked students to elaborate on the subject and provide a thoughtful joint response (e.g., "If you want to create a webpage
featuring articles in a chronological order, should you use a blog or a wiki? Why?").
During the students' discourse, the conversational agent displayed the teacher-
defi ned intervention whenever the associated domain concept (keyword or phrase)
was identifi ed. Subsequently, the students were encouraged to discuss with each
other and provide a joint response, typing their answer into the agent answer box
(Fig. 2 , C). In addition to this intervention method, which was active throughout the
students' discussion, a fi nal intervention was also made by the agent at the end of the
activity. More specifi cally, before exiting the activity, the agent reminded students
all the teacher-defi ned domain concepts that had not mentioned during their discussion, providing them with the option to continue their conversations on the suggested topics (e.g., "It seems that you have not discussed wikis. Do you want to
continue your discussion or fi nish the activity?"). If students' discussion included all
the teacher-defi ned domain concepts, the agent did not display any intervention.
Compared Conditions
The teachers assigned the students into small groups consisting of two or three
members (six dyads and four triads). Each student was given a score, which indicated students' expertise in computer science (based on the course grades and in-
class performance), and the fi nal groups emerged in such a way as to be slightly
heterogeneous. According to Rovai (2007), the above method constitutes an effective strategy for creating an educational context that facilitates peers' online discussions and promotes equal participation.

Furthermore, a combined score was also calculated for each group based on the
average of the individual students' scores. Taking this score into account, instructors
stratifi ed the student groups by their domain knowledge and assigned them to two
conditions so that both conditions were balanced in terms of the overall scores of the
groups.
Our two-condition experimental design involved (a) six groups (four dyads and
two triads) interacting with a conversational agent that enacted the role of a peer
(P condition) displaying the interventions in an informal manner (Fig. 3 , A) and
(b) fi ve groups (two dyads and three triads) interacting with an agent that enacted
the role of a tutor (T condition) employing a more formal appearance and
communication style (Fig. 3 , B). Each of the teacher-defi ned agent interventions
was tailored according to the different agent communication styles in the two
conditions (Fig. 3).
Data Collection and Analysis
Post-task Questionnaires
After the activity, students were asked to fi ll in a post-task questionnaire, which
aimed to explore students' opinions about the MentorChat interface and the agent
interventions. The questionnaire included three multiple-choice questions, two
open-ended questions, and ten Likert-scaled questions. Measures of central tendency were computed for all questionnaire items. Additionally, a series of Pearson
product-moment correlation coeffi cients was calculated to examine the relationships among the questionnaire variables.
Fig. 3 The agents acting as peer (A) and tutor (B) in the P and T conditions, respectively

Interviews
Interviews were conducted in order to record details of how the students worked or
perceived the learning activity. The interviews were semi-structured and lasted for
about 10 min each. They focused on students' opinions about (a) the collaborative
dialogue-based activity as a whole, (b) the usability of the MentorChat tool, and (c)
the pedagogical effi cacy of the conversational agent interventions. Students were
interviewed individually. All interviews were transcribed verbatim and analyzed in
search of common themes using the open-coding process of the constant comparative method (Corbin & Strauss, 1990).
Discourse Data Observations
Following the completion of the activity, the authors examined the text fi les of all
group discussions. In particular, the authors acted as independent raters assessing
the degree of formality/informality of users' responses to the agent. A scoring
rubric, deriving from Moskal's study (2000), was used to measure the formality of
students' utterances on a simple 2-point scale (0 for formal and 1 for informal). The
inter-rater reliability for the scoring process was found to be high (Kappa = 0.82;
ICC = 0.83). Following this asynchronous process, raters participated in a roundtable discussion elaborating on each group dialogue to draw joint inferences.
It should be noted that the individual student constituted the unit of analysis for
the post-task questionnaires and the interviews, whereas the discourse data observations involved both individual- and group-level analyses.
Results
Post-task Questionnaire Analysis
The post-task questionnaire results revealed that most students were familiar with
instant messaging applications (F = 70.83 %), while only some of them used them on
a daily basis (F = 41.2 %). They also rated their typing speed as slightly below average (N = 24, M = 2.38, SD = 1.27) on a 5-point scale ranging from 1 (slow) to 5 (fast).
Table 1 presents the descriptive statistics computed for the Likert-scale questions
that measured the user acceptance of the MentorChat tool. Likewise, Table 2 presents a selection of the results relating to the agent behavior.
Furthermore, a Spearman's product-moment correlation analysis revealed two
signifi cant correlations among the questionnaire variables. First, there was a negative correlation between the "learners' age" and the "system ease of use" (r = −0.51,
p = 0.01). Second, the "learners' typing skill" was found to be positively correlated

with the "comprehensiveness of the interface options" (r = 0.48, p = 0.02). These
correlations were anticipated since younger students are typically more familiar and
experienced with the interface and functionality of instant messaging applications.
Given that the normality and the homogeneity of variance criteria were satisfi ed,
we proceeded to apply parametric statistics to our individual-level questionnaire
data. More specifi cally, a series of independent samples T-tests was performed comparing the scores of the questionnaire variables in the P and T conditions. The analysis did not reveal any signifi cant difference in the scores for the two conditions.
Nevertheless, although nonsignifi cant (t [22] = 4.76, p = 0.07), worth mentioning is
the difference in how students in the two conditions perceived the content of the
agent interventions. In respect to the agent interventions appearing during students'
discussions, the students who interacted with the peer-agent (P condition) considered its interventions as more comprehensible (M = 4.00, SD = 0.35) than the students who interacted with the tutor-agent (T condition) (M = 4.92, SD = 0.05).
Interview Analysis
The qualitative data that derived from the analysis of the interview transcripts indicated fi ve common themes, as presented in Table 3 .
Table 1 The questionnaire results concerning the MentorChat tool
Question (translated) Mean (M)
Standard
deviation (SD)
The texts displayed on MentorChat are easy to read and
comprehensible.
4.21 1.25
The options available in the environment are easily understandable. 4.08 1.10
The icons and symbols used are familiar to me. 3.96 1.36
I believe that MentorChat is an easy-to-use application. 4.42 1.06
Table 2 The questionnaire results concerning the interventions of the agent
Question (translated) Mean (M)
Standard
deviation (SD)
The agent interventions, which appeared during the discussion, were
simple and comprehensible.
4.50 1.14
The agent interventions made the discussion more interesting. 4.63 0.65
The agent interventions that appeared during the discussion helped
me to recall/fi nd out valuable information about the topics under
discussion.
4.63 0.65
The agent interventions that appeared at the end of my discussion
helped me to recall/fi nd out valuable task-related information.
4.62 0.59
The agent interventions distracted me, and I would prefer they did not
appear.
1.31 0.48
I understood the subject better through answering the agent
questions.
4.25 1.07

Discourse Data Observations
The examination of the groups' discussions revealed a series of patterns regarding
the students' interaction and conversational behavior. First of all, we observed that
when participants entered their group discussions, they posted a number of messages (N = 10 groups, M = 7.10, SD = 4.15) that played a purely social function and
were not related to the task. Although a lot of students (10 out of 24) initiated their
discussion typing in "Greeklish" (i.e., writing in Greek but using Latin characters),
all of them altered their typing to Greek when at the beginning of the activity they
saw the agent prompt urging them to use Greek characters.
Furthermore, the agent interventions displayed, especially at an early stage of the
activity, seemed to have caused some confusion in half of the groups (5 out of 10).
More specifi cally, a relatively high number of task coordination contributions was
identifi ed in students' utterances after the fi rst agent intervention occurred. Taking a
closer look at the students' dialogues we found that, despite the agent explicit
instructions (Fig. 2 , C), some peers (8 out of 24) could not understand at fi rst if they
should individually answer to the agent or provide a joint response. In fact, in some
occasions, some of these students rushed to provide a response without reaching an
agreement with their partner.
It should also be noted that even though all students communicated with each
other in a friendly manner, we observed a considerable difference in the way the
learners interacted with the agent in the P and T conditions. Specifi cally, the descriptive statistical analysis of the rubric scores indicated that the student groups (N = 5) in
the P condition responded to the agent questions in a far more informal way (M = 0.84,
SD = 0.21) as compared to the student groups (N = 5) in the T condition (M = 0.18,
SD = 0.16). In particular, the students in condition P responded to the agent as they
would respond to a question of their human partner(s) (e.g., "Hi Elena! I can help you
with the webpage …") while the students in condition T engaged in a more formal
communication with the agent (e.g., "From my point of view, the webpage should be
developed …"), as a student would answer the question of a human tutor in class.
Table 3 Common themes identifi ed
No. Theme
1 Agent voice usefulness (e.g., "the agent voice drew my attention to the agent intervention")
2 Spelling auto-correction suggested (e.g., "it would be nice to have a spelling auto- correction
feature activated while composing new messages")
3 Directed agent interventions suggested (e.g., "each agent intervention should address only
one group member and explicitly state who is being addressed")
4 Interdisciplinary use of agent suggested (e.g., "besides computer science, agent
interventions could also be helpful in other learning contexts such as language
learning")
5 Agent for educational purposes only (e.g., "the use of an agent, which monitors peers'
discussions, would not be appropriate when chatting with friends out of educational
settings")

Moreover, an examination of the agent interventions made revealed that most of
the groups (seven out of ten) did not discuss all the teacher-proposed topics and,
hence, triggered an agent intervention at the end of the activity. Although these
agent interventions reminded these groups of the domain topics not discussed, only
some of them (four out of seven) decided to resume their discussion on the topics
proposed. Nevertheless, we consider that this happened mainly due to the limited
duration of the learning activity.
Discussion and Conclusions
The main goal of the study is to investigate whether the different roles (peer or tutor)
of the conversational agent may affect students' perceptions of the agent and their
utterances. The study results indicate that the different appearance and communication styles of the agent affected the formal/informal style of the students' responses
to it (discourse data observations) but did not conclude in any signifi cant difference
in students' opinions about the agent (post-task questionnaires).
More specifi cally, all students had a favorable opinion regarding the user interface and the usability of MentorChat (Table 1). Moreover, students believed that the
agent interventions were simple and comprehensible and made the discussion more
interesting. They also stated that the agent interventions helped them recall and
identify valuable points of the topics being discussed (Table 2 , rows 3 and 4) or
understand the domain subject better through answering the agent questions
(Table 2 , row 6).
Furthermore, all the students seemed to appreciate the agent interventions and
the analysis did not reveal any signifi cant difference between P and T conditions.
The students in both conditions perceived the agent as a valuable discussion facilitator whether acting as a tutor or a peer (Table 2). Students' perceptions were not
adversely affected by the different appearance or communication styles of the agent.
However, although no statistically signifi cant differences were reported in the
post-task questionnaires, there is some evidence to suggest that students in the P
condition considered the agent interventions as more comprehensible than the students in the T condition. Based on our fi ndings, we argue that the friendlier interventions of the peer agent had a more positive impact on students, making them feel
as if they were engaged in human-to-human conversation, and eventually more willing to focus on prompt information.
This result seems to support the "personalization principle" of multimedia learning theory as described by Clark and Mayer (2011). This principle suggests that
instructional designers should use a conversational rather than formal communication
style so that learners interact with the interface in a way that resembles human-to-
human conversations. Indeed, although the students interviewed reported being aware
that the virtual character was not in an actual conversation with them, they seemed
more likely to act as if the agent was their conversation partner in the P condition.
Moreover, the discourse data revealed that students in the P condition responded
to the agent questions in a more friendly/informal way as compared to students in

the T condition. This result shows that the appearance of the agent or the conversational style of the agent interventions may infl uence students' conversational behavior. Hence, the different role of the agent in a collaborative learning activity can
impact on specifi c social characteristics of students' utterances.
This fi nding appears to be consistent with the outcomes of other studies exploring
how the different roles and appearances of contextually relevant conversational
agents can affect learners' impressions, stereotypes, or engagement (Veletsianos,
2010). For instance, Rosenberg-Kima et al.'s study (2010) indicates that the strategic

use of pedagogical agents of various races and genders can provide learners with
social models that are similar to them, thus increasing their interest towards the agent.
Furthermore, Gulz et al. (2011) highlight that a key challenge for the agent
design is to manage students' expectations about the social profi le of the conversational agent. Students have expectations of both what the agent may be able to say
to them and how it will address them. Thus, a good match between the students'
expectations of the agent's social profi le can alter how the students perceive the
agent's general personal features (e.g., a humorous or a serious character, a fi gure of
authority, or a peer) as well as enhance the pedagogical objective of making the
conversation engaging and motivating.
In spite of the various study limitations, such as the small number of participants
and the limited duration of the activity, we consider that this study provides preliminary evidence and valuable insights into the potential effect of the conversational
agent roles (peer or tutor) and their subsequent appearance and communication styles
in collaborative learning settings. We consider that teacher-confi gurable conversational agents have a pedagogically benefi cial role to play in the e-learning systems of
the future. We expect this study to contribute towards exploring various agent roles
or interventions that can improve collaboration in everyday instructional situations.
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Introduction
Biofeedback is a means for gaining control of our body processes to increase
relaxation, relieve pain, and develop healthier, more comfortable life patterns (Wall,
2004). Biofeedback gives us information about ourselves by means of external
instruments. Biofeedback training familiarizes the person with the activity in his/her
various body systems, and therefore he/she may learn to control this activity to
relieve stress and improve health.
Educational activities in complex learning contexts produce negative or positive
emotions. Stress is such an emotion that might have an impact on the educational
activity and learning outcome. Therefore, student's stress-level self-awareness and
self-regulation can be a major issue in the educational process. Furthermore, student's stress-level awareness by the tutor could help the tutor to adapt the educational approach in order to promote the student's positive emotions.
The recent interest in the link between emotions and learning is based on the
assumption that deep learning is not entirely limited to cognition, discourse, action,
and the environment, because emotions or affective states are inextricably bound to
the learning process (D'Mello, Craig, & Graesser, 2009 ; Lepper & Henderlong, 2000 ;
Linnenbrink & Pintrich, 2002 ; Meyer & Turner, 2006 ; Stein & Hernandez, 2007).
A very important feature of learning environments is the support provided to students and to the instructor. The emotion recognition in many cases can be considered
as students' motivation in order to regulate their emotional state or trying to relax
their teammates in cases of high anxiety during collaborative learning activities.
This chapter aims to (a) identify whether anxiety awareness can motivate
the students to regulate their emotions; (b) explore if a well-known biofeedback
Anxiety Awareness in Education: A Prototype
Biofeedback Device
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technique (i.e., diaphragmatic breathing) supports the students supported to regulate
their stress lever; (c) evaluate students' acceptance of biofeedback device and sensors during a learning activity; and (d) examine the usefulness of student's anxiety
awareness for the teacher.
The chapter is structured as follows: The next section is devoted to student academic emotions and adaptive learners' support. Then, the biofeedback device for
anxiety awareness is described. Section "Pilot Study" presents a pilot study that has
taken place in order to evaluate the usefulness of the biofeedback device for the
educational activities. The last section presents the concluding remarks and our
vision for the next steps.
Student Emotions and Adaptive Learners' Support
Emotions are very important functions that affect students' academic motivation,
behavior, performance, health, and development of their personality. Research from
1950s (Zeidner, 1998) investigated students' anxiety and produced suffi cient knowledge in order to support the educational practice. The impact of emotions during
problem solving as well as during the participation in educational activities seems
to be very important. More specifi cally, emotions can affect positively or negatively
the learning process (Allen & Carifi o, 1995). Efforts to study diffi cult subjects at
deeper levels of comprehension involve a complex coordination of cognitive processes and affective states (D'Mello, 2012). In the academic context, the treatment
focused on the problem seems to be the most appropriate adjustment. The emotion-
oriented treatment may also be an adaptive solution. The following factors seem to
be important to students' motivation and can be controlled by the teachers in order
to stimulate positive emotions for the educational process:
• Teaching quality and motivation : Factors, such as lack of structure, lack of clarity, and excessive demands, are known to enhance the students' anxiety during
the test (Pekrun, 2006 ; Zeidner, 1998 , 2007). Also if the learning environment
meets the students' needs, then students' engagement is encouraged (Hatfi eld,
Cacioppo, & Rapson, 1994 ; Pekrun, 2006 ; Zeidner, 1998 , 2007).
• Collaboration : Collaborative activities and structures in the classroom (Johnson
& Johnson, 1974) in many cases promote positive academic emotions (Pekrun,
2006 ; Zeidner, 1998 , 2007).
• Management of students' mistakes : Managing mistakes as learning opportunities
rather than as a personal failure, and linking benefi ts to achievements, might help
students develop adaptive emotions (Pekrun, 2006 ; Zeidner, 1998 , 2007).
• Challenging and deep inquiry : The authors of the model of dynamic affective
conclude that the main pedagogical strategy that yields in deep learning is the
challenging which can inspire deep inquiry instead of comfortable learning environments (D'Mello & Graesser, 2012).
Our proposed approach to challenging and deep inquiry would systematically
scaffold the student out of the confused state which can cause frustration and high

anxiety. This approach would presumably work better for less experienced learners
with lower domain knowledge and lower ability to self-regulate their learning activities and their emotional states. Moreover, our approach might be systematically used
as adaptive support to the learning activity. This approach is a biofeedback- based
approach that aims to keep the student on the qui vive and active but not stressed.
More specifi cally, this method aims to (a) wake the student if he/she is very relaxed
or bored trying to avoid deactivation and (b) support the student to relax applying a
biofeedback technique called diaphragmatic breathing if the student is stressed.
The synthesis and analysis of emotions is an interdisciplinary scientifi c fi eld
consisting from the combination of computer science, psychology, and cognitive
science (Allen & Carifi o, 1995). In order to apply this proposed approach we have
implemented a biofeedback device for anxiety awareness. Next section presents this
device.
Description of Biofeedback Device for Anxiety Awareness
The biofeedback device for anxiety awareness is based on two physiological techniques in order to collect bio-signals from students who are engaged in a learning
activity. The techniques used are the following:
• Galvanic skin response (GSR): This is a test of the sweat function, which measures the change in conductivity of the skin during the fl ow of low-voltage current
after a stimulus. The recording of the conductivity (or the inverse of conductivity,
i.e., resistance) is based on the application of external constant voltage to the skin
(Lykken & Venables, 1971).
• Photoplethysmography (PPG): PPG is a simple and low-cost optical technique
that can be used to detect blood volume changes (BVP) in the microvascular tissue (Challoner, 1979).
The biofeedback device (Fig. 1a) collects, identifi es, and utilizes bio-signals
resulted by physiological reactions to stressful situations, such as epidermal ephidrosis (GSR, Fig. 1b) and heart rate (HR, Fig. 1c). The device is connected to a
computer and collaborates with dedicated software in order to convert bio-signals
into a stress level (relaxed, waking, or anxious) on the screen as well as to advise the
user to regulate stress through relaxation techniques.
The software (Fig. 2) has the following functionality for each person:
• Displays the student code (Fig. 2a).
• Displays the emotional state progress bar and color (green for relaxed, orange for
waking, or red for anxious) as depicted in Fig. 2b .
• Recommend the user to apply a biofeedback technique in order to relax in case
of high anxiety level (Fig. 2c).
The tutor can monitor the students' stress level in real time as depicted in Fig. 3 .
The next section presents a pilot study that has taken place in order to evaluate
the usefulness of the biofeedback device for the educational activities.
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Fig. 1 Biofeedback device for stress-level awareness
Fig. 2 User interface of biofeedback device software for personal stress-level awareness
Fig. 3 User interface of biofeedback device software for the tutor

Pilot Study
As referred before this chapter aims to (a) identify whether anxiety awareness can
motivate the students to regulate their emotions; (b) explore if a well-known biofeedback technique (i.e., diaphragmatic breathing) supports the students in order to
regulate their stress level; (c) evaluate the students' acceptance of biofeedback
device and sensors during a learning activity; and (d) examine the usefulness of
student's anxiety awareness for the teacher.
A pilot study has taken place in the postgraduate course "Educational Virtual
Environments" of the Postgraduate Studies Program of the Informatics Department
at the Aristotle University of Thessaloniki, Greece. Four educational activities were
taken place. Two of them were synchronous in the same place, and the other two
were synchronous from distance.
The activities were applied on 13 students, 4 males and 9 females, and each of
them concerns oral examination. In each session each group has been examined by
presenting its project and answering to tutor's questions. Every member of a group
has presented his/her contribution. The schedule of activities and the deliverables of
each activity were clearly stated at the beginning of the semester.
Method
The students were divided into four groups, one consisting of four people and the
others of three. Each group had identical assignments. The collaborative technique
used in educational test activities was fi shbowl (Leonard, Dufresne, Gerace, &
Mestre, 1999).
For the evaluation of the activities the students answered to 23 questions. The
questions were divided into two groups. The fi rst group included questions about
the evaluation of anxiety recognition support to the educational activity and
biofeedback- relaxing technique. The second group followed the Unifi ed Theory of
Acceptance and Use of Technology framework (UTAUT) (Venkatesh, Morris,
Davis, & Davis, 2003).
In addition an unstructured interview was taken by each student separately and

by the professor who was teaching the course in which the activities occurred. The
professor (tutor) was watching the students' anxiety measurements from the monitor application.
Materials
In all activities the students were connected to the biofeedback device for measuring
their stress level (Fig. 1). The tutor was aware about every student stress level using
the stress level-monitoring application depicted in Fig. 3 .
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The fi rst oral examination activity and stress-level measurement took place at the
classroom. It could be considered as the familiarization activity, since extra care was
taken to practical matters on using the implemented device. For example most of the
students had trouble to get connected properly to the GSR and HR sensors.
The second and third oral examination activities have taken place from distance,
and they were supported by an open-source videoconferencing tool called the Big
Blue Button (http://www.BigBlueButton.org/). The students presented the basic
technical characteristics and architecture of the systems they were implementing.
Finally the fourth activity that was the fi nal examination took place at the same
time within the classroom. Each group presented the implemented system and its
evaluation.
Instruments
Two questionnaires have been fi lled by the students. For the needs of the fi rst questionnaire the participants were separated to control and test group with identical
questions.
In both kinds of questionnaires the fi rst question was a nominal scaled about
student's gender and the others were interval-scaled questions following the fi ve-
level Likert scale (Likert, 1932) (from 1 = strongly disagree to 5 = strongly agree).
Evaluation Procedure
An important issue of this work was to identify whether anxiety awareness can
motivate the students to regulate their emotions and to explore if a well-known biofeedback technique (i.e., diaphragmatic breathing) supports the students to regulate
their stress level. In order to evaluate this issue a two-step evaluation process has
been followed. In the fi rst step of this process the students fi lled a psychometric
questionnaire (Spielberger, Gorush, & Lushene, 1970) in order to collect data about
their emotional mentality. In the second step and according to the results of this
questionnaire the students have been separated into two groups (control and test)
consisting of equivalent number of stressful and relaxant members. The control
group was consisting of two males and fi ve females, and the test group was consisting of two males and four females. The difference between these groups was that the
students belonging in the control group were not aware about their stress level and
they didn't receive any biofeedback messages.
Evaluation Results, Discussion, and Conclusions
Investigating if students' stress-level awareness can augment to educational and
leaning activities, the majority of the students (61.2 %) answered that they "strongly
agree" that this information would be useful in examination activities. Also, most of
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the students (53.8 %) agreed that stress-level awareness would be useful to workshops. However, most of the students do not think that students' stress-level awareness can augment distance-learning teleconferences and individual studying.
Furthermore, independent sample t -test was applied to identify whether anxiety
awareness can motivate the students to regulate their emotions with confi dential
interval of 95 % and assuming equal variances. The signifi cance for equal variances
was 0.290 > 0.05 satisfying the homogeneity assumption. The mean of the test group
was signifi cantly greater than the mean of the control group, and the effect size was
greater than 1. This could be interpreted that the test group having the representation
of the measurements was more motivated than the control group.
Concerning the support of the diaphragmatic breathing technique to the students
to regulate their stress-level question, most of the students answered (38.5 % of
them agree and 30.8 % strongly agree) that they believe it helped them to be more
relaxed. However, there was no signifi cant statistical difference between control and
test group.
Concerning users' acceptance of biofeedback device and sensors during a learning activity, the UTAUT model (Fig. 4) has been applied to all students with common questions.
In order to examine the reliability of the question groups, Cronbach's alpha
reliability test has been applied. For the performance expectancy group of six questions the reliability was 0.73, for the expectancy effort (three questions) was 0.712,
and for the behavioral intension (four questions) was 0.745. So the three question
groups approved reliable. Concerning the diffi culty of connecting to the sensors of
the developed device most of the participants (46.2 %) answered that they strongly
agree that it is diffi cult and 30.8 % of the participants found little inconvenience.
Also about the diffi culty of using this device in parallel with the main activity, a
strong majority (69.2 %) found it a little inconvenient. So the majority of the participants had a little diffi culty with the device without any serious problem. Then
Spearman's correlation signifi cant at the 0.05 level was applied to performance
expectancy, effort expectancy, facilitating conditions, gender, experience, voluntariness of use, and behavioral intension. Signifi cant correlation approved between (a)
performance expectancy and facilitating conditions (0.610*), (b) performance
expectancy and behavior intension (0.573*), and (c) experience and behavior intension (0.586*). Also signifi cant correlation approved between expectancy effort and
voluntariness of use (0.630*).
Behavioral
Intention
Gender, Experience,
Voluntariness of use
Use behavior
Performance
Expectancy
Effort
Expectancy
Facilitating
Conditions
External variables
Control Variables
Fig. 4 The applied
UTAUT framework
(Venkatesh et al., 2003)
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According to the teacher the screen of the monitor application, where he could
have a representation of his students' emotional state, was very useful especially at
the distance activities and supported him in regulating his behavior when he was
noticing that there were stressful conditions. The professor who was responsible for
the evaluation activities was interviewed. He mentioned two signifi cant benefi ts
from using the biofeedback device in an educational activity: (a) Awareness of the
class emotional state for both distance and onsite activities : In distance-education
activities class anxiety awareness appears to be quite helpful for the teacher. It is a
factor that reduces academic distance between the teacher and the students. In addition, the prototype device enables the teacher to better understand students' anxiety
level. Furthermore, he/she had a more comprehensive awareness about his/her class
during the examinations. (b) Teacher's self-regulation during oral examinations : As
the professor reported, the awareness about students' anxiety helped him to support
the students to relax during the oral examinations. This resulted to establish a better
academic context during the examination.
After completing the questionnaires the students were interviewed separately.
All students suggested that the anxiety measurement may be useful to learning
activities such as examinations and workshops.
In summary, all the participants were willing to reuse these sensors in other academic activities. All students admitted that stress-level awareness can motivate the
students to regulate themselves. From the teacher's point of view students' stress-
level awareness support him to realize his students' emotions and to adapt his
approach, rhythm, and pace of oral examination. Our future plans involve (a) repeating learning activities with bio-signal measurements on a larger scale with more
students, (b) examining more bio-signal measurement techniques in addition to
GSR and HR, and (c) applying these measurements in collaborative distance-
learning activities.
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Digital Storytelling: Defi nition and Benefi ts
As narration is the symbolic presentation of a sequence of events (Scholes, 1981),
storytelling might be defi ned as a form of narrative (Lathem, 2005 ; More, 2008).
Narrative is defi ned as a sequence of events which refer to a unifying subject represented in a perspicuous order in time (Carroll, 2001 ; Worth, 2005). The terms "narrative" or "story" are used interchangeably in the literature (Velleman, 2003 ; Worth,
2005). It is well known that children begin to write their stories through images that
often entail a symbolic function. They use pictures to create a sequence with a plot,
which tells a story (Gershon & Page, 2001). Children's stories may be short and serve
different aims such as telling a personal tale, recounting a historical event, or making
a presentation aimed to give information or instructions on a particular topic of interest (Robin, 2006). This process, refl ecting an old concept, has been defi ned as storytelling and has been used extensively in different levels of education (Lathem, 2005 ;
Robin, 2006 ; Valkanova & Watts, 2007 ; Yuksel, Robin, & McNeil, 2011). It refers to
Digital Storytelling for Children with Autism:
Software Development and Pilot Application
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a linear narration of a story with a starting point, a middle point (climax), and an
ending point (Gershon & Page, 2001). Storytelling may include text as well as other
forms of media such as images, sounds, animation, and videos (Lathem, 2005).
Following the expanding use of digital media, it has been developed into a different
form of creation and presentation. Digital storytelling (DS) is based on digital images,
videos, text, and sounds to form a genre in order to present a story to readers, viewers,
and listeners via a computer system (Li, 2007 ; More, 2008 ; Robin, 2006). In educational terms, new instructional methods, based on the use of technology, have been
developed for facilitating learning. As Levin (2003) argues, known concepts are
adjusted in a new pedagogical setting and enhanced with the use of technology.
There is a range of DS software such as PhotoStory3 , Animoto , ComicLife
StoryBird , Digital Vaults , VoiceThread , and Glogster , Kerpoof that offer the tools to
create a story with images and text through a computer system. Users can freely
place images in a sequence in order to create their own story, since there is no correct or wrong order. Digital storytelling offers several benefi ts to the educational
process in variable domains and for different learning groups (Heo, 2009 ; Kulla-
Abbott & Polman, 2008 ; Li, 2007 ; Meadows, 2003 ; More, 2008 ; Robin, 2006 ;
Yuksel et al., 2011). In particular, Kulla-Abbott and Polman (2008) suggested that a
digital storytelling program can assist students in improving their writing skills
because learners need to create and organize a sequence of images in correspondence to their ideas for making a story. They also concluded that DS can make students more engaged and creative, by helping them to discover different ways to
express their own ideas. Other researchers, such as Valkanova and Watts (2007),
asked children to make sound "voice over" in a narrative form for their own videos.
They reported that this procedure enabled children to express themselves verbally
and visually in an artistic, productive, and inspiring way. The same conclusion was
reached by Burgess (2006), whose research offered evidence for the positive effect
of DS for creative thinking. Sadik (2008) explored teaching and learning through
the application of digital storytelling with Microsoft Photo Story 3 and found that it
increased their comprehension and learning. Meadows (2003) observed that when
students used multimedia software to visualize their thoughts, they were more active
and engaged with the subject matter. Li (2007) evaluated the integration of multimedia technologies into higher education and concluded that through DS students

improved their learning. Furthermore, DS has been used as an educational tool for
learners with special needs. O'Neill and Dalton (2002) used successfully digital
story books for teaching literacy to students with learning diffi culties.
Overall, researchers agree that DS is an effective approach that engages learners
and supports teachers to effectively integrate technology into learning. Its application with special learning groups has been examined in a few research projects
(Daigle & Sulentic Dowell, 2010 ; Daigle and Free, 2007 ; Gal et al., 2005 ; Tartaro,
2005). Researchers have also examined the implementation of DS to students with
ASD regarding activity schedules via computer applications (Stromer & Kimball,
2004 ; Stromer, Kimball, Kinney, & Taylor, 2006). Activity schedules are defi ned as
a set of pictures or words that cues someone to engage in a sequence of activities
(McClannahan & Krantz, 1997 ; Stromer & Kimball, 2004). They can be considered
as a form of storytelling, as they are often used to display a sequence of steps for the
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completion of an activity and this could also form a story constructed with a
sequence of images and text. Moreover, the potential benefi ts of activity schedules
presented with computer support have been well documented (Stromer et al., 2006).
The Education of Students with Autism
Autism spectrum disorders (ASD) have been identifi ed as pervasive developmental
disorders with neurobiological background, causing impairments and delays in
social communication and imagination (Barry & Burlew, 2004 ; Welton et al., 2004).
Consequently, children with ASD prefer to get engaged in solitary repetitive activities, avoiding social interaction. Moreover, they have a cognitive bias to local processing, that is, they pay special attention to details which prevents them from
grasping the purpose or the steps of an activity (Carnahan, Musti-Rao, & Bailey,
2009 ; Odom, Collet-Klingenberg, Rogers, & Hatton, 2010 ; Rajendran & Mitchell,
2007). Also, persons with ASD present major diffi culties in executive functioning
(Carlson, Mandell, & Williams, 2004 ; Pellicano, 2007), which involves organizing
and following sequences of steps. Therefore, educating people with ASD can be
challenging and it is necessary to adopt appropriate educational methods matching
closely the needs of these individuals. A range of educational interventions utilizing
visual methods have been successfully applied to children with ASD for overcoming the aforementioned diffi culties. Structured teaching has been found to be an
effective educational method for increasing on-task behavior and independence of
children and adults with ASD (Ganz, 2007 ; Hume & Reynolds, 2010 ; Mesibov,
Shea, & Schopler, 2005). The main component of Structured Teaching is the application of visual structure in the physical environment, in daily schedules and tasks,
so that individuals with ASD can follow predictable routines and visual instructions
for executing tasks and participating in activities.
On the other hand persons with ASD seem to have a strong motivation to use
computers for both learning and leisure (Heo, 2009 ; Moore et al., 2000 ; O'Neill &
Dalton, 2002 ; Valkanova & Watts, 2007). The use of computers offers the opportunity to the learner with ASD to interact in a controlled environment, which is predictable and with minimal social stimuli. Besides, it has been found to be effective
for students with ASD by increasing their attention span and their performance in a
range of academic tasks as well as by developing a repertoire of social skills and
behaviors (Bernard-Opitz et al., 1994 ; Moore et al., 2000 ; Tanaka et al., 2010 ;
Williams, Wright, Callaghan, & Coughlan, 2002). There is considerable evidence
showing that the main characteristics of people with ASD related with their learning
are the following: (a) they have a different way of communication, (b) they prefer a
predictable environment, (c) they fi nd it diffi cult to understand emotional cues, (d)
in general, they prefer images to text or oral information, (e) they concentrate better
in structured environments, with clear expectations and visual instructions, (f) they
learn by repetition of learning tasks, (g) they need frequent breaks in their schedule,
and (h) they take pleasure using computers for learning and leisure (Bernard-Opitz
et al., 1994 ; Hume & Reynolds, 2010 ; Moore, McGrath, & Thorpe, 2000).
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Therefore the use of computers may entail several benefi ts associated with the
special characteristics of persons with ASD (Golan & Baron-Cohen, 2006 ; Mineo,
Ziegler, Gill, & Salkin, 2008 ; Moore et al., 2007; O'Neill & Dalton, 2002 ; Williams
et al., 2002). In particular, computers:
• Are predictable and thus they are controllable devices. They do not display
emotional behavior which can often be confusing for persons with ASD.
• Enable nonverbal or verbal expression.
• Present a less threatening environment compared to an adult or a peer.
• Can be used repetitively.
• Provide positive reinforcement.
• Are easy to use.
• Can be tailored to the individual needs of each learner.
• Are based on special hardware solutions designed to help persons with special
needs.
• Are versatile, since they use software which can be adaptive and adaptable to
users' needs and special interests.
Digital Storytelling and Students with ASD
In the fi eld of ASD, DS has been used to achieve student involvement as it relies
heavily on visual information; a suitable way of presenting stimuli to this special
group of learner (Daigle & Sulentic Dowell, 2010 ; Flippin et al., 2010 ; Stromer et al.,
2006 ; Stromer & Kimball, 2004). It also provides a framework integrating images
and text to form a story that narrates useful information. In addition, this method
could be viewed as an alternative means of instruction in an inclusive classroom.
Social Stories describe social situations aimed to share social information (Gray,
1998 ; Gray & Garand, 1993). Sansosti, Powell-Smith, and Kincaid (2004) presented Social Stories through DS to children with ASD and reported positive effects
of the intervention recommending further research on the use of DS for effective
learning. In addition, Heo's recent research (2010) indicated that DS increases children's involvement in the learning of subject matter. His research was conducted
over a period of 6 weeks with 17 fi fth graders and his analysis showed evidence that
DS can offer a motivating and effective learning experience. It is commonly agreed
that the creation of a story through digital media engages children's interest and
increases their creativity and involvement in a subject matter being taught (Meadows,
2003 ; Robin, 2006 ; Valkanova & Watts, 2007 ; Yuksel et al., 2011). However, their
use, in special learning groups, has not been thoroughly examined (Daigle &
Sulentic Dowell, 2010 ; Gal et al., 2005 ; Tartaro, 2005).
In Daigle and Sulentic-Dowell's initial study (2008) on the use of DS as an intervention to improve the academic performance and social interactions of a sixth
grade student with high-functioning autism, positive fi ndings were reported. Similar
fi ndings were documented by Gal and his colleagues (2005) and Tartaro (2007) who
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combined DS with virtual peers for children with ASD. Also, Lindsey-Glenn and
Gentry (2008) reported promising results following the use of DS to improve
vocabulary skills of students with ASD.
It is notable that the software tools that have been used in the aforementioned
studies have not been designed specifi cally to cater for the special needs of this group
of learners. Subsequently the software does not accommodate the components of
structured teaching suitable for persons with ASD. Educators use the existing software by modifying its scope and operation in order to teach social skills to children.
Special education teachers need to organize the learning material and the way information is presented in order to maximize the achievement of their instructional goals.
The Di.S.S.A. Environment
The present chapter presents a software system of DS targeting children with special needs. The software created is called Di.S.S.A., which stands for Digital
Structured Storytelling for Autism. The software is designed and developed purely
for this research work and is programmed in action script 3 in Adobe Flash authoring environment. It is an online application. This is the fi rst time such software,
based on combining visual supports with elements of structured teaching, is developed to cater specifi cally for persons with ASD. To our knowledge, similar software
development has not been published.
The system is aimed to help people with ASD learn social skills through storytelling with the use of images, as sole text is, often, not the preferred way of presenting information. It evaluates user's performance and presents information about
user's progress. Its design is imposed by the need for a structured way of presentation, which has been proposed as appropriate for this group of learners (Mesibov
et al., 2005). Persons with ASD need to be aware of the meaning of an activity and
clear guidelines on how it will be done, how long it will be, and when it will be
fi nished. If this information is missing, they get confused and cannot perform tasks
independently. The development of Di.S.S.A. accommodates the above components
and users always know where they are in the application, what the purpose of their
actions is, and what the desired outcome is.
The design of the system (Di.S.S.A.) is based on the A.D.D.I.E. (Analysis,
Design, Development, Implementation, Evaluation) model. A.D.D.I.E is a framework that lists generic process for instructional design software (Dick & Carey,
1996 ; Leshin, Pollock, & Reigeluth, 1992). The content is adapted to the individual's capabilities by providing different levels of access (users have different capabilities depending on the severity of ASD), then the content is presented accordingly
to the adaptation module, the user creates the story, the system evaluates user's
performance, and a story with different actors but with the same concept is presented so that the users can interpret in a conceptual way the given task (children
with ASD have diffi culties generalizing what they learn in one setting to another
setting) (Stromer et al., 2006).
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Stages of the Application
The system is adaptive to cater for different levels of ASD and evaluates user's tasks
in order to provide feedback for user's performance. The architecture is based on
experiential learning (Kolb, 1984) and follows the basic steps of experiential
learning; namely, experience—refl ective observation—conceptualization, and active
experimentation.
When the application launches for the fi rst time, a questionnaire is presented that
is fi lled by an adult, such as the parent or the special education teacher. The questions of this instrument refer to the student's skills and preferences, the level of ASD,
the use of text, movements, sound, or text messages, the number of pictures used
and the visibility of the evaluation model. The relevant story is picked up from the
stories pool (Figs. 1 and 2). The story is adapted to suit user's special characteristics
and preferences. The system decides how information is presented to users (adaptive
approach) with respect to the adaptivity dimensions of user and content. This is not
an intelligent system as the system follows a series of production rules (if …then).
The user logs into the system by inserting his/her name. The system "remembers" him/her next time he or she logs in and automatically starts from the last point
he or she left the application. This function is accommodated, considering that students with ASD might give up trying while using the application. As repetition helps
them comprehend the task, the system "remembers" them so that they can continue
their trials from the point they stopped last time. Users (teachers, parents, or students
with ASD) choose a social skill to learn.
Fig. 1 The design of Di.S.S.A.
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The user is presented with four levels of graduated diffi culty for constructing a
story (Fig. 3). That is, he or she can choose the length of the story by selecting either
three, four, six, or eight images as breaking an activity into distinct steps has been
proved to be an effective method for achieving task mastery and independence in
special education (Swanson & Hoskyn, 1998). In the case of the system being used
by the student without supervision he or she can choose the desired level independently. In all other cases, a professional or a parent will select the appropriate level
of diffi culty for the student with ASD, considering the student's ability level and the
severity of autism.
Next, a sequence of images making a story is presented (Fig. 4a). Following this
presentation, the images are placed randomly in the screen (Fig. 4b) and users are
asked to rearrange the pictures in the correct order to create the story. An example
of a story (i.e., "opening up a present") is presented in Fig. 4 . For a realistic representation, the images used are photographs. A step-by-step procedure is followed.
At this point, Di.S.S.A. has been designed following structured teaching which
Fig. 2 Screenshot from Di.S.S.A. showing how the user can pick up a story
Fig. 3 Screenshot of the Di.S.S.A. showing the selection of level of diffi culty for users
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places a strong emphasis on the execution of a sequence visually presented (Mesibov
et al., 2005 ; Stromer et al., 2006). Therefore, the user can move to the next step only
if he or she is correct in selecting the fi rst picture of the sequence.
Feedback/Evaluation
The system provides sound messages to refl ect user's actions such as messages that
inform the user for the correct or false completion of tasks. The messages "lead"
users to the right direction in order to make the correct decision. The system tells the
users orally and through text messages what they have to do and what the outcome
will be at all stages of performance. Text messages are presented too, in order to
build a relationship between images and words. For the evaluation of use, their performance is recorded, stored, and compared with past trials for the same story by the
same user. Depending on the outcome of the evaluation, the system executes a
refl ective function by selecting the same story theme, but with different images.
This is necessary given the need for skill generalization. Therefore, a new story is
presented to make sure the learner has acquired the social skill being practiced independently of the particular sequence of images.
A "present" is given every time the user places an image in the correct place.
Persons with ASD are very fond of animals (O'Haire, McKenzie, Beck, & Slaughter,
2013). So the system is designed to include an animated animal that places itself in
a different area in the screen other than the user's working space, to prevent any
distractions (Fig. 5). The user is given a different present for every story to avoid
boredom. In the end of the session the user can save a collection of all animated
animals in the system.
Following task completion, a result log is presented that contains the number of
errors of the user and the duration of engagement in each task. This is a useful tool
for comparing data to check for improvements between different trials of the same
user (Fig. 6).
Fig. 4 Screenshot of the Di.S.S.A. for making a sequence of images for the story
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Fig. 5 A "present" is given every time user puts an image
Fig. 6 Sample evaluation log with data about task performance
Pilot Application of Di.S.S.A.
A pilot application of Di.S.S.A. was carried out with four pupils (7-11 years) with
a formal diagnosis of mild autism. Two aspects of the application were evaluated:
user's satisfaction and user's task accomplishment. The evaluation was conducted
in collaboration with the parents of the children and took place in their homes.
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The researcher met with the parents to demonstrate the application and explain to
them how to operate it. Each child used the system on their own with instructions
and supervision by their parents, who had chosen the level of diffi culty that was
suitable for their own child. Each child learned three social skills by using DiSSA
and three social skills with DS software for typically developing children. The software used is Photo Story by Microsoft that doesn't use steps when operated and its
interface design is "busy" as it is addressing typically developing children. It is used
to create a story but it does not accommodate features as correct sequence of images
and subsequently a module of informing the user for the correct completion of
tasks. The stories were aimed to teach them: (a) to cross a road, (b) to open a present, and (c) to wash their teeth.
Interviews with the parents were carried out by the fi rst author of this chapter. In
general, the fi ndings were positive for both aspects of the application and all children with ASD reported that they enjoyed using the application. Their parents
emphasized that their child had learned the sequence—stressing that story was easier than using printed images or other educational tools. They mentioned that their
children showed a strong motivation to operate the application and seemed highly
concentrated on the tasks. Also, they repeated patiently the whole procedure as
often as it was required. In contrast, when they were asked to repeat the task with
the DS software not specifi c for ASD, their parents had to spend more time to convince them to practice the repetition. Parents reported that their children appeared to
view Di.S.S.A. much more like a game than as a learning tool. They particularly
liked the animated "presents" and this function (to collect as many animals) motivated their engagement with the application.
One of the parents described user's experience as an enjoyable activity: "My son
didn't want to leave the application. He liked using it."
The same parent added: "My son wanted to use it even after he did all three stories. He asked for more stories. I think he thought it is a game, maybe because he
was getting the animated animal as presents when he was fi nishing a task and he

wanted to get more presents."
Another parent reported that his son learned the sequence of images very easily
comparing tasks with printed images that he had tried before. Also, he reported that
his son considered the application as a game and not as a lesson. He said: "I think
that my son wanted to get as many more animals as possible, I am not sure but I
think that this is one of the reason he liked using DiSSA."
Another comment was the following: "My son got confused with the other software and I needed to stand by him all the time he was operating it. I needed to tell
him every time he was placing an image if he used the right one. When I left him
alone for a while he got upset."
In summary, the main fi ndings of the pilot application were the following: (a)
children with ASD liked more using Di.S.S.A. than the DS software nonspecifi c for
ASD, (b) the structured style of presentation helped them maintain their attention on
the task, (c) they could operate Di.S.S.A. almost independently, and (d) they learned
the tasks given faster using Di.S.S.A comparing to the DS software nonspecifi c for
ASD. The above are important fi ndings for children with ASD as they address
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crucial elements in their learning process as these children have different learning
characteristics than their typical peers. The special needs that complicate their
learning are diffi culties with verbal expression, diffi culties with remembering verbal instructions, their short attention span, the lack of organizational skills, the dif-
fi culties to generalize the skills taught and to make transitions from one task to the
other. Since Di.SS.A. utilizes greatly visual processing and presents information in
a structured way through computer systems it carries the potential to be an appropriate method for teaching social skills to this special group of learners.
Following the positive outcomes of the pilot application, an intervention with
more participants has been scheduled to take place, including persons with differing
levels of autism severity.
Conclusions
Despite the changes of the format of narrative storytelling over the years, its purpose
remains the same. Digital storytelling serves the same principles as the inclusion of
new media using computer technology. The number of available DS systems is
indicative of their importance for teaching academic and leisure skills. Their use, for
learners in the autistic spectrum, has been introduced in the last decade and systems
appropriate for this special group of learners need to be designed and evaluated.
Di.S.S.A is our fi rst attempt to design DS software specifi cally for users with
ASD. It utilizes components of Structured Teaching, such as minimal use of written
messages and extensive use of images, for the instruction of social skills to persons
with ASD which have been recommended as appropriate for students with
ASD. Another useful feature for professionals and parents is the continuous
evaluation which allows them to record user's competence in the learning task.
It is anticipated that the use of DS is a promising method for making learning
more appealing, raising students' productivity and positive affect during individual
or collaborative activities. Future work includes testing the application to a large
number of people with AS, in different settings, targeting not only social skills but
other skills too such as safety and hygiene skills and other learning subjects such as
mathematics and history.
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Codes (19)###emergent modelling, ###guided reinvention, ##human activity, ##real life, #RME, #RME theory, $$$modelling, $$mathematics education, $$teaching methods and strategies, $mathematics, $mathematics content, %role players, &&student support, &distance education, &pedagogy, @@challenges when working with ICT, @@ICT systems: virtual learning environment (VLE), @@value of using ICT, @Devices

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Document body:

Pretesting Mathematical Concepts with the Mobile Phone:
Implications for Curriculum Design
Abstract
One of the neglected elements when teaching at a distance is establishing what learners
already know at the beginning of the course or module. Unlike the face-to-face environment, in distance learning there is no opportunity for administering diagnostic activities
just before the onset of instruction. This means that both the weak and more advanced
students receive the same level of support since there is no mechanism for differentiating
their learning needs. This paper describes the characteristics of a diagnostic test aimed at
determining student understanding of the basic calculus concepts of the derivative and the
integral, using the mobile phone as the method of delivery. As a proof-of-concept exercise,
10 questions designed to test concept attributes and procedural knowledge involving the
two basic calculus concepts were given to a sample of 30 students at the beginning of the
course. The implications for curriculum design were then analysed in terms of the didactical functionalities and the communication strategy that could be developed with reference
to the mobile phone.
Keywords: Prediagnostic testing; mobile phone; distance learning; undergraduate calculus teaching; Realistic Mathematics Education; didactic functionalities
Introduction
Imagine the following scenario:
You are a student wanting to take a distance calculus
course, but you aren't sure that you are really up to it. It has
been a while since you did your high school mathematics.
You quickly access the UNISA Web site via your phone for
the pretest they offer and take it. The results indicate the
areas you need to revise before joining the course. You
contact a social network friend who has been in a similar
situation. He or she immediately texts back a message
attaching an online calculus manual that you need to go
through. Once home, you continue your preparations for
the next pretest on your computer.
The scenario is based on the observation that mobile learning is an emerging and rapidly
expanding field that provides new opportunities for learning improvement. Strictly speaking, mobile learning is a very broad term that is used in different contexts and that involves
a number of mobile devices such as mobile phones, personal digital assistants (PDAs), various forms of handheld devices, and even notebooks. For the purposes of this article, I am
using the term mobile learning to refer to handheld pocketsize technologies that can be put
in your pocket at the point of learning. In the South African distance education environment, where the majority of students simply cannot afford access to web-based learning,
the use of a mobile learning strategy seems to be a viable and rational alternative for providing instructional support.
Previously, the adoption of mobile learning was constrained by slow networks, limited services, and hesitancy on the part of organisations to invest in devices whose shelf life was too
short. Mobile phone penetration and adoption were not really matched by a parallel uptake
of mobile learning, despite predictions of a possible mobile learning revolution (Wagner,
2005). Reasons posited for the delay in adoption included limited and non-standardised
broadband distribution capacity (Wagner, 2005), device attributes such as screen size, battery life, and security (all of which hinder learning), limited resources, and a lack of organisational acceptance (Brown, Metcalf, & Christian, 2008).
The limitation of mobile content development and distribution is slowly changing as demonstrations of the potential and actual roles of mobile technologies increase. Consumers
(including distance students) who are apt mobile users have established a demand for mobile technology use. According to Traxler (2007), mobile devices are creating a new "mobile
conception of society in which we are beginning to look at new ways of creating and accessing knowledge, performance, art forms, and even new economic activities" (p. 4). However,
not enough research has been conducted to identify the actual functionalities that should be
targeted when designing instruction using the mobile device in teaching.
Within the context of distance teaching, a neglected element is the establishment of learners' prior knowledge at the beginning of the course or module with a view to offering the
learner appropriate support. This type of diagnostic testing, prior to the onset of a learning
unit or module, is slightly different from the pre-enrolment tests offered via paper (postal
system), by telephone, or through prerequisite courses. While pre-enrolment testing establishes learner readiness for participating in a course, the precourse diagnostic testing is
conducted to determine the amount and level of support a learner will require to advance
successfully within the course. Unlike the face-to-face environment, in which a teacher has
the time to briefly find out where the learners are before the course begins, in distance
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learning there are few opportunities for administering diagnostic activities once students
have enrolled for each course or module. This means that both weak and more advanced
students receive the same level of support since there is no quick mechanism for responding
to individual learning needs (as would be the case in a traditional classroom). The problem
is exacerbated by the large number of students enrolled in any distance learning module.
This paper describes the characteristics of a precourse diagnostic test aimed at determining initial student understanding of the basic calculus concepts of the derivative and the
integral, using the mobile phone as the method of delivery. The aim of the project was to try
to find out what the didactic functionalities of a simple mobile phone were and how these
could be exploited to inform supporting learning mathematics at a distance. The questions
posed were as follows:
• Can we use the mobile phone to facilitate the administration of diagnostic activities
quickly?
• If we are going to use the mobile phone, what are its didactic functionalities?
• How can these functionalities be exploited in order to inform teaching and learning as
well as curriculum design in distance education?
The Role of the Mobile Phone in Providing Learning Support
There have been a number of examples of successful mobile learning interventions. For
instance, Wei and Chen's (2006) e-book interface design allowed students to enter queries
on the text which were transferred to a discussion forum accessible through the mobile
phone. Hartnell-Young and Vetere (2008) personalised learning by letting students capture everyday aspects of their lives in order to reconstruct their lived narratives within a
classroom environment. Kinsella (2009) built a mobile application which allowed students
to anonymously post questions to the teacher, who was then able to give summarised feedback to all participating students in real time. The reports from South Africa have been
predominantly around the SMS "short message system" usage. These include Viljoen and
Carlcook's (2005) exploration of the experiences of adult learners' use of SMS as a support
tool while enrolled in the University of Pretoria's distance education unit and Van Rooyen's
(2010) investigation of student responses to the integration of SMS as support tool systems
in an accounting module at UNISA.
The current smart phones have multimedia conduits which, if exploited, are potentially
useful for teaching and learning in many subjects. Students can collect evidence and data
via audio and video recordings and by taking photographs. Students can send short text
messages between mobile telephone devices or even send messages that include multimedia objects (e.g., video images and audio recordings). All these features can make a difference to the way in which students build new knowledge (Daher, 2010).
Mobile Phone Use in Supporting Mathematics Learning
Mobile phone usage in mathematics learning is slowly increasing. For example, in their
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research, Yerushalmy and Ben-Zaken (2004) established that the mobile phone was a tool
capable of opening up fascinating opportunities for learning because of its mobility, availability, and flexibility. Professor Yerushalmy and her team at the University of Haifa developed Java-based mathematical applications that can be installed on most mobile phones.
The devices operate as minicomputers and are used for constructing graphs, solving equations, and sketching functions. Their Math4Mobile phone applications for learning algebra,
geometry, and calculus can be accessed from the Web at www.math4mobile.com. This team
has developed activities for elementary, middle, and high school students. Dynamic lessons
where students explore mathematical concepts can be accessed from mobile phones. This
means that students who have no access to classroom computers are not excluded from
these learning opportunities.
On the basis of their research study, Roschelle, Patton, and Tatar (2007) claim that using
mobile devices transforms the mathematics classroom into a student, assessment, knowledge, and community-centred entity. Genossar, Botzer, and Yerushalmy (2008) analysed
the learning processes and experiences that occurred within a mobile phone learning environment. They found that, apart from making the dynamic mathematical applications
more accessible, the mobile phone-enabled students to engage in real, authentic tasks. For
instance, students could record simple dynamic events with their cell phones' video cameras and later convert the video clips into mathematical models using the mathematics applications. The mobile phone learning environment also allowed the students to work collaboratively by sharing graphs and formulae and solving
problems collectively. This helped
establish a community of mathematics learners. In a recent study (2010), Daher confirmed
the formation of collaborative communities of budding young mathematicians. Preservice
teachers collected data that supported the finding that middle school student knowledge
building could be fostered through learning mathematics in a mobile phone environment.
Students jointly engaged in and solved authentic mathematical problems with the help of
mobile phones.
In South Africa, the Meraka Institute (a research institute) has been at the forefront of
the innovative use of mobile/cell phone technology to support the teaching and learning
of mathematics. The Institute has been able to develop a mobile tutoring system, Doctor
Maths, which runs on the MXit platform. MXit is a very popular instant-messaging service
that is accessible via cell phones and that has over three million school-age subscribers.
Volunteers from the Department of Engineering at the University of Pretoria offer real-time
mathematical support to the high school students using the MXit chat facility on students'
cell phones at reduced rates. From an initial enrolment of just 20 students, to date the
service has grown to support over 1,000 students (Van Rooyen, 2010). However, even with
this evidence, it is still difficult to clearly isolate the actual functionalities that should be
targeted when designing instruction using the mobile phone in teaching.
Diagnostic Testing in Distance Education
The aim of the precourse diagnostic testing carried out in this project was to determine,
as quickly as possible, students' prior knowledge of the subject at the beginning of an in-
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troductory distance calculus course. The type of pre-testing administered in this project is
slightly different from the multiple-choice computerised diagnostic testing administered
to determine students' mathematical skill competencies with a view to suggesting remedial strategies (Appleby, Samuels, & Treasure-Jones, 1997). Prior research done at the undergraduate level consistently reveals that students start calculus courses with a limited
view and understanding of the functional concept (Tall, 1996; Ferrini-Mundy & Lauten,
1993). In addition to this, students often exhibit cognitive difficulties when interpreting the
functional concept using algebraic and graphical representations (Schnepp & Nemirvosky,
2001). The aim of this project was to ascertain if the students who enrolled for the calculus
course at UNISA had an understanding of the two basic calculus concepts, the derivative
and the integral, and to subsequently build on that knowledge to inform future instructional design decisions.
The use of precourse diagnostic testing in order to support student learning is not a new
idea. Over the years, the recognition that students' initial beliefs and ideas about particular
phenomena form the basis of their future learning was used to design and develop diagnostic instruments to test and support student learning (Halloun & Hestenes, 1985). Currently, there are many examples of computer-assisted tutoring and testing systems (Chiou,
Hwang, & Tseng, 2009; Springer & Pear, 2008). The advent of web technology has also
provided us with a platform on which web-based diagnostic systems can be accessed by
face-to-face and distance students alike (Bälter, 2009).
For this study, it was important to identify, as a proof-of-concept exercise, how the mobile
phone could be used to enhance the learner's experience in a way that current distance
learning provision was failing to do. The main aim of the study was to establish what the
didactic functionalities of a mobile phone were and how they could be exploited to inform
prediagnostic testing design at a distance.
Theoretical Framework of the Study
The learning transaction between teacher and student in a distance learning environment
cannot occur without some form of technological mediation. The theoretical framework
used in this study therefore includes ideas borrowed from two parent frameworks, an instructional design (or teaching and learning framework) together with a framework for
analysing the role played by technology. The next sections are elaborations of these two
frameworks; note that these two frameworks were combined for the final analysis of the
project outcomes.
The Instructional Design Framework
The prediagnostic testing in this project is part of a larger project that examines the developmental efforts required to adapt the instructional design perspective of Realistic Mathematics Education (RME) to the teaching and learning of calculus via distance education.
RME was first introduced at the Freudenthal Institute in the Netherlands in the 1970s in
response to a call for reform in mathematics education. RME is based on Freudenthal's
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(1973; 1991) belief that mathematics education should take its point of departure in mathematics as a human activity, rather than mathematics as some sort of ready-made system.

The fundamental idea is that by designing instruction in which both everyday situations
and mathematical activities are mathematised, students can be helped to reinvent mathematics and (as a result) attain a better understanding of the subject.
The approach capitalises on mathematising (regarding or treating a subject or problem in
mathematical terms) as a central learning activity together with guided reinvention and
emergent modelling as central processes within the learning experience (Gravemeijer,
1994; Bakker, 2004; Zulkardi, 1999).
• Guided reinvention involves reconstructing "a natural way of developing a mathematical concept from a given problem situation" (Bakker, Dorman, & Drijvers, 2003).
• Emergent models are models that initially represent problem situations, but later on
develop into models of abstract mathematical objects and relations (Bakker, Dorman,
& Drijvers, 2003).
From a design perspective, the teacher needs to have some idea about where his/her students are in terms of the projected learning in order to design appropriate learning activities based on ideas that are less sophisticated than the mathematical concepts being developed. This is why the diagnostic test needed to be
administered quickly. In this case, the
diagnostic test was directed to getting a sense of students' initial knowledge of the concepts
of derivative and integral at the beginning of the course.
Framework for Analysing the Role Played by the Technology
Since the transaction between teacher and student in a distance learning environment cannot occur without some form of technological mediation, a type of construct was required
to analyse the role played by the technology. Cerulli, Pedemonte, and Robotti's (2005) perspective of didactical functionalities fit well with this requirement. Cerulli et al. (2005) regard didactical functionalities as those properties (or characteristics) of a given information
and communication technology (ICT) and its modalities of employment which may favour
or enhance teaching and learning processes in reaction to a specific education goal.
Another construct borrowed from Cerulli et al.'s (2005) approach was the education experiment cycle (EEC) model. This was used to identify the critical phases within the project.
The EEC model has three main phases: (a) the planning phase involving the setting up and
sequencing of activities; (b) the putting into practice phase, in which the planned activity is
implemented; and (c) the diagnostic phase, which involves evaluating the actors (teachers
and students) involved. A tool's modality of employment can be analysed in any (or only
one) of the three phases, depending on the educational goal.
In order to illustrate the universality of their theory, Cerulli et al. (2005) demonstrated how
the perspective of didactic functionalities could be combined with different teaching and
learning theoretical frameworks in order to delineate the didactical purposes of the tech-Pretesting Mathematical Concepts with the Mobile Phone: Implications for Mobile Design
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nological tools employed in the course of teaching and learning. Examples of the teaching
and learning frameworks examined were Brousseau's theory of didactic situations, Nardi's
version of activity theory (concentrating on computer-mediated learning), and the theory
of instruments of semiotic mediation.
• Brousseau's (1997) theory of didactic situations organises the teaching process into
three parts: (a) the non-didactical, which is not specifically organised to allow for
learning; (b) the didactical, in which the teachers explicitly organise tasks to teach students forms of knowledge in a specific manner; and (c) the adidactical. The adidactical
refers to the process of channelling students into solving problems that are designed to
provoke students into finding their own solutions. As a result, students acquire the desired forms of knowledge and reasoning for understanding given concepts. At the heart
of Brousseau's theory is the milieu, which describes the middle ground of the teaching and learning cognitive space in which the teacher, the learner, and all the facets
of the teaching/learning environment interact. The basis for this theory is a Piagetian
view of learning, where the learner goes through the universal processes of assimilation
and adaptation in well-defined stages of learning development until he or she reaches
the complete adidactical situation of taking full responsibility for his or her learning.
Analysis of tool use in this case focuses on interactions and retroactions or feedback.
The modality of employment involves situations where the learners interact with a tool
and receive feedback. This can be achieved in either the planning or implementation
phases. The didactical functionalities are defined in terms of the user's retroactions.
• Activity theory is a research framework entrenched in a set of viewpoints derived from
the work of the Russian psychologists Vygotsky, Leont'ev, and others in the 1920s
(Nardi, 1996; Engeström, 1999). Activity theorists seek to understand and scientifically explore individual consciousness (cognitive acts) rooted in everyday practice and
embedded in social mediums composed of people and artefacts (Nardi, 1996). Artefacts
here may be either tools or symbol systems (e.g., languages). In this project, the point
of reference is Nardi's version of activity theory as applied to human computer interaction (HCI) research. Consequently, the examination of tool use in this context shifts to
the structuring of activities. The modalities of employment are geared toward creating
activities designed to encourage the engagement of all participants. In this case, the
didactical functionalities are defined in terms of how the tools are used to structure the
activities.
• The last example Cerulli et al. (2005) refer to is one using the theory of instruments of
semiotic mediation. This approach is derived from a Vygotskian social cultural theoretical view of learning, in which the development of meaning is based on a phenomenological experience of the learner with other members of the society, with the help
of an artefact or tool. According to Mariotti (2002), the artefact serves as a semiotic
mediator because it is used by the teacher to help the learner construct meaning. For
the learner, meanings emerge as he or she participates in an activity involving the artefact. Here, the modalities of employing the tool require setting up activities involving
the tool as well as orchestrating discussions using signs developed by the tool to assist Pretesting Mathematical Concepts with the Mobile Phone: Implications for Mobile Design
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meaning making.
The Combined Framework
For this project, Cerulli et al.'s (2005) perspective of didactic functionalities and the RME
approach to instructional design were combined to create a framework for designing and
evaluating the precourse assessment activities. The RME framework was used to guide the
design of the preassessment activities. The resulting theoretical framework, consisting of
a combination of RME and the perspective of didactic functionalities, is shown in Figure
1. In terms of the perspective of didactic functionalities, the tool being considered in this
research was the mobile phone within the mobile learning platform. The goal was to establish the tool's didactic functionality. The didactic functionalities of the mobile phone were
analysed in only one modality of employment, that of diagnostic testing.
Figure 1. RME and the perspective of didactic functionalities in prediagnostic testing design using the mobile phone as a didactic support tool (diagram developed by the author).
Method
A prediagnostic test aimed at determining initial student understanding of basic calculus
concepts (the derivative and integral) was delivered using both print and the mobile phone.
Thirty volunteer students took a 10-item prediagnostic test that was delivered using both
print and the mobile phone. The students were UNISA students who had completed their
first semester in a first-year calculus course. The researcher for the study was the curriculum designer, who also played an instructor role in this pilot study.
The questions were designed to gauge the students' understanding of functions, their interpretation of graphs, and their understanding of the terms the derivative and the integral
(see Appendix). Questions 1 and 2 were designed to test students' understanding of the
function and its graphical representation. Questions 3 to 7 tested students' understanding
of the derivative, including personal definitions, graphical interpretation of the concept,
and use of simple formulae. Questions 8 to 9 tested students' personal definitions of the
integral and included one typical integral evaluation problem. Question 10 was designed to
check the students' understanding and graphical interpretation of both the derivative and Pretesting Mathematical Concepts with the Mobile Phone: Implications for Mobile Design
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integral concepts in relation to the average velocity and total distance covered by a moving
object.
The Creation of the Content
OutStart-Hot Lava Mobile, available at http://www.outstart.com/about-hot-lava-mobile.
htm, was used as the learning development and management platform. Figures 2 and 3 are
representations of two question items (1 and 4) as they appeared in the normal text version
and the mobile phone version.
Normal text version Mobile phone version
Q1 Let f be a function defined by
2sin(x + 3) and let g be the function
defined by
2sin(u + 3) for all real numbers x
and u, then,
A f and g are exactly the same functions
B f and g are different functions if x and
u are different numbers
C not enough information is given to
determine if f and g are the same
functions
Figure 2. Representation of question 1 in the normal and the mobile phone versions.
Normal text version Mobile phone version
Q4 Interpret from the graph what the quotient

$$\frac{f(b) - f(a)}{b - a}$$

means.

Figure 3. Representation of question 4 in the normal and the mobile phone versions.Pretesting Mathematical Concepts with the Mobile Phone: Implications for Mobile Design
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The Hot Lava software integrated an authoring system with a mobile delivery and tracking
system. The integrated system of components provided a fast and efficient way of designing, creating, editing, deploying, and tracking content that could be delivered to different
mobile phone models. One advantage of Hot Lava was that the content could be delivered
on a variety of mobile phones. The other main advantage was its ability to track and see
which students were logged in and when they registered. On individual screens, the instructor could see the order in which students attempted the questions and the number of
attempts at each question before a final answer was submitted. I could also see how much
time students spent on the test.
Results
Only nine students (30%) of the group preferred to take the test on the mobile phone. Those
who declined to take the test on the phone felt that they were more comfortable working
with mathematics on paper. However, all of the students who took the test on the mobile
indicated that they enjoyed it. There were no notable differences in terms of overall student
performance on tasks. There were a few differences in terms of the number of questions
completed (due to input issues), the turnaround time for receiving the answers, and the
availability or nonavailability of a system for tracking student responses. Table 1 is a comparative summary of some of the findings.
Table 1
Comparison of Students who took the Print and Mobile Versions of the Test
Print version Mobile version
Number of participants involved
21 9
Number of completed tests 17 9
Time taken to complete test Not available 7 minutes and 45 seconds
Time spent on each question Not available 46.5 sec
Questions not completed None Q5 and Q8 (which required written
inputs)
Discernible difference in performance on questions
None None
Tracking information from the Hot Lava tracking system revealed that the average time
spent on the mobile test was 7 minutes and 45 seconds.
Responses
The test sought initial student interpretations of the function, the difference quotient, and
the concepts of derivative and integral. The following questions were used to guide the inquiry in terms of the four constructs.Pretesting Mathematical Concepts with the Mobile Phone: Implications for Mobile Design
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• Could students recognise that a function represented a relationship between the variation of one quantity (input) with the variation of another quantity (output)?
• Were students aware that the difference quotient
$\frac{f(b) - f(a)}{b - a}$
represented a ratio between a change in output and a change in the input quantities?
• Did students associate the derivative with a rate of change of a quantity at a specific
instant?
• Could students link the integral to an accumulated area under a curve?
Responses from the Printed Version of the Test
From the printed version of the test, only 17% of all participating students provided an acceptable interpretation of the concept of function.
The responses related to the difference quotient revealed that the majority of students had
not yet developed a satisfactory interpretation of the difference quotient
$\frac{f(b) - f(a)}{b - a}$.
For
example, one student referred to the quotient as the "gradient of the tangent to curve," even
though the line drawn connecting the two points, (a, $f(a)$) and (b, f (b)), was not a tangent
line. Another student defined the quotient as "the region between the graph y (f (x)) and the
line." No respondent mentioned anything about the average rate of change of a function,
which was what the quotient represented. A number of students said that the value of the
quotient increased or the gradient became steeper as b moved close to a. The respondent
who defined the quotient in terms of area said that "the area covered by the curve and the
line as b moved closer to a." One student correctly stated that "as b moves nearer to a, the
gradient of the chord approaches the gradient of the curve at point (a, $f(a)$))."
The general descriptions of the derivative given by the students depicted the derivative as
an entity resulting from mathematical manipulations. Below are three examples of derivative definitions:
A derivative is a task that gets an expression out of a
function and makes that found expression be a function
on its own;
A derivative is a function or constant obtained from
differentiating a previous function one or more times;
. . . a derivative is a mathematical equation or constant
obtained after def.
All the students using the print version were able to interpret the integral sign $\int_a^b f(x)\,dx$
correctly. Responses included descriptions such as ". . . this is integration of the function (x)
with respect to x between limits a and b" and "the term that means the total area between Pretesting Mathematical Concepts with the Mobile Phone: Implications for Mobile Design
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the f (x) and the x-axis within the limits b (which is the upper limit) and a (which is the
lower limit) on the x-axis." There was also a computational or procedural definition: "to
integrate means to raise the power of the function f (x) by 1 and divide the top function by
what you get after raising the function." The majority of students carried out the graphical
evaluations of integrals correctly.
Responses from the Mobile Version of the Test
Responses from students using the mobile version were similar to those who used the print
version. Students still confused the difference quotient and the derivative. One respondent
defined the difference quotient as "the gradient of the curve found from first principles, an
approximate value of its gradient." Students gave the correct interpretation of the integral
and acceptable responses to Q9 and Q10, which involved graphical interpretations. The
multiple question items on the mobile phone were all completed, but some of the items that
required filling in were not completed. In contrast, all students completed all 10 items when
using the print-based format.
Conclusion and Future Directions

Even though the sample size was quite small, one can deduce that the important didactic
feature of the tool (the mobile phone) was mainly the speed and ease of tracking and analysing student responses. From this preliminary study, it was clear that the utility of the
mobile phone for pre-enrolment diagnostic testing in distance learning could be enhanced
if the phone was used in conjunction with a mobile phone platform such as Hot Lava. As
far as the adoption of an RME perspective in the design of instruction is concerned, the
phone would be useful in quickly determining a student's starting point before embarking on any form of learning. Simple diagnostics tests could also be interjected at different
teaching points during the course of learning to check student progress. In terms of student
precourse testing, it seemed that the most prominent didactic functionality of the mobile
phone was the speed and ease of tracking and analysing student responses.
One of the limitations of the mobile device was the students' difficulty in handling questions
that required manipulating symbols and equations. The mobile phone interfaces of the
smart phones used were unable to handle this effectively. Another drawback to using the
Hot Lava platform was that this platform is too expensive to maintain in a developing world
context. For example, the cost for the initial testing was US$1,000 for the 30 students, and
this fee did not include the additional charges required for each student registering on the
Outstart-Hot Lava Web site.
Taking into account the limitations of this study (e.g., small sample size), one can draw the
following conclusions:
• The mobile phone (embedded within a mobile phone platform such as Hot Lava mobile) can be used for precourse testing in a distance teaching and learning environment.
• The modality of employment for the tool in this project was as a diagnostic tool. The Pretesting Mathematical Concepts with the Mobile Phone: Implications for Mobile Design
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tool's didactic functionality was positively influenced by the speed and ease of tracking and analysing student responses but negatively affected by issues of availability
(related to cost) and ease of use (linked to the difficulty of manipulating symbols and
equations).
• For effective delivery and acceptance by users, one would need to find a way of using
mobile devices where issues of affordability and ease of usage are adequately addressed.
Tim O'Reilly (2008) projects similar futuristic design ideas when he imagines future mobiles that have sensor-rich interfaces (having features such as a microphone, a camera, a
touch screen, an accelerometer). These devices are also able to link to an array of cloud
services that support data recognition and retrieval in order to contain assets such as voice
recognition and location sensors (GPS or cell triangulation). The technology is certainly
advancing very quickly, and it is therefore possible that in the near future devices and affordable systems allowing easy manipulation of symbols and more opportunities for quality
student-teacher interactions will be within reach of the majority of students. For now, it
becomes more efficient to make instruction design decisions based on the required human
functions and make adjustments as the technology progresses.
While it is essential to understand the strengths and weaknesses of a particular technology
such as the mobile phone and use this information to deploy good pedagogical practices
to achieve specific learning goals, we should—as do Jay Cross, Tony O'Driscoll, and Elif
Trondsen (2007)—ask the question: "What can this technology do that will enhance the
learner's experience that my current learning provision does not?" For example, precourse
assessment provision could be leveraged by organising it around a functional set of student
needs, such as flexibility of symbol manipulation and affordability. One would then be able
to examine the affordances the mobile device allows and map it onto students' ease of use
and course instructors' priorities to develop relevant precourse assessment content.
In general, students preferred the paper-based tests, although they welcomed the increased
opportunities for communicating with their tutor. Interviews would have to be integrated
into future studies to explore in-depth perceptions of student experiences of mobile phone
use. The results from this proof-of-concept project can inform the design of projects involving precourse assessment with a larger number of students. More focused research is still
required to determine how best to turn the capabilities of the mobile phone into a leading
component of the testing environment in distance education.Pretesting Mathematical Concepts with the Mobile Phone: Implications for Mobile Design
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AppendixPretesting Mathematical Concepts with the Mobile Phone: Implications for Mobile Design
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47 quotations

Codes (18)###guided reinvention, ##mathematization, ##real life, ##RME tasks, #RME, $$$approaches to teaching Mathematics, $$$modelling, $$meaningful learning of Maths, $$teaching methods and strategies, $factors that influence teaching and learning, $learner aspects, $mathematics, %role players, &motivation, &pedagogy, &social aspects, @@ICT systems: virtual learning environment (VLE), =methodology

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From problem solving
to modelling
the emergence of models and modelling perspectives
NICHOLAS MOUSOULIDES, BHARATH SRIRAMAN
AND CONSTANTINOS CHRISTOU
More than 25 years ago, a research project in the U.S investigated the question: "What
is needed by students, beyond having a mathematical idea, that enables students to
use the mathematical idea in everyday problem solving situations? (Lesh, Landau &
Hamilton, 1983). The answer to this question has begun to emerge after 25 years of
systemic work in the domain of modeling. In this paper, we chronicle the emergence
of models and modeling perspectives (MMP) from the genre of problem solving
research via a synthesis of the major strands in the extant literature.
An increasing number of mathematics education researchers have begun
focusing their research efforts on mathematical modeling at the school
level. This is evident in numerous research publications from groups of
researchers in Australia (English, Galbraith and colleagues), Belgium
(Verschaffel and colleagues), Denmark (Niss, Blomhøj and colleagues),
Germany (Blum, Kaiser and colleagues), Netherlands (de Lange and colleagues) and the U.S (Lesh, Schoenfeld and colleagues). Among the questions that have been raised, is how well prepared are learners today to
solve the problems that they will encounter beyond school, in order to
fulfil their goals in work, as citizens and in further learning? (Christou,
et al., 2005; OECD, 2004; Doerr & English, 2003; Schoenfeld, 1992).
How can students work with problems that are less obviously linked to
school mathematics and require students to deal with unfamiliar situations by thinking flexibly and creatively (Lesh & Doerr, 2003a, 2003b)?
It is necessary to distinguish problem-solving activity as described by
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24 Nordic Studies in Mathematics Education, 12 (1), 23-47.
Schoenfeld (1992) and other researchers (for example, see English, 2003)
from the activity of solving traditional word problems. Traditional word
problems usually present simplified forms of decontextualized world
based situations. The purpose of such word problems is only the exercise for specific type of mathematical learning, such as addition or subtraction (Wyndhamm & Saljö, 1997). The use of word problem solving
in school mathematics hardly matches this idea of mathematical modeling and mathematization, which is the structuring of reality by mathematical means (Freudenthal, 1991). Student work with that type of
solving problems is absent of heuristics and mathematical strategies and
the result is a prevalence of mechanical and mindless solutions (Greer,
1997). A second, probably worse, consequence of student work in solving
traditional word problems is the absence of high level cognitive and metacognitive processes involved. This absence is forwarding students to look
for key words and employ direct translation strategies to solve a problem
(Schoenfeld, 1992). Burkhardt & Pollak (2006) take the modeling perspective of applied mathematicians in contrasting the pure versus applied
nature of mathematics relevant for today's world.
While the 'pure' and 'applied' viewpoints have many things in
common, including delight in the elegance of mathematics, they
differ in others that are important for the design of school curricula
- notably the central importance of teaching modeling. (p. 178)
This call is consistent with recent initiatives to emphasize the applied
nature of mathematics used in emergent professions. Steen (2001) argued
that in spite of the rich and antiquated roots of mathematics, mathematicians among others should acknowledge the contributions of researchers
in external disciplines like biology, engineering, finance, information sciences, economics, education, medicine etc who successfully adapt mathematics to create models and tool kits with far reaching and profound
applications in today's world. These interdisciplinary and emergent applications have resulted in the field of mathematics thriving at the dawn of
the 21
st
century (Lesh & Sriraman, 2005a, 2005b).
Another important perspective to mathematical modeling is that it
should also foster critical mathematics education (Skovsmose, 1994,
2000). Although the National Council of Teachers of Mathematics
(NCTM, 2000) calls for purposeful activities along with skillful questioning to promote the understanding of relationships among mathematical ideas, this recommendation can be pushed further and modeling
activities can be used as a way to cultivate critical thinking and critical

literacy (D'Ambrosio, 1998; Gutstein, 2006; Michelsen, 2006, Skovsmose, 2000; Sriraman & Lesh, 2006). In other words, modeling problems Nordic Studies in Mathematics Education, 12 (1), 23-47.
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situated in the real world can also be used to foster critical mathematical
literacy. Critical mathematical literacy
1
holds that the aim of mathematics learning and teaching is not simply to impart procedural skills and
functional literacy but create situations in which students are able to identify, interpret, evaluate and critique the mathematics embedded in social
and political systems and claims. The literature also reports success of the
critical approach using data from newspapers, maps, advertisements, and
ecological data etc as a means of getting students to think about the basis
for decision-making and the inequitable societal distribution of resources
(Gutstein, 2006; Sriraman & Lesh, 2006).
Lesh & Sriraman (2005a) contend that in mathematics and science:
A Modeling is primarily about purposeful description, explanation, or
conceptualization (quantification, dimensionalization, coordinatization, or in general mathematization) – even though computation
and deduction processes also are involved.
B Models for designing or making sense of complex systems are, in
themselves, important "pieces of knowledge" that should be emphasized in teaching and learning – especially for students preparing
for success in future-oriented fields that are heavy users of mathematics, science, and technology. Therefore it is important to initiate and study modeling, particularly those of complex systems that
occur in real life situations from the very early grades.
This perspective traces its lineage to the modern descendents of Piaget
and Vygotsky, but also to American Pragmatists like John Dewey, George
Herbert Mead and Charles Sanders Pierce. The philosophy of this perspective (Lesh & Sriraman, 2005a, 2005b) is based on the premise that:
- Conceptual systems are human construct, and that they also are
fundamentally social in nature.
- The meanings of these constructs tend to be distributed across a
variety of representational media (ranging from spoken language,
written language, to diagrams and graphs, to concrete models, to
experience-based metaphors.
- Knowledge is organised around experience at least as much as
around abstractions – and that the ways of thinking which are
needed to make sense of realistically complex decision making
situations nearly always must integrate ideas from more than a
single discipline, or textbook topic area, or grand theory.N. MOUSOULIDES, B. SRIRAMAN AND C. CHRISTOU
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- The "worlds of experience" that humans need to understand and
explain are not static. They are, in large part, products of human
creativity. So, they are continually changing – and so are the
knowledge needs of the humans who created them.
The models and modeling perspective adapts Zoltan Deines' instructional principles to design model eliciting activities. An example will help
better illustrate the notion of a model-eliciting activity. Suppose students
are asked to "rate" the quality of all the potential players on a volleyball
team and then select the team based on a consensus on the ratings. This
task requires students to gather/procure "objective" data related to players
speed, endurance, past performance, special abilities and reach "subjective" consensus on the criteria most valued for the selection of the team.
This is a model eliciting activity because it invokes the six instructional
principles of Lesh et al. (2003) namely, (1) the reality principle (i.e., does
the situation warrant sense making and extension of prior knowledge/
experiences?), (2) the model construction principle (i.e., does the situation create the need to develop (or refine, modify, or extend) a mathematically significant construct?, (3) the self-evaluation principle, (does
the situation require self-assessment?), (4) the construct documentation
principle (i.e., does the situation require students to reveal their thinking about the situation? (5) the construct generalization principle (i.e., is
the elicited model generalizable to other similar situations?) and finally
(6) the simplicity principle (is the problem solving situation simple?).
The construct of 'model-eliciting' circumscribes a problem solving situation, its mathematical structure, the mathematical models generated
as well as the problem solving processes that are invoked by the given
situation. Authentic examples of model eliciting activities based on the
design principles of Dienes are found in Lesh & Doerr (2003a). A recent
adaptation of the volleyball problem for the Danish context is the handball problem designed by Iversen and Larson (2006). The model eliciting
perspective is based on the premise that modeling research should take
into account findings from the realm of psychological concept development to develop activities which motivate and naturally allow students
to develop the mathematics needed to make sense of such situations.
More generally, models and modeling perspectives emphasize promising
aspects associated with both sociocultural theories and theories of situated cognition. Neither of these latter perspectives necessarily forces
researchers to re-examine their own basic assumptions about the changing nature of elementary mathematics – nor the changing nature of "real
life" problem solving situations in which some type of 'mathematical
thinking" is useful. In fact, just as in traditional research on problem
solving, both of these perspectives tend to begin with the assumption that Nordic Studies in Mathematics Education, 12 (1), 23-47.
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the researcher already possesses clear and accurate conceptions about the
nature of elementary mathematics – and about what it means to "understand" relevant concepts and processes (Lesh, personal communication,
November 23, 2005).
A modeling perspective to problem solving leads to the design of an
instructional sequence of activities that begins by engaging students with
non-routine problem situations that elicit the development of signifi-
cant mathematical constructs and then extending, exploring and refining those constructs in other problem situations, leading to a generalizable system (or model) that can be used in a range of contexts (Lesh &
Doerr, 2003a, 2003b; English & Doerr, 2006). For instance, in problem
solving activities, referred to as model eliciting activities (Lesh & Sriraman, 2005a, 2005b), the products that students produce go beyond short
answers; they include sharable, manipulatable, modifiable, and reusable conceptual tools (e.g. models) for constructing, explaining, predicting and controlling mathematically significant systems (Lesh & Doerr,
2003a, 2003b).
Students' descriptions, explanations, and justifications form an integral component of the models the students produce. In contrast to many
of the problem situations students meet in school, modeling activities
are inherently social experiences, where students work in small teams to
develop a product that is explicitly sharable. Numerous questions, issues,
conflicts, resolutions, and revisions arise as students develop, assess, and
prepare to communicate their products (English & Doerr, 2006; Lesh &
Sriraman, 2005a,2005b).
In an attempt to review the related literature and provide a coherent
state of affairs, the literature review in the present paper is organized into
three major discussion strands. The first major strand situates mathematical modeling as being a problem solving activity and issues related to the
teaching, learning and assessment of modeling are discussed. The second
major strand presents some basic principles for designing modeling activities and finally, the third strand discusses the benefits for students and
teachers in working with thought revealing modeling activities.
Mathematical modeling as a problem solving activity
Inadequacy of traditional teaching approaches
A number of researchers raise the question of the appropriateness of
current teaching approaches in teaching mathematics and in mathematical problem solving in particular (Doerr & English, 2003; Sriraman &
Lesh, 2006). The inadequacy of traditional approaches is even worse in N. MOUSOULIDES, B. SRIRAMAN AND C. CHRISTOU
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the case of students' work with problems that are less obviously linked to
school mathematics and require students to deal with unfamiliar situations by thinking flexibly and creatively (Lesh & Doerr, 2003a, 2003b). A
necessary distinction should be made between 'problem-solving' activity,
as proposed by Polya (1973) and Schoenfeld (1991) and relates to Dewey's
reflective thinking and the activity of "solving problems" and the traditional
use of word problems in a school environment. Polya (1962) stressed that
"in solving a word problem by setting up equations, the student translates
a real situation into mathematical terms; he has an opportunity to experience that mathematical concepts may be related to realities, but such relations must be carefully worked out" (p. 59). National Research Council
documented that mathematical problem solving rather deals with data
and observations from science. Mathematical modeling is far more than
just calculation or deduction; it involves observations of patterns, testing
of conjectures, and estimation of results (Schoenfeld, 1992). Word problems appear in most textbooks usually represent recontextualized forms
of decontextualized descriptions of everyday life situations. In most of
the cases, such problems are just exercises for specific types of mathematical learning, such as addition or subtraction (Wyndhamm & Saljö,
1997). Hiebert et al., (1996) pointed out that solving such word problems cannot prepare students for everyday life, since students are not
able to transfer the specific domain-related knowledge (mathematics)
and also more general problem- and solution-related skills. Their recommendation was to base instructional design on problem solving in outof-school situations, rather than on problem-solving models of limited
applicability (Hiebert et al., 1996).
A significant number of recent research studies pointed out that the
practice of word problem solving in school mathematics hardly matches
this idea of mathematical modeling and mathematization (Reuuser &
Stebler, 1997; Kaiser, Blomhøj & Sriraman, 2006; Kaiser & Sriraman,
2006; Mousoulides et al., 2006). As a result, students readily 'solve'
unsolvable, even absurd, problems if presented in ordinary classroom
contexts. Students do not have opportunities to do mathematics; students
look for key words, employ direct translation strategies when solving
stereotyped word problems and their ability in solving these problems
is influenced by contextual information (Greer, 1997, Verschaffel, De
Corte & Lasure, 1994; Yoshida, Verschaffel & De Corte, 1997).
The role of context in mathematical modeling
Although modelling problems are contextualized by nature, and authentic problems always manifest themselves in a context, it is nevertheless Nordic Studies in Mathematics Education, 12 (1), 23-47.
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important to explicitly address the role and the nature of contexts used.
The role of context is very important in mathematical modeling, since
modeling requires a context in which to "frame" the problem and "develop"
the mathematics. This meaning of contextual teaching and learning is
close to the notion of situated learning (de Lange, 1992). Researchers
have documented a number of different functions a "realistic context" can
play in problem solving. De Lange (1987) listed the following functions:
enabling concept formation, facilitating model formation, providing a
wider range of utility and more interesting practice problems. Similarly,
DaPueto and Parenti (1999) reported the following factors for using a
realistic context in mathematical problem solving:
A Facilitating collaboration, interaction and contribution of students
who have different styles of exploration/understanding/use of concepts, different levels of formalized knowledge, etc,
B helping more balanced development of reflective learning and experiential learning, and
C facilitating the design and restructuring of the schemata through
which knowledge is organized (p. 7).
A number of researchers commented on the importance of contextual
based problem solving. Bohl (1998) identified the lack of relevance as a
critical factor in engaging students in problem solving. Bohl (1998) documented that the use of contextual settings has the potential to assist
students in actively be engaged in problem solving. Pace (2000) further
pointed out that students need to experience real-world situations in the
chosen context before they can create models for solving world based
problems related to the same context. Contextual based teaching encourages the development of an active learning environment which in turn
generates better mathematical understandings by the students (Pace,
2000). In line with previous findings, Gravemeijer and Doorman (1999)
stressed that research on the design of Realistic Mathematics Education
(RME) based activities has shown that the use of personalized contexts
improved word problem solving by increasing the meaningfulness of
contexts and enhancing student motivation. Gravemeijer and Doorman
(1999) also claimed that "well-chosen context problems offer opportunities for the students to develop informal, highly context-specific
solution strategies" (p. 117).
Research has also indicated a number of possible limitations related
to the use of contexts in problem solving. The choice of context may
be a turn-off to particular students or be so familiar as to inhibit some N. MOUSOULIDES, B. SRIRAMAN AND C. CHRISTOU
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approaches judged as unreasonable (Choi & Hannafin, 1997). McNair
(2000) pointed out that if the situation and experience used are not familiar to the students, then instruction may degenerate into the presentation
of abstract ideas related to the context. Finally, Resnick (1988) addressed
that there is a need for finding ways to create in the classroom situations
of sufficient complexity and engagement that they become mathematically engaging contexts in their own right.
Affective factors and mathematical modeling
The importance of the affective domain and students' communication in
classroom work is stressed in a number of research studies (Lesh & Doerr,
2003a, 2003b; Gravemeijer, 1997; Verschaffel, De Corte & Borghart,
1997). Yoshida, Verschaffel & De Corte (1997) reported that mathematical problem solving is now also focussed to student attitudes and beliefs
and their capacity to apply their mathematical knowledge in non-routine
problems. Verschaffel and his colleagues (1997) documented that student
efficacy beliefs could act as a predictor of student achievement in solving
authentic problems. The importance of considering student conceptions
is also pointed out by Confrey & Doerr (1994), who argued that the use
of learner-centered modeling tools and approaches can create positive
beliefs in the mathematics classroom. The use of realistic mathematical
modeling problems can enhance student sense-making, while bringing
real-world situations into school mathematics is a necessary condition
to foster a positive attitude towards mathematics. In line with previous
findings, Bonotto and Basso (2001) interestingly reported that bringing
real-world situations into school mathematics is a necessary condition to
foster a positive attitude towards mathematics for many students.
While the use of modeling as a problem solving activity can positively
change student beliefs towards mathematics, a significant hazard in teaching mathematical modeling is teachers' pre-conceived beliefs, and the
projection of those beliefs onto their students. Verschaffel and his colleagues (1997) reported a strong and resistant tendency among teachers
to exclude real-world knowledge when teaching arithmetic word problems. Teachers' pedagogical beliefs stressed that non routine problems
are less important and the goal of teaching word problem solving is to
find the correct numerical answer. Verschaffel and his colleagues (1997)
concluded that these teacher attitudes and beliefs about the significance
of real world knowledge in problem solving have a negative impact of their
teaching practices and consequently on their students' learning processes
and outcomes. Nordic Studies in Mathematics Education, 12 (1), 23-47.
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A quite interesting aspect of the relation between teachers' beliefs and
implementation of authentic problems in the classroom is raised in the
work of Doerr and English (2006). They documented that selecting and
teaching appropriate activities whose features can promote powerful
student learning can be a key component in teachers' negotiation of both
mathematical knowledge and pedagogical content knowledge. The new
roles in the practice of problem solving that teachers adopt and the negotiation of their knowledge can be the first steps in changing teachers'
attitudes and beliefs towards mathematical modeling in a positive way
(Doerr & English, 2006).

Assessment in mathematical modeling
Niss (1987) pointed that assessment of modeling could be problematic,
since modeling is difficult to assess, let alone test, by traditional evaluation
tools. Niss (1993) further clarified that assessment takes time and cannot
be standardised. It does not imply that assessment cannot be exercised
on a sound foundation of reflection and reasoning and articulate criteria
and be subject to clear communication (Niss, 1993).
A number of different types of assessment being used to evaluate
students' modeling abilities and understanding of models are found in a
review of the literature. Crouch and Haines (2004) used a multiple choice
format, in developing several test questions related to modeling. Kitchen
(1993) proposed questions requiring students to set up a model, interpret
a solution, or criticize a model based on the data, while Bell et al. (1992)
and Hjalmarson (2005) suggested the use of an analytic scoring scale, by
assigning point values to various dimensions of the modeling work.
Model eliciting activities: design and research
Modeling processes in problem solving
The modeling approach to problem solving suggests that there is not a
single powerful procedure between givens and goals and a set of 'strategies' for overcoming any difficulties in this procedure. Indeed, the modeling approach indicates a number of trial procedures between givens and
goals in order to succeed a solution. This includes a number of iterative
cycles, in which students move back and forth from givens to goals, go
back and again moving towards goals to test their hypotheses, refine their
results and to improve their solutions (Lesh & Doerr, 2003a, 2003b). N. MOUSOULIDES, B. SRIRAMAN AND C. CHRISTOU
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A number of relevant works (Lesh et al., 2003; Blum & Niss, 1991) have
documented the different processes involved in mathematical modeling
as problem solving activity. In particular, students engage in the following processes: (a) Understand and simplify the problem. This included
understanding text, diagrams, formulas or tabular information and
drawing inferences from them; demonstrating understanding of relevant
concepts and using information from students' background knowledge
to understand the information given. (b) Manipulate the problem and
develop a mathematical model. These processes included identifying the
variables and their relationships in the problem; making decisions about
variable relevancy; constructing hypotheses; and retrieving, organising,
considering and critically evaluating contextual information; use strategies and heuristics to mathematically elaborate on the developed model.
(c) Interpreting the problem solution. This included making decisions,
analysing a system or designing a system to meet certain goals, and diagnosing a malfunction and proposing a solution. (d) Verify, validate and
reflect the problem solution: This included constructing and applying
different modes of representations to the solution of the problem; generalizing and communicating solutions; evaluating solutions from different
perspectives in an attempt to restructure the solutions and making them
more socially or technically acceptable; critically checking and reflecting on solutions and generally question the model (Blum & Kaiser, 1997;
Lesh & Doerr, 2003a, 2003b).
Student and teacher models
A model is an internal conceptual system plus the external representations of that system used to interpret other complex systems (Lesh &
Doerr, 2003a, 2003b; Lesh, Doerr, Carmona & Hjalmarson, 2003). Typically, this definition of model has only been used in reference to student
or teacher thinking and learning (e.g., Doerr & Lesh, 2003). To provide
a parallel construct at the researcher level, a design experiment carried
out from a models and modeling perspective (a modeling design experiment) should be consistent with this definition. The design tested by the
experiment encompasses two parts (similar to a model). Namely, the
design includes theoretical assumptions (i.e., researcher-level conceptual systems about mathematical knowledge, models, teacher development, etc.) and external artifacts
2
(Lesh & Doerr, 2003a, 2003b; Lesh
& Sriraman, 2005a, 2005b). Models consist of an internal conceptual
system and external artifacts or representations (Lesh & Doerr, 2003a,
2003b; Lesh et al., 2003). In addition, models incorporate a number of Nordic Studies in Mathematics Education, 12 (1), 23-47.
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external representations (e.g., a graph, a table). In constructing models,
students identify, select and collect relevant data, express limitations and
conditions of a model, interpret the solution in context, communicate
effectively and describe situations using a variety of representation
forms.
Characteristics of modeling activities
The different tools being designed and created to facilitate students' and
teachers' externalization of their thinking and understandings of problem
situations aim to elicit their thinking and thus researchers are referring to
these tools as model eliciting activities (Lesh et al., 2003; Lesh & Sriraman, 2005a, 2005b). As stated earlier, in model eliciting activities students need to: (a) develop a model(s) that describes a real-life situation,
(b) use their models to describe, revise, and refine their ideas; and (c)
use a number of representational media to explain (and document) their
conceptual systems. Model-eliciting activities can be designed to lead to
significant forms of learning because they involve mathematizing - by
quantifying, dimensioning, coordinating, categorizing, algebraizing, and
systematizing relevant objects, relationships, actions, patterns, and regularities. An example of a model eliciting activity for students is intended
to reveal the way students are thinking about a real life situation that can
be modelled through mathematics. The solution calls for a mathematical model to be used by an identified client who needs to implement the
model adequately. As a result, students must clearly describe their thinking processes and justify not a single solution, but rather all (or most of)
the optimal and appropriate solutions (English, 2003). Students' engagement with such mathematical tasks results in developing math concepts
through the need to develop powerful math ideas in order to solve a
problem. Thus, they are given a purpose (and End-in-View) (English
& Lesh, 2003) to develop a mathematical model that best describes,
explains, predicts, or manipulates the type of real-life situation that is
presented to them. In this way, model-eliciting activities allow students
to document their own thinking and learning development. The aim of a
modeling activity engages problem specification and validation, engage in
critical usage of modeling, participation and communication skills; foster
creative and problem solving attitudes, activities, competencies; provide
the opportunity for students to practice applying mathematics that they
would need as individuals in society; to contribute to a balanced picture
of mathematics; to assist in acquiring and understanding mathematical
concepts (Battye & Challis, 1997). N. MOUSOULIDES, B. SRIRAMAN AND C. CHRISTOU
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Types of the modeling activities product
Model Eliciting Activities include three types of products; tools, constructions and problems.
1 Product as a tool: Tools fulfil a functional or operational role and they
include:
A Models. Models are used for ranking items, people and places;
determining loan payments and may form the base of complex
systems such as company's financial operations (ranging from productive systems to administrative systems and societal systems
such as income tax and insurance models.
B Descriptions and explanations. Descriptions and Explanations illustrate and verify the results of an experiment or investigation or may
describe why something that appears superficially correct is mathematically incorrect.
C Designs and plans. Used in all walks of life, designs and plans must
meet detailed and complex criteria and must incorporate appropriate mathematical and representational systems.
D Assessment instruments. They are used in a wide range of contexts
such as assessing learner's progress, and selecting staff. They normally undergo rigorous development that incorporates cycles of
testing, refining and applying. (Lesh & Doerr, 2003a, 2003b)
2 Product as a construction. A construction normally requires students
to use given criteria to develop a mathematical item. They do not define
the nature of the product rather they set parameters for the design of the
product. A construction can be in the form of:
A Spatial constructions.
B Complex artefacts. Inventions are a good example of complex artefacts. The criteria for their design frequently focus on deficits in
existing artefacts or on perceived societal needs.
C Cases. Cases make use of persuasive discourse to adopt a stance
on an issue, to recommend one course of action over another, or to
highlight an issue in need of attention. Cases are especially effective
when they draw upon mathematical data to support their claims.
D Assessments. They are the products of applying an assessment tool.
Such products can serve a number of purposes and usually suggest
or imply courses of action. (Lesh & Doerr, 2003a, 2003b)Nordic Studies in Mathematics Education, 12 (1), 23-47.
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3 Problem as a product. The ability to pose problems is becoming increasingly important in academic and vocational contexts. During modeling cycles involved in model eliciting activities students are engaged
in problem posing, that is, they are repeatedly revising or refining their
conception of the given problem. During the model eliciting activities,
students find ways to judge strengths and weaknesses of alternative ways
of thinking and whether a given response is appropriate and good enough
(Lesh & Doerr, 2003a, 2003b; English & Lesh, 2003).
Principles for modeling activities
One defining characteristic of a design experiment is that the researchers
create, test, and modify a design within the context of use (Design-Based
Research Collective, 2003). For example, the researchers may be testing
a new curriculum or teaching method in a classroom (e.g., Erickson &
Lehrer, 1998; Verschaffel, De Corte & Borghart, 1997). This characteristic is consistent with model-eliciting activities that ask students to
develop mathematical models to explain real-life situations. The development of a design or model is also often cyclic (Lesh & Lehrer, 2003). In
a typical series of cycles, the student expresses thinking in some artifact
or product, tests the artifact, and then revises the artifact. For example,
a student creating a consumer guide for buying a car developed a spreadsheet for scoring characteristics of cars, asked other members of their
group or class to test the appropriateness of their scoring guides (to test
the product), and then revised the product based on testing results to
improve their solution (improve the product) (Hjalmarson, 2005). The
students' revisions are guided by a purpose (end-in-view) that describes
the functions the final product should be able to perform (English & Lesh,
2003). Similarly, for modeling design experiments, researchers should
have some end-in-view for the product under development. The end-inview should guide researcher decision-making about revisions that are
made to the product from research cycle to research cycle.
An important caveat is that for design experiments using a models
and modeling perspective, the assumptions and understandings of the
teachers (and researchers) may change throughout the study. It is imperative to document those changes as they are made (Lesh & Sriraman,
2005b). Often, researchers are interested in the students' development
of responses or in how student models change within a session or between
modeling sessions. So, rather than studying fixed constructs or examining snapshots of constructs in isolation, researchers may be studying changes in constructs over time and across problems and individuals. Capturing change and the effects of change can be a goal of design N. MOUSOULIDES, B. SRIRAMAN AND C. CHRISTOU
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experiments with a models and modeling perspective. So, both components of the design (theoretical assumptions and artifacts) will change
just as for students' models both the internal conceptual system and the
external representations change. This characteristic is another example
of how the researcher-level design experiment should be consistent with
the student-level.
For model-eliciting activities, a crucial component is the local context
that situates the task. The context guides the students' development of
solutions, aids in their decision-making about whether a way of thinking is 'bad' or 'good', and helps them place the end-in-view in a context
that is real to the students (English & Lesh, 2003). The context situates the usefulness of the design and aids development since the final
product should be useful in that context (Design-Based Research Collective, 2003). However, this does not suggest that the products are not
generalizable to other situations (or contexts). As with model-eliciting
activities where students develop a product for a particular client that
is generalizable to other (similarly structured) situations, designs should
also be generalizable to other educational situations. This proviso means
that the researcher needs to outline precisely the conditions under which
the design was used and possible modifications that may need to be made
for the design to be appropriate for different situations (Design-Based
Research Collective, 2003).
Collaboration is also a component of design experiments following the
modeling perspective that parallels assumptions about student learning.
Collaborators may include researchers, teachers and students proceeding along multiple levels of development similar to multi-tiered teaching experiments (Kelly & Lesh, 2000; Lesh & Kelly, 2000; Schorr &
Lesh, 2003). Researchers need teachers to help design, test and implement products. Products should be developed with teachers' questions
about their own practice in mind (e.g. personal meaningfulness), and
researchers can provide resources to aid teacher development (DesignBased Research Collective, 2003). There may also be multiple teachers or
researchers involved in the development of any product. This characteristic can aid the triangulation of interpretations about results and the generalizability of results if products have been tested in multiple contexts.
Collaboration also aids the documentation of results by requiring that
strategies or tools need to be communicated to other people for comment
(e.g., individual teachers develop a ways of thinking sheet or concept map
to share with the group) (e.g., Koellner Clark & Lesh, 2003).Nordic Studies in Mathematics Education, 12 (1), 23-47.
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Appropriateness, usefulness and benefits of modeling activities
It is imperative that mathematics educators take students beyond the
traditional classroom experiences, where problem solving rarely extends
their thinking or mathematical abilities or where modeling competencies
are developed. There is a strong need to implement worthwhile modeling experiences in the elementary and middle school years if teachers
are to make mathematical modeling a successful way of problem solving
for students.
Modeling activities have been found appropriate to enhance students'
and teachers' capacities to engage in problem solving, thereby laying the
foundation for exploring complex systems (Lesh et al., 2003). These
activities are highly innovative learning experiences (English, 2003). A
number of related features have emerged, indicating a number of benefits
of modeling activities, both for students and teachers. Modeling activities
provide a pathway in understanding how students approach a mathematical task and how their ideas develop; these activities appear to provide
a strong basis for teachers to interact with students in ways that would
promote their learning (Doerr, 2006). In the following part of the literature review we summarize the benefits for students and teachers while
working with thought revealing modeling activities, including benefits in
student mathematical literacy and conceptual understanding, in student
social development, in student metacognition, and in teacher pedagogical
approaches and teaching practices.
Mathematical literacy and student conceptual understanding
Related research in mathematical modeling indicated that student work
with modeling activities assisted students to build on their existing understandings and to be successfully engaged in thought-provoking, multifaceted complex problems (Lesh & Doerr, 2003a, 2003b; English, 2003).
Modeling activities set within authentic contexts, allow for student multiple interpretations and approaches, promoting intrinsic motivation and
self regulation. A number of related research studies showed that the use
of modeling activities encouraged students to develop important mathematical ideas and processes that students normally would not meet in
the traditional school curriculum (English & Watters, 2004; Zawojewski,
Lesh & English, 2003). The mathematical ideas are embedded within

meaningful real-world contexts and are elicited by the students as they
work the problem. Furthermore, students can access these mathematical
ideas at varying levels of sophistication. Student work in modeling activities facilitates student development of generalizable conceptual systems.
Students move beyond just thinking about their models to thinking withN. MOUSOULIDES, B. SRIRAMAN AND C. CHRISTOU
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them for solving an important world based problem. English (2003)
reported that there was considerable evidence that students' mathematical ideas had improved after they worked in a sequence of modeling activities. Mathematical language improved but also considerable fluency with
the use of tables and data were acknowledged (English, 2003). However,
there was an acceptance that students needed to know basic operations
to be effective in these activities. Gravemeijer and his colleagues (2000)
related their work in connection with Freudenthal's (1971) comprehension of mathematics as an activity that involves solving problems,
looking for problems, and organizing subject matter resulting from prior
mathematizations or from reality.
Lesh and Doerr (2000) have pointed out that modeling activities
can promote students' conceptual understanding. They clarified that in
modeling activities students are not simply working with ready made
models. Since models are interacting systems based in more complex
conceptual systems, Lesh and Doerr (2003a) claimed that models must
be constructed in a meaningful way. This construction leads to conceptual understanding and mathematization (Lesh & Doerr, 2003a, 2003b;
Sriraman & Lesh, 2006).
Harel and Lesh (2003) further stressed the importance of modeling
activities by highlighting that student work on modeling activities can
enhance student conceptual systems. They documented that conceptual
systems are developed first as situated models that apply to particular
problem solving situations. Then, these models are gradually extended
to larger classes of problems as they become more sharable, more transportable, and more reusable. The aforementioned features of modeling
activities helped students be successful beyond problem situations for
which models were created (Harel & Lesh, 2003).
Lesh and his colleagues (2003) and English (2003) investigated the
role of modeling activities with regard to student algebraic reasoning.
Lesh et al., (2003) reported that modeling activities provide opportunities for students to explore quantitative relationships, analyze change,
and identify, describe, and compare varying rates of change, as recommended in the Grades 3-5 algebra strand of the Principles and Standards
for School Mathematics (NCTM, 2000). In addition, English (2003)
pointed that elementary probability ideas emerging when young students
linked the conditions and constrains of problems (e.g., drug pain relief
activity). The above research studies have also highlighted the contributions of these modeling activities to young students' development of
mathematical description, explanation, justification, and argumentation.
Modeling activities are inherently social activities, and as so, students
engage in numerous questions, conjectures, arguments, conflicts, and Nordic Studies in Mathematics Education, 12 (1), 23-47.
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resolutions as they work towards their final products. Furthermore, when
they present their reports to the class they need to respond to questions
and critical feedback from their peers (English & Watters, 2004; Zawojewski, Lesh & English, 2003; Lesh & Doerr, 2003a, 2003b; Sriraman
& Lesh, 2006).
An important parameter in students' work in modeling activities is
students' use of their informal knowledge. Researchers have observed
the interplay between students' use of informal, personal knowledge and
their knowledge of the key information in the problem (Zawojewski,
Lesh & English, 2003; Mousoulides et al., 2006). In a number of modeling activities, students' informal knowledge helped them relate to and
identify the important problem information (e.g., understanding and
interpreting the conditions for the solution of a problem). Doerr (2006)
and Doerr & English (2003) also documented that students embellished
their written reports with their informal knowledge and most importantly, many students recognized when their informal knowledge was
not leading them anywhere and thus students reverted their attention to
the specific task information (Doerr, 2006; Zawojewski, Lesh & English,
2003; Lesh & Doerr, 2003a, 2003b).
Concluding points
As the literature review has pointed out, it is vital that the mathematics education research community continually revisit the fundamental
question: What does it mean for a younger student to understand models
and modeling? Another important point we stress is that the models and
modeling perspectives in North America and other parts of the world
(like Cyprus and Australia) have evolved out of the limitations of problem
solving research and the need to take into account the claim (which we
hear from many) that the nature of problem solving (and 'mathematical
thinking') has changed dramatically in the past 20 years (see Lester &
Kehle, 2003; Lesh, Hamilton & Kaput, in press).We also think there is
a real need for research about: (i) the nature of new 'real life' situations
where some type of mathematical thinking in needed for success, (ii)
what it means to understand relevant knowledge and abilities, (iii) how
these ideas and abilities develop, and (iv)how development can be documented and assessed. There is also a need to focus on a call for research
which takes into consideration what we already know about concept
development in children. Sriraman & Lesh (2006) argue that today, when
some kind of mathematical thinking is needed to solve real problems,
the products that need to be produced often involve much more than
short answers to pre-mathematized questions. For example, they often N. MOUSOULIDES, B. SRIRAMAN AND C. CHRISTOU
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involve developing conceptual tools (or other types of complex artifacts)
which are designed for some specific decision maker and for some specific decision-making purpose – but which seldom are worthwhile to
develop unless they go beyond being powerful for a specific purpose to
being sharable with others and re-useable beyond the immediate situations in which they were first needed. Consequently, solution processes
often involve sequences of iterative development → testing → revising
cycles in which a variety of different ways of thinking about givens, goals,
and possible solution steps are iteratively expressed, tested, and revised
(e.g., integrated, differentiated, or reorganized) or rejected. That is, the
development cycles often involve a great deal more than simply progressing from pre-mathematized givens to goals when the path is not obvious.
Instead, the heart of the problem often consists of conceptualizing givens
and goals in productive ways.
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Notes
1 It is important to distinguish between mathematical literacy for critical citizenship (D'Ambrosio) and the formatting power of mathematical models
in society (Skovsmose). This indicates a need for critical competencies
towards modeling and the use of models.
2 By this we mean representations of the researcher-level conceptual system
in the form of interventions, curriculum, etc
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Constantinos Christou
Constantinos Christou received his PhD from the University of Toledo,
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effects of integrating technology in the teaching of mathematics on the
cognitive development of students.
Sammandrag
Ett forskningsprojekt i USA undersökte för mer än 25 år sedan följande
fråga: "What is needed by students, beyond having a mathematical idea
that enables students to use the mathematical idea in everyday problem
solving situations? (Lesh, Landau & Hamilton, 1983). Efter 25 års
systematiskt arbete inom området "modellering" har svaret på frågan
börjat framträda. I artikeln beskriver vi hur utvecklingen av "models and
modeling perspectives" (MMP) har skett utifrån forskning om problemlösning. Beskrivningen sker med hjälp av en syntes av de viktigaste linjerna
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17 quotations

Codes (19)###HLT, ##human activity, ##real life, #RME, $$learner reasoning, $$mathematics education, $$meaningful learning of Maths, $$teaching methods and strategies, $factors that influence teaching and learning, $learner aspects, $mathematics, $mathematics content, %role players, &pedagogy, &TPCK, @@ICT systems: virtual learning environment (VLE), @Devices, @Tools, =methodology

Memos (0)

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Arit UYOUKO PREFACE
This volume contains the Doctoral Student Consortia Proceedings of the 20th
International Conference on Computers in Education (ICCE 2012). For this year, the
Doctoral Student Consortia (DSC) brings together PhD students working in the broad
research areas of computers in education in the following seven sub-themes: (1) Artificial
Intelligence in Education/Intelligent Tutoring Systems and Adaptive Learning
(AIED/ITS/AL); (2) Computer-supported Collaborative Learning and Learning Sciences
(CSCL/LS); (3) Advanced Learning Technologies, Open Contents, and Standards
(ALT/OC/S); (4) Classroom, Ubiquitous, and Mobile Technologies Enhanced Learning
(CUMTEL); (5) Digital Game and Digital Toy Enhanced Learning and Society (GTELBS); (6)
Technology Enhanced Language Learning (TELL); and (7) Technology, Pedagogy and

Education (TPE).
The DSC aims to provide an opportunity for a selected number of PhD students to present, discuss and receive feedback on their dissertation work-in-progress from a panel of established researchers with expertise in the same research areas. The DSC is meant to help students to shape their research methodologies, and to develop analysis methodologies at the early stages of their PhD research using the guidance and feedback that they get from the invited mentors. Mentors can also provide insights into fruitful areas for future research. The DSC also hopes to nurture a supportive learning community and promote interactions among the young researchers who come from a number of different institutions and across different countries in the Asia-Pacific region and beyond. It also provides opportunities for theme-based forums to discuss evolving methodological and theoretical issues of central importance to the community. The DSC and the related social events are financially supported by the Asia-Pacific Society for Computers in Education (APSCE).
A group of senior PhD students (Kevin Kai-Wing CHAN, Chia-Jung CHANG, Jaime GALVEZ, Anita HUANG, Wai Ying KWOK, Rose Ru-Whui LEE, Tuck Leong LEE, Mengmeng LI, Larry Lei Chun LIN, Boon See TAN, Arit UYOUKO) who were highly recommended by the APSCE Special Interest Group Chairs (SIG) Chairs/Co-Chairs were invited to be the organizers of this prestigious event. This group of senior PhD students is guided by the DSC Chairs (Weiqin CHEN, Chen-Chung LIU, and Hiroaki OGATA). The DSC chairs helped oversee the whole process of organizing the DSC and provided guidance along the way. With a strong sense of responsibility and enthusiasm, this highly dynamic group has been successful in organizing the DSC for ICCE 2012. It is clear that by entrusting this group of senior PhD students with the responsibility of organizing this important event and editing the DSC Proceedings, they were able to form a vibrant and supportive research community within a short period of time; which is one of the main goals of the APSCE. This year we received a total of 10 submissions, and 9 of these papers were selected and have been included in the Proceedings. Each selected paper went through a rigorous blind review by independent peer reviewers to ensure the work was of high quality. We hope that the papers in the proceedings on various research topics will stimulate additional research ideas and discussions among the young researchers.
We would like to thank all the invited mentors in making this year's DSC a highly successful event. Finally, we would like to take this opportunity to record our sincerest appreciation to the local organizers in Singapore for their valuable support and arrangement in organizing the DSC.
On behalf of DSC Co-Chairs
Weiqin CHEN
Hiroaki OGATA
Chen-Chung LIU TABLE OF CONTENTS
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Unsupervised Learning Environments
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Abstract: Technology has made it convenient for students today to learn outside of the classroom. However, learning in this kind of setting introduces challenges such as proper time management, maintaining motivation and avoiding distractions. This research investigates how student learning progresses in this unsupervised learning environment and proposes an automated support mechanism that promotes self-regulation to help students learn. The proposed system is not bound to any domain and its methodology supports learning in naturalistic settings. Results gathered from students' use of a system prototype confirmed the problems students encounter when learning in this environment and their need for support. It also showed that helping students self-regulate made them aware of their learning behavior and helped them identify ways to improve their learning efficiency.
Keywords: learning, unsupervised learning environment, feedback, self-regulation
Introduction
Student learning is not bound to formal learning settings in classrooms. Students continue to learn on their own in the library, in their homes or with their peers. Technology has made it even easier for them to learn because it gives them easy access to information anytime and anywhere. However, informal learning settings introduce challenges such as the need to self-motivate, manage time and resist distractions to learn effectively.
In formal learning settings, students are discouraged from engaging in non-learning activities such as chatting with friends or playing games. Students are also required to use certain skills and solve problems within given time frames. However, informal learning settings give students more control over their time and goals. Students can engage in non-learning activities freely which may benefit them by helping them de-stress and disengage from feelings of frustration and helping them move into more positive emotional states which allow them to regain motivation and resume learning [5]. The duration, frequency and type of non-learning activity will have different effects on students thus there also has to be a certain configuration for non-learning activities to be helpful. Students know themselves best so they are also the best persons to identify what is most effective for them. Our goal is to improve students' learning efficiency as they learn in informal learning settings by helping them self-regulate.
1. Literature Review
Self-regulation is a systematic process for promoting the attainment of goals and is composed of the cyclic phases of forethought, performance and self-reflection [6]. Forethought involves the formulation of goals for the learning session which serve as guides 2 on what activity to do. Performance involves doing activities that will accomplish the previously defined goals and monitoring progress to identify if there is a need to adjust goals or strategies. Self-reflection involves the assessment of the learning session, attribution of the causes of unfavorable behavior during the learning session and the identification of possible strategies to increase learning efficiency in succeeding learning sessions.
Self-regulation facilitates good learning behavior and its benefits have not only spawned research in traditional classroom settings, but also in more modern settings such as in computer-based learning environments. Systems like MetaTutor [1] and Process Coordinator [4] allow students to keep track of the self-regulation strategies they use and also suggest strategies that can help them accomplish their goals as they learn in computer-based formal learning settings. However, these systems expect students to engage only in learning related activities during the session. Research has shown that students use and actually prefer learning activities outside of the classroom apart from classroom-based activities [2]. Students' preference for learning in this domain and the challenges they encounter highlight the need to support them.
2. Proposed Research and Preliminary Research Questions
This research investigates the interplay between students' learning and non-learning activities while learning in an informal learning setting and the support that can be given to help students self-regulate and increase learning efficiency. We limit our work to informal learning settings that involve the use of a computer because work done on or in front of a computer can be logged automatically and because computer systems can provide automated support to help students self-regulate.
We use the term unsupervised learning environment to refer to informal learning settings wherein students learn by using a computer without supervision from a teacher. In this environment they need to identify their own goals, manage their learning, avoid distractions and self-motivate to learn efficiently. We aim to accomplish the research's goals by answering the following general and specific questions:
How do we help students self-regulate in an unsupervised learning environment to improve learning efficiently?
1. How can we help students define and prioritize their learning goals?
2. How can we help students become more aware of their activities while learning and encourage them to change their behavior that is not helpful to learning?
3. How can we help students self-reflect and identify ways to improve their learning?
4. Does helping students self-regulate when they learn in an unsupervised learning environment improve their learning efficiency?
3. Architectural Framework
The framework we are using for this research consists of three cyclic phases with elements based on the three aforementioned self-regulation phases and is illustrated in Figure 1. In the interaction phase, students are asked to identify and input their goals at the beginning of a learning session. After which, students start the learning session wherein information about their actions such as applications used, timestamps and screenshots of the desktop and webcam are recorded and stored in an interaction database. The system provides timely feedback to help students keep track of their activities and progress and to help them identify if there is a need to adjust them. In the annotation phase, students review their learning session with the help of the desktop and webcam screenshots recorded. Students 3 then annotate their activities on the computer (e.g., browsed a web page, read a textbook), identify if it is related to their learning goals or not and indicate when they felt distracted from their learning goals. Students are also asked to comment on the system's feedback during the learning session and rate their learning efficiency for the session. In the modeling phase, a machine learner will be used to create a predictive model from the annotated data to identify which conditions cause students to become distracted. Another machine learner will be used to incrementally adjust a feedback model to fit the students' feedback preferences. Feedback adjustments result in changes in feedback type, frequency and conditions. The updated distraction and feedback models are used by the system in the next learning session to decide when and what type of feedback should be given to the student. Over time, the adjustments will result in better models of the students' behavior and feedback preference thus, improving the feedback mechanism.
Figure 1. Architectural Framework of the Proposed System
4. Preliminary Results
An initial prototype of the system with both the interaction and annotation phases was created and tested on four students doing academic research [3]. The subjects were mostly graduate students working on different domains (i.e., computer science and physics) but were all required to do research and present results to their respective supervisors once a week. Academic research was considered an unsupervised learning environment because students did their work on a computer, managed their own learning and didn't receive direct guidance from their supervisor. Each student used the system for approximately two hours in one learning session per day over the period of five days resulting in 10 hours of data per student. After each learning session, students spent about an hour to annotate their data, which is around half of the learning session's duration. Surveys and informal interviews were also conducted to further verify and understand the students' learning behavior.
A transition likelihood metric was used on the annotated data to identify the likelihoods
1
Student System Interaction DB
• Goals
• Activities
Feedback
Interaction
Information
2 Annotation Interface
• Annotations
• Comments on Feedback
• Learning Efficiency
Measure
Interaction
Information
Annotation DB
Annotated
Interactions
3
Machine
Learner
Annotated
Interactions
Distraction
Model
Feedback
Model
• Comments on Feedback
• Efficiency Measure
Machine
Learner

Interaction Phase
Annotation Phase
Modeling Phase4
of transitioning between states, wherein each state consisted of a student's emotion and intention (i.e., learning or non-learning). The results showed that students transitioned between feelings of engagement and confusion which were indicative of cognitive disequilibrium, a state commonly experienced by students when learning. Students were also constantly shifting to non-learning activities which may have served as distractions. However, students shifted back to learning which indicated that non-learning activities may also have served as rewards and de-stressors that helped motivate students to resume learning at a later time. Both effects of non-learning activities were also reported by students in the survey results collected. During the interview, we found out that students used varying measures for learning efficiency which included the number of completed learning goals, the amount of time spent in learning activities and the difference between what they learned from the session and their self-expectations. Furthermore, students said that they were able to identify what caused them to learn inefficiently and also what they could do to improve their learning behavior.
The next step in our research is to use our preliminary results to design and automate feedback provision based on the students' annotations, learning efficiency ratings and proposed solutions for improving their learning behavior. This feedback also needs to be verified if it translates to actual learning improvements through pretests and posttests.
5. Conclusion
This research investigated learning in an unsupervised learning environment and proposed a self-regulation based automated support mechanism to help students learn. The designed methodology allows students to learn in a naturalistic environment and helps them become more aware of how often, how long and which non-learning activities were helpful or harmful to their learning. The methodology also allows students to self-reflect and identify how they can improve their learning efficiency. The kind of support given by the system is not domain dependent so it is able to provide support for learning as long as it is done on or in front of a computer. Self-regulation is an important skill for life-long learning which the system promotes through honing students' self-regulation skills outside formal learning settings without the need of human supervision.
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Exploring Discourses: Third Culture Kids'
digital lives
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Abstract: An important question for research is the ways in which children from globally mobile families negotiate the different Discourses they encounter through mass-media digital technologies, the home, and the classroom. The study reported on here took the form of a multiple case study, which was undertaken at a European International School over the course of a school year. This paper focuses on students' digital game play documented within the different contexts of classroom and out-of-classroom environments. It reports on the social dynamics and the productive outcomes made possible in spaces such as these, and the ways these children negotiated, appropriated or embraced the Discourses that they found as part of their ongoing work in establishing social identity.
Keywords: Identity, digital play, third culture kids, literacy education
Introduction
Issues of identity are of intense significance for the globally mobile child, and so too is digital play, as a forum for both pleasure and community. To inform contemporary pedagogy and curriculum in school, more information is needed about the ways in which issues of identity present themselves in on and offline contexts, and the ways in which children in turn negotiate them. To explore children's integration of digital games into their lives, for my Master's research study I held a series of after school computer clubs in the International School in which I was teaching. Through these clubs, I investigated the children's imaginative engagement with internet games [1] [2]. This research prompted more questions, particularly concerning identity and digital play, and the challenges these distinctive young people face in negotiating the worlds of the internet, home and school. For the last two years I have explored this topic as a full-time PhD student.
1. Related work with key references
My research draws on a wide selection of literatures, briefly summarised here in the following three topics:
1.1 Identity
Questions of identity and subjectivity are central to this research. So too is language use in both print and multimodal contexts for establishing these constructions of identity. Sociolinguists such as James Gee point to strong connections between language use, specifically how terms and concepts are framed and the everyday language used to speak about them, and the processes by which people perceive themselves and are positioned 6 within society. Gee calls on the notion of Discourse to explain key distinctions and operations here [3]. In his writings about identity, Gee uses a capital letter to distinguish between the familiar use of the word 'discourse' to mean a dialogue, and 'Discourse', by which he refers to a socially-formed grouping embedded with cultural practices. Gee theorises that we all have experience of a unique primary Discourse, our formative social communication learning from our parents, which is shaped by cultural influences such as national and religious Discourses, as well as a comprehensive range of secondary Discourses, our necessary ways of communicating differently in different situations, such as may be encountered within digital environments and school communities.
1.2 Third culture kids
The site of the research is an International School that prioritizes the children of intergovernmental organizations' employees. The majority of these children are expatriates, many of whom live a peripatetic life and thus live as outsiders from their home country, and to a large extent from the country where they currently reside. Consequently, these children have quite different formative experiences than children who have matured primarily within one culture. Such children have been labelled as "global nomads" [4] or "third culture kids" [5], and as Sears notes, these children can be seen as "exemplars" of an increasingly mobile world [6 p.79]. The growth of new digital technologies has presented a new range of options for the ways these children live their lives. Such changes consequently have important implications for the educational institutions that serve these children.
1.3 'New' Classroom Literacies
In 1996, the New London Group collaborated to produce a manifesto that seeks to broaden the scope of literacy pedagogy to accommodate both technological changes, and changes in the social landscape. Literacies occur within a broad social context, and within a wider range of spaces and Discourses than is dealt with by traditional classroom settings. Educational research therefore needs to move beyond classroom doors and into students' wider communities [7]. The research undertaken here adopts such a sociocultural approach, and is grounded in the idea that communicative production can best be understood within the social and cultural practices from whence it is formed.
2. Research Questions
The central research question in this study is 'how do third culture kids negotiate the Discourses of home, school and new media'. In researching this question, it is also necessary to identify some of the key Discourses these culturally diverse children encountered in the different intersecting spaces and places of digital culture, the classroom, and the home, and to explore the ways in which these children position themselves in relation to these Discourses. To refine the focus, this topic is being broken down into a number of specific directions.
One direction concerns the social dynamics established by the participants when playing or working in a group, both in school, and in out-of-school contexts. This includes group dynamics within virtual spaces. This focus works towards understanding the central research question as each student's differing Discourse backgrounds can be glimpsed within social interactions such as these. Questions being explored in this direction are: 7
When the children play internet games in a room together, or work in collaborative learning situations utilizing digital technologies, what are the characteristics of the group dynamic that develops? And then:
How do the individual children negotiate their relationships within these different places and/or within virtual group settings?
A second direction is concentrated on the need to know more about how game play, and other uses of digital technologies, factor in the performance of, and development of, these children's language and literacy skills. Students produce and share meaningful texts via internet games and social networking sites used at home, and games and other digital platforms used in the classroom. Exploring this production works towards understanding the broad topic as the research is grounded in a sociocultural position that language is central to identity; that identity is one and the same as its expression or performance through literate practices. Specific questions being explored in this direction are:
What are the characteristics of the texts which children produce utilizing mass-media digital technologies in the home?
How do they differ from texts produced in classroom situations when digital technologies are utilized?
What evidence do these literate productions provide of the students' negotiation of the Discourses that surround them?
3. The contribution of the proposed research
Young people in the 8-13 year age group are at a stage of vital importance in how they see themselves, and in the development of their points of view and values, yet this age group remains under-investigated in most areas of educational research. This study specifically contributes to the literature due to the unique nature of its third culture kid participants, and aims to bring investigations of identity and the third culture kid into a digital-age focus. This broad research topic directly addresses Jackie Marsh's declaration of the "urgent need to map children's engagement with cultural texts in a global context" [8 p.12].
4. Research methodology
The research is guided by case study procedures. Over the course of the past school year, twenty International School students aged from 8 to 13 years kept a journal recording their at-home digital play. These journals were then used as an aide-memoire at weekly meetings where the students discussed their use of digital games and other media. The students were also observed in the classroom, focusing on the children's experiences with digitally enhanced learning. An opportunity to attend an after-school computer club during which students played on their own choice of internet games was also offered. Interviews were held at the beginning and the end of the school year with the students. Their teachers and parents were also interviewed.
The research employs qualitative research methods to ensure the authentic, "messy complexity of human experience" can be seen and contemplated [9 p.3]. The design encompasses two different kinds of triangulation to impart validity, a methodological triangulation and a data triangulation [10]. First, three methods have been employed to collect information from the child participants; taped interviews and discussions, observation notes, and journals. Secondly, three different data 'categories' have been engaged as interviewees: children, parents, and teachers. 8
Appadurai [11] maps contemporary global fluidity within a set of five different categories of flow: ethnoscapes, mediascapes, technoscapes, financescapes, and ideoscapes. These "scapes" provide an illuminating framework for exploring the contemporary childhood of the globally mobile child. The data, which is now in the initial stages of analysis, is being initially codified according to these categories. This involves selecting representative sections of dialogue and image as the data is transcribed and collated. These selections highlight typical and atypical responses, as well as characteristic responses for different age and/or gender subgroups.
5. Preliminary results
The data is in the early stages of analysis and it is not yet possible to draw conclusions. Two papers utilizing early findings have been written and are currently under consideration for publication. One centres on the experiences of one family who had recently moved into the school community, and considers the complex and conflicting pressures, expectations and possibilities they encountered at home, at school, and in their online play, at this moment of cultural transition. The other reflects on the differing attitudes observed between the school community and the children's families, with respect to the role of digital play in children's lives, and suggests that schools need to understand parental positions around technology and digital culture, and reflect on these as they plan their digital futures.
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Inspiring Creativity through a Creativity

Game of Players as Game Story Designers: A
Metacognitive Approach
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Abstract: There have been tremendous ways of developing creativity, yet instead of traditionally formal instruction, games are the best media to spur creativity, as creativity can easily and spontaneously be evoked through games. On the other hand, metacognition is a good way to promote creative thinking, as metacognition which contains the abilities such as self-planning, self-monitoring, and self-evaluation can foster creative-relevant cognitive processes. Games are exactly the ideal platform that provides a playful context for metacognitive awareness prompting creative processes. Based on the advantages of games and that design is a creativity-relevant cognitive and constructive process, this research adopts the means of digital game-based learning (DGBL) to propose a metacognitive approach to inspiring creativity through a creativity game of players as game story designers.
Keywords: creativity assessment, game story design, game story creativity, creativity
Introduction
Creativity has been widely acknowledged as one of appreciated 3Cs today-creativity, curiosity, and courtesy. Many studies revealed that creativity is associated with and also facilitates divergent thinking, critical thinking, flexibility, and so forth, which are the vital elements to promote learning. However, is creativity inborn or acquired? Creativity, like other potentials, demands inspiration and practice to receive its whole-scale development. There have been tremendous ways of developing creativity, yet instead of traditionally formal instruction, games are the best media to spur creativity, as creativity can easily and spontaneously be evoked through games.
1. Purpose and Significance of the Proposed Research
Based on the advantages of games and that design is a creativity-relevant cognitive and constructive process, this research adopts the means of digital game-based learning (DGBL) to propose a metacognitive approach to inspiring creativity through a creativity game of players as game story designers. The purpose of this research contains three aspects. First of all, this research attempts to explore what kind of creativity assessment is appropriate for game story design. Second, this research investigates the effects of inspiring creativity through a game of players as game story designers. Third, this research tries to understand in which game story constructs were the students good at and short of creativity? 10
The significance of this research is that this research proposes an entertaining and metacognitive approach to inspiring creativity in replace of traditionally formal instruction of creativity. Besides, the devised creativity game can prompt players' creativity-awareness for distinctive game story constructs and creativity dimensions. This research provides valuable information for game design course instructors as well as game story design evaluators.
2. Literature Review
This research is based on the integrated rationales of game, creativity, and metacognition. Some selected literatures are briefly discussed as follows.
2.1 Game and Creativity
The literature has widely discussed the relationship between games and creativity. Some major advantages of games are described as follows. First of all, game play is a wonderful source of relaxation and stimulation for the brain and body-a sure and fun way to develop the ability of creativity [3]. Second, game play can arouse curiosity which leads to discovery and creativity [3]. Third, creativity can be activated by intrinsic motivation through games [5]. Fourth, games can stimulate divergent thinking which promotes creative learning [2].
Noticeably, game play as engagement fosters the cognitive and affective dimensions of the creative process, as well as the development of domain-relevant skills and creativity-relevant skills [5]. There are five creativity-relevant cognitive processes: problem framing, divergent thinking, mental transformations, practice with alternative solutions, and evaluative ability. Through game play, these creativity-relevant cognitive processes can be facilitated in learning. Evidently, games not only a brilliant means to mediate learning but also a fantastic catalyst to boost up creativity. Thus, games are the best approach to an enjoyable learning environment prompting creativity which in return inspires creative learning.
2.2 Game and Metacognition
The literature has extensively discussed over metacognition in the past decades. Metacognition refers to the ability or awareness of understand and monitor one's own thoughts [4]; simply put, thinking about one's thinking. Many studies have reported that metacognitive awareness can be fostered with proper instructional strategies. Likewise, the conditions that have been proven to foster metacognitive awareness through instructional strategies are similar to the conditions established in games [6]. Differently, games provides a context in which players can enjoy a playful activity. Using games to prompt metacognitive awareness can encourage motivation, promote engagement, and increase pleasure-an ideal means much better than formal instruction.
2.3 Metacognition and Creativity
Creativity involves f innovation, imagination, uniqueness, etc. Many studies have explored how to promote creativity through a variety of means and strategies. It has been demonstrated that metacognition is a good way to promote creative thinking, as metacognition which contains the abilities such as self-planning, self-monitoring, and self-evaluation can foster creative-relevant cognitive processes [1, 5]. 11
3. Implementation of Game Story Players as Designers Approach
Grounded on the rationale of the triangular relationships of creativity, metacognition, and games, game story constructs and six types of creativity are presented in a metaphorical way in the devised creativity game, requiring players as game designers for inside-game- story players. Through designing, it stimulates players' metacognitive awareness that simultaneously prompts creativity awareness.
4. Method
4.1 Research Design
This research consists of three phases. In the first phrase, a creativity assessment rubric for game story design is developed and tested its validity and reliability. In the second phrase, a standard creativity test is selected and used to pretest and posttest a control group that is treated with a learning system containing selected instructional materials on creativity. For the experiment group, a questionnaire is used to pretest the participants' knowledge of creativity for game story design, and then the creativity game proposed by this research is conducted on the participants. In the third phase, the game story design projected done by the experiment group at the end of the semester will be assessed with the devised assessment rubric by three raters. Two groups will be compared in terms of different instructional treatments.
4.2 Preliminary Research Questions
Three research questions are formulated as follow: First, in which game story constructs are the students good at and short of creativity? Second, whether the proposed game treatment better promotes creativity? Third, how do the students view such a metacognitive approach to creativity inspiration?
5. Preliminary Research Work Achieved and Future Work
This proposed research has come to the phase of pilot study. An analytic rubric of game story creativity has been devised and the paper of rubric design has been accepted in the 2012 International Conference of Computers in Education. Based on the analytic rubric, initial ratings have been conducted and the validity and reliability of the analytic rubric are being examined. Besides, the game structure diagram of the creativity game as well as a questionnaire has been initially drawn. Future work will focus on the development of the creativity game and the evaluation of its effectiveness.
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Exploration of Signature Pedagogy Using ICT
for English Grammar Learning in Elementary
School Education
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Abstract: This research aims to explore a signature pedagogy that addresses the trends toward the pedagogical goals and the technological integration for English grammar learning in the twenty-first century. Based on a twofold literature review, a potential signature pedagogy with three variations in the use of two selected types of information and communication technology (ICT) resources has been designed for supporting English as Second Language (ESL) learners at the elementary school level to learn English grammatical knowledge. An empirical study which combines qualitative and quantitative methods is planned to evaluate the effectiveness of the designed signature pedagogy in a real classroom environment. Three classes of Grade 4 ESL learners will be invited to learn the target topic with the selected ICT resources during a one-month trial teaching, under three settings: inclining to form-focused approach; balancing form-focused approach and communicative approach; or inclining to communicative approach. The empirical study will conduct attainment tests, classroom observations, questionnaire surveys and semi-structured interviews to investigate the impact of the designed signature pedagogy on the achievements, processes and perceptions of students in the learning of elementary English grammar. This research will finally advise a signature pedagogy that fully exploits the use of the two selected types of ICT resources for supporting ESL learners to learn elementary English grammar.
Keywords: Elementary school education, English, grammar, ICT, signature pedagogy
1. Research Motivation
In many Asian cities such as Hong Kong where English language is not the native language of the majority of the population, this language is one of the major subjects that receive much attention in elementary school curriculum. There is a decade-long emphasis on the pedagogical integration of ICT into the delivery of elementary English language curriculum [1, 2]. In Hong Kong, in this curriculum, grammatical topics such as tense and agreement are considered to be the very important but most difficult component due to the inherent differences in the language systems between Chinese and English languages.
A signature pedagogy refers to a collection of discipline-specific teaching methods that focus on preparing learners to think like the experts in the target discipline for a skilful demonstration of discipline-specific knowledge for learning and daily pursuits [3]. Through the routinized approach to classroom teaching, a signature pedagogy could promote learners to habitually bridge abstract knowledge and regular practice for the deep learning of core knowledge in the target discipline. The implementation of signature pedagogy is regarded as important in the learning of rule-based topics. It is therefore good to implement 14 significant pedagogy in the teaching of English grammar, which is a topic comprised of language rules. This research is motivated to address the emergent need in the field of ESL education for more investigations into signature pedagogies which meaningfully use ICT resources for supporting English grammar learning in ESL classrooms.
2. Research Context
This research aims to explore a signature pedagogy that maximizes the potential of ICT resources for supporting ESL learners at the elementary school level to learn English grammar. This research is comprised of two stages for progressively designing, evaluating and then establishing a signature pedagogy that fully exploits the use of ICT resources for learning elementary English grammar.
2.1 The First Stage: Designing a Potential Signature Pedagogy Using ICT for English Grammar Learning
The first stage of this research focuses on a twofold literature review for designing a potential signature pedagogy that coordinates the pedagogical goals and the technological integration for English grammar learning in the twenty-first century. Based on an in-depth review of research literature on the pedagogical approaches for teaching elementary English grammar in the recent decade, ESL teachers over the world are found to mainly adopt two pedagogical approaches in English grammar lessons. The first approach is the form-focused approach, which focuses on inculcating learners with knowledge about variation forms and usage patterns of diverse types of linguistic units to generalize prescriptive grammar rules. The second approach is the communicative approach, which focuses on engaging learners in authentic learning tasks in the collaborative learning setting for applying grammatical knowledge in classroom interactions.
Based on an in-depth review of research literature on the use of ICT for learning and teaching English grammar system in the recent decade, ESL teachers over the world are found to mainly adopt two types of ICT resources in English grammar lessons. The first type is online learning websites specific for grammar learning. This type of ICT resources typically provides a set of static grammar notes, multimedia illustrative animations and dynamic multimedia simulations for rules explanation. The second type is online communication tools general for daily interaction. This type of ICT resources typically offers a range of written or oral communication software for interpersonal interaction in the synchronous or asynchronous mode.
The findings of the twofold literature review, concurred with the advocacy by established academics (such as [4]), suggest that a potential signature pedagogy for English grammar learning should make a strategic combination for leveraging the effect of the form-focused approach and the communicative approach. On the one hand, there should be inquiry learning activities under the form-focused approach, in which learners browse the selected online learning websites with topic-specific notes and worksheets to realize and

conceptualize usage conditions of grammatical rules. On the other hand, there should be collaborative learning under the communicative approach, in which learners use the selected online communication tools to share and discuss target knowledge, and then to integrate and apply knowledge about grammatical rules. It should be noted that there is flexibility for varying the proportion of these two types of learning activities during English grammar lessons according to the learning needs of students. 15

2.2 The Second Stage: Evaluating the Designed Signature Pedagogy for English Grammar Learning

The second stage of this research will be an empirical study which focuses on evaluating the designed signature pedagogy in a real classroom environment. Three research questions are made for the empirical study:

(i) What are the achievements of elementary ESL learners toward English grammar learning under the designed signature pedagogy?

(ii) What are the processes of elementary ESL learners toward English grammar learning under the designed signature pedagogy?

(iii) What are the perceptions of elementary ESL learners toward English grammar learning under the designed signature pedagogy?

An elementary school that has rich experience in ICT in education will be purposefully sampled in Hong Kong as the partner school for the empirical study. A trial teaching in the form of one-month supplementary course, which amounts around 20 one-and-a-half-hour lessons, will be arranged in the partner school. Three classes of Grade 4 students, each consisting of around 30 students with similar learning ability, will be randomly selected and then assigned to one of the following three settings varying in the proportion of learning activities during lessons:

Setting 1: The proportion of inquiry learning activities under the form-focused approach is higher than that of collaborative learning under the communicative approach.

Setting 2: The proportion of inquiry learning activities under the form-focused approach is the same as that of collaborative learning under the communicative approach.

Setting 3: The proportion of collaborative learning under the communicative approach is higher than that of inquiry learning activities under the form-focused approach.

The proportion of inquiry learning activities under the two approaches will be further delimited by taking the empirical findings of research on digital classrooms (such as [1] and [4]) and the current practice in ESL classrooms in Hong Kong into consideration. During the trial teaching period, students in each of the three settings will be grouped into pairs to learn English grammar, with the use of the two selected types of ICT resources and tailor-made learning worksheets. In each lesson, students will first be taught the target knowledge by the teacher and then, they will explore this topic with the assigned learning materials in pairs. The author will take charge of the classroom instruction for the one-month trial teaching.

Four methods will be adopted in the evaluation work to investigate the effect of the designed signature pedagogy. First, students in all the three settings will sit for identical pre-test and post-test [5] before and after the trial teaching, respectively. The test papers will include a series of questions that assesses knowledge of the students about key grammatical knowledge of English language, such as tense and agreement. Second, an interaction analysis [6] will be conducted to investigate the effect of the designed signature pedagogy on the learning process of students in each of the three settings. Two student groups will be randomly selected in each setting. The six selected student groups will be observed and videotaped in all lessons throughout the teaching period for collecting data on the learner-learner and learner-technology-learner interactions in the learning activities. A systematic analysis of the interactions for constructing target knowledge will follow to trace the learning process among students under the designed signature pedagogy. 16

Third, a questionnaire survey [5] will be conducted at the end of the trial teaching. All students will be asked to complete a self-administered questionnaire to indicate their perceptions of the implementation of the designed signature pedagogy for English grammar lessons. Fourth, one-fifth of the students in each of the three settings will be randomly selected for the semi-structured, individual interviews [5] to further investigate their perceptions of the implementation of the designed signature pedagogy. The selected students will be asked to describe the changes in their process, motivation and achievement in the learning of English grammar through the lessons that implement the designed signature pedagogy.

The empirical study is going to be conducted in a real classroom environment. The evaluation results obtained at this stage will contribute to the establishment of a signature pedagogy that uses ICT resources for maximizing the learning effectiveness among ESL learners in English grammar learning in elementary school education.

3. Research Contribution

In the Asian cities where English language is not the native language of the majority of the population, local ESL teachers in elementary school education, in general, lack sufficient evidence-based recommendations which support them to plan and implement effective and efficient methods for the pedagogical use of ICT in English grammar lessons. In addition, there is a lack of comprehensive research on technology-mediated signature pedagogy for the field of ESL education [4, 7].

This research contributes to the pedagogical advancement in the use of ICT in ESL education. It will not only help to enhance the learning of grammatical knowledge among ESL learners, but also help to cover the lack of comprehensive research in the field. The outcome of this research will be a potential signature pedagogy to maximize the effectiveness of ICT resources on the learning of elementary English grammar among ESL learners. This can provide ESL teachers with insights into the innovative design of technology-mediated pedagogies for the teaching of the target topic.

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Impact of Students' Perceptions of ICT-Supported Learning Environment on Approaches to Learning for Principles of Accounting in Secondary Schools

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Abstract: This research aims to study the impact of students' perceptions of ICT-supported Learning Environment and their personal characteristics on approaches to learning in the context of Malaysian accounting education at secondary school level. Students are facing the learning problem where they lack the opportunity to experience deep approaches to learning due to the teacher-centred teaching approach and immaturity of the curriculum and assessment design. In view of these problems, the Ministry of Education (MOE) has revised the curriculum and assessment from content-based to skilled-based with the aim to enable students to practise deep approach to learning with ICT. However, to what extent the new environment is able to stimulate deep approach to learning is the main issue underpinning by the present study which will be investigated through a correlational research design employing the Structural Equation Modeling (SEM) analysis method. Contributions and preliminary results of this study are discussed.

Keywords: Perceptions of ICT-supported learning environment, deep approach to learning, surface approach to learning, students' characteristics, Principles of Accounting

Introduction

A business or organisation communicates its results and position to its stakeholders through the use of financial statements. Given that financial statements are among the deliverables of an accounting task, accountancy is widely referred to as the "language of business". Basically, accounting is seen to involve the process of identifying, measuring, and communicating economic information to permit informed judgments and decisions by users of the information [1]. Fundamental to the identifying and measuring processes, is the application of the double-entry book-keeping system to record financial transactions which conducted under the procedures named as accounting cycle. It involves a set of steps in preparing the financial statements for a given period. All these steps are inter-linked and it requires students to employ deep approach to learning in order to master the whole set of accounts throughout the accounting cycle rather than fragmentally studying on each of the accounting procedures. The deep approach to learning represents a personal commitment to learning through seeking meaning of the contents, trying to relate parts to each other, associating new information with existing prior knowledge or to personal meaningful context. It is contrasted with surface approach to learning which is characterised by motivation to acquire only sufficient knowledge to complete the task or pass the subject through rote learning [2, 3, and 4]. 18

In 2010, the Malaysian Ministry of Education (MOE) has committed an effort to transform the curriculum and assessment of the fundamental accounting subject in secondary schools, the Principles of Accounting, from content-based to skilled-based with the aims to enable students to practise deep approach to learning with ICT [5]. The effort undertaken is to overcome the learning problems which will be discussed in the next section. However, to what extent the effort is able to impact positive changes in students' learning is the main issue underpinning the present study.

1. Research Motivation

Unlike other disciplines, accounting is a technically-oriented subject and hence, the teaching of accounting was found to be dominated by the objective of training students to know facts and solve problems from a procedural perspective [6]. Thus, it was observed by few researchers that students always perceive that learning accounting is simply about learning a set of rules and evidences suggest that they tend to adopt a surface learning approach compared to other subjects [7, 8].

Similarly, in the Malaysian context, students' learning for the subject of Principles of Accounting has yet to achieve deep approach as it was found that most of the accounting teachers tended to use the teacher-centred teaching methods such as lecture, drill and practice and demonstration of problem solving by teachers [9]. Such methods could lead to surface learning where the lower-level procedural skills are acquired without processing information for meaning. This problem could be attributed to the former curriculum and assessment of Principles of Accounting (before year 2010) which were arranged in such a way that the process of accounting cycle was fragmented by its steps. Students learnt each part of the process through the method of drill and practice where they were exposed to exercises or problems which were related to that portion of process only. In other words, there were different sets of exercises or problems for each of the steps of accounting cycle e.g. exercises for journal-entry were unrelated to exercises in trial balance. The division of the book-keeping procedures negates the students' skills and knowledge to relate every aspect of accounting into a coherent whole [10]. Under this learning content structure, students were more oriented towards surface approach to learning as they treated parts of the subject as separate entities and failed to master the full set of accounts. Moreover, students lacked exposure and were not encouraged to engage in deep learning approach through technology [11]. It further proves that students are lacking ICT skills to handle a full set of accounts where these skills have long been heralded as a crucial element in both professional accountancy and accounting education [12].

The launch of the revised curriculum and assessment signifies the commitment of MOE to improve learning quality which enables students to have a coherent understanding of accounting through enhancing their competencies in preparing a full set of accounts with the assistance of ICT. The overarching question now is how effective is the ICT-supported learning environment as perceived by students in influencing their deep approaches to learning? What are the other factors which influence students' approaches to learning? Is the perceived ICT-supported learning environment a mediator factor between individual personal characteristics and approaches to learning? All these stand as the general motivation for the present research.

2. Research Questions

The research questions for the present study are suggested as below: 19

I. Do the students' perceptions of ICT-supported learning environment (in terms of Personal Growth, Assessment, and Relationship) influence their approaches to learning?

II. Do the students' characteristics (in terms of Prior Accounting Knowledge, Academic Ability, Preconceptions of Accounting, and ICT Proficiency) influence their approaches to learning?

III. Is the students' perceived ICT-supported learning environment a mediator between their characteristics and approaches to learning?

3. Contribution

This study adds to the body of accounting education and student learning literature by investigating students' approaches to learning in the ICT-Supported Learning Environment through examining their perceptions of the environment and personal characteristics. It was commented that the direct investigations of students' perceptions are scant, particularly in relation to technological innovation in the learning environments [13]. Thus, this study makes a prima facie contribution for accounting educators to obtain the captured reflective voice of students. Furthermore, the findings of the present study could provide important information for educators on which aspect of the learning-teaching context can be altered for the sake of improvements in teaching and learning, curriculum, and assessment that encourages deep learning approach.

4. Methodology

This research employs a correlational research design. It aims to establish the empirical relationships of students' perceptions of ICT-Supported Learning Environment, their personal characteristics, and aproaches to learning through using of Structural Equation Modeling (SEM) analysis method. A closed-ended questionnaire will be distributed to secondary school students who have completed their school-based assessments of Principles of Accounting. The questionnaire will be adapted from the instruments of Learning Process Questionnaire (LPQ; reliabilities for the subscales: .58-.75)[14] for measuring approaches to learning; Technology-Rich Outcomes-Focused Learning Environment Inventory (TROFLE; reliabilities for the subscales: .77-.95)[15] and Course Experience Questionnaire (CEQ; reliabilities for the subscales: .67-.88)[16] for assessing Perceptions of ICT-Supported Learning Environment; and Expectations of Learning

Accounting Inventory (ELAcc; reliabilities for the subscales: .67-.92)[17] and ICT Skills Audit Scale [18] for appraising the individual student's characteristics.

5. Preliminary Results

To ensure the successful implementation of the revised curriculum by 2010, a "try-out" project was conducted in 150 technical and national secondary schools nationwide in Malaysia in 2007. Studies on students' receptiveness on the ICT-supported learning environment have been conducted in the "try-out" project which involved 1,322 respondents. The results revealed an overall positive tendency of all the dimensions of receptiveness (with mean values above 3.5 of a 5-point Likert scale) by students with the under-achievers demonstrating significantly higher receptiveness in terms of the dimensions of Skills Acquisition (*M* = 3.818, SD = 0.522), Teaching Competency (*M* = 20 3.755, SD = 0.707), and ICT Liking and Utility (*M* = 3.819, SD = 0.587) than the high-achievers (*M* = 3.714, SD = 0.768), t = 2.913, p = 0.002 (one-tailed); (*M* = 3.606, SD = 0.832), t = 3.464, p = 0.0005 (one-tailed); and (*M* = 3.618, SD = 0.776), t = 5.316, p = 0.0000 (one-tailed) [19].

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Scaffolding Teachers' Construction of a Learning Trajectory for Mathematics Supported by ICT

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Abstract: In this paper, we present preliminary results related to a developmental project where we explore how technology can be used to improve teachers' teaching practices and their students' learning of mathematics. We have adopted a teacher-centered, collaborative approach that challenges the participating researcher to develop strategies for finding different forms of scaffolding to support the teachers' participation in this project. So far, we have used the software GeoGebra to provide competence development for the teachers in terms of mathematical representations. This software will also have a central role when investigating how the teachers perceive and make use of different affordances provided by technologies when addressing a specific learning goal related to the teaching of algebra at a lower secondary school.

Keywords: Affordances, co-design, GeoGebra, Hypothetical Learning Trajectory (HLT)

Introduction

We are currently involved in a developmental project with two mathematics teachers at a lower secondary school in Sweden where we explore how to make use of ICT to improve the teachers' mathematical teaching practices and their students' learning of mathematics. Drawing inspiration from the methodology of co-design [8] our objective is to develop technology-enhanced mathematical learning activities. Co-design provides a user-centered, collaborative approach to design research, suitable when requiring expertise from different areas, in our case mathematics, pedagogy and technology.

Involving different stakeholders and especially teachers in the design process is a key factor for an innovative learning activity to find its way to teachers' everyday practice [10]. The teachers' role in the co-design process should be regarded as dynamic and their influence should be increased in latter design iterations [10] with particular focus on refining the interplay among various forms of scaffolding required when implementing the activity in order to provide favorable conditions for its integration in the regular curriculum [11]. This methodological approach puts increasing demands on the teachers as the design process progresses and challenges the researchers to identify and provide appropriate theoretical and methodological underpinnings to support the teachers in the continued design process.

1. The Hypothetical Learning Trajectory ZZ

Within the iterative process of design research different phases of preliminary design, teaching experiment and retrospective analysis can be identified [2]. The preliminary design includes the development of a Hypothetical Learning Trajectory (HLT) which can be described as "the consideration of the learning goal, the learning activities, and the thinking and learning in which students might engage" [9, p. 133]. Although the notion of HTL has a constructivist origin [9] it has been used from other perspectives. Gravemeijer, Bowers, and Stephan have for example used the instructional theory Realistic Mathematics Education (RME) to guide the formulation of a HLT to develop classrooms activities that reflect the view of mathematics as a human activity and learning as a process within a social context [4]. For our objectives, this property makes the notion of HLT a highly adaptable and flexible tool suitable when discussing different aspects of teaching and learning within the co-design group.

The effects of integrating technologies in educational settings, are not necessary transparent. Even if technologies can be used to address some problems related to teaching, there is a possibility that they introduce new unexpected problems [5] [6]. Lindwall and Ivarsson claim that a common problem is that students tend to "exclusively focus on the operational aspects of the task without actually approaching the subject matter content" [6, p. 376]. This suggests that even if the intention of a learning activity is to afford students to work towards a predetermined learning goal, students might still focus on other aspects and other affordances. Another challenge for the designer of a HLT is that, each tool, although seemingly similar in many aspects, can have divergent effects on how students interact in an activity [6]. The result could be unexpected forms of interaction in conflict with the intentions of the HLT.

Formulating a HLT that integrates technologies requires knowledge about the affordances provided by the tool and also knowledge of how these affordances are perceived and used by teachers and students in a learning activity [1] [6]. Addressing these kinds of issues with a collaborative approach challenges the researcher to create conditions for the teachers to discuss and consider different affordances provided by technology as well as create awareness among the teachers on how digital tools might affect their educational settings [1] [5].

2. Key Idea

Our goal is to develop a HTL that is grounded in theory and attuned to the various prerequisites imposed by different stakeholders, including the possibilities and constraints that may rise when using technologies in mathematics instruction. The collaborative approach challenges the researcher to create appropriate conditions for the teachers to be able to make informed decisions. Therefore, we also seek to explore how to develop, provide and evaluate different forms of scaffold for teachers' participation in the design process. Our strategy is to monitor and qualitatively analyze the outcomes from the discussions held with the teachers and, based on the analyses, provide support for enhancing the teachers' knowledge base. So far, our efforts include two meetings with the teachers.

3. Preliminary Results

Two teachers from the current school volunteered to participate and the purpose of the first meeting was to discuss the goals of the project. At this meeting, the teachers brought forward that they were interested in addressing some of the difficulties their students had concerning algebra, e.g. the students' inability to make sense of the distributive law. The 23 analysis of the discussions rendered the following research question: "What mathematical knowledge requirements does a teacher need in order to participate in the co-design of a mathematical learning activity supported by ICT?"

A synthesis of two complementary models was used suggest that "teachers' mathematical knowledge requirement would be to consider and be able to judge and compare the didactical value of various mathematical representations" [7]. This requirement includes also the use of technologies in the sense that technologies provide affordances for multiple and multi-modal representations, simulations, manipulation of data, and conversions of representations [5]. Furthermore, the analysis of the discussion indicated that there was a need for competence development among the teachers regarding mathematical representations. For this purpose a second meeting was held where the teachers where shown some activities, implemented in GeoGebra, where numerical expressions are connected with their corresponding geometrical representations (Figure 1).

Figure 1: The activities implemented in GeoGebra

The dynamics of GeoGebra provides possibilities to move and rearrange geometrical figures with ease. This affordance is used in the activities to illustrate how algebraic treatments of a numerical expression can be interpreted and represented geometrically. The possibility to display and hide text and figures in Geogebra is used to create freedom of choice to display the two different mathematical representations either simultaneously or one at a time. Although figures and expressions are organized in a determined order in the activities, the teacher still needs to consider how the different representations should interact within the activities. The activities are designed to address students' conceptions of the distributive law by affording transformations of representations which can be regarded as a characteristic of mathematical proficiency [3]. The activities are also designed to provide affordances for different forms of interaction between the teacher, the students and the activities.

4. Reflections on the Outcomes of the Meetings

The teachers had never used GeoGebra before the second meeting and they became interested in the features of this software and in the activities presented. Furthermore, they 24 wanted immediately access to the activities in order to use them with their students. During this meeting the teachers adopted the mathematical ideas presented in the activities and they came to realize the didactical potential in interpreting and representing algebraic laws geometrically. This was clearly a new insight for them and they recognized the limitations of alternative explanations that they normally used that where exclusively based on instructions on how to manipulate different variables.

The teachers were encouraged to use the activities with their students but they were not provided any details on how to create conditions for using the activities in instruction. Instead they were invited to modify and use the activities in GeoGebra in any preferable way and thereby challenging the teachers to create their own HLT. This last meeting has not yet been followed up and the outcome of these efforts has therefore not been analyzed yet.

5. Future Work

The next step is to discuss with the teachers how they have used the activities and also attend lessons where the teachers have implemented the activities as part of their daily work. The analysis of these sessions has two objectives. First we are interested in understanding and discussing how the teachers coordinate the different parts of the hypothetical learning trajectory when using GeoGebra. Secondly, we are interested in evaluating our efforts of providing competence development in terms of mathematical representation. We want to see if and how the teachers perceive and make use of affordances for representation and communication provided in the specific activities implemented in GeoGebra.

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Towards Improving Reading Comprehension
Skills in Third Graders with a Serious Game

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Abstract: The results of different evaluations about reading comprehension in Mexico are not positive. Different methods or techniques have been implemented to counteract such numbers, and the support of information technologies (IT) in reading comprehension, is one of them. During the last years, video games demand increased in Mexico, which indicates that they are already part of the culture, and consequently, those must be harness to obtain better results. The purpose of this research is to improve reading comprehension skills of third graders through a serious game.
Keywords: Reading comprehension, serious games, third graders

Introduction

Reading is a process that each person does by themselves and it allows to examine the content of what is being read, analyze each of its parts to stress the essential and compare already existing knowledge with the one that has been just acquired [1]. A wide range of reading skills, including digital reading, are essential for an individual's personal and social fulfillment. In essence, acquiring the ability to read and understand well is a basic requirement for the social and economic demands in the society. Proficiency in reading literacy is not only one of the principal goals of schooling, but is also one of the principal means of learning [2].

A report made by Entertainment Software Association [3] explains that many children, teenagers and young people, spend part of their free time playing video games, which suggests that most of such groups enjoy using and interacting with such technologies [4]. New generations show the impact of growing inside this digital wave; technologies are an integrated aspect in their lives [5]. Students in this group use technologies to learn, and to communicate in new ways.

1. Theoretical Background

1.1 Reading Comprehension

Reading comprehension evolves due to the social, cultural and economic changes that society experiences day by day. Reading comprehension is not just the mere decoding of meanings of a text, but the construction of meanings that result from the interaction of the reader with the text [6]. Reading competence is defined as the ability of a person to understand, use, reflect and gain interest in texts written with the purpose to achieve their own goals, develop their knowledge and their personal potential, and participate in the society [7]. However, according to Nation, Cocksey, Taylor, and Bishop [8], students at 26 young age need to be competent in two specific abilities in order for them to become good readers. They need to learn how to recognize and decipher words from a piece of writing, and also understand the message that each word transmits. Reading comprehension is an act that takes time to develop, it is impossible to demonstrate a lack of reading comprehension in children who have not yet learned to read with enough precision and fluency. Therefore, researches with greater load in reading comprehension report that children at the age of 8 and older are the ones that start suffering the lack of such activity.

1.2 Serious Games

One of the ways to support the development of reading and to encourage the reading in general, is through IT. Some authors [4] think that video games help to solve problems as the ones mentioned before, partly, because of the stories of success in the use of video games in the business and military training in some countries. The results of research studies show that video games encourage the acquisition of certain cognitive skills and improve the comprehension of students in the subjects presented [4]. According to Zapušek, Cerar, and Rugelj [9] serious games are video games used for training, publicity, simulation or education and they are designed to be executed in computers. They allow student to simulate and experiment with situations that are impossible or difficult in the real word for different reasons, such as security, cost or time. Results presented about learning show a positive impact in the development of new knowledge. Similarly, serious games are receiving interest from the researchers and the industry because of the advantages that education and formation present [10]. Besides, some pedagogues consider these types of games motivating and attractive, when the characteristics that make them convincing are incorporated, of quality and even addictive sometimes [11].

1.3 Video Games and Education

Brown [12] explains that as the adoption of video games by the military as devices of training and recruiting, the appearance of video games in classrooms represents an important institutional credit to the new media. Second Life, for example, a multiuser virtual environment, is used by many universities and institutions (e.g. Texas University, Hong Kong Polytechnic University, Harvard Law School, among others) as a virtual classroom or virtual campus, where professors and students interact online. Squire [13] states that video games encourage a way of learning that goes beyond the traditional disciplinary limits and emphasize the problem solving. For Brown and Squire, video games encourage achievements that are attained through the competence, cooperation and teamwork. Nowadays, there are new researches like Guillén-Nieto and Aleson-Carbonell [14] that explain advantages of the use of video game as an educational tool.

2. Problem Statement

Currently, there is a huge problem regarding reading comprehension in Mexico. It is been known that people do not have a reading habit, conclusions that emerged based on results which were obtained in the first survey about reading [15], in which it is revealed that the Mexican's reading average is 2.9 books per year. According to report [16], Mexico is in the 48th place among the 66 countries of the Organisation for Economic Co-operation and Development (OECD) in reading comprehension and analysis. This report places Mexico in Level 2 where there is a minimum and insufficient competence for the execution of complex cognitive activities. Meanwhile, in Spanish language in reading comprehension and 27 reflection on language at a national level, one of every four third graders is on the "Under basic" level, while the 56% is on the Basic level (See figure 1-A).

3. Niche Opportunities

The video game industry in Mexico had an average annual growth of 18.7% during 2007-2010 [17]. In 2010, the Mexican market was worth $757 million, placing the country among the top 15 markets worldwide video game being the first in Latin America (See figure 1-B). So, taking into consideration all this to think about the impact that video games will have in the field of reading comprehension, as proved by several studies on how they have been applied for education in different branches [4].

Figure 1. The first graphic (A) shows the percentage of students by level of educational achievement and school stratum in Spanish/Reading Comprehension and Reflection on laguange [16]. The second graphic (B) shows the distribution of the video game market in Latin America 2009-2013 [17].

Also, it is known that there is correlational evidence that students who read more have higher achievement [18]. An example of this is the correlation between the reading of elementary school students and their score on the Excale test. Excale are tests that measure the school achievement of students from preschool education, primary education and secondary education in different fields. In Gaytán-Lugo and Hernández-Gallardo [19] the results of their study shows that the number of books that an student of elementary school read is directly proportional to the score of Spanish in Excale, inasmuch as the correlation coefficient (r) was 0.608.

Based on the above, two research questions are generated: How to design and implement a methodology in a serious game to raise the level of third graders in reading comprehension? How a serious game promotes reading comprehension of third graders?

4. Method

We are going to use thinking aloud mixed with Solomon experiment. We have to design four groups. Two groups are going to be experimental samples and two groups are going to experience no experiment manipulation. Two groups are going to receive a pretest and posttest, and the other two are going to receive a posttest. This is because the control of the effects of the pretest [20]. In order to collect information and evidence about the progress of reading comprehension that third graders have at a specific moment, during the year they have to play with the serious game, every week we are going to go with the students to apply thinking aloud. This method allows students to speak about what they think and feel at the same time that, in this case, they play with the serious game. 28

5. Current Progress

Right now we are working in the story that will be fantasy genre as [11] recommended. At the same time, we are looking for techniques that improve reading comprehension. These ones have to be in the script and in the challenges of the serious game in an implicit way.

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Development of Number Sense in Third Grade
of Elementary School Using Serious Game

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Abstract: This research focuses on instrumentation of information technologies that become psychological tools, designed to develop number sense of third graders. The theoretical background is supported by the Cultural-Historical approach, and considers that, using a serious game; emphasize the student's attention and perception for constructing mathematical concepts. A preliminary outcome identifies numerical relationships and counting, as a field of opportunity to increase math skills associated with number sense.
Keywords: Number sense, serious game, artifact

Introduction

In early years at school is perceived the difficulty of students to build mathematical concepts and it is reflected in numeric skills. Different components of cognitive processes have been associated to a poor performance on that field. The aim of research is to identify new standards that foster cognitive processes of higher psychological functions of attention and perception, as a strategy to develop number sense in the third graders. To achieve this goal, we propose designing a serious game as psychological tool, that enables students relate the abstract operations of numerical entities with concrete objects, and navigate through different training paths, according student' cognitive level. It is hoped, from these mathematical learning experiences, that students develop their number sense.

1. Literature Review

1.1 Number Sense
The study of number sense is documented in several studies, which analyze the concepts and components of the skill needed to build mathematical concepts [1, 2, 3, 4, 5, 6]. Malofeeva et al. [3], establish that number sense is the ability to understand the meaning of numbers, relations between them and the ability to perform mental calculation, numerical estimation, and quantitative reasoning; furthermore, includes conceptual understanding of numbers, knowledge of place value, number relationships, counting and sequencing, understanding and meaning of addition, subtraction, multiplication and division.
1.2 Serious Games and Mathematics
Serious games (SGs), according Ling, Xiaoqiang and Dandan [7], are a category of video and computer games to train or educate users while giving them an enjoyable experience. SGs provides a simulated environment with experiential learning activities [8]. SGs can be 30 used as tools in schools to improve students' efficiency to learn [9], moreover, emphasize the student' attention and perception, to overcome the limits of their sensory field by doing a voluntary effort. The imaginary context of the game teaches the child to guide their conduct by the perception, situations that affect him, and the meaning of the situation. As result children learn to do, consciously, an activity, and acquire a development. Researches evidence the potential positive impacts of using games and serious games, with respect to learning, skills enhancement [10], therefore the advantages of use them in mathematics [11].
1.3 Artifacts as Psychological Tools
The Cultural-Historical approach of mediations, centered on the use of cultural artifacts, is widely used in mathematics learning [12, 13, 14, 15]. According to Vygotsky [12], the artifacts are cultural mediations, transformed as psychological tools, by the use of signs as language. The use of psychological tools promotes development of higher psychological processes as perception and attention, which are the source of a specifically human behavior [12, 16], necessaries to construct mathematical concepts [17]. The use of information technologies allow to exploit the potential of artifacts [14]. With SGs, the child could perceive its environment, and language help him to perceive objects immersed in it and the relationships between them. Through SGs and language, the child reorganizes its spatial and visual field, and creates a temporary field, which is real and perceptible for him. He captures changes and reconstructs the separate activities, therefore he develops his attention. These processes are necessary to construct mathematical concepts, and enables students relate the abstract operations of numerical entities with concrete objects [17].
2. Problem Statement
By the logical character and level of abstraction necessary to construct mathematical concepts, the students demonstrate cognitive difficulties, which are perceived by indicators of educative achievement [18, 19, 20]. In México, the problem is accentuated in third graders, whom must develop mathematical skills associated with number sense. However, statistics shows a score of 419 in math, when Organisation for Economic Co-operation and Development (OECD) countries has an average of 496 [20]; 52% of students are at levels considered unsatisfactory and show difficulty identifying numerical relationships, recognition of magnitudes and simple operations with numbers [19].
3. Contribution of the Proposed Research
Identifying new benchmarks in the transformation of higher psychological processes, for develop number sense, will be the major contribution of the proposed research. The incorporation of dynamic assessment and learning gradients, into a serious game, will be the second contribution. From these approaches it is expected to contribute to developing of skills of counting and numerical relationships as components of the number sense.
4. Proposed Research Methodology
4.1 Objective31
To design a serious game that develops number sense, through transforming of psychological processes of perception and attention in mathematical contexts, looking to improve the skills of counting and numerical relationships in third graders.
4.2 Research Questions
The research will address these questions: How to enhance the development of the psychological processes of perception and attention by an Instrumental approach? How a serious game develops number sense skills of third graders?
4.3 Hypothesis
The use of serious game transforms perception and attention, and develops the skills of counting and numerical relationships of number sense, in third graders.
4.4 Research design
The research will adopt a combination of qualitative and quantitative methods to explore the development of number sense. At the first stage, the transformation of the psychological functions of perception and attention will be studied by the Genetic method of Vytgotsky [12], whose function is to study these processes in its functionality. At the second stage, two classes of third grade, will be randomly selected and assigned to the experimental and control groups. Students of both the experimental and control groups will sit for identical pre-test [21]. For six months, control group will use the SGs designed. After that, both the experimental and control groups will be subject to post-test, to measure development of the skills of counting and numerical relationships [21].
5. Preliminary Outcomes
A pilot study was conducted to identify mathematical abilities of third graders. During first part of the study, a test was designed on the basis of number sense concept, and the Mexican curriculum. After applying test, Coefficient alpha was measure (α=0.8) and the test's structure was reviewed to improve internal consistency. The figure 1 shows results of the study. In the second part of the study (See Figure 2), same test was applied to different students, and they expressed difficulty in properly resolving the items associated with numerical relationships (36%), counting (38%) and exercises for applying basic concepts of arithmetic operations (46%). These findings allow to identify areas of opportunity associated to number sense's elements. This can lead to design of strategies to exercising these specific skills, to develop number sense, looking for increase mathematics learning in third graders.
Figure 2. Results of the number sense assessment.
Figure 1. Analysis of internal consistency, of the number sense test. 32
6. Conclusion and Future Work
Currently, we have been identifying areas of opportunity to improve mathematical skills, related with number sense. This allows designing strategies, for instance a serious game, to develop number sense. As future work of this study, the author is planning the following:
• Instructional design for SGs based on numerical relationships, counting and arithmetic operations, as categories of the number sense.
• Implementing a SGs with math problematizations, according student' cognitive level.
• Pre-test and post-test to measure skills of counting and numerical relationships.
From these processes it is expected to contribute to development of skills of counting and numerical relationships as components of the number sense of third graders.
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31 quotations

Codes (28)###design of activities, ###didactical phenomenology, ###emergent modelling, ###guided reinvention, ##advantages of using RME, ##characteristics of RME, ##human activity, ##mathematization, ##principles of RME, ##real life, #RME, $$approaches to teaching Mathematics, $$$modelling, $$learner understanding, $$mathematics education, $$meaningful learning of Maths, $$teaching methods and strategies, $factors that influence teaching and learning, $learner aspects, $mathematics, $mathematics content, %role players, &pedagogy, &social aspects, @@ICT systems: virtual learning environment (VLE), @@value of using ICT, @Devices, =methodology

Memos (0)

Document body:

Virtualmatriks: A Conceptual Mathematization Process in Virtual Learning Environment
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Abstract: In the Virtual Learning Environment, students interact with the teacher and the other students. By interacting, students can enhance the understanding ofmathematical concepts and its application in the real life. This study aims to develop a model of virtual learning environment based on Realistic Mathematics Education. The research questions are how to develop and implement this model and its effect on the learning outcomes. The results of development in this study are the model and device of VirtualMATRIKS. Students simulate the conceptual mathematization process by create the movie about problem solving. Then students share and discuss it through virtual-learning device. After validated by experts, the model and device were implemented in Semarang Senior High School. And the result is the model and devices qualify the effectiveness and practicality. This model gives a positive impact on student learning outcomes, and students can reach the Upper Limit of Zone of Proximal Development. This model can be applied and developed further in learning mathematics by considering into environment, conditions and students interest.
Key words: RME ,Virtual-Learning Environment, Educational Movie, Conceptual Mathematization
Introduction
Teaching and learning mathematics are complex tasks. According to Freudenthal (1991), mathematics as human activity and mathematics must be connected to reality. Therefore, students should be able to understand the concepts of mathematics and and its application in the real world. But in fact, there are some problems in learning mathematics, such as some students are difficult to understand the concepts and problem solving, some students have low motivation in learning mathematics, and some students do not understand the application of mathematics in real life. By considering the expectations and problems, it is necesary to develop a model of learning environment that makes the learning activities become easier and more enjoyable so the students can reach the optimal learning outcomes. When referring to the Vygotskian perspective, learning outcomes and students' understanding can be increased as a result of interaction in learning. The interaction between teacher-student and student-student in learning, illustrates that social interaction in the form of discussion is able to provide students with opportunities to optimize the learning process. This interaction allows teachers and students to share and modify their ways of thinking. There is also a possibility for some students to showcase their own arguments as well as for other students the opportunity to try to capture the thought patterns of other students.
Activity is believed to be able to increase knowledge and understanding of the object was learned from the previous stage to a higher stage. Students can interact around the difficult task and share their effective problem-solving strategies, if there is an arrangement of classes and a form of learning environment. There are two forms of learning environment, namely physical learning environments and virtual learning environments. Virtual learning environment (VLE) developed by making the design of information spaces as learning environments. In the VLE, students interact with teacher and the other students, wherever and whenever. Teachers and students are familiar with the use of the system. Hardware and software that supports this system has been available. However, to achieve the purpose of learning mathematics, the development of the learning environment must be based on the characteristics and principles of mathematics and learning mathematics.
Theoretical Background
Vygotskian Perspective
According to Vygotsky (1978), there are two important concepts in the sociocultural theory, the Zone of Proximal Development (ZPD) and scaffolding. ZPD is the distance between the actual developmental level as determined by independent problem-solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers (Vygotsky, 1978). It is a dynamic developmental state. At the lower limit of the zone are the tasks that children can accomplish independently, while at the upper limit is the space where more complex tasks can be realized by children through interactions with more knowledgeable others. Learning can evoke a variety ofmental processes that can only be operated stored when a person interacts with an adult or collaborate with their peers. With these interactions, students can complete a complex task that can only be solved if students are given assistance by adults or

71collaborate with peers, the level is referred to as the upper limit of the ZPD. The process of giving assistance in this process is called scaffolding.
Scaffolding means providing large amounts ofaid to students during the early stages oflearning and then reducing the assistance and provide opportunities for children is to take over greater responsibility as soon as he can do it (Slavin, 1994). Vygotsky (in John and Thornton, 1993) explains that learning occurs in two stages: first stage occurs when collaborating with others, and the next phase done on an individual basis in which occurs the internalization process. During the process of interaction occurs (teacher-student and student-student), the following capabilities should be developed: mutual respect, to test

the truth ofthe statement of others, negotiate, and adopt the other opinion.
Virtual Learning Environment
A Virtual Learning Environment is a collection of integrated tools enabling the management of online learning,
providing a delivery mechanism, student tracking, assessment and access to resources (Dillenbour & al, 1999). In a virtual
learning environment, students and teacher can interact anywhere, anytime and easy to find and share informations. Interaction
can take many forms, including synchronous or asynchronous communication, one-to-one or one-to-many or many-to-many,
text-based or audio and video, or even indirect communication such as sharing objects. Researchers have introduced the notion
of "place" (Dourish & Chalmers in Dillenbour & al, 1999) to emphasize that space has a social impact. Places are settings in
which people interact. (Munro, Höök & Benyon in Dillenbour & al, 1999). The representation of the learning environment
ranges from text-based interfaces to the most complex 3D graphical output. 3D graphical representation is often used, because
it can increase motivation, and triggering a positive attitude towards the environment. Dillenbour & al (1999) observed that
virtual space imparts on users behaviour even when space is only described by text. In virtual environments, learning activities
ranging from multiple-choice questionnaire for simulation and problem solving. The idea of learning activities in a virtual
learning environment refers to something richer than the individual, closer to the idea of the project.
The difference between other constructivist environments and what virtual environments potentially offer can be
described as making students not only active, but also actors, i.e. members and contributors of the social and information
space. (Dillenbour & al, 1999). A virtual learning environment integrates a variety of tools supporting multiple functions:
information, communication, collaboration, learning and management (Peraya & al. in Dillenbour & al, 1999). Virtual learning
environments do not only integrate a variety of software tools but also integrate all the physical tools that can be found in a
classroom. According Carabaneanu (2006), a learning environment is considered adaptive if it is capable of: monitoring the
activities of its users; interpreting these on the basis of domain-specific models; inferring user requirements and preferences
out of the interpreted activities, appropriately representing these in associated models; and, finally, acting upon the available
knowledge on its users and the subject matter at hand, to dynamically facilitate the learning process.
Realistic Mathematics Education
Mathematics as human activity and mathematics must be connected to reality. These statements are the root of the
Realistic Mathematics Education (RME). RME refers to the mathematics education approach which has been developed and
applied in the Netherlands since 1971. Mathematics should be undertaken as an activity in the students experience mathematics
as a meaningful subject and can better understand it (Freudenthal, 1991). Freudenthal emphasizes real activity in mathematical
activities. Activities under consideration should consist largely of organizing or mathematization of subject matter and is taken
from reality. Learners must learn mathematics with mathematization of subject matter from the real context and activity of
mathematics, not learning from the traditional presentation of mathematics to the students as a ready-use system that is
generalized. Real situations can include contextual problems or authentic Mathematically Contexts for students where they're
having problems which are presented in relevant and real. The process of developing mathematical concepts and ideas that
started from the real world called conceptual mathematization (de Lange, 1993). Schematic model of learning is described as
follows.
Picture 1. Conceptual Mathematization(de Lange, 1993)
Through a process of progressive matematization, learners are given the opportunity to rediscover the insight,
knowledge and mathematical procedures. Thus learners do the stages which in RME called horizontal and vertical
mathematization. Gravemeijer (in Sembiring & al., 2008), suggests that there are three main principles of RME, namely: (a)
Guided reinvention / progressive mathematizing, (b) the didactical phenomenology and (c) self-developed models. For the
operationalization of the three main principles of RME, according Panhuizen (in Gravemeijer, 1994: 114-115), RME has five
characteristics, namely: a) The use of contextual problems, b) The use of various models, c) Student contributions, d)
interactivity, and e) intertwining (integrated). Strategy, which could be applied in the implementation of RME by LoucksHorsley (1998): (1) a short learning process, (2) curriculum development, and (3) the use of technology. The results of research
in the Netherlands showed that the RME has shown satisfactory results. RME has the potential to improve students'
understanding of mathematics (Streefland, 1991).

All human actions are called activities. An activity involves an object. The object is to be transferred to the output of
the activity. A subject performs an activity using a tool. The tool can be a physical or an abstract tool. The interactions between
subject, object and community can all be mediated. Rules mediate the activity between the community and a subject. The
activity may be collaborative, i.e. several subjects jointly do the activity using tools and dividing the work between each
subject (Multisilta, 2008). In general, tools, rules and division of work mediate the relationship between the subject,
community and object. Tools, rules and division of work are artefacts that are used to achieve the outcome. Experiences and
attitudes that influence one another (Uden, 2007). Activity can furthermore be divided into actions and an action on operations.
In general, activities are based on high-level goals (for example, documenting a work process with images and video clips).
The Shared Activities and Experiences framework originates from a need to describe sharing and experiences in social media
in theoretical terms (Multisilta, 2008).
Figure 2. The Shared Experience and Activities Framework (Multisilta, J., 2008)
Conceptual Framework and Research Question
Mathematics as human activity and mathematics must be connected to reality. Therefore, students shoud be able to
understand mathematics concepts and its application in the real world. Students' understanding can be optimized through social
interaction in the learning environment. There are two forms of learning environment, namely physical learning environments
and virtual learning environments. In a Virtual Learning Environment (VLE), students can interact with the other wherever and
whenever. In this learning environment, students do activities to learn and gain new knowledge and experience with solving
problems, sharing information, and discuss. Thus, students are expected to reach the upper limit of the zone of proximal
development (ZPD). Activities should be adapted to the characteristics of mathematics. Mathematical activity of solving
contextual problem starts from the real world problem, mathematizing and reflecting the problem, abstraction and
formalization, mathematizing in application and then bring back to the real world at the end process. In Realistic Mathematics
Education (RME) approach, this process is called Conceptual Mathematization
This research questions are (1) how to develop and implement the model of VLE based on RME in Semarang Senior
High Schools? and (2) Is the implementation of the model has a positive effect on student learning outcomes?
Research Method and Procedures
This research is a development research that continued from previous research, MATRIKSMovie: building the nation
character through movie-based RME (See also Cahyono, 2011). The purposes of this research is to develop the model of VLE
based on RME in Semarang Senior High School. According to Cahyono (2006), model is defined as a conceptual framework
used as guidance to do certain activity. Pouwer (1974: 243) explain about model as assumption like metaphor formulized
explicitly containing unsures which depends each other. As a metaphor, model has never been seen as part of data represented.
Model explains phenomena in a form which is not usual. Every model is needed to explain something more or different from
data. This requirement is fulfilled by presenting data in forms of summary (type, diagram), configuration (structures),
correlation (pattern), idealization and the combination of the four. Thus, model is metaphor which is solid and useful for the
comparation of relation between the choosen data and the relation among the choosen unsures of a logical construction.
In this study, validity, practicality and effectiveness criteria are defined as follows: validity refers to the extent that the
design of the intervention should include "state of the art knowledge" (content validity) and the various components of the
intervention are consistently linked to each other (construct validity), practicality refers to the extent that users (teachers and
pupils) and other experts consider the intervention as appealing and usable in normal conditions, effectiveness refers to the
extent that the experiences and outcomes from the intervention are consistent with the intended aims.
Following the work of Nieveen (1997) and Ottevanger (2001), the development and research activities in this research
were conducted in three stages. The first stage is called the front-end analysis, the current situation of mathematics learning in
Semarang Senior High School were analyzed. The second stage of the study is called the prototyping stage. This stage
consisted to develop and validate the prototype, namely virtual-learning device, the lesson kit and formative evaluation. The

implemented in Senior High School, and then the learning process and result are reflected and evaluated with the instrument.
Results and Discussion
The first stage is called preliminary stage or front-end analysis, that the current situation of mathematics learning in
Semarang senior high schools was analyzed. Result of the analysis in this stage are used as the basic for the development of
model with valid, practical, and effective criteria. There are some problems in learning mathematics, such as some students are
difficult to understand the concepts and problem solving, some students have low motivation in learning mathematics, and
some students do not understand the application of mathematics in real life. Students and teachers need a medium to interact
with easier, more enjoyable, and adjust to the development of information and communication technology. The results of the
analysis indicate that some of the high school students still require visualization of abstract concepts and application of
concepts in the real world. One material that is considered difficult by students is a matter of trigonometry. Student difficulties
understanding the concepts and solving problems related to trigonometry, including equations and trigonometric functions and
applications of trigonometry in real life. Students at this age also tend to be less motivated in carrying out learning activities in
mathematics and more interested in the entertainment and social networking because of the influence of age, social,
environmental, facilities, and technology. Technological development encourages innovation in teaching and learning of
mathematics by adjusting to conditions. By considering the expectations and problems, it is necessary to develop a model of
learning environment that makes the learning activities become easier and more enjoyable so the students can reach the
optimal learning outcomes. Learning outcomes and students' understanding can be increased as a result of interaction in
learning.
In the second phase (prototyping stage), the result of analysis in first stage used to develop the prototype. Based on
the preliminary analysis, the prototype being developed is a model and device of virtual learning mathematics based on RME,
called VirtualMATRIKS. The prototype that need to develop, i.e virtual-learning device, lesson plan, teachers guide, students
workbook, and the evaluation sheet. Preliminary description of the learning process is the students simulate of the conceptual
mathematization process by creating the movie about problem solving. Then students experiencing by share and discuss the
movie with other through virtual-learning device. It is based on the fact that mathematical activity of solving contextual
problem starts from the real world problem, mathematizing and reflecting the problem, abstraction and formalization,
mathematizing in application and then bring back to the real world at the end process. The activities of students in these
activities is to demonstrate the process of solving problems related to the rediscovery of the mathematical concepts and
applications of mathematical concepts in real life. Process in the production of educational films starting from a given theme as
a problem (in the form of the rediscovery of the concept of mathematics, or mathematical applications), then the students
discuss in a production team to develop problem-solving scenarios to be presented in the drama of human activity in problem
solving, setting up properties which is needed in problem solving and presentation, to present the steps solving the problem by
acting in front of the camera (as actrist or as a presenter), the product is presented in real video and/ or graphic animation
according to student creativity and made simple with simple tools, so implementation of these activities in accordance with the
allocation of time that has been planned in the lesson plan. The results of this learning process are portofolios and movie. Each
team share the learning experience by uploading the movie in the Virtual Learning Environment device, then students discuss
together about material with the guidance of teachers in the virtual learning environment. This activity can be done anywhere
and anytime, with rules specified by the teacher.
Draft of prototype validated by experts (education, mathematics, media, evaluation) to obtain a valid prototype and
ready to be implemented in learning process on the third stage. After validated, the prototype is corrected based on the
suggestions given by experts, so getting a second prototype that is better than ever. Then the second prototype was simulated in
the learning process. Simulation is used to obtain data on the implementation of models and tools in learning. The data come
from teachers, students, and the observer. Simulation concludes with discussion and reflection to obtain feedback and know the
lack of model and device as the basis for subsequent improvement. After repair and produce the final version of the prototype,
and ready for use, then executed the third stage. The result of prototyping are the model of virtual learning environment and
device of VLE. Learning environment that developed through this research is a hybrid learning environment (face-to-face
learning and learning through elearning devices). Learning processes and devices were developed based on the philosophy,
principles, and characteristics of RME and using IT-tools in the form of e-learning device that focus on providing support for
teacher(s) and students in the learning process. E-learning device was developed with the software and computer networks.
Syntax of the model are (1) Teachers provide instruction, convey learning goals, students in the class divides into groups,
distribute the student activity sheet, (2) The teacher gives problems to be solved by the students, so students are able to
construct the concept of matter being studied and/ or apply the concepts in real life, (3) Students discussing with his team to
share tasks and create problem-solving scenarios in accordance with the creativity of each. (with teacher guidance). (4)
Accordance with their respective duties, students creating a simple short film production process of solving the problem (the
construction of concepts and/ or application of concepts). (5) Each team sharing the learning experience by uploading a movie
in the VLE device (6) Students discuss their work together with the guidance of teachers in the virtual learning environment.
This activity can be done anywhere and anytime, with rules specified by the teacher, (7) Each student gives a conclusion. (8)
Teachers give clear, straighten the concept (if there is a less precise), and give final task of learning (projects, pop quizzes and/
or homework) using VLE device. Virtual learning device was developed and implemented by using software, hardware, and
computer network, that supports text files, images, interactive media, mathematical formulas and video. The device also
provides both synchronization and asynchronization communication feature.

share and discuss in VLE device
The third stage of the research is called the assessment stage. In this stage the final version of the VirtualMATRIKS
model and device are implemented in Senior High School, and then the learning process and result are reflected and evaluated
with the instrument.VirtualMATRIKS was implemented in a Semarang Senior High School class 10 in 2008 in the subject
matter of Trigonometry.
The research result suggest that the average of reliability of learning equipment (virtual-learning device, lesson plan,
teachers guide, students workbook, and the evaluation sheet) is 95,30%, that is a high reliability category, the capability of
teacher in organizing the learning could be categorized as very good, with average score 3,52. The average of teacher's
activities in the learning process is 3,54 or it reaches 88,48 % (category of very good). The average of students' activities in the
learning process is 3,54 or it reaches 88,57 % (category of very good). Seeing the average of students' involvement of each
meeting, it shows that there is an increase of students' involvement from 3.45 to 3.54 at the last meeting. The average of
students' skill of process in the learning is 3.57 or it reaches 89.26% (category of very good).
According the calculation,
2
x
calculation
= 1,296 which is significant at 0,069 > 0,05 , thus, Ho is accepted which
means the data has normal distribution. Therefore, the statistic used is statistic of parametric. The result shows that the
mathematics learning result data coming from upper, middle and lower groups in the experiment class and control class are
normally distributed and are homogenous. By using Univariate One Way Anova to test the difference of averages between the
upper, middle and lower groups. Based on the calculation, at the row group, the significant value is 0,020 < 5%, it means that
the learning outcomes between experiment class and control class is different. At the column group, the significant value is
0,000 < 5%, it means that there is no difference of the learning outcomes among the three groups (upper, middle, lower). Then,
there is no interaction because the value of significant is 0,091 > 5%.
By using Regression Analysis, F
calculation
= 25,643 with significant rate is 0,000. It shows that there is strong effect of
skill of process toward the learning outcomes or Ho is rejected which means there is influence between skill of process and
learning outcomes. From the calculation, the correlation between skill of process and learning outcomes is 63,5 % and the skill
of process gives contribution toward learning outcomes 40,3%. Therefore, the skill of process strongly correlates with the
learning outcomes. The regression equation between skill of process and learning outcomes is
^
Y = 32,371 + 2,657 X , where Y
= learning outcomes, and X = skill of process. From the calculation, F
calculation
= 27,616 with significant rate is. It shows
that there is strong effect of students' activity toward the learning outcomes or Ho is rejected which means there is influence
between students' activity and learning outcomes. The correlation between students' activity and learning outcomes is 64,9 %
and the students' activity gives contribution toward learning outcomes 42,1%. Therefore, the students' activity strongly
correlates with the learning outcomes. The regression equation between students' activity and learning outcomes is
^
Y = 13,968
+ 1,987 X , where Y = learning outcomes, dan X = students' activity.
From Compare Mean One Sample t Test , we find that t
calculation
= 2,963. By using right side test, for α = 5% and df
= n - 1 = 39-1 = 38 we gain t
1(α)(n--)1
= 2,021. Because t
calculation
> t
table
, then Ho is rejected. Therefore, it can be
concluded that the average of learning outcomes ≥ 65,0, thus, the students have mastered the material because they reach
material comprehension. Therefore, learning process using this model can reach the purposes of learning (reach the material

comprehension) with the average of learning result is 80,57.
The results of implementation show that the learning of mathematics through VirtualMATRIKS showed the good results. Activities and process skills of students in this learning is making a positive impact on student learning outcomes. The learning outcomes in this study referred to three aspects: cognitive, affective, and psychomotor. The cognitive aspect included pupils' achievement and reasoning, the affective aspect involved pupils' motivation, activity, and creativity, while the psychomotor included skills in problem-solving process. Scaffolding for students to reach the ZPD occurs when students interact with each other in processing of problem solving.
In the activities, students are trained to be creative and caring in problems solving, both social and environmental problems. Students work together in solving the problem. Students are able to communicate their ideas and dare to be responsible. This learning can build motivation, appreciation, contribution, interest, beliefs, creativity, confidence and
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75perseverance and a sense of responsibility and communication skills. RME approach can build self-reliance, democracy, tolerance, humanism and honesty. Mathematical learning occurs during the film production process and during the discussion and interact with teacher or the others using VLE device. Activities and process skills of students in mathematics learning through VirtualMATRIKS in the high school has provided a positive effect on the results student learning (cognitive, affective, and psychomotor). Students learning outcomes in this learning environment are better than students learning outcomes in the conventional leagrning environment. There is a difference between understanding and student learning outcomes between before and after the learning process in that environment. After interacting via the VLE devices, understanding and student learning outcomes are higher than ever. Students can reach the upper limit of the ZPD.
Conclusion and Suggestion
Based on the results from the three stages of this study, it has been concluded that: The result of this research is a VLE model based on RME, namely VirtualMATRIKS. In this learning environment, the students simulate of the conceptual mathematization process by creating the movie about problem solving. Then students experiencing by share and discuss the movie with other through virtual-learning device.Model and device are effectively implemented in mathematics learning in Senior High School, and this model gives a positive impact on student learning outcomes (cognitive, affective, and psychomotor). Students can reach the Upper Limit of Zone of Proximal Development.
Based on the conclusions of this study, it can be suggested that this model can be applied and developed further in learning mathematics by considering into environment, conditions and students interest.
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10 quotations

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ME-427
THE DEVELOPMENT OF REALISTIC MATHEMATICS EDUCATION-BASED
BLOG AT LINEAR ALGEBRA COURSE IN UIN SUSKA RIAU
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Abstract
More sophisticated technology tools from laptops, smartphones, tabs and others make students dependence to use them and bring it everywhere they are. It would be better if they are used by the students to learn, not just for social networks such as Facebook, Twitter and others, but can be used for student's learning. The purpose of this research is to develop a blog based on the Realistic Mathematics Education (RME) in Algebra courses so that the students can access courses they need anywhere and at any time, review the effectiveness of blogs in supporting learning process algebra, review the practicality blog to use by students to have lecturing processes, use of sophisticated technology tools to support the learning process.
The development model employed the model of Dick and Carey (2009). This development model steps was started by writing targetted competences, to analyze learning, to identify basic ability, to write performance objectives, develop instrument assessment, to select and develop learning materials, to design and conduct formative evaluation, to revise, and design and develop summative evaluation.
The research results showed that validation of Educational Technology Expert of 88.7% and 89.71% by Mathematics Content Expert stated that the blocks with the RME approach was effective and practical, and could make the students understand the concepts of Algebra related to the real world. The results of testing a small group was 83.25%, and 85.50% for test large groups. From the tests that were done through pre-test and post-test, through the test of t> t table, it was obtained calculation results of t count = 6.48, while the values of the t table with significant levels of 5% and 1% were 2.09 and 2.84 respectively. Thus there were differences in the average value of learning achievements prior to and after the use of the blog-based learning (RME). In other words, the development of RME-based blog was effective to be used as a learning media and source to improve students' understanding of the Algebra materials.
Key Words : Blogs, Algebra, Realistics Mathematics Education (RME)
A. Introduction
As the times growing and more sophistication of technology tools from laptops, smart phones, Tab and others make students dependence to use and always bring them everywhere they are. The existences of these technological tools have a lot of benefits, ranging from as a means of communication and social networking, up to as a learning medium.
Math learning is not quite with learning process begins with lecturers to explain the algorithm with some examples, and then students do the exercises in accordance with the

example given by lecturers. The students are almost never given the opportunity by the lecturers to understand the rationale behind the algorithms that are taught to them. Lecturer focused the students to remember "the ways" they teach in solving the problem rather than to stimulate them to construct knowledge, so that the knowledge gained by the students is less meaningful and quickly forgotten .
In mathematics learning in higher education, there are a few things that tend to be ignored by the lecturers, which is characteristic of the topic being taught and student characteristics. From many mathematical topics that are taught, there is an easy one, so that the subject can be understood by the students just by listening to the explanation from the lecturer. Most of other topics will be understood if the presentation is accompanied by illustrations and drawings. Meanwhile, there are also topics to understand that requires hands-on activities or learning by doing. These conditions indicate that lecturers "should not" use the same learning methods from time to time. Lecturers need to select and use models, approaches, methods, or various media, making it easier for students to study a topic, and make the students actively in learning.
In addition, in terms of media usage, lecturer should use a simple, attractive, easily accessible media and close to the student life. Additional tools biasly used by lecturers as media are computer technology. Because through a computer program (internet), the users allows to save, organize and manipulate information, including text data and numerics (Eggen & Kauchak, 2012). In this modern era, the students are not strange with the internet. With the Internet, students can access a variety of information, ranging from the world of entertainment, social networking sites, even learning materials. However, many students today who prefer to use the internet to chat on social networks such as facebook, twitter, and others . It would be nice if they are used by the students to learn, not just on social networks such as Facebook, Twitter and others, but can be used for students' learning, such as Linear Algebra material.
Linear algebra is one of the mathematical material that is easy to understand if the presentation is accompanied by illustrations and drawings. In addition, many Linear Algebra material that can be associated with contextual problems. Therefore, researchers are interested in creating a Realistic Mathematic Education-based blog on Linear Algebra in UIN Riau Suska.
Realistic Mathematics Education (RME) approach is one approach that uses real-world context in the delivery of learning, where students are expected to be more motivated because they feel that math is very close to the real world. It's the real thing will be easier to remember than abstract ideas. Math will be easy to learn if you can see its application in the real world.
According to John Dewey (Uno, 2012), linking learning to one's life makes more meaningful learning, can learn something through experience, attention, and change; find it easier to run other activities conducted by a real and meaningful situations, so that students feel that learning mathematics has a great advantage because it can be applied in everyday life. Beside it with the blog Linear Algebra courses, students can access courses they need where anytime and anywhere, review the effectiveness of blogs in supporting learning process algebra, review the practicality of use student blogs to conduct the lecture, as well as the use of sophisticated technology tools to support the learning process. According to Sukiman (2012) through blogs bring significant benefits for students. Lecturers can lead the student to focus on the ideas or writing, or both. Thus, students can develop analytical abilities and faculty can learn to be a better writer and communicator.
Based on the background of the problem, this study focused on answering several questions as follows : 1) how is the expert response to the development of an EMR-based blog on Algebra course? 2) how is the media expert response to the development of an EMR-based blog on Algebra course? 3) how the test responses of learners in small groups towards the development of an EMR-based blog on Algebra course? 4) how the user response (educators) in field tests against the development of an EMR-based blog on Algebra course? 5) how is the Proceeding of International Conference On Research, Implementation And Education
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effectiveness of research products, which can be measured by looking at the difference between the scores of pre-test and post-test learners in the learning achieved by using RME-based blog ?
B. Method
This type of research used is the development (research and development/R & D). The use of this development research is based on the opinions Richey (Richey & Nelson, 2000) that facilitates the development of a research study of new models, equipment, and procedures that can reliably anticipate their effectiveness and efficiency. In this way, it can be determined the relevance from specific-context of the findings to other teaching and learning environment as well as to identify new general principles of design, development, and evaluation. When associated with this research that wants to create a blog based on the RME Linear Algebra courses that meet the valid, practical and effective criterias.
The design and procedures used in this study is a model of Dick and Carey (2009) with system oriented This model has ten following steps (Dick and Carey, 2000) . Picture 1
Figure 4.
The first step, second and third model of Dick and Carey are categorized into preliminary research activities. These activities include the identification of competencies to be achieved in a Linear Algebra course, learning analysis, and analyze the basic skills of students. The fourth, fifth, sixth, seventh steps are to write the specific purpose of learning or better known as the indicators of achievement of learning achievements of Linear Algebra course, the development of standard reference tests the development of learning strategies, and developing and selecting instructional materials is a further blog development steps. The next activities of development steps are evaluated in the eighth step which was to develop formative evaluation. Formative evaluation is done to see the advantages and disadvantages of the development of the

blog.
Picture 1. Development Model of Dick and Carey (2009)
Formative evaluations are conducted with the mechanism of one to one evaluation,
small group evaluation, and field trial evaluation. One to one is done by experts of Educational
Technology and Mathematics. After technology and maths experts provided materials'
improvement and certain considerations, and put them in a blog to be tested by the user as much
as 3 people from each of the different levels of ability and small group evaluation was
conducted to 6-9 users, as well as field trial evaluation conducted on 20 - 30 users. Field Risnawati, Annisa & Elviza /The Development of ... ISBN.978-979-99314-8-1
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evaluation or referred to by the term field trials was conducted on students of Mathematics
Education Program. Data collection instrument used in this study was a questionnaire and
observation sheet. Questionnaire was designed for experts and for field trials. Meanwhile, the
observation sheet was designed and used when we conducted field trials . Revisions were made
at each step .
In this study, it was used two data analysis techniques : 1) a qualitative descriptive
analysis techniques, 2) quantitative analysis techniques. Descriptive analysis techniques are
used to process data from the test results matter experts, media specialists, and educators a small
group of participants in the field test, whereas quantitative analysis is used to process the data
obtained from the questionnaire in the form of descriptive percent.
The data was processed on the field test data from the pre-test and post-test. Pre-test
achievements were given to students before the lecture with a problem-based learning blog,
while the post-test was given after they understood the blog with the problem-based learning.
C. Discussion
Based on the steps undertaken, EMR-based blog development product, the blog can be
unlocked through http://risna-wati-math.blogspot.com. Product development of the blog was
further validated and tested. The results were as described as follows :
1. Data of Validation Results from Educational Technology Specialists.
As a whole based on the indicators assessed by the Expert Technology from design blog, as well
as the suitability of the content of the expected performance, experts recommended that the
development of an EMR-based blog was well categorized, and it deserved to be used with the
percentage of 88.7%.
2 . Data of Validation Results from Matter Expert.
Validation that was performed by material experts recommended that RM -based blog deserved
to be used. Indicators of material truth and relevance to the basic competences and learning
indicator reaches 100% accuracy rate. This means that the competency has been referred to the
necessity of learning. Ease of understanding the material on blogs, attractiveness, clarity of
material, systematic presentation of the material presented had been very good. Matter expert
assessment of the components related to the design of the blog; subject's contents that include
affective, cognitive and psychomotoric, which means it has to meet several capabilities; quality
of the learning interaction; appropriateness of instructional media learning of life skills. Overall
data that was obtained from validation results of matter experts on a blog based on the category
of RME were very good around 89.71%.
3 . Results Data of Small Group Test.
Small group trials of 3-6 people as a whole was obtained a percentage rate of users reached
86.56 %. This quality means developing an EMR-based blog was in the category of fit for use.
Users expressed that blog can be studied either independently or in groups, anytime and
anywhere. This approach can make the students learn to understand and meaningful.
4 . Result Data of Field Test
The next test is to lecturers, before being tested, it was priorly socialized to lecturers of Linear
Algebra course, this blog was being piloted to two lecturers, field trial data on lecturers of
Linear Algebra course, the lecturer stated that this blog was easily understood by the students,
because it is directly related to their real life. Then the blog was directly tested on a large group
of about 20 people, from the processed data it was obtained the percentage of 85.50 % with
good qualifications. Based on the results of the assessment and expert reviews on both media
and materials, as well as individual testing, both small groups, to field test, the problem-based
learning-based blog was with good category.
5 . Different test of Pre-test and Post-test
Based on the tests that were tested through pre-test and post-test, through the test of t
count> t
table
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it was obtained the calculation resultsof t = 6.48, while the value of the t
table with significant
levels of 5 % and 1 % were 2.09 and 2.84 respectively. Thus, there are differences in the
average value of learning achievements before and after using blogs based on problem-based
learning. In other words, the development of RME-based blog was effective to be used as a
media source and learning to improve students' understanding of the material Algebra.
Of the overall aspects evaluated and tested, it can be said that the development of this
blog was well categorized, in terms of fit to use as a medium of learning. Blog is a website that
is easy to use. This facility can quickly post thoughts of someone, interact with others, to
publish the papers, announcements and other much more advantages. Because it is easy and
practical, blogs can be used not only for communication but also used as a medium of learning
that allows everyone toaccess. Huette (2006) also described the benefits of using blogs in the
classroom, among other things : 1) to promote critical thinking and analytical, 2) to encourage
creative thinking, intuitive and associational thinking, 3) can encourage analogical thinking, 4)
has the potential to increase access and exposure to quality information, and 5) combining
solitary and social interaction .
RME incorporates the views of "what is mathematics, how students learn and how
mathematics should be taught. Mathematics education should lead students to use a variety of
situations and the opportunity to re-invent mathematics in their own way (Sutarto Hadi, 2005).
Through the blog, combined with real objects to make students learn more relaxed, less anxious,
because students can adjust the problem or understand an answer discovery materials at the time
the ability to think and learn. Wang, Haertel, and Walberg (1992) emphasized the importance of
reducing the anxiety of learning, particularly fear to do wrong, and to make students confident
(Harmin and Toth, 2012:68). Furthermore Harmin and Toth stated that all students have
individual learning styles, so the student can do and learn in accordance with the time he needs,
without any pressure. Learning without the pressure will be more meaningful.
Other advantages of developing blogs with RME based are as follows : 1) the student
can ask for the help of people who preferred to give understanding to those who are less
problems when they understand ; 2) can evoke the spirit of learning, because it is followed by
an attractive display, good presentation of the material, in fact the problem. 3) make it easier for
students to reinvent mathematics concepts in accordance with the characteristics of realistic
mathematics ; 4) to train students to solve math problems, with the understanding that has been
obtained in advance ; 5) makes it easy for lecturers to check student understanding
authentically, in accordance with the instructions in the blog. In addition to advantages also has
its limitations include 1) not all students have an internet connection ; 2) material developed is
limited to materials related to the real world.
D. Summary
Based on the formulation of the problem, the results of the data analysis and discussion
in this study, conclusions can be drawn. 1) learning technology experts responded that the
development blog RME -based learning was very good and it deserved to be used; 2) learning
materials experts responded that the block-based EMR was appropriate and feasible for use in
learning. 3) students as subjects for a small group test commented that the -based math blog
was good, because it could improve our understanding of Linear Algebra subjects; 4) lecturers
of Linear Algebra course in field tests provide a response that blocks mathematics through
problem-based learning was good and fit for use in the learning process; 5) based on the t test
conducted got the results of t
count (6.46), thus it can be said that t
count > t
table either on
significance levels of 1% (2.84) and 5 % (2.09). This means that there were differences in the
average value of student results after using blog trough RME-based learning.Risnawati, Annisa & Elviza /The Development of ... ISBN.978-979-99314-8-1
ME-432
Based on the research results, discussion and conclusions can be recommended : 1)
educators ought to have a strong desire and willingness to learn and make creative design that is
appropriate with learning system as learning needs in the classroom, in addition to the educators
also need to have creativity using ICT, both in the individual learning condition and study
group. 2) based on the data obtained, this study is limited to the pre-experimental phase with a
significant t-test results, the research can be carried out by quasi-experiments involving the
control group.
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Codes (16)#principles of RME, #real life, #RME, $$approaches to teaching Mathematics, $$mathematics education, $$meaningful learning of Maths, $$teaching methods and strategies, $factors that influence teaching and learning, $learner aspects, $mathematics, $mathematics content, &pedagogy, @@challenges when working with ICT, @@value of using ICT, @Devices, @Tools

Memos (0)

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Does the use of technology make a difference in
the geometric cognitive growth of pre-service
mathematics teachers?
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This study investigated the geometric cognitive growth of pre-service mathematics
teachers in terms of the Van Hiele levels in a technology-enriched environment, as
opposed to that of students in a learning environment without any technological
enhancements. In order to investigate this, a quasi-experimental non-equivalent
comparison group design was used. Similar course content was used for both the
control and experimental groups. The students worked through a series of geometry
activities and problems. The difference between the groups was that dynamic
geometry software was integrated into the teaching of the experimental group. The
Cognitive Development and Achievement in Secondary School Geometry (CDASSG) Van
Hiele geometry test was used to determine all the students' level of geometric thinking
before and after the course. The study found that the use of dynamic geometry
software enhanced student teachers' geometric visualisation, analysis and deduction,
but not their ability to informally justify their reasoning and to understand the formal
aspects of deduction.
Background and literature study
Despite large amounts of money having been invested in equipping schools with
technology, there is limited evidence of positive effects on student achievement
(Wenglinsky, 1998; Zhao, Pugh, Sheldon & Byers, 2002). The assumption was that
increased availability of technology in the classroom would lead to increased use, and
increased use would then lead to not only efficient teaching and better learning, but
also better student achievement (Cuban, 2001). According to Brown-L'Bahy (2005),
researchers voiced uncertainty about the benefits of school technology use as early as
in 1987. In fact, Myhre, Popejoy and Carney (2006, p. 1002) pointed out that there is
"considerable uncertainty within the educational community regarding the value of
technology in teaching and learning". From these different arguments it is clear that
researchers themselves struggle with the question of whether and how technology can
improve teaching and learning. There are however researchers who believe that
technology, if correctly used, can enhance teaching and learning. Ching, Basham and
Planfetti (2005, p. 226) found from research that "student-centered, technologyintegrated learning environments help to produce students who are better able to
think critically, solve problems, collaborate with others, and engage deeply in the
learning process". According to Sanders (1998), the appropriate use of dynamic
geometry software can enhance mathematics teaching and conceptual development,
and enrich visualisation, while also laying a foundation for deductive proof.1234 Australasian Journal of Educational Technology, 2012, 28(7)
Wong (1998) contended that the instructional objective with graphing software is to
develop and reinforce concepts, to rectify common errors, to check graphical solutions,
to solve equations graphically, to test conjectures through problem posing, to
encourage users to become metacognitive, to help users to acquire information
technology skills, and to enhance the desire to learn. A study by Usun (2007) suggested
that if technology is used for higher-order learning, it can result in increased
mathematical achievement. Wang (2008) also argued that technology-supported
collaborative learning has a positive effect on students' performance in problem-based
tasks.
Guven (2012) explained that the contribution of technology to the teaching and
learning of geometry has been associated with the dynamic nature of software such as
Cabri, GeoGebra and Geometer's Sketchpad. The power of the dynamic software does not
stem only from the possibility of making constructing, it also allows interactive
explorations by the dragging of points, vertices and objects:

Once a construction is completed, the user can drag certain elements of it, and the whole construction behaves in such a way that specified constraints are maintained, providing an environment in which students can experiment freely. They can easily test their intuitions and conjectures in the process of looking for patterns … (Guven, 2012)
The current research aims to investigate whether and how the use of dynamic geometry software, such as Cabri 3D, GeoGebra and Geometer's Sketchpad, influences the cognitive growth of geometry students. The Van Hiele theory was used to measure the cognitive growth of these students. Idris (2009) also used the Van Hiele theory in a study among Form 3 students in one of the secondary schools in Malaysia and found that using Geometer's Sketchpad has promising implications for the potential for teaching geometry at the secondary school. This study, however, did not explain to what extent and how the use of the dynamic software influences the specific Van Hiele levels.
Theoretical framework
The Van Hiele theory was used to measure the cognitive growth of these students. This theory has made a significant impact upon geometry education, particularly after it became known internationally what its impact had been on Russian mathematics education. Following in the footsteps of Piaget, Pierre and Dina Van Hiele identified five hierarchical, sequential and discrete levels of geometric development that are dependent on a learner's experience. In contrast with Piaget's theory, development is not dependant on age but rather on experience and the quality of instruction. In this context, it is useful momentarily to consider the basic tenets of both Piaget and Van Hiele. Battista and Clements (1995, p. 50) summarised the two theories as follows: Piaget's theory, on the one hand, describes how thinking in general progresses from being non-reflective and unsystematic, to empirical, and finally to logical-deductive. The theory of Van Hiele, on the other hand, deals specifically with geometric thought as it develops through several levels of sophistication under the influence of a school curriculum.
According to Kotze (2007, p. 22), Piaget's argument can be put like this: there is a "maturation process" that takes a learner through acquisition, representation and characterisation of spatial concepts. Van Hiele, however, suggested progress throughStols 1235 thinking on sequential levels as a result of experience. This experience is almost entirely dependent on instruction (Larew, 1999). According to their model, learners have to master a level to be able move to a higher level. The levels, as described by Mason (2009, pp. 4-5) are as follows:
Level 1 (Visualisation): Students recognise figures by appearance alone, often by comparing them to a known prototype. The properties of a figure are not perceived. At this level, students make decisions based on perception, not reasoning.
Level 2 (Analysis): Students see figures as collections of properties. They can recognise and name properties of geometric figures, but they do not see relationships between these properties. When describing an object, a student operating at this level might list all the properties he/she knows, but may not discern which properties are necessary and which are sufficient to describe the object.
Level 3 (Abstraction): Students perceive relationships between properties and between figures. At this level, students can create meaningful definitions and give informal arguments to justify their reasoning. Logical implications and class inclusions, such as squares being a type of rectangle, are understood. The role and significance of formal deduction are, however, not understood.
Level 4 (Deduction): Students can construct proofs, understand the role of axioms and definitions, and know the meaning of necessary and sufficient conditions. At this level, students should be able to construct proofs such as those typically found in a high school geometry class.
Level 5 (Rigor): Students at this level understand the formal aspects of deduction, such as establishing and comparing mathematical systems. Students at this level can understand the use of indirect proof and proof by contrapositive, and can understand non-Euclidean systems.
The Van Hieles considered the levels to be discrete, but other researchers (Clements & Battista, 1992; Burger & Shaughnessy, 1986; Crowley, 1987) argue that since learners develop several Van Hiele levels simultaneously and continuously, it is problematic to assign a learner to a particular level. This is an important reason why this research focuses on the mean scores of the students per level, rather than to assigning students to particular levels.
Research question
The current study investigates whether the use of dynamic geometry software as an integrated part of instruction is beneficial in increasing students' geometric cognitive growth, measured in terms of the Van Hiele levels.
Research design
In order to address the above question, a quasi-experimental, non-equivalent comparison group design was used. The reason for this decision was that practically it was not possible to assign the students randomly into two groups because of their timetables. Two geometry classes of a one semester geometry module were used (Table 1).1236 Australasian Journal of Educational Technology, 2012, 28(7)
Table 1: Non-equivalent comparison group design

Group	Pre-test	Treatment	Post-test
Experimental/ technologyenriched group	CDASSG Van Hiele Test	Geometry course and tablet PC GeoGebra and Cabri 3D	CDASSG Van Hiele Test
Control group	CDASSG Van Hiele Test	Geometry course and tablet PC	CDASSG Van Hiele Test

The study used descriptive statistics, a McNemar Test on each individual test item, and independent t-tests to investigate whether the use of dynamic geometry software as an integrated part of instruction was beneficial in increasing students' level of understanding, as measured in terms of the Van Hiele levels.
Intervention
The researcher offered a six-month geometry module to two groups of third-year preservice student teachers at the University of Pretoria, South Africa. This module dealt with various aspects of geometry, including Euclidean. Problems and activities were presented and discussed in class, thereby gradually and almost imperceptibly taking students from one level to the next in terms of the development of their thinking. The idea was to start with activities on Level 2 and gradually increase the level of thinking so as to reinforce Levels 3 to 5. The design of the geometry course was loosely based on the five tenets developed by the Realistic Mathematics Education (RME) movement, as described by Treffers (1987, cited in Bakker, 2004). The first tenet states that "[a] rich and meaningful context or phenomenon, concrete or abstract, should be explored to develop intuitive notions that can be the basis for concept formation" (p. 6). These activities were meant to develop the students' understanding of geometry while solving real-world and theoretical problems. Although there was ample opportunity to integrate and use dynamic geometry software in this course, the course was not designed specifically for a technology-enriched environment. Both the control and experimental groups were taught by the researcher/lecturer with the same amount of contact time.
The researcher used GeoGebra and Cabri 3D in class when teaching the experimental group. These programs allow users to drag vertices and points to investigate and explore mathematical relationships. GeoGebra is an open-source dynamic mathematics software that combines arithmetic, geometry, algebra, statistics and calculus. Cabri 3D enables students and teachers to construct and investigate three-dimensional geometry. It encompasses the benefits of interactive geometry and enables the user to construct, measure, unfold and rotate objects. The experimental group also spent two weeks using GeoGebra in the computer lab. The students in the control group were not exposed in class to either of these software programs. It is however possible that some of the students in the control group had 'informal' access to GeoGebra outside the classroom because it could be downloaded for free.
Examples of activities and how GeoGebra and Cabri 3D were used
• Activity: If you have a piece of land that is a quadrilateral, what kind of quadrilateral will be formed if you take the midpoint of each of its four sides and join these midpoints? Explain and justify your answer.Stols 1237
The idea of the activity was firstly to explore and discover that the new quadrilateral EFGH is a parallelogram. The experimental group used GeoGebra to make the construction. The advantage of using GeoGebra is that the mouse could be used to drag the quadrilateral vertices A, B, C, and D in order to observe the behaviour of quadrilateral EFGH.
• Activity: If a line is drawn parallel from one side of a triangle, it will divide the other two sides proportionally.
GeoGebra was used by the experimental group to make the construction and to measure the segments accurately. The advantage of using GeoGebra is that the mouse could be used to drag the vertices A, B and C to create more special cases. GeoGebra will measure the segments immediately and also update any calculations.
• Activity: Calculate the surface area of the rectangular pyramid with a perpendicular height of 5 cm and base of 8 cm by 6 cm.
The idea of the activity was to unfold the pyramid and to draw the net and use Pythagoras Theorem to determine the height of the different triangles. This unfolding was also illustrated in the class of the experimental group by using Cabri 3D.1238 Australasian Journal of Educational Technology, 2012, 28(7)
• Activity: What is the sum of the measures of the interior angles of a 1000-gon (a triangle is a 3-gon)?
The idea of the activity was to force students to use inductive thinking. The students could find the pattern for a 3-gon (1 triangle), 4-gon (2 triangles), and 5-gon (3 triangles) and generalise the pattern. GeoGebra was used by the experimental group to sketch the quadrilaterals and to measure the interior angles.
A typical lesson consisted of a short introduction, after which students worked on a series of problems and spontaneously bounced ideas off their classmates. When the majority struggled too much, the researcher intervened and worked with them on specific problems. The nature of the struggling was threefold, in some instances it was limited knowledge of the software, sometimes a lack of content knowledge, and in some instances, limited problem-solving skills. For example, some students struggled to solve the following problem: What is the sum of the measures of the interior angles of a 1000-gon (a triangle is a 3-gon)? Some students lack problem solving skills and tried to construct a 1000-gon using GeoGebra. This approach does not work because it is too tiresome and will take hours. Other students tried to use inductive reasoning but did know how to measure the interior angles in GeoGebra.
Participants
The students in both the control and experimental groups were third-year BEd preservice teachers who were enrolled for two geometry courses at the University of Pretoria, South Africa. The study used a control group of 55 pre-service teachers and an experimental group of 53 pre-service teachers who were enrolled for the geometry module. These students were selected for the sake of convenience.Stols 1239
Research instrument
The research instrument that was used was the Cognitive Development and Achievement in Secondary School Geometry (CDASSG) Van Hiele Geometry Test that forms part of the CDASSG project developed by Usiskin (1982). The Van Hiele Geometry Test consists of 25 multiple-choice test questions (five questions on each Van Hiele level). This instrument was selected because it was easy to analyse, well tested, and widely used (permission was obtained from Professor Usiskin to use the test). By using the KuderRichardson Formula 20 in his study (1982), he found that the pre-test reliability of this test was 0.31, 0.44, 0.49, 0.13, 0.10 and 0.39, 0.55, 0.56, 0.30, 0.26 in the post-test for the questions on Van Hiele levels 1, 2, 3, 4, and 5 respectively. According to Larew (1999, p. 37), the construct validity of the instrument was established by Senk in 1989: Senk found that achievement in writing geometry proofs was positively correlated with Van Hiele level (.50 in the fall and .60 in the spring). Senk also found that in spring the percentage of students who had mastered proof from each spring Van Hiele level was as follows: Level 0 –17%; Level 1 –22%; Level 2 – 57%; Level 3 – 85%; Level 4 – 100%. These data seem to support that Level 3 (as tested by the CDASSG test) is the level at which students master proof and that Level 2 is a transitional level … These results established the predictive validity of the CDASSG Van Hiele Geometry Test.
Data analysis procedure
Both the experimental and the control group wrote the same pre-test and post-test, namely the CDASSG Van Hiele Geometry Test, before and after their courses were presented. During the first and last contact session students were given 30 minutes to complete the five multiple-choice questions on each of the five Van Hiele levels, thus in total 25 questions.
As a first step, descriptive statistics were used to determine the average scores of students in both the control and experimental groups on each Van Hiele level before and after the course. Secondly, the study employed an independent t-test for each Van Hiele level, to investigate whether the difference between the pre- and post-test was as a result of the use of GeoGebra and Cabri 3D during instruction.
Results and discussion
The results of the study suggest that the pre-service mathematics teachers did not have a sound understanding of more advanced Euclidian geometry. The most problematic areas were the construction of proofs, understanding the role of axioms and definitions, and an understanding of non-Euclidean systems. There was a definite descending trend from Level 1 (Q1 to Q5) through to Level 4 (Q16 to Q20), as predicted by the literature (see Figure 1). This was, however, not the case with the Level 5 (Q21 to Q25) questions. The students performed slightly better on the Level 5 questions compared to the Level 4 questions. This was contrary to the Van Hiele model, which suggests that mastering on one level is a prerequisite for mastering on the next level.1240 Australasian Journal of Educational Technology, 2012, 28(7)
0
20
40
60

80
100
120
Q1 Q2 Q3 Q4 Q5 Q6 Q7 Q8 Q9 Q10 Q11 Q12 Q13 Q14 Q15 Q16 Q17 Q18 Q19 Q20 Q21 Q22 Q23 Q24 Q25
Control group (N= 55) pre-test
Control group (N= 55) post-test
Technology group (N=53) pre-test
Technology group (N=53) post-test
Figure 1: Percentage of students who answered the questions correctly
Although Question 4 was poorly answered compared to the other Level 1 questions
(Q1 to Q5), the experimental group did perform slightly better than the control group.
The use of dynamic geometry software probably improved students' understanding of
the basic definitions and properties of the different quadrilaterals. Question 4 was
"Which of these are squares?"
A) None of these are squares
B) G only
C) F and G only
D) G and I only
E) All are squares
Figure 2 analyses the selection of distracters as selected by the students. Although only
a few students selected distracters A and C, distracters D and E attracted quite a
number of students. A possible reason for selecting options D and E could be that
some students were under the impression that the inclusion property works in two
directions. Students who selected option D defined a square as a quadrilateral with
four equal sides.
Students found it difficult to grasp the idea of class inclusion or hierarchical
classification, such as a rectangle being a type of a parallelogram. The idea of class
inclusion falls typically in the abstraction level of geometrical thought, which is Level
3. This may be the reason why Question 14 was also poorly answered, compared to the
other questions (Q11-Q15) on Level 3.
F G H IStols 1241
0
10
20
30
40
50
60
70
80
A B C D E
Answers to Question 4
%
Pre test Control
Pre test Technology
Post test Control
Post test Technology
Figure 2: Percentage of students who selected the different distracters (Question 4).
Which of these can be called rectangles?
A) All can.
B) Q only
C) R only
D) P and Q only
E) Q and R only
The same phenomenon has been identified by other researchers. A study done by
Smith (1987) revealed that students find these questions about class inclusion or
hierarchical classification consistently more difficult than some higher-level questions
(on Levels 4 and 5).
Other outlier questions that were poorly answered compared to other questions on the
same level are 10, 19, 21 and 22. These questions will be discussed briefly. Question 10
(Van Hiele Level 2) concerned diagonals, sides and the angle properties of a rhombus.
Question 19 (Van Hiele Level 4) focused on the properties of an axiomatic system
while questions 21 and 22 (Van Hiele Level 5) dealt with non-Euclidean geometry:
P Q R1242 Australasian Journal of Educational Technology, 2012, 28(7)
In F-geometry, which is different from the geometry
you are used to, there are exactly four points and six
lines. Every line contains exactly two points. If the
points are P, Q, R and S, and the lines are {P,Q}, {P,R},
{P,S}, {Q,R}, {Q,S}, and {R,S}. Here is how the words
"intersect" and "parallel" are used in F-geometry. The
lines {P,Q} and {P,R} intersect at P because {P,Q} and
{P,R} have P in common. The lines {P,Q} and {R,S} are
parallel because they have no points in common. From
this information, which is correct?
A) {P,R} and {Q,S} intersect.
B) {P,R} and {Q,S} are parallel.
C) {Q,R} and {R,S} are parallel.
D) {P,S} and {Q,R} intersect.
E) None of the above answers (A–D) is correct.
In order to determine the impact of the use and non use of dynamic geometry software
on students' understanding of individual questionnaire items, a McNemar test was
applied (see Table 2).
Table 2: McNemar Test applied to each individual questionnaire item
Chi-square results (Sig. 2-sided)
Question
Control group Experimental group
Q1 1.000 0.070
Q2 1.000 1.000
Q3 0.500 1.000
Q4 1.000 0.180
Q5 1.000 0.774
Q6 0.832 0.017
Q7 0.607 0.774
Q8 0.383 0.167
Q9 0.754 1.000
Q10 1.000 1.000
Q11 1.000 0.017
Q12 0.664 0.664
Q13 0.004 0.065
Q14 0.096 1.000
Q15 0.185 0.424
Q16 1.000 0.167
Q17 0.508 0.078
Q18 1.000 0.549
Q19 1.000 1.000
Q20 0.238 0.815
Q21 0.063 1.000
Q22 0.065 0.424
Q23 0.001 0.629
Q24 0.180 0.791
Q25 0.000 0.332
Only two items showed that the use of technology has a statistically significant (below
the 5% confidence level) impact on students' conceptual understanding. That was Q6,
which is about the properties of quadrilaterals, and Q11, which is about mathematical
logic. Three items in the control group showed that the non-technological intervention
had a statistically significant (below the 5% level) impact on students' conceptual
Q
R
P
SStols 1243
understanding. These questions were Q13, Q23 and Q25. Both Q23 and Q25 focused on
mathematical rigor and the students' understanding of formal aspects of deduction.
The means were computed to summarise the scores (out of 5) for each Van Hiele level
for both the pre- and post-tests (see Table 3). The majority of students did not reach the
Van Hiele levels 4 or 5 in both groups and only about half reached Van Hiele level 3. It
came as a surprise that not all pre-service students scored full marks on Van Hiele
levels 1, 2 and 3 in both the control and experimental groups – not even after the
course had been presented.
Table 3: Pre-test and post-test mean scores per Van Hiele level
Mean
Van Hiele Group N (out of 5)
pre-test post-test
Mean
improvement
(out of 5)
SD
Level 1 Control 55 3.87 3.93 0.05 1.161
Experimental 53 4.06 4.28 0.23 0.847
Level 2 Control 55 3.56 3.42 –0.15 1.161
Experimental 53 3.13 3.49 0.36 1.272
Level 3 Control 55 2.09 2.71 0.62 1.459
Experimental 53 1.62 2.11 0.49 1.476
Level 4 Control 55 0.80 0.98 0.18 1.073
Experimental 53 0.89 1.30 0.42 1.232
Level 5 Control 55 0.40 1.33 0.93 1.215
Experimental 53 1.25 1.21 –0.04 1.224
Comparing the mean improvements of the control and experimental groups in Table 3,
it appears that the technology-enriched environment improved the conceptual
geometric growth of students in the experimental group on Van Hiele levels 1, 2 and 4.
However, there was also evidence of cognitive growth when technology was not used
on Van Hiele levels 1, 3, 4 and 5. The negative mean improvement score on Van Hiele
level 5 suggests that the technology-enriched environment did not enhance students'
understanding of the formal aspects of deduction, such as proofs. In the "rigor" (Level
5) category (questions that covered the more formal aspects of deduction), the score of
the experimental group declined, while the control group improved on average by 0.93
(out of 5) on Van Hiele level 5 questions.
An independent samples t-test was conducted to compare the improvement in scores
of the control and the experimental groups for each Van Hiele level. The results of the
mean (maximum 5) improvement on each level are presented in Table 3. The results of
the independent t-test show that the only statistically significant difference in the preand post-test scores of the control group was on Van Hiele levels 3 and 5 at $p < .05$ (see
Table 4). In the case of the experimental group (technology-enriched) there was a
significant difference in the pre- and post-test scores on Van Hiele levels 1, 2, 3 and 4 at
$p < .05$.
Conclusion
This study sought to use the Van Hiele theory to investigate the geometric cognitive
development of students in a technology (dynamic geometry software) enriched
environment, compared with students in a learning environment without any
technological enhancement. The results suggest that the technology enriched
environment helped to improve the conceptual geometric growth of students on Van1244 Australasian Journal of Educational Technology, 2012, 28(7)
Hiele levels 1, 2 and 4 which is about geometric visualisation, recognition of properties
of geometric figures, and the construction of proofs. This finding about the
improvements on Van Hiele levels 1 and 2 resonates well with the literature, which
suggests that technology can help to create an active learning environment in which
students can discover, explore, conjecture and visualise.
Table 4: Paired samples t-test
Van Hiele Group Test Mean N SD t df p
Control Pre 3.87 55 0.904
Post 3.93 55 1.034
–3.48 54 0.729
Pre 4.06 53 .969
Level 1
Experimental
Post 4.28 53 .769
–1.947 52 0.050*
Control Pre 3.56 55 1.014
Post 3.42 55 1.117
0.929 54 0.357
Pre 3.13 53 1.241
Level 2
Experimental
Post 3.49 53 1.137
–2.051 52 0.045*
Control Pre 2.09 55 1.309
Post 2.71 55 1.370
–3.142 54 0.003*
Pre 1.62 53 1.259
Level 3
Experimental
Post 2.11 53 1.354
–2.420 52 0.019*
Control Pre 0.80 55 .931
Post 0.98 55 .952
–1.257 54 0.214
Pre 0.89 53 .934
Level 4
Experimental
Post 1.30 53 .972
–2.454 52 0.018*
Control Pre 0.40 55 .627
Post 1.33 55 1.156
–5.660 54 0.000*
Pre 1.25 53 1.090

Level 5
Experimental
Post 1.21 53 .968
0.224 52 0.823
* indicates a significance at the 0.05 level (two-tailed)
Dynamic geometry software cannot improve Van Hiele level 4 reasoning directly
because deductive reasoning is about the understanding of axioms and the
construction of proofs. The question then is why this improvement on Van Hiele level
4? According to the Van Hiele theory the use of dynamic geometry software is not
rendered worthless on this level. Larew (1999) argues that in order for students to
reason at the higher levels, there must be sufficient mastery at the lower levels. De
Villiers (2007, p. 55) explained:
This is why I have frequently argued that it is far more meaningful to INTRODUCE
proof within a dynamic geometry context, NOT as a way of making sure, but rather as
a means of explanation, understanding, and discovery before dealing with the more
formal and abstract functions of verification and systematisation.
Although the experimental (technology-enriched) group outperformed the control
group on Van Hiele levels 4, this advantage did not materialise in terms of a better
understanding on Van Hiele level 5. The results also show that the use of dynamic
geometry software, as opposed to a traditional learning environment of the control
group, may have a negative impact on the geometric development on Van Hiele levels
3 and 5. These levels are about informal argumentation and the formal aspects of
deduction. This result suggests that geometric reasoning at the Van Hiele level 5
(deduction) differs in nature compared to the first 4 levels, which concern
visualisation, relationships between properties, informal justification, and proof.Stols 1245
In fact, several researchers found the last Van Hiele level problematic. Even Pierre Van
Hiele (1986, p. 64) himself, writing subsequent to the demise of his wife, admitted that:
"It takes nearly two years of continual education to have the pupils experience the
intrinsic value of deduction, and still more time is necessary to understand the intrinsic
meaning of this concept". Weber (2001, p. 102) found that: "While this research has
provided rich data, there is a large and important class of failed proof attempts that it
cannot explain. Students often fail to construct a proof because they reach an impasse
where they simply do not know what to do." The achievement or lack thereof cannot
be accounted for on the grounds of Van Hiele levels alone. Weber (2001, p. 116)
concluded that students also need strategic knowledge, but is fully aware of the
problematic consequence of the fact: "Since strategic knowledge is heuristic, designing
activities that will lead students to acquire this knowledge will be a formidable task."
De Villiers (1987, p. 24) pointed out that "two aspects of the Van Hiele theory need
clarification and further research, namely a refinement with regard to the levels at
which deduction (as justification, explanation and systematisation) is supposed to
occur, as well as the relationship between hierarchical thinking and deduction".
Against this background it is understandable that the use of dynamic geometry
software does not support the development of the formal aspects of deduction. The
results of this study therefore contradict some of the findings of other studies, as
mentioned in the literature review, which suggested that the use of technology always
supports the development of higher-order thinking.
Limitations
A limitation of the present study was the fact that the Van Hiele test that focuses on
Euclidean geometry was used. The course offered covered not only Euclidian
geometry, but also analytical geometry, transformation geometry, and volumes and
surface areas. However, this particular test still gave us insight into students'
understanding of one of these topics. A possible idea for future research is the
development of a new and more comprehensive geometry test.
Another possible limitation of this study in terms of the Van Hiele module is identified
by Salomon (2005, p. xvii), who warned that:
Related to this is the assumption that wisely integrating technology into instruction will
yield better learning outcomes assessed by the same yardsticks. But this totally ignores
the observation that the use of different instructional means afford different kinds of
learning activities which, in turn, facilitate the attainment of different kinds of outcomes.
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Effects of worked examples in a primary school mathematics curriculum

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A large body of research has shown that for novice learners, instruction that relies
more heavily on worked examples than on problem solving, is more effective for
learning as shown by higher test performance. Moreover, this beneficial effect is
often obtained with less acquisition time and lower cognitive load during acquisition
and test phase. However, most of this research has been conducted in laboratory
settings with college or university students and a control condition consisting of
problem solving without any additional support. The present study, using a quasiexperimental design, investigated the effects ofimplementing worked examples in an
existing primary school mathematics curriculum in which a realistic mathematics
teaching method is used, during a 3-week period. The results showed no significant
differences in test performance or cognitive load; however, the worked examples
group attained this level of performance with significantly less acquisition time.

Keywords: worked examples; problem solving; cognitive load; mathematics;
primary education

Introduction

The present study investigated the effects of implementing worked examples in an
existing primary school mathematics curriculum in which a realistic mathematics
method is used, over a 3-week period. Despite the popularity of ''learning by doing''
in contemporary education, a large body of research has shown that for students
who lack prior knowledge of a task, self-directed problem solving is not the most
efficient way to acquire knowledge. It is far more effective and efficient for novice
learners to study a good example (see Atkinson, Derry, Renkl, & Wortham, 2000;
Sweller, 2006; Sweller, Van Merrie¨nboer, & Paas, 1998). This has become known as
''the worked example effect'', which cognitive load theory has explained as the
effectiveness of worked examples in terms of reduced ineffective load on working
memory (Sweller, 1988; Sweller et al., 1998).

Cognitive load theory

Cognitive load theory (Sweller, 1988; Sweller et al., 1998) is concerned with the
design of instructional methods that efficiently use people's limited cognitive

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Downloaded by [North West University] at 00:46 28 May 2015 processing capacity. The theory distinguishes three types of cognitive load: Intrinsic,
extraneous, and germane load. Intrinsic load depends on the complexity of a task,
that is, on the number of interacting information elements the task contains
(Chandler & Sweller, 1991). These elements have to be processed simultaneously in
order to successfully learn to perform that kind of task. Because working memory
capacity is limited to seven plus or minus two elements (or chunks) of information
when merely holding information (Miller, 1956) and even fewer (ca. four) when
processing information (Cowan, 2001), the higher this number of interacting
information elements is, the higher the intrinsic load imposed on working memory is.
However, intrinsic load does not only solely depend on the nature of the task but
also on the level of expertise of the learner. As expertise increases, information
elements contained in the task become incorporated into cognitive schemata stored
in long-term memory. Because a schema retrieved from long-term memory can be
handled in working memory as a single element, the number of interacting
information elements a task contains, and hence, the intrinsic load it imposes,
decreases for more knowledgeable learners. With extended practice, certain
schemata can be automated and no longer require controlled, effortful processing,
which further reduces the load on working memory (Schneider & Shiffrin, 1977).
Extraneous and germane load both depend on the way the task is designed, with
extraneous load being imposed by processes that are evoked by the design of the task
that are ineffective for learning (e.g. unnecessary visual or mental search processes)
(Chandler & Sweller, 1991) and germane load being imposed by processes evoked by
the design of the task that do contribute to learning (e.g. imagining a solution
procedure) (Cooper, Tindall-Ford, Chandler, & Sweller, 2001). The central tenet of
cognitive load theory is that in order to be effective and efficient, instruction should
be designed in such a way that intrinsic load is optimized, that is, tasks should be at
an appropriate level of complexity for trainees (i.e. challenging, but feasible; cf.
Vygotsky's (1978) concept ofthe Zone ofProximal Development), extraneous load is
minimized, and germane load is optimized so that the available cognitive capacity is
optimally used (Sweller et al., 1998). An effective instructional technique to
accomplish this is to implement a heavier reliance on worked example study rather
than problem solving.

Effectiveness of worked examples

Conventional problems contain only a description of some ''givens'' (e.g. how fast a
car accelerates and its average velocity) along with a goal statement (e.g. ''calculate
how far the car has traveled''). Instruction that consists mainly of solving
conventional problems forces novices to resort to weak problem-solving strategies
such as means-ends analysis, which impose a high cognitive load but are not effective
for learning: even though such weak strategies may allow learners to succeed in
solving the problem eventually (i.e. performance), they have been shown to
contribute very little to learning (Sweller, 1988), that is, to building a cognitive
schema ofhow such problems should be solved. Worked examples prevent the use of
such weak problem-solving strategies, by presenting the learner not only with the
givens and a goal statement but also with the worked-out solution steps that are to
be taken to reach the goal state. The learner can devote all his or her available
cognitive capacity to studying the given solution and constructing a cognitive schema
for solving such problems. As such, compared to instruction consisting of

Downloaded by [North West University] at 00:46 28 May 2015 conventional problem solving, instruction that relies more heavily on studying
worked examples reduces ineffective cognitive load on working memory, and
enhances learning, as well as transfer. That is, students not only learn how to solve
identical problems but also (partly) novel problems, for which they have to flexibly
adapt the learned solution procedure (see Sweller et al., 1998). In addition to
enhanced learning and transfer (i.e. higher test performance), a heavier reliance on
worked examples has also been shown to have beneficial effects on required
acquisition time (i.e. lower) (see e.g. Sweller & Cooper, 1985; Van Gog, Paas, & Van
Merrie¨nboer, 2006; Zhu & Simon, 1987) and cognitive load experienced by students
during acquisition (i.e. lower) (see e.g. Paas & Van Merrie¨nboer, 1994; Van Gog
et al., 2006) as well as during the test (i.e. lower) (see e.g. Paas, 1992; Paas & Van
Merrie¨nboer, 1994).

It should be noted though, that these beneficial effects of worked examples on
learning, acquisition time, and cognitive load, seem to apply primarily to novice
learners, and only when the examples are well-designed. To start with the latter,
following the early studies on worked examples by Sweller and Cooper (1985) and
Cooper and Sweller (1987), it was soon discovered that studying worked examples
was not always more effective for learning than problem solving. Rather, the design
of the examples played a crucial role in their effectiveness (Tarmizi & Sweller, 1988).
If worked examples induced split attention, by presenting mutually referring
information sources such as text and picture/diagram separately, a worked example
effect (i.e. benefit over learning by problem solving) was not obtained. Presumably,
this was due to extraneous cognitive load imposed by the processes of dividing
attention (i.e. visual search) and mental integration that were required in order to
learn from the separate sources of information. When these information sources
were presented in an integrated manner, the worked example effect was again
obtained (Tarmizi & Sweller, 1988); reduction of extraneous load due to split
attention can also be done by providing spoken rather than written text with the
pictorial information in the example (Mousavi, Low, & Sweller (1995). On the
other hand, when information sources are not mutually referring and can be
easily understood in isolation, extraneous cognitive load is imposed by the need
to process redundant information. In this case, one of the information sources
should be left out (see Chandler & Sweller, 1991). Next to these basic
requirements for the design of worked examples, there are some design measures
that can be applied to enhance the effectiveness of examples even further, by
stimulating more active processing of the examples or emphasizing important
aspects of the procedure (i.e. increase germane load) (Paas & Van Gog, 2006),
such as making subgoals explicit through labeling or visually isolating sets of
steps (e.g. Catrambone, 1995), requiring students to complete steps in partially
worked-out examples (e.g. Paas, 1992), asking students to explain the principles
behind the worked-out solution steps for themselves (e.g. Chi, Bassok, Lewis,
Reimann, & Glaser, 1989; Renkl, 1997), or providing them with instructional
explanations (e.g. Lovett, 1992; Renkl, 2002) or problem-solving process
information (e.g. Van Gog, Paas, & Van Merrie¨nboer, 2008) that help them
understand the principles behind the solution steps.

Next to design, prior knowledge also plays a role in the effectiveness of worked
examples. As mentioned above, with increasing expertise, the intrinsic load imposed
by the task becomes lower. As a consequence, extraneous load might no longer have
negative effects, as enough cognitive capacity is available to deal with the task

Downloaded by [North West University] at 00:46 28 May 2015 aspects that impose extraneous load. Moreover, techniques that may successfully
reduce extraneous load for novice learners, such as studying worked examples, may
increase extraneous load for more knowledgeable learners. That is, because of their
prior knowledge, studying the worked-out solution is a redundant, unnecessary
cognitive activity. As a consequence, for advanced learners, worked examples may
no longer be effective, or may even start to hamper learning (e.g. Kalyuga, Chandler,
Tuovinen, & Sweller, 2001). This is an example of what is known as the ''expertise
reversal effect'' (for a review, see Kalyuga, 2007; Kalyuga, Ayres, Chandler, &
Sweller, 2003).

In sum, the worked example effect indicates that when examples are well-designed,
instruction that relies more heavily on example study is more effective, efficient, or
both, for novices than instruction consisting of problem solving. A ''heavier reliance''
on example study has been implemented differently in different studies, though. Some
authors have compared instruction consisting only of studying examples to instruction
consisting of problem solving (e.g. Van Gerven, Paas, Van Merrie¨nboer, & Schmidt,
2002; Van Gog et al., 2006, 2008), whereas others have used example-problem pairs,
arguing that this would be more motivating for students (e.g. Cooper & Sweller, 1987;
Sweller & Cooper, 1985), or two examples-one problem triplets (Paas, 1992). Others
have used problem-example pairs, arguing that this allows students to experience where
their difficulties with problem solving occur, so that they would subsequently study the
example more closely (e.g. Stark, Gruber, Renkl, & Mandl, 2000; for a discussion of
their study in English see Stark, Gruber, Hinkofer, & Mandl, 2004). Finally, a heavier
reliance on example study has been implemented by providing students with examples
that are available for students to use as an analogy while they are solving a problem
(e.g. Reed, Dempster, & Ettinger, 1985), which is also the approach taken in this study.

The present study

Although the effects of worked examples on mathematics learning have been widely
studied (e.g. Carroll, 1994; Cooper & Sweller, 1987; Mwangi & Sweller, 1998; Reed
et al., 1985; Sweller & Cooper, 1985; Zhu & Simon, 1987), research consists mostly of
short-term experimental studies (i.e. conducted in one or two sessions); there are very
few studies done in actual school curricula over a longer period of time (for an
exception, see e.g. Zhu & Simon, 1987). Moreover, most of the research on learning
from worked examples has been conducted with college or university students, and very
little with primary school age children (for an exception, see e.g. Mwangi & Sweller,
1998). Finally, the control condition in previous research usually consisted of problem
solving by oneself without any additional instructional support or guidance, not of the
teaching methods regularly used in schools, which might be a stronger control
condition (cf. Schwonke et al., 2009).

Therefore, the present study investigated the effects of implementing a heavier
reliance on worked examples by providing them as analogies for problem solving in
an existing primary school mathematics curriculum in which a realistic mathematics
teaching method is used, over a 3-week period. The focus during this period was on
subtraction problems with borrowing 10s and 100s. Such problems contain many
interacting information elements, so they impose a high intrinsic cognitive load for
students, and – probably as a consequence – many errors may occur, such as
subtracting digits in the wrong order, trying to borrow from zero rather than
the column to the left, not carrying 10s or 100s, etc. (see Peltenburg, Van den

Downloaded by [North West University] at 00:46 28 May 2015 Heuvel-Panhuizen, & Doig, 2009). Based on previous research, it is hypothesized
that the availability of worked examples as an analogy for problem solving, will lead
to lower acquisition time, lower cognitive load during acquisition and test phase, and
higher test performance.

Method

Participants and design

Participants were 45 fourth-grade Dutch primary school students (age M ¼ 9.9
years; SD ¼ 0.34). They were students from two existing classes of the same school.
A quasi-experimental design was used in which one class (n ¼ 23) was assigned as
experimental group and one class (n ¼ 22) as control group.

Materials

Subtraction instruction

This primary school uses a realistic mathematics teaching method called ''RekenRijk''

(Bokhove, Kuipers, & Postema, 2003). This method consists of 12 blocks per year. This study focused on block 6, in which subtracting with borrowing 10s and 100s is taught. This block consists of 12 lessons (four teaching lessons, six self-study lessons, one repetition lesson, and one ''deepening understanding'' lesson). Six of those lessons were devoted to subtracting with borrowing 10s and 100s according to the problem format 787 7 492 ¼ . . . (i.e. in the other lessons, other problem types such as word problems were used), so data from these lessons are used in the analyses.
The control group engaged in their regular instruction and self-study using the RekenRijk method materials. The experimental group also engaged in their regular instruction and self-study with the RekenRijk method and materials, but additionally received two worked examples on subtracting with borrowing (one on borrowing 10s and one on borrowing 100s), which were explained by the teacher in the first lesson and were handed out to each student so that they could consult these examples as an analogy for problem solving throughout the acquisition phase.
Cognitive load rating scale
The most common method ofmeasuring cognitive load in educational research is via subjective rating scales (Paas, Tuovinen, Tabbers, & Van Gerven, 2003). An illustrated 4-point cognitive load rating scale was developed for this age group, asking students to indicate how ''heavy'' they found the tasks they just completed, ranging from (1) not at all (a drawing of a child carrying one block above her head and smiling), to (4) very heavy (a drawing of a child carrying four blocks above her head looking sad).
Time-on-task
Students were asked to write down their start and end times on the pretest and posttest and for each lesson according to a digital alarm clock that was placed in such a way that it was clearly visible for all students.

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Pretest andposttest
To measure participants' knowledge gain, the regular ''Block Test'' for Block 6 from the ''RekenRijk'' method (Bokhove et al., 2003) was used. Twenty items from this test concerned subtracting with borrowing 10s and 100s similar to the problems that were worked out in the examples, so these 20 items were used to determine test performance in the context of this study.
Example experience questionnaire
To get an impression of how students in the worked examples condition experienced working with the examples, a short self-report questionnaire was used, with answer options ''no'', ''don't know'' or ''yes'', that asked students: (1) whether the examples were clear, (2) whether they felt problem solving was easier when using the examples, (3) whether they felt they required less assistance during problem solving when using the examples, (4) whether they used the examples with every problem, (5) whether they would want to use examples also with other types of problems.
Procedure
Prior to the study, the teachers received information about how to proceed in their respective condition. In the control group, this meant engaging in instruction as usual, except for the pretest and the cognitive load and time on task measures that were added for this study. In the experimental group, this additionally included information for the teachers on how to implement the examples. Students were also acquainted with the cognitive load and time on task measurement procedure prior to the study, and they were informed that the pretest (which was added for this study) would not be taken into account in determining their progress report scores. At the start of the 3-week period, participants first completed the pretest, indicating their perceived cognitive load immediately after completing the test. Then, for 3 weeks, they engaged in the learning activities of Block 6 of the RekenRijk method, either with or without the additional worked examples depending on their assigned condition. For each lesson, students indicated their start and end time according to the digital clock, and at the end of each lesson rated their perceived cognitive load. Finally, all students completed the posttest, again indicating their perceived cognitive load immediately after completing the test. In the week after the posttest, the students in the worked examples condition completed a short questionnaire about their experience working with the examples.
Results
Means and standard deviations of performance, cognitive load, and acquisition time are presented in Table 1. A repeated measures analysis on pretest and posttest performance showed - as one would expect - a significant reduction in the amount of errors made (max. ¼ 20) from pretest (M ¼ 15.98, SD ¼ 6.26) to posttest (M ¼ 5.91, SD ¼ 4.85), $F(1,42)$ ¼ 101.94, MSE ¼ 21.89, p 5 .001; however, there was no significant interaction with group $F(1,42)$ 51, ns. That is, in contrast to our hypothesis, the worked examples group did not outperform the control group on the posttest.

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A repeated measures analysis on perceived cognitive load during pretest and posttest showed - as one would expect given the increase in knowledge - a significant reduction in the perceived cognitive load from pretest (M ¼ 2.70, SD ¼ 1.03) to posttest (M ¼ 1.61, SD ¼ .78), $F(1,42)$ ¼ 39.69, MSE ¼ .66, p 5 .001; however, there was no significant interaction with group $F(1,42)$ 51, ns. That is, in contrast to our hypothesis, the worked examples group did not experience lower cognitive load than the control group on the posttest.
An ANOVA showed that in contrast to our hypothesis, perceived cognitive load during the acquisition phase in the six lessons on subtraction with borrowing did not dijer between the experimental group and the control group, $F(1,38)$ 5 1, ns (four participants were excluded from this analysis due to missing data on one or more of the lessons). However, in line with our hypothesis, there was a significant dijerence in acquisition time during those lessons (five participants were excluded from this analysis due to missing data on one or more of the lessons), with acquisition time in the experimental group being lower than in the control group, $F(1,37)$ ¼ 6.28, p ¼ 0.017.
Descriptive statistics of the examples experience questionnaire data are provided in Table 2. Not taking into account the ''don't know'' answers, it can be concluded that all students found the examples clear, and the majority of students reported they found problem solving easier when using the examples, felt they required less assistance from a teacher or peer when using the examples, would like to work with examples also with other types of math problems, and that they had not used the examples as an analogy with every problem they solved.
Table 1. Means and standard deviations of performance, cognitive load, and acquisition time.

	Control		Worked examples	
	M	SD	M	SD
Pretest performance (number of errors, max. ¼ 20; lower ¼ better)	15.90	6.43	16.04	6.24
Pretest cognitive load (max. ¼ 4)	2.76	1.00	2.65	1.07
Cognitive load acquisition (max. ¼ 4)	1.62	0.50	1.66	0.52
Acquisition time (min.)	33.80	5.40	29.44	5.43
Posttest performance (number of errors, max. ¼ 20; lower ¼ better)	5.52	4.43	6.26	5.27
Posttest cognitive load (max. ¼ 4)	1.62	0.87	1.61	0.72

Table 2. Answers to example study experience questionnaire (n ¼ 22, data from 1 participant missing due to absence).

Question No	Don't know	Yes	Proportion Yes/Yes þ No (%)
Were examples clear?	0	2	20 100.00
Problem solving easier when using examples?	4	5	13 76.47
Less assistance required when using examples?	5	5	12 70.59
Examples used with every problem?	10	10	2 16.67
Want examples also with other problems?	3	7	12 80.00

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Discussion
This study aimed to investigate the ejects of implementing a heavier reliance on worked examples in an existing primary school mathematics curriculum over a 3-week period. It was hypothesized that a heavier reliance on worked examples would lead to lower acquisition time, lower cognitive load during acquisition and test phase, and higher knowledge gains (i.e. dijerence between pretest and posttest scores). This hypothesis was only partially confirmed: No dijerences in test performance or cognitive load were found between the groups, but the worked examples group did require less acquisition time.
Even though there were no dijerences in perceived cognitive load between the two groups, the measure that was developed for this study seems to be adequate for use with relatively young children, and seems to show construct validity. That is, for people without any prior knowledge, as was the case at the time of the pretest, problem solving tasks impose a much higher intrinsic load than for people who do have prior knowledge of the tasks, as was the case at the time of the posttest. As indicated in the introduction, with increasing practice, interacting information elements that a task contains are combined into cognitive schemata that can be handled in working memory as a single element, which reduces the amount of intrinsic load imposed by a task. Moreover, with increasing practice, problemsolving procedures can become automated, which further reduces working memory load. The measures of perceived cognitive load used in this study reflect the decrease in cognitive load one would expect with increasing knowledge/practice. However, a scale with more answer options might have provided more sensitivity to detect potential subtle dijerences in cognitive load during the acquisition phase.
There are several possible explanations for the finding that the only dijerence between the groups lay in the required acquisition time, whereas a lot of previous studies have also shown beneficial ejects on learning. First of all, the realistic mathematics teaching method used by this school as well as the ecologically valid setting of this study which included not only self-study but also teacher instruction may have provided a stronger control group than normally used in studies on the ejectiveness of worked examples. That is, in most studies, the control condition usually consists of solving problems by oneself without any additional instruction or guidance for students. The study by Schwonke et al. (2009) which also used a stronger control condition, that is, problem solving guided by a cognitive tutoring system, also showed positive ejects of worked examples mainly in terms of lower acquisition time. Secondly, students were given only two worked examples to use as analogies for problem solving and even though these two examples were illustrative for both problem categories (i.e. borrowing 10s and borrowing 100s), the fact that they were always the same may have reduced their motivation to actively process them over the course of the 3 weeks. Finally, and this is a limitation of this study, we do not have any data on the extent to which the students used the examples and when they used them. The self-report data on the questionnaire suggest that students did not use them with every problem they solved. Given that we know that worked examples are more ejective in the initial stages of skill acquisition, and tend to become redundant when students have gained prior knowledge of the task (the expertise reversal eject) (Kalyuga, 2007; Kalyuga et al., 2003), and that our students' knowledge levels did increase in the course of this study, as shown by the performance and cognitive load data on the pretest and posttest, it would have been

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interesting to have data on whether students indeed consulted the examples more in the beginning ofthe study. It would be interesting to complement the self-report data with observation data in future studies, to get information on when and how students use worked examples as an analogy for problem solving.
Nonetheless, the finding that providing worked examples as an analogy for problem solving in an ecologically valid setting reduces the time required for attaining the same level of performance is interesting for educational practice: By implementing examples, time can be gained that can be used for repetition or other activities. Moreover, providing some examples as analogies for problem solving is very easy to implement and does not require a lot of development time from the teachers; indeed, if students' self-report data on their reduced need for assistance are accurate, providing students with worked examples might even save teachers some time. As mentioned in the introduction, there are other ways of implementing a heavier reliance of worked examples than providing them to be used as an analogy during problem solving (in which case they are an addition to problem solving). An interesting question for future research would be whether replacing part of the problems from the teaching method with worked examples would enhance learning. Research suggests that a completion or fading strategy (Renkl & Atkinson, 2003) might be a good way of doing this. In a completion or fading strategy, completion problems (i.e. partially worked-out examples that leave some solution steps open for the learner to complete) (Paas, 1992) provide a bridge between studying fully worked-out examples and solving the entire problem oneself. So, this strategy takes into account the learners' developing expertise, by starting instruction with studying worked examples, and proceeding via completion problems with increasingly more steps to be completed by the learner, to having learners solve problems on their own.
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Comment: *Possibilities for interview with teachers*

87 quotations

Codes (28)###design of activities, ###didactical phenomenology, ###emergent modelling, ###guided reinvention, ##advantages of using RME, ##characteristics of RME, ##disadvantages of using RME, ##human activity, ##mathematization, ##real life, ##recommendations for RME based lessons, ##RME tasks, #RME, #RME theory, $$$approaches to teaching Mathematics, $$$modelling, $$meaningful learning of Maths, $$teaching methods and strategies, $factors that influence teaching and learning, $learner aspects, $mathematics, $mathematics content, §role players, &pedagogy, @@ICT systems: virtual learning environment (VLE), @@value of using ICT, @Devices, =methodology

Memos (0)

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How a Realistic Mathematics Education Approach and
Microcomputer-Based Laboratory Worked in Lessons on Graphing
at an Indonesian Junior High School
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Amsterdam, The Netherlands
This study focuses on the applicability of ICT-supported lessons based on a Realistic
Mathematics Education (RME) approach in an Indonesian classroom context. The topic chosen
for the research experiment was pupils' graphing skills with emphasis on graph interpretation
and with application especially to kinematics. One class was involved in this study and 13 – 14
years old pupils used the specially designed materials and activities. This included the use of
motion detectors in pupils' experimental work. The authors investigated the expectations,
performances, and opinions of the pupils and the teacher with respect to the activities and
analyzed pupils' thinking and understanding in the reformed setting. The results of the classroom
experiment indicated that the pupils made remarkable progress in their performances that can
be attributed to the chosen approach. The pupils' and the teacher's opinions on the teaching and
learning activities in general also tended to be positive.
Introduction
Although mathematics is considered important in all stages of education, Indonesian pupils'
performances in this subject is generally still poor: for instance, Indonesian 8
th
graders ranked
34
th
among 38 participating countries in the TIMMS-R assessment (Mullins et al., 1999). The
poor performance on the international stage also held for science and technology. It does not
seem likely that the intelligence and capacity of Indonesian children are worse than those of
children from other countries. We are of the opinion that the main cause lies in the current
instructional setting.
Indonesian mathematics education faces another problem: most pupils' attitudes towards
mathematics are negative. Most of them perceive mathematics as difficult and boring. This is not
surprising when we look closely at the common practice of teaching and learning mathematics in
Indonesian classrooms. A diagnostic survey conducted by the Ministry of Education and Culture
in 1996 revealed that despite many reform efforts, teacher-centered learning focusing on
procedural knowledge is still the norm. Teachers actively explain materials and provide
examples and exercises, while pupils only listen, write, and perform the tasks initiated by the
teachers. Discussion, interaction, and communication are seldom conducted. Mathematical goals
and curriculum materials are still based on the mathematics of mathematicians, and they lack real
life application. More recent research that was conducted to identify Indonesian problems
(Zulkardi, Nieveen, van den Akker, & de Lange, 2002) confirms this educational picture and
mentions various causes, including inaccurate learning materials, inadequate mechanistic
teaching methods, poor forms of assessment, and pupils' anxiety about mathematics.Published in: Journal of Science and Mathematics Education in Southeast Asia, 2003, Vol. 26, No 2, pp. 1-51. 2
Another problem of Indonesian mathematics education is its lack of touch with Information
and Communication Technology (ICT). This is a pity because research indicates that ICT is a
good catalyst for the realization of alternative setups such as an activity-based approach or a
realistic approach to mathematics and science instruction.
One of the ongoing developmental research efforts is to adapt the instructional design
perspective of RME to the teaching and learning of mathematics in Indonesia (Fauzan,
Slettenhaar, & Plomp, 2002; Hadi, Plomp, & Suryanto, 2002). In the concept of RME,
mathematics is a human activity connected with reality. Context problems, i.e. problems in
which the problem situation is experientially real to the pupils, are used as starting point in
learning. By giving pupils opportunity to use their own informal strategies in solving the
problems and to discuss with teacher and fellow pupils, the process of understanding is
stimulated. RME seemed to us a promising approach to tackle some problems of mathematics
education in Indonesia. The authors also thought that ICT tools could be used to support the
process for pupils of coming to grips with mathematics.
The authors decided to design and test ICT-supported teaching and learning activities that
were based on the RME approach. The design was used as a replacement of part of the chapter
'Time, Distance, and Velocity' in a popular Indonesian mathematics textbook. The focus was on
graphing skills with emphasis on graph interpretation and with application especially to
kinematics. One of the main reasons for choosing this topic is the frequent complaint of teachers
in various fields about the lack of pupils' competence in interpreting graphs. The instructional
design concerned some characteristics of the RME approach and used a Microcomputer-Based
Laboratory (MBL) environment – in our case, the "Coach computer learning environment"
(Heck, 2002) – in the teaching and learning activities. Besides the focus on enhancing students'
learning, the design was also tested for the purpose of ascertaining how this teaching and
learning model worked in an Indonesian classroom context. This included organizational
aspects, teacher and pupil classroom behavior, and teacher and pupil opinions of the activities.
Moreover, the teaching and learning activities and the materials designed and developed in this
study might become a model for later use by teachers. Results of the study might also reveal
some aspects that should be considered when implementing RME based and/or technology
supported lessons.
The Research Project
The major concern in this research study is the applicability of the designed teaching and
learning activities. The research questions are:
a) How applicable is the teaching and learning activity based on an RME approach to a
graphing topic with application to kinematics in an Indonesian second grade junior high
school classroom? In particular, The authors were interested in the way the pupils and the
teacher behaved during the activity and in their opinions about the RME approach.
b) How does the implementation of MBL-based instruction work in lessons on kinematics
graphs in an Indonesian second grade junior high school classroom? In particular, The
authors wanted to find answers to the following questions about the MBL approach: Does it
help to correct pupils' alternative conceptions in graphing? How do the pupils and the teacher
behave in the MBL environment and what difficulties do they experience? What are their
opinions about the MBL approach?Published in: Journal of Science and Mathematics Education in Southeast Asia, 2003, Vol. 26, No 2, pp. 1-51. 3
Regarding the research questions, The authors expected the following to occur during the
teaching and learning activities:
a) Concerning the RME approach: Pupils will talk more actively and become more aware of
their mathematical thinking. The situational problem as the starting point helps pupils to
relate what they learn to problems in daily life. The class environment becomes livelier. The
teacher might have difficulty in the beginning to adjust to his new role in the activities; e.g.,
he might have problems with guiding the class discussion because this is new to him.
b) Concerning MBL: It enables pupils themselves to construct and interpret graphs that are
related to real situations, and this helps to correct their alternative conceptions in graphing.
The laboratory set-up encourages pupils to become engaged and to be more enthusiastic in
the work. There might be technical problems related to the MBL environment or to the
teacher's capability to master it in a short time.
Previous Research and Theoretical Framework

The research background of this study consists of three elements: results of research on graphing, the philosophy of Realistic Mathematics Education, and the theory and practice of the use of Microcomputer-Based Laboratory in mathematics and science education.

Graphing

In the world where fast, accurate, effective, and efficient exchange of data and information is needed, methods of displaying trends and relationships between variables are very important. Weintraub (as cited in McKenzie & Padilla, 1986) stated, "Graphs present concepts in a concise manner or give at a glance information, which would require a great deal of descriptive writing. They often distill a wealth of information into a small amount of space" (pp. 571 - 572).

Graphing skills are also thought to be one of the critical topics that are important and fundamental to other more sophisticated parts of mathematics as well as to other disciplines (e.g., see Beichner, 1994; Clement, 1985; Lenton, Stevens, & Illes, 2000; McKenzie & Padilla, 1986). Of these skills, graph construction and interpretation have been identified as important. However, pupils are likely to have difficulties in graphing, especially when dealing with interpretation tasks. Research conducted by Lenton et al. (2000) on 14-15 year old pupils showed that construction of graphs was generally done well by this year group, but that interpretation of data was often less accurate. There have been many similar findings, most of which deal with kinematics, i.e., the motion of objects (e.g., Beichner, 1994; Clement, 1985; McDermott, Rosenquist, & van Zee, 1987; Mokros & Tinker, 1987).

Research has identified the following alternative interpretations and conceptions that pupils have in graphing:

• Variable confusion. Pupils do not distinguish among variables. They often believe that graphs of these variables should be identical; they appear to readily switch labels on axes from one variable to another without recognizing that the graphed line should also change (Beichner, 1994).

• Interval/point confusion. In interpreting graphs, pupils often narrow their focus to a single point, even though a range of points (an interval) is more appropriate. This is most apt to occur if the wording of a question is ambiguous, especially concerning use of the wordPublished in: Journal of Science and Mathematics Education in Southeast Asia, 2003, Vol. 26, No 2, pp. 1-51. 4 "when". Nevertheless, their focus on a single point seems to be part of an overall tendency to interpret graphs point-wise (Leinhardt, Zaslavsky, & Stein, 1990).

• Height/slope confusion. Pupils have been found to confound these two graphical features on both interpretation and construction tasks (Leinhardt et al., 1990). They often confuse gradients with maximum (or minimum) values – reading values off the axis and directly assigning them to the slope. An example of such an error is the use of graphical feature of height instead of slope to represent speed in a distance-time graph.

• Iconic interpretation ('graph as a picture' error). Pupils sometimes interpret a graph of a situation as a literal picture of that situation. A frequently cited finding in this regard is pupils' interpretation of travel graphs as the paths of actual journeys. Clement (1985) divided it into two categories:

1. Global correspondence: when the shape of an entire problem scene is matched to the shape of the entire graph in a global manner.
2. Feature correspondence: when a specific visual feature of the problem scene is matched to a specific feature of the graph.

Realistic Mathematics Education (RME)

Realistic Mathematics Education was first introduced and developed by the Freudenthal Institute in the Netherlands where it is still under active development. It is based on the view of Freudenthal (1991) that mathematics is a human activity and that reality can be used as a source for mathematization. Zulkardi (2002; after Treffers, 1987) provides the following characteristics of RME:

• Use of contextual problems. It is important to use real contexts that are meaningful and natural to pupils as a starting point for their learning, allowing them to become immediately engaged in the situation. Instruction should not start with the formal mathematical system and end with an application or related context problem as a kind of add-on to be studied after the appropriate mathematics has been learned, perhaps in order to conclude the learning process. On the contrary, the phenomena in which the concepts appear in reality can be taken as anchoring points for concept formation. Also, 'real contexts' and 'reality' should not be understood primarily as problem situations from real life or the real world; rather, they refer more to situations that are experientially real to pupils and/or something real in their minds.

• Use of models or bridging by vertical instruments. In solving problems, pupils develop and use models as a bridge between abstract and real. At first it is a model of a situation that is familiar to the pupils. By a process of generalizing and formalizing, the model eventually becomes an entity on its own and is used as a model for mathematical reasoning.

• Use of pupils' contributions. Pupils should have the opportunity to produce more concrete things themselves and to develop their own informal problem solving strategies. The teacher and the instructional materials guide a bottom-up reinvention process of the pupils. The process by which a given piece was invented in the history of mathematics may be a source of inspiration for the teacher and for the designer of the instructional materials.

• Interactivity. Interaction among pupils and between pupils and teacher is an essential part in RME because discussion and collaboration enhance reflection on the work (Gravemeijer, 1994; de Lange, 1995). In an interactive instruction pupils are engaged in explaining, justifying, agreeing and disagreeing, questioning alternatives, and reflecting.Published in: Journal of Science and Mathematics Education in Southeast Asia, 2003, Vol. 26, No 2, pp. 1-51. 5

• Intertwining of learning strands. In the philosophy of RME, various mathematical topics should be integrated in one curriculum. Pupils should develop an integrated view of mathematics as well as the flexibility to connect to different sub-domains and/or to other disciplines. This demands connection among strands, to other disciplines and to meaningful problems in the real world.

Thus, RME is more than "using real life contexts in mathematics education". Its main points are guided reinvention, didactical phenomenology, and emergent models (Gravemeijer, 1998). We restrict the further discussion of RME here to two aspects, viz., the mathematization and the use of pupils' informal strategies and own productions. More detailed descriptions of RME can be found in Freudenthal (1973, 1991), Gravemeijer (1994, 1998), Treffers (1987), and at the website www.fi.uu.nl/en/rme.

In mathematics education the focus is quite often on mathematics as a closed system. But organizing phenomena by means of progressive mathematization, i.e., by considering mathematics as a human activity, seems more important in the learning of mathematics. Treffers (1987) distinguished two kinds of mathematization: horizontal mathematization refers to modeling experientially real situations into mathematics and vice versa, whereas vertical mathematization refers to the process of attaining a higher level of abstraction within mathematics. In other words, horizontal mathematization is a process in which pupils translate problem situations that they perceive as real or realistic into some mathematical system. They invent or use mathematical tools to organize and solve their mathematical problems. In the end they translate mathematical results back into underlying statements of the problem situation, and they reflect on the work they have done. Vertical mathematization, on the other hand, concerns reorganization within the mathematical system itself like, for example, finding shortcuts, generalizing methods, and making connections between concepts and strategies. Freudenthal (1991) stated it in the following way: "horizontal mathematization involves going from the world of life into the world of symbols, while vertical mathematization means moving within the world of symbols". In RME mathematization takes place in both directions by means of a reinvention process that is guided by the teacher and by the instructional materials.

Treffers (1987) classified mathematics education into four types with regard to the presence (showed by a checkmark) of horizontal and vertical mathematization, as presented in Table 1.

Type	Horizontal mathematization	Vertical mathematization
Mechanistic	-	-
Empiric	Y	-
Structural	-	
Realistic		

Table 1. Classification of Mathematics Education Type According to Treffers (1987)

The mechanistic approach is based on drill practice. Mathematics is seen as a system of rules and algorithms; doing mathematics is analogized with machinery works which means verifying and applying these rules to problems that are similar to previous problems (Wubbels, Korthagen, & Broekman, 1997). This approach is dominant in Indonesian schools. The empiric approach sees the world as reality. Pupils are confronted with materials from the real world in which they have to do horizontal mathematization, but they are not prompted to extend a situation in order to come up with a formula or model. According to the structural point of view, mathematics is anPublished in: Journal of Science and Mathematics Education in Southeast Asia, 2003, Vol. 26, No 2, pp. 1-51. 6 organized, deductive system. The learning process in mathematics education is guided by the structure of this system and has nothing in common with the learner's living world.

The realistic mathematics education approach is based on a different point of view of mathematics education. The main difference with the mechanistic and structural approaches is that RME does not start from abstract principles or rules with the aim to learn to apply these in concrete situations (Wubbels et al., 1997). On the contrary, much importance is attributed to informal strategies and constructions that pupils develop themselves. They form the most natural way for pupils to attack problems and RME makes use of this in the instructional design of lessons. The art of teaching according to RME is to give pupils the opportunity to produce more concrete things themselves and to take their own productions as starting points for the mathematization and the gradual formalization of the informal strategies. The general idea behind this is that by making free productions, pupils are forced to reflect on the path they themselves have taken in their learning process and, at the same time, to anticipate its continuation (de Lange, 1995). Thus, in lesson work pupils are encouraged to realize and identify mathematical aspects in their daily life and to give meaning to problems from a real world context. They are challenged to develop their own strategies and approaches for solving the problems and to discuss them with fellow pupils. The teacher's role here is more one of facilitating pupils' learning through the so-called "guided reinvention" process in which pupils reflect on their own strategies in a follow-up discussion facilitated by questions from the teacher. In essence, the pupils are in a RME approach very actively involved in the development of their relational networks of mathematical concepts.

Microcomputer-Based Laboratory (MBL)

Microcomputer-Based Laboratory is equivalent to the UK English term "computer-based data logging". Computer-based data logging consists of computer tools for measuring data in practical work. One needs sensors, an interface, and a computer with an application that can present the data as tables and graphs. Physical quantities such temperature, force, current, voltage, light intensity, motion, sound, pressure, and so on, are measured with sensors whose signals are digitised and fed into the computer through an interface. After data are collected and stored in the computer, they can be processed, analyzed and presented in tables and graphs. One major goal of this data manipulation is to aid in the interpretation of graphs. Data processing and data analyses are often considered as a part of MBL.

[1]

Many researchers have already done research studies on using MBL-based instruction to enhance pupils' graph interpretation skills in various contexts. According to reviews done by Lapp and Cyrus (2000) and Lapp and Moenk (2001), some research has suggested positive results in this respect. Various contexts were used in the research projects including chemical concepts (Nakhleh & Krajcik, 1991), heat energy and temperature (Linn, Layman, & Nachmias, 1987), and kinematics (Brasell, 1987; Mokros, 1985; Thornton, 1996; Thornton & Sokoloff, 1990; Svec, 1999). Mokros and Tinker (1987) identified the following characteristics of the MBL-tools that are important to enhance pupils' interpretation of graphs:

• MBL uses multiple modalities. In learning through MBL pupils experience the materials in many different ways. They have kinesthetic experience by manipulating physical lab materials and probes and sometimes using their own physical features or movements as data. These are reinforced by visual (and sometimes audio) as well as analytical experience. The

[1]

We refer to Tinker (1996) for a broad overview of the use of MBL in science education.Published in: Journal of Science and Mathematics Education in Southeast Asia, 2003, Vol. 26, No 2, pp. 1-51. 7 multi-modal approach enables pupils to use their strong intelligence or learning styles and encourages them to build up learning modalities that are weak.

• MBL provides a real-time link between a concrete experience and the symbolic representation of that experience. It may form a bridge between concrete and formal operations. Barclay (1985) mentioned that the fast feedback allows pupils to relate immediately the graph to the event. Linn et al. (1987) and Thornton and Sokoloff (1990) made similar statements.

• MBL provides a genuine scientific experience for pupils. For instance, pupils gather and analyze real data. Mokros and Tinker (1987) viewed learning about graphing within real contexts and pupil-controlled experiments as analogous to learning about grammar by writing.

• MBL eliminates the drudgery of graph production. This advantage allows pupils to focus more on exploring the data and its representations. Rather than doing time-consuming tasks associated with data collection and display, pupils can spend more of their laboratory time in observing physical phenomena and interpreting, discussing, and analyzing data. Thornton and Sokoloff (1990) also supported this point.

Especially in a kinematics context it appears that the immediate display of the real-time graph production, along with pupils' ability to control the environment and the kinesthetic experience, play a vital role (Lapp & Moenk, 2001). It is speculated that experiencing the movement while watching the graph helps pupils to form a link between the two.

The computer application used in the activity design of this research was Coach 5,

[2]

which is

under continuous development at the AMSTEL Institute in the Netherlands. This computer learning environment provides several possibilities for active, self-responsible study of natural and mathematical phenomena by pupils, such as:

• carrying out measurements, immediately displaying results of measurements, processing data;
• controlling apparatus (e.g., a lamp or a motor);
• setting up computer models, investigating relationships between physical quantities, making predictions and comparing these to real life data;
• collecting position-time data from moving objects in digital video clips or in a sequence of pictures

The sensor used by the pupils in this study was the Ultrasonic Motion Detector (UMD) which measures distance by emitting ultrasonic pulses and by determining the length of time it takes for the reflected pulses to return. It was connected to the computer via the Coach Lab II interface.

Design of Classroom Activities

Before developing the instructional unit we analyzed the chapter "Time, Distance, and Velocity" of the mathematics textbook (Adinawan & Sugijono, 2000) that teacher and pupils used in class. We restricted ourselves to the part that discusses motion graphs, i.e., the section on drawing distance-time graphs and the section on graph reading. We identified the following graphing skills regarding distance-time graphs:

[2]

For further details about the Coach 5 environment see: http://www.science.uva.nl/research/amstel/ or http://www.cma.science.uva.nlPublished in: Journal of Science and Mathematics Education in Southeast Asia, 2003, Vol. 26, No 2, pp. 1-51. 8

• Skills required for creating graphs:

1. choosing scales;
2. drawing and labeling axes;
3. plotting points (from table or context) in a coordinate system;
4. understanding the connection between a graph and a table

• Skills required for analyzing and interpreting graphs:
5. point reading and the meaning of coordinates;
6. interval reading and what it means;
7. knowing the meaning of zero slope;
8. interpreting the steepness of parts of a graph;
9. determining the slope of a straight line and relating it to average velocity;
10. determining the intersection of graphs and relating it to the situational context
The textbook supports the teaching of these skills through drill and practice.
The authors also applied the classification of tasks on graphing of Leinhardt et al. (1990). For each task found in the examples and exercises we wrote down its type (prediction, classification, translation, or scaling), its category (construction or interpretation), its perspective (local or global), its character (quantitative or qualitative), and its focus (e.g., point reading, interval reading, graph feature, and so on). Our conclusions were:
• All tasks in the section on drawing distance-time graphs are combinations of translation and scaling, belong to the construction category, have local perspective and quantitative character, and focus mainly on the coordinate system.
• All tasks in the section on graph reading are translations of interpretation type and quantitative character. Two-thirds of these tasks relate to local properties and focus on one quantity through point reading. The other tasks mostly focus on interval-reading and deal with global properties such as stopping time and average velocity on a certain time interval, i.e., involve the use of two or more values extracted from the graph.
So, tasks of prediction or classification type and tasks of qualitative character were not present in the textbook. The findings from the analysis also indicated some omissions in this widely used textbook that we consider as important aspects of creating and viewing a graph:
• sketching a graph, e.g., from a narrative description of the situational context;
• describing a given graph in simple words, e.g., using a graph for communication purposes;
• global understanding of a graph and connecting it to the situational context, i.e., seeing a graph not as just a nice picture, but realizing that it is a concise representation of information and a useful mathematical tool;
• relating a distance-time graph to a velocity-time graph (as a first example of different types of graphs that describe the same phenomenon)
A teaching and learning activity was then designed to take into account the instructional goals already found in the textbook (skills 1 to 10) and the extra skills listed below:
11. sketching a graph, given a textual situational context;
12. making a textual description of a given graph in a related real situation;
3
13. knowing that in reality time cannot move backward to zero and that actions do not happen instantaneously;
14. drawing a graph related to a given graph, in particular, making the connection between a distance-time graph and a velocity-time graph

3
This is supposed to be achieved by giving a predicted graph of motion and asking pupils to describe the motion.Published in: Journal of Science and Mathematics Education in Southeast Asia, 2003, Vol. 26, No 2, pp. 1-51. 9
With the instructional goals in mind and guided by the theory of RME we developed an instructional unit on motion graphs. In the lessons pupils would start with practical works using their own movements as a source of data (made possible by the MBL environment) and as a starting point in their learning. This experimental set-up was chosen to allow pupils to become engaged immediately in the situation that connects "real life" to mathematical concepts. It involves horizontal mathematization. The practical works were meant to be done in-groups of two or three pupils and to be followed by classroom discussion to ensure interaction among pupils and between teacher and pupils in order to encourage talking about mathematics and mathematical thinking. For the same reason, the pupils were asked to write lab reports of their practical works. All this reflects the RME ideas of "interactivity", "guided reinvention", and "use of pupils' own contributions". The end point of the unit was a small investigation task for which pupils were asked to give a presentation of their results. This reflects the RME idea that pupils need to have room for exploration and for construction of their own products and strategies in order to build up their own theory. The intertwining of learning strands was taken into account in the instructional unit by making connections to other disciplines and to meaningful problems from the pupils' real world.
The instructional unit was designed for four classroom meetings of two hours. In the first meeting the teacher would introduce the pupils to the MBL-environment Coach and to the ultrasonic motion detector (UMD). The idea was to actually invite the pupils to get involved in the introductory experiments, thus giving them the opportunity to become acquainted with the hardware and software. Classroom discussions about the way things work and about the meaning of the graphs produced would play a key role in the first meeting. Pupils would practice in this introductory meeting the already taught skills required for creating graphs. The experiments would also serve the purpose of preparing the pupils for their practical works in the next two meetings. The compulsory practical works in these meetings are listed below, together with the graphing skills that are addressed in them (using the above numbered list of skills).
1. Investigate the difference of the appearance of the distance-time graph and of the velocitytime graph while you walk steadily and while you walk with changing speed (faster, slower, and at random). Skills: 5, 8, 11, 14.
2. Find the characteristics of the distance-time graph and velocity-time graph when you walk fast, slow, and when you stop. Skills: 6, 7, 8, 14.
3. Walk the graph, i.e., simulate a motion that produces (as similar as possible) some given graph. Skills: 5, 6, 7, 8, 9, 12, 14.
4. Walk the graph (continued) or explain why the given graph cannot be a motion graph. Skills: 12, 13.
5. Explore (with Z UMD) how the diagrams will look if you record the simultaneous motion of two objects and plot their distance-time graph in one picture. Skills: 5, 10.
Pupils would conclude their practical work with a short investigation task that they could choose from a list of five:
1. Investigate the motion graph of a bouncing ball.
2. Draw and describe the distance-time graph of your trip from home to school.
3. Compare the distance-time graph of a boy and girl in a walking race.
4. Compare the height vs. age diagram of East-Asian boys and girls (Internet data).
5. Compare graphs of temperature in two different regions during a week (Internet data).Published in: Journal of Science and Mathematics Education in Southeast Asia, 2003, Vol. 26, No 2, pp. 1-51. 10
Pupils would be asked to prepare a presentation of their results for the final classroom meeting in which their lab notes would be discussed and the results of the investigation tasks would be presented.
To support the teacher and pupils during the teaching and learning activities, we developed a Teacher's Guide and a Pupils' Guide. The section in the Pupils' Guide for Practical Work 3 is shown below and is representative of the kind of tasks given. Note that this text was also made electronically available to the pupils in the form of a Coach activity, which allowed us to prepare the graphs to be matched beforehand.
Goal of Practical Work 3:
In this activity, you are asked to simulate the motion to
produce (as similar as possible) the graph to the right:
Steps of Practical Work 3:
First, describe what you think you should do by
answering the following questions:
1. How far from the detector should you start?
2. How far from the detector should you finish?
3. How many times should you stop and how long each time?
4. How long should you walk before the first stop?
5. When (in which time interval) do you walk at highest speed?
6. Determine the velocity in each time interval (0 to 3, 3 to 6, 6 to 9, 9 to 13, 13 to 18)
Now, try to walk that way. Is your prediction correct? Try several times until you get the best result. Copy down the resulting x-t and v-t graph on the experiment worksheet or save it in Word.
Have a look on the v-t graph. Compare the result with your answer to question 6.
7. If there is any difference, why do you think that is?
8. If you continue to walk with the same velocity as the last time interval, when do you think you will be 6 meters away from the motion detector?
Research Design and Methodology
Participants
The research took place from January to March 2002 at SLTP Ciputra,
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a junior high school
located in Surabaya, East Java, Indonesia. One of the reasons for choosing this school was that it is well equipped for conducting the classroom experiment. It has a computer laboratory with enough computers for a complete class and with a beamer that can be used for demonstrations and for pupils' presentations of their results.
One class of the second grade level of junior high school with 23 pupils (13 males and 10 females, all between 13 and 14 years of age) was involved in the study. There were four parallel classes at this grade level, but the grouping of these classes was not based on the pupils' performance level. So, in the particular class, pupils' performance varied from low, middle, to high achievement level.
The class had one mathematics teacher who was also the mentor of this class. He was and is a young teacher, who at the time of the research was in his second year of teaching at the school. Before that, during his teaching training period, he had given private mathematics tutorials to

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For further information about Ciputra School, see their web site: http://www.sekolahciputra.com/Published in: Journal of Science and Mathematics Education in Southeast Asia, 2003, Vol. 26, No 2, pp. 1-51. 11
high school pupils. At the time of the research study, he taught mathematics in four classes, two classes at the first grade level and two classes at the second grade level, and he taught the particular topic for the second time in a school setting. The teacher was acquainted with the RME idea, but he admitted that the notion was still not clear to him and his other colleagues. There were some differences in their understanding of the RME idea. For the ICT aspect, he reported that he was quite familiar with it, but seldom used it in his mathematics lessons. The only time he used ICT was by asking his pupils to search for some information about a particular mathematical topic on the Internet.
Methods and Instrumentation
The research experiment was set up in a classroom situation. The pupils made use of the designed classroom activities. The validity of these activities (e.g., whether their contents indeed reflect the school curriculum) was discussed one month before with Indonesian mathematics teachers. The activities were also tried out with some mathematics and science teachers at SLTP Ciputra shortly before the experiment took place. This led to some changes in the Pupils' Guide to make it less difficult and more suitable for the pupils.
The authors gave pre- and post-tests to the pupils to investigate the effect of the ICT-rich RME-based instructions on the pupils' graphing skills. The designing of the pre- and post-tests was based on some sources, e.g. Test of Graphing in Science (TOGS) developed by McKenzie and Padilla (1986) and some research papers by Mokros and Tinker (1987) and Lenton, Stevens, and Illes (2000). These tests contained 8 items and were scheduled for one hour. Answer sheets were prepared so that pupil would not only select an answer from various options but also explain the reasoning behind choosing or not choosing an option. The pre-test was tried out on some non-participating pupils of the same age in order to ascertain the understandability and the time aspect of the test. The pre- and post-test results were scored in terms of a percentage using a predetermined weighting scheme. It should be noted that the marks depended strongly on the reasoning given by the pupils and that, therefore, subjectivity in marking was unavoidable.
Since we were more interested in the improvement of graphing skills of individual pupils rather than in the comparison of pupils by their marks, we considered the pre- and post-tests as useful tools for finding out the effects of our teaching and learning design on the pupils' performance in graphing.
The authors selected three pupils, one each from low, middle and high achievement levels (according to the teacher), to do the pre- and post-test in a special setting in which they were invited to think aloud and were interviewed separately. In this way we hoped to obtain a better overview and understanding of the pupils' mathematical thinking.
The authors developed pre- and post-questionnaires to learn more about pupils' opinions on teaching and learning activities in mathematics before and after the actual classroom meetings. These questionnaires contained both Likert-type items and open-ended questions regarding RME and ICT aspects. Results of Likert-type items were reported in a percentage format, whereas open-ended items were analyzed and summarized.
The authors interviewed the teacher before and after the instructional unit had been used. We did this to ascertain his expectations before the pupils' activities and his evaluation and suggestions after the classroom meetings. We used the interviews with the teacher in our assessment of the applicability of the teaching and learning design.Published in: Journal of Science and Mathematics Education in Southeast Asia, 2003, Vol. 26, No 2, pp. 1-51. 12
Whiteboard
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Other sources of data were the pupils' reports of their practical works and the observations made during the classroom meetings and the pupils' presentation sessions. The observation protocol consisted of taking field notes about each group's method and progress in doing practical works, pupils' discussions, and interesting events in each meeting.
The Classroom Meetings
In this section The authors give a short impression of how the classroom meetings went. For a written meeting-by-meeting account we refer to the extended research report (Widjaja, 2002).
The very first meeting suffered severely from two things. Firstly, due to a school event, the teacher had found little time to prepare the lesson and to acquaint himself well enough with the software and equipment. Secondly, the computer lab could only be used for one lesson hour of 40 minutes, which was really too short for this introductory lesson. Although the pupils evinced some concern about this, they nevertheless showed great interest in the software and equipment as well as in the upcoming practical works. The first meeting was repeated one week later, but now for the full 80 minutes and with a better preparation by the teacher. This time everything went well with one exception: it turned out that the computer laboratory was not spacious enough for 8 groups of 23 pupils in total and 3 adults to work simultaneously in an effective way. The motion detector was limited to short distances, the surrounding tables and computers were in the way, and the groups could not avoid hindering each other during motion experiments. We made a design for optimal use of the space in the computer lab (see Figure 1), but after trying this out the following class meeting, it became clear that the only viable solution was to split the class into two.
Figure 1. Set up of the computer laboratory (dimension: 9m × 8m). The rectangles represent computers (Coach 5 was installed in the shadowed ones) and the dashed lines represent the

pupils' walk paths. The numbers indicate which group was assigned to which computer and
which walk path.Published in: Journal of Science and Mathematics Education in Southeast Asia, 2003, Vol. 26, No 2, pp. 1-51. 13
While doing their practical works, some groups took measurements by placing the ultrasonic
motion detector (UMD) on top of a stool, table, or monitor. Others did them by asking one of
their group members to hold the UMD while another member walked in front of it. The difficulty
most pupils faced in the beginning was how to adjust their movement so that they could produce
a good and smooth graph. In the beginning they tended to walk too fast. Then they realized that
by doing so, either they could not walk any further even though the preset measuring time was
not over, or that they moved too far from the detector so that it could not detect them anymore.
Some pupils changed the experiment setting and shortened the measuring time; others just
walked more slowly. One unexpected problem remained; the UMD seemed to function only with
a pupil whose body was broad enough to be detected easily. This forced groups with only skinny
members to 'borrow' a fatter person from another group to do the motion.
The teacher usually walked around in the classroom, observing the groups and having
discussions with them. Almost every time he came to a group, questions arose from group
members, mostly about what had to be done. This indicated that pupils still were not used to, or
felt a bit insecure about, the new role of active, investigative learners. They still seemed to prefer
that the teacher instruct them on what to do, rather than to read the Pupils Guide and find out by
themselves. The teacher tried to refrain from intervening in pupils' experiments, giving
instructions, or giving hints and answers to questions too easily.
The activity which pupils found most difficult was Practical Work 3: "Walk the Graph"
(shown on the example of practical work). There were two reasons for this: First, before going
on to this particular activity, pupils got a separate text explaining the velocity concept. After
studying this text, they could continue with the Pupils' Guide and do another experiment. Here
we made too strong an appeal to pupils' ability to read theory and the practical guide
independently and to link them with one another. As a result, pupils felt insecure and tended to
ask what to do, what the meaning of something was, and so on. Second, producing a graph
similar to a given one is not easy. One has to pay attention to many things, like whether to move
forward or backward, whether to move slow or fast, and in the meantime, has to pay attention to
the distance. Our pupils had the same expected difficulties, even more so because of lack of
space and, sometimes, technical problems with the equipment.
During the practical works, pupils not only collaborated with members of their group but also
with members of other groups. Encouraged by the fact that one group had finished, some groups
asked for extra hours to complete their own tasks. This interest in other pupils' work was also
revealed when some groups did their practical work in the Internet corner of the school library
because of scheduling problems for the computer lab. Hence pupils from other classes and even
from senior high school were very interested in what happened, had a look, and commented that
they would very much like it if they also got that kind of activity in their mathematics lessons.
Because of a shortage of time, the tight school schedule, and pupils' various stages of work,
the teacher and the first author decided to modify the last part of the teaching and learning
design. As an assessment, the groups would do their presentation of the 6th task in pairs instead
of in a class discussion format. Two groups would be assessed at a time, one of them presenting
their results and the other giving comments, and vice versa. The presentation session was held in
the computer laboratory. The teacher assessed all presentations, and the researchers were present
merely as observers. Some groups gave good presentations and were quite confident when
reporting their results. They also made good comments that showed their understanding of
features of graphs. Others were a little clumsy when making their presentation and had some
confusing discussions with the teacher because of misunderstandings. These pupils used quite aPublished in: Journal of Science and Mathematics Education in Southeast Asia, 2003, Vol. 26, No 2, pp. 1-51. 14
lot of improper terms when presenting results and answering questions. In general, the teacher's
questions to pupils during the presentation session were mainly asking them to describe what
they had been doing in the activity and how they had gotten the required data for the tasks. Then
he would inquire about some details, and based on the question in the activity, he would demand
explanations on how they found the answers. During the presentation, sometimes it seemed that
the teacher asked unrelated questions, and unfocused discussion followed as a result. In spite of
this, in general the assessment ran smoothly.
Findings and Analysis
Information from all data sources was examined in relation to the concerns and questions of the
study. Results are presented in this section. Readers interested in more details can find them in
the extended research report (Widjaja, 2002).
Pupils' and Teacher's Behavior in the RME-Based and MBL-Supported Activities
The most noticeable new aspect in the classroom setting was its interactivity, one of the main
RME characteristics. It was no longer the teacher who actively took control of everything and
determined what to do, but it was the pupils who in doing this became more responsible for their
own learning process. Nevertheless, this was new for pupils and they were not yet accustomed to
it. They tried to read and follow the Pupils' Guide, but most of the time they tended to ask for
exact guidance on what to do. Instead of inquiring and finding out by themselves, they asked the
teacher what some things were meant to be, what the answers were for the questions, and so on.
That they were accustomed to teacher-centered learning was also seen from comments made in
the open-ended items of the post-questionnaire: some pupils expected the teacher to explain the
topic to them in a detailed way.
When working on their tasks, some pupils were more persistent than others. They insisted on
doing several trials again and again until they got satisfactory results, whereas others tried to
manipulate the situation to get expected results. Nevertheless, it is of interest that, when given
more self-responsibility, pupils were more concerned to do their tasks well. The pupils were
actually the ones who proposed extra time to continue and finish their work.
In group-work, most pupils could collaborate well and were willing to do so. At the least,
they divided the tasks and did their own favorite part, but mostly they did all the tasks together
and took turns in doing the motion, handling the software, and typing the report. When giving
the presentation, they also divided the duties (who would give the introduction, who would
present the graphs, and so on), and they assigned a fair time for each member to present his/her
part. There was a sort of competition between some of the groups, but not in a negative manner.
They just liked to do the tasks faster and better than the others did. On the other hand,
cooperation not only happened within groups, but it also happened between members of different
groups. This consisted of helping other groups with their tasks when necessary, of discussing
among one another, and debating answers to the questions.
The discussion and debate between pupils and the teacher was also something noticeable.
Not only during the practical work sessions, but also during the presentation session, pupils did
not hesitate to express their own views or their own thinking, even when they differed in opinion
with the teacher. On the other hand, discussion between the teacher and pupils was not as intense
as we expected beforehand.Published in: Journal of Science and Mathematics Education in Southeast Asia, 2003, Vol. 26, No 2, pp. 1-51. 15
Pupils were quite comfortable with using the software and equipment. In the beginning they
asked a lot of questions and demanded explanations on how to do things. Because of limited
answers from the teacher (partly on purpose, partly because there were too many pupils asking
questions at the same time), they tried out the hard- and software themselves. In quite a short
time, they could already find useful tools and commands on their own. They were able to make
changes in measurement settings or in the graphical format to fulfill their intentions. During the
last classroom meetings, some pupils could set up the whole equipment by themselves.
Pupils' and Teacher's Difficulties in the Activities
In the open-ended post-questionnaire results, the pupils reported mostly technical problems.
They had difficulties with the motion detector, the software, and the computers. These were
revealed in many different ways by pupils, e.g., "not reaching the furthest distance", "not
working", "not measuring" (for the UMD), and "malfunctioning", "broken", "not satisfactory",
"out of date", "not saving well" (for the computers). Another problem mentioned by pupils was
the lack of space to work on the practical tasks. Concerning the RME approach, some pupils
mentioned difficulties in preparing and conducting the presentation and in the cooperation within
the group.
The teacher's difficulties were also mostly related to technical issues such as mastering the
software in time and his knowledge limitation in technical matters. This was revealed during the
post-interview:

According to the teacher, he did not have problems in conducting the discussions in the
introductory activities or during the pupils' presentation. The problems in the first lesson were
mainly because the material was new for him and for the pupils, so that sometimes the
explanation and communication during introductory activities did not run smoothly. Another
problem that he mentioned was about encouraging pupils' collaboration during group work.
When asked about this matter, he stated:

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Pupil Performance
Pupils' performances were evaluated by their results on the pre- and post-tests, their report on the
practical works, and by the teacher's judgement.
Question No. 1 2 3 4 5 6 7 8
Pre-test Mean 42 64 21 23 34 54 3 43
Post-test Mean 61 70 61 38 69 69 23 63
Table 2 Pre- and Post-Tests ResultsPublished in: Journal of Science and Mathematics Education in Southeast Asia, 2003, Vol. 26, No 2, pp. 1-51. 16
time
B. Distance
from mark
In general, pupils showed good progress on the post-test. Table 2 shows that the mean value
for each question's mark increased between the pre- and post-test. This means that in the posttest session more pupils gave more satisfactory answers to the questions. Not only could they
give more correct answers, but they also could reason better in their answers. The mean value of
scores of each individual increased from pre-test to post-test up to an increase of 40%.
Comparison of the pre- and post-test results also revealed that pupils who got lower marks on the
pre-test made greater progress in their post-test scores compared to their fellow pupils who got
higher marks on the pre-test.
The pre- and post-test results indicate that pupils made progress mostly on items that were
related to the distance-time graph (numbers 3, 5, 6) and to the graph with a similar conception in
a different context (number 8). In particular, pupils showed more confidence in giving reasons
for their answers. They were more to the point and clearer in stating their explanations, not
circling around giving unclear and confusing reasons. Apparent improvements dealt with the
graph as picture error that many pupils revealed on the pre-test and with the correspondence
between a zero slope in the distance-time graph and cease in motion. These misconceptions did
not show up in the post-test anymore. This increase of performance in pre- and post-test is
illustrated by two examples:
In question 3 in the pre-test, pupils were given a description of motion:
Adit walks away from a mark on the floor at a steady rate and then directly walks
back toward it.
Some options of graphs were given and pupils were asked to answer and give their reason
for each graph whether or not it could describe the motion. One of the options was:
Kendy's answer and reasoning for the particular item in the pre-test:
Wrong, because the graph displays that Adit from where he started was back by turning
around but not going back on the same path.
Kendy's reasoning for the associated question and option in the post-test:
Impossible, because in the graph, the time goes back. It is impossible that the time is
repeated or back and a person cannot be at a different place or distance at the same time.
In question 5 in the pre-test, pupils were given a description of trips done by three people
and six options of graphs. One of the persons (Valentina) described her trip by:
I take the train every day to go to school. On the way, the train stops twice.
Stefanus assigned option B for Valentina's graph (graph on the left).
Distance
from
time
Distance
from
time
Distance
fromPublished in: Journal of Science and Mathematics Education in Southeast Asia, 2003, Vol. 26, No 2, pp. 1-51. 17
His reason was:
Valentina, because it stops twice.
The associated question in the post-test was a trip description made by a person, Victor:
I take the bus every day to school. There are three bus stops on the way.
Stefanus' answer for the associated option (graph in the middle) in the post-test:
Wrong, because in this graph, it's back to initial place three times with constant velocity.
He assigned the correct graph for Victor instead (graph on the right) and gave reason:
Victor, because there are three flat lines which means he stopped.
Pupils still revealed difficulties dealing with velocity, either in determining velocity from a
distance-time graph or in interpreting a velocity-time graph (Questions 4 and 7 in the pre- and
post-tests). The teacher also mentioned this in the post-interview. This came to no surprise:
pupils were only asked in Practical Work 3 to determine a velocity from a distance-time graph.
Nowhere else in the teaching and learning activity were they asked to derive a velocity-time
graph from a distance-time graph (it was just implicitly done in some of the practical works).
Velocity-time graphs were not covered in the textbook, either. So, the limited discussion of this
topic during the classroom activities was clearly not enough for pupils to fully understand the
concept. This is also supported by the fact that there were pupils who showed confusion between
the distance-time and velocity-time graphs in the post-test. This could well be because they dealt
mostly with distance-time graphs during the teaching and learning activity. So they mixed them
up, especially in certain features of graphs. To describe this confusion, consider question 4 in the
post-test (see Figure 2).
A ball was rolled by somebody on a flat surface. It moved along the surface, went up a small bump and
finally glided down. See the picture below:
Which graph best describes the speed of the ball? Explain your reason (for instance, why you chose or do
not chose a certain graph)!
Figure 2. Question 4 in the Post-test
time
C
Speed
time
D
Speed
time
B
Speed
A
Speed
timePublished in: Journal of Science and Mathematics Education in Southeast Asia, 2003, Vol. 26, No 2, pp. 1-51. 18
Olivia's answers for option A and B were:
Wrong, because there is a flat line which shows that the ball stopped.
Nevertheless, there was still progress in pupils' reasoning when dealing with a velocity-time
graph. Remarkable progress on the item concerning the transformation of a given distance-time
graph to a velocity-time graph (see related question from the post-test in Figure 3) was shown by
Shanty, as described below.
Below is a distance – time graph of a trip.

Which of the four graphs below most likely represents the velocity - time graph of the trip? Explain your reason!
A B.
C. D.
Figure 3. Question about graph transformation in the post-test.
Shanty's answer and reasoning for each option of question 7 in the pre-test:
Option A, B, D: Wrong, because the graph is not suitable with a distance-time graph of a
trip.
Option C: True, because the graph is suitable with a distance-time graph of a trip.
Her answer for the associated question in the post-test were:
Option A: Wrong, because in the beginning, the vehicle was not moving in a velocity of
12 km/hour, but it actually stopped.Published in: Journal of Science and Mathematics Education in Southeast Asia, 2003, Vol. 26, No 2, pp. 1-51. 19
Option B: Wrong, because the graph shows changing velocity but increasing distance,
the distance continuously increased from the initial position.
Option C: Wrong, because the velocity of 10 km/hour should be negative, since the
numeral of its final position is smaller than the numeral of its initial position.
Option D: True, because the velocity-time graph shows 2 hours of stop, 2 hours of
moving nearer in a velocity of 10 km/hour, then stopped for 1 hour, and
moving further for 2 hour in a velocity of 5 km/hour and moving further for 2
hour in a velocity of 20 km/hour. (she also wrote her calculation of the
velocities)
This progress was shown not only in the written post-test, but also during the interview with
some pupils. For example, Welim suffered with the graph-as-a-picture error and did not reason
well during the pre-test. In the post-test however, Welim (W) could interpret the velocity-time
graph in question 4 (see Figure 2) correctly and he could reason meaningfully, as shown in the
discussion with the first author (Y) below.
Y: Which one is correct?
W: B
Y: B? And the reason?
W: Yeah, at first the surface was flat, it moved fast. After that ascending (pause) it ascended then
(pause) it was a bit slow, wasn't it? Nah, a bit slow means a bit down (pointing at the part
where the graph descends). Then, the ball descended (pause) descended (pause) nah,
descending means fast (pointing at the part where the graph ascends). After descending, the
surface is flat again, yeah ... a bit normal again.
Y: Why didn't you choose C?
W: (he thought and mumbled for a while). Eh, C (pause) in C the first motion was directly fast.
Here (pointing at the picture in the question) it was not fast, was it? First it should be
normal. Here it's suddenly fast. After moving fast, then slow, and then fast again. It was not
the case, was it? In my opinion, the velocity is the same. So (pause) not directly fast.
Y: If, for instance, when rolling the ball, I pushed it like this? (I made a gesture of pushing with
my palm)
W: Yeah, that is C.
Y: So, that is now C. Why didn't you choose A and B, either ?
W: Ehm ... firstly, A here (pointing at the part where the graph ascends) is bulging. When it
bulges, it means faster, doesn't it? It should be slower, though. And B here (pause) it is
suddenly fast (pause) drastically fast.
In general, pupils' answers to questions in the practical works were as expected. Some
groups seemed to be imprecise when answering questions although they actually could have
done better on this. For instance, pupils were often imprecise when doing point reading. Another
prominent point was the use of improper terms in answers. For instance, one group stated, "when
walking fast, the graph yields a sloppy line 45 degrees up, and when walking slowly, the graph
yields a sloppy line less than 45 degrees". Some groups described intersection as: "crossing likePublished in: Journal of Science and Mathematics Education in Southeast Asia, 2003, Vol. 26, No 2, pp. 1-51. 20
a multiplication sign", "yielding the letter X", "meeting each other". This may be due to the fact
that pupils lacked guidance from teacher when doing group discussions.
According to the teacher, the performance difference between pupils taught by this method
and the traditional method was not really sharp. However, he admitted that the difference might
be greater if the approach was applied to more complicated contextual problems. He believed
that those pupils who had the opportunity to apply the topic to practical problems must have
more experience in solving them, compared to pupils who only imagined and solved paper and
pencil exercises. The teacher also mentioned the fact that in the school's final assessment, the
pupils involved in this research outscored those in the other class who he had taught the same
topic in the traditional way and who he considered to be stronger in mathematics. This is an
interesting side outcome, despite the fact that we could not validate it in this research.
Pupils' Opinions on RME-Based and MBL-Supported Activities
It seemed that pupils of this age enjoy very much being able to carry out experiments during
lessons instead of just listening to the teacher and solving written exercises in the classroom.
This could be seen from their involvement and their willingness to try out new things during the
practical works. On the one hand it resulted in a messy and noisy classroom, but on the other
hand everyone seemed to enjoy the work. The teacher supported this view and said that the
pupils tended to follow the lessons more enthusiastically and that they were not easily bored. The
classroom work seemed to interest them more.
Pupils' expectations about the teaching and learning activity in general tended to be positive.
As much as 61% of the pupils had positive or very positive opinions about RME aspects, only
14% were negative or very negative beforehand. This positive tendency was seen in almost every
item, except the ones concerning the presentation of one's work. It seems that many pupils did
not think that doing a presentation would improve their understanding. They did not feel ready
for the presentation and feared that they might face difficulties. Most pupils' opinions about the
usage of ICT in the lessons were positive beforehand, too (54% of pupils). However, somehow
they were not really sure whether they would have problems with the software and the
equipment, or not.
After the classroom meetings, it seemed that fewer pupils (47%) had positive opinions about
the RME approach. Among all items related to RME, pupils still did not seem to think that the
presentation encouraged them to understand the material thoroughly and more deeply, and they
seemed to think that the tasks and the workload might not have been divided fairly amongst
group members.
On the other hand, pupils were still positive about the ICT usage. This was especially shown
by the fact that 77% agreed or strongly agreed that doing activities using a computer in the
lesson was interesting and exciting, and that 73% of them looked forward to more use of ICT in
the next mathematics lessons.
In the analysis of individual pupils, they generally tended to be positive before the classroom
meetings, but their opinions afterwards were more variable and had mostly moved towards the
less positive. This might be due to difficulties they faced during their work: further analysis
revealed that pupils whose opinions afterwards were less positive than before were the ones who
had suffered from many problems and troubles in doing their tasks. Another reason might be that
the expectations of some pupils were too high in the beginning and just became more realistic
afterwards.Published in: Journal of Science and Mathematics Education in Southeast Asia, 2003, Vol. 26, No 2, pp. 1-51. 21
The Teacher's Opinion
During the post-interview, the teacher revealed his opinions about the teaching and learning
process in ICT-supported and RME-based activities and about his new role. In general, the tone
of the interview was positive. The instructional unit had made a good impression on the teacher,
in the sense that he liked his new role and would like to have more teaching and learning
activities designed in this way.
According to the teacher, the chosen approach offers both parties (teacher and pupils)
advantages. Pupils obtain results more by their own efforts. They experience the process of
achieving results by themselves, instead of receiving descriptive materials from the teacher.
Moreover, pupils become acquainted with new technology, which is an advantage for them.
Their abilities and skills are explored and encouraged more in an open computer-learning
environment. They are also trained in doing presentations, speaking in front of an audience, and
so on. For the teacher it is an advantage that he or she does not have to spend too much energy
on explaining things all the time. It should be noted that this last comment was tentative. If there
are ready materials and a fixed arrangement or scheduling of lessons, the teacher might indeed
not face many difficulties, but this would certainly not be the case in every situation. The
teacher found that the assessment method was appropriate. It matched his idea that the pupils'
capacity should not be measured only by written tests.
Discussion and Recommendations
This study focussed on the applicability of ICT-supported lessons based on a RME approach in
an Indonesian classroom context. In this section we summarize and discuss our findings. This
leads to recommendations to teachers who want to apply this approach in their own lessons.
Discussion
Many of the findings obtained from the research experiment were in accordance with what we
had expected before. Interactivity is the keyword to describe the classroom setting. The
classroom became livelier with pupils' experimental work as well as with discussions and
debates among the pupils and between the pupils and the teacher. Encouraged by the open
computer-learning environment, pupils tried to find out things themselves, talked more actively,
and they seemed to become more aware of their own mathematical thinking. Pupils were also
more involved and more enthusiastic in doing the lesson work. The experimental set-up
encouraged them to explore more phenomena. They made predictions and were quite persistent
in proving their hypotheses. We would like to state that one of the most important changes in the
RME-based and ICT-supported instructional unit was the way pupils were stimulated to think, to
question, and to wonder about phenomena. Given a chance to explore their ideas and to discuss
them freely, they became more critical and more involved in the learning process, and they were
more willing to understand things better. But also unexpected things happened. For example,
pupils appeared to be concerned more about their tasks when given more self-responsibility.
The technical problems that occurred were also expected beforehand, including the extra
time needed by the teacher to master the software. Only the lack of space was not foreseen.
Nevertheless, pupils' ability to cope with the software and the equipment was quite admirable.Published in: Journal of Science and Mathematics Education in Southeast Asia, 2003, Vol. 26, No 2, pp. 1-51. 22
On the other hand, pupils did feel that they had problems in collaborating within their own group
and in conducting the presentation of their work.
The teacher had fewer difficulties during the lessons than expected. He did not have crucial
problems either in adjusting to his new role or in conducting class discussions. He felt
comfortable with the different role that he took, with the teaching and learning activities, and
with the assessment. This might have been due to his nature and attitude in teaching, which was
quite progressive, and the fact that he as a mentor of the class had a good relationship with the
pupils. His problem, however, lay more in encouraging good cooperation among pupils during
their group work.
Pupils showed progress in their performance between the pre- and the post-tests. On the posttest they could give a greater number of correct answers and they could reason better in their
answers. Some concepts were mastered and some alternative conceptions were corrected. The
topic related to velocity-time graphs had not been mastered well enough, but this might be due to
the lack of discussion concerning this topic in the teaching and learning activities.
Evidence was not sufficient to verify some expectations very strongly. Did the materials and
activities help pupils to relate what they had learned to problems in daily life? However, by using
real situational contexts such as a bouncing ball, the trip from home to school, and human
growth, it is hoped that pupils' realizations of mathematical aspects in daily life were
encouraged. Although the results from this research experiment do not reveal great differences in
pupils' understanding of the mathematical concepts between the traditional lessons and the RME
approach, it is the pupils' learning process that might make a difference in the time ahead. It
seems logical to expect that pupils who obtain understanding of a concept by their own struggle
and through their own way-of-thinking process will stay with it much longer than pupils who
receive ready lesson materials, formulas, and so on, without going through the process of
deriving and manipulating them. It is our strong belief that pupils should not only receive, but
also should be given a chance to question and process the information they get. This is one of the
corner stones of the RME approach.
The use of the MBL environment also helped pupils to elaborate their ideas to their own
satisfaction and at their own pace. It served as an open learning environment in which pupils
could do trials and experiments, and test their own hypotheses. All of these would otherwise
have been more difficult to realize. The MBL environment also enabled the development of
materials using daily problems, which were real and meaningful to pupils, and which encouraged
them to realize mathematical aspects in nature and in their daily life. Hence, we can see
indications that technology might serve as a catalyst for a realistic approach to mathematics
education.
It should be noted that starting with real situations and real data is not at all easy. This might
even look more complicated as far as real phenomena are concerned. For instance, the graphs
produced in the activities in this research were messier and not as ideal as the ones pupils usually
are introduced to in their mathematics textbook. This is something to be taken into consideration
by teachers. This is just one of the many things that must be paid attention to when adopting the
approach. But most important, the teacher must be willing to change his/her role in the
classroom. The teacher must allocate the more active parts of the learning activities to pupils and
must be able to design materials and activities that can stimulate pupils to think, to dare to draw
their own conclusions and to express their opinions about the topics they are learning. A teacher
also needs to be prepared in technological literacy - at least in basic technical things. On the
other hand, pupils should also be prepared, e.g., for more independent self-responsible learning.Published in: Journal of Science and Mathematics Education in Southeast Asia, 2003, Vol. 26, No 2, pp. 1-51. 23
The roles of policymakers in education and of the school's management are not less important.
The progressive changes in pupils' and teachers' roles, and in the educational approach, crucially
need the support and openness of both policymakers and school management. Provision of
facilities and access to technology are also things to be taken into consideration nationwide by
schools and the government.
Limitation of the Study and Suggestions for Further Research
One of the limitations in the research experiment was the size of observational data and the lack
of comparative data. One experiment in one class is a weak basis for making generalized
conclusions.
The problem of lack of space in the computer lab forced the teacher and us to split the class
in two for most of the classroom meetings. As a consequence, the teacher was not always present
during all practical works. This and also the changes to some parts of the lesson design caused
weakening of some characteristics of RME that were actually supposed to be a highlight, i.e., the
guided discussions.
In further research concerning RME-based materials and also in the implementation of an
RME-based lesson, this matter should be given more attention. In retrospect, we would like to
pay more attention in the learning materials to understanding velocity-time graphs and to
transformations from a distance-time graph to a velocity-time graph and vice versa.
Another limitation is the fact that the school was not representative of Indonesian schools in
general. At the moment, most schools do not have their own computer laboratories and other
sophisticated facilities. Pupils' and teachers' familiarity with technology is also not something
that can be presupposed for Indonesian pupils and teachers in general. Further research done in
this area would be very useful if it involved more representative Indonesian schools and also
teacher training faculties. Development of materials in other topics that use the RME approach
and that are supported by ICT also must be considered.
There were some questions still left to be explored, to which the limited data gained in this
research and the restricted time have obscured finding answers to them. One of them is the
indication that low achievement pupils might benefit more than high achievement pupils (see the
section on pupils' performance). Pupils' and the teacher's changing roles would also have been
interesting research topic and could have been given more attention. The concepts of distance
and velocity might also be addressed more in further research. Distance and velocity are known
to be difficult concepts for pupils, and more attention to these concepts would have been

valuable to them.
Recommendation for Teachers
Our research experiment leads us to the following list of recommendations for teachers who want
to implement RME-based and ICT-supported lessons:
• In general, teachers must be willing to change their role in the classroom towards more pupilcentered learning.
• Activities and tasks should be chosen properly and carefully. Completion of tasks should
guide pupils to acquire the concepts they are supposed to master. Too few activities might
hinder the process, but too many activities might result in bored and exhausted pupils.Published in: Journal of Science and Mathematics Education in Southeast Asia, 2003, Vol. 26, No 2, pp. 1-51. 24
• The organizational things should be paid much attention. It is always better to make sure far
in advance the availability of laboratories, other facilities, and so on.
• Especially from the implementation of our instructional unit, we can learn that it is useful to
prepare enough laboratory space for pupils to do their practical works, for instance, by a
good set up of the laboratory or a good arrangement of groups.
• Teachers should sharpen their skills in conducting class discussions, and in addressing
questions that challenge pupils and encourage them to think critically. This might need some
practice, training, and customization of teachers' behavior.
Final Remarks
From what happened during the research experiment in an Indonesian classroom, it seems that in
practice many things still have to be worked out. It was still difficult, in this case especially for
pupils, to adjust to the pupils' more active role in learning. We suppose that encouraging pupils
to be more active will be even more difficult in a typical Indonesian school, less progressive than
the one involved in the research. It will be more difficult, as well, to change traditional-minded
teachers' attitudes. But, this surely can be done. Some Indonesian mathematics educators have
already begun such efforts by conducting research, by introducing the RME idea at every
opportunity, and by influencing policymakers in education. Developing the idea within teacher
training institutes, providing more material for teachers, and establishing good networks among
teachers, student teachers, and teacher educators are also efforts in progress.
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Codes (10)###IHLT, ##RME tasks, #RME, $$$approaches to teaching Mathematics, $factors that influence teaching and learning, $mathematics, $mathematics content, &pedagogy, @@value of using ICT, @Devices

Memos (0)

Document body:

The use of Information and Communication Technologies in the first grade of primary
school for teaching rectangles based in Realistic Mathematics Education
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Abstract—The aim of this study is to investigate the question:
"Can Information and Communications Technology (ICT)
helps to improve primary school students' geometry
achievements for rectangles?" The applied educational
software consisted of a story and several activities with and
without the use of computers which were designed following
the background of Realistic Mathematics Education for
geometry concepts; referred to as the 'Primary Shape Model'
(PSM). In addition, we use a mathematical test referred to as
the 'Primary Shape Test' (PST) to explore the use of ICT in
the primary classroom for teaching basic geometrical concepts.
It is a task orientated test which attends to measure the level of
geometry competence developed for students of the first grade.
The study dealt with students in the first grade of primary
school of Athens and Crete, who were divided into two teams
(experimental and control). The experimental group consisted
of 113 students, who were taught of rectangles by the support
of computers. There were 121 students in the control group,
which received only the traditional science instruction for
rectangles. Students in both groups were pre-tested and posttested for their geometry performance for rectangles. The
results of the study indicated that teaching and learning
through ICT is an interactive process for children at primary
school level and has a positive effect for geometry education of
rectangles.
Keywords - ICT; educational software; geometry teaching;
realistic mathematics; primary school; rectangle.
I. INTRODUCTION
The innovations which have been occurring through recent
decades in Information and Communications Technology
(ICT) have awakened the interest of the educational
community which in turn has felt the need to evolve in its
theoretical areas and pedagogic practices. Especially the
integration of ICT into primary education has become a high
priority for everybody involved in the learning process.
Educational researches, which have been occurring through
recent decades, suggest that the mathematical difficulties
students encounter later are connected with insufficient
development of mathematical thinking in the early years [1],
[2].
Moreover, various research results relate the appropriate
use of computers with the ability of students to more
efficiently understand the different mathematical notions [3],
[4], [5], [6]. Thus it becomes obvious that primary school
emerges as a very attractive environment of investigating the
computer use in education of mathematics. Indeed, a vast
number of studies show a positive correlation between the
use of computers and the development of mathematical
thinking in school [7], [8], [9].
II. THE LEVELS OF UNDERSTANDING GEOMETRY
Realistic Mathematics Education (RME) is a theory of

teaching and learning mathematics which started as a movement in the late 60s in the Netherlands in an effort of reconsidering mathematics education, where the mechanistic approach was dominant until then. Starting in 1968 with Wiskobas [10], a program initiated by the Dutch government, a new approach on mathematics education made its first steps offering an alternative to the American "New Mathematics," threatening at the time to invade the Dutch mathematics education [11]. RME is an active and constantly evolving theory of teaching and learning mathematics [12]. Indicative of this is the learning teaching trajectories with intermediate attainment targets which was first conducted for calculation with whole numbers in primary school and then extended to measurement and geometry [13], [14].

In the whole trajectory, four levels of understanding geometry concepts are involved. These should be the main focus of the learning and teaching procedure concerning geometry in primary school. The four levels were developed from the van Hiele theory, which is one of the basic axis of RME. The theory of van Hiele model deals specifically with geometric thought as it develops through several levels of sophistication under the influence of a school curriculum [15]. The van Hiele model uses five levels, however, for the first grade students only the first two levels were used. In the Visual Level, students were to identify figures such as squares and triangles as visual gestalts, for instance, they might say that a figure is a rectangle because 'it looks like a door.' In the Descriptive/Analytic Level, students were to identify shapes from their properties, for example, a student shows a rhombus as a figure with four equal sides [2].

Following the above model, the four level model designed for this research was inspired by the framework of the levels of van Hiele model for geometrical concepts, referred to as the 'Primary Shape Model' (PSM). In the first level, young students were introduced to visual recognition, verbal presentation and drawing skills of the shapes. They were also implicated with problem situations in which questions of comparisons or 'what is this' questions were posed in a way to be meaningful for students. Examples of such activities could be as follows: 'Draw a rectangle.' 'Bring me something that is a rectangle from the classroom.' 'What is the shape of the door?' etc.

In the second level, students should be able distinguish shapes from their figure, separating them from a set of objects and fitting together several figures under certain conditions. This can be realized by performing activities such as playing with constructing blocks, working with cutouts and tangram. Examples of such questions could be 'make a dog from these rectangles', 'find the shape by touching it', 'find the correct shape' etc.

In the third level, students should be able to face direct questions such as 'find the properties of rectangle' and 'find the shape with four sides' and answer them. Attention is focused to quantificatıon where the involvement of properties is immediate. As a result, questions should relate to distinctive objects or quantities such as 'how many angles does a rectangle have' or 'which shape has four sides'. At this level students should be able to organize the properties of objects using clear patterns as to avoid mistakes.

In the last level students do not need the physical presence of objects any more, but the 'mental representations,' of the objects like their properties. Also they understand the concept of shared properties, like 'the square is a rectangle' etc.

III. METHODOLOGY

The study was carried out during the 2011-12 school year in ten public primary schools located in the cities of Rethymno (Crete - Greece) and Athens. The sample consisted of 234 first graders (123 girls and 111 boys). There were two groups in the study, one control and one experimental. Five schools were selected as the control group (n=121) and the others as the experimental group (n=113). In the control group there was not a computer available for the students' use, while in the experimental there was one computer in the class and a video projector, which were used from children as part of the teaching procedure.

The present research was conducted in three phases. In the first phase the pre-test was given to the class during the beginning of December 2011 to isolate the effects of the treatment by looking for inherent inequities in the geometry achievement potential of the two groups. The first twenty questions of the test were based on the first level of the Van Hiele model and the last twenty two questions were from the test that was used in the research by Clements et al. [16]. The whole test is referred to as the 'Primary Shape Test' (PST). Due to the young age of the students the pre-tests were administrated in the class with explicit and detailed instructions of the teacher. These were pencil-and-paper tasks in which the children were asked to mark the shapes that are rectangles, squares, triangles, and circles on an page with various geometric figures "Fig. 1". The test lasted about 25 minutes. The responses were scored and coded and the data set was created by scoring the students' selections for correctness.

In the second phase the control group coordinated with traditional teaching. Group and individual activities were given to children every day. The experimental group covered the same material at roughly the same time, according to the PSM. The content of the four week syllabus was about shapes including a variety of topics concerning fundamental geometry concepts and focus on rectangles.

The first level of the PSM started with the educational software which was designed using Flash CS3 Professional Edition. It was a story about 'The Family of Shapes.' In this story Miss Circle met Mr. Rectangle "Fig. 2", they got married and had many children that looked like their parents. Then a drawing activity was performed by the children in which the students drew the 'The Family of Shapes' "Fig. 3". Next the teacher asked the children to bring things which look like rectangles and asked questions like the following: 'What is the shape of this door?', 'What is the shape of the clock?' etc.

The second level of the PSM included an activity where a child had to draw a shape from a bag and to find out the name of the shape without seeing it. Then, the students split in groups and each group had to construct a dog or a man from a set of shapes. At the end of this level a computer activity followed where the student had to recognize shapes and to choose the correct shape among triangles, circles, squares or rectangles "Fig. 4".

Figure 1. Student marks rectangles. The third level of the PSM had a software activity in which the properties of each shape are represented "Fig. 5". Next, the children were separated into groups and made a shape with their bodies on the floor "Fig. 6". In addition, they were given cut-outs with all kinds of shapes and shared them to the rest of the children.

In the last level of the PSM the students played card games "Fig. 7". There were two kinds of card games: the white cards with the properties of sides and the yellow cards with the properties of angles. The cards had shapes on them and at the end the child who had the majority of cards was the winner. Afterwards, there were computer activities where the children had to recognize the shapes from their properties "Fig. 8".

Figure 2. First level, student saw the 'The Family of Shapes'.
Figure 3. First level, student drew the 'The Family of Shapes'.
Figure 4. Second level, the student had to recognize shapes.
Figure 5. Third level, rectangle with its properties.
Figure 6. Third level, the students made a shape with their bodies.

Regarding the educational software represented above, once an activity is selected, the problem is announced and directions are given to the user through a recorded message. The feedback users get after following these directions is represented by two different screens, one positive and one negative, followed by corresponding audio messages from which the user can navigate out of by clicking on the arrow at the bottom right side of the screen. In the first case the user gets a 'well-done' message and on the second a 'try it again' one. In both cases, though, an effort was made to keep these messages to be as little emphatic as possible, so that children's interest is focused more on the mathematical procedure of the application rather than their result or the competition. However, drill and practice applications, as the described software, can be beneficial on promoting competence on certain mathematical skills while they encourage taking turns and competition [17], [18].

During the third phase of the study after the teaching intervention, the same test (post-test) was given to all students to measure their improvement at the beginning of March 2012. This was used to test the geometry achievements for the experimental and control groups. The independent variable was the use of educational software. Figure 7. Fourth level, the students played card games.
Figure 8. Fourth level, the educational software.

The main purpose of the software is to foster the first grade students' geometry concepts and to engage them in self-regulated learning. The dependent variable was the students' PST scores.

IV. RESULTS

The PST was taken by 234 students. One hundred and eleven of the students were male and 123 were female. Analysis of the data was carried out using the SPSS (ver. 19) statistical analysis computer program. An independent sample t-test was conducted. The independent variable had two levels: exposure to educational software (experimental group) and no exposure (control group).

The dependent variable was the student's pre-PST score. Levene's Test for equality of variances was not significant ($F = 0.521$, $p = 0.471$). The t-test for equality of means was also not significant ($t = 0.035$, $p = 0.972$), indicating no significant differences initially, in numeracy achievement between the experimental and control groups. Though the experimental group had a mean score slightly higher than the control group, the mean difference in the pre-PST scores was 0.01887. The results of this test are summarized in Table I and Table II.

In order to determine if the performance of the experimental group is significant, a paired t-test was performed using the grades of this group for a comparison between pre and post-PST scores. The mean grade for the pre-test in the study was 35.0354 (SD=3.93457) compared to 38.3451 (SD=4.66708) for the post-test. At $a = 0.05$ and $df = 112$, the critical value of the t ratio ($t=-12.954$) was less than 0.001. Therefore, the post-PST score was significantly different from the pre-PST score in the experimental group, as represented by Table III and Table IV.

Similarly, to determine if the performance of control group is significant, a paired t-test was performed using the grades of this group for a comparison between pre-test and post-test scores. The mean grade for the pre-test in the study was 35.0165 (SD=4.19917) compared to 35.5868 (SD=3.95110) for the post-test. At $a = 0.05$ and $df=120$, the critical value of the t ratio ($t=-2.743$) was 0.007. Therefore, the post-PST score was significantly different from the prePST score in the control group, as depicted in Table V and Table VI.

Finally, an independent sample t-test was conducted. The independent variable had the same two levels as in the previous test: experimental and control. The dependent variable was the student's post-PST score. Levene's Test for equality of variances was not significant ($F = 2.594$, $p = 0.109$). The t-test for equality of means was not significant ($t = 4.890$, $p = 0.000$) indicating significant differences, in post-PST scores between the experimental and control groups, as summarized in Table VII and Table VIII.

Results of this study expand the research on the effects
of appropriate educational programs embedded in a
computerized environment as a tool for visualization and
mathematical reasoning [6], [14], [18].
TABLE I. GROUP STATISTICS OF PRE TEST
Group N Mean Std. Dev. Std. Error
Experimental
Control
113
121
35.0354
35.0165
3.93457
4.19917
0.37013
0.38174
TABLE II. INDEPENDENT SAMPLES TEST OF PRE TEST
Pre-test t df Mean difference Sig. (2-tailed)
t-test 0.35 232 0.01887 0.972
TABLE III. PAIRED SAMPLES STATISTICS OF PRE AND POST
TESTS IN THE EXPERIMENTAL GROUP
Experimental
Group
N Mean Std. Dev. Std. Error
Pre-test
Post-test
113
113
35.0354
38.3451
3.93457
4.66708
0.37013
0.43904
TABLE IV. PAIRED SAMPLES TEST OF PRE AND POST TESTS
IN THE EXPERIMENTAL GROUP
Experimental
Group
t df Mean Sig.(2-tailed)
Pair 1 pre-test
post-test
-12.954 112 -3.30973 0.000 TABLE V. PAIRED SAMPLES STATISTICS OF PRE AND POST
TESTS IN THE CONTROL GROUP
Control Group N Mean Std. Dev. Std. Error
Pre-test
Post-test
121
121
35.0165
35.5868
4.19917
3.95110
0.38174
0.35919
TABLE VI. PAIRED SAMPLES TEST OF PRE AND POST TESTS
IN THE CONTROL GROUP
Control Group t df Mean Sig. (2-tailed)
Pair 1 pre-test
post -test
-2.743 120 -0.57025 0.007
TABLE VII. GROUP STATISTICS OF POST TEST
Group N Mean Std. Dev. Std. Error
Experimental
Control
113
121
38.3451
35.5868
4.66708
3.95110
0.43904
0.35919
TABLE VIII. INDEPENDENT SAMPLES TEST OF POST TEST
Post-test t df Mean difference Sig. (2-tailed)
Test 4.890 232 2.75836 0.000
V. CONCLUSIONS
The purpose of the study was to investigate the impact of the
intervention of PSM in regards to the geometry competence
of the first grade students. The results obtained from the
statistical analysis support our initial hypothesis, considering
that students of the experimental group as opposed to those
of the control one, displayed a greater improvement on the
post-test measurement.
During the study observational notes were taken by the
teacher who reported that students in experimental group
were eager to take their turn at computer and all of the
children were generally attentive. The students in the
computer group were able to use the computer with relative
ease and exhibited mutual consulting, cooperation, and
collaboration. Moreover, the computer may not be as
effective with students who were just at the first van Hiele
level but may of greater value with children who have a
better understanding of geometric concept at the second level
van Hiele model. The observations suggest that these
students are more comfortable with the computer than
children with less geometrical understanding. These results
tend to support the conclusion that computers may not be
equally beneficial to all students. For the students with less
representational competence concrete instructional activities
may be more appropriate.
The current findings add to a growing body of literature
supporting the effective role of educational software in
education and more specifically in mathematics [2], [6], [4],
[18]. Furthermore, the undertaken computer assisted
educational procedure revealed an extended interest for the
tasks involved from the part of the students which
transformed the whole procedure into a thorough, focused,
quiet, independent learning environment reinforcing
previous research conclusions [2], [3], [4]. Our findings in
this report are subject to certain limitations though. We have
to acknowledge that a larger and more representative sample
across all of Greece would have provided us with more solid
results.
Considering the above limitations of this work and the
fact that a research trend of developing and implementing
specific, theory-based and effective software tools, a
necessity to expand research towards this direction emerges.
The results of our research show that teaching and learning
through ICT is an interactive process for of the first graders
and has a positive effect for the teaching of rectangles.
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Codes (19)####HLT, ###guided reinvention, ##characteristics of RME, ##human activity, ##real life, ##recommendations for RME based lessons, #RME, #RNE theory, $$$approaches to teaching Mathematics, $$$modelling, $$meaningful learning of Maths, $$teaching methods and strategies, $factors that influence teaching and learning, $mathematics, $mathematics content, $pedagogy, @@challenges when working with ICT, @@value of using ICT, @Devices

Memos (0)

Document body:

The use of ICT in the first grade of primary school for teaching circles, triangles, rectangles and squares

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ABSTRACT
The purpose of this study is to investigate if information and communications technology (ICT) helps to improve first grade students' basic geometry achievement regarding circles, triangles, rectangles and squares. Our research compares the level of geometrical competence of the students taught using our ICT oriented learning method specifically targeting 'Realistic Mathematics Education' (RME) for geometry concepts, as opposed to traditional teaching methodology. The designed software consisted of a story and several activities with and without the use of computers for circles, triangles, rectangles and squares. It was designed following the background of RME theory for geometry concepts. The study dealt with first grade students in Crete and Athens, who were divided into two groups (experimental and control). The experimental group consisted of 113 students who were taught about circles, triangles, rectangles and squares with the support of computers. There were 121 students in the control group which were not exposed to the computer oriented curriculum. Students in both groups were pretested and post-tested for their geometry performance. The results of the study indicated that teaching and learning through ICT is an interactive process for students at the first grade level and has a positive effect for the learning of circles, triangles, rectangles and squares using the background of RME theory. Also, the most important outcome of this study was that this teaching intervention seemed to be more effective for triangles and rectangles than for circles and squares.

Keywords
ICT, Realistic Mathematics, geometry, circles, triangles, rectangles, squares, primary education.

1. INTRODUCTION

Nowadays, it is widely accepted that information and communication technologies are used in every steps of education [1], [2], [3]. Technology acts as a catalyst in children, provides social interaction and moreover provides additional opportunities for a rich learning environment that is consistent with modern times [4]. Although concerns about developmental convenience of technology have been announced, many studies have determined that technology has had positive influences on children cognitive and sensual developments [5], [6].

A growing body of literature provides increasing evidence of the effectiveness of using computer technologies to facilitate instruction and learning across a variety of school subjects [7], [8], [9], [10]. Particularly, studies have demonstrated that computers have supported the development of the abilities in children's memory, problem-solving, literacy and math [11], [12], [13], [14], [15], [16]. Indeed, ICT can play an essential role in achieving the objectives of the kindergarten curriculum if supported by appropriate software applications embedded in appropriate educational scenarios [17], [18], [19].

2. ICT AND MATHEMATICS

In the most ideal setting, information communication technologies are treated as a tool for teaching and learning [20], [21], [22]. They are used as a tool for the students to become more familiar with new technology and to integrate investigation, communication and understanding across the full range of the curriculum. Particularly, in the cognitive field of mathematics an evaluation of learning outcomes regarding computer based mathematical teaching in students showed that computer-assisted learning can significantly help in developing proper mathematical skills and the cultivation of deeper conceptual thinking in comparison to the traditional mathematical teaching method [23], [24], [25].

Researchers suggest that the mathematical difficulties students encounter later are correlated with insufficient development of mathematical thinking in their early years [26], [27]. Various researches' results relate the appropriate use of computers with the ability of students to more efficiently understand the different mathematical notions [28], [29]. Thus, it becomes obvious that in the primary school level a very attractive environment of investigating the computer use in mathematics education emerges. Indeed, a vast number of studies show a positive interrelation between the use of computers and the development of mathematical thinking in school [8], [30]. Nonetheless, computer based activities should reflect the theoretical ideas behind them [31], [32].

Following this principle, the software designed and the students' activities developed and examined for the purposes of the current study were inspired by the framework of Realistic Mathematics Education (RME). RME was based and developed principally upon Freudenthal's [33] view of mathematics as a "human activity." According to his perspective, in order for mathematics to be of human value, it has to be taught so as to be useful, it has to be closely related to reality, close to children and relevant to the society [33], [34]. RME is an active and constantly evolving theory of teaching and learning mathematics [35]. Indicative of this, the learning and teaching trajectories with intermediate

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DOI: http://dx.doi.org/10.1145/2643604.2643607attainment targets were first conducted for the subject of mathematical calculation at the primary school level and extended to the subject of geometry [36].

In the whole trajectory of the RME teaching theory, five main characteristics of understanding geometry concepts are involved: (a) introducing a problem using a realistic context; (b) identifying the main objects of the problem; (c) using appropriate social interaction and teacher intervention to refine the models of the problem; (d) encouraging the process of reinvention with the development of the problem; and (e) focusing on the connections and aspects of mathematics in general [35]. These should be the main focuses of the learning and teaching procedure concerning geometry in primary school.

Following the theoretical framework that blends together Realistic Mathematics Education (RME) and the use of ICT in primary school, we designed a new model referred to as the First Class Primary Shape Model (FCPSM) which consisted of five levels.

The majority of previous studies aggregately examined the effects of various teaching on the geometric shapes. However, a small number of studies have found that in the first levels of primary school, various shapes are understood differently by students [16], [37]. These studies examined the impact of various teaching interventions on each shape distinctively. Specifically, in the case of circles and squares, only the thickness and size may be varied. Rectangles and triangles may also share differences in characteristics of thickness and size with the addition of other variable characteristics such as length of sides and orientation. Moreover, triangles may vary in the degree of their angles, resulting in a greater variety of shapes. Thus, from the perspective of characteristic features, circles and squares are the simplest. On the contrary, triangles are the most complex, and rectangles are of intermediate difficulty [38], [39].

Our study was based on the above mentioned international literature; we set out to investigate the following research question: Is the concept of triangles more difficult to understand than circles for the first grade students?

In addition, based on the previous studies, we set out to investigate the following hypotheses:

1. The students who will be taught circles with educational intervention based on FCPSM will have a significant improvement in comparison to those taught using the traditional teaching method according to the first grade curriculum.
2. The students who will be taught triangles with educational intervention based on FCPSM will have a significant improvement in comparison to those taught using the traditional teaching method according to the first grade curriculum.
3. The students who will be taught rectangles with educational intervention based on FCPSM will have a significant improvement in comparison to those taught using the traditional teaching method according to the first grade curriculum.
4. The students who will be taught squares with educational intervention based on FCPSM will have a significant improvement in comparison to those taught using the traditional teaching method according to the first grade curriculum.

The present study makes an important contribution to the literature; we examined and compared the effects of a new model which combines computer and non-computer activities for teaching circles, triangles, rectangles and squares separately.

3. METHODOLOGY

The present research was conducted in three phases. In the first and third phases, the pre-test and post-test were given to the classes respectively. In the second phase, the teaching intervention was performed.

3.1 Subjects

The study was carried out during the 2011-12 school year in ten public primary schools located in the cities of Rethymno in Crete (two classes) and Athens (eight classes). It was an experimental research which compared the FCPSM teaching process to traditional teaching based on first grade curriculum. The sample included 234 first graders consisting of 123 girls and 111 boys age six to seven years old. There were two groups in the study, one control (n=121) and one experimental (n=113). In the control group there was not a computer available for the students' use. The classes in the experimental group, in order to participate, were required to have a laptop computer and a video projector available for use by children as part of the teaching procedure. For the uniformity of the survey, instructions were given to the teachers who taught in the experimental or control groups. Teachers who participated in the study had university degrees in education. The teachers in the experimental group ranged from 40 to 50 years old with in-service teaching experience ranging from 18 to 28 years. The teachers in the control group were of ages 38 to 48 years old with in-service teaching experience ranging from 15 to 25 years.

3.2 Instructional Intervention

In the second phase the control group taught with traditional teaching according to first grade curriculum. Group and individual activities were given to children every day. The experimental group covered the same material at roughly the same time according to the FCPSM teaching. The content of the four week syllabus of the FCPSM divided in five levels. It comprised of shapes including a variety of topics concerning fundamental geometry as circles, triangles, rectangles and squares.

The first level of the teaching intervention was according to the first characteristic of the teaching theory of RME which introduces a problem using a realistic context. In this level a story called 'The Family of Shapes' was presented to the students. Various geometrical shapes including circles, triangles, rectangles and squares are presented in this story in a fictional family setting where each member of the family represents a particular shape. This story was designed using Flash CS3 Professional Edition and presented with the video projector in the classroom. In this story *Mrs.* Square and *Mr.* Triangle had a daughter *Miss* Circle. Later in the story, *Miss* Circle met *Mr.* Rectangle, they got married (see Figure 1) and had many children that looked like their parents and grandparents.

The second level started with an activity which identifies the main objects of the problem as underlined by the RME theory. According to the second characteristics of the RME theory, the students (a) identified various geometrical shapes by examining objects in their classrooms, (b) drew certain geometrical shapes

(see Figure 2) and (c) named the shape of a set of various objects presented to them by the teacher.
The third level of the teaching procedure included activities using appropriate social interaction to refine the models of the problem. Firstly, a child had to draw a shape from a bag and to find out the name of the shape without seeing it. Then, the students split into groups and each group had to construct a dog or a man using a set of shapes. The final part of this level involved a computer-based group activity in which the student had to recognize and choose the correct shape among other shapes (see Figure 3).
In the fourth level of the teaching procedure, according to the RME theory, we implemented a process of reinvention with the development of the shapes in a higher cognitive level where the properties of an object were presented in a software activity. Next, the children were separated into groups and cooperated with one another to make a shape with their bodies on the floor. Then, they divided into groups and each group was assigned a specific shape. Each of these groups was instructed to make fake cookies from plasticine in the form of the groups' assigned shape (see Figure 4). At the end of this activity, the students in each group shared their cookies with the other groups.
Figure 1. Students watched the story 'The Family of Shapes' (first level).
Figure 2. The student did a drawing activity (second level).
Figure 3. The student had to recognize shapes (third level).
Figure 4. Each student was instructed to make fake cookies from plasticine (fourth level).
In the last level of the teaching process, the students played card games with the properties of shapes. The goal of this card game is to focus on the connections and aspects of the properties of circles and triangles as implied by the fifth characteristic of the RME theory. There were two kinds of card games: the white cards with shapes and their properties of sides and the yellow cards with shapes and their properties of angles. The students are separated into groups and each student in a group starts with a number of cards in their hand. Each student takes his/her turn throwing a card on the table. The purpose of the game is to match cards with identical properties and to collect those cards. At the end of the game the child who had the majority of cards was the winner.
Afterwards, there were computer activities where the children had to recognize the shapes from using only the properties of the shapes. The computer presents a picture as a puzzle. The students then must drag and drop the corresponding shape, based the geometrical property displayed on the screen, into the box (see Figure 5).Figure 5. Students completed a software activity (fifth level).
In the software represented above, once an activity is selected, a problem is announced verbally and directions are given to the user through a recorded message. The feedback users get after following these directions is represented by two different screens, one positive and one negative. This feedback is accompanied with the corresponding audio messages. In the first case, the user receives a 'well-done' message (see Figure 6) and in the second case the user gets a 'try it again' message. In both cases, though, an effort was made to keep these messages to be as little emphatic as possible. This way, the children's interest is focused more on the mathematical procedure of the application rather than their result or the competition.
Figure 6. The user receives a 'well-done' message.
3.3 Educational Measures
In the first phase, the pre-test was given to the classes of the experimental and control groups during the beginning of December 2011 to isolate the effects of the treatment by looking for inherent inequities in the geometry achievement potential of the two groups. The pre-test was referred to as First Class Primary Shape Test (FCPST) and it contained thirty tasks in total. The first twenty tasks of the FCPST were based on the first level of the Van Hiele model and the last ten were from the test that was used in the research by Clements et al. [40] based on the second van Hiele level.
The theory of the van Hiele model deals specifically with geometric thought as it develops through several levels of sophistication under the influence of a school curriculum [41]. The van Hiele model uses five levels, however, for the first grade students only the first two levels were used. In the Visual Level, students were able to identify figures such as circles and triangles as visual gestalts. Conceptualization at this level includes being able to name, reproduce and group similar geometric objects by visual recognition. For instance, they might say that a figure is a rectangle because it looks like a door. In the Descriptive/Analytic Level, students were able to identify shapes from their properties. Conceptualization at this level includes identifying geometric objects according to their properties. For example, a student sees a rhombus as a figure with four equal sides [42].
Due to the young age of the students, the pre-tests were administrated in the class with explicit and detailed instructions of the teacher. These were pencil-and-paper tasks in which the children were asked to select shapes including rectangles, squares, triangles and circles on a page with various geometric figures (see Figure 7). Each task had a weighted grade that was computed from the student's answers. Particularly, for each correct answer the student was given one grade and lost one grade for each incorrect. Scores were computed for each of the individual geometrical tasks of the FCPST. Since the numbers of problems varied across tasks, a mean proportion of correct responses for each of the thirty tasks were produced by dividing the number of correct responses by the total number of problems on that task (possible range of standardized scores, 0-1.00). Moreover, for the purposes of the present study, we measured the score of circles and the score of the triangles separately. The scale used to measure the unstandardized scores for circles is based on a scale from 0 to 69 and for squares is from 0 to 67. The scale used to measure the unstandardized scores for triangles is based on a scale from 0 to 40 and for rectangles is from 0 to 41..
Figure 7. Student marks circles.Similarly, during the third and final phase of the study, after the teaching intervention, the same test (FCPST) was given to all students in both the experimental and control groups as a post-test at the beginning of March 2012 to measure their improvement on circles and triangles separately.
3.4 Research Design
The present study was a quasi-experimental design with one experimental and one control groups. Eight first grade classes from Athens and two from Rethymno participated in this study. From the eight classes located in Athens, we randomly assigned four classes to the control group and the remaining four classes were assigned to the experimental group. We then randomly assigned one of the two classes from Rethymno to the control group and the other class to the experimental group. Therefore, five classes were in the control group (n=121) and the remaining five classes were in the experimental group (n=113).
4. RESULTS
A set of analyses was conducted to determine the effects of the mathematics intervention on first grade students' geometry knowledge for circles and triangles. The pre-test and post-test were taken by 234 students. Analysis of the data was carried out using the SPSS (ver. 19) statistical analysis computer program.
The independent variables were the group (experimental group and control group) and the shape (circles, triangles, rectangles or squares). The dependent variable was the students' FCPST posttest score.
4.1 Comparing the difficulty of understanding among shapes
The first analysis was a paired t-test among the students' FCPST pre-test scores of circles, triangles, rectangles and squares. This was done in order to examine whether the concept of triangles is more difficult to understand than the other shapes. Also it was examined whether the concept of circles are more easy to understand than the other shapes. Moreover, the difficulty of squares is higher than circles and lower than rectangles. There were significant differences in the students' FCPST pre-test scores among circles (M =.951, SD =.035) and triangles (M=.837, SD = .057), t(233) = 29.033, p <. 001, rectangles (M=.859, SD = .100), t(233) = 15.067, p <. 001, and squares (M =.897, SD =.080), t(233) = 11.021, p <. 001. Also, there were significant differences in the students' FCPST pre-test scores among triangles (M=.837, SD = .057) and rectangles (M=.859, SD = .100), t(233) = -3.431, p <. 001, and squares (M =.897, SD =.080), t(233) = -11.653, p<.001. Moreover, there was a significant difference in the students' FCPST pre-test scores among rectangles (M=.859, SD = .100) and squares (M =.897, SD =.080), t(233) = -6.452, p<.001.
The descriptive statistics for students' scores for circles and triangles are presented in Table 1.
Table 1. Descriptive statistics for students' FCPST pre-test scores

Circles	Triangles	Rectangles	Squares
M, SD	M, SD	M, SD	M, SD
Control and Experimental	.951, .035	.837, .057	.859, .100	.897, .080

4.2 Evaluate the effectiveness of FCPSM for circles
Before conducting the analysis of ANCOVA on the students' FCPST post-test scores for circles to evaluate the effectiveness of the intervention, checks were performed to confirm that there were no violations of the assumptions of homogeneity of variances and homogeneity of regression slopes [43]. The result of Levene's test when pre-test for circles was included in the model as a covariate was not significant, indicating that the group variances were equal, F(1, 232) = .342, p = .559; hence the assumption of homogeneity of variance was not been violated.
Table 2. Comparison of student scores for circles in post-test:

ANCOVA analysis Sources	Type III Sum of Squares	df	Mean Squares	F	Sig.	η2
Pre-test	.069	1	.069	102.0	.000	.306
Group	.030	1	.030	44.38	.000	.161
Error	.156	231	.001			

After adjusting for FCPST scores for circles in the pre-test (covariate), the following results were obtained from the analysis of covariance (ANCOVA). A statistically significant main effect was found for type of intervention on the FCPST post-test scores for circles, F(1, 231) = 44.384, p = .001, η2 = .161 (Table 2); thus the experimental group performed significantly higher in the FCPST post-test for circles than the control group.
4.3 Evaluate the effectiveness of FCPSM for triangles
Then, the analysis of ANCOVA on the students' FCPST post-test scores for triangles was performed to evaluate the effectiveness of the intervention. The result of Levene's test when pre-test for triangles was included in the model as a covariate was not significant, indicating that the group variances were equal, F(1, 232) = 2.743, p = .099; hence the assumption of homogeneity of variance was not been violated.

After adjusting for FCPST scores for triangles in the pre-test (covariate), the following results were obtained from the analysis of covariance (ANCOVA). A statistically significant main effect was found for type of intervention on the FCPST post-test scores for triangles, F(1, 231) = 66.558, p < .001, η 2 = .224 (Table 3); thus the experimental group performed significantly higher in the FCPST post-test for triangles than the control group.

Table 3. Comparison of student scores for triangles in posttest: ANCOVA analysis

Sources	Type III Sum of Squares	df	Mean Squares	F	Sig.	η 2
Pre-test	.106	1	.106	30.05	.000	.115
Group	.234	1	.234	66.55	.000	.224
Error	.812	231	.0044.4			

Evaluate the effectiveness of FCPSM for rectangles

Then, the analysis of ANCOVA on the students' FCPST post-test scores for rectangles was performed to evaluate the effectiveness of the intervention.

After adjusting for FCPST scores for rectangles in the pre-test (covariate), the following results were obtained from the analysis of covariance (ANCOVA). A statistically significant main effect was found for type of intervention on the FCPST post-test scores for rectangles, F(1, 231) = 58.108, p < .001, η 2 = .201 (Table 4); thus the experimental group performed significantly higher in the FCPST post-test for rectangles than the control group.

Table 4. Comparison of student scores for rectangles in posttest: ANCOVA analysis

Sources	Type III Sum of Squares	df	Mean Squares	F	Sig.	η 2
Pre-test	1.869	1	1.869	565.1	.000	.710
Group	.192	1	.192	58.10	.000	.201
Error	.764	231	.003			

4.5 Evaluate the effectiveness of FCPSM for squares

Then, the analysis of ANCOVA on the students' FCPST post-test scores for squares was performed to evaluate the effectiveness of the intervention. The result of Levene's test when pre-test for squares was included in the model as a covariate was not significant, indicating that the group variances were equal, F(1, 232) = .044, p = .834; hence the assumption of homogeneity of variance was not been violated.

After adjusting for FCPST scores for squares in the pre-test (covariate), the following results were obtained from the analysis of covariance (ANCOVA). A statistically significant main effect was found for type of intervention on the FCPST post-test scores for squares, F(1, 231) = 3.367, p = .001, η2 = .068 (Table 3); thus the experimental group performed significantly higher in the FCPST post-test for squares than the control group.

Table 5. Comparison of student scores for squares in post-test: ANCOVA analysis

Sources	Type III Sum of Squares	df	Mean Squares	F	Sig.	η 2
Pre-test	.252	1	.252	115.6	.000	.334
Group	.120	1	.120	55.26	.000	.193
Error	.503	231	.002			

Results of this study expand the research on the effects of appropriate software embedded in a computerized environment as a tool for visualization and mathematical reasoning used alongside with specially designed activities [26], [23], [32], [27], [28], [44], [29], [37], [39], [42]. Also, the outcomes of the present study create a new teaching model with computer and non-computer activities based on the theoretical framework that blends together Realistic Mathematics Education of geometric thinking in primary school.

5. DISCUSSION

The general purpose of the study was to investigate the impact of instructional intervention using the First Class Primary Shape Model (specially designed mathematics activities and software based on Realistic Mathematics Education) for the purpose of teaching the basic geometrical concepts of circles, triangles, rectangles and squares in regards to the geometry competence of the first grade students of primary school. In this research, we found that it is more difficult for first grade students to understand the concept of triangles than the other shapes. These students carried a relatively great amount of geometric knowledge from kindergarten and as a result, it is possible that the students understand circles and their properties better than triangles, rectangles and squares because triangles are more complex shapes than the other shapes [26], [31]. Similar behavior we have with rectangles, which are more complex than squares and circles. Finally, squares are more complex than circles. Our findings agree with similar researches [38], [16], [37], [39] which implied that circles are simpler and triangles are most complex.

Moreover, we found that the students that were taught circles with educational intervention based on FCPSM had a significant improvement compared to those taught using the traditional teaching method according to the first grade curriculum (hypothesis 1). Therefore, the first hypothesis was confirmed. Also, our findings suggest that the students were taught triangles with educational intervention based on FCPSM had a significant improvement compared to those taught using the traditional teaching method (hypothesis 2). As a result, the second research question answered positively.

Moreover, we found that the students that were taught rerctangles with educational intervention based on FCPSM had a significant improvement compared to those taught using the traditional teaching method according to the first grade curriculum (hypothesis 3). Therefore, the third hypothesis was confirmed. In addition, we found that the students that were taught squares with educational intervention based on FCPSM had a significant improvement compared to those taught using the traditional teaching method according to the first grade curriculum (hypothesis 4). Therefore, the fourth hypothesis was confirmed. Our results overlap with the results of other analogous studies which indicate the positive effects of a computer based-model of teaching geometry [26], [32], [37], [42]. Also, our results tend to confirm that other similar studies which indicate the positive effects of a computer based teaching model for geometric shapes [23], [28], [44], [29], [39], [45].

Moreover, an important statistical outcome of the present study was that the partial eta squared for triangles (η2 = .224 - Table 3) is higher than it was for rectangles (η2 = .201 - Table 4). Also, the partial eta squared for rectangles (η2 = .201 - Table 4) is higher than it was for squares (η2 = .193 - Table 5). Finally, the partial eta squared for squares (η2 = . 193 - Table 5) is higher than it was for circles (η2 = . 161- Table 2).

This outcomes supports that, our teaching intervention had a somewhat greater impact in learning triangles than of learning circles for first grade students. Also, the present study demonstrates that our first grade teaching process, the noncomputer based mathematics activities and the computer based mathematics activities, may support the learning of both difficult and simple mathematics in the primary school level. Our findings agree with other similar researches supporting the effective role of ICT in education and more specifically in mathematical reasoning [31], [23], [45], [24], [28], [25], [30].

The above discussion should be referenced in light of some of the limitations of this study. The first limitation of the study is that the data collected was from the participants residing the cities of Athens and Rethymno. The second limitation was the generalizability of this study which was limited to participants attending public schools. Therefore, the results from this research can be generalized only to similar groups of students. The results may not adequately describe students from other regions of Greece. However, as the study was of small scale and context specific, any application of the findings should be done with caution.

Furthermore, the undertaken computer assisted educational procedure revealed an extended interest for the tasks involved from the part of the students. It is an ongoing challenge for the reflective teacher to decide how this technology can be best utilized in education; especially in light of the current researches on the effects of such an implementation. This study is simply one small piece in the puzzle of geometry education in primary schools.

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Codes (18)###HLT, ###guided reinvention, ##characteristics of RME, ##human activity, ##real life, #RME, #RME theory, $$approaches to teaching Mathematics, $$$modelling, $$meaningful learning of Maths, $$teaching methods and strategies, $factors that influence teaching and learning, $mathematics, $mathematics content, %role players, &&learning environment, @@value of using ICT, @Devices

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Introduction
Nowadays, it is becoming increasingly accepted that various ICT applications are
developmentally appropriate technological resources for children of primary and
secondary school age (Druin & Fast, 2002 ; Passey, 2006 ; Plowman & Stephen,
2003 ; Zaranis, 2011). It is obvious now that the original "aphorism" and the ignoring of ICT are no longer effective strategies and are gradually replaced by more
realistic solutions as infl uenced by related research (Remtulla, 2010).
Technology, in this enlarged view, acts as a catalyst in their social interaction and
moreover provides additional opportunities for a rich learning environment that is
consistent with the modern era. International research confi rms that the implementation of computers in education can have positive results in different subjects (Chou
& Liu, 2005 ; Finegan & Austin, 2002 ; Livingstone, 2012). These technologies can
therefore play an essential role in achieving the objectives of the primary curriculum
in all sectors and subjects if supported by developmentally appropriate software
applications embedded in appropriate educational scenarios (Dwyer, 2007 ; Fisher,
Denning, Higgins, & Loveless, 2012 ; Lee, 2009 ; Zaranis & Oikonomidis 2009).
ICT and Mathematics
In the most ideal setting, information communication technologies are treated as a
tool for teaching and learning (Burnett, 2009 ; Sife, Lwoga, & Sanga, 2007 ;
Sutherland et al., 2004). They are used as a tool for the students to become more
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in Primary School
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familiar with new technology and to integrate investigation, communication, and
understanding across the full range of the curriculum. Particularly, in the cognitive
fi eld of mathematics an evaluation of learning outcomes regarding computer based
mathematical teaching in students showed that computer-assisted learning can signifi cantly help in developing proper mathematical skills and the cultivation of deeper
conceptual thinking in comparison to the traditional mathematical teaching method
(Dimakos & Zaranis 2010 ; Hardman, 2005 ; Keong, Horani, & Daniel, 2005).
Researchers suggest that the mathematical diffi culties students encounter later
are correlated with insuffi cient development of mathematical thinking in their early
years (Bobis et al., 2005 ; Gersten, Jordan, & Flojo, 2005). Various researches'
results relate the appropriate use of computers with the ability of students to more
effi ciently understand the different mathematical notions (Howie & Blignaut, 2009 ;
Trouche & Drijvers, 2010). Thus, it becomes obvious that in the primary school
level a very attractive environment of investigating the computer use in mathematics
education emerges. Indeed, a vast number of studies show a positive interrelation
between the use of computers and the development of mathematical thinking in
school (Clements, 2002 ; Vale & Leder, 2004). Nonetheless, computer based activities should refl ect the theoretical ideas behind them (Clements & Sarama, 2004 ;
Dissanayake, Karunananda, & Lekamge, 2007).
Following this principle, the software designed and the students' activities developed and examined for the purposes of the current study were inspired by the framework of Realistic Mathematics Education (RME) and the van Hiele model. RME
was based and developed principally upon Freudenthal's (1968) view of mathematics
as a "human activity." According to his perspective, in order for mathematics to be
of human value, it has to be taught so as to be useful, it has to be closely related to
reality, close to children and relevant to the society (Freudenthal, 1968 ; Van den
Heuvel-Panhuizen & Wijers, 2005). RME is an active and constantly evolving theory of teaching and learning mathematics (Van den Heuvel-Panhuizen, 2001).
Indicative of this, the learning and teaching trajectories with intermediate attainment targets were fi rst conducted for the subject of mathematical calculation at the
primary school level and extended to the subject of geometry (Van den Heuvel-
Panhuizen & Buys, 2008).
In the whole trajectory of the RME teaching theory, fi ve main characteristics of
understanding geometry concepts are involved: (a) introducing a problem using a
realistic context; (b) identifying the main objects of the problem; (c) using appropriate social interaction and teacher intervention to refi ne the models of the problem;
(d) encouraging the process of reinvention with the development of the problem;
and (e) focusing on the connections and aspects of mathematics in general (Van den
Heuvel-Panhuizen, 2001). These should be the main focuses of the learning and

teaching procedure concerning geometry in primary school.
Moreover, the theory of the van Hiele model deals specifi cally with geometric
thought as it develops through several levels of sophistication under the infl uence of
a school curriculum (Clements & Battista, 1992). The van Hiele model uses fi ve
levels, however, for the fi rst grade students only the fi rst two levels were used. In the
Visual Level, students were able to identify fi gures such as circles and triangles as
visual gestalts. Conceptualization at this level includes being able to name, reproduce
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and group similar geometric objects by visual recognition. For instance, they might
say that a fi gure is a rectangle because it looks like a door. In the Descriptive/
Analytic Level, students were able to identify shapes from their properties.
Conceptualization at this level includes identifying geometric objects according to
their properties. For example, a student sees a rhombus as a fi gure with four equal
sides (Zaranis, 2012). Following the theoretical framework that blends together
Realistic Mathematics Education (RME), the van Hiele model of geometric thinking and the use of ICT in primary school, we designed a new model referred to as
the First Class Primary Shape Model (FCPSM) which consisted of fi ve levels.
The majority of previous studies aggregately examined the effects of various
teaching on the geometric shapes. However, a small number of studies have found
that in the fi rst levels of primary school, various shapes are understood differently
by students (Walcott et al., 2009 ; Wong et al., 2007). These studies examined the
impact of various teaching interventions on each shape distinctively. Specifi cally, in
the case of circles, only the thickness and size may be varied. Rectangles and triangles may also share differences in characteristics of thickness and size with the
addition of other variable characteristics such as length of sides and orientation.
Moreover, triangles may vary in the degree of their angles, resulting in a greater
variety of shapes. Thus, from the perspective of characteristic features, circles are
the simplest. On the contrary, triangles are the most complex, and rectangles are of
intermediate diffi culty (Satlow & Newcombe, 1998 ; Wong et al., 2011).
Our study was based on the above mentioned international literature; we set out
to investigate the following research question: Is the concept of triangles more dif-
fi cult to understand than circles for the fi rst grade students?
In addition, based on the previous studies, we set out to investigate the following
hypotheses:
1. The students who will be taught circles with educational intervention based on
FCPSM will have a signifi cant improvement in comparison to those taught using
the traditional teaching method according to the fi rst grade curriculum.
2. The students who will be taught triangles with educational intervention based on
FCPSM will have a signifi cant improvement in comparison to those taught using
the traditional teaching method according to the fi rst grade curriculum.
The present study makes an important contribution to the literature; we examined and compared the effects of a new model which combines computer and non-
computer activities for teaching circles and triangles separately.
Methodology
Subjects
The study was carried out during the 2011-2012 school year in ten public primary
schools located in the cities of Rethymno in Crete (two classes) and Athens (eight
classes). It was an experimental research which compared the FCPSM teaching
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process to traditional teaching based on fi rst grade curriculum. The sample included
234 fi rst graders consisting of 123 girls and 111 boys age 6-7 years old. There were
two groups in the study, one control (n = 121) and one experimental (n = 113). In the
control group there was not a computer available for the students' use. The classes
in the experimental group, in order to participate, were required to have a laptop
computer and a video projector available for use by children as part of the teaching
procedure. For the uniformity of the survey, instructions were given to the teachers
who taught in the experimental or control groups. Teachers who participated in the
study had university degrees in education. The teachers in the experimental group
ranged from 40 to 50-year-olds with in-service teaching experience ranging from 18
to 28 years. The teachers in the control group were of ages 38-48 years with in-
service teaching experience ranging from 15 to 25 years.
Instructional Intervention
In the second phase the control group taught with traditional teaching according to
kindergarten curriculum. Group and individual activities were given to children
every day. The experimental group covered the same material at roughly the same
time according to the FCPSM teaching. The content of the 4-week syllabus of the
FCPSM divided in fi ve levels. It comprised of shapes including a variety of topics
concerning fundamental geometry concepts and focus on circles and triangles.
The fi rst level of the teaching intervention was according to the fi rst characteristic of the teaching theory of RME which introduces a problem using a realistic
context. In this level a story called "The Family of Shapes" was presented to the
students. Various geometrical shapes including circles and triangles are presented in
this story in a fi ctional family setting where each member of the family represents a
particular shape. This story was designed using Flash CS3 Professional Edition and
presented with the video projector in the classroom. In this story Mrs. Square and
Mr. Triangle (Fig. 1 -left) had a daughter Miss Circle. Later in the story, Miss Circle
met Mr. Rectangle, got married, and had many children that looked like their parents and grandparents.
The second level started with an activity which identifi es the main objects of the
problem as underlined by the RME theory. According to the fi rst level of the van
Hiele model and the second characteristics of the RME theory, the students (a) identifi ed various geometrical shapes by examining objects in their classrooms, (b) drew
certain geometrical shapes (Fig. 1 -right), and (c) named the shape of a set of various
objects presented to them by the teacher.
The third level of the teaching procedure included activities using appropriate
social interaction to refi ne the models of the problem. Firstly, a child had to draw a
shape from a bag and to fi nd out the name of the shape without seeing it. Then, the
students split into groups and each group had to construct a dog or a man using a set
of shapes. The fi nal part of this level involved a computer-based group activity in
which the student had to recognize and choose the correct shape among other shapes
(Fig. 2 -left) as implied by the fi rst van Hiele level.
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In the fourth level of the teaching procedure, according to the second level of the
van Hiele model and the RME theory, we implemented a process of reinvention
with the development of the shapes in a higher cognitive level where the properties
of an object were presented in a software activity (Fig. 2 -right). Next, the children
were separated into groups and cooperated with one another to make a shape with
their bodies on the fl oor (Fig. 3 -left). Then, they divided into groups and each group
was assigned a specifi c shape. Each of these groups was instructed to make fake
cookies from plasticine in the form of the groups' assigned shape. At the end of this
activity, the students in each group shared their cookies with the other groups.
In the last level of the teaching process, as the second level of van Hiele theory
implied, the students played card games with the properties of shapes. The goal
of this card game is to focus on the connections and aspects of the properties of
circles and triangles as implied by the fi fth characteristic of the RME theory.
There were two kinds of card games: the white cards with shapes and their properties of sides and the yellow cards with shapes and their properties of angles. The
students are separated into groups and each student in a group starts with a number
of cards in their hand. Each student takes his/her turn throwing a card on the table.
Fig. 1 Students watched the story "The Family of Shapes" (left —fi rst level) and did a drawing
activity (right —second level)
Fig. 2 The student had to recognize shapes (left —third level) and students viewed the properties
of a triangle on the computer (right —fourth level)
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The purpose of the game is to match cards with identical properties and to collect
those cards. At the end of the game the child who had the majority of cards was the
winner. Afterwards, there were computer activities where the children had to recognize the shapes from using only the properties of the shapes. The computer presents
a picture as a puzzle. The students then must drag and drop the corresponding shape,
based the geometrical property displayed on the screen, into the box (Fig. 3 -right).
In the software represented above, once an activity is selected, a problem is
announced verbally and directions are given to the user through a recorded message.
The feedback users get after following these directions is represented by two different screens, one positive and one negative. This feedback is accompanied with the
corresponding audio messages. In the fi rst case, the user receives a "well-done"
message and in the second case the user gets a "try it again" message. In both cases,
though, an effort was made to keep these messages to be as little emphatic as possible. This way, the children's' interest is focused more on the mathematical procedure of the application rather than their result or the competition.
Educational Measures
The present research was conducted in three phases. In the fi rst phase, the pretest
was given to the classes of the experimental and control groups during the beginning of December 2011 to isolate the effects of the treatment by looking for inherent
inequities in the geometry achievement potential of the two groups. The pretest was
referred to as First Class Primary Shape Test (FCPST) and it contained thirty tasks
in total. The fi rst twenty tasks of the FCPST were based on the fi rst level of the Van
Hiele model and the last ten were from the test that was used in the research by
Clements, Swaminathan, Zeitler-Hannibal, and Sarama (1999) based on the second
van Hiele level. Due to the young age of the students, the pretests were administrated in the class with explicit and detailed instructions of the teacher. These were
pencil-and-paper tasks in which the children were asked to select shapes including
Fig. 3 Students made a shape with their bodies on the fl oor (left —fourth level) and completed a
software activity (right —fi fth level)
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rectangles, squares, triangles, and circles on a page with various geometric fi gures
(Fig. 4). Each task had a weighted grade that was computed from the student's
answers. Particularly, for each correct answer the student was given one grade and
lost one grade for each incorrect. Scores were computed for each of the individual
geometrical tasks of the FCPST. Since the numbers of problems varied across tasks,
a mean proportion of correct responses for each of the thirty tasks were produced by
dividing the number of correct responses by the total number of problems on that
task (possible range of standardized scores, 0-1.00). Moreover, for the purposes of
the present study, we measured the score of circles and the score of the triangles
separately. The scale used to measure the unstandardized scores for circles is based
on a scale from 0 to 69. The scale used to measure the unstandardized scores for
triangles is based on a scale from 0 to 40.
Similarly, during the third and fi nal phase of the study, after the teaching intervention, the same test (FCPST) was given to all students in both the experimental
and control groups as a posttest at the beginning of March 2012 to measure their
improvement on circles and triangles separately.
Research Design
The present study was a quasi-experimental design with one experimental and one
control groups. Eight fi rst grade classes from Athens and two from Rethymno participated in this study. From the eight classes located in Athens, we randomly
assigned four classes to the control group and the remaining four classes were
Fig. 4 Student marks circles
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assigned to the experimental group. We then randomly assigned one of the two
classes from Rethymno to the control group and the other class to the experimental
group. Therefore, fi ve classes were in the control group (n = 121) and the remaining
fi ve classes were in the experimental group (n = 113).
Results
A set of analyses was conducted to determine the effects of the mathematics intervention on fi rst grade students' geometry knowledge for circles and triangles. The
pretest and posttest were taken by 234 students. Analysis of the data was carried out
using the SPSS (ver. 19) statistical analysis computer program. The independent
variables were the group (experimental group and control group) and the shape (circles or triangles). The dependent variable was the students' FCPST posttest score.
Comparing the Diffi culty Between Understanding
Circles and Triangles
The fi rst analysis was a paired t -test among the students' FCPST pretest scores of
circles and triangles in order to examine whether the concept of triangles is more
diffi cult to understand than circles. There was a signifi cant difference in the students' FCPST pretest scores for circles (M = 0.94, SD = 0.04) and triangles (M = 0.83,
SD = 0.07); t (233) = 25.93, p <0.001.
The descriptive statistics for students' scores for circles and triangles are presented in Table 1 . Observing the standardized values, students scored higher for
circles and triangles after receiving the intervention than before receiving it.
Moreover, the score of the students in triangles was lower than those in circles after
the teaching intervention (Table 1).
Evaluate the Effectiveness of FCPSM for Circles
Before conducting the analysis of ANCOVA on the students' FCPST posttest scores
for circles to evaluate the effectiveness of the intervention, checks were performed
to confi rm that there were no violations of the assumptions of homogeneity of
Table 1 Descriptive statistics for students' standardized scores of experimental and control groups
Circles Triangles
Pretest Posttest Pretest Posttest
Group M SD M SD M SD M SD
Control 0.94 0.04 0.95 0.03 0.82 0.05 0.85 0.06
Experimental 0.95 0.04 0.97 0.03 0.83 0.08 0.91 0.06
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variances and homogeneity of regression slopes (Pallant, 2001). The result of
Levene's test when pretest for circles was included in the model as a covariate was
not signifi cant, indicating that the group variances were equal, F (1, 232) = 0.476,
p = 0.491; hence, the assumption of homogeneity of variance was not been violated.
The value of the covariance by dependent variable interaction (group × score for
circles) was not signifi cant, F (1, 230) = 2.042, p = 0.154, η^2 = 0.009; therefore, the
assumption homogeneity of regression slopes was tenable. After adjusting for
FCPST scores for circles in the pretest (covariate), the following results were
obtained from the analysis of covariance (ANCOVA). A statistically signifi cant main
effect was found for type of intervention on the FCPST posttest scores for circles,
F (1, 231) = 46.513, p < 0.001, η^2 = 0.168 (Table 2); thus, the experimental group performed signifi cantly higher in the FCPST posttest for circles than the control group.
Evaluate the Effectiveness of FCPSM for Triangles
Then, the analysis of ANCOVA on the students' FCPST posttest scores for triangles
was performed to evaluate the effectiveness of the intervention. The result of Levene's
test when pretest for triangles was included in the model as a covariate was not signifi cant, indicating that the group variances were equal, F (1, 232) = 0.044, p = 0.834;
hence, the assumption of homogeneity of variance was not been violated.
The value of the covariance by dependent variable interaction (group × score for
triangles) was not signifi cant, F (1, 230) = 0.214, p = 0.644, η^2 = 0.001; therefore, the
assumption homogeneity of regression slopes was tenable. After adjusting for
FCPST scores for triangles in the pretest (covariate), the following results were
obtained from the analysis of covariance (ANCOVA). A statistically signifi cant
main effect was found for type of intervention on the FCPST posttest scores for
triangles, F (1, 231) = 69.715, p < 0.001, η^2

= 0.232 (Table 3); thus, the experimental
group performed signifi cantly higher in the FCPST posttest for triangles than the
control group.
Table 2 Comparison of student scores for circles in posttest: ANCOVA analysis
Sources Type III sum of squares d f Mean squares F Sig. Partial eta squared
Pretest 0.105 1 0.105 155.990 0.000 0.403
Group 0.031 1 0.031 46.513 0.000 0.168
Error 0.156 231 0.001
Table 3 Comparison of student scores for triangles in posttest: ANCOVA analysis
Sources Type III sum of squares d f Mean squares F Sig. Partial eta squared
Pretest 0.197 1 0.197 69.645 0.000 0.232
Group 0.197 1 0.197 69.715 0.000 0.232
Error 0.654 231 0.003
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Results of this study expand the research on the effects of appropriate software
embedded in a computerized environment as a tool for visualization and mathematical reasoning used alongside with specially designed activities (Bobis et al., 2005 ;
Dimakos & Zaranis, 2010 ; Dissanayake et al., 2007 ; Gersten et al., 2005 ; Howie &
Blignaut, 2009 ; Starkey, Klein, & Wakeley, 2004 ; Trouche & Drijvers, 2010 ; Wong
et al., 2007 , 2011 ; Zaranis, 2012). Also, the outcomes of the present study create a
new teaching model with computer and non-computer activities based on the theoretical framework that blends together Realistic Mathematics Education and the van
Hiele model of geometric thinking in primary school.
Discussion
The overall purpose of the study was to investigate the impact of instructional intervention using the First Class Primary Shape Model (specially designed mathematics
activities and software based on Realistic Mathematics Education and the van Hiele
model) for the purpose of teaching the basic geometrical concepts of circles and
triangles in regard to the geometry competence of the fi rst grade students of primary
school. In this research, we found that it is more diffi cult for fi rst grade students to
understand the concept of triangles than circles. These students carried a relatively
great amount of geometric knowledge from kindergarten and as a result, it is possible that the students understand circles and their properties better than triangles
because triangles are more complex shapes than circles (Bobis et al., 2005 ; Clements
& Sarama, 2004). Our fi ndings agree with similar researches (Satlow & Newcombe,
1998 ; Walcott et al., 2009 ; Wong et al., 2007 , 2011) which implied that circles are
simpler and triangles are most complex. As a result, the research question answered
positively.
Moreover, we found that the students that were taught circles with educational
intervention based on FCPSM had a signifi cant improvement compared to those
taught using the traditional teaching method according to the fi rst grade curriculum
(hypothesis 1). Our results overlap with the results of other analogous studies which
indicate the positive effects of a computer based-model of teaching geometry (Bobis
et al., 2005 ; Dissanayake et al., 2007 ; Wong et al., 2007 ; Zaranis, 2012). Therefore,
the fi rst hypothesis was confi rmed.
Also, our fi ndings suggest that the students were taught triangles with educational intervention based on FCPSM had a signifi cant improvement compared to
those taught using the traditional teaching method (hypothesis 2). Our outcomes
overlie the results of other similar studies which indicate the positive effects of a
computer based teaching model for geometric shapes (Dimakos & Zaranis, 2010 ;
Howie & Blignaut, 2009 ; Starkey et al., 2004 ; Trouche & Drijvers, 2010 ; Wong
et al., 2011). Thus, the second hypothesis was answered positively. In addition, as it
mentioned in the results section, students in both the experimental and the control
groups scored higher for shapes of circles or triangles after receiving the teaching
intervention than before the teaching process.
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Moreover, an important statistical outcome of the present study was that the
partial eta squared for triangles (η
2
= 0.232, Table 3) is higher than it was for circles
(η
2
= 0.168, Table 2). This outcomes supports that, our teaching intervention had a
somewhat greater impact in learning triangles than of learning circles for fi rst grade
students. Also, the present study demonstrates that our fi rst grade teaching process,
the non-computer based mathematics activities and the computer based mathematics activities, may support the learning of both diffi cult and simple mathematics in
the primary school level. Our fi ndings agree with other similar researches supporting the effective role of ICT in education and more specifi cally in mathematical
reasoning (Clements & Sarama, 2004 ; Dimakos and Zaranis 2010 ; Furner &
Marinas, 2007 ; Hardman, 2005 ; Howie & Blignaut, 2009 ; Keong et al., 2005 ; Vale
& Leder, 2004).
The above discussion should be referenced in light of some of the limitations of
this study. The fi rst limitation of the study is that the data collected was from the
participants residing the cities of Athens and Rethymno. The second limitation was
the generalizability of this study which was limited to participants attending public
schools. Therefore, the results from this research can be generalized only to similar
groups of students. The results may not adequately describe students from other
regions of Greece. However, as the study was of small scale and context specifi c,
any application of the fi ndings should be done with caution.
Furthermore, the undertaken computer assisted educational procedure revealed
an extended interest for the tasks involved from the part of the students. It is an
ongoing challenge for the refl ective teacher to decide how this technology can be
best utilized in education; especially in light of the current researches on the effects
of such an implementation. This study is one small piece in the puzzle of geometry
education in primary schools.
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Document body:

The use of ICT in kindergarten for teaching addition
based on realistic mathematics education
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Abstract The purpose of this study is to investigate if information and communications technology (ICT) helps improve kindergarten students' basic mathematical
achievement regarding addition. Our research compares the level of mathematical
competence of the students taught using our ICT oriented learning method which
specifically takes advantage of 'Realistic Mathematics Education' (RME) for the
concept of addition, as opposed to traditional teaching methodology. The designed
software consisted of a story and several activities with and without the use of
computers for addition. It was designed following the background ofthe RME theory.
The study dealt with kindergarten students in Crete, who were divided into two groups
(experimental and control). The experimental group consisted of 165 students who
were taught addition with the support of computers. There were 170 students in the

control group which were not exposed to the computer oriented curriculum. Students in
both groups were pre-tested and post-tested for their mathematical achievement. The
results of the study indicated that teaching and learning through ICT is an interactive
process for students at the kindergarten level and has a positive effect for the learning of
addition using the background ofthe RME theory. Also, the most important result of
this study was that this teaching intervention appeared to be more effective for the
kindergarten students with medium-level mathematical abilities.
Keywords ICT. Realistic mathematics education . Math . Addition . Kindergarten
1 Introduction
Nowadays, it is widely accepted that information and communication technologies are
used in every steps of education (Druin and Fast 2002; Passey 2006; Zaranis 2011).
Technology acts as a catalyst in children, provides social interaction and moreover
provides additional opportunities for a rich learning environment that is consistent with
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e-mail: nzaranis@edc.uoc.grmodern times (Plowman and Stephen 2003). Although concerns about developmental
convenience of technology have been announced, many studies have determined that
technology has had positive influences on children cognitive and sensual developments
(Chen and Chang 2006; Zaranis and Oikonomidis 2009).
A growing body of literature provides increasing evidence of the effectiveness of
using computer technologies to facilitate instruction and learning across a variety of
school subjects (Bayraktar 2002; Clements 2002; McKenney and Voogt 2009; Trundle
and Bell 2010). Particularly, studies have demonstrated that computers have supported
the development of the abilities in children's memory, problem-solving, literacy and
math (Dodge et al. 2003; Morrow et al. 2003; Ihmedieh 2010; Judge 2005; Clements
and Sarama 2003; Walcott et al. 2009). Indeed, ICT can play an essential role in
achieving the objectives of the kindergarten curriculum if supported by appropriate
software applications embedded in appropriate educational scenarios (Dwyer 2007; Lee
2009; Fisher et al. 2012).
2 ICT and mathematics
A child requires particular skills to start learning basic mathematical calculations in
kindergarten such as counting, recognizing number symbols, subitizing small quantities, discerning number patterns, and comparing numerical magnitudes and estimating
quantities (Bobis, et al. 2005; Desoete, et al. 2010; Gersten et al. 2005; Kroesbergen,
et al. 2007). Information communication technologies in the cognitive field ofmathematics help in developing proper mathematical skills and the cultivation of deeper
conceptual thinking in comparison to the traditional mathematical teaching method
(Dimakos and Zaranis 2010; Hardman 2005; Keong et al. 2005).
Various researches' results relate the appropriate use ofcomputers with the ability of
students to more efficiently understand the different mathematical notions (Howie and
Blignaut 2009; Trouche and Drijvers 2010). Also, a vast number of studies show a
positive interrelation between the use ofcomputers and the development ofmathematical thinking in school (Clements 2002; Vale and Leder 2004).
Although computers can be a valuable tool, there are disadvantages for the incorporation of them in a classroom. When computers and other technological tools used
continuously, students can develop a dependency on these tools. Just as students who
are never required to do math without a calculator lose their ability to solve math
problems by hand, students who use computers for almost every activity experience a
decrease in their ability to spell even write by hand. Students should practice these
simple skills without technological tools regularly to ensure that their skills do not
atrophy (Sáinz, and Eccles 2012).
Also, computers present an added challenge for teachers because educators have to
deal with students of different levels of computer literacy. Some students enter the
classroom fully versed in computer applications, while others come with no previous
experience. It is difficult for teachers to deal with this huge difference and ensure that
they provide to students who need help with that aid while require from capable
students to slow their academic progress and wait (Dina, and Ciornei 2013). While
computer games are just as, ifnot more, engaging than board or card games, they do
not afford the same degree ofplayer interaction. Many computer games are individual
Educ InfTechnolpursuits. When students complete these digital games and activities they miss out on
the interaction practice that they receive through the playing of traditional games
(Sampayo-Vargas et al. 2013).
Nonetheless, computer based activities should reflect the theoretical ideas behind
them (Clements and Sarama 2004; Dissanayake et al. 2007; Zaranis and Oikonomidis
2009). Following this principle, the software designed and the students' activities that
were developed and examined for the purposes of this study were inspired by the
framework ofRealistic Mathematics Education (RME).
RME, founded from the work of Hans Freudenthal (1973), places an emphasis on
making mathematics education a relatable and usable experience for students. This is
achieved by emphasizing the connections between the real world and the mathematical
world. By staying close to reality, educators are able to give mathematics a sense ofvalue
inwhich students can findreal life associations (Freudenthal 1973). Moreover, RME is an
active and constantly evolving theory of teaching and learning mathematics (Van den
Heuvel-Panhuizen 2001). Indicative of this, the learning and teaching trajectories with
intermediate attainment targets were first conducted for the subject of mathematical
calculation at the kindergarten level (Van den Heuvel-Panhuizen and Buys 2008).
In the whole trajectory of the RME teaching theory, five main characteristics of
understanding mathematical concepts are involved: (a) introducing a problem using a
realistic context; (b) identifying the main objects ofthe problem; (c) using appropriate
social interaction and teacher intervention to refine the models of the problem; (d)
encouraging the process ofreinvention with the development of the problem; and (e)
focusing on the connections and aspects ofmathematics in general (Van den HeuvelPanhuizen 2001). These should be the main focuses of the learning and teaching
procedure concerning addition in kindergarten.
Following the theoretical framework that blends together Realistic Mathematics
Education (RME) and the use of ICT in kindergarten, we designed a new model
referred to as the Kindergarten Addition Model (KAM) which consisted offive levels.
The Kindergarten Addition Model (KAM) was designed by the group ofkindergarten
teachers and developed in cooperation with the researcher in the Department of
Preschool Education ofthe University of Crete according to the kindergarten curriculum during the 2012. The main purpose of the model is to foster young students'
mathematical concepts and skills about addition and to engage them in self-regulated
learning, using computer and no computer activities. In this article the abbreviation
KAM is used for the Kindergarten Addition Model. Our study was based on the above
mentioned international literature; we set out to investigate the following research
questions:
1. Will the students who will be taught with educational intervention based on KAM
have a significant improvement on their general mathematical achievement in
comparison to those taught using the traditional teaching method according to
the kindergarten curriculum?
2. Will the students who will be taught with educational intervention based on KAM
have a significant improvement on addition in comparison to those taughtusing the
traditional teaching method according to the kindergarten curriculum?
3. What is the mathematical level of the students who had the highest benefit from
KAM according to their general mathematical achievement?
Educ InfTechnol4. Will the students who will be taught with educational intervention based on KAM
have a significant improvement on counting in comparison to those taught using
the traditional teaching method according to the kindergarten curriculum?
5. What is the mathematical level of the students who had the highest benefit from
KAM according to their mathematical achievement on counting?
The present study makes an important contribution to the literature; we examined
and compared the effects ofa new model which combines computer and non-computer
activities for teaching addition and counting.
3 Methodology
The present research was conducted in three phases. In the first and third phases, the
pre-test and post-test were given to the classes respectively. In the second phase, the
teaching intervention was performed.
3.1 Research design
The present study was a quasi-experimental design with one experimental and one
control groups. Twenty-fourkindergarten classes from Rethymno participated in this
study. From these classes, we randomly assigned twelve classes to the control group
(n=170) and the remaining twelve classes were assigned to the experimental group
(n=165). In the control group there was not computer available for students to use in
the classroom and students were taught mathematics as usual. The classes in the
experimental group had a computer for everyday use by children as part of the
educational process and they were taught mathematics with the help of computer.
Forthe uniformity ofthe survey, instructions were given to the teachers who taught in
the experimental or control groups. The teachers who participated in the study were
students from the University of Crete ofthe Department ofPreschool Education in
4th year oftheir studies and they did the teaching procedure during their final thesis
work.
3.2 Participants
The study was carried out during the 2012-13 school year in 24 public kindergartens
located in the city of Rethymno, Crete. It was an experimental research which compared the KAM teaching process to traditional teaching based on the kindergarten
curriculum. The sample included 335 kindergarteners consisting of 159 girls and 176
boys aged 5-6 years old. There were two groups in the study, one control (n= 170) and
one experimental (n=165). The experimental group has 85 boys (51.50 %) and 80 girls
(48.50 %). The control group has 91 boys (53.50 %) and 80 girls (46.50 %).
3.3 Instructional intervention
In the second phase, the control group taught with traditional teaching according to the
kindergarten curriculum. Group and individual activities were given to children every
Educ InfTechnolday. The experimental group covered the same material at roughly the same time
according to the KAM procedure.
3.3.1 Experimental group
In the experimental group the content ofthe 4 weeks syllabus ofthe KAM was divided
into five levels. It comprised of mathematical activities focusing on addition and
counting. The content of the 4 weeks syllabus of the KAM was divided into five
levels. It comprised ofmathematical activities focusing on addition and counting.
The first level ofthe teaching intervention was according to the first characteristic of
the teaching theory of RME which introduces a problem using a realistic context. In
this level a story called 'An adventure in the forest' was presented to the students using
a computer. It involved several characters from favorite fairy tales ofchildren like Little
Red Riding Hood, Hansel and Gretel, Snow White, the Seven Dwarfs and the Big Bad
Wolfand it included addition and counting problems. It was designed using Flash CS3
Professional Edition and presented with a computer in the classroom. In this story Little
Red Riding Hood and the Seven Dwarfs (Fig. 1-left) were friends and played games.
Later in the story, the Big Bad Wolfkidnapped some ofthe Seven Dwarfs because he
was hungry. In the end they became friends and everybody was happy. Then a child
repeated the story and the students painted some oftheir favorite characters ofthe story.
The second level started with an activity which identifies the main objects of the
problem as underlinedbythe RME theory. According to the second characteristics ofthe
RME theory, the students identifiedthe main objects ofthe additionby counting objects.
The teacher ofthe kindergarten gave three candies to George and two candies to Maria.
Then, she asked the children, "How many candies have George and Mary together?"
Another activity was a dice game and the children had to add the numbers by counting
(Fig. 1-right). Finally, a computer activity took place with problems ofaddition.
The third level ofthe teaching procedure included activities using appropriate social
interaction to refine the models of the problem. Firstly, the children played the super
market game. One student played the customer who wanted to buy something and the
other student acted as the employee. For instance, the kindergarten teacher said, "John
had five euro and wanted to buy two things." Next, the students had to construct the
problem ofthe using mathematical symbols e.g. numbers, the plus sign and the equal
sign (Fig. 2-left). The final part ofthis level involved a computer-based group activity
in which the student had to construct the mathematical problem (Fig. 2-right) as implied
by the third characteristic ofthe RME theory.
Fig. 1 Students watched the story 'An adventure in the forest' (left -first level) and use math symbols for
addition (right - second level)
Educ InfTechnolIn the fourth level of the teaching procedure, according to the RME theory, we
implemented a process ofreinvention with the development ofthe problem in a higher
cognitive level. In this level the objects were hidden. For instance, the kindergarten
teacher gave some chocolates to the children and hid them in their bags (Fig. 3-left).
Then the children had to construct the mathematical problem of the hidden objects.
Next, the children played a card game where each card had an addition problem which
was showed to the students only for 10 s. Finally, a software activity followed with
hidden objects (Fig. 3-right).
In the last level ofthe teaching process, the activities focused on the connections and
aspects of mathematics in general as implied by the fifth characteristic of the RME
theory. The students are separated into groups and played the game "the super market".
One student was the buyer and the other was the employee ofthe super market (Fig. 4-
left). The buyer wanted to buy two things. The students knew how much money had
the buyer and the cost ofthe first thing that he bought. The children had to find how
much money the second thing cost. Afterwards, there were computer activities where
the children had to find the missing number of an addition problem (Fig. 4-right).
In the software represented above, once an activity is selected, a problem is
announced verbally and directions are given to the user through a recorded message.
The feedback users get after following these directions is represented by two different
screens with a happy cartoon and another with a sad cartoon.
3.3.2 Control group
The control group covered the same material at the same time with similar activities as
the experimental class. The content ofthe teaching was 4 week syllabus according to
the kindergarten curriculum. It comprised of mathematical activities focusing on
addition and counting. Additional activities were given to the student ofcontrol group
Fig. 2 The student had to construct the mathematical problem (left and right - third level)
Fig. 3 Students played games with hidden objects (left and right -fourth level)
Educ InfTechnolin order to cover the time corresponded to the computer activities of experimental
group.
Activities was assigned and collected daily, there were quizzes given periodically, to
be carried out individually and in small groups. In the zoo activity, the teacher of the
kindergarten asked Maria to take two animals for the basket and George three. Then,
the teacher asks a child, "How many animals have George and Mary together?" (Fig. 5-
left). Another activity was the fisherman. The children had to fish tow fishes from the
"lake." Each fish had a price on it and the child had to find out how much money the
fishes cost (Fig. 5-right).
A common activity was a game with abacus and the child had to add the numbers by
counting (Fig. 6-left). Moreover, the activity with the "blocks of house" was very

attractive to the children (Fig. 6-right). The teacher said, "The Little Red Riding Hood lived on the fifth floor and the Snow White on the first. One day the Little Red Riding Hood visited Snow White. How many floors Little Red Riding Hood must went on in order to go her house?"
3.4 Educational measures
In the first phase, the pre-test was given to the classes ofthe experimental and control groups during the beginning ofDecember 2011 to isolate the effects ofthe treatment by looking for inherent inequities in the mathematical achievement potential of the two groups. The pre-test was the Test ofEarly Mathematics Ability third edition, TEMA-3 (Ginsburg and Baroogy 2003).
The TEMA-3 is a norm-referenced, reliable, and valid test of early mathematical ability that is appropriate for children ofages 3 years and 0 months through 8 years and
Fig. 4 Students played games with missing numbers without a computer (left) and with computer activities (right -fifth level)
Fig. 5 Students played the "Zoo activity" (left) and "The fisherman" (right) in exercising in addition
Educ InfTechnol11 months. The form ofTEMA-3 contains 72 items. The purposes ofTEMA-3 are to: a) identify the children who are significantly behind or ahead of their peers in the development ofmathematical thinking, b) identify specific strengths and weaknesses in mathematical thinking, d) suggest instructional practices appropriate for individual children, e) document children's progress in learning arithmetic and f) serve as a measure in research projects. Also, one of the purposes for developing the TEMA-3 was to provide researchers with a statistical test that was based on current research and theories about mathematical thinking. In particular, TEMA-3's availability would stimulate the study of mathematical thinking in young children (Song and Ginsburg 1987).
Due to the young age ofthe students, the pre-tests were administrated in the children individually. These were pencil-and-paper tasks in which the children were asked to select shapes including: reading and writing numbers, verbal counting, enumeration, cardinality rule, produce sets, choosing the greater and the lesser number, addition, subtraction, multiplication (Fig. 7).
Each task had a grade that was computed from the student's answers. Scores were computed for each of the individual mathematical tasks of the TEMA-3. Since the weight of problems varied across tasks, the total correct responses for each of the seventy two tasks were produced by adding the all correct responses of each problem on that task (possible range ofscores, 0-202). Moreover, for the purposes ofthe present study, we measured the total score, the score of the addition tasks and the score of counting tasks separately. The scale used to measure the scores for addition is based on a scale from 0 to 31. The scale used to measure the scores for counting is based on a scale from 0 to 19.
The THEMA-3 is not a timed test; therefore, no precise time limits are imposed on the children being tested. On average, children will be able to complete the relevant portion ofthe test in 45-60 min.
Fig. 6 The games "abacus" (left) and "The blocks ofhouse" (right)
Fig. 7 Student marks the larger number (left) and does addition (right)
Educ InfTechnolSimilarly, during the third and final phase ofthe study, after the teaching intervention, the same test (TEMA-3) was given to all students in both the experimental and control groups as a post-test at the beginning ofMarch 2012 to measure their improvement on general mathematical ability, addition and counting.
4 Results
A set ofanalyses was conducted to determine the effects ofthe mathematics intervention on kindergarten students' mathematical achievement fortheir general mathematical knowledge, addition and counting. The pre-test and post-test were taken by 335 students. Analysis of the data was carried out using the SPSS (ver. 19) statistical analysis computer program. The independent variable was the group (experimental group and control group). The dependent variable was the students' TEMA-3 post-test score.
4.1 Evaluate the effectiveness ofKAM for general mathematical achievement
The first analysis was a paired t-test among the students' TEMA-3 pre-test scores of mathematical achievement in order to examine whether the experimental and control group start from the same level. There was not a significant difference in the students' TEMA-3 pre-test scores for experimental (M=45.65, SD=20.51) and control groups (M=43.68, SD=23.76); t(328.56)=.814, p=.416.
Before conducting the analysis of ANCOVA on the students' TEMA-3 post-test scores for general mathematical achievement to evaluate the effectiveness of the intervention, checks were performed to confirm that there were no violations of the assumptions of homogeneity of variances (Pallant 2001). The result of Levene's test when pre-test for general mathematical achievement was included in the model as a covariate was not significant, indicating that the group variances were equal, F(1, 333)=.223, p=.637; hence the assumption of homogeneity of variance was not been violated.
After adjusting for TEMA-3 scores for general mathematical achievement in the pretest (covariate), the following results were obtained from the analysis of covariance (ANCOVA). A statistically significant main effect was found for type of intervention on the TEMA-3 post-test scores for general mathematical achievement, F(1, 332)= 18.104, p<.001, η2=.052 (Table 1); thus the experimental group performed significantly higher in the TEMA-3 post-test for general mathematical achievement than the control group.
Table 1 Comparison ofstudent scores forgeneralmathematical achievementinpost-test: ANCOVA analysis
Sources Type III sum ofsquares df Mean square F Sig. Partial eta squared
Pre-test 159084.136 1 145332.316 1194.764 .000 .783
Group 496.752 1 2202.192 18.104 .000 .052
Error 40995.912 332 121.641
Educ InfTechnol4.2 Evaluate the effectiveness ofKAM on addition
Then, an analysis was a paired t-test among the students' TEMA-3 pre-test scores of addition in order to examine whether the experimental and control group start from the same level. There was not a significant difference in the students' TEMA-3 pre-test scores ofaddition for experimental (M=5.15, SD=3.576) and control groups (M=4.79, SD=3.819); t(333)=.898, p=.370.
Also, the analysis of ANCOVA on the students' TEMA-3 post-test scores for addition was performed to evaluate the effectiveness ofthe intervention. The result of Levene's test when pre-test for addition was included in the model as a covariate was not significant, indicating that the group variances were equal, F(1, 333)=.672, p=.413; hence the assumption ofhomogeneity ofvariance was not been violated.
After adjusting for TEMA-3 scores for addition in the pre-test (covariate), the following results were obtained from the analysis of covariance (ANCOVA). A statistically significant main effect was found for type of intervention on the TEMA-3 post-test scores for addition, F(1, 332)=14.320,p<.001, η2=.041 (Table 2); thus the experimental group performed significantly higher in the TEMA-3 post-test for addition than the control group.
4.3 Evaluating the stratification of students according to their success in TEMA-3
Moreover, a stratification ofexperimental and control group according to their success in TEMA-3 was divided into three equal categories: less than 32 (33.33th percentile - low), 32 to 55 (33.33 th to 66.66th percentile - medium), and more than 55 (66.66th percentile - high). In the following Table 3 the students' performance is presented including both groups (i.e. the experimental and the control group) before teaching.
The Table 3 shows that 34.5 % ofthe students ofthe experimental group exhibited high grading, 37 % exhibited medium grading, whereas 28.5 % exhibited low grading. Likewise, 32.4 % of the control group exhibited high grading, 29.4 % medium and 38.2 % low. In other words, students' performance in the medium category of the experimental group appeared to be superior (i.e. 37.0 % compared with 29.4 % ofthe control group).
A two-way ANOVA was conducted that examined the effect ofclass (experimental versus control) and the students' level of mathematical achievement (low versus medium versus high) on their improvement of TEMA-3 on addition (post-test minus pre-test score). There was not a significant interaction between the effects ofclass and mathematical level on students' according to their success in TEMA-3, F(2, 329)=.955, p=.386, η
2
=.006. On the contrary, the effect of mathematical level was
Table 2 Comparison ofstudent scores on addition in post-test: ANCOVA analysis
Sources Type III sum ofsquares df Mean squares F Sig. Partial eta squared
Pre-test 3713.838 1 3713.838 391.296 .000 .541
Group 135.914 1 135.914 14.320 .000 .041
Error 3151.051 332 9.491
Educ InfTechnolsignificant (F(1, 329)=59.678, p=.002, η
2
=.038), with the improvements of addition
in the low and high levels were lower (low- M=2.38, S.D.=2.926, high- M=2.19, S.D.=3.278) than those in the medium level (M=3.64, S.D.=3.077) after the teaching intervention (Table 4, Fig. 8). Also, the effect ofgroup was also significant (F(1, 329)= 111.539, p=.001, η
2
=.035), with children in the experimental group scoring higher
(M=3.36, S.D.=3.208) than those in the control group (M=2.12, S.D. 2.986) after the teaching intervention.
The Bonferroni post hoc tests indicated that students' improvement in addition ofthe medium-level group differed significantly from students' improvement ofthe low-level (p=.019) and the high-level (p=.002) group.
4.4 Evaluate the effectiveness ofKAM for counting
Initially, a paired t-test analysis was preformed among the students' TEMA-3 pre-test scores ofcounting in order to examine whether the experimental and control group start from the same level. There was not a significant difference in the students' TEMA-3
Table 3 Frequencies ofthe two groups in the pre-test ofaddition
Pre-test Experimental group Control group
Grading N f% N f%
Low 47 28.5 65 38.2
Medium 61 37.0 50 29.4
High 57 34.5 55 32.4
Total 165 170
Table 4 Mean and Standard Deviation ofmathematical improvement on addition according to the levels of general mathematical achievement
Level Class M S.D. N
Low Experimental 2.70 2.999 47
Control 2.15 2.874 65
Total 2.38 2.926 112
Medium Experimental 4.39 2.734 61
Control 2.72 3.245 50
Total 3.64 3.077 111
High Experimental 2.81 3.593 57
Control 1.55 2.808 55
Total 2.19 3.278 112
Total Experimental 3.36 3.208 165
Control 2.12 2.986 170
Total 2.73 3.154 335
Educ InfTechnolpre-test scores of counting for experimental (M=8.90, SD=3.889) and control groups (M=8.66, SD =4.680); t(325.317)=.520, p=.603.
Also, the analysis of ANCOVA on the students' TEMA-3 post-test scores for counting was performed to evaluate the effectiveness of the intervention. The result of Levene's test when pre-test for counting was included in the model as a covariate was not significant, indicating that the group variances were equal, F(1, 333)=1.511, p=.220; hence the assumption ofhomogeneity ofvariance was not been violated.
After adjusting for TEMA-3 scores for counting in the pre-test (covariate), the following results were obtained from the analysis of covariance (ANCOVA). A statistically significant main effect was found for type of intervention on the TEMA-3 post-test scores for counting, F(1, 332)=8.882, p=.003, η
2
=.026 (Table 5); thus the
experimental group performed significantly higher in the TEMA-3 post-test for counting than the control group.
4.5 Evaluating the stratification of students according to their success in counting
Moreover, a stratification ofexperimental and control group according to their success in the counting portion ofTEMA-3 was divided into three equal categories: less than 7 (33.33th percentile - low), 7 to 11 (33.33 th to 66.66th percentile - medium), and more
Fig. 8 Mathematical improvement on addition after the teaching intervention according to the levels of general mathematical achievement
Educ InfTechnolthan 11 (66.66th percentile - high). In the following Table 6 the students' performance is presented including both groups (i.e. the experimental and the control group) before teaching.
The Table 6 shows that 26.06 % ofthe students ofthe experimental group exhibited a high grade, 43.03 % exhibited a medium grade, whereas 30.91 % exhibited a low grade. Likewise, 29.41 % ofthe control group exhibited high grades, 34.12 % medium and 36.47 % low. In other words, students' performance in the medium-level category ofthe experimental group appeared to be superior (i.e. 43.03 % compared with 34.12 % ofthe control group).
A two-way ANOVA was conducted that examined the effect ofclass (experimental versus control) and the students' level ofmathematical achievement in counting (low versus medium versus high) on their improvement of TEMA-3 in addition (post-test minus pre-test score). There was not a significant interaction between the effects of class and mathematical level on students' in counting, F(2, 329)=2.203, p=.112, η
2
=.013. On the contrary, the effect of mathematical level in counting was
significant (F(2, 329)=4.990, p=.007, η
2
=.029), with the improvements of addition
in the low and high levels were lower (low- M=2.79, S.D. 3.827, high- M=2.04, S.D.= 2.586) than those in the medium level (M=2.33, S.D.=2.993) after the teaching intervention (Table 7, Fig. 9). Also, the effect of group was also significant (F(1, 329)=9.839,p=.002, η
2
=.029), with children in the experimental group scoring higher
(M=3.43, S.D.=3.074) than those in the control group (M=2.61, S.D.=3.683) after the teaching intervention.
The Bonferroni post hoc tests indicated that students' improvement on addition of the medium group differed significantly from students' improvement of the low (p=.005) group.
Table 5 Comparison ofstudent scores for counting in post-test: ANCOVA analysis
Sources Type III sum ofsquares df Mean squares F Sig. Partial eta squared

Pre-test 2539.590 1 2539.590 500.863 .000 .601
Group 45.035 1 45.035 8.882 .003 .026
Error 1683.383 332 5.070
Table 6 Mean and standard deviation ofmathematical improvement on addition according to the levels of counting skills
Pre-test Experimental group Control group
Grading N f% N f%
Low 51 30.91 62 36.47
Medium 71 43.03 58 34.12
High 43 26.06 50 29.41
Total 165 170
Educ InfTechnolResults of this study expand the research on the effects of appropriate software embedded in a computerized environment as a tool for mathematical reasoning used alongside with specially designed activities (Bobis, et al. 2005; Desoete, et al. 2010; Dimakos and Zaranis 2010; Dissanayake, et al. 2007; Gersten, et al. 2005; Kroesbergen, et al. 2007; Howie and Blignaut 2009; Starkey, et al. 2004; Trouche and Drijvers 2010; Zaranis 2012). Also, the outcomes ofthe present study create anew teaching model with computer and non-computer activities based on the theoretical framework that blends together Realistic Mathematics Education in the kindergarten level.
5 Discussion
The general purpose of the study was to investigate the impact of instructional intervention using the Kindergarten Addition Model (specially designed mathematics activities and software based on Realistic Mathematics Education) for the purpose of teaching the mathematical concept of addition in regards to the mathematical competence ofthe kindergarten students. In this research, we found that the students who were taught with educational intervention based on KAM had a significant improvement on their general mathematical achievement in comparison to those taught using the traditional teaching method according to the kindergarten curriculum. Our findings agree with similar researches (Judge 2005; Keong, et al. 2005; Walcott, et al. 2009; Zaranis 2011) which implied that ICT helps students to understand mathematical notions more effectively. As a result, the first research question answered positively. Moreover, we found that the students that taught with the educational intervention based on KAM had a significant improvement in addition in comparison to those taught using the traditional teaching method according to the kindergarten curriculum. Our results overlap with the results of other analogous studies which indicate the positive effects of a computer based-model of teaching math (Bobis, et al. 2005;
Table 7 Frequencies ofthe two groups in the pre-test ofcounting
Level Class M S.D. N
Low Experimental 2.51 2.774 51
Control 4.32 2.857 71
Total 2.79 3.827 43
Medium Experimental 3.36 3.208 165
Control 1.66 2.374 62
Total 2.33 2.993 58
High Experimental 2.46 3.587 50
Control 2.12 2.986 170
Total 2.04 2.586 113
Total Experimental 3.43 3.074 129
Control 2.61 3.683 93
Total 2.73 3.154 335
Educ InfTechnolDissanayake, et al. 2007; Kroesbergen, et al. 2007). Therefore, the second research question was confirmed.
Also, our findings suggest that students with the medium level of general mathematical achievement, who were taught addition with educational intervention based on KAM had a significant improvement, compared to those with high and low level of general mathematical knowledge students who were taught with the same method. Our outcomes overlie with the results of other similar studies which indicate the positive effects of a computer based teaching model for mathematical notions (Keong, et al. 2005; Zaranis 2011). Thus, the third research question was addressed.
In addition, as it mentioned in the results section, students that were taught with educational intervention based on KAM had a significant improvement on counting in comparison to those taught using the traditional teaching method according to the kindergarten curriculum. Our outcomes overlap with the results of other similar studies which indicate the positive effects ofa computer based teaching model for mathematical notions (Dimakos and Zaranis 2010; Howie and Blignaut 2009; Starkey, et al. 2004; Trouche and Drijvers 2010). As a result, the fourth research question answered positively
Also, our findings suggest that students with a medium-level ofmathematical achievement in counting, who were taught addition with educational interventionbased on KAM had a significant improvement, compared to those with high-levels and low-levels of mathematical achievement in counting who were taught using the traditional teaching
Fig. 9 Mathematicalimprovementonadditionaftertheteachinginterventionaccordingtothelevelsofcountingskills
Educ InfTechnolmethod. Our outcomes overlie with the results ofother similar studies which indicate the positive effects ofa computer based teaching model for mathematical notions (Plowman and Stephen 2003; Zaranis 2012). Thus, the fifth research question was addressed.
Moreover, the important outcome of the present study was that our model (KAM) helps the children in the medium-level of math achievement on addition to have the greater improvement on math understanding than those students in the low and high levels. Also, we faced the same results when we measured the students' level of understanding of counting. The medium level students had the greatest improvement on counting with the suggested model (KAM). We see that the students had the same behavior in counting and addition. That is because, as it is mentioned on the theoretical background session, counting is the first level of addition according the RME theory. Thus, the students who had as good background with counting were treated in the same manner with those who had a similar achievement on addition.
During the study observational notes were taken by the teachers who reported that children in experimental group were eager to have their turn in computer, and all ofthe children were generally attentive. The children in the computer group were able to use the computer with relative ease and exhibited mutual consulting, cooperation, and collaboration. Subjects did not explore the keyboard and few of the children in the experimental group consistently requested assistance as to whether their answer was correct. These children seemed unwilling to allow the computer to "tell them" iftheir answer was correct or incorrect.
Finally, we wantto emphasize thatthe enthusiasm ofthe children and theirwillingness to engage with computers during the research should serve as a reference for teachers of preschool education. Teachers are required to build up the excitement this and to give children a richpedagogical framework, which includes the computer as an integral part of the program so that education cannot be divorced from the real, everyday life.
The above discussion should be referenced in light ofsome ofthe limitations ofthis study. The first limitation of the study is that the data collected was from the participants residing the city of Rethymno, Crete. The second limitation was the generalizability of this study which was limited to participants attending public schools. Therefore, the results from this research can be generalized only to similar groups of students. The results may not adequately describe students from other regions of Greece. However, as the study was ofsmall scale and context specific, any application ofthe findings should be done with caution.
Furthermore, the undertaken computer assisted educational procedure, which is an ongoing challenge for the reflective teacher to decide how this technology, can be best utilized in kindergarten education. Especially in light ofthe current research which is simply one small piece in the puzzle of mathematical education in the kindergarten level.
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Using Mobile Devices for Teaching Realistic Mathematics in Kindergarten Education

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New ICT tools allow children to take advantage of new learning platforms as well aiding them effectively in attaining new knowledge through activities related to their immediate interests and real life scenarios. Nowadays, computers and digital applications are a part of the daily life of children. In kindergarten education, properly designed digital educational activities can become a very powerful educational tool for efficient and effective learning. The utilization of interactive activities may contribute towards the growth of learning incentives as well as proper mental development in particular areas; such as mathematics and science. Mobile devices have new attractive features and provide considerable advantages in the teaching of mathematics in kindergarten education. Our study proposes the integration of mobile devices, running our own specially designed learning activity applications, in kindergarten classrooms. These applications are based on the three levels of Realistic Mathematics Education (RME) targeting fundamental mathematical concepts for the kindergarten level. We intend to gather information on effectiveness of the incorporation of these devices and applications as leaning tools for kindergarteners.
Keywords: Preschool Education; ICT; Mobile Learning; Digital Learning Activities; Realistic Mathematics

Introduction

The introduction of ICT (Information and Communication Technologies) in the field of Preschool Education has been a point of debate and controversy among researchers for an extended period of time. Due to the highly dynamic introduction of computers in the educational reality among the Western society and the influence of related research, computer aided learning is more realistic than it once was; thus gradually replacing any initial resistant reactions towards the use of ICT in a learning environment. A wide variety of ITC applications have become increasingly accepted as developmentally appropriate education resources for children of preschool and primary school age (Druin & Fast, 2002; Plowman & Stephen, 2003; Zaranis & Kalogiannakis, 2011a; Zaranis, 2012). Moreover, modern researchers have expanded the content of the term ICT to deal not only with the desktop computer but well as mobile technologies embedded in various devices (Plowman & Stephen, 2005; Cjeleaj, 2013) such as e-toys, digital cameras, smart games, electronic tables and smart mobile devices (Voithofer, 2005). Via the Internet, children are now able to surf and search for information, play games, watch videos and even make use of numerous online services such as making and receiving VoIP calls using Skype. There have even been reported cases of kindergarten children creating and maintaining their own blog and engaging in online social networking. Now digital media and interfacing with new technologies is a part of day-to-day life for children of all ages (Finitsis, 2012).

Computers not only isolate children but also act as a catalyst in their social interaction and provide children with additional opportunities for rich learning activities that are relevant to their growth characteristics regardless of age. International research studies (Yelland, 2002; Druin & Fast, 2002; Plowman & Stephen, 2003; Musawi, 2011; Zaranis, 2011; Zaranis, 2012) on the introduction of computers in preschool education show that computers in the classroom have positive results in regard to the learning of different subjects.

ICT can play an essential role in achieving the goals of the kindergarten curriculum in all areas and subjects (Plowman & Stephen, 2003; Chronopoulou & Riga, 2012) if the provided developmentally appropriate software applications (Haugland, 1999; Lee, 2009) are embedded in appropriate educational scenarios (Zaranis & Kalogiannakis, 2011b). In addition to the educational purpose, the use of computers in kindergarten is characterized mainly as a learning activity while the common child will see it as a game of sorts. Like many researchers, in formal and informal settings, we strongly endorse the need to focus our investigations on meaningful learning through playing. It is now clear that the use of this learning process is more than adequate in achieving given educational goals for children at the preschool and primary school levels (Plowman & Stephen, 2005; Zaranis, 2012).

In the classroom, ICT is treated as a learning tool (Zaranis & Kalogiannakis, 2011a). For the students, ICT is a means for achieving familiarity with new technologies as well as being a N. ZARANIS ET AL.
tool of investigation, communication and understanding across the full range of the curriculum.

In recent years, with the emergence of smart mobile devices, several researchers (Lam & Duan, 2012) have proposed the use of mobile devices for learning. There are multiple advantages with respect to the use of said devices in the learning process including stimulus, motivation, ease of use, availability, etc. (Vavoula & Karagiannidis, 2005). Such devices are becoming highly valuable tools in the educational process because of their attractive features (Klopfer, Squire, & Jenkins, 2002). Mobile devices are portable, more affordable, in relation to desktop computers, offer the opportunity for learning without local restrictions, use the possibilities offered by the wireless mobile technologies for easy access to information, promote the development of digital literacy, provide opportunities for independent learning, facilitate people with disabilities etc.

Shuler (2009) stated that applications on smart mobile devices have the potential to become the new means of providing educational content to students. As mobile technologies play an increasingly prominent role in the lives of children worldwide, national ministries and schools are experimenting with the use of these popular devices for a wide range of alternative methods of teaching and learning objectives. Smartphones and tablets are among of the six new emerging technologies that may have a major impact on teaching, learning, and research in primary education (Johnson, Smith, Willis, Levine, & Haywood, 2011).

Particularly in the subject matter of mathematics, Zaranis (2011) made a comparison between the learning outcomes of computer based teaching and mathematical thematic teaching; more specifically, targeting the teaching of "Realistic Mathematics" for children ages 4 to 6 years old in Greek Kindergarten. Compared to the traditional thematic teaching method, results showed that computer-assisted learning may significantly enhance the development of mathematical skills and the cultivation of a deeper perceptual ability for the pupils (Zaranis, 2011; Zaranis & Kalogiannakis, 2011b).

The main purpose of this study is to provide a better understanding of the characteristics and the effect of ICT, especially mobile learning, in the context of kindergarten education. We will try to achieve the above through a systematic literature review on the introduction and use of ICT and smart mobile devices in kindergarten education. In particular, we will study the use of educational applications in the form of digital activities for preschool education. This paper will also present the first phase of an instructional intervention that utilizes a series of educational applications designed according to the principles of Realistic Mathematical Education. Our research materializes in the form of digital activities performed on smart mobile devices. These activities consist of comparison, sorting, matching, structured counting, efficient counting and the general knowledge of numbers exercises.

The originality of this work lies in the fact that it is the first study carried out in the field of kindergarten education in Greece. It attempts to take advantage of the children's existing familiarity with the use of smart mobile devices (tablets) to provide a benefit to the learning process. In the context of this document, a detailed and systematic effort will be made towards achieving a theoretical justification for the implementation of reformed teaching proposals.

ICT and Children as "iLearners"

Researchers Christie and Johnson (2009) indicate that digital media has now permanently settled in the lives of young children. As pointed out by Lavidas, Komis, & Gialamas (2012), many times children surpass adults in their modern technological knowledge even before attending kindergarten. Hertzog and Klein (2005) define a distinct line between the current generation of children and their parents. They very aptly report that children do not need to adapt to the new technological society because they were born in it; unlike their parents who have acquired their technological knowledge as somewhat of a foreign language at an advanced age. Consequently, according to Prensky (2001), young children can be described as "digital natives" since they are growing up in the digital world. Most recently, Prensky (2010) refers to today's students as "iLearners" deriving from the fact that digital devices such as smartphones and tablet computers dominate the daily lives of children in Western societies from the age of 6 months.

The Lieberman, Bates and So (2009a) report that several studies have shown that digital media can introduce children to abstract concepts that were previously considered too advanced for their age. In her research, Yelland (2005) has shown that activities entailing the use of digital media, within the school environment, facilitate collaborative learning for young children and the development of logical thinking while reinforcing their ability to solve problems. Digital learning activities may encourage children to work together. They have been found to be more effective than traditional learning activities (Zaranis, 2011; Zaranis & Kalogiannakis, 2011a).

Young children have access to new technologies not just in the school environment but also out of school, at home (Plowman & Stephen, 2005; Somekh, 2007). For some children, the first educational experience with computers begins in the kindergarten age, between 4 to 6 years old. Existing studies have shown that children of this age can successfully handle computers with the appropriate instruction (Zaranis, 2011; Zaranis & Kalogiannakis, 2011a). In a study of the comparison between learning at home and in the kindergarten classroom, Plowman, Stephen and McPake (2010) found out that ICT is used to promote three main areas of learning. The extension of knowledge about the world (cognitive objects), the acquisition of functional skills (such as the operation of the mouse) as well as the development of the propensity for learning (by strengthening a range of emotional, social and cognitive functions of learning). Recent studies have identified a number of emerging digital devices, such as tablets, as being appropriate for the education and entertainment of children (Vereniki & Kervin, 2011).

Lieberman et al., (2009a) commented that young children age three to six years old play with a vast variety of digital learning activities, now available on desktop monitors and portable screens, spending a constantly increasing amount of time doing so.

Furthermore, Couse and Chen (2010) state the advantages offered by the use of mobile devices with touch screens compared to typical computer applications controlled by a mouse. The significant potential offered by tablets should also be noted, as they are not only useful and portable, but also more affordable compared to laptops (Leoni, 2010). The acquisition cost of applications is very competitive and often includes the ability of a user to download a "light," often free, or trial version of applications. Fuegen (2012) identifies tablets as compact laptops with optimized operating systems, devoid of a physical keyboard, however, provide interactive capabilities via built-in functionality and the use of third party applications (mobile
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apps). Similarly, the term smartphone describes mobile phones which, besides the usability of voice data over cellular networks, have similar features and characteristics of tablets but with a smaller screen size.
Orlando (2012) commented that portable devices fit perfectly in the lifestyle of young children as they do not need to sit at a table or an office to use the device, they do not need to handle a mouse, while the interface offered with a single touch on the touchscreen is irresistible. Children now use the new technology at a younger age than ever before (Common Sense Media, 2011). A study including Australia, New Zealand, USA and Great Britain showed that most children ages two to five years are more able to interact with a tablet than to tie their shoes (Orlando, 2012). Moreover, the possession of smart mobile devices by children ages four to fourteen years old has doubled since 2005 (NPD Group, 2008).
According to a survey by the MDG Advertising Company (2012), preschoolers today make use of mobile devices more than ever in the past. The same survey indicated that more than half of all first time iPad and iPod device users are children under the age of five. However, not all researchers explicitly support the use of smart mobile devices by young children. Banister (2010) notes the lack of research on the use of tablets in education as they have only recently been introduced into the market. In addition, Marsh (2010) advocates the "dichotomy" that arises between the activities at hand and the technology supporting it. He claims that the emerging phenomenon of "digital activities" differs from that of the spontaneous children's play because the digital activity depends largely on, and is often limited by, the actual design of the software and hardware.
Learning via Mobile Devices
Sharples, Taylor and Vavoula, (2007) define mobile learning as any kind of learning that takes place in learning environments and areas which take into account the mobility of technology, the mobility of learners and the mobility of learning. Mobile learning through the use of tablets and smartphones presents new opportunities for strengthening the learning experiences in ways that simply other devices cannot achieve (Lam & Duan, 2012).
High-resolution screens allow tablet users to share static content and resources such as images and videos in an easy way. Most tablets have no phone features making them ideal tools for education since disruptive elements for the learners' attention such as incoming text messages or unwanted calls that are present in phones and smartphones alike, are absent. Tablets are able to offer the benefits of mobile applications in a broader context in all levels of education, not only as an affordable solution for one-to-one learning but also as a feature rich tool for work inside and outside the classroom.
Wakefield and Smith (2012) indicate that the technological analysts characterize the tablet as an ideal tool for all levels of education. After the introduction of tablets in the classroom of several schools in the United States, students report that they want to participate in learning activities due to the novelty of the medium, the visual characteristics and ease of use. In the classroom, tablets can be used to create text, audio or video notes. Students, independent of age, can store educational materials in a digital portfolio. Using a tablet, learning can be achieved through the active participation of students and the use of interactive activities and animations. Additionally, the attractive appearance of the working environment and the innovative touch interface are considered key learning facilitators for young children and students with learning disabilities. The characteristics of tablets including light weight, portability, touch screen, large icons, speakers, voice commands, zoom features and the cognitive simplicity are some of the key advantages that make them an affordable and more than efficient tool for the education of young children (Buchanan, 2010).
At least 1000 universities and colleges create and distribute free educational content in a format suitable for reproduction on smart mobile devices (MDG Advertising, 2012). According to McManis and Gunnewig (2012), this fact creates new requirements from the devices users as well as other opportunities for innovative services through these devices.
Several governments worldwide have given special attention to the development and implementation of mobile learning in order to improve teaching and learning effectiveness. Lam and Duan (2012) state that pioneers in this initiative are not only the governments of developed countries but also of developing countries that have begun to adopt policies and measures for the development of mobile learning. Characteristically, they indicate that the Royal Thai Government plans to provide tablets to 800,000 students of all educational levels by the next school year. As early as 2011, an initiative of the Indian government was to build the cheapest tablet in the world. It was sold to students at a subsidized price of 35 dollars, implementing the transition from the traditional to mobile learning in a faster and easier way. Simultaneously, the Turkish government announced that it plans to spend 8 billion dollars for the purchase of 15 million tablets, which will be distributed among school-age children throughout the country.
Finally, as reported by Kenny and McDaniel (2011), while overcoming past misconceptions that perceived the use of technology only as electronic gaming and play among students, many teachers and schools united are beginning to allow students to bring their own smart mobile devices into the classroom; especially since recent studies have shown that video games, including learning activities, have become an integral and essential part of the way in which young people learn.
Mobile Devices in Kindergarten Education
Egan and Hengst (2012) show that, even though educational software has been available almost 30 years now, the educational community of kindergarten education initially resisted the use of computers in the teaching of young children. Many educators, inspired by Piaget's theory of the developmental stages of children, considered that young children need only the physical activity and the ability to handle tangible objects in order to consequently achieve understanding of the various abstract concepts. However, since the mid-1990s, researchers found that virtual manipulations which are facilitated by computer software are similar to physical manipulations and therefore, the use of ICT could effectively support the learning process, particularly in mathematics, and the educational development of children as a whole (Yelland, 1998; Clements, 2000).
Nowadays, the use of ICT in the education of young children, as well as the development of educational software and relevant activities in preschool education, has already spread in many educational systems (Druin & Fast; 2002; Plowman & Stephen,
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2003; Zaranis, 2011; Zaranis, 2012). Research results point out the potential of ICT as a teaching tool (Segers & Verhoeven, 2005) and as a cognitive tool (Clements, 2000; Klein, Nir-Gal, & Darom, 2000). Such results seem to formulate a common agreement at the level of educational policy and school practices that promote the rational integration of ICT in early childhood education (NAYEC, 1996; Yelland, 2002; Plowman & Stephen, 2003; Bolduc & Lefebvre, 2012). Several studies indicate that computers are developmentally appropriate technological resources for pupils of preschool and primary school education and show how technology can be used to support and encourage the development and learning in these age groups (Musawi, 2011; Zaranis & Kalogiannakis, 2011a).
The integration of mobile devices into the preschool curricula aims at strengthening the interest of young learners and in enhancing their participation and cooperation with their classmates and teachers (Lindahl & Folkesson, 2012). According to Nix (2005) this type of learning attracts the interest of young students and it is considered both an enjoyable experience and an attractive learning environment. Rushton, Juola and Larkin (2009) point out the need for proper learning conditions and indicate that children learn more easily in positive and encouraging learning environments. These said learning environments allow the students to make their own decisions regarding which topic to study out of their own free will. Rushton (2008) states that the creation of a classroom environment that enhances student expression and selection opportunities provides a welcoming atmosphere towards helping children to learn at their own pace. Learning environments, which through various mediums provide differentiated teaching to young learners, allow children to be actively responsible for their learning, by simultaneously stimulating various areas of their brain thus resulting in better intellectual development (Rushton & Larkin, 2001).
The portability of tablets allows children and kindergarten teachers to make use of various locations in the classroom enabling the creativity and collaboration of small groups of students. Additionally, the sophisticated user interface of tablets allows both individual interaction and often, mutual interaction, depending on the application at hand, between two or more children (Wakefield & Smith, 2012).
Research findings (Abrams, 2009; Jenkins, 2009; Dickens & Churches, 2011 as cited in Peluso, 2012) correlate the effectiveness of learning through educational content delivery based on touch screen technology. A study by Chiong and Shuler (2010) in the United States involving iPod touch devices and audiovisual material properly configured for children ages three to seven years old showed that children drew remarkable gains in vocabulary and phonological awareness. Another recent survey by Bebell, Dorris and Muir (2012) on kindergarten students selected randomly to use tablets for the study of literacy, showed that infants who used tablets recorded remarkably strong performance in their phonological awareness and their ability to represent sounds with letters in relation to other children that did not.
Sandvik, Smørdal, and Østerud (2012) point out that as well as tablets are highly portable technology, learning can take place in a variety of everyday activities in the kindergarten, in a formal and informal context, inside and outside the classroom. As reported by McManis and Gunnewig, (2012), the new digital tools, such as smart mobile devices and their accompanying applications, can offer unique opportunities for kindergarten children to participate in useful and targeted activities; either individually or collectively with their peers and their kindergarten teachers.
Digital Activities and Children
The introduction of smart mobile devices in the daily life of children positively supports integration of digital applications in childhood education. There is a positive continuum between the everyday life of children and their school life. Moreover, it is important that the technological instruction offered by the school equips children with the necessary knowledge and skills for later life as well.
Digital learning activities on mobile platforms are very popular among students, especially at younger ages. Particularly, in the last few years, the accession bid of educational digital activities as teaching tools in the educational process has increased. The term electronic or digital activity refers to all categories of activities that are implemented through digital

technology. It concerns activities and games that run on large classic game consoles, specialized consoles, personal computers, smartphones and tablets (Chiong & Shuler, 2010). Researchers such as Klein et al., (2000), Lewin (2000), Segers and Verhoeven (2005), Comaskey, Savage and Abrami, (2009). Zaranis and Kalogiannakis (2011b) document the positive impact of digital applications in the educational development of preschool age children. Other studies show that electronic activities and games attract children and seem to stimulate them in a more constructive manner than the standards the conventional education has adopted so far (Klawe, 1999; Prensky, 2001; Zaranis & Kalogiannakis, 2011a).

Since the early 1990s, the trend has been that of edutainment. This refers to the blend of gaming and educating based on the logic of activities developed with clear educational orientation (Shaffer, Squire, Havelson, & Gee, 2005). The contribution of digital activities lies in the fact that they allow children to participate in worlds in which they learn to think, speak and act in new ways. Being rich in features, these virtual worlds constitute a very promising framework for learning, as children can play many different roles (Squire & Jenkins, 2003; Shaffer et al., 2005; Squire, 2006). Digital activities are considered to be particularly effective when they are designed to examine a specific problem or to teach a specific skill. For example, encouraging learning in the thematic areas of the curriculum such as mathematics, natural sciences and language where the specific objectives can be determined and when it is selectively developed within a context relevant to the learning activity and the specific target (Johnson et al., 2011).

For children aged three to five years old, educational digital activities often focus on readiness skills for the kindergarten, including reading (letter recognition, letter formation, correlation of sounds and letters, simple spelling), mathematics (recognition of numbers, formation of numbers, counting, grouping), thinking and reasoning skills, perceptual skills, daily life skills (hygiene), social skills, creativity and self-expression; as well as the understanding of concepts such as family relations, emotions, professions, etc. (Lieberman, Fisk, & Biely, 2009b). Lieberman et al., (2009b) distinguish the quality of digital learning activities in the following categories:

1. Well-designed activities—provide powerful interactive experiences which can enhance the learning of young children, fostering skills development, as well as their healthy development;

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2. Poorly designed activities—simple sedentary activities that contribute little in children's learning, skills development or their healthy development while potentially associated with obesity and poor physical condition;
3. Very poorly designed activities—can potentially cause considerable damage to children either through strengthening the aggressive or antisocial behaviour, approbating ethnic or transgender stereotypes and promoting bad eating standards.

A finding of several studies in children three to six years old correlates the benefits of using digital activities in different sectors (Figure 1).

More specific benefits are identified in the following sectors:

1. Learning—Digital activities can provide considerable educational services to children. Comparative studies have suggested that well-designed educational activities provide potentially more motivation and result in encouraging learning compared to traditional teaching methods (Swing & Anderson, 2008);
2. Cognitive skills—Using digital activities, children learn cognitive skills through repetition, as relevant studies have found improvements in operating memory, spatial ability, visual attention, etc. (Thorell, Lindqvist, Bergman, Bohlin, & Klingberg, 2009);
3. Social interaction—In preschool education, young children often engage in cooperative social interaction when playing digital activities (Christie & Johnson, 2009).

If materials for play and learning (including computer activities) are designed to indulge the interests and abilities of children, as well as their instinct to learn, students are more likely to develop and strengthen their initiative, their attention, their industriousness and their love for learning (Kagan & Scott-Little, 2004; Sadowski, 2006).

However, as pointed out by Garrison and Christakis, (2005) not all digital activities are developmentally appropriate or meticulously designed to enhance the procedure according to which young children play and learn.

Realistic Mathematics

The foundations of children's mathematical thinking form in the early years of their life. They gradually acquire their first experiences with time and space then with basic mathematical processes (measurement, calculation, sorting and comparison) (Gadzichowski, 2012). Research findings prove the existence of logical principles for counting from the age of three (Geiman & Meck, 1983). Other researches indicate that numbers and numerical phenomena sparks the intense interest of children from an early age (Zaranis & Kalogiannakis, 2012). Mathematics in preschool education is approached as a foundation that will help children learn about the world around them and to adapt to this world (Tzekaki, 2007; Zaranis, 2012). Egan and Hengst

Figure 1.

Benefits of the use of digital activities by young children.

(2012) report that a large body of research shows that the time spent in the kindergarten is a crucial period for children to develop fundamental numeric skills. In their research on the development of mathematical concepts in kindergarten, Clements and Samara (2007) concluded that the numerical abilities of children, developing before they enter the first grade of primary school, are the best predictors of their subsequent mathematical progress; more than other skills. When monitoring 200 students from the kindergarten up to the third grade, Jordan, Kaplan, Ramineni and Locuniak, (2009) concluded that the conquest of mathematical concepts in the kindergarten is positively correlated with the achievement of high mathematical performance at the end of the third grade.

Consequently, the teaching of mathematical concepts in the kindergarten has a huge significance, as it assists children to acquire the necessary mental condition and the necessary cognitive foundation for the systematic learning of "real" mathematical concepts, later on, in primary school. The aim of kindergarten is not just to teach mathematics but to initiate children in ways of thinking that characterize the mathematical science while providing a parallel realization of their social dimension.

For many years, the central theme for scientific debate was the formulation of a comprehensive mathematical theory towards interpreting phenomena related to the teaching and learning of mathematics. Such a theory aimed to provide educators with the ability to help students understand mathematics as a subject and as a tool for solving everyday problems. Over the last few years and in the context of the above considerations, various reform proposals for the teaching of mathematics have been developed internationally, posing their focus on the problem solving procedure. One of these proposals is the Realistic Mathematical Education (RME) which was developed in the Netherlands and supports the view that the same phenomena through which the mathematical concepts acquire content, should be used as a basis for a teaching process that aims to impart these concepts (Freudenthal, 1983).

According to Freudenthal (1983), mathematics is a human activity and therefore it must constitute a human value, must be close to reality of fact, be close to children and have a relationship with society. The central idea of the Realistic Mathematical Education is when saying, "I know math" really means "I know how to do math" (Streefland, 1991). The student becomes able to easily handle the mathematical language, to solve and construct problems, but mainly, to recognize mathematical concepts within specific situations. The term "Realistic Mathematics" refers to mathematics, which relate to problems of the real world as well as to phenomena, which appear in our daily life (De Lange, 1996). This form of mathematical education is characterized as "realistic" because it relates to the real world, in addition, to that special emphasis given in situations in which the students can envisage with their imagination.

The digital learning media can contribute to the learning of mathematical concepts by young children. Many aspects of early informal learning of mathematical concepts, such as enumeration, arithmetic problem solving, and spatial syllogism along with general geometrical knowledge, are developed dramatically during the preschool age. Therefore, digital technologies and accompanying software that teach early learning of mathematical concepts have received considerable attention from the scientific community (Lieberman et al., 2009a). The use of mathematics-related computer programs in preschool

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children leisure activities enhanced the mathematical knowledge of those children (Starkey, Klein, & Wakeley, 2004). Jordan et al., (2009) report in a similar study, that young children were able to develop mathematical skills and relevant reasoning and thinking skills while using developmentally appropriate mathematical software.

Today, the use of mathematical technology in kindergarten education and the first primary grades is a standard. It is included as one of eight Standards for Mathematical Practice in the Common Core State Standards for Mathematics (Common Core State Standards Initiative, 2010). The principles that underpin the Realistic Mathematical Education constitute a framework in which technology could offer significant assistance in the teaching and learning of mathematics (Clements, 2000).

A Proposal for Teaching Realistic Mathematics through Mobile Devices

Currently, within the Faculty of Education, Department of Preschool Education, University of Crete, systematic research is being carried out in order to investigate whether there are compelling benefits to using tablet computers in preschool education in an attempt to implement teaching reform proposals, such as Realistic Mathematical Education and the Natural Sciences. This proposition is carried out by creating educational applications for the mobile device platform in the form of digital learning activities, using specialized software (App Inventor). These applications intend to help preschool children in learning mathematics by following the principles of RME, focusing primarily on the existence of rich thematic frameworks associated with a child's experiences. For this reason, digital applications are combined with group and individual activities without the use of a tablet including board games, dice, activities with given and hidden objects, etc. This form of digital application was chosen because studies (Nix, 2005; Vavoula, Pachler, & Kukulska-Hulme, 2009) have shown that digital educational activities are not only appealing to the students' interest but are also considered to be a pleasurable pastime while establishing a new and attractive learning environment. Today's students are more likely to enjoy a learning experience embedded with digital activities because these games are based on the primordial form of learning "play and learn" from which they derive their benefits as an educational tool (Squire, 2006). Specifically, we have developed sixteen different activities that cover four levels of mathematical intervention (ground or zero, first, second and third) based on the principles of RME for preschool education (Van Den Heuvel-Panhuizen, 2008; Zaranis, 2011, 2012). The purpose of the next step of the research is to extensively implement such applications and other educational activities in kindergarten classrooms in order to systematically evaluate their integrity and educational use compared to the traditional method of teaching.

According to Van Den Heuvel-Panhuizen (2008), the knowledge of formal mathematical concepts may differ considerably from child to child when entering into kindergarten. To elabo-

rate, some of these children are quite familiar with simple counting while other children are not. Likewise, the extents to which they are capable of distinguishing between the different meanings of numbers may vary. The understanding of resultative counting children have developed upon entering to kindergarten differs significantly too. The somewhat variable initial situation in which the children's different initial levels of mathematical knowledge upon beginning the kindergarten could be described as the ground level. In general, an elementary numerical sense develops before and during the preschool education, in four general levels (Figure 2).
In addition to the pre-existing ground level, the three general levels that are developed during kindergarten education are as follows:
1. First level—The level of context-bound counting and calculating simple addition and subtraction;
2. Second level—The level of object-bound counting and calculating simple addition and subtraction problems where the objects are displayed and then hidden;
3. Third level—The level of pure counting and calculating addition and subtraction with the use of a missing variable.
The first level consists of context-bound situations in which pupils are instructed to count up to ten, organize numbers in the proper order, and make reasonable estimations or comparisons of numbers through concepts of more than, less than or equal too. Basic addition and subtraction problems are introduced at this level.
The relevant object-bound counting and calculating of the second level, which occurs in problem situations, focus directly on the quantitative aspect. In contrast to the first level, questions asking that the students count and state the number of given objects, in a given scenario, are presented and understood in the second level. This applies, however, only if the questions asked are associated with specific objects involving natural numbers. The second level also consists of activities in which pupils able to choose an appropriate strategy for solving simple addition and subtraction problems in which the objects at hand are displayed for a brief period of time before they are hidden.
The following are several examples of level two questions:
How many pieces of candy are in the box? How many chairs are there? How many people are waiting in line? Which box has the most sweets? How many of the seven candles will remain lit if the wind blows out three?
Finally, the third level consists of pure counting and calculating, questions using real numbers, rather than objects, focusing on the aspect of a missing variable. For instance, "What will remain if you subtract three from seven?" By previously hiding objects after a brief period of time (second level), children were obliged to use their fingers or other representations for more the complex tasks as the visual objects were being removed from the equation. This way, the enumeration is no longer dependant on the objects themselves, and is instead put into perspective using physical or mental representations of these objects. These representations can occupy different levels of abstract problems including the use of real numbers in addi-
Figure 2.
Levels of realistic education in kindergarten education.
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tion and subtraction problems with a missing variable.
As Van Den Heuvel-Panhuizen (2008) claims, this can occur by using a "birthday hat" but there are other possibilities, such as the kindergarten teacher asking the children to describe their age without using words. At the third level, children can represent the numbers one to ten using their fingers, as well as by using lines and dots. They are then able to use these skills for addition and subtraction activities. The use of numbers and the execution of the arithmetic operations used in the third level are not generally one of the targeted objectives of the kindergarten curriculum but there are usually children who are able to work at this level.
At the ground or zero level, students are instructed to use various portable applications. In one of the applications, for example, they have to place a certain number of chicks in their nest or to count the number of animals that appear on the screen (Figure 3). Variables are randomized numbers from one to ten for each round.
The other tasks at hand are to help a climber pass over mountains, counting each mountain on the way and to a help a monkey down a ladder, counting each rung as it descends (Figure 4).
At the first level the children are expected respond to questions such as, "How many of the apples in her basket can Helen buy with 5 euro if each fruit costs 1 euro?" Another example problem is, "Each child needs a ticket to enter the castle. How many children will not be able to enter?" (Figure 5).
At the second level the children are asked to solve problems such as, "Place as many eggs as you would like in the fridge," after the selection is made the eggs are concealed. "Helen came and took 1 egg from the fridge. How many eggs are left?" (Figure 6).
Finally, for the third level applications, pupils are told to answer more complex questions such as, "Johnny has 5 pieces of candy. His uncle came and gave him some more candy. Now Johnny has 8 pieces of candy. How many pieces of candy did his uncle give him?" Alternatively, in the case of subtraction, the children are told that a monkey has cut a number of bananas from the tree. Some of his animal friends suddenly appear and

Figure 3.
Examples of ground level portable applications.

Figure 4.
Examples of ground level portable applications.

Figure 5.
Examples of first level portable applications.

Figure 6.
Example of a second level portable applications.
eat an unknown number of those bananas. The remaining bananas are displayed and the children are asked to find the number of bananas that were eaten (Figure 7).
Considering the categorization of Naismith et al., (2004), we can claim that from a pedagogical point of view, the mobile applications we created follow either the behavioral or the constructivist teaching method. Some applications follow the behaviorist perspective by providing fast and immediate feedback while utilizing reinforcing learning elements like the confirmation of the correctness of an answer. Others utilize the constructivist perspective as they provide open type learning experiences for the students. All applications provide feedback both in the case of error as well as the successful completion of an activity. This feedback is in both visual and audible form so that it is easily understandable by pupils. Simple feedback is provided through graphic characters (e.g. the appearance of a happy or sad face), while more complex feedback is provided by the narrator through an audio message (e.g. "Try again," or "Well done, you did it!").
The applications were created using the application development software App Inventor (AI) and were tested on the Android operating system. Android was preferred over the iOS operating system (Apple) mainly because of its flexibility for the creation of applications as well as the existence of a very large variety of available devices with various price ranges and features. AI was first announced as a small project of Google Labs at the end of 2010. In late 2011, it was moved to the center of mobile learning in MIT (Massachusetts Institute of Technology) for public use as open source software (http://appinventor.mit.edu/). AI is a new free online block based visual development environment for creating mobile applications of the Android operating system (Hsu et al., 2012).
The design process of the educational software had three main components:
1. The research and development of the original applications prior to their implementation in kindergarten;
2. The modification of the existing applications during the pilot test in order to respond more effectively to the needs of children and kindergarten educators;
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Figure 7.
Examples of third level portable applications.
3. The creation of new applications that meet the demands of preschool educators.
The produced software should have the following characteristics: 1) It should be user-friendly so that children can easily handle it without any special assistance from an adult; 2) It should not require any reading and writing knowledge for its use so that it is suitable for the preschool age; 3) It should combine animation and sound. At the preschool age, sound is essential, as speaking is the only way to give instructions to children.
In order to examine the children's knowledge in the comparison, classification, matching, structured counting, efficient counting, basic addition and subtraction and general knowledge of numbers, we will most likely use the Utrecht Early Mathematical Competence Test (for children aged 4.00 - 7.05). This criterion was designed by a team of scientists from the department of special education at the University of Utrecht in the Netherlands, in order to be used for the assessment of early mathematical competence in preschool age children (4.06 - 7.05) (Van de Rijt, Van Luit, & Pennings, 1999). In 2008, the weighting of this criterion in Greece led to the expansion of the age range for six months. We will use all of the worksheets from this test, because we assume that it is suitable for the purpose of our research.
Conclusion
As already mentioned, the purpose of this paper was to provide a better understanding of the characteristics and the effect of ICT and mobile learning in the context of preschool education. In addition, we will try to examine children's knowledge by comparing the learning outcomes of teaching using specially designed applications for smart mobile devices (tablets) and the thematic teaching in mathematics; specifically targeting "Realistic Mathematics", for children ages 4 to 6 years attending Greek kindergartens.
Although the research is still in a preliminary stage, initial results of the statistical analysis of pre tests (which were held in a small sample of kindergarten students in specific geographic regions of Greece) allow us to assume the following:
1. The tablet aided learning compared to the traditional teaching method produces better learning outcomes for the students;
2. The teaching of Realistic Mathematics with the use of educational software for tablets produces better learning outcomes in relation to teaching based on the Greek Cross Thematic Curriculum Framework (CTCF, 2003).
With great interest, we look forward to the learning outcomes that result from the teaching intervention using a large and properly weighted sample of kindergarten students. We intend to expand our research in kindergartens located in all geographical regions of Greece in order achieve a larger possible representative sample distribution.
Discussion
Quite often, the debate on the use of ICT in mathematical education is limited by and focused on the tool itself rather than on the activities and in the specific problem situation that the student is presented. In contrast, the activities and applications that we have created refer to real world problems and situations. The next stage of the research will be carried out in kindergarten education in order to systematically evaluate the integration of ICT and its educational benefits compared to the traditional method of teaching.
This study is preliminary in its nature, as our research serves to demonstrate a potentially greatly beneficial installation of

mobile learning in childhood education. With the execution of a systematic study, we believe that our findings will contribute to the acceptance that the positive effects derived from the use of mobile devices and ICT in kindergarten in general, do not exist only in the presence of a computer in the classroom, but also by the way in which the technology is used by teachers. The primary concern, therefore, is how to maximize the benefits of these new instruments in order to enhance, rather than hinder, children's ICT experiences for the benefit of learning.

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Design And Understanding

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We explore ways in which access to technological tools can support new approaches to the design of pedagogical tasks and at the same time is providing us with new insights about the nature of mathematical understanding. We describe a novel approach that situates the challenge of designing pedagogic tasks in the same framework as that of locating mathematical understanding. An example of the use of this design approach is explored.

Introduction

In this paper we explore ways in which access to technological tools can support new approaches to the design of pedagogical tasks and at the same time is providing us with new insights about the nature of mathematical understanding. It is not new to argue that technology offers opportunities to re-think both the content and the implementation of the curriculum, not only allowing the same curriculum to be taught and learnt in new ways, but fundamentally challenging the current sequencing of some topics. However, we describe a novel approach that situates the challenge of designing pedagogic tasks in the same framework as that of locating mathematical understanding. Such a framework promises to point designers towards the search for purposeful tasks which are linked to aspects of mathematical understanding that are under-researched.

The Current Context

More than twenty years ago the Cockcroft Report into the learning and teaching of mathematics in the UK expressed the widely held view that 'Mathematics is only 'useful' to the extent to which it can be applied to a particular situation' (Cockcroft, 1982, para 249). Although this view is not unproblematic, the issue of 'applying' mathematics, or of making links between school mathematics and the 'real world', continues to concern mathematics educators, researchers and curriculum designers. At first glance, the need for mathematical learning to include an understanding of how mathematical ideas can be useful may seem obvious. However, even a brief look at the typical content of the school mathematics curriculum makes it clear that the view which is presented of the uses of mathematics in the real world can be highly contrived:

• Ravi bought a pack of 30 biscuits. He ate one fifth of them on Thursday. He ate one eighth of the remaining biscuits on Friday. How many biscuits did he have left?(DfEE 1999) 11

There has been a considerable amount of research in mathematics education into the difficulties which children have in applying mathematical knowledge, and particularly in combining mathematical and 'real-world' knowledge appropriately when tackling problems set in real-world contexts (see for example Boaler, 1993). Cooper and Dunne's (2000) detailed study of pupils answering such contextualised questions in tests has indicated that in order to engage appropriately with the mathematical focus of such questions, pupils have to understand complex but implicit rules about the extent to which they should attend to features of the realworld setting. This suggests that the apparent difficulties which pupils have in 'applying' mathematical ideas may in fact be a product of pedagogic approaches and assessment.

A different approach to linking school mathematics to the 'real world' is to design tasks that offer 'authenticity' by resembling out-of-school activities, such as setting up a classroom shop. However, the structuring resources provided by this situation will be very different from those offered when the child really goes shopping: the prices of items on sale may be simplified to an unrealistic extent, getting the correct change will not be of the same level of concern, and even with an element of roleplay, the social interactions of the classroom shop will not provide the structure and constraints experienced in a real shopping trip (Brenner, 1998). In the classroom shop the 'shopper' cannot make real choices about what to buy, or how much to spend, or indeed choose not to buy anything. In Walkerdine's (1988) words: " everything about the task is different from shopping … the goal of the task is to compute the answer rather than to make a purchase" (p. 146).

In contrast to these school-based approaches, we turn to situated cognition research which has studied mathematical practices in the real-world contexts of shopping and employment (for example, Lave and Wenger, 1991, Nunes et al, 1993), often referred to as street mathematics. From such studies we identify the purposeful nature of the activity as a key feature which may be transferred to the school context, rather than the superficial characteristics of the setting (Ainley, Pratt & Hansen, 2006). Lave and Wenger (1991) claim that in out-of-school contexts, "learners, as peripheral participants, can develop a view of what the whole enterprise is about". This overview of the purposes on the activity is generally absent in school mathematics. A significant difference between learning street mathematics, and learning school mathematics, is that the ideas that you learn in school do not enable you to use mathematics to get things done in the ways that adults do: indeed children rarely see adults using the sorts of mathematics that they learn in school. We argue therefore that an important challenge for pedagogic task design is to create tasks which are purposeful for learners within the classroom context, rather than attempting to make links to (supposedly) real-world settings. Whilst the use of technology is not essential

to such design, it has enormous potential to provide opportunities for learners to use mathematical ideas in powerful and meaningful ways.12
One source of ideas that has influenced our response to this challenge can be found in the Constructionist literature (Harel and Papert, 1991). Although the Constructionist ideals have evolved out of early work with the programming language Logo, we have found them illuminating in a more general sense. Many authors in this tradition have reported on the significance of allowing students control over their own decisionmaking as apparent when they are allowed to build a project through the use of Logo. In our interpretation, the key notion here is that the students are making decisions for themselves in ways more akin to engagement with street mathematics than is typically found in classrooms, where control is often strictly in the hands of the teacher. Tasks involving building and mending computer-based artefacts are specific examples of how control can be transferred from the teacher to the student.
Purpose and Utility
We conjecture that engaging purposefully in the use of mathematical ideas in a welldesigned task leads to learning which is different from that which might arise when practicing an associated technique or exploring why that technique works in more traditional classroom tasks. Based on our previous research, we have developed a framework for pedagogic task design which offers a new perspective on the issue of creating opportunities for pupils to learn about the ways in which mathematical ideas are useful, using the linked constructs of purpose and utility.
Purpose, as we use the term, refers to the perceptions of the pupil rather than to any uses of mathematics outside the classroom context. There is considerable evidence of the problematic nature of pedagogic materials which contextualise mathematics in supposedly real-world settings, but fail to provide purpose to which the learner can relate, either in terms of the overall task, or the ways in which mathematical ideas are used within it (see for example Ainley, 2000; Cooper and Dunne, 2000). The purpose of a task, as perceived by the learner, may be quite distinct from any objectives identified by the teacher, and does not depend on any apparent connection to a 'real world' context. The purpose of a task is not the 'target knowledge' within a didactical situation in Brousseau's (1997) sense. Indeed it may be completely unconnected with the target knowledge. However, the purpose creates the necessity for the learner to use the target knowledge in order to complete the task, whether this involves using existing knowledge in a particular way, or constructing new meanings through working on the task. Movement towards satisfactory completion of the task provides feedback about the learner's progress, rather than this being judged solely by the teacher (Ainley et al, 2006, Ainley and Pratt, 2005).
Within such purposeful tasks there is the possibility of creating opportunities to understand the utility of mathematical ideas. We define the utility of a mathematical idea as how, when and why that idea is useful. Traditional approaches to teaching mathematics in school address instrumental understanding of procedures, and relational understanding of mathematical concepts (Skemp, 1976), but generally fail to address the utility of these ideas. The pedagogic tradition, embodied in textbooks around the world, is to begin with procedures and relationships, and to address 13 utilities as the final stage in the pedagogic sequence (if at all). We conjecture that utility is not merely an application of a concept but a separable dimension of mathematical understanding, alongside the instrumental and relational components. The potential of technological tools to allow learners to use powerful ideas before they need to learn the detail of how to perform calculations greatly expands the possibility to introduce ideas of utility early in the pedagogic sequence.
In order to illustrate the place of utility in mathematical understanding, we shall reflect on approaches to teaching proportion. We choose this not only because it is a highly significant concept in the school curriculum but also as it was the focus of a recent research experiment.
An illustrative example based around the concept of proportion
Proportion lies at the heart of mathematical curricula and is commonly regarded as one of the most significant challenges for the child's cognitive development during the secondary phase of education. In fact the concept of proportion is, like all powerful ideas, a synthesis of many component notions and it is part of the design challenge to decide which of those notions to foreground when offering experiences to the learner.
In the UK curriculum for the early years of secondary schooling (DfES, 2001), children are expected to:
"Compare ratios by changing them to the form m : 1 or 1 : m.
For example: The ratios of Lycra to other materials in two stretch fabrics are 2 : 25 and 3 : 40. By changing each ratio to the form 1 : m, say which fabric has the greater proportion of Lycra." (p. 81)
Using an instrumental approach, a child might memorise a procedure for tackling problems of this kind in which the 2 is divided into the 25 to give 1 : 2
1
12 and
similarly 3 : 40 is transformed into 1 : 3
1
13 . A further routine might then be needed to
decide which of these ratios indicates a higher proportion of Lycra. Of course, there are many places where the child's memory could fail, leading to errors of one type or another.
Skemp (1976) has contrasted such an approach with one which is based on relational understanding. Then a child would have a range of strategies that could be used, which might include reducing the ratio to a unit as above but might also include recognising that other approaches are equally useful. For example, having reduced the first ratio to 1 : 2
1
12 , the problem can be solved by multiplying this by 3 to give 3
: 2
1
37 and making the comparison with 3 : 40. Indeed the child who has relational understanding might be able to use one method to confirm another.
Most teachers would recognise the superiority of relational understanding over instrumental understanding for most mathematical situations. Nevertheless many teachers and text books appear to adopt approaches which are likely to reinforce instrumental rather than relational understanding. We argue that a major reason for 14 this is that the kinds of pedagogic tasks which are regularly used, both in teaching resources and in assessment, are ones which can be completed using instrumental approaches, and provide little incentive to explore the concept relationally. Furthermore, the Lycra task, although set in a 'real world' context, fails to provide any purpose for making the comparison of the composition of the two fabrics, and thus offers no opportunity for pupils to appreciate the utility of ideas of ratio and proportion.
We now consider an alternative approach to the design of a pedagogic task for proportion. Our starting point is to consider contexts in which the utility of the mathematics becomes apparent, and to use such a context to design a task which has a purposeful outcome for pupils.
In a recent experiment, we gave the following task to 11-12 year olds:
Children in a primary school want to make a 'dolls' house classroom'. Use the piece of furniture you have been given to work out what size they should make some other objects for their classroom.
Each pair were given an item of dolls' house furniture, and also had available measuring tapes and a spreadsheet. The role of the spreadsheet here is highly significant: it provides the calculating power to allow pupils to work with real data, however 'messy' the results, and at the same times offers a visual space in which to record their explorations.
The Dolls' House Classroom task focused on scaling, a key idea in ratio and proportion. The outcome of the task was to be a set of instructions for another group of children to make items for the dolls' house classroom. The activity of comparing the item of dolls' furniture with its full-size equivalent in the classroom involved measuring and discussion, as the pupils decided on which were the most important measurements to use.
We report here on the activity of one pair of boys. Initially the boys tried to relate the task to their own experiences. One boy told the teacher about how his grandfather used to make dolls' furniture. The other talked about scaling in maps in response to the teacher's mentioning of the term scale factor. From an early stage, the boys questioned the nature of the task that they had been set. (Figures in brackets indicate time elapsed in minutes.)
[6:06] Is this real? Are a Year 6 class really going to do this?
The researcher admitted that this was not actually going to happen.
[6:35] Why can't they just buy the dolls' house?
What do we make of these questions? Are they challenges that suggest the boys are resisting the invitation of the teacher to engage with the problem? If so, it would be hard to explain the subsequent activity, which was marked by the boys' considerable intent and persistence. Rather, we believe that these questions indicate a process in which the boys were beginning to take ownership of the task, 15
When students take ownership of a task, the levels of engagement can be very high; it is our belief that the opportunity to make choices is influential in helping students to make a problem their own. Furthermore, a well-designed task will also enable students to follow up their own personal conjectures when they try to make sense of the task.
The boys used the spreadsheet confidently as a tool to support their exploration. Their spreadsheet shows several different attempts at ratio. In one set of cells, they divided the height of the real table by that of the dolls' table (68.5 / 4.3 = 15.93). But when it came to the width of the table, they divided the dolls' table by the real table (5.5 / 134.2 = 0.040983607). In another part of the spreadsheet, they divided the width of a real shelf by the width of a real table (75.5 / 134.2 = 0.562593). Each of these calculations has possible utility for their task but whether any particular approach has explanatory power depends on how exactly the boys wanted to use the result and what sense they could make of the feedback from the spreadsheet. The nature of the task allowed them to explore all three routes, and to compare them, rather than following a route defined prescriptively by the teacher.
Such explorations enabled the boys to construct meanings for the divisions being carried out on the spreadsheet. The spreadsheet both handled calculations which would have been beyond the boys' competence, and displayed a complete record of their work, allowing them to focus on whether the ratio was actually useful to them in their task. The purposeful nature of the task produced an emphasis on how the scale factor might be useful, admittedly in a situated narrative, rather than on technical aspects of calculating a scale factor.
We claim that these boys were connecting to what we recognise as the concept of proportion not through instrumental or relational understanding but by developing a sense of the utility of scaling for pursuing their problem. We use the term "their" advisedly. The construction of a utility for scaling was dependent upon them adopting the problem for themselves and this would only be achieved if they found the problem purposeful.
We see the design of tasks that are likely to be purposeful and yet at the same time are likely to yield utility-based understanding as key to resolving the teacher's predicament of how to promote deeper understanding. At the same time, we note how poor the Lycra problem was, despite its apparent reference to a real world scenario, as a vehicle for promoting such understanding.
Final comments
Within our framework, purpose and utility are closely connected. Indeed we see purpose as an element of designing for abstraction whilst we frame utility as an element of abstraction in context. Appreciation of the utility of mathematical ideas can best be developed within purposeful tasks. A focus on purpose in isolation may produce tasks which are rich and motivating, but which lack mathematical focus. However, if the tasks are designed appropriately, learners may use a particular mathematical idea in ways that allow them to understand how and why that idea is 16 useful, by applying it in that purposeful context. This parallels closely the way in which mathematical ideas are learnt in out-of-school settings.
It is generally acknowledged that pedagogic approaches that focus mainly, or exclusively, on instrumental learning of procedures will result in impoverished learning. However, even approaches that emphasise relational learning tend to give little attention to utilities. We suggest that this results in mathematical knowledge becoming isolated as weak connections are made to the pupil's existing knowledge of the contexts in which it may be usefully applied.
Pedagogic design based on the framework of purpose and utility inverts the pedagogic tradition of school mathematics by placing the emphasis primarily on the utilities of a new mathematical idea, and the use of technology greatly increases the scope for this. Thus the learner is able to construct meanings that are shaped by strong connections to the application of that idea: in Lave and Wenger's terms, to develop a view of what the whole enterprise is about.
This inversion is made possible by the power of technology to offer opportunities for using a mathematical idea before you learn about its procedures and relationships. Technology affords the possibility of pursuing purposeful tasks by working with mathematical tools, instantiated on the screen, whilst coming to appreciate the utility of those tools, in ways which lead to powerful mathematical learning.
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Mathematics Investigations: Towards curriculum design and implementation
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A teacher's perspective on the nature of mathematics, the potential of a technology, and the training that they receive determines their effectiveness in the integration of that technology. How to teach for transfer is yet another crucial piece of teacher knowledge for creating and sustaining technology-based learning environments. Sense-making, self-assessment, and reflection on what worked and what needs improving are teaching practices congruent with metacognitive approaches to learning. These practices have been shown to increase the degree to which students transfer their learning to new settings and events. The course Mathematics Investigations is designed to deal with current demands of digital technologies integration and inquiry-based approaches to teaching and learning mathematics. The course has five components (of unequal weight): Problem sets, Reflections, Self Evaluations, Readings and Final Presentation. As a final product, each student compiles a Digital Resource File that consists of five problem sets, a final presentation, and additional resources relevant to their future work. Students are guided and encouraged to develop their fluency in dynamic geometry, spreadsheets, selection and use of virtual manipulatives, and other Web resources. University-wide available courseware is used to support complementary online activities, group discussions, and the virtual classroom.
Each problem set focuses on one mathematical idea or concept and begins with an open-ended, real-life-related and challenging problem. The problem set consists of 5-7 additional problems scaffolding "down" the main concept. The final product demonstrates a gradual development of a selected concept through a sequence of word problems. Although students are encouraged to collaboratively discuss their work, each student works on a unique collection of problems and submits their work individually. Each problem set utilizes technology tools in an essential way. At the end of the problem set, a required metacognitive reflection reports about students' thinking during the process of problem set design. Two self-evaluations per semester each consist of (a) a self-report to inform the instructor how the student is progressing in the class; (b) dispositions (both for student and instructor), (c) grade records (spreadsheet kept by student), and (d) student's plans for future work related to the course.
Weekly discussions are carried out through the use of online courseware. The classroom learning network includes discussion groups, Reflective pods. Each pod consists of approximately four students. Students are asked to reflect on certain questions that are supporting ongoing classroom activities. On a rotating basis, one member of the group summarizes. Summaries are brought in for face-to-face class discussions. More details about this course are available at the following address: http: //www.education.wichita.edu/alagic/319spring06/319spring06.asp. 19
Instrumented techniques in tool – and object perspectives
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The aim of this paper is to report from a study of the role of instrumented techniques in the students' learning process. The paper analyses an episode from a case study of students solving differential equations in a CAS environment. The analysis demonstrates how tasks can be designed with the aim to encourage the students to change between the perspective of tool on a mathematical conception and the perspective of object on the conception. Reasons are given in the paper for the assertion, that changing between these two perspectives supports the instrumental genesis as well as the conceptual development.
Instrumental genesis and instrumented techniques
The French theory of instrumental genesis is based on the idea that an artefact, for example a CAS calculator, does not in itself serve as a tool for the student. It becomes a tool, referred to as an instrument in this notion, only by the student's formation of (one or more) mental utilisation scheme(s). The term instrumental genesis denotes the process in which the artefact becomes an instrument. (Drijvers and Gravemeijer 2005 pp 165-169). The formation of utilisation schemes and the building up of instrumented action schemes proceed through activities in 'The twosided relationship between tool and learner as a process in which the tool in a manner of speaking shapes the thinking of the learner, but also is shaped by his thinking'. (ibid. p 190). The French framework is underlying the theory of instrumental genesis: according to Luc Trouche the scheme concept, encompassing utilisation schemes and instrumented action schemes, was introduced by G. Vergnaud as 'an invariant organization of activity for a given class of situations. It has an intention and a goal and constitutes a functional dynamic entity. In order to understand its function and dynamic, one has to take into account its components as a whole: goal and subgoals, anticipations, rules of action, of gathering information and exercising control, operational invariants and possibilities of inference within the situation. (Trouche 2005 p 149)
The formation of utilisation schemes and instrumental action schemes, thereby, is pivotal for the instrumental genesis. Since the utilisation schemes are mental, they are not directly accessible for study and analysis. The concept of instrumented techniques, taken as the external, visible and manifest part of the instrumented action scheme, therefore, is of special interest. An instrumented technique is 'a set of rules and methods in a technological environment that is used for solving a specific type of problem.' (Drijvers & Gravemeijer 2005 p 169). An instrumented technique includes conceptual elements as far as the technique reflects the schemes. This leads me to two crucial points: 20
A student's development of an instrumented action scheme can be studied by inquiry of the student's development and use of instrumented techniques related to the scheme.
Development of mathematical conceptions cannot be studied if use of technology is considered separate from the student's other activities.
The first point stresses the importance of empirical studies of students' work. The second point opposes my research to the standpoint, that teaching may be performed independently of what tools the students have at their disposal. This is in line with Jean-Baptiste Lagrange who stressed, that 'the traditional opposition of concepts and skills should be tempered by recognising a technical dimension in mathematical activity, which is not reducible to skills. A cause of misunderstanding is that, at certain moments, a technique can take the form of a skill.' (Lagrange 2005 pp 131-132).
Tool – and object perspectives.
During a recently concluded research project (Andresen 2006) on the teaching of differential equations in upper secondary school in laptop-classes, I have constructed and tested a conceptual tool, flexibility. This notion of flexibility encompasses the tool – and object perspectives subject to this paper. In the following, the definition of flexibility is reproduced without further explanations. For a discussion of details and examples, see (Andresen 2006). Definition: The flexibility of a mathematical conception constructed by a person is the designation of all the changes of perspective and all the changes between different representations the person can manage within this conception. The changes of perspective considered are divided in three groups:
a) Dualities of perspectives intrinsic to mathematics: 1. Local – global, 2. General – specific 3. Analytic- constructive
b) Dualities of perspectives linked to the construction of epistemic knowledge: 4. Process – object, 5. Situated – decontextualised
c) Dualities of perspectives linked to the construction of pragmatic knowledge: 6. Tool – object, 7. Model – reality, 8. Model of – model for
Three main representations are considered: graphic representation, analytic representation (or formal language), and natural language. A fourth, called technical representation (or computer language) is included as well, caused by the use of laptops. There is no symmetry between the four representations.
The conceptual tool flexibility serves to capture and conceptualise certain learning potentials experienced by teachers and students, for instance, when using the laptops in a modelling-context. One element of flexibility with special relevance for the theme of this paper is changes between a tool perspective on mathematical conceptions and an object perspective on the same conception. The notion of a tool perspective on a mathematical conception is opposed to a pure skill understanding of mathematical activity and the notion includes the technical dimension mentioned by 21 Lagrange. The duality composed by a tool perspective on a mathematical conception and an object perspective on the same conception appears to be appropriate in problem-solving settings, in the same way as Anna Sfard's process – object duality (Sfard 1991) is useful to frame aspects of learning mathematics. The term tool perspective here refers to the mathematical processes, carried out to serve a concrete purpose. This resembles the use of the term tool synonymously with instrument in contrast to artefact. This notion of tool perspective on a given mathematical conception is in accordance with Régine Douady's definition: 'We say that a concept is a tool when the interest is focused on its use for solving a problem. A tool is involved in a specific context, by somebody, at a given time. A given tool may be adapted to several problems; several tools may be adapted to a given problem.' (Douady 1991 p 115)
The distinction between the pair of process – and object perspective and the pair of tool – and object perspective can be illustrated by the following example: a tool perspective on the conception of derivative of a function could be the derivative seen and used as a means for finding out how the function changes over time. The corresponding object perspective could be the derivative, characterised or categorised by its merits and demerits when it was assessed in the context of solving a specific problem. In contrast, a process perspective of derivative could be focusing on the actual determination or calculation of the derivative in question. The corresponding object perspective could be the derivative, generally characterised or categorised by its qualities within in a structure of functions. Mathematical activities, then, are considered from a tool perspective when they are part of a (problem solving) technique, regardless of its being instrumented or not. The generation of the instrument, then, is in a crucial way linked to the change to object perspective: From the object perspective corresponding to a tool perspective, a unit is considered which may encompass intension, goal, conditions and prerequisites, restrictions, function and dynamics. Like in the case of process – object, the object perspective implies an encapsulation of the conception as a tool. So for the student, the development of an object perspective gradually leads to master the techniques in which the conception is embedded and to complete the formation of the connected instrumented schemes.
Change of perspective to support learning
Basic to the research, which lead to the construction of the conceptual tool flexibility was the idea that learning is supported by alternating diving into the process of solving a problem and taking a distant look upon the activities and efforts (Andresen 2006). Edith Ackermann presents this idea in (Ackermann 1990) as a mean to integrate, roughly speaking, Jean Piaget's and Seymour Papert's views on children's cognitive development. In her paper, Ackerman combines the Piaget'ian construction of invariables with Papert's situated learning in her dynamical approach to cognitive growth. Flexibility incorporates this basic idea in the form of the aforementioned changes within dualities of perspectives on given mathematical conceptions. One aim of the research project was to inquire how the teacher can provoke and support the 22 students' change of perspective in both directions within these dualities, and to interpret the role of such changes for the students' ongoing mathematical activities. In the actual case, a group of three students used several instrumented techniques during three episodes. The episodes were analysed and interpretations of the techniques' role in the students' learning process are presented.
Case
The case presented in this paper is part of the data from my Ph.D. project. These part of the project's data were produced from a small scale, qualitative inquiry which encompassed classroom observations in four classes, 50 lessons in all, field notes, students' written reports and teaching materials prepared for a sequence of teaching differential equations from a dynamical point of view using the software Derive. A group of three students were working with a differential equation model of the transformation of cholesterol in the human body. The students were in third year of an experimental class in upper secondary school, where all the students had their own laptops at their disposal from first year on. The CAS software Derive was installed on the laptops. This case is based on group's work during one lesson which was video recorded. The students' written report and the teaching materials were examined in relation to the analysis of the case.
The students were preparing a written report on a series of tasks, which concerned exploring a model for transformation of cholesterol, presented in the textbook. The tasks aimed to stimulate the students' learning about equilibrium point and general as well as specific solutions to differential equations. Further, the tasks concerned relations between general and specific solution and connections between analytic and graphic representation, both mediated by computer language. In the case, the group was in an early phase of their work, concentrating on this text from the teaching materials (Hjersing et.al. 2004):
... another handy form is:

$$\frac{dC}{dt} = !(8.2)$$

(Bubba changes his diet at t0 = 0, with C0 = 180 mg/dl, the new daily cholesterol intake is E = 250 mg/day.)
If we let t0 = 0 be the time where Bubba starts eating at the grill and if Bubba's level of cholesterol at that time is supposed to be C0 = 180 mg/dl, then Bubba's cholesterol level is expressed:

0.1(265)
(0) 180
dC
C
dt
C
= !
= (8.3)
Tasks 23
1. Find the equilibrium point for (8.2) and analyse the variation of the sign of the
right side.
Is the equilibrium point a sink or a source? Use the answers to sketch (in hand)
more solutions to this differential equation.
2. Find the general solution to the differential equation (8.2) (Show calculations)
3. Find the specific solution to the initial value problem (8.3)
4. If Bubba keeps this high cholesterol level diet for a very long time (one year or
more), at what level will he end? Explain how you reach the conclusion?
During the case, the students used several instrumented techniques: First, in episode
1, they used the Derive command RK
1
to obtain a graph of the solution to the
differential equation (8.2) with the initial conditions t0=0, C(t0)=180. In episode 2,
they used their compendium of formulas supplied by paper and pencil techniques to
find the general solution to the equation. The solution was typed into the computer
and the students used the Derive command VECTOR to get a family of graphs of
solution curves, as kind of an intermediate between general and specific solution. To
answer the next question, they substituted the initial values in the formula for the
general solution, calculated the constant d (determined by the initial values) and
substituted it into the expression. To answer question 4. in episode 3, the students
repeated graphing the same solution curve as they graphed in the first episode, but
this time based on the expression obtained from the preceding answer. Their answer
to question 4., then, was based on visual inspection of this later graph.
Episode 1
To answer question 1., the students sketched the graph and wrote:
' The function nears 265, so, 0.1 is the rate of growth and 265 is the point of
equilibrium.
The right-hand side is positive if his start C is below 265 and negative if it is above. The
equilibrium point is a sink, that is, a stable equilibrium.
The students made at least one guess before they reached this result: their first try in
the written report was a RK command, which was impossible to graph because the
capacity of the computer-memory was exceeded. So, their strategy implied a trial-and
error use of an instrumented technique that can be described as follows: 1) substitute
the left side from the differential equation into the RK command, 2) type in the
names of the independent and the dependent variables, 3) type in the initial values
and 4) try to find values for the x-increase and the number of tangent-segments,
which allows for: 5) graph the solution. Apparently, the students identified the
horizontal asymptote by inspection of the graph and then graphed the function y=265
to verify the result visually. Afterwards, the equilibrium point was identified with
this horizontal asymptote. So, since the graph with its asymptote was used to

1
stands for the 4.order Runge Kutta method of numerical solution 24
determine the equilibrium point, the graph with asymptote was in this case seen in a
tool perspective and it was obtained using the instrumented technique sketched
above.
The second part of the answer must be obtained from analysis of the differential
equation. Therefore, the graphic method used in the first part of the solution serves to
link graphic and analytic representations closely.
Episode 2
To answer question 2, the students wrote:
General solution:
The equation for cholesterol is of the type dy/dx=b-ay and may be solved as follows: (b
is a constant)
First, the students used paper and pencil and they looked in their compendium of
formulas to find the general solution. They tried to identify the type of equation.
The paper and pencil technique implied to 1) identify the type of equation, 2)
recognise it in the compendium, 3) identify and substitute the actual values of the
constants in the expression for the solution. The students typed the results into the
computer stepwise, as they were asked to show the calculations. Apparently, they
then wanted to graph the result, which is, obviously, impossible. The students used
the command VECTOR to graph a family of solution curves, which could be seen as
kind of an intermediate between general and specific solutions. The report reveals,
that they did not completely manage this instrumented technique at that stage of their
work so they must have made more than one trial: The command VECTOR(C = 265-
....) would not result in graphs as shown, as far as "C = 265..." is evaluated logically.
To succeed in graphing that family of curves it is necessary to delete the "C=".
Intermezzo
The students answered question 3 by 1) substituting the initial values in the formula
for the general solution, 2) calculating the constant d and 3) substituting it into the
expression. Though, the dialogue in the group revealed no clear signs of having
developed a general perspective of solution to the differential equation (Andresen
2004).
Episode 3
When starting to answer the last question in this task, question 4, it was clear from
the dialogue in the group that the students did not try to estimate the result, based on
the preceding answers. Apparently, the fact that the students found equilibrium for
the general solution earlier in the lesson did not 'ring a bell' when they were asked to
argue for their latest result.The students spent some time in the group discussing how
long time they had to take into account. Two of the three refused to consider the fact,
that they found an asymptote.
In the final report, the students wrote:25
"Based on the graph we conclude, that the equilibrium point does not change even if the
starting point is different. The general as well as this solution therefore near to the same
equilibrium point and whatever long he keeps the high level, the equilibrium point does
not change."
In the final version of the report, the students simply graphed the specific solution
from episode 1 once more. Since the window was changed it is obvious that they regraphed it. The written comment reveals that the students did not expect the
coincidence between the equilibria points for the general solution and the specific
one in question. This fact questions the students' adoption in advance of the general
perspective on solution to differential equation. In line with this the last statement, in
my interpretation, reveals unfamiliarity with the conceptions of asymptotic behaviour
and of equilibrium.
Conclusion
In the case, the students' work with the task concentrated on two mathematical
conceptions, represented by the example of one differential equation: 1) equilibrium
point for differential equations and 2) solutions to differential equations. The
equilibrium point was closely connected to asymptotic behaviour of the solution
curve. So, an instrumented technique of solving and graphing the solution curve,
encompassing the RK command and seeing the curve with its asymptote in a tool
perspective, was used by the students to build and strengthen their conception of
equilibrium point.
Determination of the general solution was carried out with a combined paper &
pencil- and computer-instrumented technique, where the last part concerned change
to graphic representation. Especially, the computer-instrumented part of the
determination served to link between a family of solution curves, on the one hand,
and the specific solution curve, examined earlier, on the other hand. The family of
solution curves served as pseudo-graphing the general solution.
The experiences of asymptotic behaviour and of coincidence between the asymptotes
of these solution curves, provoked by the task, supported the students' change of
perspective on the two conceptions in question: Realising that "whatever long he
keeps the high level, the equilibrium point does not change" is one step to adapt an
object perspective on equilibrium interpreted by horizontal asymptote. Likewise, the
family of graphs are visually convincing about the fact, that the general solution
should encompass the specific solution.
The case illustrates genesis of Derive-commands as an instrument in an ongoing
process. The first use of RK had the character of trial and error in episode 1 (omitted
from the data presented in this paper). The fact, that changes to graphic
representation were not carried out with full routine, is revealed in the report in the
case of VECTOR. But it was very clear, that especially the possibilities of graphing
shaped the students' thinking. So, the tool influenced: 1) Their strategy, which
implied to choose asymptote as the tool for finding the equilibrium, 2) Their thinking
of general solution by making it tangible by pseudo-graphing into a family of 26
solution curves and 3) Their idea of verifying the asymptotic behaviour by visual
inspection and comparison with the graph of y=265.
The idea of provoking changes between tool and object perspective can be realised,
for example by the asking of questions and tasks which involves ready-made
procedures as well as self-developed instrumented techniques for solving modelling
problems. The analysis of the case shows how the idea can facilitate proceeding of
students' work as part of their learning process.
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Connecting Grade 4 students from diverse urban classrooms:
virtual collaboration to solve generalizing problems
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We have been investigating the potential of a web-based collaborative workspace,
Knowledge Forum (Bereiter & Scardamalia, 2003), to support Grade 4 students in
generalizing with patterns as part of our research in early algebra. Our hypothesis
was that incorporating Knowledge Forum, with its underlying knowledge building
principles, might offer an authentic platform for developing students' mathematical
discourse. We present analyses of the Knowledge Forum database from a recent
study in which three diverse urban classrooms were linked electronically to
collaborate on solving a series of generalizing problems. Analyses of contributions to
the database revealed that the opportunity to work on a student-managed database
supported students in developing a community practice of offering evidence and
justification for their conjectures. The database also provided students with the time
and software capability to revisit and revise their notes and to develop a level of
discourse that elicited high-level mathematical problem solving.
Research context the mathematics of patterns and functions
Patterning activities have been heralded as an important foundation for the
development of mathematical functions. It has been proposed that patterning
activities can support students in understanding functional relationships and provide
a rich context for generalizing. In fact in recent years the inclusion of patterns can be
seen in elementary curriculum documents and text books in many countries.
However, substantial evidence from past research suggests that with current
instruction the "route from perceiving patterns to finding useful rules and algebraic
representations is complex and difficult" (Noss et al, 1997). Further, even when
students find rules, they do so with an eye to simplicity rather than accuracy, commit
to their first conjecture even in the face of invalidating data, and do not attempt to
support or justify their conjectures (e.g., Stacey, 1989; Mason, 1996; Lee, 1996).
The research that we have been conducting has focused on a study of new
approaches to support students in working with patterns. In line with suggestions of,
for example, Mason (1996) and Lee (1996), we have been working to broaden
students' conceptualization of patterns as a means of understanding the dependent
relations among quantities that underlie mathematical functions, and further, as a
means of developing students understanding of generalizing by seeing "the general
through the particular, and the particular in the general" (Mason, 1996).
An important part of our work with grade 4 students has been our investigation
of whether Knowledge Forum (Bereiter & Scardamalia, 2003), and its underlying
knowledge building principle of epistemic agency, can promote inquiry-based
mathematics learning. Knowledge Forum (KF) is a networked multimedia 28
community knowledge space created by community members. Our conjecture was
that incorporating KF would allow students access to multiple pattern "seeings"
(Mason, 1996) and that the discourse structure would provide an authentic context
for collaborative problem solving and extended discussions that would necessitate

the provision of evidence and justifications. Although the scope of this paper does not allow for a discussion of KF, we will briefly describe how KF works and outline the theoretical principle of epistemic agency, which we believed would contribute to our research goals.
Knowledge Forum: How Does It Work?
Knowledge Forum was developed as an online forum for discussion and knowledge building by learning theorists Bereiter & Scardamalia based on their early work in intentional learning (please see Bereiter & Scardamalia, 1989). When students work on KF they have the opportunity to contribute individual ideas or to build onto the ideas of others by writing and posting "notes". A note (Figure 1) contains a space for composing text, and metacognitive scaffolds designed to encourage students to engage in theory building while they write their notes (Scardamalia, 2002). These scaffolds include
my theory, I need to understand, new information,
a better theory, and putting our knowledge together.
Students can also use the graphics palate to create
illustrations, or scan drawings, function tables or
photographs. When notes are contributed to the
database, the notes are automatically labeled with
the author's user name (usually first and last initial)
and the note's title.
Students' notes are contributed to problem
spaces called "views'. Figure 2 presents a view of the Perimeter Problem, one of six generalizing problems used to create the six views in our database. The small squares represent student notes and the connecting lines represent discussions created as students read and respond to each other's
contributions. Some of these notes have
small circles that are referred to as "buildons", i.e., responses to notes posted by other students. A unique feature of KF
that distinguishes it from other CSCL
(computer supported collaborative learning)
environments is the physical layout of the
problem space as students contribute their
ideas. The database views are continuously
evolving interactive discourse spaces,
where each thread of conversation is documented, webs of interchanges graphically displayed, and collective understandings captured as they progress. 29
Epistemic Agency and Higher Order Mathematizing
While we believed that establishing a database to house discourse and promote collaboration would contribute to our research goals, the theoretical frame underpinning KF, particularly the emphasis on student agency and the centrality of student ideas, was a fundamental reason for incorporating KF in our study. Bereiter and Scardamalia (e.g., Bereiter, 2002; Scardamalia, 2002) use the term epistemic agency to characterize the responsibility that the group assumes for the ownership and improvement of ideas that are given a public life in KF. In this discourse structure, it is not the teacher who asks for clarification and revision of the ideas or conjectures that the students have contributed, but rather the students themselves who take on this responsibility with an eye towards moving the theorizing forward. We wondered if this responsibility would result in an increased engagement in the language of and disposition for mathematical discourse.
We developed three specific research questions to allow us to determine how the principle of epistemic agency was manifest in this study, and the extent to which it underpinned the progression of students' mathematical thinking.
Our research questions
1. Will students provide evidence and justifications for their conjectures, and will this disposition develop as students gain experience working through problems on KF?
2. Will students revise their ideas?
3. Given that the database is entirely student-managed, will students take on the responsibility of developing their mathematical understanding by moving thinking forward through working collaboratively?
Procedure
Students from three different Grade 4 classrooms (8-9 year olds) in two different schools (n=51) were linked electronically and invited to collaborate in solving six generalizing problems, for which students were asked to discern a functional relationship between two sets of data and express this as a "rule". There were three pairs of problems, which were matched for structural similarity and increasing difficulty. The first two problems (Linear Pair 1) both had an underlying functional rule of y=mx+b. The second two problems (Linear Pair 2) had an underlying function of y=mx-b. The final two problems (Quadratic Pair) were based on the quadratic function, x
2
-x/2, which posed an unfamiliar challenge to the students since they had previously worked only with linear functions. The KF database was available to students over an eight-week period, and on average each student had approximately 30 to 45 minutes per week to work on-line. The time that the students had to work on the database varied depending on the classroom and the availability of computers. No teacher or researcher posted notes on the database, so that it was clear to students that it was their responsibility to work together to find and "prove" the solutions to these problems. 30
The data for the present study come from the 247 notes that the students in the three classes posted in response to the six generalizing problems. Each note was read and coded by the researcher (first author) and one or more research assistants.
Evidence and Justifications
All notes were rated for the level of evidence and justification offered and were coded as Level 3 (high), Level 2 (medium), or Level 1 (low). Notes designated Level 1 were those in which only a conjecture was offered - My theory is that the rule is x4 -4. Notes coded as Level 2 offered a conjecture with some sort of explanation of problem solving strategy. I figured out the rule and it is the number times 4 (the four sides) minus 4(because you use one twice at each corner). Notes coded as Level 3 were those that offered a conjecture and evidence within the context of the particular problem, and included multiple representations, and/or a detailed account of a problem-solving strategy.
The note below (PP19) titled "I got it!" was offered by a student, JF, as an explanation for the Perimeter Problem (Steele & Johanning, 2004). In this problem students are asked to find a rule that will allow them to ascertain how many squares are in the perimeter of an nxn grid. This note is as an example of a note coded as Level 3 because it includes a proposed rule and contextualized explanations based on this student's particular way of perceiving the problem:
.
I got it! - JF
I got x+x+(x-2x2)and i tried it out for a lot of them and it worked:
5+5+(5-2x2)
5+5=3x2
5+5=6
10+6 = 16 so that means the rule is x+x+(x-2x2) and that's it! if you seperate the question into two you have 5+5 and +(5-2x2) so let's focus on 5+5 and that equals 10 and that means that we can do this
So now we can focus on this part: =(5-2x2) so 5-2=3 and the space left in the square is 6 and 3x2 is 6 so it's x+x+(x-2x2)
Our analyses revealed that there was an
even distribution of the three levels of
notes throughout the database, with 82
notes coded as Level 1, 81 notes coded as
Level 2, and 84 notes coded as Level 3.
However, when we looked at the level
note as a function of the three problem
pairs, we found that the proportion of
[illegible]
high-, medium-, or low-level notes was different for each of the problem pairs with the highest proportion of Level 3 notes posted for the third pair of problems (40%), and the lowest for the first pair (13%). Our analysis of the database revealed that the majority of students moved through the problems in chronological order starting with the first pair of linear problems and finishing with the quadratic problems. Thus, as students became more experienced at working in KF, they became more sophisticated and more mathematically oriented in their offering of evidence and justification. Table 1 shows the levels of notes as a function of Problem Pair.
The students established a community practice of routinely offering notes that included justifications for their answers using language, tables of values, symbols and/or images to justify their conjectures.
Revision of notes
An important feature of KF is that notes can be revisited and revised at any time. To answer our second research question we counted the number of times that students revised their own notes in each problem view. Each time a student added or modified an idea in their note, it was counted as a revision. There were 194 revisions, with a spread of 0-11 revisions per note. Although it might seem self-evident that revisions would lead to a higher level of note, we wondered about the nature of these revisions given that this was a student-managed database. We discovered that there was a relationship between number of revisions and level of note. The Level 3 notes went through an average of 3 revisions (M=3.1, min 0- max11), the Level 2 notes went through an average of 1 revision (M=1.4 min 0 max 5) and the Level 1 notes went through on average less than 1 revision (M=0.6, min 0 max 2). This indicated to us that when students revised their notes, they did so by improving upon and adding to their previous ideas, thus adding to the overall level of mathematical knowledge.
When we examined notes that had been revised by the author, we could see that the impetus to revise was either based on other students' responses to an offered conjecture, or based on the student's own dissatisfaction with their solution. To illustrate the former, we present the following discussion, which began when a student, SR, posted a note containing his strategy for solving the Perimeter Problem in the form of a table of values. In response to SR's note, two students posted notes (PP24 and PP25) requesting that SR provide a rule to explain his data. SR then posted a revised note in which he described his rule using diagrams, a table of values, and an explanation for his rule.
PP23 My thinking - SR
square grid shaded squares
3x3 8
4x4 12
5x5 16
6x6 20
PP24 rule? - SV 32
I need to understand what is your rule, evidence and how did you get your rule.
PP25 Finish please - BA, MD,
We do not think that is a rule , so can you explain the rule as well as you can show it. We also agree with you because your T-table is right. But your rule needs improving.
PP26r4 My thinking - SR
3x3
8
4x4 12
5x5 16
6x6 20
My theory is that the rule is that the rule is x4-4
Rule: Output equals Input x4-4Evidence: It is x4 in the rule because the number of sides a square has is 4. It is -4 because when you are multiplying 4 each corner you are repeating the number of squares one more.
SR was prompted by others to refine his ideas. In contrast some students recognized that their original conjectures contained only a partial solution, and used the time to think provided by working on an asynchronous database to continue to revisit and revise their initial ideas. The following note entitled "Relationships" was written by a student, NS, who posted a solution to the Perimeter Problem but recognized that her solution was not fully explained. As she commented in her note; "I think I still have to think a little more to explain my theory".
PP19 Relationships - NS
My theory is that the Output # is = to the input # times 4 - 4.
My evidence is that 3x4=12 and -4 is 8 which is the output. You need to multiply the input x 4[only one side] because without multiplying you wouldn't get 12. Then when you minus another # besides 4 the output wouldn't be 8. The same rule applies for all the other numbers.
Like: 100x4= 400-4=396 10x4=40-4=36 14x4=56-4=52
I think I still have to think a little more to explain my theory.
When we analyzed NS's contributions we could see that she revised this original note

a total of five times over the course of three weeks. In her final revision (PP19r5), posted three weeks after her original entry, she contributed an addition to her original idea that included a more conceptual orientation to the problem structure.
PP19r5 A better theory - NS 33
You need to x 4 because you need 4 sides to make a square. Like 3x3 means 3 is the length and the other 3 is the width so one length or width x 4 = to the whole perimeter.
This kind of revision and rethinking was typical of many of the efforts of other students as well.
Progressive Mathematical Discourse
In knowledge building communities members make progress not only in improving their personal knowledge, but also in developing collective knowledge through progressive discourse (Bereiter & Scardamalia, 2003). We found many instances of students making concerted efforts to move the understanding of the group forward. A numerical count revealed that 65% of notes were written in response to other students' notes, indicating that students were reading each other's contributions and building onto the ideas of others, rather than simply posting individual ideas. There were many instances of exchanges in which individuals extended the ideas of others. Also, there were strings of discussions that revealed commitment to the notion of moving the discussions forward to include broader issues of mathematical understanding. To conclude we present two discussions taken from two different problems, both of which involved students from all three classrooms.
The first is from the Perimeter Problem. This discussion began with a solution posted by a student, AW, (PP 38) which she titled 'Eureka!'. The solution strategy that she proposed involved calculating the total number of squares in the grid, calculating the total number of squares in the centre of the grid, and subtracting the centre blocks to find the number of squares in the perimeter.
PP38 Eureka! AW
for the 5x5 question you do 5x5=25 the square of 25 is 5 and you minus two from the square and square that then minus it from your original number and you have your answer!
First i drew the five by five grid and there was nine in the middle to take away - 3x3=9
so then i figured out a 6x6 square was 36 and i know that inside there would be a 4x4 square to take away so the difference between 6 and 4 is 2 - so it was 36-16=20
nxn=nsquared -(n-2)squared -
so minus (n-2)squared from nsquared
In response to AW's note, SI offered a note in which he proposed a different rule for the problem based on multiplying one side of the grid by 4 then subtracting out the overlapping corners .
PP40 Another rule - SI
I have another rule for you and it is the output x4-4.In the rule it is x4 because
there is 4 sides in a square. It is -4 because when you multiply 4 you are 34 repeating the
corners twice so you -4.
In her response to SI, (PP41), AW postulated that there may be more than one solution to the problem.
PP41 2 rules - AW
but there might be two rules because we got the same answer so i think there is more then 1 way to figure the problem out
Another student, GA, then questioned AW's rule not on the basis of whether it yields the correct answer, but rather on the basis of elegance. She titled her note "Both right?"
PP42 Both right? - GA
I agree with you and disagree with you because you've got the answer but in a complicated way. I disagree with you because there's an easier way than taking the square of 25, subtracting 2 from it and square that and then subtract that from your original answer. I got the rule times 4 - 4 because a square has 4 sides and you don't count the corners twice. I agree with you because for the first few questions you got it right.
At this point in the discussion AW became firm in her conjecture that there is more than one solution to the problem.
PP43 Two Ways - AW
Why can't there be two ways. There are different ways to do lots of different problems i think you can have two ways nxn=nsquared -(n-2)squared -
so minus (n-2)squared from nsquared works and x4-4 works
This last discussion comes from the Handshake Problem. This problem, which has been shown to be difficult for much older students (e.g. Cooper & Sakane, 1986), asked students to find the quadratic rule to determine how many handshakes there would be if everyone shook hands with everyone else in any size group. Initially many students came up with a recursive numeric pattern, as the note HP5 exemplifies. However, in contrast to findings of other research with older students, a number of notes were posted by students who realized the limitations of this recursive approach, and questioned one another as to how to find the explicit functional rule. As M and J state in their note (HP8), "the thing about math is to figure out the fastest and most accurate way to do things."
HP5 Follow the Next Output Number - TH
If there is 3 people there are 3 handshakes. If there is 4 people there are 6 handshakes. So I think that it is like 2+1=3 that is the next handshake output. So 3+3=6 that is the next handshake. My rule will be the input+output=next output.
35
Number of people Number of handshakes
2 1
3 3
4 6
5 10
HP8 M and J's Theory - MT
I was wondering could you do this rule for 149 378 people? Because the thing is in our theory you have to know the number of handshakes before the one you're doing...but the large numbers are so big it would take forever to figure it out! The thing about math is to figure out the fastest and most accurate way to do things.
HP9 I need to Understand - VT
I need to understand if there is another way to get the answer? I want to solve how many handshakes would 10 people do without making a Tchart.
HP10 Thinking - SR
My theory is that the rule is the number x the number - 1 divided by 2. I think that it is the number x the number - 1 because a person can shake with the number of people 1 less than the person because the person cannot shake with itself. It is divided by 2 because 2 people make a handshake.
Discussion
Throughout the database we found evidence of what Bereiter and Scardamalia have termed epistemic agency. Students revised their own notes, and encouraged others to revise and reconsider their initial theories. Epistemic agency was also seen in students' disposition for offering not just solutions for problems but offering evidence and justifications as well. Furthermore, our analyses revealed that the level of justifications increased as the students gained more experience on KF. By the third pair of problems students routinely offered evidence to support their theories and frequently contextualized their justifications within the structure of the problem. Finally, it was clear in this student-managed database that students worked collaboratively not just to find the solutions of the problems given, but that their discussions broadened to include such themes as the elegance of finding explicit functions from recursive numeric patterns, and the possibility of multiple rules for mathematical problems.
How our work addresses the themes of the ICMI Study 17
Although our work is aligned with the general questions of classroom connectivity in particular Theme 7, in addition our research has raised several interesting issues that relate to other themes of ICME Study 17. 36
Theme 2 - In assessing the notes posted in the database, we have come to understand that KF has the potential to capture significant moments of learning.
Theme 3 - Not only does KF allow teachers to step back and assess student thinking, whilst simultaneously allowing the progression of ideas to become the responsibility of the students, teachers in our study were also introduced to a new view of student learning in general and of mathematics in particular. Some questions that have arisen centre on how to support teachers as they implement a knowledge building pedagogy, and how to organize classrooms to optimize a knowledge building pedagogy.
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Providing mathematics e-content
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Mathematics learning seems to be hard and exhausting task for many learners. Mathematics educators and teachers always try to stimulate the public and specially students for studying mathematics.
Certainly ICT is an effective tool for providing an interesting atmosphere for mathematics learning. Using this tool, one can make some virtual spaces such as exact diagrams and figures, attractive animations, and most important, making games and parametric programs to provide mutual interactions between learners and teaching media, such that they can change the parameters and see the results in figures or in the processes of the programs and much easier understand the concepts.
Isfahan Mathematics House(IMH) was trying to organize content provider teams of these specialists and professionals as its member: Mathematics educator, Mathematicians, Scenarists, Graphic experts and Programmers and multimedia experts. The team was making up some mathematics contents, but it faced to a big problem. It was the lack of communication between these people since many of them don't understand others with different background. For example the art experts don't understand mathematics and vice versa.
As a solution we tried to train some "interpreters" who can understand or have more feeling of both sides, and finally some successful results raised. In this article we are trying to report these activities with many useful experiences for all interested in the process of providing mathematical e-content. 38
Designing Didactical Tools And Microworlds For Mathematics Educations
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This paper reports on the extensive design activities by the Freudenthal Institute over the last few years in the field of small didactical tools and microworlds (java applets). New technologies have led to new ideas on visualising and learning mathematics. The effort made in several development and implementation projects have resulted in a collection of robust and well-tested didactical tools. Many of these applets have found their way to the practice of mathematics education and many schools work with these new tools. More research is still needed though (and planned) to obtain a detailed image of the possibilities and constraints of these new tools and their role in longer learning trajectories.
Introduction
The objectives of the Freudenthal Institute are research and curriculum development for innovation and improvement of mathematics education. From the very start of the introduction of the computer in mathematics education the possibilities of this medium were seriously investigated, and software development has always been practised at the institute. About 8 years ago, new technologies and programming tools that facilitate faster computer graphics led to a glut of new ideas for visualising and new ways of learning mathematics and creative programmers and designers produced some quite exciting prototypes. Of course this was just the first stage in the development of robust, well-tested and researched didactical tools. In the next stage further development, field testing and refinement were performed in several research, development and implementation projects, in real educational settings and in close

co-operation with teachers. A considerable collection of applets have found their way to the practice of mathematics education and there is much appreciation for these new tools in schools. Yet more research is needed (and planned) to obtain a detailed image of the possibilities and constraints of these new tools, especially regarding their role in longer learning trajectories.
To give an insight in the design activities and the development process, some paradigmatic examples of applets are described. First a distinction is drawn between three different kind of applets. The borders between these three categories are not very rigid, but this classification helps to structure the underlying design choices.
• Applets that offers a 'virtual reality'. These applets are used for representing and simulating real-world objects and processes that form the basis of mathematical reasoning. 39
• Applets that facilitate the use of 'models'. These applets offer interactive models that can be helpful in building and understanding the more abstract mathematical objects and concepts.
• Applets that offer a mathematical microworld. In these applets mathematical objects like formulas, equations and graphs can be constructed and transformed.
For each category one or more examples are described and used to illustrate and discuss typical issues and problems in the design process.
Theoretical Framework
The theoretical foundation is mainly based upon the theory of Realistic Mathematics Education (RME). The main idea of RME is to create opportunities for students to come to regard the knowledge they acquire as their own knowledge. Therefore, contextual problems which students recognise as relevant and real, and which also evoke productive solution strategies, play an important role. In RME, the intended goals are built progressively upon the students' informal ideas and strategies (Freudenthal, 1991).
For RME to work, it is necessary to know how students model new situations. Students are confronted with problem situations for which they do not have the appropriate models at their disposal; i.e. models which describe possible structures or patterns in the situations. At this point, theories on symbolising - individual as well as social aspects - are useful (Gravemeijer et al., 2002).
Applets that offer a 'virtual reality'
In the RME theory, real-world situations and experiences form the starting point of mathematical activities. Educational designs should for that reason contain carefully chosen real-world situations that are suitable for being mathematised. Software that simulates these situations can be used to recall old experiences and extend these with new ones by purposefully provoked interactions.
3D-Geometry has shown to be a very fruitful subject for this approach. We developed a number of applets on this subject for students of all levels.
The term 'virtual reality' is possibly somewhat misleading. In common virtual reality software the aim is to create a 'world' that should be experienced to be as real as possible. In the applets the aim is to make a firm connection with real-world experiences, but also to make a step to more model-like representations, in which certain mathematical concepts can emerge. The design challenge lies in finding a balance between giving the user freedom in his constructions and explorations in the virtual environment and imposing constraints to guide the user to the intended experiences. For example, the block building environment (figure 1) gives the user freedom in making his own constructions, but the environment also enforces a cubic structure that draws the attention more easily to orthogonal co-ordinates as a means to model space. 40
A quite different example of this category is the applet Fruit Balance (figure 2). It offers a real pair of scales that can be used to compare the weights of three different pieces of fruit. The task is to find the weight of all of them, when one is given. It is a nice example of the RME approach. A meaningful problem is introduced, without offering ready-made mathematical tools like standard solving algorithms for a set of equations with multiple variables, but offering a tool to explore the problem situation. The often informal strategies that students put forward can become the start of a learning trajectory on solving equations, based on 'guided reinvention', one of the important aspects of RME. Of course the most important thing is the design of such a trajectory, in which the student's explorations, reasoning and solutions are shared, discussed and generalised to other problems. Otherwise the applet activity would be just a nice puzzle without a substantial role in the learning of mathematics.
Applets that facilitate the use of 'models'
Geometrical shapes and objects are not only used in the mathematical domain of geometry. Often they appear as model for representing more abstract mathematical concepts and processes. A nice example of such a model is the number line that can be used in arithmetic. Processes such as addition and subtraction can be modelled by making step or jumps on the number line. For a developer of it-tools is a challenge in making those models dynamic, and in developing interactions or games in which especially the dynamic processes can be made visible and controllable. The applet "Jump tool" offers such a dynamic and interactive version of the more static representations of the number line.
Figure 1
Figure 241
Another example of a geometrical way to model mathematical objects is the 'area model' for algebraic quadratic expressions. Variables are used in the width and height of rectangles, and quadratic expressions represent their areas. Several applets were designed using this model. 'Geometric Algebra 2d' is one of them (figure 3). In this applet, line
segments of both constant and variable length can be combined to compound line segments and rectangles. By constructing these rectangles, the resulting algebraic expressions for the area are produced. Other, equivalent expressions can be made by splitting and joining the rectangles. Some interactive game activities were designed, in which students have to make factorisations by combining a number of small rectangles to a single one (figure 4).
Although students liked to work with these applets, it appeared that their skills in manipulating the rectangles did not automatically enhance their abilities with and insight in algebraic manipulations. Explanations of these shortcomings can be found in the RME theory. Especially the part on emergent modelling is useful (Gravenmeyer, 1994). In the RME view it is important that formal mathematics can be developed as a tool for solving problems that are real for students. One should
separate
join split
split
completely
join
samen
separate
Figure 3 42
start with investigating phenomena and problem situations that can be structured and solved by the mathematics that has to be learned. In working on these problems models can be used as mediating tools that bridge the gap between informal solving strategies and the formal mathematics. So a model is more than a (visual) representation of a piece of abstract mathematics. First, it has to be developed as a tool for structuring a real situation or problem. The strong relation between the rectangles in the applet and the algebra is obvious to experts, but for learners for whom algebra is still a unknown area, the world of rectangles can become isolated and not transferable. This is especially the case if there is no proper attention for the problems for which the model, and in the end the algebra, offers a generic tool. The sketched model development requires a carefully designed learning trajectory of which the applet activities are just a part.
Another applet, 'Algebra Arrows', is based on the well-known 'machine' model for functions and formulas, in which functions are represented as input-output machines. In this applet the designer has tried to facilitate the model development, as described above, more extensive. The applet offers students a tool to build their own models to structure and solve problems. It has several levels of application and is meant to facilitate a step by step development of the function and formula concept and to get acquainted with commonly used mathematical objects and representations (Boon, 2005).
Applets that offer a mathematical microworld
The third category contains applets that work on formal mathematical objects, like formulas, graphs equations etc. In this sense they are comparable with mathematical
Figure 5
Figure 4 43
tool like the graphic or symbolic calculator or CAS. The difference is that the applets are didactically designed and do not aim at merely carrying out the standard operations and algorithms. Most of them allow interactions and transformations on the objects that are not common in the standard mathematical tools. They often have a built-in task for students to perform. Some of the routine work is performed by (CAS) facilities of the applet, other parts, always carefully chosen, are left for the student to do, supported by proper feedback. In this way it is possible to focus on certain concepts and skills.
An example of an applet in this category is 'Solving equations with the coverup strategy' (figure 6). Using this applet students can solve a certain type of equation by selecting a part of the expression with the mouse and giving it a value. (Figure 6 shows a solving scenario.) The applet doesn't impose a blind algorithm. It lets the student interact with the expression and gives insight in its structure. With growing insight, it becomes possible to make larger and more efficient steps in the solving process.
In the design of these environments it is important to find a balance in directing the student to desirable skills and giving room to personal strategies and exploration. When the latter is missing the applet becomes just a training tool for ready-made algorithms. For understanding the features of this kinds of tools, it is important to reflect on the role of (solving) algorithms in mathematics education. In a mechanical approach of mathematics education, algorithms are seen as the starting point of a learning trajectory. The learning objective is a proper application of these given algorithms. In the RME approach algorithms themselves and the understanding of how they work are products of the learning trajectory as well. So it is important to work on problems that involve mathematical objects in a way that permits explorations and student's own alternative strategies. Of course there should be guidance and negotiation towards the known and most efficient strategies, but not by imposing them from the start. Another important requirement of the RME approach is that the elementary mathematical objects that are the 'building blocks' of the activities in the applets are 'real' for students, that is to say they already have a meaningful foundation, based on processes of a lower order. 44
Challenges and future development
In the different development and research projects carried out at the Freudenthal Institute, the applets discussed in this paper to have additional value in RMEdesigned learning trajectories. It was noticed too that a proper design of the whole learning trajectory is vital for exploiting the benefits of single applets. Our division in three different categories also defines more or less their possible places in such trajectories.
Especially learning trajectories with applets that help students in developing the founding models for 'inventing' the formal mathematical tools are interesting, but also difficult to design. One of the problems is that applets that support model development often tend to impose a ready-made model that is not flexible enough to evolve. For applet designers it is a challenge to exhaustively explore the dynamics of this new medium and apply them to overcome this problem. For development researchers the challenge lies in the design of convincing learning trajectories that integrate these new tools.
Applets are distributed over the world wide web, which makes them accessible quite easily. But the web offers other features as well, such as easy to use and placeindependent communication and registration facilities, that can also be used to enhance education as well. At the Freudenthal Institute an environment was developed that can be used by schools and individual students to store the results of their applet activities. It has been noticed that working with the applets could be fleeting because of the lack of persistence of the achieved results, the impossibility to reflect on them later and to share them with the teacher or other students. In a new planned research project on tool use, these new storage and registration features will be integrated with the applets and other activities in a new and innovative learning arrangement for mathematics.
Note
Figure 6 45
At www.fi.uu.nl/wisweb/icmi2006 a collage of some of the Freudenthal applets is shown. The entire collection can be found at www.wisweb.nl
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Developing a joint methodology for comparing the influence of different theoretical frameworks in technology enhanced learning in mathematics: the TELMA approach
TELMA (KALEIDOSCOPE) European Research Team
2
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This contribution deals with the work of the European Research Team TELMA (Technology Enhanced Learning in Mathematics) of the Kaleidoscope network towards understanding the role played by theoretical frames in design and research in that area, and building tools to improve communication between researchers from different cultures. We present two facets of TELMA work: a "cross-experimentation" project in which each TELMA team experimented with an Interactive Learning Environment (ILE) for mathematics designed by another team; and the design of a methodological tool for systematic exploration of the role played by theoretical

frames in the design and analysis of uses of ILEs. We focus on the methodological dimension of this work, showing how we employ the construct of didactical functionalities as a means of comparing and integrating the research conducted by the teams. We provide some preliminary results of the joint experiment and use of the methodological tool.
Introduction
This contribution originates from TELMA, a European Research Team (ERT) established as one of the activities of Kaleidoscope, a Network of Excellence (IST-507838) supported by the European Community (www.noe-kaleidoscope.org). The contribution reports on the work developed within TELMA for analysing the influence of different theoretical frameworks in the design and/or use of digital technologies for shaping mathematics teaching and learning activities and discusses some of its outcomes that we think of interest for this ICMI Study. It is especially related to theme 4 of the ICMI Study: "Design of learning environments and curricula". Indeed it addresses some of the research questions raised in this theme: "How can theoretical frameworks be helpful for understanding how design issues impact upon the teaching and learning of mathematics?", "What kind of mathematical activities might different technologies and different theoretical

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University of Paris, France; National Kapodistrian University of Athens, Greece; Consiglio Nazionale delle Ricerche, Italy; MeTAH and Leibniz - IMAG, France; University of London, UK; University of Siena, Italy. 47
backgrounds shape and how can learning experiences (including the tools, tasks and settings) be designed to take advantage of these affordances?", and, last but not least, "Which methodologies can be developed and applied to understand and compare different approaches, theoretical frameworks, and backgrounds?".
Background: Kaleidoscope and TELMA
Kaleidoscope's central aim is to address the lack of harmonised research in the field of Technology Enhanced Learning (TEL) in Europe by integrating various existing initiatives and research groups. The aim is to develop, on the one hand, a rich and coherent theoretical and practical research foundation, and, on the other hand, new tools and methodologies for an interdisciplinary approach to research on learning with digital technologies at a European level. The network is doing this by supporting a range of integrating actions, including European Research Teams (ERT) such as TELMA. ERT are integrating activities, which aim to network European excellence through specific research challenges. The key idea of creating an ERT is to stimulate the mutualisation of knowledge and know-how of a number of recognized research teams on the identified issues, and to favour the construction of shared scientific policy, building complementarities and common priorities.
TELMA is specifically focused on Technology Enhanced Learning in MAthematics. It involves six European teams with the aim of building a shared view of key research topics in the area of digital technologies and mathematics education, proposing related research activity, and developing common research methodologies. Each team has brought with it particular focuses and theoretical frameworks, adopted and developed over a period of time. Most of the teams have also contributed learning environments integrating digital technologies for use in mathematics learning, designed, developed and tested in accordance with their own theoretical perspectives
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. We will refer to these as Interactive Learning Environments (ILEs).
The starting phase of TELMA was very challenging, requiring six teams with different backgrounds, work methodologies, and ILEs, to begin to share knowledge, developing a common language and common topics of interest. This demanding task was addressed by working on a number of topics considered important for mutual knowledge and comparison (including research areas and goals, theoretical frames, ILEs implemented or used, contexts, work methodologies). Each team had responsibility for one topic and, on the basis of materials sent by the other teams, produced a report analysing the different contributions and developing them into an integrated presentation (the result is available the TELMA web site (www.itd.cnr.it/telma).
This first effort, based on the descriptions provided by the teams and analysis of papers they had published, helped to identify some common sensitivities to, for example: the contextual, social and cultural dimensions of learning processes; the

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For instance Ari-Lab2 (CNR-ITD), Pepite and Casyopée (DIDIREM), Aplusix (MeTAH), E-slate (ETL), L'Algebrista (CNR-ITD and UNISI). 48
role played by semiotic mediation; instrumental issues. It also made it evident that the diversity of the theoretical frames we employed
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affected the ways we dealt with
these common sensitivities in the design or use of ILEs. But reading and exchanging descriptions and research papers left us unsatisfied as we felt that our understanding of the underlying processes and their possible effects on practice remained too superficial. For that reason, we decided to develop a strategy allowing us to gain more intimate insight into our respective research and design practices. This strategy consisted of a 'cross-experimentation' project and simultaneous development of a methodological tool for systematic exploration of the role played by theoretical frames in teaching and learning in mathematics using digital technologies.
In this contribution, we focus on this second phase of our collaborative work, introducing first the idea of "didactical functionality", which we used as a tool for approaching the relationships between theory and practice.
The notion of "didactical functionality" of a tool
The notion of didactical functionality of an ILE (see Cerulli, Pedemonte, Robotti, 2005) was developed as a way to link theoretical reflections to the real tasks that one has to face when designing or analysing effective uses of digital technologies in given contexts. It is structured around three inter-related components:
• a set of features / characteristics of the ILE;
• an educational goal;
• modalities of employing the ILE in a teaching/learning process related to the chosen educational goal.
These three dimensions are inter-related: although characteristics and features of the ILE itself can be identified through a priori inspection, these features only become functionally meaningful when understood in relation to the educational goal for which the ILE is being used and the modalities of its use. We would also point out that, when designing an ILE, designers necessarily have in mind some specific didactical functionalities, but these are not necessarily those which emerge when the tool is used, especially when it is used outside the control of its designers or in contexts different from those initially envisaged. In the second phase of TELMA work, this notion of didactical functionality thus took a central and unifying role:
• on the one hand, the cross-experimentation aimed to explore the didactical functionalities the different teams involved would associate with ILEs they had not designed, and how their particular educational contexts and the theoretical frames they used would influence their constructions;
• on the other hand, this notion was also used to structure the methodological tool for exploration of the role played by theoretical frames.

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These were mainly: activity theory, socio constructivism, Vygotskian theories of semiotic mediation, social semiotics, theory of didactic situations (TDS), anthropological theory, Rabardel's theory of instrumentation, situated abstraction, AI theories. 49
In what follows we present these two facets of TELMA work, focusing on the methodological dimension.
The cross-experimentation
The idea of cross-experimentation is a new approach to collaboration, seeking to facilitate common understanding across teams with diverse practices and cultures and to progress towards integrated views.
Some important methodological choices:
There are three principal characteristics of this cross-experimentation:
• the design and implementation by each research team of a teaching experiment making use of an ILE developed by another team;
• the joint construction of a common set of guidelines expressing questions to be answered by each designing and experimenting team in order to frame the process of cross-team communication;
• the specific role given to PhD students and young researchers.
Each team was asked to select an ILE among those developed by the other teams. This decision was expected to induce deeper exchanges between the teams, and to make more visible the influence of theoretical frames through comparison of the vision of didactical functionalities developed by the designers of the ILEs and by the teams using these in the cross-experimentation.
The cross-experimentation involved a rich diversity of ILEs, educational contexts and theoretical frames, but important attention was paid to the control and productive exploitation of this diversity, especially through the joint construction of guidelines, developed through an on-line collaborative activity. On-line collaboration allowed the teams to communicate the results of their within-team discussions and resulted in an agreed joint set of guidelines (http://www.itd.cnr.it/telma/documents.php), negotiated between the teams to be as relevant as possible to their interests and theoretical frameworks, while remaining feasible in light of the constraints of time and empirical settings. These guidelines structure and support a priori and a posteriori reflective analysis of the crossexperimentation.
In order to allow as much comparability as possible between the research settings, it was also agreed to address common mathematical knowledge domains (fractions and algebra), with students between years 7 and 11 of schooling in experiments lasting approximately one month. Table 1 summarises the ILEs chosen, the teams who developed the ILEs and the teams conducting the experimentation.
An important role has been given to young researchers and PhD students. Starting from three draft sets of questions addressing the issues of contexts, representations and theoretical frames, they developed the guidelines through the Kaleidoscope Virtual Doctoral School platform, and have taken charge of the experimentation. This role is coherent with the general philosophy of TELMA and Kaleidoscope. It also has the benefit of allowing "fresh" eyes to look at the teams' 50 approaches, theoretical frameworks, and consolidated practice in order to make explicit those factors that often remain implicit in the choices made by more experienced researchers.

ILE Developer's team	Experimenting team(s)
Aplusix	MeTAH-Grenoble CNR-ITD, UNISI
E-Slate	ETL-NKUA UNILON
ARI-LAB 2	CNR-ITD MeTAH, DIDIREM, ETL-NKUA

Table 1: The tools employed by TELMA teams in the cross experiment
The selection of research questions, experimental settings and modes of use of the ILE, methods of data collection and analysis were all determined by each experimenting team after a period of familiarisation with the ILE itself, following the common guidelines developed through the on-line activity. Each team thus conducted an independent study of the use of an ILE. At the same time, however, the framework of common questions provided a methodological tool for comparing the theoretical basis of the individual studies, their methodologies and outcomes.
The current state of the project
The experimentations took place during the first term of this academic year. A priori analysis of the experiment, according to the guidelines, has been produced by each team. The a posteriori analysis, following the guidelines, is in progress. Comparison and discussion of similarities and differences between the reflective analysis carried out by the PhD students and young researchers and the analyses, expectations, and results obtained by the ILE designers is planned in the following months. However, preliminary comparison of each team's results has highlighted interesting issues and indicates directions for future investigation, as described in the following section.
Some preliminary results from the cross-experimentation
In order to point out and compare the preliminary results of the cross experiments a meeting was held. Each team reported on its own experiment focusing on the defined/employed didactical functionalities of the ILE used, trying to make explicit the relationship between such didactical functionalities and the team's theoretical assumptions. In order to structure this preliminary analysis each team was asked to complete a form before the meeting, focusing on the three dimensions of didactical functionalities. The form followed the principle of "necessary conditions" in the sense that not all the details of the experiments needed to be given, but only those that the team believed to be necessary conditions for the experiment to be successful according to the team's theoretical assumptions (http://www.itd.cnr.it/telma/documents.php).
Comparison of the forms completed by each team, and of the oral reports of the experiments, highlighted a set of issues that seem promising in terms of future investigations and in relation to the key question: How can the use a given ILE within a specific context be characterized by specific theoretical frameworks and by cultural and/or institutional constrains? 51
During the cross experiment some difficulties arose when teams attempted to use a given ILE in a context (both in the sense of school and of research context) different from that in which it had been developed. For example, the software Aplusix has been designed (by the French team MeTAH) to facilitate the teacher's work, and to offer him/her a good level of autonomy with respect to standard algebra curricular activities. The software allows students to build and transform algebraic expressions freely and to solve algebra exercises by producing their own steps as on paper; for each step the system gives an indication of correctness as feedback. Aplusix was designed to support the standard activity of algebraic manipulation, based on the solution of calculation tasks like expand, factorize, solve the equation, etc. However, the CNR-ITD team, adopting a socio-constructivist approach, faced the problem of planning open-ended tasks. According to this theoretical framework, open-ended tasks favour pupils' construction of meanings through exploratory activities. This was achieved by interpreting the feedback concerning the correctness of steps as feedback concerning the equivalence of expressions and/or statements. This change of perspective implied also that Aplusix was no longer used autonomously by students, but required the teacher to orchestrate the activity by asking the students to make their strategies explicit, to justify them and to discuss them with their classmates.
Adapting the way in which an ILE is used to a changed context, even if

possible, may also be complicated by the role played by different curricular constraints and school praxis. As an example, we consider ARI-LAB2 (developed in Italy by the ITD team). ARI-LAB2 is composed of several microworlds designed to support activities in arithmetic problem solving and in the introduction to algebra. One of these is the "fraction" microworld, which provides a graphical representation of fractions on the real line, allowing the user to build fractions by means of commands based on Thales theorem. Some teams encountered difficulties using this microworld in their school context due to the fact that Thales theorem is usually introduced in the curriculum later than fractions. The MeTAH team tried to use it as a "black box" but found this caused problems when pupils needed to make sense of feedback. Similarly, the DIDIREM team decided to switch to other microworlds of ARI-LAB2 because they judged it was not realistic to ask a teacher to change the mathematics organisation of the school year.
During the cross-experimentations another aspect has been highlighted related to the influence of theoretical frameworks on the use of ILEs. This is related to the role assigned to feedback by different teams. For example, the DIDIREM team, drawing on the theory of Didactic Situations, found the feedback provided by ARILAB2 too limited with respect to what is generally expected from a "milieu" offering an a-didactic potential for learning. On the contrary the ITD team, who had developed the ILE, drawing on a more general constructivist framework, considers the feedback sufficient because the teacher's role and feedback are considered as fundamental as those of the ILE. 52
These examples show how, in order to employ an ILE in a context different from that taken as reference by its designers, one has to face a set of problematic issues. To sum up, the highlighted issues include:
• educational goals
• typologies of tasks proposed to the students
• computer's feedback and autonomy of the ICT tool and/or of the pupils
• settings and role of teachers
These issues point to significant investigative directions to be addressed in the next year of the work of TELMA. In fact, in order to refine the comparison between the experiments, our preliminary analysis raises the need to refine the lenses through which the experiments are analysed and compared. Starting from the idea of didactical functionality, we need to address its three dimensions in more detail, and in order to do so, a first methodological tool has been developed in parallel with the preliminary analysis and will be refined and employed in the next year. Below we present the tool and indicate the kinds of analysis it can help to bring forward, showing how it was employed by one of the TELMA teams.
A methodological tool for systematic exploration of the role played by theoretical frames
Some preliminary remarks
In the design of this methodological tool, we were inspired by the work already developed in TELMA about theoretical frames, the common sensitivities and the evidence of differences in the ways these were dealt with, and also by the meta-study previously developed by DIDIREM researchers involved in TELMA (Lagrange & al., 2003). This led us to consider this methodological tool as a multidimensional tool structured around the notion of didactical functionality, and to associate to each component of the notion of didactical functionality a set of 'concerns', expressed in the most neutral way. Analysis using the tool will try to determine for each of these concerns whether and how it is addressed, and to elucidate the role played both explicitly and implicitly by theoretical frameworks. It is assumed that such a tool will help to establish productive links between different frames, and will support partial integrative views when these appear accessible and possibly productive, keeping in mind that a global integration is certainly out of reach, and even not desirable, the strength of any approach being attached also to the specific lens it chooses for approaching the complexity of the reality we study.
The methodological tool
a) Analysis and identification of specific ILE characteristics :
The analysis of an ILE using the definition of didactical functionalities generally involves two different dimensions, questioning on the one hand how the mathematical knowledge of the domain is implemented in the ILE, and on the other hand the forms of didactic interaction provided by the ILE. Both the implementation of the knowledge of the domain and the didactic interaction can be approached 53 through different perspectives, which are neither independent nor mutually exclusive. We have thus selected according to this dimension the following concerns:
- concerns regarding tool ergonomy
- concerns regarding the characteristics of the implementation of mathematical objects and of the relationships between these objects
- concerns regarding the possible actions on these objects
- concerns regarding semiotic representations
- concerns regarding the characteristics of the possible interaction between student and mathematical knowledge
- concerns regarding the characteristics of the possible interaction with other agents 5
- concerns regarding the support provided for the professional work of the teacher
- concerns regarding institutional and/or cultural distances
b) Educational goals and associated potential of the ILE
It is more the relationship between potentialities and goals rather than each of these considered separately which can contribute to illuminate the role played by theoretical frames in relation to this dimension, complementing what is offered by the information provided by the analysis of the ILE. It seems interesting to investigate the relative importance given in the definition of educative goals to considerations of an epistemological nature - referring to mathematics as a domain of knowledge or as a field of practice, considerations of a cognitive nature - focusing on the student in her relationship with mathematical knowledge, considerations focusing on the social dimension of learning processes, and finally institutional considerations. Thus the concerns we selected for this dimension are:
- Epistemological concerns focusing on specific mathematical content or practices
- Cognitive concerns focusing on specific cognitive processes or difficulties
- Social concerns focusing on the social construction of knowledge and on collaborative work
- Institutional concerns focusing on institutional expectations and on compatibility with the forms and contents valued by the educational institution
c) Modalities of use
The design of modalities of use and the a priori analysis of their implementation supposes a multiplicity of choices of diverse nature. It is reasonable to hypothesize that only a small part of these are under the control of theoretical frames, explicitly or even implicitly, many other being dictated consciously or unconsciously by the educational culture and the particular context within the realization takes place. The categories are:

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Other agents can be other students, the teacher, tutors as well as virtual agents such as the companions implemented in some ILEs. 54
- Concerns regarding contextual characteristics
- Concerns regarding tasks proposed to students, including their temporal organisation and progression
- Concerns regarding the functions given to the tool including their possible evolution
- Concerns regarding instrumental issues and instrumental genesis
- Concerns regarding social organisation, especially the interactions between the different actors, their respective roles and responsibilities
- Concerns regarding interaction with paper and pencil work
- Concerns regarding institutional issues, especially the relationships with curricular expectations, values and norms, the distance from the usual environments
Using the methodological tool
The methodological tool has been used first in order to explore the role played by methodological frames in the preparation of one experiment: that carried out by DIDIREM. There is no space here for a detailed presentation of this analysis which is accessible on the TELMA website (Artigue, 2005). It shows that, in this experimentation involving two different microworlds offered by ARI-LAB2, nearly all the concerns mentioned above were addressed, but with evident differences in emphasis (four different levels of emphasis were distinguished). The analysis also shows how the three main theoretical frames used: instrumental approach, theory of didactic situations, anthropological approach (together with didactic knowledge about the mathematical domain at stake) influenced the choice of the selected microworlds and the design of the experiment. However, an interview with the young researchers of the team involved in the project also shows that many of their choices were not under the explicit control of these theoretical frames. Some of the choices can be explained a posteriori by referring to theoretical frames but had been used as naturalized and implicit conceptual tools. Others were dictated by institutional and cultural habits. We can hypothesize that, from one team to another one, differences in the implicit theoretical tools and cultural habits will be made visible by the exchanges organized around the cross-experimentation, offering us insights into the real influence of theory on research and design practices that the reading of papers hardly offers.
Conclusion
In this short text, we have tried to make clear the kind of contribution we can offer to this ICMI Study. This work tries to overcome the difficulties generated by the existing diversity of theoretical frames and the lack of communication between these, through a better understanding of the role played by theories, the development of methodological tools and cross-experiments. This is on-going work and, by the time of the Conference associated to this study, all the analysis being completed, we hope to be able to offer a sound contribution on these difficult but crucial issues. 55
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Looking through zones at the teacher's role in technology-rich teaching and learning environments (TRTLE's)
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The equilibrium of teachers and teaching is inevitably altered by the availability of electronic technologies. It is imperative to establish what it is that enables teachers to perceive, attend to, and exploit affordances of the technology salient to their teaching practice and likewise for students in their learning about function. This paper focuses on one teacher and his teaching where technology use is expected by curriculum authorities. The aim is to show how Valsiner's zones and Gibson's affordances have been used as a theoretical framework to document the teacher's role in integrating electronic technologies into his teaching. The students' subsequent use of technology to support their learning is also examined. Thematic matrices have been used to identify manifestations of affordances, affordance bearers used, and the conditions enabling perception or promoting enactment of particular manifestations of affordances. These conditions, the latter indicative of the Zone of Promoted Action, have been used to identify teacher's role as he canalises students' current and future thinking about concepts and methods taught. The teacher's approach also impacts on the Zone of Free Movement/Zone of Promoted Action complex of the future.
Introduction
A broad variety of electronic technologies has the potential to be used in mathematics classrooms today. The presence of these in the classroom can fundamentally change how we think mathematically and what becomes privileged mathematical activity. Classroom tasks can include those that are transformed by electronic technology use rather than technology being used to produce or check results but not necessarily contribute to the development of understanding, and concept and skill development. Whilst acknowledging that teaching and learning are inseparable, this paper focuses on the teacher and teaching by reporting a case study of a teacher and the Technology-Rich Teaching and Learning Environments (TRTLE's) within which he is a participant, these being environments where electronic technologies are readily available. Affordances and Valsiner's Zone Theory are being used "to analyse the role of the teacher in orchestrating technologyintegrated mathematics learning" (ICMI 17 Discussion Document, p. 7).
Affordances
A technology-rich teaching and learning environment affords new ways of engaging students in learning mathematics. Teachers and students equally have to learn to become attuned to the affordances of such environments. The construct, affordances, is used here following Gibson (1979) who made up the term to explain what is 57 perceived, and Scarantino (2003), whose more recent elaborations of the construct support its interpretation in an educational setting. Affordance to Gibson is a construct to help explain what motivates human behaviour (Reed, 1988). Brown, Stillman, and Herbert (2004) outline some of the different uses and meanings of the term and it potential usefulness as a framework to support analysis of TRTLE's. More specifically, affordances of a TRTLE are the offerings of the TRTLE for facilitating and impeding teaching and learning. They are potential relationships, involving interactive activity, between the teacher and/or students and the environment. In line with Gibson, the environment includes both animate and inanimate objects such as technology. To take advantage of opportunities arising, both teachers and students need to perceive affordances and act on them (Drijvers, 2003). This action may well be the rejection of a particular affordance in favour of another. This perception and subsequent enactment depends not only on "the technological tool, but [also] on the exploitation of these affordances embedded in the educational context and managed by the teacher" (Drijvers, 2003, p. 78).
Following Gibson and Scarantino, specific objects within the environment, the TRTLE, that enable an affordance to be enacted are affordance bearers. The manifestation of an affordance in a TRTLE in background circumstances [C]

involves an event [E] in which both the affordance bearer [AB] and the actor [A] are involved (e.g., a teacher [A] brings pictures of real world examples into the classroom for illustration of quadratic functions [E] on a laptop computer connected to a data projector [AB], given the relevant photographs were inserted into a PowerPoint presentation [C]). The affordance described here is communicate-ability a 'technological - communication affordance' (Kaput, 2004).

Situating the Research

The research reported here is situated within the Australian Research Council Linkage Project 2004-2006, Enhancing mathematics achievement and engagement by using technology to support real problem solving and lessons of high cognitive demand (the 'RITEMATHS' project). Within and across project classrooms the uptake and use of technology has been different. However, every teacher and student involved has the opportunity to use a selection of technology ranging from hand held graphing calculator, calculators including computer algebra systems, computers applications available on laptop or desktop computers to applets available on the internet. The broad aim of RITEMATHS is to discover the most effective ways to use technology to stimulate higher order thinking in mathematics classrooms, in the context of using real world problems. Within the project a design research methodology is being used, thus the teachers and researchers are partners working together toward this aim. This methodology values contributions of teachers equally with those of researchers, giving the practitioners a voice and further acknowledging that it is ultimately the teacher who orchestrates the learning of their students. The study reported here is part of a larger study (Brown, 2004; 2005a; 2005b) within RITEMATHS to establish what it is that enables teachers to perceive, attend to, and 58 exploit affordances of the technology salient to their teaching practice and likewise for students in their learning about function. This paper focuses on one teacher and the aim is to show how Valsiner's zones have been used to document his role in integrating electronic technologies into his teaching. The students' subsequent use of technology to support their learning is also examined. Particular emphasis is placed on determining what use students make of technology when given opportunities to choose the type of technology and the purpose they make of the technology selected.

Valsiner's Zone Theory

The theoretical underpinnings of this study also draw on zone theory from developmental psychology. Elsewhere, Valsiner's Zone Theory has been applied to the development of algebraic reasoning in primary settings (Blanton & Kaput, 2002), technology enriched teaching and learning environments (Galbraith & Goos, 2003), teacher education focusing on the learning of pre-service teachers (Blanton, Westbrook, & Carter, 2005; Evans, Galbraith, & Goos, 1993), and to preservice and beginning teachers using technology in mathematics classrooms (Goos, 2005). It is applied here to teaching and student actions in TRTLE's where the learning focus is mathematics in an attempt to elucidate the teacher's role.

Valsiner (1997) expanded on Vygotsky's Zone of Proximal Development (ZPD) and proposed two additional zones describing the structure of "the environment of the developing child" (p. 186) both between people in the environment and in terms of regulating an individual's "own thinking, feeling, and acting" (p. 188). "The zones are always temporary, constantly changing structures that organise the immediate construction of the future state out of a here-and-now setting" (p. 319). The Zone of Free Movement (ZFM) is described by Valsiner as dynamically "providing a structural framework for the child's cognitive activity and emotions … when internalized ZFMs regulate the relationships of the person with the environment" (p. 189). Valsiner (1984) links the affordance and Zone Theory frameworks.

Within the field of objects and affordances related to them in the environment of the child, the zone of free movement (ZFM) is defined for the child's activities. The ZFM structures the child's access to different areas of the environment, to different objects within these areas, and to different ways of acting upon these objects. (pp. 67-68)

In TRTLE's this includes affordances, allowable actions, and technology and other learning artefacts available to a student acting in the TRTLE at any given time. The ZFM "has a counterpart oriented toward the promotion of new skills" known as the Zone of Promoted Action (ZPA) (Valsiner, 1997, p. 192). Importantly, particularly in a secondary school setting, the ZPA is non-binding in nature. The ZPA describes "the set of activities, objects, or areas in the environment, in which the person's actions are promoted" (p. 192) by the teacher in the TRTLE as they attempt to guide their students' "actions in one, rather than another, direction" (p. 317). Valsiner argues that these two zone concepts should not be separated, rather they should be considered as a ZFM/ZPA complex working together "by which canalisation of 59 children's development is organized" (p. 195). Following Valsiner, the ZPD is "a narrowed down extension of Vygotsky's concept made subservient to the ZFM/ZPA complex" (Valsiner, 1997, p. 199). It "entails the set of possible next states of the developing system's relationship with the environment, given the current state of the ZFM/ZPA complex and the system" (p. 200). In the TRTLE the notion of ZPD helps capture those aspects of the student's learning that are in the process of being actualized.

Methodology

Capturing technology use as enacted by teachers and students is no simple task. Close scrutiny of TRTLE's where access to, and substantial use of, technology by both teachers and students is assumed, enables a comprehensive picture of what is occurring in such environments to be obtained. Increasing understanding of such situations subsequently supports others becoming more effective users of technologies in mathematics classrooms where the use of technology is expected by curriculum authorities. A qualitative approach provides such a picture. Technology is not only a focus of what is being studied, but also it is through the use of digital technologies for data collection, such as audio and video recordings of the TRTLE's, that a detailed picture of the environment is constructed and subsequently analysed. In addition, the use of a Key Recorder (available www.fi.uu.nl/wisweb/en/) enabling keystrokes of graphing calculators to be recorded and the subsequent reconstruction of actions taken and screens viewed adds to the richness of the collected data. An instrumental approach (Stake, 1995, p. 3) is being followed. The case presented here is part of a collective case study (p. 4). The larger study aims to construct a grounded theory establishing what conditions enable students and teachers to perceive and enact affordances offered by TRTLE's for the teaching and learning of functions. Evidence is presented here for one case detailing the role of the teacher in manifestations of affordances within the TRTLE, and the conditions that enabled or promoted their enactment. In this study the environment, that is, the TRTLE, includes electronic technologies and other objects, ways of acting with these objects by the teacher and students, and the teacher and students themselves.

The case being studied includes one teacher, James, a very experienced teacher of mathematics, and two of his classes, his Year 11 Mathematical Methods students (16-17 year olds) and his Year 10 Mathematics class (15-17 year olds). Students had access to both laptop computers and Texas Instruments graphing calculators (83/84 Plus) on a daily basis and this was the case in the previous year of their schooling. Use of electronic technologies is expected by the statutory authority overseeing the curriculum at both levels. Functions was one of the curriculum foci in both TRTLE's, but of a more introductory nature for the Year 10 class.

Within the larger study conducted by this researcher, data have been collected from teachers at all six project schools with more focused data collection from four of the schools, including eight teachers and nineteen TRTLE's over the first two years of the three year study. These data and subsequent analysis inform the analysis of the 60 two TRTLE's that are discussed here. Data collection included field notes, audio and video recordings, photographs, student scripts, graphing calculator key recordings, post-task student interviews, teacher interviews and reflections, and documentary materials from lesson observations.

For the Year 11 TRTLE, for example, observation included nineteen 50 minute lessons, two teacher designed task implementations (Quadratic Function Task [QF], Cubic Function Task [CF]), and a researcher designed task implementation, Platypus Task [PP]. The teacher designed tasks included students taking physical measurements involving hanging chains, curved sticks, and chromatography as shown in Figure 1. CF involved the use of only flexible sticks shaped into functions. Both tasks required students to relate coordinates of key features to particular forms of the given function type and make links between algebraic and graphical representations, refining the final unknown (dilation) parameter to identify the equation of the function being modelled. In the researcher designed task, students were presented with two sets of data representing a platypus population before and after an intervention project and asked to find a model to represent platypus numbers over time for both data sets. Students then considered questions such as, did the intervention improve the situation, what was the predicted population a decade later, and when would the population return to the initial value? The task required students to make use of functions once identified and promoted the use of a broader understanding of functions, including various manifestations of function calculateability, and function view-ability.

Figure 1. Data collection by students for Quadratic Functions Task

The intent of the diagrams in Figure 2 is to portray the differences between the ZFM/ZPA for the teacher and researcher designed tasks. Representing a zone whose boundary is fuzzy, partly indeterminate, and dynamic is clearly fraught with difficulties. The outer circle represents the ZFM, and given its later timing in the function unit, one would expect that the ZFM existing at the time of PP was greater than for the teacher designed tasks. The focus is however on the relationship between the two zones constituting the complex at the time of task implementation. Both teacher designed tasks involved repetition of the same sub-task, but for different general forms of the function. The ZPA for students when undertaking these two tasks was much smaller than their ZFM with students being told what actions they were to undertake. In the researcher designed task, the range of actions was greater and there was little repetition across sub-tasks. In addition, students had to decide what methods they should undertake to solve any sub-tasks. Hence, students had to select actions that they believed were appropriate from within their individual ZFMs. 61 The promotion by the task of student choices of actions resulted in the ZPA being significantly closer to their ZFM than was the case for the teacher designed tasks.

Figure 2. The ZFM/ZPA complexes for a) Quadratic and Cubic Functions Tasks b) Platypus Task

Analysis and Results

The identification of manifestations of affordances, affordance bearers, and the circumstances in which these occurred was facilitated by preliminary coding after the data were entered into a NUD*IST database (QSR, 1997). To identify conditions enabling and promoting enactment of affordances subsequent examination of both the coded data and re-analysis of the case record was undertaken. A thematic conceptual matrix (Miles & Huberman, 1994, p. 131) was developed to show manifestations of affordances, affordance bearers, and conditions enabling perception or promoting enactment of particular affordances of this TRTLE for student understanding of function. Table 1 shows one row of the matrix, one manifestation of the particular affordance: Function View-ability, that is, how particular views of a function can be observed. For illustrative purposes only the manifestation resulting from set viewing window to given values is reported. Other manifestations of this affordance included those resulting from using current window settings, editing viewing window to include key features or get a better/global view, editing viewing window to allow key feature to be clearly visible, using TABLE/TRACE to provide information regarding appropriate viewing window, and using context to select appropriate viewing domain, viewing range or an endpoint of either of these.

Table 1.

Manifestations of the affordance: Function View-ability (Affordances of TRTLE's allowing particular views of functions to be observed)

Manifestations of the Affordance	Affordance Bearers	Conditions Enabling Perception	Conditions Promoting Enactment
Setting Viewing Window to given values	S108 sets viewing window of graph of WINDOW	S108 adjusts his WINDOW Settings to those given. [Obs 16Mar05 36]	Lesson Element, Window Settings given
Slide 15: … WINDOW. Set Xmin = -2, Xmax = +2 and Ymin = -9 and Ymax = +9. Quadratic function test	(b) Sketch a graph [of a function given for T in terms of S] between Smin = 0 and Smax =360. ZFM ZPA ZFM ZPA 62	y = (x - 1)(x + 2)(x - 4) to -3 < x < 6, -12 < y < 12. [Obs 16Mar05 18]	GRAPH. [Obs 16Mar 44] [Quadratic function test]

The first column in the matrix includes summary phrases describing the

manifestation of the affordance and representative illustrative actions. The second column indicates the affordance bearers utilised in the enactment of the affordance, the WINDOW settings that allow direct manipulation of the viewing window. The final two columns describe the conditions existing either during or prior to the manifestation of the affordance being considered. The conditions are those either enabling perception (column 3) or promoting enactment (column 4) of a particular affordance. A condition enabling perception is a circumstance where a teacher or student action allows a particular affordance to be perceived. Conditions enabling perception include those where a learning experience is provided during which the student experiences a particular affordance, as is shown in Table 1 where students are expected to follow the instruction and experience the particular affordance thus facilitating future enactment. A condition promoting enactment is a circumstance where a teacher or student action promotes enactment of a particular affordance. In this case it is through the wording of the task that the teacher promotes the direct setting of the viewing domain of the graphing calculator.

To consider data at a more conceptual level, content analytical summary tables were constructed. Table 2 shows an example for the affordance function view-ability. The final column of Table 2 is indicative of the ZPA. The teacher's organisation of the students' ZPA canalises (Valsiner, 1997) the students' current and future thinking about the concepts and methods being taught.

For, example, James observed that determining settings for the graphing calculator allowing particular views of a function to be observed is no simple task, echoing earlier findings by Brown (2003) who found diverse initial actions and subsequent views as teachers began looking for a global view of a 'difficult' function. Many of the resulting views would be potentially problematic for students. James observed a similar difficulty with his students.

James: Interestingly, a lot of kids find the notion of setting a WINDOW to a particular graph [difficult], especially if you are doing real, in inverted commas, applications where you do some linear modelling and you might have so many books sold for so many dollars which ... is a problem that kids can relate to. And inevitably [you] see them with a graph with the four quadrants. And when you say to them, 'Now is it realistic to have a negative number of books?' 'No', or 'A negative amount of dollars?' and, 'No'. 'Well then, are those values realistic to have on your graph?' 'No'. 'Well, you would have more efficient use of your graph if you deleted those bits and use your WINDOWs'. 'But I don't know how to use WINDOWs, I don't understand'. [Interview 21Feb 05]

Table 2 6 3

Affordances of TRTLE's allowing particular views of functions to be observed

Manifestations of the Affordance	Conditions Enabling Perception	Conditions Promoting Enactment
Using current settings Serendipity Task - find graph of data/function		
Task - identify model of physical curve		
Setting viewing window to given values		
Lesson Element, Window Settings given		
Quadratic function test, sketch function over a specified domain		
Edit Viewing Window to include key feature, get a better/global view		
Lesson Element focused on setting of a 'good' window		
Functions task requiring exploration of graphs of families of functions		
Teacher Promotion - Can you Show me a bit more of your graph?		
Contextualised task requiring a suitable WINDOW		
Functions task requiring students to explore graphs of families of functions		
Editing viewing window to allow key features to be clearly visible		
Teacher scaffolding - understanding of the effect of c		
Lesson element, adjusting WINDOW Settings to view key features.		
Contextualised task requiring a suitable WINDOW		
Functions task requiring exploration of graphs of families of functions		
Using TABLE/TRACE to provide information regarding appropriate Viewing Window		
Did Not Observe Lesson element- find global view of a function		
Using context to select appropriate viewing domain, viewing range or an endpoint of one of these		
Did Not Observe Lesson element, context (biggest box) used to determine WINDOW Settings		
Contextualised task requiring a suitable WINDOW		
Functions task requiring students to explore graphs of families of functions		
Teacher belief in importance		

Thus he sees function view-ability as an essential affordance for his students to enact and actively enables its perception and promotes its various manifestations. On one occasion James was observed promoting use of context (finding the biggest box volume) to select an appropriate viewing domain.

James: When you do those cut-outs, of x, what is the biggest value of x that you can cut-out? If you have a look at your picture, what did you put for your diagram? If you started to make those corner cut-outs bigger, what would be the biggest cut-out that you could make?

Cam: Five.

James: Five. That is right. So for your WINDOW, you would set Xmin to be zero. And Xmax to [pause]?

Cam: So that, is that [wrong]?

James: No, you are right up to there.

Cam: So we didn't have to do that much?

James: But, beyond here [x = 5] it is not a realistic part of the problem. Because the biggest value of x you could ever cut out is 5. Okay?

Cam: Yeah.

James: So you would set your WINDOW to? [Obs30Mar05TB5 177-187]

Here James has deliberately organised the learning experience so that this particular manifestation of the affordance is promoted. Thus through the ZPA he hopes to canalise the students' future cognitive actions as they internalise the conditions he has orchestrated for their learning. In promoting use of the context to support the determination of a suitable viewing window for the function and the subsequent 64 successful problem solution, James is also hoping that in future situations his students will be able to independently perceive and enact this affordance where appropriate.

Concluding Remarks

Valsiner's zones together with Gibson's affordances have provided a suitable vehicle for elucidating the role of the teacher in technology-rich teaching and learning environments. The teacher makes use of the ZPA to promote particular student actions involving technology in an effort to give them experience of particular affordances of the TRTLE and to increase the future ZFM of the students.

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A model for teaching mathematics via problem-solving supported by technology

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A model for teaching mathematics is developed based on problem-solving and the use of technology in education. The research model stems from a ten year-old intensive Calculus project containing workshops designed over Computer Algebra Systems (CAS). Emphasis is placed on distance learning attributes such as creativity, critical thinking, autonomous learning, group work and the capacity to evaluate results, study errors, and contextualize the study area. The pedagogical model is centered on the student's talents for deep learning via the solution of problems with real applications that require understanding, creativity, the use of technological tools, and the development of an appropriate language for documentation, communication and socialization. Technological tools for education include systems that allow the visualization of concepts, simulation and experimentation, operation strength and self-evaluation environments.

Teaching centered around student learning

The model studies the elements of teaching based on the student's ability to learn under the premise that the student assumes responsibility for the learning process. Accordingly the student must be responsible and motivated to learn and furthermore be willing to perform group work, utilize technology and appreciate self-evaluating environments.

A dynamic distance learning environment is generated by first considering what pedagogical activities must be developed. This is the study of the student's

interaction with three learning instances, as shown in Figure 1 below.
Working alone: The student must be able to work alone and develop autonomously the necessary cognitive abilities to learn and apply the acquired knowledge to practical tasks and open problems.
For this aspect tools must be provided for effective self-evaluation and result verification. 67
Figure 1. Relational model
Interaction with the teacher: The student must learn to rely on the teacher's orientation and professional knowledge.
For this the teacher must guide the student in a personal fashion by evaluating the progress and providing the necessary indications, materials and support.
Collaborative work: The student must learn to appreciate groupwork as a strategy for the construction of knowledge and problem-solving. Interaction with other students provides another way to answer arising questions while interacting in a workplace environment.
For this collaborative tools must be developed as well as verification and self-evaluating environments.
The study of the relational model leads to the choice of appropriate learning theories and pedagogical strategies to support the teaching model. These are shown around the relational model in Figure 2 and figure 3.
Finally, the model must include stages of evaluation where the instructor may evaluate the implementation of the strategies and their effect on student learning to further vary the activities accordingly. 68
Figure 2. Pedagogical model
Figure 3. Stages of the strategy
Structural design of the laboratory workshop
Once the pedagogical components are established a laboratory workshop is designed. The model relies on certain talents from the interested student some of which are:
Discipline: Students must be able to maintain their own learning rhythm without the physical guidance of a teacher.
Responsibility: Students must acquire a sense of responsibility of being the principal element of the learning process. They must understand the objectives of the learning program and take them as their own. 69
Scheduling: the effective management of time is essential to reach the objectives of every course.
Curiosity, research and analysis: Students must learn to obtain and discern what is the pertinent information needed at any particular instance.
The relationship between the didactic model and the structure of the lab as a distance learning element is presented in Figure 4.
Figure 4. Learning components of the workshop.

Components of the workshop:
Problem-solving: The solution of theoretical and applied problems is the pedagogical basic tool of learning assumed throughout the work. Problems related to the area of knowledge and to predetermined standards are chosen. Emphasis is placed on both theoretical concepts and solution methods and techniques.
Individual work and hand calculations: Students must internalize the structure and the meaning of the problems by manipulating symbols with pencil and paper and performing the appropriate operations and calculations. Designing the strategy of the solution autonomously is very important. Afterward, the student will be allowed to verify results with technology.
Group work: Discussion is the basis of collaborative problem-solving. It allows for the presentation of different points of view and strategies of solution present in different levels of understanding. Above all it permits the creation of a language and of an abbreviated system of symbols the pertains to communication.
Research: Comprises various talents such as search techniques, information confrontation and discernment of core elements, translation skills and use of 70 technology, among others. For this probing activities are developed for the enhancement of knowledge and problem-solving.
Technological components with Maple:
The technological platform used was the symbolic mathematical software Maple V. Activities were presented in a standardized format. These activities will be explained further:
Guided exercises: A problem is presented to the student with all phases of solution explicitly designed in Maple. The student must actively perform the operations and learn the method, algorithms, arguments and styles as well as the Maple commands. The guided problems offer guided examples of methods of proof. The student learns to do and structure mathematics while solving itemized parts of a bigger complete solution of a complex problem. Real problems are presented and their solutions are explored step by step by using Maple commands. The student must analyze the arguments and answer both conceptual and operational questions at each step. Other problems are proposed to check the apprehension of procedure. The technological platform allows the student to perform and check difficult operations.
Interactive exercises: Problems presented where the student must provide the answer by reasoning, hand calculations or use of technology. These may include steps towards a solution or simply a specific answer to a problem. Feedback in provided for self-evaluation and knowledge enhancement. Feedback includes generally the complete solution to the problem sometimes with other examples. Feedback for errors may include partial solution to the problem and possibly examples and counterexamples.
The interactive problems allow the student to appreciate and solidify what has been learned by offering an opportunity of self-evaluation. The tools permit the construction of solutions whose validity can be checked without the presence of the teacher. The problem is called interactive because the student gets feedback even if the solution is wrong and hints are given towards the real solution or reasons are given for the mistakes. The problem can be solved any way and only solutions or steps of solutions are checked. This was done by programming over the software program Maple is built on. Programming allows for the prediction of errors of many types and the presentation of corresponding correct solutions to operational problems, procedure problems or conceptual problems.
Interactive solutions to problems: Explicit versions of the previous type of exercise where solutions are framed against arguments of solution.
Use of Internet: Mostly utilized in a bibliographical or communications context. For example, workshops can be downloaded and worked on collectively through chats or message boards.
Feedback and self-evaluation: Classical feedback provided by an instructor is itemized and studied via explicit indicators of the actual relationship between the objectives wanted through problem-solving and the activities presented in the 71 workshop. Some of the feedback is classical via examinations, papers or simple instructor observations. Yet the core idea is to provide the tools necessary for selfexamination.
Finally, the feedback and evaluations are studied so the material can be restructured and emphasis can be placed on what the student needs to reorganize and enhance knowledge. The teacher is an important player in this part of the process and must be able to guide new understanding and abilities to the effective crystallization of basic ideas and knowledge in the area.
The evaluation phase
Evaluation allows for consistency and quality of the learning process and provides the teacher with the appropriate feedback on the learning of each individual student. In Figure 5 evaluation processes and components are established which will allow to design and efficient pedagogical model for self-learning.
Standards are taken as the bases of the evaluation process. These are defined as follows:
Standards are clear and public criteria that provide knowledge of what students should learn and apply in solving problems of their environment and constitute a reference point of what should be known and applied in a particular area of knowledge and a specific level of proficiency. These may depend on social, cultural and political needs of a community. Especially created in order to change the emphasis on concepts in education, mathematical and geometrical standards are included in the laboratory workshop design.
Figure 5. Conceptual structures of evaluation 72
Indicators: Are characteristics or components of the standards that allow for the determination of the level of proficiency of the student in that particular task associated with the standard. The breakup of abilities allows for the evaluation of strength of knowledge and of weakness and errors. The classification of levels of abilities permits the evaluator to understand and measure the attainment of objectives in a specific fashion.
Rubric definition: A mathematical rubric is the specific talent or ability that must be attained by the student in order for learning or acquiring knowledge in an area of mathematics and a certain level of proficiency.
Rubric evaluations study certain aspects or indicators and assign well-determined levels in such a way that:
Two evaluators must reach the same results of evaluation, in this work this should include one evaluator being the student.
The aspects are well-defined and do not allow for the consideration of irrelevant factors.
The result must correctly and pertinently evaluate the status of the student with respect to the standards.
Evaluation: It is important to utilize both formative and summative evaluation. On the one hand, it is important to determine not only the tools that allow for the learning of a particular subject area but those that allow for the correct feedback. On the other hand, evaluation of the problem solving abilities and strategies must be included with that of the conceptual understanding and the operational proficiency in the production of a correct solution. The study of the abilities as a whole: the correct utilization of concepts learned, the abilities of creating problem solving strategies, and the abilities to assess the validity of a solution change the emphasis of a solution from that based on simple procedural and operational proficiency to one that allows the use of technology correctly in every day life. Diagnostic evaluation must be utilized for the correct planning of the activities of the course and can be applied at various stages. Well-chosen rubrics allow the ERRORS to be determined diagnostically also. Therefore every evaluation can be used as feedback for the present and future design of the course.
To conclude, the study focuses on learning via problem solving intending to use technology as a support for self-learning in providing self-evaluation tools such as: well-defined and easy to determine rubrics consisting of specific indicators of abilities tied to each problem and subject area and which are presented at an appropriate time and in a pertinent setting of a workshop; tools for communication, development of language, and research; operation, procedure and solution verification; and through these tools the personalized guidance of the teacher.
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Evolving technologies integrated into undergraduate mathematics education
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This submission focuses on the design of learning environments and curricula and describes a twenty-five year evolution of integrating digital technology
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in the
teaching and learning of mathematics at Brock University. It provides information on actual uses of technology in university level programs for students, majoring in mathematics, or taking mathematics for their major in another discipline, or aiming to be teachers. A brief history explains the ever increasing use of established mathematics and statistics computer systems in courses and programs until the Department had gained enough experience with technologies to institute a new core mathematics program MICA (Mathematics Integrating Computers and Applications). Student interest in the MICA program is demonstrated by a threefold increase in mathematics majors. The submission pays special attention to the role of the teacher. First, a new faculty member reflects on the teaching adjustments she made to teach in a department that has built an array of technologies into its courses. Second, it explains how technology, in a first year core mathematics course, helps to shift the mediator responsibilities from the teacher to the student. Of particular significance is the students' enthusiasm and willingness to work beyond all expectations on their main project in which they construct Learning Objects.
Introduction
There are many publications (for examples Kallaher (1), and Baglivo (2)) that describe the integration of established Computer Mathematical Systems (for example Maple) and Computer Data and Statistical Analysis Systems (for example SAS (4)) into mathematics and statistics education at the university level. Because of this wealth of publications and because the Department of Mathematics at Brock University had, by the mid 1990s, integrated such systems in the majority of its courses, we will focus our discussion on the Department's next evolution. We describe how the Department integrated communication technology (e.g. Internet)

and environment building technology (e.g. VB.NET (3)), into an innovative core mathematics program called MICA (Mathematics Integrating Computers and Applications). For us it is evident that this step was only possible because the majority of faculty had substantial teaching experience in courses that integrate

6
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7
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In what follows the term digital will be understood whenever we mention technology. 75 technology in a significant way. The MICA courses provide working examples of mathematics learning environments that integrate technologies. Furthermore, within these courses, students learn how to construct technological environments to explore mathematics. Future teachers have an important place in these courses as they learn to develop technological learning environments that focus on the didactical development of mathematical concepts.
This submission is made by two practitioners who, in the words of the Discussion document, 'can make solid practical and scientific contributions to ICMI Study 17'. The reader will find; in Section 1 a brief 25 year history of the integration of technologies in mathematics programs at Brock; in Section 2, a discussion of one role that evolving technologies can play in mathematics education; in Section 3, a summary of important aspects of the MICA courses; in Section 4, a description of some didactical considerations that were introduced in MICA specifically for future mathematics teachers, and; in Section 5, a reflection by Buteau on the challenges and adjustments that were required in her teaching in order for her students to achieve the learning expectations of MICA.
Section 1: A brief history of the evolving integration of technologies
In 1985, at the time of the first ICMI Study, the Department of Mathematics was making innovative use of technology in some of its courses. In large enrolment service courses, some faculty (3) were generating individualized sets of problems for each student and Muller (4) was assessing an experimental Calculus course with over 100 students who worked with Maple in a laboratory setting. In this presentation we reflect on the Department's subsequent sustained development of the use of evolving technologies in its undergraduate mathematics programs. Although one can point to certain years when major changes were implemented, the reality is that evolution and innovation in university mathematics education is a slow process. One reason for this is that few mathematics doctoral programs require teamwork or provide opportunities for reflection on the teaching and learning of mathematics. Yet these experiences are necessary for faculty in a department to work as a team and for its faculty to critically redesign a mathematics program. There is much evidence that technological innovations that are instituted in a course by a single faculty member rarely survive when the course is taken over by another colleague. Therefore, for changes to permeate beyond a set of courses, a consensus needs to be built with the majority of faculty in a mathematics department. The changes that occurred at Brock required many hours of discussions between colleagues and demanded that they approach the subject with open minds. In retrospect, a major argument for the use of technology and for a complete review of the mathematics programs was generated from faculty experiences in Maple laboratories. There they observed student activity and involvement. In general they found that students in laboratories were much more engaged than in the traditional tutorials and that they were also asking more significant mathematical questions.
By 1995 a majority of students in all mathematics programs were using technology in a significant way. In general students were working with Computer 76 Mathematical Systems or Statistical Analysis Systems. By this time, faculty who were keeping up with the evolution of technologies, especially in the areas of communication and computer environment building, were convinced that learner experiences in mathematics could be further enriched and that these experiences could be structured so as to lead students towards more independence in learning. Over the next five years an innovative core program in mathematics was developed and approved. The philosophy and aims of this program, MICA are described in the Brock Teaching journal (5). Student interest in the MICA program is demonstrated by an increase in mathematics majors from 52 in the first year of the program, 2001, to 140 in 2005. In the following sections we explore how the faculty worked to meet MICA guiding principles, including 1) encouraging student creativity and intellectual independence, and 2) developing mathematical concepts hand in hand with computers and applications.
Section 2: Evolving roles of technology in mathematics education at Brock
In this discussion we describe the evolution of the use of technologies in Brock's mathematics programs. In order to facilitate our points of view we shall use the following definitions :
• Digital information – data, algorithms, responses, etc. that are available through technologies;
• Knowledge – the acquaintance of information obtained through experience or instruction;
• Understanding – the power or ability to acquire and interpret knowledge.
A principal aim of integrating technologies into mathematics programs at Brock is to teach students how to transform information into understanding. Initially the teaching and learning process matched the one that the Department used before the birth of digital information, namely
Figure 1
This model also works well with mathematical technologies such as Maple, Mathematica, Minitab, SAS, etc... These are more than repositories of information, they are intelligent
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, in the sense they are capable of generating new information. A challenge for undergraduate mathematics education continues to be that such systems can, for the knowledgeable user, provide solutions to most well structured problems that arise in the first three years of a traditional university mathematics program. The integration of these technologies into the Brock mathematics programs changes the

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In this text we use the term 'intelligent' to distinguish from technologies that are passive, i.e., strictly provide data.
Information Student Knowledge
Mediator (teacher)
Proposes questions
Student understanding
Mediator (teacher)
Provides examples77
first box in Figure 1 and adds digital information to traditional forms of information (texts, lecture notes, etc.). This addition provides many ways in which to enrich the base of student knowledge, for example: faculty can spend more time on the development of mathematical concepts because they and the students can rely on the technology to provide technical information; alternative representations are often easily generated; students can work on problems and applications that are not bound by traditional course information, and; learners can explore conceptually advanced mathematical concepts which are normally deferred until all the analytical skills have been addressed. In summary, by the mid 1990s, information technologies were well established in a majority of mathematics Brock courses.
By that time some faculty became aware of the great potential of communication environment building technologies. Their vision was that a program would be developed to integrate these and to motivate its students to take on, more and more, the responsibility of mediator in their own mathematics learning (second and fourth box in Figure 1). Ralph (7) summarizes the situation as follows: "The central challenge of any mathematics program is to create an environment in which students become internally directed and personally invested in moving themselves along the long road to mastery. The problem with traditional undergraduate mathematics programs in this regard is that if students try to take the initiative in creating and investigating problems and applications of their own devising, they quickly come up against difficulties that they cannot handle with purely analytical tools. For this reason, traditional programs must be very tightly choreographed around the problems that can be solved by hand and over the years this approach can become "canned" and regimented. Technology can offset the rigidity of a traditional mathematics program by providing students with access to an endless supply of problems and applications that can be investigated both computationally and analytically." Therefore the aim of the proposed MICA program was to change the model in Figure 1 to the following:
Figure 2
In many ways this parallels the development that one would hope for in a mathematics program that has a core of modelling courses. However students in the MICA program build on a knowledge base that is more extensive as the information they have access to includes both passive and intelligent digital sources.
How does one educate a student to become her own mediator and how is this done as early as possible in their university mathematics experience? At Brock students take, in their first semester, a course in Calculus and one in Linear Algebra. Both of these courses include extensive experiences with Maple and with Journey
Information Student Knowledge
Mediator (student)
Proposes questions
Student understanding
Mediator (student)
Provides examples78
Through Calculus (8). A brief discussion on the first MICA course that students take in the second semester and upper year MICA courses will highlight the approach that the Department has taken to encourage creativity and intellectual independence as the students develop mediator skills.
Section 3: MICA courses – directions and the integration of technologies
In the first part of the MICA I course, students are exposed to a rich context for conjectures: prime numbers and Collatz conjecture. During lectures
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, students work in small groups of 3 or 4 and raise original questions and conjectures about the topics. These are written on the board and a discussion on their testability follows. For their first assignment, each student designs a program (vb.NET) in which they explore a conjecture of their own. In the second part of the course, students are introduced to modular arithmetic leading to the theory behind RSA encryption. The speed of the theoretical presentation is determined by the students as they lead the way by making observations and conjectures from explorations, computations and theorems. Of course the lecturer guides students but importantly he/she reacts to class questions/ observations/ conjectures that are constantly encouraged and raised. Students then implement the complete algorithm of RSA encryption. The last topic in the course is discrete and continuous dynamical systems. Each student designs a program that outputs numerical values and graphs the cobweb diagrams of the logistic function. This topic concludes with an exploration, in the lab, of the system stability which students simultaneously test and visualize the theory with their own program.
A major part of MICA I is the original final project that encompasses a computer program and a written report. Each student selects a mathematical topic in which they are particularly interested and intrigued. Mathematics students focus on a mathematical investigation. Future teachers design a learning program about an elementary or high school mathematical concept. Students from another core discipline investigate a mathematical application to their own discipline. In this project students essentially construct and implement a Learning Object -- an instructional component that focuses on one or two mathematical concepts and that is designed for another person. These objects are interactive, engaging, easy to use, and are designed to mediate the user from information to understanding. In the MICA program Learning Objects may include exploration of a mathematical conjecture or of a mathematical application. The main goal in MICA I course is to bring students to experience becoming the mediator through the design of original Learning Objects.
The first experimental project on Learning Objects at Brock was undertaken in the summer of 2002. It involved a team composed of mathematics professors, practicing mathematics teachers, future teachers, and mathematics and computer

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Presently the course runs with two hours of lectures and two hours of labs per week. The experience of the Department is that this type of course works best with a maximum enrolment of 35 students in each section. 79
science students. Examples of finished products can be viewed on the departmental website (9). Other examples of Learning Objects developed by students in the MICA courses are also available (10).
It is our experience that the construction of a Learning Object not only builds on the designer's mathematical and didactical knowledge but it reveals these understandings in a visual and interactive way.
In the MICA II full year course the focus is on mathematical modelling of diverse types including, for example, discrete dynamical systems, stochastic models, Markov chains, empirical models, and queuing models. These topics, covered in the MICA way, are all implemented (mainly in VB.NET and Maple) by students and are concluded with simulation and conjectures. For example, students design (VB.NET) a Learning Object to explore the distribution histograms and graphs of random variables. This is done before the students see the Central Limit Theorem in their Statistics course. Therefore students are guided through different computations, and are asked to develop conjectures based on their observations. Not all students are able to conjecture the theorem on their own, but after a full class discussion about plausible conjectures they are able to identify examples of the theorem in their results. When students finally see the theorem in their statistics class, it is no longer a theorem outside them, but indeed, it is somehow internalised since they personalized it within the design and use of their Learning Object. MICA II students work on two main original projects for which they personally decide on a topic. Their projects are

significantly more sophisticated than in MICA I, since they have a better mastery of the technology and importantly, they have become more confident in their role of mediator.
The MICA III full year course is focussed on partial differential equation modelling including for example heat flow and wave propagation. Guided assignments and projects (mainly in Maple and C++) each include an original part in which students have to fully use their role of mediator. For example, students were assigned to extend and improve some MAPLETs that animate solutions of particular PDEs. Two students presented their remarkable MAPLET extension at the Maple Summer Workshop in Summer 2004. With mastery of technology and with their ability to mediate their own learning, undergraduate students can contribute to the development of new mathematics.
Section 4: Technologies and the education of future mathematics teachers
Teacher Education in Ontario follows a consecutive model. This means that individuals interested in teaching must first graduate with a university degree and then apply to a Faculty of Education for a one-year program. After completing this additional year they receive a teaching certificate. For future elementary and middle school teachers and for future mathematics teachers at the secondary level, the consecutive model clearly places important responsibilities on departments of mathematics. How can these future teachers be best educated in mathematics to meet their specific and desired goals? Unfortunately many universities do little more than to pay lip service to this population of students. Within the Brock community, Muller 80 has been proactive in negotiations across different Faculties in order to develop and establish concurrent education programs. In these programs, students follow integrated studies between a Faculty that offers a teachable subject
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and the Faculty
of Education. For those students who enter university with a desire to become teachers, concurrent programs provide opportunities to reflect on didactical issues starting from their first undergraduate year.
The Mathematics Department at Brock has taken its responsibilities for future teacher education very seriously and has developed programs or courses for all levels of school teaching. Appropriate technologies such as Geometer's SketchPad and other Ministry of Education school licensed programs are used in appropriate courses. Concurrent education students who aim to specialize in mathematics and to be certified for teaching at the middle and high school levels, take a majority of the MICA courses which play a fundamental role for them. They provide a unique opportunity for these future teachers to reflect on their own development as a mediator. Furthermore in their MICA projects they construct Learning Objects which have strong didactical components.
Section 5: Reflections by a new faculty, Chantal Buteau
I am currently in my second year as faculty in the Department and I'm coming from a rather traditional mathematics education. I knew that I was joining a department that makes extensive use of technologies in its courses. Therefore I had mixed feelings, anxiety, insecurity and excitement. In service courses (Calculus and Statistics for large classes) my main concern was and is to focus on concepts rather than on computational techniques that can easily be handled using technology. I admit that it is a constant battle for me. When I was taught these concepts there was equal emphasis on concepts and computation abilities. Diverse and rich discussions with colleagues help me to find a good balance. Also, my class preparations keep changing as I rethink what should be first discovered by students in a guided assignment using technology rather than directly presented to them. My conception of assignments and exams also had to be changed. As a new lecturer, it has been a genuine and enriching challenge not to copy the teaching model I had experienced. In the MICA I course I faced teaching an innovative course in which the how to present the theory was more important than the what. On top of this, the how was supported by a programming environment. Fortunately, during my PhD, I had experienced some experimental mathematical investigations supported by technology. This was my beacon together with uncountably many discussions with my colleague Bill Ralph who has been teaching this course since the MICA program was first launched. It did take me some time to understand my role in the course. How can I best assist the students to become the mediator of their own mathematical development? What mediation should or should not be provided at any particular time? I had to adjust to the fact that a class can sometimes take a direction different
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Teachable subjects are specified by the Ministry of Education as being appropriate major disciplines for future teachers 81
from what I had planed. This is not a secure position for a fairly new lecturer, as I had to build on class interactivity and not reject it. I challenge students to explore mathematics on a personal level. Students challenge my traditional education of mathematics teaching.
The astonishing pride of MICA I students for their final projects confirmed that the department is for me a great environment for learning how to teach mathematics in the XXI century.
Conclusions
Technologies are evolving so rapidly that there are many avenues that mathematics departments can take to integrate them into their mathematics programs. This submission describes one route that the Department at Brock has taken to structure technological environments to help students engage in abstract mathematics. We have found that the approaches, activities, and experiences in the MICA courses are able to harness the students' motivations thereby empowering them to become their own mediators in the development of their mathematical knowledge and understanding.
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Varying pedagogical media: How interaction with spreadsheets might mediate learning trajectories
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This paper is concerned with the use of spreadsheets within mathematical investigational tasks. Considering the learning of both children and pre-service teaching students, it examines how mathematical phenomena can be seen as a function of the pedagogical media through which they are encountered. In particular, it shows how pedagogical apparatus influence patterns of social interaction, and how this interaction shapes the mathematical ideas being encountered. Notions of conjecture are considered, and the trajectories learners negotiate as they settle on subgoals, reflect on output, and further develop their emerging theory. The particular faculty of the spreadsheet setting is examined with regard to the facilitation of mathematical thinking. Employing an interpretive perspective, a key focus is on how alternative pedagogical media and associated discursive networks influence the way that students form and test informal conjectures.
Introduction
The pace of development in digital technologies is rapid, with an understandable lag between innovation and well documented classroom-based, educational research. There is benefit from examining more easily accessed software over extended periods, through a multitude of perspectives, so as to inform educational communities in different ways. It allows, for instance, the potential to move beyond the influence of a particular software package on a particular mathematical content area, to more generic pedagogical issues. Compared to some digital tools, spreadsheet software is relatively accessible and could therefore be considered to offer a more equitable digital environment, in both local and global situations. The study described was part of an ongoing research programme exploring how spreadsheets might function as pedagogical media, as compared with pencil and paper methods. In being used as a tool for investigation, how might spreadsheets colour the learning experience and, in particular, how might this influence learner's perceptions and understandings of mathematical phenomena? One aspect of this programme, to be pursued here, was to identify the ways in which participants approached mathematical investigations, from how they negotiated the requirements of the tasks, to how they produced their conjectures and generalisations. The paper then is positioned in the theme of learning and assessment, and has as its most prevalent approaches the contribution to learning mathematics and theoretical frameworks.
Literature Review 83
Greater emphasis on inquiry methods, activates interplay between the task of the individual learner, and the way in which that is understood as an engagement within a more social frame. While the introduction of a social frame is inevitable, this will vary according to how the activity is constructed and the perceived environment within which this takes place. The mathematical activity is inseparable from the pedagogical device as it were, derived as it is from a particular understanding of social organisation, and hence the mathematical ideas developed will inevitably be a function of this device (Brown, 1996). A hermeneutic, phenomenological perspective is concerned with interpretation where the subject views the world by means of a variety of cultural forms through which understandings are filtered. In this context, particular pedagogical media can be seen as cultural forms and different forms model different ways of knowing (Povey, 1997). Ricoeur's (1983) notion of the hermeneutic circle emphasises the interplay between understanding and the narrative framework within which this understanding is expressed discursively, and which helps to fix it. While these 'fixes' are temporary, they underpin the understanding that follows and the way this comes to be expressed. The internalisation is manifest in what they say and what they do. This enables the contention that examining the participants' social interaction and output, will give insight into the ways they internalize mathematical understandings. This can be reconciled with other theoretical perspectives (Mariotti, 2002) that position language, being cultural artifacts, as semiotic mediators.
The prevalence of digital media generally has begun to challenge the map of mathematical ideas encountered in schools. Access to many key elements of school mathematics has been altered, as different technologies offer new ways in which certain constructs are created and understood. Studies involving the dynamic geometry software, Cabri-geometre, (Laborde, 1999, Mariotti, 2002) assert that conceptualization of mathematical phenomena, will be different when engaged through the particular software lens. Meanwhile spreadsheets have been found to offer an accessible medium for young children tackling numerical methods. Researchers have highlighted their suitability for an investigative approach as students learn to pose problems and to create explanations of their own (Ploger, Klinger and Rooney, 1997), and simultaneously link symbolic and visual forms (Baker and Beisel, 2001). Other characteristics, including their interactive nature (Beare, 1993), and the capacity to give immediate feedback (Calder, 2004) appear to give the learner the opportunity to develop as a risk taker; to make conjectures, and immediately test them in an informal, non-threatening, environment. This permits the learner opportunity to reshape their conceptual understanding in a fresh manner. The capacity to provide instantaneous feedback also allows for conjectures to be immediately tested and perhaps refuted. Lin (2005) claims that refuting is an effective learning strategy for generating conjectures. Mathematical conjectures often have speculative beginnings and as Dreyfus (1999) implies, have elements of logical 84 guesswork. Other researchers often consider them as generalised statements, containing essences distilled from a number of specific examples (Bergqvist, 2005). In their embryonic form they emerge as opinions, mathematical statements, generalisations, or positions. These can then be challenged or confirmed with explanation, leading to mathematical thinking. Suggesting counter-examples, or exposition of how two mathematical explanations are similar, indicate a more robust form of examination of the conjecture (Manouchehri, 2004). The learner's perturbance, when gaining immediate access to counter-intuitive outcomes to inputted data, and the subsequent influence of that tension on the investigative process, also influence the investigative trajectory

Approach
The paper considers two settings where investigation takes place in a spreadsheet environment. The first situation located groups of three, first year, primary, preservice students in a typical classroom setting while groups, from the same class, worked in an ICT laboratory, doing the same investigation using spreadsheets. Their discussions were recorded and transcribed, each group was interviewed after they had completed their investigation, and their written recordings were collected. This data, together with informal observation and discussions, formed the initial basis for the research. Five weeks after the first data was gathered, a similar approach for data collection was used, with the students using the same medium, but a different investigation. The second situation involved ten-year-old students, attending five primary schools, drawn from a wide range of socio-economic areas. There were four students from each school, eleven boys and nine girls, of mixed ethnicity. The data was produced in the same way as the first situation. The transcripts from both were then systematically analysed for patterns in the dialogue, within and between the settings.
Results and Analysis

Two aspects were considered in the formation and testing of conjectures. Firstly, the data is examined for differences the pedagogical media may have evoked, with particular regard to the pre-service teaching students. An episode with the ten-yearold learners is then analysed with regard to the notion of subgoals
Comparison of two pedagogical media
The dialogue in each situation demonstrated a contrast in the initial approach to engaging in the mathematics. In the classroom situation it began with a group member initiating the negotiation of the meaning and requirements of the task with a single discrete numerical example. For example, group one.
Kari: Lets try each number one at a time. One times 101 is 101. 85
Group two likewise used this to begin the process of solving, but also to help determine the nature of the task; what it was asking them. For example:
Sarah: So if we had twenty three times a hundred you would have twenty three hundred...Lets say we do twenty three times a hundred and one, we would get twenty three hundred plus twenty three ones
As they made further sense of what the problem was about, they began to predict, verify and reflect in a discrete numerical manner.
Rachel: We went through one at a time and solved them. We solved them on paper and we solved them with a calculator.
In contrast, those groups working in the spreadsheet setting used the spreadsheet to get a broad picture; they utilised the formulae and copy down functions to create a numerical table that could then be examined for any pattern. They used more algebraic language, while the pencil and paper groups had more numerical reference.
For example:
Kylie: I haven't predicted. I was just going to put in A1 times 101 and drag it down.
Josie: So we're investigating the pattern of 1 to 16 times 101
This appears a more direct path to the patterning approach, and these groups quickly recognised a pattern, and explored further based on visual aspects. It also introduced a difference in terms of the technical language utilised. "Drag it down" is functioning language rather than mathematical, but the inference is clearly that there is a pattern, which might possibly lead to a generalisation; and that the spreadsheet by nature will enable users to quickly access that pattern. Most significantly, the social interactions appear to shape the analysis of the patterns in distinct ways. Given that the path to, and manifestation of, the patterns differs, the dialogue indicates a different approach once the patterns are viewed. Those using the spreadsheet used a more visual approach. They were observing and discussing visual aspects eg the situation of digits or zeros. For example:
Jo: With two digits you just double the number. You take the zero out.
Those using pencil and paper were more concerned with the operation aspects that generated the patterns. For example:
Sarah: Basically, if you times your number by a hundred, and then by one, you would add them together, and get your answer.
To generalise a pattern in terms of the sequence of digits is significantly different to generalising in terms of an operation. In this aspect, the different settings have certainly filtered the dialogue and approach to forming conjectures, and by inference the understanding.
The influence on sub-goals
The characteristic of spreadsheets to produce immediate responses to inputted data permitted new sub-goals to be promptly set, assisting the emergence of a theory. The data produced relates to an investigation involving exploring terminating and 86 recurring rational numbers, when one is divided by the counting numbers. In the first case they negotiated to gain some initial familiarisation of the task.
Sara: One divided by one is one - it should be lower than one.
They entered 1 to 5 in column A and =A1/1 in column B to get:
1 1
2 2 etc
This posed an immediate tension with their initial thoughts. After exploring various formulae and associated output, they settled on a way to easily produce a table of values to explore. The spreadsheet environment shaped the sense making of the task and the resetting of their sub-goal. Critically, it had enabled them to immediately generalise, produce output, then explore this visually. They generated further output:
1 1
2 0.5
3 0.333333..
4 0.25
5 0.2;
6 0.166666..etc.
Sara: So that's the pattern. When the number doubles, it's terminating. Like 1, 2, 4, 8 gives 1, 0.5, 0.25, 0.125.
Jay: So the answer is terminating and in half lots. Lets try =0.125/2; gives 0.0625- which is there. (Finds it on the generated output from)
The structured, visual nature of the spreadsheet prompted the children to pose a new conjecture, reset their sub-goal and then allowed them to easily investigate the idea of doubling the numbers. The table gave them some other information however.
Jay: 1 divided by 5 goes 0.2, which is terminating too. (Long pause)
After further exploring, they reshaped their conjecture, incorporating their earlier idea.
Sara: If you take these numbers out they double and the answer halves.
Jay: That makes sense though, if you're doubling one, the other must be half. Like 125 0.008; 250 0.004
Sarah: What's next. Let's check 500
Jay: Let's just go on forever!
They generated a huge list of output; down to over 4260.
Jay: 500 0.002; 1000 0.001. When you add zero to the number you get a zero after the point
Although this particular group didn't fully explore the relation of the base numbers to the multiples of ten, they have made sense of, explored, and generalised aspects of the investigation. The pedagogical medium through which they engaged in the task has influenced the contextualization and approaches they have taken. When asked: 87 "When you saw the problem, how did you think you would start?" the children's responses corroborated this perspective.
Sara: Re-read to get into the math's thinking, then straight to a spreadsheet formula.
Greg: I type what I think and try it
As well, the spreadsheet groups progressed more quickly into exploring larger numbers and decimals. This appears to indicate a greater propensity for exploration and risk taking, engendered by the spreadsheet environment. It seems the spreadsheets have not only provided a unique lens to view the investigation, but have drawn a distinctive investigative response.
Fran: Using a spreadsheet made it more likely to have a go at something new because it does many things for you. You have unlimited room. You can delete, wipe stuff out.
Chris: Columns make it easier- they separated the numbers and stopped you getting muddled. It keeps it in order, helps with ordering and patterns.
Conclusions
This study demonstrated that the different pedagogical media provided a distinct lens to contextualise the mathematical ideas, frame the formation of informal mathematical conjecture, and condition the negotiation of the mathematical understanding. As Brown (1996) argued, the mathematical understanding is a function of the social frame within which it is immersed, and the social frame evolves uniquely in each environment. The data supported the supposition that the availability of the spreadsheet led the students to familiarise themselves with, then frame the problem through a visual, tabular lens. It is clear also that it evoked an immediate response of generalisation, either explicitly through deriving formulas to model the situation, or implicitly by looking to fill down, or develop simple iterative procedures. Tension, arising from differences between expected and actual output, and opportunities, arising from possibilities emerging from these distinctive processes, led to the setting and resetting of subgoals. These, in turn, further shaped the understanding of the investigative situation, and the interpretation of mathematical conjectures.
The spreadsheet approach, perhaps due to the actual technical structure of the medium, led more directly to an algebraic process, with the language interactions containing both algebraic and technical terminology. It seemed, in fact, that the spreadsheet setting, by its very nature, evoked a more algebraic response. The participants in these groups were straight away looking to generalise a formula that they could enter and fill down. Their language reflected this, but the interactions also contained more language of generalisation, and it took them generally less interactions to develop an informal conjecture. Those working in the classroom setting used a discrete numerical example to engage in the problem; to make sense of its requirements as well as initiating the process of solving. They tended to try, 88 confirm with discussion and then move more gradually into the generalisation stage. Their conjectures were slower to emerge not only due to variation in computational time, but also due to the approach the spreadsheet evoked. The way they thought about the problem was different. Their initial dialogue seemed more cautious, and contained comments requiring a degree of affirmation amongst group members before moving into developing their conjecture. As a consequence the descriptions of the process undertaken and the mathematical thinking were more fulsome. This may be evidence of more fulsome understanding too.
The children also identified speed of response, the structured format, ease of editing and reviewing responses to generalisation, and the interactive nature as being conducive to the investigative process. While this particular medium has unfastened unique avenues of exploration, it has as a consequence fashioned the investigation in a way that for some learners may have constrained their understanding. The approaches and outcomes, as reflected in the dialogue, are different. If the dialogue between learners filtered the mathematical thinking and formation of conjectures in different ways, according to pedagogical media, then perhaps complementary approaches would give opportunity to enhance mathematical understanding. As well, how gestures might mediate the learning, and how the findings of this ongoing study might resonate with those investigating other software, for example Cabri-geometry, are key aspects to also consider.
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An emergent field of enquiry:
the introduction in the classroom of E-Exercise Bases
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Investigating on E-Exercise Bases (EEBs) is a necessity. This presentation successively focuses on teachers, students and mathematics. The anthropological approach and the methodological individualism frame account for some regularities and disparities in teachers' and students' attitudes towards these tools and specify the kind of mathematical work accomplished. The results presented here are based upon observations in various French high schools and universities.
The purpose here is to enquire about the use of E-Exercises Bases (EEBs) in teaching and learning mathematics. It is especially related to theme 2 of the ICMI Study "learning and assessing mathematics with and through digital technologies". Indeed, some aspects of "how students learn mathematics with -these- digital technologies and the implications of the integration of -these- technological tools into mathematics teaching for assessment practices" are investigated. According to the discussion document of the ICMI Study, several approaches are given. After exposing the nature of EEBs and why we should inquire about their use, the three usual poles (teacher, student and mathematics) are successively analysed in an EEB environment. This study is supported by emblematic examples taken from quite an important number of observations in ordinary classes in France.
The nature of EEBs
EEBs mainly consist of classified exercises and propose in addition to these an associated environment which can include advice, solutions, corrections, explanations, tools for the resolution of the exercises, score and even sometimes corresponding courses etc. They differ from microworlds or computer algebra systems (CAS) which are open environments in which generally no specific tasks are predefined.
As an example, let us look at a particular EEB : Wims (http://wims.auto.u-psud.fr). This software is a collaborative one, available in six languages and initially developped by French Profesor Xiao Gang. All examples given hereafter will stem

from this EEB. Wims is a library of on-line interactive mathematics resources which includes exercises for all levels: from primary to tertiary education. Teachers can choose some of these exercises and build their own on-line worksheets for their students. Students can do the same exercise several times in order to improve their marks. In such case, the structure of the exercise will remain the same, but its numerical values will differ. 91
Faced with the variety of EEBs, some tools were designed to describe and to evaluate them, under various aspects. In a recent paper, Tricot (2003) uses three key concepts that are linked: usefulness, usability and acceptability. To show this variety, here are some variable characteristics of an EEB. Each time, we classify Wims in this variety.
• The EEB may be « opened » or « closed » to the teacher. Wims is opened because teachers may choose the exercises and the parameters linked and design the exercise sheet wanted. To create exercises, some computer knowledge is required.
• Expected answers differ from one EEB to another. For Wims, answers are numerical values, MCQs or short mathematical expressions but not sentences.
• Feedbacks are another EEB characteristic. Wims's feedbacks are: "right" or "wrong" with the right answer given and a mark. In several exercises feedback can be more interesting as we will see later. Wims does not provide any proof.
• Help or advice is also variable. Wims suggests some generic tools such as formal calculators and graphic tools but no specific helps adapted to any particular exercise.
Why investigate exercises set on line?
An inventory of didactic research in the area of ICT tools has shown the importance given by researchers to microworlds and CAS (Lagrange & al, 2000). On the contrary, research works especially devoted to questions raised by the use of EEBs are scarce. However, the number of EEBs is growing every day and EEBs themselves evolve very quickly. Moreover, it is not relevant to apply the results obtained with other ICT tools to EEBs. A lot of studies with microworlds and CAS consist in the building and the study of didactic situations where the "milieu" produces contradictions, difficulties and disequilibria (Brousseau, 1997); whereas, in EEB situations, the "milieu" is often friendly. Therefore, the economy of the mathematical work is altered.
The institutional encouragement to integrate the digital technologies into the curricula and educational practices is another reason to investigate EEB's use. For instance, in France, the network connection of educational establishments has allowed the introduction of new pedagogical tools called "Espaces Numériques de Travail" (global digital learning environments). Each student has access to a set of personal and general resources through the network. Some of these pedagogical resources can be EEBs. Another example of French institution commitment is a regional project focusing on the use of EEBs at high school level. Some specific results presented further stem from this project.
This trend is general. In their survey regarding the use of technology in mathematics courses in England, Ruthven and Henessy (2002) also observe that working with EEBs is often mentioned by teachers as a help to organise sessions where the students can work at their own pace. 92
The questions are: how to work with an EEB in a classroom? How do teachers manage the computer sessions? How do students work with this tool? And (part 4) what kind of mathematics are they doing?
How do actors (teachers and students) cope with the use of an EEB?
The use of an EEB brings modifications in the classroom. After having specified the observed sessions and the framework, we report in this part on teachers' and students' adaptations to these modifications.
Observed sessions
We observe ordinary classes and we are not concerned with the building and the analysis of didactical engineerings. All observed sequences in high school or university are training exercises and most are organized in the common following way. Before the sequence, the teacher builds a digital exercise sheet or designs a path in the EEB. During the sequence, students work, alone or in pair, in computer and paper/pencil environments. According to the EEB and the exercise, they may or may not put some advices into practice or ask the teacher. They enter their answer on the computer. The EEB then provides a mark, a comment, the correct answer or all the details of the proof. During the sequence, the teacher helps students individually. Most of the time, the program is long so that no student can finish it during the sequence. The end of the sequence is the end of the time granted: the teacher either says "you have to finish at home" or "we shall finish next time" or even nothing concerning the completion of the exercise sheet.
Framework
In part 2, we saw, that the institution must be recognised as essential in the process of ICT integration. However, students can have different paces during the observed sessions, and can also follow different paths among exercises. It is thus necessary to choose a framework to articulate these two aspects: institutional pressure vs individual approach. This is why we first turn our attention to the anthropological approach developed by Chevallard (1999). This approach with its institutional basis gives a proper place to institutional issues. To articulate this frame with the actors' individuality the analysis of EEB integration is supported by methodological individualism. The later is a philosophical method aimed at explaining and understanding broad society-wide developments as the aggregation of decisions by individuals. In sociology, Jon Elster among others follows this lead: "to explain social institutions and social change is to show how they arise as the result of the actions and interaction of individuals. This view, often referred to as methodological individualism, is in my view trivially true." (Elster, 1989). According to Bernoux (2000), the viability of a modification in a community is the result of collective rules and of the individual meanings linked to them. Notice that these individual meanings are not easy to identify.
The teacher's role 93
Analysis discussed here stem from observations at high school level. As mentioned previously all the observed sequences are training exercises in which teachers help students individually. However, there are some differences from one class to another and from one teacher to another. In our view point, the chosen framework can help to understand both regularities and diversities.
For teachers, rules come from the institution which encourages the use of ICT tools and allows the teachers some flexibility. For example, teachers involved in the regional project can, from a panel, choose their preferred EEB and the Region will finance the EEB which is not free. Another important rule exists even if it is not directly linked to ICT tools: the mathematics syllabus for high school requires teachers to manage specific sessions with pupils in difficulty. Observations show that teachers often use EEB to manage such sessions. EEBs allow them to respect the working pace of each student and to help them individually if necessary. So, it is a way to satisfy this double injunction which comes from the institution: to use ICT tools and to support individually students. It may be a way for teachers to make sense of the integration of ICT tools. Therefore EEBs are easier to integrate into their practice than other kind of ICT tools. This idea can be linked with Cuban's results (2001): « the point is that teachers change all the time. It is this kind of change that needs to be specified. Champions of technology wanted fundamental change in the classroom practice. The teachers that we interviewed and observed engaged mostly in incremental change." However, to keep this teacher's individual meaning, it is necessary for the institution to guarantee a good technical material and to maintain the student's individual help injunction.
Among these regularities, some disparities exist. First of all, teachers do not choose the same EEB. Interviews show that their expectations are also different. They do not choose the same didactical organisation. For instance, they may organize EEB sessions with weak students or with strong ones to have more time to work face to face with weak ones. They may also design different paths in the EEB according to the students. A difficulty observed and expressed by students and teachers, is to keep written tracks of the work done and, more generally, to articulate paper support with support screen. Teachers also have different choices about this point. For instance, some of them ask students to use a specific notebook for EEB sessions, others insist on the same notebook and a third group- frequently teachers recently involved with EEB- will have no notebook for EEB sessions.
The student's role
In this part, the discussion is based on observations at tertiary level. The chosen framework allows to discuss both observed regularities and diversities.
As in any classical exercises' session, the rule for students is to solve the exercise sheet and to ask for help if needed. However, students seem to work really hard during the observed sessions, may be more than in classical exercises sessions. Our explanation is that the rule is not exactly the same. Students know that if they do nothing, nothing will happen. In particular, there is no common solving moment they 94 can wait for. Actually, students have more responsibilities. They may choose their own path among the exercises: decide to use or not helps or to have a glance at solutions, decide what to keep on their written notes etc. Hence there is an obligation to act and a multiple choice of actions. We think that these two points help students to give personal meaning to these sessions. So the hypothesis to explore here is: working with an EEB introduces some liberty which enhances students' activity by adding more meaning to their work. Notice that these new responsibilities require them to produce more difficult cognitive efforts.
However, if all students seem more active in EEB sessions, the type of activity is different. We, sometimes, observe strategies such as scoring (students always solve the same kind of exercises to have a good mark) or random answering (especially in MCQ). Log files when available also show that, on the contrary some students work a very long time at home on the EEB. These different attitudes could also be linked with another rule: to pass the exam. In some interviews, some students said "working with an EEB is funny but we will never have such exercises at the exam". So a consequence from the dialectic between rules and individual meaning is: the more EEB sessions are linked to the exam, the stronger the students' efforts are. Thus to ensure a successful integration the first challenge for the teacher should be to build didactic organisations linking together classical sessions, EEB sessions and exams.
We have seen that EEB sessions can increase students' activity. Now let us inquire about what kind of activities.
Mathematical activities developed by an EEB use
Personal and institutional practices can be explained in terms of praxeologies in Chevallard's approach. They are described by three main components: the type of tasks, the techniques used to solve these tasks and the technologico-theoretical components that is to say the discourses and the mathematical basis which are used in order to both explain and justify the techniques. The advance of knowledge requires the routinisation of some techniques and in a more general idea, a work on the techniques. In this part we would like to stress out that EEBs can make this process easier. Moreover, we notice that EEBs enable to highlight some specific techniques.
The importance of the mathematical work dedicated to the technique
As previously mentioned, all observed sessions with EEBs are training exercises and students work on already learned techniques. The process of routinisation may require a critical quantity of exercices and EEB sessions is one way for students to reach their own necessary quantity. For instance, EEB sessions may allow to learn technical solving methods step by step. In the following example, students must choose the successive adequate operations and the computer executes them, giving the new equation after each step. At the end the computer assesses both the result and the number of steps to obtain it. 95
Fig. 1
This exercise is interesting because it allows to train oneself in one specific part of the technique: the choice of the operations. The computer takes in charge the calculation. Some of these exercises could not be suggested to students without EEB and would otherwise be sent back implicitly to a possible personal work.
Mathematical techniques highlighted
Due to the MCQ form of some exercises, student may use techniques such as testing or eliminating. For instance, here is a Wims exercise to work in algebraic and geometric frames. According to Douady (1986), this kind of tasks is known to be useful in the learning process because it creates a link between algebraic frame and graphic frame.
Fig. 2
It is necessary to discuss all these techniques: they are mathematical' ones and not classical' ones. Because they ask students to have a control over their result, it can be very interesting to train them into this type of reasoning.
Of course, it is also possible to search such exercises in a paper and pencil environment but, with EEB, it seems easier to generate as many exercises as 96 necessary for each student. Sometimes, EEBs also offer tasks which are impossible to work with in a traditional environment as in the following Wims' example.
Students must find an algebraic expression whose curve is the closest to the given one (here a lower red line). The student's answer is the formula g(x)=(x/2)+1. EEB's feedback is the representative line corresponding to this answer. The student may then realise that his/her answer is wrong and try to adjust it with a better one. The answer is wrong however it is not far off and the student may find the right answer on his/her own, making links between algebraic and geometric modifications. One can examine this exercise, in terms of didactical situation theory (Brousseau, 1997). In this example, the milieu is antagonistic and the student may acquire knowledge thanks to the feed-backs.
Fig. 3
Conclusion
EEBs are specific ICT tools consisting of classified exercises with an associated environment. The contribution wants to explain that specific research on EEBs' use is necessary for three reasons. More and more EEBs exist. Due to institutional injunctions in particular, their use will increase. Previous studies are frequently irrelevant as they differ from other ICT tools, for instance they provide mostly an allied "milieu". Based upon different EEBs' use observations in French context, this contribution focuses successively on teachers' and students' attitudes and on mathematics embedded in EEBs sessions. The anthropologistic approach and the methodological individualism frame allow us to focus on the actor's attitude.
Regarding teachers, EEBs appear as a way to organise individual support. This organisation may differ from one teacher to the next or even from one class to another. That is the result of the dialectic action of norms coming from the institution, and personal meaning coming from teachers. Students' norms, passing the exam and solving the exercises, and their individual meaning linked to the EEB session imply some regularities and singularities in their attitude. At last, EEBs seem particularly 97

adapted to work on techniques. Some techniques like testing, eliminating, solving step by step, or solving by trials and errors are highlighted by EEBs' use.
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Role of Digital Technologies in supporting Mathematics Teaching and Learning: rethinking the terrain in terms of schemas as epistemological structures

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In this discussion I wish to tackle the issue of how digital technologies (DTs) shape teaching and learning of mathematics. Teachers and students of mathematics use DTs in a multitude of ways to enhance mathematical understandings but there is limited information about the directions and nature of shifts in the conceptual ground gained by the learners and its relationship to pedagogical strategies adopted by teachers. The issue will be examined within the framework of schemas as epistemological structures. Working on the view that schemas provide visual representations of what is learnt, I propose to analyse a series of activities by students and teachers that involve the active use of a variety of DTs. The impact of DTs on students' and teachers' prior knowledge, and the extension of that knowledge will be a principal consideration.

Introduction

The nature of mathematics and mathematics learning continues to be a dominant theme in current debates about reforms in mathematics teaching and curriculum. In so doing, the mathematics education community at large is focusing on issues concerning how individuals come to understand mathematics and how teachers can better scaffold deep learning. Against this background, the perceived and actual role of digital technologies in mathematical pedagogies of practitioners has received particular attention.

In this discussion paper my aim is to a) characterize the development of mathematical knowledge by drawing on schemas as epistemological structures and b) examine the potential and actual role of digital technologies (DTs) in supporting the growth and transfer of such structures for learners and teachers of mathematics. The above aims are expected to address three themes that are the focus of the 17 th
ICMI
Study: Roles of different digital technologies, Contributions to learning mathematics and Role of the teacher.

Theoretical framework

Learning mathematics can be seen as the continuous process involving the assimilation and accommodation of new understandings into existing understandings. This Piagetian notion as it relates to mathematical learning is consistent with the view that mathematics constitutes a corpus of knowledge constructed and used by members of a community. This knowledge and the accepted conventions may appear to be static or 'given' at a particular point in time. However, these conventions and 99 facts about mathematics are subject to change as the community evolves, and other models are developed to make sense of the environment. Thus, there is a need to consider frameworks about learning and teaching that would provide windows into how DTs mediate thinking and construction of new forms of meanings.

Proponents of Activity Theory argue that DTs can be seen as cultural tools as that mediate thinking and thereby create new forms or levels of understanding. How can we characterize the trajectories of understandings that learners follow? While the notion of trajectories of understandings seems to impose boundaries to the path of understanding, we need some way to characterize changes in learner's attempt to make sense of the mathematical knowledge and its utility. The construct of schemas provides a solid base from which to examine this issue.

Schemas and mathematical knowledge development

One group of cognitive psychologists adopt the network perspective in making judgments about mathematical knowledge development (Anderson, 1977; Marshall, 1995; Sweller, 1989). According to this view conceptual growth and mathematical understanding can be interpreted in terms of conceptual nodes and relations between nodes. As students' experiences with a concept or a set of concepts increase, they come to form organised meaningful wholes called schemas. Schemas can be visualised as knowledge structures or networks having one or more core concepts that are connected to other concepts by relational statements. The relations that are found between concepts that form a schema could denote a number of features including information about (a) similarities and dissimilarities between those concepts, (b) procedures for using the concepts for solving problems and (c) affective factors about those concepts. Chinnappan (1998), for example, provided data that showed that schemas in the domain of geometry could be organised around axioms or theorems about Euclidean geometry.

According to Anderson (2000), two variables determine the quality of a schema: the spread of the network and the strength of the links between the various components of information located within the network. A qualitatively superior schema can be characterised as having a large number of ideas that are built around one or more core concepts. Further, the links between the various components in the network are robust, a feature which contributes to the accessing and use of the schema in problem-solving and other situations. A high quality schema can also benefit students by helping them assimilate new mathematical ideas because such a schema has many conceptual points to link with. As a theoretical construct schemas provide a useful way to interpret the growth of advanced mathematical knowledge by identifying pedagogically important relations.

Role of schemas in problem representation

It is assumed that performance in mathematical tasks is to a large measure dependent on accessing and using prior knowledge that is organised in the form of schemas. A major advantage of having knowledge stored in clusters or chunks is that they facilitate retrieval of the required knowledge from the long-term memory into the 100 working memory during information processing. In problem-solving contexts schemas play an influential role during the construction of a representation for the problem. Cognitive psychologists argue that the solution of mathematical problems can be greatly enhanced if students are taught to construct useful representations of problems (Frederikson, 1984, Kaput, 1987).

Building a problem representation can be a deliberate process in which students attempt to establish meaningful links between bits of information in the problem statement and knowledge embedded in their schemas that can be related to the problem. Students' repertoires of problem-related schemas could include, but not are restricted to, (a) knowledge of procedures and strategies associated with tackling a group of problems that are similar to the problem in question, (b) mathematical concepts and (c) knowledge about previous experiences with similar problems. Hence, building a representation of the problem involves, among other things, making decisions about what to select from the above range of schemas. This point was made by Hayes and Simon (1977) who have suggested that "the representation of the problem must include the initial conditions of the problem, its goal, and the operators for reaching the goal from the initial state' (p.21).

The construction of representations is a cyclic event where students continue to refine one representation or change to a different one until the correct match is found between schemas that have been accessed and the goal. The goal could be unknown value that has to be determined or a mathematical result that has to be proved via a chain of reasoning.

The above model suggests that instructional methods that would help students decompose problems into sub-problems would benefit them in three ways. Firstly, students might be expected to access previously acquired schemas from their memory by examining what is given in the problem. Secondly, the accessed schemas could be deployed in solution of sub-problems. Thirdly, students could relate the subproblems in ways that would help them reach the problem goal. The net effect of teaching for problem representation is that students are encouraged to access and use a greater proportion of their previously learnt knowledge.

In order for students to develop a sophisticated schema, say, about functions, they need to increase the number and quality of connections between the definition of functions, families of functions and use of functions among others. As the schema expands one might expect information about related concepts such as derivatives and optimisation become more easily incorporated. In other words, an existing schema that has the relevant prerequisite knowledge supports students' understanding of derivatives and optimisation. In this way schemas can be argued to provide a measure of the depth of understanding students develop about mathematical ideas. More critically, schemas provide a useful tool for the analysis of conceptual links between university and secondary mathematics.

Teaching based on schema analysis101

Once we have some information about the gaps and weaknesses in the students' schemas, we are in a better position to devise strategies to help students develop appropriate schemas or modify existing schemas by drawing on a range of digital technological tools. Let's us assume that students are experiencing difficulty in grasping the concept of functions. We could adopt the following three strategies in order to develop schemas and facilitate the transition to more advanced schemas involving a system of three or more linear functions.

Firstly, tutorial-type classes can be effective in 'reteaching' the target ideas such as systems of equations, variables, solution of equations and geometric interpretation of solution of two equations. The nature of the target ideas will naturally depend on the prerequisite knowledge that lecturers consider as necessary for the next level course.

Secondly, students could be encouraged to work in groups on a series of activities that are developed in response to improving the above schema. For instance, we could provide a practical problem that requires the generation and solution of a system of two linear equations. Student could attempt to solve this as a group, after which they could brain storm the problem and their solution in terms of the three concepts above. This activity has the potential to facilitate the construction of new links that were non-existent in the schema of the students in the first instance. Teachers could act as critical friends during this exercise.

The teacher's own understanding of the focus concept (functions) and his knowledge about how the DTs can be used effectively to aid students' thinking about the concept are key to the successful orchestration of the lesson and the above instructional strategies.

Digital technologies and mathematical problem representation

Teachers can use DTs such as computer softwares in multiple ways during the course of problem solving. Firstly, softwares could be used as an evaluative tool to check the quality of students' prior knowledge schemas. A useful strategy here would be to ask students to compare and contrast concepts and procedures that are found in the solutions produced by computers with that of their own. For instance, we could ask students to find the limit of a rational function,f(x) with and without the use of computers. Students' own solution attempts would reveal attributes of their schema in this area. Students could then be required to find the limit of the same function with the aid of computer programs. These programs have in-built facilities that help them visualise the function as well generate a table of values that demonstrate the link between values of x and the limit off(x). That is, softwares provide relatively easy and rapid access to multiple representations of the problem and associated concepts. Palmiter (1991) advocated this technique of using technology to help students build and refine schemas that are rich in conceptual information about calculus.

The comparison of students' answers and that produced by the computer softwares could thus be used as an important learning and diagnostic activity. The 102 more enriched interpretation of the problem provided by the computer solution has significant pedagogical value in that it would help students not only understand the limitations of their schema, but more importantly, demonstrate in a dynamic manner the relationship between the x , f(x) and the limit of f(x). We can go a step further by asking students to justify their solutions and computer-generated solution to peers, and explain any apparent contradictions. This activity would further enlarge their schema for the concept of limit.

Teacher and student knowledge schemas

The learning environment is one in which teachers and students engage in ways that would help teachers share his or her understandings with the students. This engagement can be mediated by the use of DTs with the aim of getting students to build the range of connections that are present in the teacher's schema As an illustration, some students tend to develop a limited understanding of the concept of fractions particularly as these relate to making sense of real-life applications. Teachers are expected to have a wider network of fraction schemas including the

prevalence of fractions in the interpretation of gradient and/slope of an inclined plane. Students could be encouraged to access this feature of fractions by examining contexts where the idea of gradient comes into play. Students could search these contexts on the Web, and teachers could ask them to engage in discussions, say about part-whole relations. The ensuing discussions could aid in the Aid in the discovery of new ways of thinking about fractions and contribute to improved connectedness, the construction of robust schemas. Here is a case of how DTs mediate thinking and active involvement of the learners. The ways tools mediate thinking and knowledge construction has been a central issue for mathematics educators (Gutierrez, Laborde, Noss & Rakov, 1999).
Teacher knowledge and student knowledge involving DTs
Engaging teachers
In a study of use of graphic calculators, Chinnappan and Thomas (2003, 2004) showed that an experienced teacher used digital tools to model the more abstract and complex areas of algebraic understanding. In a related study, beginning teachers engaged in online discussions to discuss the intricacies of teaching multiplications to young children (Chinnappan, 2003). These discussions were conducted via WebCT and other resources available on the Internet.
Engaging students
Chinnappan's (2001) work with young children's understanding and use of fraction concepts during problem solving also demonstrated the pedagogical value of softwares in supporting learners to reflect on their prior understandings. In this study, a software based on JavaBars mediated children's cognitive actions. Ekanayake, Brown and Chinnappan (2003) investigated the conceptual terrain of secondary students as they attempted to solve a series of geometry proof-type problems. The researchers developed a software to guide students to better access their prior knowledge of theorems as well reflect on the areas they need to work in order to 103 make progress with their problem-solving efforts. The need to develop softwares that would scaffold schema development was also supported in another study by Chinnappan, Lawson, Gardner, (1998).
Conclusion
How DTs support deep learning and teaching of mathematics is an important issue for the 17
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ICMI study. My analysis here suggests that we need to examine the way we can map the learning trajectories of individual students, and what teachers do in using DTs to scaffold learning in the process. Schemas as epistemological structures, it would seem, provide a powerful alternative model to analyse the issue.
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In Singapore, the revised junior college mathematics curriculum implemented in 2006 has specifically identified the graphic calculator as an important tool in the teaching and learning of advanced level mathematics topics (MOE, 2004). The study described here, which is part of my PhD thesis, investigates teacher change, in a time of transition from a classroom without graphic calculator use, to teaching in a classroom where graphic calculator has the potential to be an integral part of students' learning of mathematics. This study carried out in 2006 specifically seeks to describe how the concerns of teachers, the teaching strategies of teachers and the roles of teachers change when they integrate graphic calculator into the junior college mathematics curriculum. The study also aims to identify important features among teachers who are successful in integrating graphic calculator into the curriculum. This study is anticipated to complete by end 2006. The contributions from this study will be discussed in anticipation to theme B on teachers and teaching.
Introduction
The graphic calculator is a powerful handheld device that is becoming increasingly affordable and accessible to students and teachers in the classroom. The capabilities of the graphic calculator include drawing of graphs and the execution of numerical, matrix and statistical calculations. There is a large amount of research supporting the use of calculators in teaching and learning of mathematics (Dunham & Dick, 1994; Heid, 1997; Husna, Munawir & Suraiya, 2005; Penglase & Arnold, 1996). The graphic calculator reduces the drudgery of applying arithmetic and algebraic procedures when these procedures are not the focus of the lesson. Students are free to spend more time on problem solving. The graphic calculator also makes it possible for students to visualize data in more than one way. With graphing calculators, students can switch between graphical and numerical representation of data (Waits & Demana, 2000).
The graphic calculator has brought about changes in the curriculum, the assessment mode and the way teachers teach mathematics in various parts of the world. In Singapore, with the revised mathematics curriculum in 2006, graphic calculator will form an integral part of the teaching and learning process in schools. The use of graphic calculators will be expected for all three Advanced Level 106 mathematics papers (H1, H2 and H3
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) offered at junior colleges (MOE, 2004). This
provides a rare opportunity to investigate teacher change in a time of transition.
Rationale and Purpose of the Study
Since the 1980's, many countries have realized the potential of graphic calculators and have integrated or have made recommendations for its integration into the mathematics curriculum. The availability of graphic calculators has resulted in the teaching of mathematics to be reexamined at both the secondary and collegiate levels (Dunham & Dick, 1994). The National Council of Teachers of Mathematics (NCTM) has long advocated the use of calculators at all levels of mathematics instruction, and graphic calculators are no exception (NCTM, 1989, 2000). In 1989, in the Curriculum and Evaluation Standards for School Mathematics, the National Council for Teachers of Mathematics (NCTM) made the following recommendations: "Scientific calculators with graphing capabilities will be available to all students at all times" (p.124). NCTM's most recent standards document, Principles and Standards for School Mathematics (2000), placed greater emphasis on the implementation of technology in the teaching and learning of mathematics by making technology one of its main principles. This principle states: 'Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning' (NCTM, 2000, p. 24).
At present, there is much research being done on graphic calculator usage (Kor, 2004; Noraini, 2005). The majority of research on graphic calculators seems to fall into two categories; namely, student performance, and attitudes and beliefs. Most research studies on graphic calculators involve the investigation of the teaching of a certain mathematics topics using graphic calculator and report on its impact on student performance and understanding of concepts (Burrill, 2002). Authors of various reports have concluded that benefits in student achievement can be derived from appropriate graphic calculator use (Heid, 1997; Husna, Munawir & Suraiya, 2005). The other category of research investigates how teacher attitude, belief and conception of mathematics affect the use of graphic calculator in the classroom (Jost, 1992; Simmt, 1997; Tharp, Fitzsimmons & Ayers, 1997). So far, there is no study done to investigate how the concerns of teachers change as they integrate graphic calculator into mathematics curriculum at secondary or tertiary school levels. The revised mathematics curriculum in 2006 provides a rare opportunity for me to investigate teacher change.
There is limited research on teaching strategies employed by teachers when they integrate graphic calculator into mathematics curriculum (Barton, 1995; Fox, 1997). The relationship between teachers' knowledge and pedagogical strategies and their use of graphic calculator is largely unexamined (Doerr & Zangor, 2000).

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H1 level: Half of H2 in breadth but similar to H2 in depth; H2 level: Equivalent to current 'A' level subjects; and H3 level: Allows for a greater range of learning and research options. Must offer subject at H2 level. 107
However, a recent study was conducted by Ball and Stacey (2005) to describe the teaching strategies that teachers can use to produce students who are judicious users of technology. The four teaching strategies mentioned are (a) to promote careful decision making and technology use, (b) to integrate technology into curriculum, (c) to tactically restrict the use of technology for a limited time, and (d) to promote habits of using algebraic insight for overview and monitoring. This study aims to describe and analyze how the teaching strategies of teachers change when they integrate graphic calculators into junior college mathematics curriculum.
There are a few studies which investigate the role of teachers teaching with graphic calculator in the classroom (Barton, 1995; Doerr & Zangor, 2000; Farrell, 1996; Simmt, 1997). Doerr and Zangor (2000) conducted a qualitative classroombased research study on role, knowledge and beliefs of a precalculus teacher. Five patterns and modes of graphic calculator tool use were identified, supported by rich field notes. The results of the study suggested that nature of the mathematical task and the role, knowledge and belief of the teacher influenced the emergence of rich usage of the graphic calculator. The descriptions of various modes of graphic calculator use seem to illuminate certain roles of teachers like being an explainer and interpreter of results. Thus, this study aims to investigate how such roles of teachers change when they integrate graphic calculator into mathematics curriculum in junior colleges.
There is limited research on the factors that impact the integration of graphic calculator into the mathematics curriculum (Arvanis, 2003; Bynum, 2002). Arvanis (2003) investigated the extent Illinois high school Algebra I teachers used graphic calculators and what factors impacted this use. Algebra I teachers reported that the factors that most influenced their use were personal beliefs, 'offers something different to do', workshops and other teachers. The factors that limited their use of graphic calculators were emphasis on basics, cost, availability, not enough time, lack of training, and lack of materials. This study aims to further investigate factors that impact the successful integration of graphic calculator into the junior college mathematics curriculum.
Research questions
The purpose of this study is to pursue answers to the following research questions:
1. How do the concerns of teachers change when they integrate graphic calculator into the junior college mathematics curriculum?
2. How do teaching strategies of teachers change when they integrate graphic calculator into the junior college mathematics curriculum?
3. How do the roles of teachers change when they integrate graphic calculator into the junior college mathematics curriculum?
4. What features seem common among teachers who are successful in integrating graphic calculator into the junior college mathematics curriculum? 108
Significance of the study
This study aims to contribute findings and knowledge of change in teacher concerns, teaching strategies and teacher roles when they integrate graphic calculator in the junior college mathematics curriculum. From this research, the changes in

teaching strategies and changes in roles of teachers identified will serve to inform the wider community of mathematics educators resulting in improved pedagogy and practice in the mathematics classrooms. Knowledge of teaching strategies and teacher roles can also be used as a base for meaningful pre-service and in-service programmes. Another significant contribution will be the development of a framework which describes factors identified from findings in the Singapore context that results in the successful integration of graphic calculator into junior college mathematics curriculum. The success factors identified will serve to inform policy makers what factors demand greater attention at various stages of implementation of new technology in mathematics curriculum.
Research Methodology
The methodology used is case-study approach. Following Merriam's (1997) suggestions for case study research, data will be collected by means of classroom observations, interviews and document analysis.
Subjects
A formal letter will be drafted and sent to principals of junior colleges to request for mathematics teachers who would like to participate in this study. The mathematics teachers have to teach the revised syllabus mathematics (H1, H2) in 2006. A total of 9 subjects from 3 junior colleges agreed to participate in the study.
Instrumentation
Every subject in this study will be visited by me once a school term for three terms. Every school term consists of 10 weeks of study. The duration of study is from January 2006 to September 2006. During each visit, the sequence of events will be lesson observation, administering Teacher Concern on Graphic Calculator Use (TCGCU) questionnaire and interview. Data collection will involve the following aspects: lesson observations, teacher self-reflection of other lessons, a Teacher Concern on Graphic Calculator Use (TCGCU) questionnaire and interviews.
A significant part of the data collection is by means of classroom lesson observations. Only lessons that involve teachers using graphic calculators as part of their instructional strategy will be observed. Every lesson observation will be audiotaped. Detailed field notes about how each lesson is conducted will also be made. The times at which activities change and the times at which significant classroom events occur will be noted in the lesson observation checklist. After checking the audiotape, a comprehensive set of observations about the lesson will be made, describing up to 20 characteristics of the lesson. Characteristics that are monitored include lesson preparation, lesson proper, teaching strategies, classroom management and technical issues. The teacher interactions with individual students and the whole 109 class will be recorded. The teachers' use and students' use of graphic calculator will also be recorded. Thus the teaching strategies and roles of teachers are carefully monitored through examination of the types of instructional activities planned, their questioning techniques and how teachers explain concepts.
Based on three sources, a Teacher Concern on Graphic Calculator Use (TCGCU) questionnaire will be constructed: concerns of teachers found in the pilot study, concerns identified from relevant literature research and concerns found in Stages of Concern Questionnaire by Hall and Hord (2001). Care will be taken to attempt to fit concerns into seven different stages proposed by Hall and Hord. The seven stages are: Awareness, Informational, Personal, Management, Consequence, Collaboration and Refocusing. In each stage, the items which are statements of concern typical of that stage are obtained by adapting items from Stages of Concern questionnaire and writing as appropriate some new items to suit the local context. The Teacher Concern on Graphic Calculator Use (TCGCU) questionnaire will have a total of 35 statements of concern.
A preliminary version of the interview protocol has been developed based on review of selected literature (Simonsen & Dick, 1997). This interview protocol will be piloted by three mathematics teachers who have experience in teaching mathematics with graphic calculators and appropriate changes will be made. The final format of the interview protocol will be derived after additional input from two authorities in mathematics education research. The interview protocol contains primarily open-ended questions grouped into four areas comprising: (a) teacher concerns, (b) teaching strategies, (c) teacher roles, (d) success factors. Specifically, teachers in the interview will be asked twelve questions. Some of the questions are adopted from Simonsen and Dick (1997). For example, under teaching strategies, the teacher will be interviewed on how the presence of graphic calculator has helped them teach the mathematics topic differently to illuminate students' learning of mathematics. The teachers will also be interviewed if there are any specific functions in the graphic calculator that deliberately made them enthusiastic about their teaching.
Conclusion
This study carried out in 2006 specifically seeks to describe how the concerns of teachers, the teaching strategies of teachers and the roles of teachers change when they integrate graphic calculator into the junior college mathematics curriculum. The data collection is anticipated to complete by end 2006. The findings from this study will be discussed in anticipation to theme B on teachers and teaching. Being offered an opportunity to participate in the discussion will definitely be beneficial and enriching to me as a new researcher and as a PhD student.
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Graphs 'N Glyphs as a Means to Teach Animation and Graphics to Motivate Proficiency in Mathematics by Middle Grades Urban Students
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The Graphs 'N Glyphs mathematics education initiative aims to provide a model for filling the need of under-resourced urban students to become proficient both in the mathematics necessary to successfully pursue high school and advanced mathematics, and in electronic technologies required for robust economic and employment prospects. Grounded in learning progressions and modeling approaches to multiplicative reasoning, the multi-representational software provides a microworld-type environment in which students learn the mathematics underlying 2-D and 3-D animation and computer graphics, in order to produce their own increasingly realistic and complex computer animations. Ultimately the project aims for students to build explicit mathematical proficiency with rational numbers, ratio, proportion, fractions and decimals, as well as periodic functions and early trigonometric reasoning, in a motivating context of a computer animation and graphic design. Level one of the project focuses on object construction on the coordinate plane; congruence, similarity, reflection and scaling through tessellations; ratio as the foundation of both translation and scaling; and, finally, designing original animations.
Introduction
Students are typically told that they must study mathematics in order to keep open their options to pursue quantitatively-oriented careers in math, science, technology, or engineering. For most of them, this is a very distant and abstract motivation, especially for students whose familial network does not include members who currently engage in such work. Indeed, it is estimated that only 10% of students in the United States complete the prerequisites necessary to take Calculus (Roschelle et al., 2000), which provides evidence that these long-term motivational statements are not very successful in convincing students to persist. Yet, these same students live in a world permeated by the use of technology—the Internet, satellite communications, cell phones, and the management of virtually all the systems within which they live (economic, transportation, demographics, medicine, etc.). In the 2003 U.S. census it was estimated that approximately 55% percent of people in the U.S. have cell phones (U.S. Census Bureau, 2004, 2005) and ever-increasing access to related digital technologies, video, cameras, etc. In order to secure even a middle-class income, students must be competent in the use of these new technologies (Murnane & Levy, 1998). We refer to this as a key technology-knowledge gap, especially ironic in that those countries with the most access to the products of these revolutions are often demonstrating the least progress in developing the underlying necessary student proficiencies. 113
We are currently developing an exciting new software environment that addresses this gap while teaching basic transitional mathematical ideas to students in grades 5-8. The research is specifically targeted to encourage and engage the participation of urban students, many of whom live in poverty and whose access to adequate preparation for advanced mathematical study is severely limited. The environment, called "Graphs 'N Glyphs" (Confrey & Maloney, 2006), is designed to introduce students to how computer animations are produced and to permit them to create, edit, share, and publish their own animations. Thus, through this software and project, we invite them to participate in a compelling animation microworld while making the underlying mathematical and computational elements visible and comprehensible. In doing so, we aim to teach students the fundamental mathematical ideas of integers and rational numbers operations, similarity and scaling, graphing and tables, basic geometric concepts, transformations, and ratio reasoning. Other targets include angles, elementary trigonometry, percents, and decimals. The context of animation provides opportunities to strengthen and connect students' numerical and geometric knowledge, and to build on the foundations that can be established in early childhood, as synthesized recently by (Clements, 2004). The software environment, when fully developed, will also teach students about optics and acoustics, permitting them to explore further ideas in geometry, trigonometry and periodic functions, and the science that underlies onscreen modeling that produces realistic objects and animations.

While our presentation for this work is limited to level one (simple 2-D animations), we developed a short movie to excite students about Graphs 'N Glyphs, with our collaborators at Virtual Ed, Inc. The movie introduces students to Fritz the Robot who, initially, is consigned to the 2-D world (http://www.virtualed.biz/wu/applet_1_framework.html). As the movie progresses, Fritz is brought into the 3-D world and students are confronted with the question "How do animators use math and science to make a robot such as Fritz look and act real?" We use this question throughout our work with students to try to get them to understand why mathematics is needed to make objects look realistic. In short, we are trying to make them understand that mathematizing the (3-D) world is how animators model the real world on a (2-D) television screen or computer monitor.

Theoretical Approach

Our software design draws on four major thematic approaches from mathematics learning theories: a) modeling, b) project-based instruction, c) learning progressions, and d) microworlds. The work extends these four theoretical themes by linking the software directly to professional use software for animation and graphics, i.e. a tools and professions-based approach. In this way, the work draws upon the study of communities of practice (Lave & Wenger, 2002) and on how their practices can be useful in drawing students into the pursuit of quantitatively linked careers (Hall, 1999). 114

The theme of modeling through the development and revision of inscriptions (graphics, tables, transformation records) that permit one to render graphical and acoustic animations on the computer is the underlying philosophy behind the work (Latour, 1990). We use the definition of modeling by Confrey & Maloney (in press), and build on the work of Lehrer and Schauble (2000, in press), in which one conceives of a student learning via a continuum of models from physical microcosms to hypothetical-deductive.

Our intervention consists of a combination of elements on and off the computer; we emphasize the importance of building in different levels of abstraction in the software (Lehrer et al., 2002).

The activities and their individual tasks form a learning progression akin to learning trajectories described by Simon (1995), (Gravemeijer et al., 2004), and Clements & Sarama (2004), and conceptual corridors as described by Confrey (in press).

These elements are drawn together with the development of the concept of microworlds defined first by Papert (1980) and extended by Weir (1987) and then Hoyles (Hoyles et al., 1991). In Graphs 'N Glyphs, we draw upon the changed definition of Microworld, from "teaching computers to solve problems" to "designing learning environments for the appropriation of knowledge and, as a consequence of this change in focus, the transitional object takes on a central role" (Hoyles, 1993, p. 2). In our microworld, students explore the potentially rich environment of animation and use it as a means for mathematical inquiry. By working through the activities and tasks, they begin to explore their own definitions of mathematizing environments, distance, scaling, and so on.

Software Design

The software interface comprises four primary windows and a graphical display for the animations. Students build objects in a graphing window that consists of a local and global plane, use a table for displaying the point values in relation to the local and global planes, a transformational record by which the animations are enacted, and an object palette for saving and reproducing objects and their characteristics. The windows are linked dynamically and can be adjusted to support predictions, data gathering, and feedback. Feedback consists not only of interface and usability hints but also allows students to assess their progress on the various activities and tasks in the curriculum. Students who feel they have not mastered certain concepts can choose to keep the detailed feedback available even as they move on to more advanced topics.

The software is collaboratively designed by a team of mathematics educators and game and graphics designers. The design team has sought to build software that acts as a genuine transition to the use of professional animation and graphics tools such as Photoshop, Freehand, Flash, and 3-D Studio Max. At the same time, the mathematics educators sought to ensure that the (usually invisible) mathematics that underlies animation and graphics packages would become visible to the student. 115 Mastery of this mathematics accompanies and is required for creation of the animations. Pedagogically, the designers also sought to implement conceptual corridors in the way tasks are sequenced. Practice is required, and assessments are continuously and periodically gathered. The design, finally, depends for its final form on implementation in classrooms where student interactions are encouraged and teacher guidance and monitoring are assumed.

Research

Research on the use of the software is underway through the use of clinical interviews. These interviews consist of working with urban students in St. Louis through a series of tasks, many of which themselves develop the basic skills necessary for working with the software. The interviews will heavily inform eventual evolution of the software from its current state as a developmental alpha to a beta version that we will use during the spring and summer of 2006 on groups of students. Results of these interviews will be incorporated into our conference presentation.

Conceptual Trajectories Embedded in Task Design and Articulation

We outline the conceptual trajectory embedded in the first level of our materials and describe how we have developed and sequenced the tasks to incorporate increasing resources for building animations and to complete subtasks along the way. Each subtask must be related to both the goal of proficiency in mathematics, as evidenced by performance on embedded and external assessments, and in the use of the software as a tool for potential transition to professional graphics and animation software.

The overall trajectory of level one of the project can be described in five phase as follows:

I. Introduction to the Cartesian plane and to building objects on the plane

II. Introduction to Congruence, Similarity, Reflection and Scaling through Tessellations

III. Translations along a diagonal line as a manifestation of ratio

IV. Mazes: Ratio in translations and scaling

V. Designing original animations

Example 1: Introduction to Cartesian plane and constructing objects (phase I). In the first set of activities using Graphs 'N Glyphs, students learn about graphing on the 2-D plane (introducing metrizability of the Cartesian Plane and reinforcing wholenumber addition and subtraction), a simple dot-to-dot model for reproducing and modifying basic shapes, and "Taxicab Geometry", a non-Euclidean geometry closely related to Euclidean geometry (differing by one axiom and arguably developmentally accessible earlier as a better model for human-created cities) (Krause, 1975). Activities to reinforce operating, locating, and graphing on a coordinate plane (and screen animation space) include a Battleship-type game and predicting geometric shapes from point coordinates. Integer addition and subtraction are reinforced through navigation, using both magnitude and direction, to destinations on 116 a one-dimensional map (a street). These integer operations are extended to two dimensions via Taxicab Geometry, which combines horizontal and vertical movements to get from point to point, rather than diagonal definition in Euclidean geometry). Thus integers (and, shortly thereafter, rational numbers) are interpreted graphically. The graphical representation sets up the use of rational numbers in multiplicative operations, including ratio and similarity.

Students then combine these skills in creating their own objects using a dot-todot representation. Not only will they need to master coordinate graphing (in order to place their dots) but they are asked to make an existing drawing "more realistic" with additional points.

Example 2: Congruence, Similarity, Reflection, and Scaling through Tessellations (phase II). Fundamental to this part of the level one learning progression is the understanding of the multiplicative (ratio) relationships underlying the concepts of similarity (Lehrer et al., 2002) and scaling (Confrey & Scarano, 1994). Students identify and develop their concept of what constitutes a similar figure in 2-D space, and are introduced to ratios as mathematical tools to preserve the similarity of figures. Students are challenged to construct and use congruent copies of screen objects to build tessellations, and then to scale those objects to build different tessellations. Students distinguish between additive incrementing and multiplicative change of objects' side lengths, and to recognize the role of ratio multiplication in preserving similarity. The scaling of figures by means of whole-number ratios, followed by composition of ratios to generalize the utility or the concept and improve their proficiency with and understanding of ratio is supported by the software as students build tessellations with larger versions of the initial objects, and experiment with decimal values for the scaling factor. This conceptual development then supports more sophisticated use of the animation tools in the next activity.

Example 3: Mazes (phases III and IV). Graphs 'N Glyphs maze activities complement the development of the ratio concept in scaling by employing it in translations as well. Students are provided (and later design) characters that must traverse a maze—moving their character (object) across the screen, through or around obstacles, and finishing at a goal with particular spatial or action requirements. The activities comprise increasingly complex animation tasks. To avoid or pass through obstacles, students must scale their characters, which reinforces and expands the scaling proficiencies established in the Tessellations activity. Moving their characters across the screen, however, promotes a different recognition and use of ratio, though still within the context of animation. The initial mazes can be negotiated solely with vertical and horizontal translations. Subsequent mazes, and the goals of the tasks, however, provide a need for diagonal movements. Some mazes can be traversed with a combination of horizontal and vertical translations, but this is time-consuming. Other mazes include channels or paths that are themselves diagonal. Students construct schemes to utilize ratios to combine the vertical and horizontal components of slope (Confrey & Scarano, 1995), and then use 117 similarity to recognize the ratio relationship implicit in a long straight diagonal and accomplish a long diagonal translation in a single step. Students' fluency and flexibility of ratio use is promoted, in part, through the students challenging each other with mazes they construct themselves.

At the end of these three activities, the students will possess the necessary mathematical and software skills to create animations using Graphs 'N Glyphs (phase V). Thus, the final project, worked on intermittently throughout the project, will be to create an original animation and share it with other students, teachers, and parents. Students will be given a project description that includes criteria on which their projects will be assessed. In addition, they will take pre- and post- assessments on their knowledge and understanding of the relevant mathematical concepts. Finally, they will be asked a set of questions, tailored to their animation, to explain the interrelationships among the three representational spaces, and will be assessed on their ability to apply mathematical concepts to explain the representations.

Conclusions

Computational environments represent a powerful link between the use of the mathematics and the deeper understanding of how the use of (ubiquitous) animation and computer graphics work. Traditionally, we speak to students about the utility of studying mathematics without providing them compelling illustrations. Microworlds like those created in Logo (Papert, 1980) provided interesting ways to link the graphical display from the turtle to concepts of programming and geometry, and in doing so, strengthened students' understanding of all three ideas. In this way, it did provide a form of a career trajectory, as a means to transition to other computer languages.

The vast changes in graphics capabilities, driven by both game environments and simulations capabilities (flight simulations, military applications, engineering environments) have largely failed to influence the teaching of mathematics at the elementary and middle grades. Some exceptions include gaming environments like Sim City (Bos, 2001; Squire, 2005) and, to a smaller effect, Civilization (Squire, 2006).

As an extension of these efforts, our research team has chosen to concentrate on providing students access to intermediate or transition tools for animation to draw students into the question of how images on computers are made lively, realistic, or 3-dimensional on discrete, pixel based, two-dimensional video screens. In pursuing this question, we have created two kinds of multi-representational symbolic inscription tools, (1) tabular displays of objects and the effects of the transformations on ordered coordinate pairs, and (2) a transformational record which creates the sequence of animation actions.

Through this research, we will assess how successful the conceptual trajectories are in building the students' mathematical understanding, in assessing the sophistication and comprehensiveness of the student-created animations and their use of various resources and tools, and the relationship between the two, as well as the 118 ways in which students' sense of themselves as potential students of math, science and technology changes, over the course of the participation in the interviews and subsequent workshops.

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Developing Dynamic Sketches for Teaching Mathematics in Basic Schools

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Reflecting on the actions and activities that are enabled by a new technology can catalyze a reconceptualization of the content and methods of teaching mathematics. Software might provide tools that enhance students' actions and imagination. The five years long research has been developed on two phases. The first phase was to analyze problematic dimensions of teaching mathematics in schools using computerbased technologies and searching the most suitable software for the National curriculum of mathematics. The next step was to investigate (also to localize) the Geometer's Sketchpad and to built the various sets of dynamic sketches for teaching and learning mathematics in basic schools. More than 800 dynamic sketches have been developed within 9 th and 10 th grades (years 16 and 17) mathematics curriculum. Two CDs and descriptions have been prepared and published. The paper explores the main questions of developing dynamic sketches for mathematics curriculum of basic school in Lithuania.

Introduction

Together with the rapid increase in the number of computers in schools a similar increase in the number of software products of the new technology has come. Mathematics is one of the main subjects in schools which require a lot of students' efforts. Using technology while teaching mathematics is not only necessary but rather inescapable (Balacheff, 1996; Hoyles & Jones, 1998; Posmastier, 2003). Regarding the Strategies of Information Technologies Implementation of Education in Lithuania a wide attention is paid to educational software's implementation to curricula of various subjects: schools are supplied with such software and prepared methodical materials on it, besides different kind of workshops to teachers to introduce them the software are being held. That is one of the most important means to direct the computerization of schools towards the positive direction of upbringing improvement.

Mathematical literacy in school is continuously gaining the stronger emphasis – that is one of the aims suggested by the politicians of the European Union. All pupils have to perceive the basic elements of mathematics. That's why they need a fair motivation. In this case the implementation of information technologies is one of such inviting solutions. "Computers have much more to offer than drill and practice; in fact, they can be used in conjunction with all parts of the constructive learning process, when embedded in classroom culture where there is communication and cooperation. There are several ways in which computers can be used; for example in 121 practicing of skill in a way that incorporated understanding or in simulations that enhance concept building" (Becker, 1996).

Dynamic sketches created by computer provide the possibility of a deeper acquaintance with mathematical definitions, theorems, and properties. Often geometry is presented in static form in which the true and deeper meaning of a theorem does not get the true exposure it should. Thus the sketches developed by using computer-based technology help to look deeper to theorems of mathematics of secondary education (Jackiw, 1988). D. Tall in his paper stresses that students using paper and pencil drawings of graph saw them as geometric shapes rather than a process of inputting x and outputting y (Tall, 1996). The field of dynamic experimentation has been opened to new objects, the conics, where one can manipulate hyperbolas, ellipses, parabolas or their equations.

The main properties of software that supports teaching of mathematics

When implementing IT in Lithuanian schools the computer applications that could be helpful for the implementation of the purposes, aims and didactical attitudes that are introduced in National Curricula and Educational Standards, and at the same time simple to use and handy for introducing of the wider scope of mathematical topics were searched.

With reference to these criteria in 2001 the educational software "Geometer's Sketchpad 3.11" [www.keypress.com/sketchpad/] was bought to all Lithuanian schools; it was localized, and in 2004 the localization was cardinally updated presenting the Lithuanian version of "Geometer's Sketchpad 4" (Jackiw, 2006). This educational software helps to implement the purposes, aims and didactical attitudes that are introduced in National Curricula and Educational Standards.

According to Jackiw (2006) Geometer's Sketchpad 4 is "a software system for creating, exploring and analyzing a wide range of mathematics. You can construct interactive mathematical models ranging from basic investigations on shape and number to advanced, animated illustrations of complex systems." It allows an organized set of primitive actions to be turned into complex one using macroconstructions. The drawing produced at the surface of the screen can be manipulated by grapping and dropping around any point having sufficient degrees of freedom (Jasutiene et al, 2005). Therefore teaching pupils to draw sketches helps to develop their creativity, algorithm thinking, carefulness, accuracy, and mathematical skills. The sketches created by pupils or teachers may be used for demonstration or research purposes. Dynamism of sketches created by the software may replace multiplex actions of drawing geometrical shapes on paper or on the desk. 122

Modeling and using dynamic sketches

Drawing of sketches

The experience of other countries indicates that sketches are often used to solve particular task, for example, to demonstrate Pythagorean theorem, the Golden Rectangle Revisited, and so on (Key Curriculum Press). There is, however, a group of sketches created to study particular mathematical or even physical topics.

In order to construct a meaningful sketch you need: 1) to choose a topic which visualization can be supported by "The Geometer's Sketchpad" possibilities, 2) to model the sketch, and 3) to construct the sketch. The Geometer's Sketchpad is convenient to introduce approx. 50% of math topics introduced in secondary schools, i.e. plane geometry, plane analysis, basics of geometrical functions and their charts, basics of mathematical analysis, trigonometry, part of stereometry (part since the software does not support 3D system), differential equations' directional fields, number line and basic arithmetic operations, as well as vectorial algebra and complex numbers.

After the proper topic of mathematics is chosen it is needed to analyze for which purpose the sketch will be created, i.e. which definition, property, or theorem must be demonstrated by the sketch. Special regard has to be paid in order to avoid the situations when the created sketch could serve as an obstacle of learning process or could bring confusion to pupils' minds.

The modeling of sketch should begin after these considerations are taken into account. The modeling of sketch is quite difficult stage and requires different skills and knowledge: 1) knowledge on mathematics theory, 2) skills of methodology of mathematics teaching, 3) deep sophistication on software, and 4) ability of information structuring. Possibilities of the Geometer's Sketchpad allow creating sketches that universally approach geometrical objects and their relations. For example, functions may be analyzed regarding two different aspects: changing functions' coefficients as parameters, selecting concrete and definite values or using the scroll bar for changing coefficients and observing function's chart as well as changing function's chart (dragging one of the chart's points) at the same time observing how the coefficients of function are changing.

Therefore when modeling the sketch on paper the common picture of mathematics is needed to be seen and the context of the future sketch has to be anticipated. After getting acquaintance with the software, the creation of dynamic image of the modeled sketch becomes not so complicated. However, programmer, who is intended to create the sketch, besides the understanding of the software, has to obtain deep knowledge of algebra, geometry and methodology of teaching mathematics as well as skills on algorithm approach.

Principle of the Geometer's Sketchpad is rather simple: we have an empty sheet of paper, ruler, pencil, calculator, and several drawing commands, thus we have to create. Very often quite complex dynamic images have to be created by using the merest means. In such case quite a few steps have to be performed. For example, to 123 create a decision model of inequality the algorithm of approx. 200 has to be implemented. The Geometer's Sketchpad does not limit the possible number of algorithm steps. It rather depends on the computer facilities as well as person's invention.

When creating a sketch the ordinary means that are not included in the software often may be necessary, e.g. the scroll bar for changing coefficients or angles' marking arc. Such means may be created by user and then implemented in various sketches. However, when looking at the final sketch all drawing steps remain invisible. In most cases just desired result, i.e. the complete image, is displayed. Thus, to create a sketch there is a need of time, knowledge of the theory and teaching of mathematics as well as familiarity with software possibilities.

Set of dynamic sketches for mathematical lessons

In 2003 the research on software implementation in Lithuanian comprehensive schools has been performed and it has revealed that just 27% of schools are actually using the Geometer's Sketchpad during lessons (Ministry of Education, 2003). It is not an easy task to the teachers themselves to develop sketches. The main reasons of this are the following: lack of time to properly prepare (teachers have many lessons), fear of technology and insufficient computer skills. Therefore the decision was made to help teachers to create sketches that are needed according to the National mathematics curriculum and to provide instructions on implementing those sketches in their lessons.

Senior grades (16-17 year) were selected as target group of the research since: 1) the curricula of these grades embrace the most part of mathematics' topics that can be visualized by dynamic sketches, 2) in these grades the major part of new definitions, properties, and proofs (although most of them do not need to be demonstrated, their sense is still obscure to students; students remain not persuaded in their correctness) are introduced, 3) in these grades the summarized course, which has influence to further studies, is provided.

Regarding these criteria the curricula of mathematics in 9-10 grades was analyzed and the topics that can be directly visualized by the Geometer's Sketchpad were selected. For 9 th grade the following topics were chosen and sets of sketches were developed (the number of sketches are presented in brackets): linear function (146), quadratic function (90), systems of linear equations (21), similarity of triangles (78), solution of quadratic equations (19), and circle and circular disk (116). Similar actions were applied to 10 th grade curricula – after analysis the following topics and sketches were selected: graph of a function (95), set of equations and inequalities

(Z3), quadratic inequalities (39), trigonometry functions of acute angles (83), and exploration of triangles (74).
Although the Geometer's Sketchpad possibilities depend on invention of user, there was no visualization provided for combinatorics, probabilities, solid geometry, and percentage. Visualization of these topics using the Geometer's Sketchpad is too 124 complicated and in some cases even impossible or inadequate in order to introduce certain topics in understandable way for students.
All sketches were developed implementing solid methodology: 1) a short description, containing the information on what to do with the sketch and where attention should be paid, was provided together with sketch, 2) sketches were dynamic, i.e. it's possible to drag objects, change parameters and therefore the possibility to go back to the initial state always remains, 3) there is a help provided to user and upon the demand the answers can be given as well.
All sketches are provided in CDs with descriptions, that help to use the sketches, theoretical material of a textbook, and recommendations on how to solve certain tasks regarding mathematics' textbook in efficient way.
Two types of dynamic sketches have been developed: 1) visualizing theory and 2) visualizing problems. The dynamic sketches that visualize problems have several properties: 1) one dynamic sketch embrace whole group of problems and 2) in many cases they widen the problems' conditions. The dynamism of Geometer's Sketchpad's sketches provides an opportunity to visualize a whole group of problems by using one sketch. Solution of quadratic equation may serve as an example. By solving a parametric quadratic equation the whole set of such equations may be solved.
By creating sketches the complete image of mathematics had to be demonstrated and therefore sketches of problems provide more information than it is required regarding the condition. For example, a student is asked to calculate length and width of the rectangle when the area and the perimeter of the shape are provided; in this case the dynamic sketch provides the graphs of functions related to rectangle's perimeter and area as well. This is the way how the simple problem of solid geometry becomes related with function and the relation between different mathematical topics appears.
Thus, The Geometer's Sketchpad helps to look at mathematics as an entirety rather than jumble of separated topics. However, when developing sketches their creators were avoiding to "overweight" them and were trying to organize material in such way that it wouldn't bother the main idea of a problem.
In 2003-2005 more than 800 dynamic sketches were developed: compact disks "Mathematics 9 with Geometer's Sketchpad" and "Mathematics 10 with Geometer's Sketchpad" (Jasutiene et al, 2003; 2005).
Teaching by using dynamic sketches: an example
There is an example of dynamic sketch that illustrates the 10
th
grade topic "system
of equations when one equation is non-linear". According to the National mathematics curriculum [4] a student should: 1) approximately calculate the solution of system of linear equations containing two variables, 2) solve ordinary equation 125 systems where one equation is linear and another quadratic (either in graphic way or in alteration way).
For this topic three dynamic sketches have been developed: 1) graphical solution of the system of two linear equations, 2) graphical solution of the system of two equations where one equation is linear one and another's graph is a circle, and 3) graphical solution of the system of two equations those graphs are circles. The first sketch analyses how the changes of the coefficients a, b, and c, result the graph of the equation ax+by=c and provides the graphical interpretation of system of two linear equations.
By changing the coefficients of linear equations students examine the whole group of linear equations' systems and may find the answer to the following sketch's problem question: how many solutions the system of two linear equations may have? This is the way to remind students the graphical method of solving the system of two linear equations, that was introduced them in the 9
th
grade.
The second sketch is intended to analyze: 1) how the change of the coefficients result the graph of equation ax+by=c, 2) how the change of the values d, e, and f result the graph of equation (x-d)
2
+(y-e)
2
=f
2
, and 3) the graphical interpretation of the solutions of the equation system

+ =
" + " =
(x - d) (y e)- f
a x b y c,
2 2 2
(Fig. 1).
Fig 1. The graphical solution of sets of equations: a) system of two linear equations, b) system of linear and circle equations, c) system of two circle equations
Such dynamic sketch is not complicated to develop - approx. 30 steps is enough, since The Geometer's Sketchpad possibilities provide an opportunity to develop graphs of functions directly (Jackiw, 2006). The short description on what should be changed or moved and what to notice is provided together with the sketch.
The sketch illustrates the whole group of systems of equations where one equation is linear one and another's graph is circle. When changing the values of equations' coefficients-parameters, the obtained systems and the interpretation of their graphical solution may be observed. That is the most important feature of such sketch [8, 9]. 126
To provide the entire picture of the topic the third dynamic sketch is developed; it illustrates: 1) how the change of values of the coefficients a, b, and c result the equation graph (x-a)
2
+(y-b)
2
=c
2
, and 2) the graphical interpretation of the solutions of equation system

+ =
+ =
(x - d) (y e)- f
(x a)- (y - b) c ,
2 2 2
2 2 2
. In fact, when examining the given
examples in the sketch the whole trick is to answer the question how many solutions such system can have.
Developed dynamic sketches help to reveal the whole picture of mathematical topic and the sketch of extended course may incorporate itself in the common context of the given topic. Dynamic sketches are easy to control since all of them have a similar structure; by examining the first sketch it's already possible to guess what the second and the third one will be about. Usage of these dynamic sketches helps to examine the systems of equations and it brings valid convenience as well, i.e. teacher doesn't have each time to draw new examples of graphs of systems of equation on the desk.
Conclusions
ICT enhance teaching mathematics and motivate students for investigation. The first step is to provide schools with the software proper to teach mathematics. The further step of particular importance is to prepare action plan that would embrace teacher training, information dissemination, development of methodological and educational aids, guidance for assessment, etc.
In Lithuania The Geometer's Sketchpad country license was purchased in 2001 and right after that teacher training and information dissemination were initiated, however that was not enough. Just small part of teachers began to use the software during their lessons. The main explanation of this was lack of time to prepare for the lessons.
Regarding this the decision to reconsider the mathematics curriculum for basic school and relate it with scripts developed by using the Geometer's Sketchpad (dynamic sketches and proofs) was made.
For 9
th
and 10
th
grades (16-17 years) more than 800 dynamic sketches and scripts according to the mathematics curriculum were prepared. Two CDs and descriptions of them have been developed.
The model and examples described in the paper emphasize educational aids for basic school curriculum. The developmental model should be modified to other levels and grades. However, the basic procedures would be quite similar. 127
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A pedagogy-embedded Computer Algebra System as
an instigator to learn more Mathematics
13
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The constraints of a Computer Algebra System are generally classified as internal constraints, command constraints and organization constraints. In fact, a fourth kind of constraints exists, namely motivating constraints. These constraints consist in features or commands of the CAS whose understanding demands sometimes from the user to acquire more mathematical knowledge than what has been taught in a standard course. Theorems can appear which necessitate learning beyond the syllabus framework. Such 'new' theorems appear generally in two situations, namely when using a pedagogy-embedded feature of the CAS (either a posteriori help, or a priori hints), or when using certain commands and trying to analyze the results. We describe a research frame in the first year Foundation Courses in Mathematics, in our Engineering College. With this research, we wish to understand more deeply the instrumentation processes at work with the students and to check motivations for a change in the institution's culture.
Levels of intervention of a Computer Algebra System.
As an assistant to mathematical learning, a Computer Algebra System (a CAS) offers three levels of help:
a technical tool performing technical tasks;
a tool whose performances help to develop more conceptual understanding;
a technological help to bypass a lack of conceptual knowledge, where such knowledge is out of reach, at least in "the next future".
The first level is the blackbox level and has no great pedagogical value. Maybe it allows the teacher to save time for reflexion and theoretical understanding, but a perverse effect is the loss of manual computation skills, as noted by (Herget et al.,

2000). Integration techniques, techniques for solving equations, either linear or nonlinear, are abilities which could disappear. The following claim has been heard in a professional meeting: "nowadays, there are computers who make the computations;

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thus there is no need anymore to teach integration techniques". We disagree with this
claim, and wish to show that, on the contrary, Computer Packages enable to learn and
to understand more Mathematics than expected.
We can distinguish a level 1 , where the student uses the CAS for verifying results.
There are at least two kinds of verifications:
Verify either a numerical result or a "closed" algebraic expression. The mathematical
correctness of the verification is not always evident. For example, two different CAS
or even two different commands of the same CAS, or a CAS and hand-work, can
provide different algebraic expressions, both valid. As inert expressions, they are
different, but when defining functions, which are dynamical objects, different
expressions can define the same function. The verification issue has been addressed
by Lagrange (1999) and Pierce (2001).
Perform the passage from n to n+1 in a recurrence, after the CAS enabled to
conjecture a formula (see (Garry 2003) page 139).
Steiner and Dana-Picard (2004) commented aspects of level 2. Low-level commands
are important for cognitive processes attempting to afford a good conceptual insight.
A CAS command is called a low-level command if it performs a single operation,
while a macro is a command programmed to perform a sequence of low-level
commands. Low-level commands act as the atoms of every computerized process for
solving a problem.
Because of syllabus limitations and of time limitations, level 3 is less commonly
considered. It appears close to the frontier of the syllabus, either for exercises aimed
to broadening knowledge beyond this frontier, or for problem solving when the
necessary theorems have not been taught and will not been taught "in the next
future". Technical use of the CAS fills the gap; see (Dana-Picard 2005b).
A fourth level exists: a CAS is a device whose performances may incite the user to
acquire more mathematical knowledge. The reason can be one of the two following:
Multiple commands are available for seemingly the same purpose. For the user to
make an intelligent decision which command to use, he/she must have a good
knowledge of the Mathematics implemented in the algorithms.
There exist situations where a unique algorithm is available, either because of the
theoretical state-of-the-art or because of the decisions of the developers. This limits
the diversity offered by the CAS; this issue is studied by Artigue (2002), page 265. In
such a case, the theorem transformed either into an algorithm or into a command is
not always a standard theorem taught in a standard course; see the example with
Derive in section II. 130
In every case, the implemented Mathematics has to be understood. In order to afford
a real understanding of the process, the user has to learn new Mathematics. We called
this occurrence a motivating constraint of the software (Dana-Picard 2006).
Generally, the word constraint evokes a limitation, an impossibility to go beyond a
certain borderline. For a software package, this can be a limitation on the size of
numbers, on the number of successive parentheses, etc. Among the most documented
internal constraints are the finiteness of the screen for graphical applications, and the
fact that the real numbers are always approximated by rational numbers. Following
Balacheff (1994), Guin and Trouche (1999) distinguish three types of constraints of
the artifact, called respectively internal constraints (linked to hardware), command
constraints (linked to the existence and syntax of the commands), and organization
constraints (linked to the interface artifact-user).
The constraint that we meet here is of a totally different nature: instead of limiting
the user within the borders of a certain topic, the CAS demands from the user to go
further, to learn a new theorem, a new technique. It is a motivating constraint, which
leads to a broadening of the student's mathematical landscape. After its apparition,
the mathematical knowledge is not supposed to be only shown anymore, the student
is incited to learn the new theorem, and then becomes able to manipulate this
knowledge, either with or without the help of the technology.
Pedagogy-embedded CAS.
Until recent times, the CAS did not give hints in order to find a pathway towards the
solution of the given problem. This is not true anymore: pedagogical features have
been implemented into Computer Algebra Systems. We call such systems pedagogy
embedded CAS.
Derive 6 has a step-by-step feature, well developed for Calculus commands. Every
step corresponds to one low-level command, as it implements one single theorem
such as an integration formula. There exist surprising situations, e.g. the following
formula is a central item:
f x dx ( ) f x f a b x dx
b
a
b
a
= ( ) + (+ !)
2
1
(*) ( )
As an example, look at the following integral:
+ !
=
a
p p
p
x a x
x dx
I
0
( )
, where p is a nonnegative real parameter. For p=0,1,2, the computation is straightforward, but for
larger integer values and for non integer values of the parameter, the work is nonilluminating. For given a, and for p=1/3, other CAS, where this formula seem not to
be implemented, cannot generally compute the integral in a reasonable amount of 131
time. Knowing the formula (*) enables to compute the integral with paper and pencil,
within a few steps, and last but not least, for the general parameter:
! + ! !
!
+
+ !
=
+ !
=
a
p p
p
p p
p
a
p p
p
dx
a x a a x
a x
x a x
x
x a x
x dx
I
0
0
( ) (( ))
( )
2 ( )
1
( )
-
2
1
2
1
-
0
a
dx
a
= " =
Formula (*) is not trivial; it is commented, and examples are given, in (Dana-Picard
2005b). An experienced lecturer, working in another institution, told to one of the
authors: "I would not dare to ask my students to know such a theorem". We think that
this implementation is a good opportunity to teach the theorem and some of its
applications. As A. Rich says: "The transformation rules Derive displays are those it
uses to simplify an expression. They may or may not be the same as those currently
taught to students. However, if teachers see an advantage to an unfamiliar rule used
by Derive, they may want to ask their students to verify the validity of the rule and
then the students will have an additional tool in their arsenal" (Böhm et al., 2005,
page 36).
This parametric integral has been proposed to an average student, named Ori, during
the preparation to an oral examination. At first glance, as he thought that the
parameter is a non negative integer, he proposed to decompose the integrand into a
sum of partial fractions. The tutor showed him the Derive's step-by-step-solution.
Tutor: Do you recognize a known formula?
Ori shows Formula (*), then says: No, actually we have not been taught this.
Tutor: Can you apply the formula?
Ori: Yes. (works for a while); oh, I never saw this, you must teach this!
During another session, Ori is proposed the integral
+ !
=
4
0
4
dx
x x
x
! . He says: this
is the same case I saw last time; let us apply the formula.
Finally, at the end of the same tutorial session, he "receives" the following integral:
+
=
2/
0
cos sin
cos
#
dx
x x
x
K .
Ori: It's not a power, but it must work the same way, as it's the same structure.
Tutor: And what about
+
=
2/
0
cos sin
cos
#
dx
x x
x
L
p p
p
?
Ori: Surely the same thing.
He makes the work and claims: Oh yes! You must teach this in classroom!" 132
Maybe that without the step-by-step, the user would not have discovered the formula.
Therefore, we consider this feature as part of the software's motivating constraints.
Maple is pedagogy embedded (via the Student package); here the conception is
different from Derive's step-by-step philosophy, and the learning process induced by
them develops otherwise. We present here an example in a different context.
Consider the following initial value problem:

=
! =
y)0(1
xy ×
dx
dy
. Working with paper and
pencil, a student is generally taught to use an integrating factor.
When using Maple's assistant for Ordinary Differential Equations, the student can
choose the method: Lie methods, Classification methods, etc., but the usage of an
integrating factor is not available for this exercise. A noticeable fact is that the
pressing a button is accompanied by the (optional) translation of the command in
Maple's language. The option "Laplace Transforms" leads to a much more
complicated form. Therefore the student is incited to learn what these methods are,
how they work and which benefit he/she can afford from their usage instead of what
has been taught in regular class.
Tools shape the learning environment (Trouche 2004b), and the last influences the
mathematical contents. The two embeddings of pedagogical features that we saw
above, and the learning processes spanned by them are different. Note that each kind
of software follows general algorithms, starting from pattern recognition, and whose
sequential steps are based on the implementation of general theorems. The human
brain works less sequentially, therefore intuition can lead to other pathways towards
the solution of the exercise. With the integral of section II, we presented earlier an
example of such a situation. This does not mean that technology has not been
programmed properly: a proper usage of technology does not require from the
technology to mimic human actions.
Let us compare briefly the two ways. On the one hand, Derive's step-by-step feature
gives an indication on how the software works; if the student did not know how to
solve the exercise, he/she has now an opportunity to understand by some kind of
"post-mortem" analysis. Maybe an unknown theorem appears, as in our example, and
the student can wish not only to discover it and to use it afterwards, but to try to have
a more profound insight in its proof and its mathematical meaning. On the other
hand, Maple's assistant lets some freedom of choice to the student, by offering
different options before the computation is performed. This is still more evident
when using the tutor for computing integrals. In this case, all the rules are presented
as "buttons"; after a button has been pressed, an immediate indication is given 133
whether the rule can be applied or not. If not, the student is invited to choose another
rule, and so on.
Finally, we wish to note that even without a specific pedagogical feature, a CAS can
be an instigator to further mathematical learning. This is the case in (Kidron 2003)
for the conceptual understanding of the limit notion in the derivative, and deep
learning of the theory is motivated by the usage of Mathematica.
Instrumentation.
At the beginning, we saw Derive's step-by-step as providing the student with "a
posteriori assistance", in order to understand what he/she would have been required
to do. Actually, the usage of the step-by-step feature of the software can be
considered as an "a priori" usage, in one of the following fashions:
The user can discover a way of solving the problem either different from his/her way;
Suppose that the student did not find how to solve the problem; he/she can ask for
the first step (pressing the appropriate button) and the CAS opens a pathway. At
every step, the student can abandon the step-by-step session. This is based on general
theorems that the student does not automatically know.
When such a situation occurs in classroom, the teacher can build various activities,
enriching by a large amount the mathematical knowledge and culture of the learners.
If at the beginning, the student influenced the software's behavior in order to obtain
the needed result, in the second scenario the software forces the educator to teach and
the student to learn a new topic, a new theorem.
We have here elements of an instrumentation process (Chevallard 1992, Lagrange
2000, Artigue 2003 page 250, Trouche 2004a): "Les potentialités et les affordances
d'un artefact (en occurrence le CAS) favorisent le développement de nouveaux
schèmes (ou font évoluer les schèmes antérieurs) de résolution d'un type de tâches
(ici le calcul d'une intégrale définie)" (Trouche 2005; private e-mail). More briefly:
"Instrumentation is precisely this part of the process where the artifact prints its mark
on the subject" (Trouche 2004b, page 290).
Of course, this process is not reduced to the acquisition and internalization of one
single theorem; the present examples are only one occurrence of the mechanisms
involved.
Contribution to the institution's culture.
We use the word "institution" in the sense of (Artigue 2002). Each institution has to
decide whether to introduce the usage of a CAS in Mathematics courses or not to do
so; not to deal with this issue is also a kind of decision. For example, the institution
named JCT decided to teach MatLab and to use it in every engineering cursus. 134
Both authors act as coordinators of first year Foundation Courses in Mathematics, i.e.
courses in which all Engineering students at JCT are involved. In a small subset of
classes, which can also be viewed as an institution, the authors adopted other
packages; for example, a course in Ordinary Differential Equations has been given
last year together with practice sessions based on the usage of a CAS. The
"institution culture" has already changed in certain classes, and is susceptible to
change the institution's culture in a larger scale (e.g., all the first year Foundation
Courses in Mathematics at JCT):
"Tools are not passive, they are active elements of the culture into which they are
inserted."(Noss and Hoyles 1996), page 58).
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Teachers & Teaching:
Technology and Teacher Education — Some Thoughts
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Today as in earlier times, there is much rhetoric about the revolutionary impact on
students' learning that will result from brining a new technology into the classroom.
In the past, it was the motion picture, or the radio — the usual teaching aids and
instructional television. None of these fulfilled expectations. Now, it is the computer
that is believed to herald a new era of more effective learning. With respect to the
mathematics classroom, computers are claimed to have the potential to change
pedagogical approaches radically and to improve students' learning. Traditional
classroom teaching methods related to mathematics have been associated with direct
teaching, black-board demonstration, use of textbooks or work-books, drill and
practice activity, homework and so on — a positivistic, behaviourist model. Teachers
who generally teach mathematics this way will most likely use technology similarly.
But this traditional classroom and teaching techniques are creating an environment
that tend to undermine higher order learning skills, such as creativity, independent
thought, inquiry and innovation. It is fundamental to our homogeneous medium for
learning that we allow others to tell us what to learn, how to learn and even why we
are learning. These mean pupils are being deprived of the scientific approach of
teaching — learning systems being the generation of this technological era. Basic
differences in the new evolving paradigm will be put the learner in charge. To create
such an environment, it is necessary to introduce new technologies like computers,
websites on the Internet and DVDs in classroom teaching. So, the objective of this
paper is to discuss how the would-be teachers and teachers can be oriented with the
use of new technology. What kind of programme should be arranged for the trainees
in teacher education courses; to familiarise them with the uses and importance of
these technologies? The importance of a mathematics laboratory will also be
discussed in this context.
Mathematics has always occupied an important position in the school curriculum.
Mathematical skills and applications form an indispensable tool in our daily life. In
the present era of technology, mathematics plays a very important role. There is
hardly any discipline that does not owe anything — directly or indirectly — to
mathematics.
However, mathematics is still characterised as a 'dull' and difficult subject,
primarily due to its hierarchical concepts, deductive approach in proofs and abstract
nature of content. A phobia about mathematics has been created in the minds of 137
children; a fear that mathematics is 'difficult to learn'. As a result, students seem not
to express enough interest in the subject and thus become weak.
Its abstract nature and hierarchical concepts are not the only reasons
contributing to distaste for mathematics. Traditional methods of teaching the subject
in schools are also responsible for this. Every year, at the time of the practiceteaching period, I supervise 45 to 50 mathematics lessons from standards VI-VIII,
executed by my trainee students, visiting different schools in Kolkata. My
observation is that the traditional 'chalk & talk' method cannot be a solution to the
inherently abstract nature of mathematics. On the other hand, if trainees use various
types of teaching aids, students respond better than usual to the typical mathematics
lesson. These aids, including the use of concrete objects and visual presentation, have
a hugely positive impact on teaching and learning mathematics. This approach, and
similar ones, will motivate students towards learning mathematics. However, to
interact with the present-technology based educational system, we need something
more — aware, and able teachers. But more on that later. School students should be
provided with a dynamic system of teaching — a learning facility to achieve the goal
of mathematics.
So, our first objective is to find out how the new/digital technology can be
blended with the 'chalk & talk approach' in schools. To bring in computers, CDROMs, DVDs and the Internet into classroom teaching, we have to first familiarise
our teachers with the uses and importance of these.
Secondly, we have to remove the 'fear' of handling computers or any other
computer-related technology. All these are possible through short-term
courses/training/re-training programmes on the use of technology and its implications
in the teaching-learning process, for teacher-trainees and freshers who plan to take up
teaching as a profession.
An alternative approach is the inclusion of some technology-based concepts
and their uses in the curriculum of the Teacher Education Course. With this view, the
Technology Based Teaching Strategy (technology based instruction, CD-ROM,
websites, DVD) has been introduced in the Teacher Education course of Calcutta
University from the year 2004. But no provision has been allotted to develop
practical experience. Only theoretical concepts would not inspire teachers to adapt a
technology-based instructional approach in the classroom.
The proposed Programme:
In teacher education colleges, there should be provision for a mathematics
laboratory with the following equipment:
a) Tables, Charts on different topics of Arithmetic, Algebra, Geometry,
Trigonometry, Mensuration, Calculus and Coordinate Geometry.

b) Models of different mathematical shapes. 138
c) Experiments to verify different results/formulae in different branches of mathematics.
d) Computer, CD-ROMs, an Internet facility, Data-Projector to prepare lessons (courseware).
e). Books and Journals
f). Portraits of Mathematicians
Not just the facilities; the adequate practice of these tools is important.
In this laboratory, trainees should also be provided with the practical experiences. Like other scientific subjects, laboratory experience in mathematics is important in providing direct access to teaching accessories, so that these can be used confidently by trainees to make the teaching lesson effective and also enhance their own concepts.
The type of programme that can be adopted in the teacher education course to acquaint mathematics teachers with the use of digital technology, and to train them to enhance their concepts using the mathematics laboratory to prepare courseware, is elaborated below.
Details of the program:
1. Trainees (trainee-teachers and would-be-teachers) have to prepare 5 or 6 simulated lessons in his/her total training period.
2. Concept & Purpose :
The dictionary meaning of the word simulation is 'pretence' or 'imitation'. A simulated lesson is a micro-lesson that is taken up for teaching in a make-believe classroom situation. The primary purpose is to enable a teacher to practice teaching with the focus on a single teaching skill, as well as to develop the skill of operating the computer as a teaching tool.
At this stage, trainees will take the help of charts/models and can perform experiments to develop a clear concept to prepare the simulated lesson. The prepared simulated lesson has to be presented on the computer to his/her peergroup.
3. The characteristics of a simulated lesson are as follows:
a) The roles of student, teacher and observers are to be played by the trainees (by rotation). 139
b) The lessons have to be short, aiming to maximise the use of a particular teaching skill
c) The ultimate aim is to make the teacher aware of the behaviour he/she displays in the class, of its purpose, and relevance to teaching learning.
d) Suggested skills for practice:
•Skill of Introducing a lesson
•Skill of Questioning.
•Skill of Explaining.
•Skill of Reinforcement.
•Skill of Closure.
•Skill of using the black-board where necessary
4. Preparation of courseware:
In the mathematics laboratory, trainees will be provided with the following experiences:
a) The teacher educator will explain the concept of a simulated lesson and will explain the components of each skill before preparing the lesson.
b) Before preparing a lesson, trainees will select a topic and will try to get more ideas using related charts or model or performing hands-on experiments. If necessary can collect information from the books and journals .
c) Trainees will then prepare the courseware as per the guideline of simulated lesson and then offer a presentation.
d) Working strategies:
•Trainees in the mathematics group are to be divided into small batches of 8 to 10 members each as per the strength of the class. This will be called a practice cycle.
•In a practice cycle, all the trainees will play the role of teacher one by one and two others will be the observers and the rests will be students by rotation.
•Subject teacher (teacher educator) may check the lesson before commencement of the practice.
5. Suggested distribution of time:
a) Concepts developed from charts, models or hands on experiment: 15-20 minutes (common for all).
b) Preparation time for courseware: 30 minutes (common to all).
c) Teaching time: 7-8 minutes.
d) Feedback by observers: 4-5 minutes. Teacher supervising the practice may comment on the skill after one complete cycle.
e) Feedback will be provided by observers in a given format. Observations will also be explained. 140
6. Evaluation:
a) Trainees will maintain a notebook that will be presented at the final examination and duly signed by the Supervisor (teacher educator). The notebook will comprise the following:
•Name of the skill, and its brief description.
•Teaching mode used.
•The courseware/simulated lesson
Theme/Topic Outline content components Courseware

•Format of feedback:
Compo
nents
Excellent
(7)
V.Goo
d
(6)
Good
(5)
Average
(4)
Below
Av.(3)
Poor
(2)
V.Poor
(1)

With this thought, the Institute of Education for Women, Hastings House, Kolkata, a Teacher Education College under CU is going to start its Mathematics Laboratory.
Model of a simulated lesson:
Skill of Introducing the Lesson
For Class-IX
Before preparing the lesson, trainees should go through the available charts, models, books and journals in the mathematics laboratory to select the appropriate steps and fulfill the objective of this presentation.
Theme/Topic Out line content Component Procedural details 141
Theorem:
Triangles with
same base &
height have the
same area
Development of
concept about
the area of the
different
looking
triangles with
the same base
and height
1. Securing
attention
— a
multitude
of
triangles
are
projected
on the
screen in
front of
the
students.
The figures
will be in the
moving
state.
X C Y
2cm
A 4cm B
X C Y
2cm
A 4cm B
X C Y
2cm
A 4cm B

The height of
the triangles are
changed using
dynamic
software,
keeping the
same base.
(Fig- 2)
Students watch
the change in
real.
2.
Assessment
of
motivational
level/arousal
of
motivation
(true impact
of
technology)
E D F
3cm
A 4cm B
E D F
1.6cm

A 4cm B
E D F
2cm

A 4cm B
T: Discuss the change and ask for observations.
Since students know the area of triangle formula, and have seen the change in the area of the triangles with the change of height, they are convinced that 142 if height and base of a triangle are fixed, the area will remain unaltered, no matter what the triangle looks like.

Relation
between the
heights of the
triangles lies
within the same
parallel lines
3. Linking
past
knowledge
and
experience
T: What can you say about the pair of parallel lines given in the figures for each type of triangle in fig-1?
S: The heights of the triangles remain the same as they lie in the same pair of parallel lines.

4.
Specifying

the main
points
T: We have already seen that:
i) Triangles having same bases
and same heights will be equal
in area.
ii) Triangles in the same pair of
parallel lines have same height.
From these two properties it is
inferred that "triangles on the
same base and within the same
pair of parallel lines have same
area".
Now we will learn the formal
proof of the theorem.
5. Using
appropriate
devices
Charts, computer & CD 143
In this way, trainees will practise six skills of teaching. They will benefit in multiple ways through this programme:
• Teacher-trainees will be able to use technology comfortably in the classroom.
• Teacher-trainees will be able to integrate technology as a complementary tool for teachers in the classroom. This is a balanced approach.
• The variety of experience provided to students through this dynamic approach will go a long way in removing the abstractness of mathematics as a subject.
• This will also motivate students towards learning mathematics and provide a meaningful experience towards holistic learning.
Note: The author regrets that examples of other teaching skills couldn't be elaborated upon due to shortage of space. This will be provided at the time of a formal presentation.
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Towards guided reinvention in a multimedia learning environment for prospective teachers
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This paper discusses the role of a multimedia learning environment, MILE, for the learning processes of prospective teachers. MILE is a Multimedia Interactive Learning Environment for prospective primary school teachers, with content for primary mathematics teachers' education programs. We summarize an investigation on student-teachers' use of language and how they give meaning to mathematics and didactics. Our analysis shows: (1) the evolving and shifting nature of meanings and processes of signification; (2) the important role of experiences from the past, but in particular from their work as trainees at primary schools; (3) the use of mathematical language and the consequences for a didactical way of thinking; (4) how student-teachers' observations lead to hypotheses and local theories. In our conclusions, we related our findings to the construction of a teacher education course that allows student-teachers to use MILE to develop mathematic and didactical insights on materializing. Teacher educators need help to capitalize on crucial moments in the interaction amongst student teachers. The study showed that the teacher educator needed resources to recognize those moments and to optimize class discussions.
Introduction
MILE is a Multimedia Interactive Learning Environment for prospective primary school teachers, with content for primary mathematics teachers' education programs (Dolk, Den Hertog & Gravemeijer, 2002). It is computer based and provides a database of real classroom teaching open for didactical investigations, allowing student teachers to investigate many aspects of mathematics education in primary schools. Besides the video database, the environment consists of communication tools for learners, a search engine--allowing users to search for classroom footage in a video database, and accompanying students' and teachers' materials--, a student teacher's journal, and thematic suggestions for investigations. These investigations offer student teachers opportunities to develop conjectures about teachers' practical knowledge and teaching behavior and about the effect of the teacher's behavior. The environment allows them also to analyze students' learning over time, and to investigate parts of the process through which students develop mathematical understanding.
We want prospective teachers to learn more about the RME. Mostly, these practitioners had learned some mathematical principles in secondary school. They are used doing mathematics without learning what it is to be a mathematician. 145 Mathematical reasoning and heuristic approaches are no daily practice for them. Moreover, they mainly learn about mathematics education through experience in the full practice of primary school, first in an internship, later on as teachers. That means that the biggest part of their competences in mathematics education is acquired by teaching primary students.
Initially, the expectations were that it would be possible for prospective teachers to learn in an investigative way within the MILE-environment. Quickly, it proved that these student teachers had no proper attitude and the teacher educators had not the abilities at one's disposal to coach them in a proper way. That is the reason that mostly, one uses this environment only with forms of directed guidance and learning questions at hand. This tension between ideal and reality necessitate research.
The Study
Theoretical framework
In the following we describe parts of the theoretical framework, taking a particular look at the Sociomathematical and -pedagogical norms (Stephan 2003). Paul Cobb emphasized that learning mathematics is a matter of a process of cognitive construction and of acculturation. Prospective teachers undergo all kinds of influences during their training: on the one hand the culture of teacher education, on the other the primary school where they server their traineeship, and added to that their social group. Their system of values is therefore determined in several different areas. We look also to the Mathematical and pedagogical conceptions. Many teachers, Cobb (1987) and Gravemeijer (1989) said, tend not to distinguish between Vorstellung and Darstellung. In their view, it would be better for teachers to think from an 'actor's point of view', in which the students' reality is put in a central position. Teachers themselves also attribute meanings, and these by no means have to correspond with those students give.
We are also using a semiotic perspective (Van Driel 1995, Bakker 2004, pp. 187-198). In Charles Sanders Peirce's way of thinking, every sign points to a concept. Then there is also the form the sign assumes (since a sign does not have to be a material), that which he called the representamen. This can occur in any number of forms: words, images, sounds, odors, flavors, acts or objects. But such things have no intrinsic meaning and become signs only when we invest them with meaning. "Nothing is a sign unless it is interpreted as a sign', declares Peirce. And then there is the meaning the sign evokes: the interpretant. How does the idea arise that a specific meaning belongs to a sign? At first similarities and experiences will stimulate that. In Peirce's words that lends a kind of iconicity. On the second level the term indexicality is used. This experience is emphatic, because it is based on the experience of the actual proximity of two phenomena. The third level of experience leads to symbolization. All kinds of conventions and agreements play a part in symbolic meanings, such as the scale in the drawing of the tiled square.
Aim of the study 146
This research study, of which this report is a part, is an ongoing one. It is an explorative research and we want to know more about:
the evolving and shifting nature of meanings and processes of signification;
the important role of experiences from the past, but in particular from their work as trainees at primary schools;
the use of mathematical language and the consequences for a didactical way of thinking;
how student-teachers' observations lead to hypotheses and local theories.
Participants and data sources
This pilot case study involved four students in their third year of education. They have finished most of PABO's standard program, and now only have to do their practical, in-service, training. Their participation in this study was voluntary. All these students are interested in mathematics education. The group consisted of one man and three women, all of them 21 years old.
The observations were partly participating, we interviewed also. In accordance with the hypothetical learning trajectory of Martin Simon (1995) the activities were developed in advance and we made hypotheses about the learning processes if the student teachers. During the research we audiotaped and we made reports.
Task Design
The study took place over four meetings which had a cyclical character, allowing for in between evaluation of the data and preparation of the next cycle. The cycles could be typified as follows:
Orientation on materialization. The students became aware of the formal and informal knowledge they possessed on the topic.
2. Observing some situations from practice and giving meaning from the materialization approach.
3. Putting more detail on the observations and their interpretation.
4. Evaluating newly gained knowledge.
It had been agreed in advance with the participants that they would spend twenty hours on the study: four two-hour meetings, an hour and a half of preparation and an hour and a half for processing the results. In preparation for a meeting they were given information about materializing with some questions as guidance. They afterwards made an evaluative report as well. Every meeting consisted of a number of parts: an initial activity, a discussion of the article and the preparations, followed either by work in MILE, working on the computer in pairs or using the video projector with the whole group. Every activity ended with a moment of evaluation and reflection, during which we analyzed our findings and observations.
Analysis of student activity
When we asked the student-teachers to make multiplication problems and support their 'thinking' with materials we saw that they took an observer's point of view.
When the question arose "Why do you use mathematical materials in daily practice? 147 One of the participants gave as her opinion that it is done to make education more fun and because it helps to explain. Another participant wonders whether materials would still be necessary in grade 6. You should not use a material, such as the arithmetic rack, if children can do without. In the end, together the students reach a kind of "definition" of material:
You can touch it.
It is 3D
Visualizing is what matters: the students can see what they are doing.
When the participants afterwards observed in MILE in pairs we saw that their view of the situation was fairly global, and afterwards their main interest is in understanding the greater picture. Questions such as 'Have the students worked with tiled squares before?' and 'What materials are normally used in grade 2?' emerge. Based on their observations, they made an hypothesis: 'Every time the students learn something new, they fall back on materials. The materials visualize what you are doing, and you will visualize the materials later on'.
In the evaluation the students explain they find observing in MILE a difficult part. Roelien thinks she chose the wrong fragment: 'This is about organization'. The other students explain to her that the teacher makes remarks about the way of thinking to solve the problems. Together, we conclude that teaching a mathematics lesson does unavoidably lead to things being interlaced. The students say they have so little insight into the order of the lessons, that they did not have time to reflect on the aspect of materializing in the video-clips.
It seems that if these four prospective teachers put the iconic interpretations of the material first. As yet, they have little experience and little theoretical knowledge, which means they are not inclined to making indexical and symbolic interpretations. Yet they are open for a more conceptual way of thinking. They have learned in the past about strategies and about the use of contextual situations and they are using that kind of knowledge all the time. They barely shift to an interpretant that is a part of the network of mathematical and didactical knowledge.
The students tended to keep talking from an observer's point of view and their interpretants mostly suited their earlier experiences. Which reflections lead to thinking of the mental processes going on in the children's heads? In this meeting we established how curious these students were about 'good' mathematics teaching. To find it, they mainly zoom in on the 'what' and the 'how' of the activities, rather than the 'why.' Especially with MILE, it is noticeable that they want to know how the lesson is set up, though we had hoped for a deeper analysis, which was not yet possible. That raises the question of how we can get students to really research something. What indicators are needed? How do we deepen their insight into existing theories about the use of material?
Personal experiences 148
In the various conversations the four participants gave direct (and sometimes implicit) information about their needs, values and thoughts. Mostly, they centralize their own experiences, both those from their primary school past and from their traineeship. We missed in such situations a careful reflection and an attitude to learn from this kind of experiences. The four prospective teachers showed that emotions govern their way of thinking. Maybe, their learning style is reproductive (we didn't test that). In that case, they want examples of good practice, ready-made solutions for practical situations.
Dutch teacher training colleges develop more and more programs with studentcentered approaches. Learning from and by experience is an important approach. We see the influence of the ideas of Donald Schön (1987) who emphasized that practitioners require forms of action learning. Reflections on and conversations with the situation are important constituents of learning to master complex situations in the future. How can we encourage prospective teachers to construct new knowledge? In our opinion there are two possible entrances. We want to discuss with them the

mathematical language in relation to the thinking processes. Second, we try to let them formulate hypotheses and local theories.
Mathematical language
The importance of language in the development of mathematical thinking has recently been acknowledged by many authors (Van Oers, 2002). The precise relation between language and mathematics however is not a straightforward one. The adoption of mathematical language does not automatically amount to genuine mathematical thinking. Van Oers (2002) observed that the development of children's mathematical activity can stay on the line of symbol development for given meanings, or the line of meaning development. By treating terms in a mathematical way, reflecting and structuring their meanings, these gradually evolve into personalized mathematical meanings.
Prospective teachers thinking about their own language in relation to the language of the young students can be helpful to give a change to an actor's point of view. Then the teachers realize that their language - especially in mathematics - differs from the way in which students are talking about mathematical topics. So, to observe real life educational situations - and there are a lot of them in the MILE-environment - can be very helpful in understanding that different meanings of the same concept exist. It will be helpful to explain the dialogues between teacher and students and to hypothesize on mental processes and concepts.
By discussing a MILE-situation where a teacher who is instructing a grade 2 class, encourages the children to think of tiled squares the question arose: what exactly did the teacher say, and what did she mean by it? In the evaluation of the observation it emerges how difficult it is to describe that precisely. The student teachers think they know what the teacher said, but when we listen again, it turns out to be slightly different. One question that emerges is whether the teacher said to make problems for the tiled square or of it. This led to a discussion about aspects of mathematical 149 language and how children interpret it. What formulation is needed to make children grasp the intention? There are two options, 5x7 and 7x5. If you want children to explore these, how should you say it? Bas concludes: 'If you say "make problems 'of' it," the children will do 7x5 and 5x7. If you say "make problems 'for' it," they will just make a whole list of problems.'
The next possible question is: 'What are the thinking processes of the children in these specific cases?' A way in which we stimulate prospective teachers to take an actor's point of view with a more language based orientation is to let them describe a primary student's inner dialogue based on a situation from MILE. This assignment's purpose is to make the students put themselves in the child's place and realize everything that is involved in a mathematical situation.
Hypothesis and local theories
Students can create their own educational narratives based on the situations in MILE. Such narratives are more than stories about the situation. Mason claimed that description is a cornerstone of all research. 'Any description is based on making distinctions and drawing attention to relationships, through the process of stressing some features and consequently ignoring or down-playing others.' Mason formulated the following as being problematic in writing: 'If what we perceive is what we are prepared to perceive, and what we are prepared to perceive is what we have perceived in the past, how then do we ever come to perceive anything new?' This is a question about constructing knowledge. Within communications among educators these considerations will contribute to the development and sharing of educational knowledge (Gudmundsdottir, 1995; Mason, 2002). Constantly we challenge the student-teachers to develop local and global theories. On the one hand, they have to limit their investigation and the justification of their understanding to the multimedia cases; on the other, they have to extend their understanding into their future profession in the form of hypotheses. During their investigations they had to switch continually between these two perspectives. Here we give an example.
In the evaluation of a discussion where a teacher, instructing a grade 2 class, encourages the children to think of tiled squares the four prospective teachers started asking all kinds of questions, such as: 'why is a tiled square used for multiplication problems?' and: 'what is needed first when children are learning to multiply: "pencils in trays or tiled squares?" Apparently their schooling did not include an overview of the learning trajectory. One of the guys finally comes up with a 'theory,' in which notions of materializing, context and the use of models pass in review. She relates how young children start with counting, but quickly move on to ways of counting. Gradually they start to see the possibility of using groups. This kind of shortening leads to jump counting. Then shortening reaches the point where applying 'times' becomes more and more obvious. You mainly use the tiled square to show on the one hand the jump character, and on the other to also show that 7x5 and 5x7 give the same result. One of the other participants summarized this in his report:
What is the purpose of the problem?150
We discuss the problem and what its purpose is. At what point in the learning trajectory for tables and area/content will this form of mathematics be on offer. Roelien's theory: Counting for kindergartners -> ways of counting (2, 4, 6, 8, 10 of 3, 6, 9) -> groups -> 'times' problems -> squares -> area -> content.
Of course, posing questions, developing local theories, etc. are no miracles. But when there is a good environment with experienced teacher-educators, a shifting interpretant is within reach. For us the program of the teacher education is more successful if the prospective teacher is used to discuss his own opinion. We can learn a lot of Descartes' methodic emphasis on doubt, rather than on certainty. We observed that prospective teachers are directed to teacher thinking most of the time. But does that yield enough profit? We hope that prospective teachers will see that their didactizing is more successful if they know more about the mental processes in the heads of their young students.
Concluding Remarks
In this study we focused on the evolving and shifting nature of meaning and processes of signification. We saw that the four prospective teachers tended to put the iconic interpretations of the material first. It was also clear that our small group gave their personal experiences as first points of reference: from their school career, but in particular from their working as trainees. In the Dutch situation the programs of the teacher educations are more and more student-centered. We want the student teachers to do forms of action learning, but when their own experiences are too dominant, there are no good conditions for growth.
We proposed two methods for an effective approach: to discuss the use of mathematical language and see the consequences for a didactical way of thinking and to let them make observations in the MILE-environment and lead them to hypotheses and local theories. In that process we try to bring the student teachers to an actor's point of view instead of their usual observers' point of view. Of course, we discussed a very small part of the math education, for example, we did not talk about the way in which the student-teachers use theoretical knowledge. We reflected upon potentials to prepare prospective teachers for their role in creating opportunities for students to reinvent mathematics. The final goal is that they build up knowledge of their understanding of students' developmental possibilities, so that they can make plans for instruction and interact with students in the classroom.
Guided reinvention is a leading characteristic behind the development of a curriculum for primary school mathematics. To guide student-teachers into their reinvention of didactical theories, teacher-educators need to have deep insights into student-teacher's thinking, feelings, beliefs, theories, assumptions, etc. This study forms the groundwork for this latter knowledge.
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Using Robots to Learn Functions in Math Class
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In this paper we present and discuss an activity realized with K-8 level students using robots to learn functions in the mathematics classroom. Research presented in this paper is framed by project DROIDE which is a three years project. We are now in the first one. The aims of DROIDE are:
• to create problems in Mathematics Education/Informatics areas to be solved through robots;
• to implement problem solving using robotics in three kinds of classrooms: mathematics classes at K-9 and K-12 levels; Informatics in K-12 levels; Artificial Intelligence, Didactics of Mathematics and Didactics of Computer Science/Informatics subjects at high level;
• to analyze students activity during problem solving using robots in this different kinds of classes.
In spite of we are just beginning the research, first data collected show them as promising and we can already point out some implications for mathematics teaching and learning when robots are used as mediators between students and Mathematics.
Introduction
In late years it become widely acceptable that learning is not a merely individual activity, isolated from social, cultural and contextual factors (Lave, 1988; Collins, Brown & Newman, 1989; Cobb, 1994; Confrey, 1995, in Núñez, Edwards and Matos, 1998). Learning occurs in social contexts that influence (and are influenced by) kinds of knowledge and practices that are build (Lave and Wenger, 1991; Wenger, 1998 e Wenger, McDermott e Snyder, 2002).
Thus, we can not neglect the real world where actual students live - a world more and more informatized and consequently more mathematized. What is important to learn in our days is not the same at the time when students' parents were children.
The evolution of technical capacity of computing equipment and network communication possibilities brought new work dimensions and possibilities. But the great majority of classrooms does not reflect this turn that carried new pedagogical challenges (see Fernandes, 2004).
In Portugal, a lot of research, focusing in the use of information and communication technologies, has been driven, either in teachers' education programmes or with

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pupils of K-5 to K-12 levels. This research has concerned mainly with the use of a certain kind of software (e.g. GSP, Cabri-Géomètre, Modellus, etc.) and calculators (graphic or not) in the classroom.
Informatics teaching is a recent curricular area in Portugal. Thus, there is few research concerning with that problematic.
Either in Mathematics Education or in Informatics teaching and learning area there are still questions that deserve our attention, namely, the use of robots to teach and learn mathematics and informatics.
The Project and its Aims
DROIDE
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: "Robots as mediators between students and Mathematics and Informatics" is a three years project and we are now on the first year.
We place three kinds of aims for the project:
• to create problems in Mathematics Education/Informatics areas to be solved through robots;
• to implement problem solving using robotics in three kinds of classrooms: mathematics classes at K-9 and K-12 levels; Informatics in K-12 levels; Artificial Intelligence, Didactics of Mathematics and Didactics of Computer Science/Informatics subjects at high level;
• to analyze students activity during problem solving using robots in this different kinds of classes.

Thus, we established the following research problem: to describe, analyse and understand how students learn mathematics/informatics having robots as mediators between them and mathematics/informatics.
Within a perspective of interpretative nature – in which empirical work constitutes a guide to the search – we posed a set of questions that we would like to answer with this search:
(a) How do students appropriate certain mathematical concepts using robots?
(b) How do they use robots to learn how to develop algorithms?
(c) What is the role of robots in mathematics/informatics learning?
(d) In which way do robots facilitate mathematics/informatics learning?
(e) How can robots help in developing mathematical and informatics knowledge?
(f) What is the role of robots in developing students' mathematical competency?
(g) How does creating mathematical/informatics problems to be solved through robots influence upon teachers and future teachers' methodologies of work in the classroom?

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The authors of this paper would like to acknowledge the collaboration of the other two colleagues of the project: Elci Alcione dos Santos and Luís Gaspar. We also acknowledge the support from Mathematics and Engineering Department (DME) and from Local Department of Ministry of Education (SRE).154
(h) How does the use of robots in teachers' education programmes develop competencies in teaching Mathematics and Informatics?
This paper relates an activity realized with K-8 students using robots Lego® Mindstorms™ Robotic Invention System™ in a mathematics class to teach functions
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Theoretical Background
The research took into account Situated Learning Theories (Lave & Wenger, 1991, Wenger, 1998, Wenger et al, 2002). The notion of community of practice, such as it is used on theoretical perspectives of Jean Lave and Etienne Wenger, which consider learning as a situated phenomenon, are used to reflect upon emergent learning within students mathematical and computer science/informatics practices.
According to Wenger et al (2002) practice
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is constituted by a set of "work plans,
ideas, information, styles, stories and documents that are shared by community members" (p.29). Practice is the specific knowledge that the community develops, shares and maintains. Practice tends to evolve as a collective product integrated in participants work and organizing knowledge in ways that make it useful for themselves insofar as it reflects their perspectives (Matos, 2005).
Wenger (2002) proposes three dimensions of the relation by which practice is the source of coherence of a community: mutual engagement, joint enterprise and shared repertoire.
Mutual engagement: a sense of "doing things together". Sharing ideas and artefacts with a common commitment to the interactions between members of the community.
Joint enterprise: having some object as an agreed common goal, defined by the participants in the very process of pursuing it, not just a stated agenda but something that creates among participants relations of mutual accountability; that become an integral part of the practice (Matos, Mor, Noss and Santos, 2005).
Shared repertoire: agreed resources for negotiating the meanings. This includes artefacts, styles, tools, stories, actions, discourses, events, concepts.
Methodology
Methodology adopted is organized in three stages according to the aims of the project: 1
st
- problems creation; 2
nd
- classroom implementation and data collection;
3
rd
- data analyses.

16 These robots are used at classrooms because students can interact with mechanical parts and see results immediately. This allows students to employ certain theoretical concepts and to understand how they work in reality. It is important to point out that it is not necessary to have prior knowledge in robotics neither in computer programming.
17 The term practice is sometimes used as an antonym for theory, ideas, ideals, or talk. In Situated Learning theories that is not the idea. In Wenger's sense of practice, the term does not reflect a dichotomy between the practical and the theoretical, ideals and reality, or talking and doing. The paper extension does not allow the development of the idea of practice. For discussion of practice related with mathematics education see Fernandes (2004).155
First stage – analyses of School Mathematics and Informatics curriculum by researchers, to choose didactical units where robotics can be used. Creation of problems/tasks (to be solved in Mathematics and Informatics classes).
Second stage – Problems/tasks implementation in Mathematics and Informatics classes.
Data collection – Data are being collected recording, on video, the activity of students observed.
Third stage – analyses of students activity at the time when they work (in mathematics/informatics) with robots. Methodology used has an interpretative nature.
Data analysis is supported by Situated Learning Theories. The unit of analysis considered was "(...) the activity of persons-acting in setting" (Lave, 1988, p.177).
Using Robots to Learn Functions: One Problem Proposed
In this part we will present a brief description of the context, of general plan of work for the unit of functions, of problem that were solved, and of mathematical activity and mathematics involved on the problem.
Context
Basic School of Caniçal, created in 1996, is situated at the East extreme of Madeira Island, at Caniçal village, whose population is about 5500 residents. Fishing is traditionally the economic base of the village. Building construction is an alternative as well as seasonal emigration whose implications in familiar structure is visible seeing that, grand parents and close parents are who care with young people. That fact is reflected on school performance of students.
In mathematics class students worked in small groups. The work involved, in a first phase, robots construction and programming to solve simple tasks using Windows® visual environment programming that come with robots kits. Subsequently, students used robots to recognise and apply coordinates system in robots programming, understand function concept, represent one function (direct proportionality) using and analytic expression and, to relate intuitively straight line slope with the proportionality constant, in functions such as x ↦ kx .
General plan of work for functions unit
First mathematical unit to be worked was functions. For that didactical unit we prepared four sets of problems. With Problem 1 we pretend that students recognize examples and counterexamples of functions in correspondences presented in such different ways and identify functions as examples of correspondences of daily situations. With Problem 2 we aim that pupils deal with other kinds of graphs, behind straight lines and recognize then as functions as well, if it is the case. With Problem 3 we pretend students to learn direct proportionality, as a function. Direct proportionality definition as a function emerges of the mathematical activity of students using robots. Finally, Problem 4 is about topics related with affin function, such as y-intersect, slope and the relation between the graphic of that kind of 156 functions with the graphic of the function of direct proportionality 'associated' to the first.
The robot travel
Now we present part of a problem that has been solved by students:
1. We asked Pedro and João to imagine and draw a graphic that represents a robot travel from a certain start. They presented the following graphics:
Fig. 1
1.1. Now you have to study the graphics presented by Pedro and João. Describe the robot travel relatively to its distance to the starting point.
1.2. Try to programme the robot in such a way that it realizes the proposed travels. Test it, and, if it is possible, confirm the results. Write the program(s) that you have done.
1.3. Did the robot realize both proposed travels? Present the difficulties that you found in programming the robot to such travels.
1.4. What is the necessary condition in order that one graphic represents the 'possible travel'?
Description of students practice
Students solved the problems in small groups, collaborating in problem solving, arising hypothesis and discussing their viability. Below we present a transcription
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of
part of a class where students were solving question 1.1.

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In this transcription (due limitations of space) we mainly present students exhibiting the result of the discussion they had on the group to the teacher. We choose to cut the episode in this way because we can 'see' on the transcription students building, intuitively, the concept of function.
Start 157
In Rui's group the solution of question 1.1. emerged only of the analyses of the graphics. The same happens with Ricardo's group, maybe because these two groups were very close on physical space, and listened the discussions among elements of the groups and between those and the teacher. In other groups to programme Pedro's travel was important to understand the graphic; to understand that the graphic is not translating the route of the robot but the relation between time and distance in the travel done by the robot. The work with the first graphic allowed students to understand the second graphic and consequently, to capture intuitively the concept of function.
Rui: Teacher…Teacher… This graphic is not good! [Pointing to the second graphic]
Teacher: It's not good? Why?
Rui: It's not good because in this way the robot is walking backwards. And the robot has always to go ahead.
Teacher: I do not understand what you mean [trying that Rui made explicit what he was thinking]
Rui: I don't know …
Rui looked to the graphic again and goes back to the discussion with the other group elements.
Rui: Teacher… we already know. Could you, please, come here?
Teacher: Yes?
Rui: This line is not what the robot route. It's the distance.
Teacher: Distance?
Rui: The distance from the starting point. And here is time. And we can not do that.
Teacher: What can't you do?
Rui: The robot can't do this route.
Meanwhile, Ricardo [a student of another group that was close to Rui's group and very attentive to the discussion] answered:
Ricardo: Can't be because the robot cannot walk backwards in time.
Teacher: What will happen if the robot walks backwards in time?
Rui and Ricardo [almost at the same time]: I don't know.
Teacher: Observe and study the graphics.
A few minutes later…
Rui: I already know! I already know! The robot had to be at to places at the same time , isn't it teacher? 158
When every group had solved the problem, teacher discussed with all the group mathematical ideas involved in the problem trying that, they together, make a synthesis of the main mathematical concepts.
In all this process of solving the questions a shared repertoire emerged. The vocabulary they used to approach problems and questions is a mix of vocabulary of two distinct domains (mathematics and robots). They are analysing a graphic but they talk about what a robot can and cannot do. Using the robots and its programming as a taken-asshared resource allow students to negotiate meaning among them (in the group) and between the group and the teacher and give meaning to students mathematical activity.
To have a join enterprise (that can be to solve the question, to please the teacher, to understand the meaning of mathematical concepts involved or only to play with the robots) is very important to motivate students to engage in the activity and is an integral part of students practice.
The co-definition of mutual engagement is visible through:
- a growing sense of responsibility in solving the questions posed by the teacher and in understanding what they are doing together and what is the meaning of what they are doing;
- not giving up until they found the problem solution;
- a pleasure in going deeper into their ideas and in building a solution to the problem and meaning to their answers.
In these three dimensions of the practice we talk about meaning. In fact Wenger (1998) argues that the social production of meaning is the relevant level of analyses

for talking about practice.
But when meaning is discussed in the sphere of mathematics education usually it concentrates on the meaning of (mathematical) concepts. Questions such as the following become important: What sort of meaning can be associated with certain mathematical concepts? What is the meaning of particular concepts to students? What sort of meaning can be associated with this concept from a mathematical point of view? What is the meaning of this concept from the perspective of the teacher? What is the shared meaning of this concept? (Skovsmose, 2005) 159
Findings
To analyse students practice in mathematical classes is fundamental as element that helps to understand learning. It is important that students' engagement in school mathematics activity is not only to accomplish a curricular programme but that they have a genuine interest by the domains they work and the use of robots have a relevant role as a mediator element in all this process.
In spite of we are now in an initial phase of data analyses we can already foresee some findings that show themselves as promising.
• Students felt comfortable both when building robots and using programming environment.
• Using robots in mathematics class promotes an increment either in discussion between students and between students and teacher and in collaboration on the resolution of proposed problems.
• Students recognize impossibility of executing a task without assuming it as an inability of them. This fact was evident, for instance, at the time when they are solving the previous described problem.
• Function concept was apprehended in a significative way. The definition of function emerged as a final conclusion of students work and not as a starting point.
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Gender, equity, teachers, students and technology use
in secondary mathematics classrooms
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In recent years we have been researching a range of issues associated with the use of digital technologies - computers, graphics calculators, and CAS calculators - in secondary mathematics classrooms. Gender and other equity considerations were a focus in some of the work; teachers' and students' beliefs about and attitudes towards the technologies were also central. In all of the studies, comparisons were made. The views of male and female students and teachers have been examined, students' and teachers' views compared, the perspectives of teachers in different countries contrasted, teachers' views on computers and calculators distinguished; and the examination results of male and female students using different digital technologies explored. In this proposal, synopses of various dimensions of a selection of the studies are presented. Taken together the studies reveal that gender differences favouring males with respect to technology use are evident, that teachers are generally supportive of the use of digital technologies for mathematics learning, and that curricular and school factors are associated with the classroom use of technologies and beliefs about their efficacy in fostering student learning.
Introduction
In Victoria, Australia, technology use across the curriculum has been strongly advocated and financially supported by government for some years. In the curriculum document for grades P
19
-10, the Curriculum and Standards Framework II
[CSFII] (Victorian Curriculum and Assessment Authority [VCAA], 2002), the "full use of the flexibility and value for teaching and learning programs provided by the increased application of information and communications technology (ICT)"
(VCAA, 2002a) is strongly advocated. In the Mathematics section of the CSFII, the use of a range of digital technologies for mathematics learning is strongly supported and, by implication, assumed:
Recent developments in the availability of calculators, graphics calculators and computer software have led to a major re-evaluation of school mathematics curricula in terms of content and strategies for teaching and learning mathematics. The Mathematics KLA supports these developments, by placing clear emphasis upon the sensible use of technology in concept development, as well as in technologyassisted approaches to problem-solving, modelling and investigative activities... At all times, the choice of the appropriate technology and the extent to which it is

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P is the grade level prior to grade 1 161
employed should be guided by the degree to which these tools assist students to learn and do mathematics. [VCAA, 2002b]
Grades 11 and 12 in Victoria are considered the post-compulsory levels of schooling and involve a two-year program, the Victorian Certificate of Education [VCE]. Grade 12 VCE results are used for selection into tertiary institutions. At the grade 12 level, three mathematics subjects are offered and one of the outcomes stipulated for VCE Mathematics is "the effective and appropriate use of technology to produce results which support learning mathematics and its application in different contexts" (VCAA, 2005, p.7). In one of the three VCE mathematics subjects, Mathematical Methods, graphics calculators have been mandated for several years. In 2003-2005, a small number of students were involved in an pilot program, Mathematical Methods [CAS], in which CAS were used in place of the graphics calculator. In 2006, the two parallel versions of Mathematical Methods will be open to all students; at the same time, the assessment program will alter in that one of the two examinations for the subject will be calculator-free. By 2008, all students in Victoria taking Mathematical Methods will be using CAS calculators.
The studies discussed in this proposal were all conducted in Victoria. Each involved students and/or mathematics teachers from the secondary grades, 7-12. One study was focussed on the use of computers, one on computers and graphics calculators, one on graphics calculators only, and one on CAS calculators. One of the studies also involved mathematics teachers from Singapore. Each study involved comparisons being made. Students' and teachers' views were examined for gender differences as were students' mathematics achievements, teachers' and students' beliefs were compared, comparisons were made between the views of teachers from Singapore and Victoria, and teachers' views on computers and calculators were contrasted.
A synopsis of each study, with selected major findings, is presented below.
Study 1. A three-year study of computer use in Victorian grade 7-10 mathematics classes.
The overall aims of the three year study into computer use in Victorian grade 7-10 classrooms included: (i) determining perceived effects of using computers on students' mathematics learning outcomes, (ii) identifying factors that may contribute to inequities in learning outcomes (equity factors included: gender, socio-economic background, language background, Aboriginality, and geographic location) and (iii) monitoring how computers are being used for mathematics learning. The research design for the three years involved:
Year 1: surveys of mathematics students in grades 7-10 (N=2140: F=1015, M=1112, ?=13) and their teachers (N=96: F=52, M=44); survey of grade 11 students (N=519: F=237, M=281, ?=1) reflecting on previous use of computers for mathematics learning - 29 co-educational schools were involved.
Year 2: in-depth studies of 7 grade 10 mathematics classrooms at three schools - surveys, observations, interviews. 162
Year 3- repeat of Year 1 surveys in same schools - only 24 schools participated - grades 7-10 students (N=1613: F=810, M=794, ?=9), their teachers (N=75; F=41, M=34); and grade 11 students (N=376: F=166, M=210).
The findings from this study have been widely reported (e.g., Forgasz, in press a, in press b, 2005, 2004a, 2004b, 2003, 2002a, 2002b, 2002c). A brief summary of some of the main findings is presented below:
• Computer ownership by mathematics teachers was high (>80%), teachers considered themselves well-skilled with computers, and most had used computers in their mathematics teaching (≥ 70%); <10% had not and were not planning to do so. Computers were more widely used in single content areas than fully integrated across the mathematics curriculum. Teachers wanted more professional development to extend skills, increase confidence levels, and become more familiar with software applications.
• The most widely used mathematic-specific software applications were Geometer's sketchpad (22% of teachers) and Graphmatica (22%); the most frequently used generic software applications were spreadsheets (62%), wordprocessors (49%), and Internet browsers (32%). CD-ROMs accompanying textbooks were also widely used (26%).
• Teachers were generally supportive and confident about using computers in their classrooms. They identified greater access to hardware, more technical support, the availability of high quality software, and on-going professional development as the significant issues to be addressed if they are to use computers more in their mathematics teaching.
• Teachers (≤ 60%) were not fully convinced that computers aided students' mathematical understandings; males, however, were more positive about the effects than females. Students were even less convinced of this than their teachers (≤ 30%); again, males were more positive than females.
• Students attending schools in lower SES locations, government schools, and at lower grade levels, were more convinced than their respective counterparts that computers assisted their mathematical understandings.
• Compared to their respective counterparts, students from non-English speaking backgrounds, males, and those with high self-ratings of mathematics achievement, had higher computer ownership and held more positive attitudes towards computers for mathematics learning.
• Some teachers believed there were differences in the ways boys and girls work with computers. Compared to boys, girls were generally considered less confident, less competent, and less interested in using computers.
• Data from grade 11 students indicated that computers served as stronger motivators for males' than for females' enjoyment of mathematics, and in persisting with mathematics at higher levels of schooling and beyond. 163
Study 2. A comparative study of Victorian and Singaporean teachers' perceptions of computer and graphics calculator use.
The aim of this study, conducted in 2005, was to investigate and compare the use of and perceptions towards computers and graphic calculators among mathematics teachers at senior secondary levels in two settings: grades 11 and 12 in Victoria and junior college years 1 and 2 in Singapore - the two final years of schooling leading to tertiary entrance (Tan & Forgasz, n.d.; Tan, 2005).
Since 2002, only a small group of mathematically inclined students taking the subject Further Mathematics at the senior secondary levels in Singapore have been allowed to use graphics calculators in their examinations; the majority studied the subject Mathematics and graphics calculator were not permitted in the examinations. Graphics calculators will be introduced more widely into the revised mathematics curriculum at the senior secondary level in 2006.
A survey questionnaire was administered to 35 (19M, 16F) mathematics teachers from 14 independent (non-government, non-Catholic) schools in Victoria, and 33 (16M, 16F, 1?) teachers from five junior colleges in Singapore. It was found that graphics calculators were extensively used by the Victorian teachers, and that high proportions of the Singaporean and Victorian teachers had not or had hardly ever used computers with their mathematics classes. Interestingly, Victorian teachers' self-ratings of their graphics calculator skills (beginner, average, or advanced) were much higher than the Singaporean teachers'; only one Victorian indicated being a beginner and 18 claimed to be advanced, while 19 Singaporean teachers identified as beginners and only two to having advanced skills. The difference in perceived competency resonated with other findings that showed that a higher proportion of Victorian than Singaporean teachers had personal access to graphics calculators and had used them for a longer time. The Victorian teachers were found to be more strongly in support of graphics calculator use over computers, while the Singaporean teachers were more supportive of computer use over graphics calculators.
The teachers in both settings identified factors that affected their decisions to use the technologies. The data suggested that mandating technology tool use in an assessment program, as was the case in Victoria, plays an important part in explaining the extent of their use by teachers, and may also account for the Victorian teachers' preference for graphics calculators over computers.
Study 3: Perceptions of the impact of graphics calculators among teachers from Victorian Catholic schools
The aim of this study was to investigate teachers' perceptions of the impact of graphics calculators on their teaching practice, on student learning, and on the mathematics curriculum in grades 10-12 in Victorian Catholic secondary schools (Griffith, 2005).
A survey was conducted with 47 (25M, 22F) senior secondary mathematics teachers from 16 Catholic schools (6 girls-only, 6 boys-only, and 4 co-educational schools). It was found that the teachers had used graphics calculators with all of the grade 10-12 164
classes they taught, and that all students owned their own graphics calculators. The

teachers indicated that students in grades 10-12 were using the graphics calculators for in-class activities, problem-solving tasks, computational tasks, during investigations, and to do tests and School Assessed Coursework [SAC] tasks. In general, students were not using graphics calculators to write programs, play games, or to do puzzles or quizzes.
Some differences were noted in the views of male and female teachers on issues associated with graphics calculators use. While all believed that using graphics calculators had enhanced their mathematics teaching and changed how they taught, the female teachers felt more strongly than the male teachers that their students were able to solve non-routine problems on the graphics calculator that would otherwise be inaccessible using algebraic techniques, and that students were able to engage with challenging problems; they believed less strongly that students accepted answers given by the calculator and rarely checked for reasonableness. The females also believed more strongly than the males that the introduction of technology-free examinations (2006 change to the VCE mathematics assessment program) was a positive development.
In summary, the teachers generally believed that graphics calculators have had a positive impact on their teaching and on students' learning outcomes, and that the curriculum has been enriched. Female teachers tended to hold these views more strongly than their male counterparts.
Study 4. CAS calculators: Gendered patterns of achievement in a high stakes examination pilot program, and teachers' expectations of their
imminent mandated use in such courses.
The focus of this two-part study was on the introduction of CAS calculators for use in the high stakes grade 12 VCE mathematics examinations (Forgasz & Griffith, n.d.). For some years, it has been mandatory for students to use graphics calculators in some VCE mathematics examinations. Since 2002, a pilot study has been conducted involving small groups of grade 12 students using CAS calculators instead of graphics calculators in one of the three VCE mathematics courses on offer, Mathematical Methods (the subject most widely required as a pre-requisite for tertiary study). From 2006-2008 the CAS calculator will be optional, that is, there will be two parallel Mathematical Methods courses available for all students to study – one in which graphics calculators must be used, the other mandating CAS calculators. From 2008, only CAS calculators will be allowed.
Part 1 of this study involved an exploration of male and female students' results in Mathematical Methods over three years, 2002-2004. Comparisons were made between the achievements of the students in the CAS pilot study and those of the vast majority of students who used graphics calculators. Male and female results at the top two achievement levels awarded (A+ and A) in each of the three assessment tasks (one school-based, two external examinations) comprising the VCE Mathematical Methods subject for the year 2004 are shown in Table 1. 165
Table 1. Mathematical Methods and Mathematical Methods (CAS) results at top three achievement levels, by gender, 2004
Mathematical Methods Mathematical Methods (CAS)
Male Female M:F
2
Male Female M:F
N %
1
N % N % N %
Grade Task 1 (school-assessed)
A+ 1747 18 1261 15 1.20 42 17 20 13 1.31
A 2003 21 1939 24 0.88 57 23 28 19 1.21
Examination 1
A+ 1102 12 793 10 1.20 35 14 16 11 1.27
A 1306 14 1030 13 1.08 40 17 19 13 1.31
Examination 2
A+ 1013 11 593 7 1.57 33 14 12 8 1.75
A 1074 11 874 11 1.00 34 14 20 14 1.00
1
Within gender cohort percentages.
2
Male to female ratio (M:F) = Male % : Female %
3
Shaded ratios – when M:F>1 ie. greater % males than females
4
Bolded M:F ratio – higher M:F ratio for the two subjects
The data in Table 1 reveal that when within gender cohort percentages were considered, higher proportions of males than females were awarded the A+ grade for all three tasks in Mathematical Methods (CAS) than in Mathematical Methods with graphics calculators) for all three tasks; this was also true for 2 of the 3 tasks at the A level of achievement (Task 1 & Examination 1). Overall, the there was a wider gender gap favouring males in performance at the highest achievement levels for Mathematical Methods (CAS) than for Mathematical Methods with graphics calculators. Similar patterns were observed for the 2002 and 2003 results.
In Part 2 of the study, 38 teachers' views of the likely impact of the wider use of CAS calculators from 2006 and beyond were examined. None of these teachers was involved in the Mathematical Methods (CAS) pilot study. In one section of a survey questionnaire, teachers were asked to respond to the open-ended question, "From 2006 onwards students will be able to use CAS (Computer Algebra Systems) calculators in VCE Mathematics examinations. Please describe in your own words the impact you think CAS calculators will have on: your teaching, student learning, and curriculum". Separate spaces were provided for teachers to answer about teaching, student learning, and the curriculum.
In general, the teachers tended to respond positively about the introduction of CAS calculators in each of the three categories that they were asked to comment on: teaching (positive: 68%, negative: 32%), student learning (positive: 65%, negative: 166 35%) and curriculum (positive: 70%, negative: 30%). None mentioned any potentially differential impacts on male and females students.
General conclusions from the four studies
Overall, the results of the four studies indicated support among Victorian teachers for the use of digital technologies in the secondary mathematics classroom. The teachers considered themselves fairly well-skilled in using the technologies, but called for more professional development to enhance their skills and confidence. Not all teachers were convinced of the effectiveness of computers on students' mathematical understanding; students were even less convinced than their teachers. Yet, the teachers use the technologies fairly regularly, and have found that their teaching, students' learning, and the curriculum, have been affected – particularly in using handheld technologies. With respect to gender, the findings seem to support Hoyles' (1998) contention that more emphasis on computer use might negatively impact on girls. Other equity factors (e.g., socio-economic, language background, and geographic location) appeared to influence access to and attitudes towards computers. There were indications that males may be advantaged over females in using the sophisticated CAS calculators in high stakes examinations. Based on the comparisons between Singaporean and Victorian teachers, another important finding was that curricular expectations and requirements seem to influence teachers' beliefs about and use of particular digital technologies.
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Development and Introduction of the New Content of Mathematics in Secondary Schools in Latvia
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This paper examines the possibilities for and principles of developing and implementing a new math curriculum into the Latvian secondary education. To get a better understanding of the Latvian situation, we are offering a brief insight into the existing educational system. The existing standard and sample curricula for teaching math have become outdated and work is being done on developing a new standard and curriculum. The real life situation in Latvia with regard to the serviceability and availability of computers and the pupil's access to computers for learning math has been identified. The conclusion is that pupils use computers mainly in classes of computer studies, but very in little in math classes. Some research which directly deals with this particular problem is presented. When designing the new curriculum, our attention is focused on how to use the computer in teaching math while preventing its use as a means in itself.
Our approach is underpinned by objectives, tasks and philosophy of teaching math that would result in engaging the modern day technologies. In Latvia a new extensive project has started - "Curricula development and further education in science, mathematics and technology related subjects". A brief overview of the project performance with project objectives and developed products is presented. Some of the potential problems are considered that need be avoided in order to ensure the projects sustainability. The project is unique to Latvia and one of its central objectives is (through the standard, curricula and teacher support material) to demonstrate to teachers what, how and why to teach math in secondary school with the help of the computer. The project's implementation will allow us to introduce a unified concept for the teaching of science and math at the national level.
The tasks of education are to prepare a pupil for life and for further education. In order to accomplish that, the main focus should be placed on three activities: imparting knowledge, evaluating and making it practical (including creativity). Learning mathematics is essential for all three activities mentioned above.
Education System in Latvia
The Law of Education in the Republic of Latvia determines following levels of education:
- Preschool education 169
- Primary education

- Secondary education
- Higher education
For all the residents of Latvia primary education is mandatory, but secondary education and higher education are optional. The Cabinet of Ministers approves the standard of education, which determines the mandatory content of primary and secondary education. The curriculum standards are parts of the education standard. The Center for Curriculum Development and Examination for each subject develops a sample program, which serves as a guide to teacher. Each school and each teacher has rights to develop the corresponding study program for their school.
In the former Soviet Union the teaching of mathematics was centralized in the whole country, including Latvia. After the fall of the Soviet Union, as Latvia was developing as an independent nation country, the first curriculum standards were developed (in 1992). In 1998 work was begun to development new curriculum. In the process of the development of the new content of subjects, the results of various international studies were used, for example, international program of pupil's assessment in OECD countries [1]. Now we have new subject curriculum standards developed in primary education. The implementation /introduction began in the school year 2004/2005, and it is planned to completely finish this process in three years. Up to now the teaching of mathematics in Latvia has had more academic approach, and not offering the student practical tasks and not showing the connection of mathematics to real life. Now the new content of secondary education is being developed. In the summer of 2005 a new and important project, "The development of curriculum and further education in the natural sciences, mathematics and technologies in secondary education," was started, which is financed with the support of the European Social Fund.
Accessibility and the use of digital technologies in Latvia
Now in Latvian schools the computer is widely used in teaching computer studies, but only a little in other subjects, including mathematics.
In October / November 2001, at the request of the Ministry of Education and Science of the Republic of Latvia, the Center of Research of the Market and Public Opinion conducted the research "The attitude of 5
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-12
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grade pupils towards information
technologies and its use in schools, existing access to computers and the Internet". The research showed, that more than 90% of the pupils have access to computers [2], 60% of the pupils have access to the Internet. Almost all the pupils (95,6%) have acknowledged, that it is necessary to increase the number of computer classes in school.
In 1997 a project "Latvian Education Information Systems" (LIIS) was begun [3]. That ensured the introduction of information and communication technologies in 170 schools. In each school one computer classroom was set up and in 60% of the schools was provided direct access to the Internet. 70% of the teachers of Latvia have been trained in regular courses. In regions and in the larger cities regional computer centers are established, in which more knowledgeable teachers are training their colleagues. As a part of this LIIS project, training materials in various school subjects have been developed (volume - 100 000 pages) and can be found in the Internet, as well as a lot of learning programs and games. Unfortunately one should admit, that the majority of the materials related to mathematics, not always can be used daily by the pupil. The target group for those materials are the pupils, who have deeper interest in mathematics, in preparation for educational Olympics and for teachers. There is a lack of interactive materials, which are developed for the interesting independent studies in mathematics.
Presently computer is practically not being used in the learning process of mathematics in Latvia. As a part of the project "Curriculum Development and further education of teachers in the subjects of natural sciences, mathematics and technologies" the survey was conducted, in which 692 10
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graders from 12 pilot
schools participated.
Asked about "The use of the computer in mathematics classes", 73% of participants responded, that opportunity to use computer in mathematics has not been offered.
Computer usage in mathematics class
There is no choice
Liked very much
Liked very much
Did not like
Not offered
Among those pupils, to whom such option to study mathematics using a computer was given, 14 % liked it and 10 % liked it very much. Only 3% of pupils did not enjoy learning mathematics with a help of computer.
Currently, one of the most widely accessible applications of the computer in mathematics would be to work with different information sources, for example, the Internet. Responding to the question about "Gaining information from different sources", 42% of students responded, that this form of work was not offered. 171
Gaining information from different sources
There is no choice
Liked very much
Liked
Did not like
Not offered
The development of the content of mathematics for the secondary school, using the computer as one of the means of teaching
Currently in Latvia work is being done in the project of developing new subject standards in secondary education and corresponding subject sample program. In developing new curriculum for mathematics, two main approaches in teaching are being observed, which are shown in the chart [4]:
What is mathematics? The main question in a
mathematics course?
Technology based
approach
Mathematics is the
means for serving
other branches of
science and practice
How?
Research based
approach
Mathematics is also
intellectual and
cultural value in itself
Why?
In the technological approach it is considered that the student has to be given an idea about as many issues as possible at the operational level without fundamental explanations. Much attention is given to motivation: Why are we introducing one or another mathematical tool, what practical problems one can solve using that, or what can be described, etc. It is typical for this approach to use mathematical algorithms. In the content there are many rules which tell what to do in one or another situation. However, once the pupil goes beyond the stated rules, he/she becomes quite helpless. When carrying out the curricula and training methodology reform, it is
important to avoid being carried away by one approach over the other. Realising the 172 possibilities afforded by the computer, its use in mathematics classes should cater for a balance between both approaches in teaching mathematics.
In the framework of the project "Teaching contents development and further education in mathematics and technology related subjects", extensive work has been started on creating a new math teaching standard, program sample and teacher support material. This is work that has no precedence in Latvia and as a result of that we expect to obtain a completely new insight into the teaching of math in the secondary school. The project has enlisted the help of math teachers, university lecturers, members of the publishing community which altogether form a professional team for the implementation of the project. The materials produced by the project are to be approbated and implemented on a common national basis.
The methodological teacher support material in math includes: samples of classes, samples of various level problems, research assignments, test work, samples of IT use and possibilities, interactive course for pupils. The objective of our work is to provide ready for work tools: fully fledged electronic teaching aids, work sheets for PC work, and references in subject teaching curriculum on how to use the computer in teaching math.
Visualization is a key element in the teaching process. A feature which we plan to introduce into the training process is the dynamic math application GEONEXT [5] which offers a new approach to teaching and learning math. We chose this software because it is widely available, free of charge and has a translated Latvian version. Thus we are hoping to foster a proactive research based approach that would facilitate formation of math oriented thinking in pupils.
There is a plan to also include in the curriculum Internet addresses relevant to each topic. Into the specific training content we plan to include assignments and research projects that require information processing with the help of spreadsheets.
One of our project's most essential products for pupils is the interactive training material which meets the standard and curriculum requirements. It is comprised of the theoretical material, problems with solutions, practice problems with solutions, and self-testing. Although similar teaching material is available on the Internet, it unfortunately does not address the specific circumstances in Latvia and is not available in the native language of the pupils. It is important that the interactive material is developed for each secondary school topic. The pupil, when working with the interactive teaching material, may proceed at his or her own chosen pace and determine what to study at a specific point in time.
The implementation stage performed by the teacher at schools is integral to the development of all the material (class samples, samples of different level problems, research projects, tests, samples of IT use and possibilities, interactive course for pupils). Therefore it is necessary to prepare the teacher for working with the material. To avoid risks associated with insufficient teacher training, the project framework provides for a 72 hour training course. The plan is that all secondary school teachers 173 in Latvia will undergo this course. This further education course for teachers will also include training into the use of computers in teaching math.
Due awareness of the potential problems relating to the implementation of the project is also important for ensuring the project's sustainability.
These are some of the risks we have identified:
• insufficient teacher training for conducting classes in computer room and for using PC for demonstration purposes;
• underfunding for updating PC resources - the service life of computers in schools might be 5 years;
• deviation away from a balanced split between the use of technology and research oriented approach to teaching math, resulting, for example, in pointless use of computers in math classes.
Although we are now only into the six month of the project we have accomplished quite a lot. Material for form 10 is developed and approbated in 12 pilot schools. Returns form pupils and teacher evaluations on the developed material is being compiled and used in further evolution of the material. Already by September 2006, new research data will be collected, which we would be pleased to present during the December conference together with the developed material.
In our move to modernizing the teaching content we are now facing an enormous challenge and would appreciate very much the opportunity to take part in this conference, thus learning more about other countries' experience and achievements in this area.
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Learning communities with a focus on ICT for inquiry in
mathematics – affordances and constraints in development of
classroom practice
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In this paper I will report from research on use of ICT tools to promote students' learning of mathematics. The work is largely situated in a learning community with teachers and didacticians working together on planning for an inquiry approach to learning mathematics. A sketch of theoretical framework is presented with emphasis on seeing ICT as a personal technology, developing into an instrument for the learner and with considerations of affordance and constraints to analyse the activities. I will present and analyse cases of teachers and students' work on mathematics in the classrooms using ICT tools. I will focus on how the teachers plan for and support the students' learning in the classroom. In particular I will focus on how mathematical concepts and relations are represented in the work.
Background
There has been long time effort to integrate use of Information and Communications Technology (ICT) in Norwegian schools to support teaching of different subjects. Computer studies, (EDB), IT, ICT and recently "digital competence" has been an issue in several curriculum guidelines since 1987, emphasising use of computers or calculators for teaching and problem solving in mathematics. The new curriculum plan, which is in effect from august 2006, states "digital competence" as one of five fundamental competencies in every subject in the

curriculum (2005b).
However, in spite of the efforts, little has been achieved concerning integrating ICT in particular subjects, in mathematics particularly, in a way that affect the subject content and ways of teaching (Erstad, 2004). A lot is claimed to have been achieved in general concerning use of ICT in schools, but my impression from contact with teachers informally and from reports, is that the work mainly is concerned with general organisation, use of Learning Management Systems, internet to search for information, text processing and so on. Little has been implemented when it comes to utilising the potential of ICT in teaching and learning mathematical subject content except for schools involved in ICT projects or with particularly enthusiastic teachers. Recent reports support this. An evaluation of the implementation of the current plan reveals little use, and hardly anything was reported from the classes observed during the evaluation (Alseth, Breiteig, & Brekke, 2003). A recently published report (Ola Erstad, Vibeke Kløvstad, Tove Kristiansen, 175 & Morten Søby, 2005) reveals that the use of ICT-tools
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in schools has decreased
since the previous report in 2003. Thus the development of ICT use in mathematics generally in Norway is a rather slow process.
Similar experience is also documented in other countries (Hennesey, Ruthven, & Brindley, 2005). Teachers have problems to find out how to implement ICT and there is considerable resistance and constraints, for example from caution on some uses, requirements in the syllabus and assessment system, and by teachers largely using ICT to support existing practice rather than transforming into new approaches to mathematics.
On many occasions when I have met teachers at conferences, courses or at school visits I understand from the conversation and they also expressed directly that they do not really know how to utilise ICT tools in mathematics teaching. They often express interest but also claim they have not sufficient insight in the ICT tools and how to use them for mathematics. Quite often teachers ask for good examples of tasks or teaching plans in order to understand how technology can help. This can also be caused by lack of knowledge and limited experience with ICT and suitable software.
Intentions in the curriculum
Digital competence in the new curriculum plan is one of five fundamental competencies in all subjects together with ability to express oneself orally and in written, read and calculate (UFD, 2005). The notion digital competence in mathematics is further elaborated to involve use of computers and calculators for games, investigations, visualisation and publishing. On higher levels in schools it deals with using digital tools for problem solving, simulations and modelling and in addition searching for information, analyse, and present data and be critical to sources and results. Furthermore it is important to be able to judge and utilise digital tools and know their limitations. The aim described here is similar in the current plan which has been in use since 1997.
In addition to the general aims described above, more details are given in different sections in the plan; like use of a spreadsheet with simple models for calculations, presentation of statistics in tables and diagrams, experiment with geometry, and use digital tools for work on functions.
Research question
A main focus in this paper is to research how the teachers can provide a learning environment for students' work with ICT tools in the mathematics

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ICT- tools is here used in the wide sense, computer software and calculators, without specifying characteristics. 176
classroom and how teachers can support students' work to learn mathematics using ICT.
1. How are the mathematical concepts and relations visible in the students' work on computers?
2. What affordances and constraints do the students'experience with the ICT support for a mathematical task or problem?
The relation between the students' work on computers and the way the teacher provide support is a part of the questions. The teachers work is done in the context of a project working on developmental research. Development of learning environment, task design and study of the implementation is a central issues in the research, and our experiences can illuminate questions about how teachers can be supported when implementing ICT tools and what we can learn from their experiences in classes (ICMI-17, 2005). Some of the questions concerning design of learning environment can also be illuminated.
Theoretical background
Introducing ICT in mathematics classrooms aiming to influence teaching and learning have implications for the way tasks and lessons are designed. The aim is, with support from ICT tools and teachers planning for learning, the student will learn mathematics and gain deep and flexible understanding of the subject and how it can be represented with the use of digital technology.
The aim according to the curriculum, is that the students should be able to judge and decide when and which digital tools to choose for a specific task. To achieve this goal the students have to know ICT tools in a personal way, not just know the technical features, but know and appreciate what kind of mathematical representations and models can be build and make connections to the actual tasks or problems they are going to solve. This requires a deeper knowledge and sense of what opportunities the tools can provide, in Gibsons' terms (1977; Greeno, 1994), what affordances and constraints the ICT tools provide in a specific mathematical situation.
Implementing ICT tools as a part of the learning environment have implications for the way teachers and students' work. The teaching and learning environment have to be considered in relations to how computers can be regarded, as an amplifier or a reorgansator (Dörfler, 1993). Amplifier implies doing the same as before, more efficient but without changing the basic structure, methods and approaches. In this way we will not be utilising the potential of the tools. Dörfler, and also others, claim that introduction of computer tools implies reorganising. Similar arguments are expressed by Goos, Galbraith, Renshaw, & Geiger (2005): " The amplification effect may be observed when technology simply supplements the range 177 of tools already available in the mathematics classroom, for example by speeding tedious calculations or verifying results obtained already. By contrast re-organising occurs when learners' interaction with technology as a new semiotic system, qualitatively transforms their thinking, ….. ". I think of this as a personal process whereby students develop their understanding of the tools and the mathematics involved.
This personal transforming of the students' thinking with the tools, seems to me to have ideas in common with more recent ideas from the French didacticians concerning turning the artefact, ICT tools, into a mathematical instrument (Guin, Ruthven, & Trouche, 2005). The students build their own personal instruments from the artefact through his (her) activity with it. This is a two way process, named instrumental genesis, whereby the students interact with the artefact forming their own schemes in order to work on specific tasks (Trouche, 2005). The resulting instruments consist of a combination of the artefact and the students' corresponding schemes. This process is influences and supported by the way the teacher provide for students' learning, named the teachers orchestration for the learning.
ICT tools represent both affordances and possibilities for the user and constraints for the work with tools (John & Sutherland, 2005). The affordances and constraints are not just connected to the tools, but can also be due to the classroom organisation, curriculum issues or other factors influencing the teaching. Constraints are not just negative obstacles, but can also support and frame the development of the working situation (Kennewell, 2001). A task that is too open can be difficult for the students to solve. In this sense constraints rather complement the affordances than the being opposite. According to Kennewell (2001) "the role of the teacher is to orchestrate affordances and constraints in the setting in order to maintain a gap between existing abilities and those needed to achieve the task outcome, a learning gap which is appropriate to the development." The study of affordances and constraints can provide a framework for analysing the effects of ICT in combination with other factors influencing the teaching and learning (Watson, 2006; Kennewell, 2001).
I am aware that the presentation of the theoretical considerations here is short, and perhaps unclear. My intention for the suggested paper is to develop this further, and with these concepts providing a framework for the study.
Recent and on-going projects
In a development and research project named the "ICT competence project" the aim was to develop the students' competence to use ICT tools such that they would be able to judge hat ICT tools are appropriate to use for a specific task and choose for themselves, not just rely on the teacher to tell what tools to use. The projects followed seven classes from grade 8 to 10 in schools with six teachers taking part and concluding in 2004. 178
Project meetings with teachers and project leaders were arranged every term for discussion of teaching ideas, experiences and further plans. The first two years had a main focus on development, but in the last year data were collected for research in the form of observations, collection of students' work on computer files and paper, questionnaire and interviews with a selection of students and teachers. In a working period in the final part of the project the students' work were observed, and afterwards they were given a questionnaire about their experiences in the work, what tools they chose to use and why (Fuglestad, 2005b; Fuglestad, 2005a).
In an on-going project ICT and mathematics learning (ICTML), the aim is to build a learning community, involving schools, teachers and a group of didacticians at the university college. The project collaborates closely and has a common conceptual model with the project Learning Communities in Mathematics (LCM)
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A central aim in the projects is to develop inquiry communities in mathematics (Jaworski, 2004). Some schools take part in both projects. The ICTML project has a focus on use of ICT-tools, in particular computer software, and how these can be utilised as support for mathematics teaching and learning and students' inquiry into mathematics.
Teachers and didacticians collaborate in workshops at regular intervals on developing competence with ICT tools and discussing ideas for teaching in schools and experiences. Furthermore didacticians take part in meetings at the school discussing and development of tasks or draft of lesson plans using ICT.
All kinds of activities in the project are audio or video recorded and material like students' work in written and on computer files are collected. This gives rich opportunity to study the development in the different parts of the projects.
Data for analysis
The focus of this paper is to analyse some representative cases of teachers' implementation of ICT tools to support mathematics learning and how the students' work on the mathematical content in the tasks or problems provided.
I will present cases of students' work on mathematical tasks in order to highlight different ways of setting the tasks and orchestrating for students' learning environment.
In the final part of the ICT competence project the students had a working period where they could choose what tasks and tools to work with, either ICT tools or others. I observed a group of students working on a task that they found rather difficult. I partly acted as a helping teacher, but gave only limited help. The task was to find the smallest surface area of a rectangular box with length the double of the width, and volume 500. I observed their work, which was started in one lesson and taken up again a few days later. The students chose to work on a spreadsheet and set

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Both projects are funded by the Research Council of Norway, http://www.forskningsradet.no 179
up formulae for the surface and volume, but did not use a formula for the connection between length and width of the box. There were a lot of arguments and frustrations, they needed a lot of time but after some hints they succeeded.
In a school taking part in the ICTML project one of the teachers made a draft, a collection of small tasks, on different sheets in Excel. On the first sheet, the students were asked to fill in two numbers, which are numerator and denominator of a fraction. Then as a result some equal valued (similar) fractions emerged for the students to observe and reflect on. The students were asked to write about their observations and comment on what they found. On the next sheet the students moved on to a similar more challenging task. The teacher had planned for increasing openness and difficulty of the tasks as the students moved on to the following sheets in the file.
At a meeting with the other teachers and didacticians the task was presented and discussed. New elements in the task were discussed and the participants learned from the others about ways of implementing and improving parts of the spreadsheet tasks. So both didactical and technical issues were discussed. A few days later the tasks was used in a class and observed. The task was discussed in a new project meeting. In discussing the experiences and the task several changes was made and new ideas for tasks developed.
The planned paper will present and analyse these and other cases in order to illuminate the research questions and hopefully reveal some critical elements in the teachers' planning and the students' work on mathematics using computers.
Conclusion
The proposed paper will focus on students and teachers' work in the class with use of a computer with suitable software. The cases that will be presented will highlight critical points in the teachers' orchestration for students' learning and development of the ICT tools into an instrument for learning and solving mathematical problems.
The analysis is ongoing and more will be completed during the spring term 2006. Since the work is not finished I will not provide any conclusions.
The main contribution from the proposed paper will be on the two themes Teachers and teaching, and Design of learning environments.
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More than Tools: Mathematically Enabled Technologies as Partner and Collaborator

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This paper theorises an extension to a framework that describes students' use of technology when engaged in mathematical activity and discussion. The framework describes students' interaction with technology through a series of metaphors: technology as master; technology as servant; technology as partner; and technology as extension of self. These metaphors allow illustration of potential relationships between students' intentions, technological engagement and actions. The framework is conceptualized from within a socio-cultural perspective of learning/teaching mathematics and extends the Vygotskian principle of Zone of Proximal Development (ZPD) by elevating computer and graphing calculator technologies beyond that of simple cultural tools to that of quasi-partner or mentor. The framework was developed through an ethnographic case study of a single class of students over a two year period. It describes different types of interaction between technology and students as they are challenged by new ideas and concepts or as they explore nonroutine, contextualized mathematical problems. A component of the framework is used to analyse two episodes of student/student and student/technology interaction while working on a specific mathematical task. Implications are discussed for the use of the extension of this framework as a means for the promotion of more sophisticated uses of technology in mathematics classrooms.

Through their inclusion as mandatory elements in Australian state syllabuses and assessment regimes, policy makers have authenticated the arguments of researchers (e.g., Morony & Stephens, 2000) in favour of the inclusion of mathematically enabled technologies and applications (META) because of their potential to transform learning and teaching in Australian mathematics classrooms. Similar change is evident outside of Australia in response to the influence of bodies concerned with curriculum reform in mathematics (e.g. NCTM, 2000).
Considerable research effort has been directed towards understanding how META can be used to enhance students' learning in mathematics (Dunham & Dick, 1994; Weber, 1998; Barton, 2000) and, in particular, how these technologies can act as catalysts for more active engagement in learning and, consequently, greater conceptual understanding (Barton, 2000). Other proponents (e.g. Asp, Dowsey, & Stacey, 1993; Templer, Klug, & Gould, 1998) have argued that these technologies can allow students the freedom to explore new ideas and concepts.
This paper outlines a framework that describes students' action and interaction with technology from a socio-cultural perspective and identifies META as an element, 183 within this framework, that plays a role beyond that of a simple tool. In doing so, this paper addresses the theme of Learning and Assessing Mathematics With and Through Digital Technologies. This theme will be approached through the perspectives of Theoretical Frameworks and Contribution to Learning Mathematics.

Theoretical Framework

Sociocultural perspectives on learning emphasise the socially and culturally situated nature of mathematical activity, and view learning as a collective process of enculturation into the practices of mathematical communities. A central claim of sociocultural theory is that human action is mediated by cultural tools, and is fundamentally transformed in the process (Wertsch, 1985). The rapid development of computer and graphical calculator technology provides numerous examples of how such tools transform mathematical tasks and their cognitive requirements. From a sociocultural perspective, technology can be regarded as a cultural tool - sign systems or material artefacts - that not only amplify, but also reorganise, cognitive processes through their integration into the social and discursive practices of a knowledge community (Resnick, Pontecorvo and Säljö, 1997). Amplification takes place when a tool provides a more efficient procedure or pathway for engagement in a task, for example, the use of a calculator or spreadsheet to deal with a series of tedious numeric calculations. Cognitive reorganisation, on the other hand, occurs when the use of technological tools mediates a qualitative change in an individual's way of thinking about a mathematical idea or concept, or their approach to a problem solving task. This type of transformation is evident when students are encouraged to develop the capacity to take a multiple representational approach to solving nonroutine problems. The freedom to assign equal privilege to different problem solving approaches provides opportunity to break free of the straightjacket of traditional algebraic reasoning and represents a completely different way of thinking about how to initially engage with a problem solving task and then how to progress after this engagement.
While the approach taken here is essentially Vygotskian, Galbraith, Goos, Renshaw and Geiger (2001) have argued previously that the widely known definition of Vygotsky's Zone of Proximal Development (as the distance between what a child can achieve alone and what can be achieved with the assistance of a more advanced partner or mentor) can be extended to conceptualisation of the ZPD in egalitarian partnerships and by the way the ZPD concept creates a challenge of participating in a classroom constituted as a community of practice. The first extension suggests that peer groups of equal expertise can promote new learning via contributions from individuals with incomplete, though relatively equal, expertise that sum to something greater than their individual parts, and so through interaction collectively progress knowledge and understanding. This is different from the purely Vygotskian view in which productive learning arrangements require that at least one individual in a group possesses greater expertise in an area of learning endeavour. The second extension argues that through the establishment of a small number of repeated participation 184 frameworks such as teacher-led lessons, peer tutoring, and individual and shared problem solving, students are challenged to move beyond their established competencies and adopt the language patterns, modes of inquiry, and values of the discipline. Such a classroom environment, representative of an active community of learners, is augmented by the availability of technology as a means to amplify and reorganise ways of communicating within the community; for example, by allowing students to contribute to collective discussions either as private individuals (via a computer screen) or publicly (via a display available to all participants). An important observation of this study was that students who were less prone to contribute to more conventional classroom discussion did so readily through electronic media.
This paper will argue that while technology plays a role as a cultural tool, as outlined above, it can, in the minds of students, assume a more active and interactive role in the process of cognitive reorganisation - that of an almost peer with expertise that can be drawn upon in the same way as other members of a learning community.

A Framework for Analysing Students' use of Technology

There are a number of studies that have sought to develop taxonomies of student behaviour in relation to the use of technology while learning mathematics. Doerr and Zangor (2000), for example, in a case study of pre-calculus classrooms identified five modes of graphics calculator use: computational tool, transformational tool, data collection and analysis tool, visualisation tool, and checking tool. Alternatively, Guin and Trouche (1999) developed profiles of behaviour in relation to students' use of graphing calculator technologies. The modalities outlined in the profiles were characterised by random, mechanical, rational, resourceful, or theoretical behaviours in terms of their ability to interpret and coordinate calculator results.
It is from the perspective of learning as a sociocultural experience, however, that Galbraith, Goos, Renshaw and Geiger (2001) have developed four metaphors for the way in which technology can mediate learning. These metaphors, technology as master, technology as servant, technology as partner, and technology as extension of self, describe the varying degrees of sophistication with which students and teachers work with technology. While these metaphors are hierarchical in the sense of the increasing level of complexity of technology usage teachers and students may attain, it does not represent a developmental progression where once an individual has shown they can work at a higher level they will do so on all tasks. Rather, the demonstration of more sophisticated usage indicates the expansion of a technological repertoire where an individual has a wider range of modes of operation available to engage with a specific task. This means, for example, that a very capable individual may well use technology as a servant if the task at hand is mundane and there is no reason to invoke higher levels of operation.
A description of these metaphors is outlined in below:
Technology as Master. The student is subservient to the technology - a relationship induced by technological or mathematical dependence. If the complexity of usage is high, student activity will be confined to those limited operations over which they 185 have competence. If mathematical understanding is absent, the student is reduced to blind consumption of whatever output is generated, irrespective of its accuracy or worth.
Technology as Servant. Here technology is used as a reliable timesaving replacement for mental, or pen and paper computations. The tasks of the mathematics classroom remain essentially the same - but now they are facilitated by a fast mechanical aid. The user 'instructs' the technology as an obedient but 'dumb' assistant in which s/he has confidence.
Technology as Partner. Here rapport has developed between the user and the technology, which is used creatively to increase the power that students have over their learning. Students often appear to interact directly with the technology (e.g. graphical calculator), treating it almost as a human partner that responds to their commands - for example, with error messages that demand investigation. The calculator acts as a surrogate partner as students verbalise their thinking in the process of locating and correcting such errors. Calculator or computer output also provides a stimulus for peer discussion as students cluster together to compare their screens, often holding up graphical calculators side by side or passing them back and forth to neighbours to emphasise a point or compare their working.
Technology as an Extension of Self. This is the highest level of functioning, where users incorporate technological expertise as an integral part of their mathematical repertoire. The partnership between student and technology merges to a single identity, so that rather than existing as a third party technology is used to support mathematical argumentation as naturally as intellectual resources. Students working together may initiate and incorporate a variety of technological resources in the pursuit of the solution to a mathematical problem.

The Study

The research reported here describes one aspect of a three year longitudinal study although the data analysed in this paper are sourced from a single mathematics classroom over a two year period (Years 11 and 12, the final two years of secondary schooling; students are aged 16-17 years.). The author was also the teacher of this class. The students were studying a challenging mathematics subject designed for students intending to pursue serious study of mathematics at a tertiary level. The intended culture of this classroom is one consistent with the sociocultural perspective of learning and teaching (see Goos, Galbraith, & Renshaw, 1999) including the acceptance of emergent uses of technology. This means a variety of interactions that involve mutuality are encouraged, including: student/student interaction; student/teacher interaction; subgroup and whole class investigation and discussion of specific tasks or of a variety of projects simultaneously. Interactions between participants and artefacts such as texts and more importantly electronic technologies also characterise the way students explore and investigate new mathematical ideas and concepts.

Data Sources 186

On average a lesson was videotaped every one to two weeks, or more frequently if a technology intensive approach to a topic was planned. Audiotaped interviews with individuals and groups of students were conducted at regular intervals to examine factors such as the extent to which technology was contributing to the students' understanding of mathematics, and how technology was changing the teacher's role

in the classroom. At the beginning of the course and at the end of each year students completed a questionnaire on their attitudes towards technology, its role in learning mathematics, and its perceived impact on the life of the classroom. A final class interview/discussion reviewing the two-year program was videotaped. This paper presents two vignettes, drawn from observation and transcript data, to illustrate the role technology can play as a Partner in the generation and repair of new knowledge and understanding.
Vignette 1
Students (Year 12) were asked to develop programs for their calculators that found the angle between two three dimensional vectors as an application of the scalar product of vectors and as a means of validating results found from pen and paper techniques. The teacher provided only minimal instruction in basic programming techniques, and expected individual students to consult peers, who had varying degrees of knowledge, for assistance. Volunteers then demonstrated their programs via the calculator viewscreen, and examined the wide variation in command lines that peers had produced. This public inspection of student work also revealed programming errors that were subsequently corrected by other members of the class. For example, the class disputed the answer obtained by executing the program shown in the first part of Figure 1.
Fig. 1. Correcting errors in a student program
His actions guided by fellow students, the presenter scrolled down through the program and replaced the plus sign in the denominator with a multiplication sign (Figure 1, second screen). The amended program again produced an incorrect answer, and yet another correction (Figure 1, third screen) – suggested by students, not the teacher – was required before the correct output was obtained.
Vignette 2
In this episode we observed how one student consistently rejected the teacher's invitations to discuss his thinking with peers, participate in whole class discussions, and generally take some responsibility for advancing his mathematical understanding. This situation began to change when the student participated in the activity described in Vignette 1. The student presented the program which included the initial screen illustrated in Figure 2 and then the second and third last screens (Figure 2, second and third screens). 187
Fig. 2. Presentation of the Dodge program
The choice of option 1, in response to the question posed in screen 3, resulted in the display of the answer to the calculation the program was designed to deliver (screen 4). The choice of option 2 resulted in no answer being provided (screen 5) in the manner of a taunt.
Now while the student had used the task to demonstrate dissent in relation to his current experience of mathematics, his clever use of the very method of discourse the teacher has been encouraging the student to use persuaded the teacher not to issue a reprimand of any type. The student responded, over subsequent lessons, by increased involvement in classroom presentation whenever technology was used to mediate discussion. This included the presentation of improved, and increasingly sophisticated, versions of his initial program. This was followed by an animated program he had created that depicted the adventures of mathematical objects (various irrational numbers) as human-like characters – Dodge: The Movie. The enthusiastic and admiring response to his "movie" (and the sequel – Dodge II: The Revenge of Dodge) was significant in drawing this student into the kind of mathematical discussion he had previously resisted, and he became a willing participant in subsequent discussions both technology-focused and otherwise.
Discussion
Technology is often viewed as a neutral tool useful for the illustration of mathematical ideas and concepts but with little potential for mediating interaction. Doerr and Zangor (2000), for example, found that the use of the graphics calculator as a private device led to the breakdown of small group interactions. The two vignettes above demonstrate the potential of technology, including associated presentation tools, for drawing in students who are initially reluctant to engage in, or in some cases resist, the social and cultural norms of a community of learners. In the first vignette, technology supported the interaction of peers of roughly equal expertise in repairing a faulty calculator program. The technology not only provided the medium in which the students worked but it also stood alone to make public a particular student's work; holding it up for scrutiny and providing the opportunity for supportive critique. In this case, technology has assumed the role of almost equal partner in the interaction. It has offered to the group a skill or expertise that they lacked to get the job done. Once injected into the interaction it allowed the group to progress the development of an individual who had encountered difficulty in progressing by himself. The significant contribution technology has made in this 188 episode is more than that of a simple tool; it is an instance where the boundaries between tools and human learners is blurred.
The second vignette features a student who uses the method of discourse he had previously resisted to register dissent in relation to the way mathematics classes were conducted in this course. Having received positive reinforcement from his peers (and no negative feedback from the teacher) he is slowly drawn into the ways of interacting with his learning community that he has previously shunned; initially when technology was involved and then, eventually, at other times. Technology, again, has acted as more than a simple tool. Firstly, it acted as a partner that assisted him to express the personal frustration that lay in a conflict between his view of how to learn and do mathematics and the social and cultural norms for doing the same in this particular classroom. Technology had acted as a partner in crime in this instance. Secondly, however, technology has almost acted as a supportive go-between that has encouraged him to move from the fringes of his learning community into the mainstream.
Implications for Learning and Teaching
The conceptualization of technology as a quasi-peer offers the following insights into the process of integrating META into mathematics classroom.
Technology can be regarded as more than a passive cultural tool to be appropriated by teachers and students to enhance mathematics learning and teaching. Rather, META can make contributions to social and cultural activity beyond that of merely mediating interaction. The vignettes presented above highlight instances where technology has played a far more important role that that of a simple presentation tool that assists in mediating discussion and interaction. In these instances, technology has almost taken on its own persona and has offered contributions to these learning episodes that students, by themselves, would not have been able to replicate. This is emphasized, particularly in the second vignette, where a student is empowered by a technological partner to voice a controversial view via a method of expression sanctioned by the learning community from which he had excluded himself. Further, he was then led by the same technological partner back into a human community from which he had previously dissociated himself.
The notion that META can be regarded as quasi-peer within a community of practice extends Vygotsky's notion of a ZPD to include technology as contributing member to a group of learners rather than merely a cultural tool. This implies that META should be afforded even greater attention in terms of its pedagogical power than has been previously assigned and so has implications for both teacher pre-service instruction and for in-service professional development.
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Understanding technology integration in secondary mathematics:
Theorising the role of the teacher
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Previous research on computers and graphics calculators in mathematics education has examined effects on curriculum content and students' mathematical achievement and attitudes while less attention has been given to the relationship between technology use and issues of pedagogy, in particular the impact on teachers' professional learning in specific classroom and school environments. This observation is critical in the current context of educational policy making, where it is assumed – often incorrectly – that supplying schools with hardware and software will increase teachers' use of technology and encourage more innovative teaching approaches. This paper reports on a research program that aimed to develop better understanding of how and under what conditions Australian secondary school mathematics teachers learn to effectively integrate technology into their practice. The research adapted Valsiner's concepts of the Zone of Proximal Development, Zone of Free Movement and Zone of Promoted Action to devise a theoretical framework for analysing relationships between factors influencing teachers' use of technology in mathematics classrooms. This paper illustrates how the framework may be used by analysing case studies of a novice teacher and an experienced teacher in different school settings.
Mathematics, science and technology education in Australia are currently experiencing major impetus for innovation and reform. The Australian Government's policy statements on educational innovation and teacher quality (Commonwealth of Australia 2003) emphasise that Australia's future lies in its potential as a knowledgebased society built on the intellectual capabilities and creativity of its people. Teachers and students are expected to become partners in a learning society underpinned by science and mathematics and successful schools are portrayed as those drawing on the resources of technology to facilitate learning. Throughout Australia there are moves to encourage – and in some cases mandate – the integration of digital technologies into school education through curriculum initiatives, funding for infrastructure, and the development of professional standards for teachers. In the current context of educational policy making it seems to be assumed that supplying schools with hardware and software will increase teachers' use of technology and encourage more innovative teaching approaches that produce improved learning outcomes for students. Yet internationally there is research evidence that that improving teachers' access to educational technologies has not, in general, led to increased use or to movement towards more learner-centred teaching practices (Cuban, Kirkpatrick & Peck, 2001; Wallace, 2004). 191
Windschitl & Sahl (2002) have identified two factors that appear to be crucial to the ways in which teachers might embrace, ignore, or resist technology. First, teachers' use of technology is mediated by their beliefs about learners, about what counts as good teaching in their institutional culture, and about the role of technology in learning. Second, school structures – especially those related to the organisation of time and resources – often make it difficult for teachers to adopt technology-related innovations. Clearly, there is a need to interrogate assumptions about relationships between access to technology and its use by teachers. This paper does so by offering a framework for theorising interactions between pedagogical knowledge and beliefs, school structures and other institutional constraints, and professional learning opportunities, together with analyses of examples of teacher learning and development drawn from a series of sociocutturally oriented research studies carried out in Australian schools. The paper addresses the ICMI Study 17 theme of Teachers and teaching by considering the role of the teacher viewed through the lens of this theoretical framework.
Theoretical Framework
Early research in this area examined the effects of technology use on students' mathematical achievements and attitudes and their understanding of mathematical concepts, often using quasi-experimental designs that compared technology and nontechnology users (Penglase & Arnold, 1996). However these studies did not distinguish between the use of technology and the context of that use, and little attention was given to issues of pedagogy and the nature of teachers' professional learning within and beyond the school environment (Windschitl & Sahl, 2002). To address some of these issues my colleagues and I have carried out studies informed by sociocultural theories of learning involving teachers and students in Australian secondary school mathematics classrooms (e.g., Galbraith & Goos, 2003; Goos,

2005). Sociocultural theories view learning as the product of interactions with other people and with material and representational tools offered by the learning environment. Because it acknowledges the complex, dynamic and contextualized nature of learning in social situations, this perspective can offer rich insights into conditions affecting innovative use of technology in school mathematics.
In this research program Valsiner's (1997) zone theory, originally designed as an explanatory structure in the field of child development, was adapted to apply to interactions between teachers, students, technology, and the teaching-learning environment. This framework extends Vygotsky's concept of the Zone of Proximal Development (ZPD) - often defined as the gap between a learner's present capabilities and the higher level of performance that could be achieved with appropriate assistance - to incorporate the social setting and the goals and actions of participants. Valsiner describes two additional zones: the Zone of Free Movement (ZFM) and Zone of Promoted Action (ZPA). The ZFM structures an individual's access to different areas of the environment, the availability of different objects within an accessible area, and the ways the individual is permitted or enabled to act 192 with accessible objects in accessible areas. The ZPA represents the efforts of a more experienced or knowledgeable person to promote the development of new skills. For learning to be possible the ZPA must be consistent with the individual's potential (ZPD) and must promote actions that are feasible within a given ZFM. When we consider teachers' professional learning, the ZFM can be interpreted as constraints within the school environment, such as students (their behaviour, motivation, perceived abilities), access to resources and teaching materials, and curriculum and assessment requirements, while the ZPA represents opportunities to learn from preservice teacher education, colleagues in the school setting, and professional development.
Previous research on technology use by mathematics teachers has identified a range of factors influencing uptake and implementation. These include: skill and previous experience in using technology; time and opportunities to learn (pre-service education, professional development); access to hardware and software; availability of appropriate teaching materials; technical support; institutional culture; knowledge of how to integrate technology into mathematics teaching; and beliefs about mathematics and how it is learned (Fine & Fleener, 1994; Manoucherhri, 1999; Simonsen & Dick, 1997; Walen, Williams & Garner, 2003). In terms of the theoretical framework outlined above, these different types of knowledge and experience represent elements of a teacher's ZPD, ZFM and ZPA, as shown in Table 1. However, in simply listing these factors, previous research has not necessarily considered possible relationships between the teacher's setting, actions, and beliefs, and how these might change over time or across school contexts. Zone theory provides a framework for analysing these dynamic relationships.
Table 1. Factors affecting technology usage
Valsiner's Zones Elements of the Zones
Zone of Proximal
Development
Skill/experience in working with technology
Pedagogical knowledge (technology integration)
General pedagogical beliefs
Zone of Free
Movement
Access to hardware, software, teaching materials
Support from colleagues (including technical support)
Curriculum & assessment requirements
Students (perceived abilities, motivation, behaviour)
Zone of Promoted
Action
Pre-service education (university program)
Practicum and beginning teaching experience
Professional development
Background to the Research Program
The research program referred to above has used Valsiner's (1997) zone theory to investigate relationships between factors influencing how novice and experienced teachers use technology in the mathematics classroom. Examples from two separate studies are analysed later in the paper. A brief outline of the research design and methods for each study is provided below. 193
The first study, conducted in 2001, aimed to analyse processes through which mathematics teachers learned to use technology as an educational resource (Galbraith & Goos, 2003). Participants were a group of ten experienced teachers who volunteered for a training program, conducted intensively over a single week-end, that prepared them to deliver professional development workshops on the use of graphics calculators. These sessions engaged participants as learners in technologyrich activities that could be used in secondary school classrooms, and in discussion of associated teaching and learning issues. We followed the progress of three teachers who subsequently delivered professional development workshops at conferences or in their own schools, and interviewed them on how their views about technology had been affected by the training program.
The second study followed successive cohorts of pre-service teachers into their first years of teaching from 2000-2004. Its main aims were to identify factors that influence how beginning teachers graduating from a technology rich pre-service course integrate computers and graphics calculators into their mathematics teaching practice (Goos, 2005). One element of the research design involved individual case studies that captured developmental snapshots of experience during the final practice teaching session and towards the end of the first year of full-time teaching. Participants were visited in their schools for lesson observations, collection of teaching materials and audio taped interviews.
Case Study of a Novice Teacher Learning to Integrate Technology
Sandra was one of the pre-service participants in the second study selected for individual case study. Her practicum placement was in a large school in the State capital city. At this time the mathematics syllabuses merely encouraged teachers to use computers and graphics calculators, although new syllabuses to be introduced the following year would make technology use mandatory. The school was well equipped with computer laboratories and had recently purchased its first class set of graphics calculators. However, none of the teachers had yet found time to learn how to use the calculators. Sandra was very familiar with computer applications such as Excel and regularly searched the internet for teaching ideas and resources. She used both these technology resources in her mathematics teaching during her practice teaching sessions, although she had not observed other teachers in the school use any kind of technology with their classes. Before starting the pre-service course Sandra had no experience with graphics calculators but she was now keen to explore the possibilities this technology might offer for developing students' understanding of mathematical concepts.
Sandra was teaching linear programming, a topic that deals with the kind of optimisation problems commonly encountered in engineering and economics. As graphical methods are usually used to solve linear programming problems in secondary school treatments of this topic, Sandra decided this presented an ideal opportunity for students to use the graphics calculators instead of drawing graphs by hand. She adapted an activity from the internet that asked students to work out the 194 optimal quantities to be produced of two different kinds of pasta, using three different varieties of cheeses, so as to ensure maximum profit for the manufacturer. Because the students had never used graphics calculators before, she also devised a worksheet with keystroke instructions and encouraged students to work and help each other in groups.
Unexpectedly, Sandra encountered strong resistance from the students, which seemed to stem from their previous experiences of mathematics lessons. Other mathematics teachers in the school tended to take a very transmissive approach and focused on covering the content in preparation for pen and paper tests, so the students were not interested in learning how to use technology if this would be disallowed in assessment situations. According to Sandra, the students' attitudes could be summed up as: "Just give me enough to pass ... I don't want to know how to do group work, I don't want to know how to use technology".
In theoretical terms, the Zone of Promoted Action offered by the teachers in the school was not a good match with the ZPD defined by Sandra's pedagogical beliefs and her knowledge and skills in using technology to teach mathematics. Neither did her supervising teacher's ZPA provide a pedagogical model consistent with the technology emphasis of the pre-service course. Some elements of Sandra's Zone of Free Movement, such as her easy access to calculators that no other teacher knew how to use, presented favourable opportunities to use technology. However, most other aspects of her ZFM - students' attitudes and lack of motivation, curriculum and assessment requirements that excluded technology - represented constraints. Yet Sandra was not discouraged by this experience and remained committed to enacting her pedagogical beliefs about using technology.
After graduation Sandra moved from the city to a smaller rural school that was much better resourced with respect to graphics calculators but lacking in experienced teachers who knew how to use them effectively. All Grade 11 and 12 mathematics students had continuous personal access to graphics calculators via a hiring scheme operated by the school, and there were two additional class sets available for teachers to use with other classes - although Sandra was the only teacher to use these with younger students. She was also beginning to use temperature probes and motion detectors which could be used in conjunction with graphics calculators to collect and analyse data from experiments.
Compared with her practicum experience, Sandra's first year of teaching offered a more expansive Zone of Free Movement: motivated and cooperative students, good access to technology resources, and new syllabuses that mandated use of computers and graphics calculators in Grades 11 and 12. Yet there was no Zone of Promoted Action within her school environment, and geographical isolation, compounded by a very slow internet connection, made it difficult for her to access professional development and teaching materials (an external ZPA). While she was still able to draw on the knowledge gained during her university program (the pre-service ZPA), Sandra recognised her need to gain new ideas via collaboration with other more 195 experienced teachers beyond the school in order to further develop her identity as a teacher for whom technology was an important pedagogical resource.
Case Study of an Experienced Teacher Learning to Integrate Technology
Teachers who completed their pre-service education before computers and graphics calculators were introduced into school classrooms rely on formal or informal professional development to learn how to use technology. By comparison with Sandra, Lisa was a very experienced teacher but a relative novice in the use of technology when she participated in the research study associated with the graphics calculator training program described earlier. When reflecting on her initial professional development experiences in this field, she commented that she "got lost in the first ten seconds, and was really turned off so didn't touch them again for a while". After several more workshops she felt confident enough to use graphics calculators in her teaching, "but not confidently and not proficiently. Not really realising how much they improved the thinking, more just as a tool to do graphs and things".
The training program proved to be a turning point for Lisa as it emphasised the impact of technology in developing students' understanding of mathematical concepts and in facilitating classroom discussion, something that had been missing from her previous professional learning experiences:
It was out of that week-end that I really understood the impact that [graphics calculators] had on the pedagogy. Up to then I saw it as a tool to draw graphs and analyse statistics. But at that workshop, just one little thing from that workshop, how we were working in groups, and they explained to us how kids start trying to help. So when we were doing that we were grabbing somebody else's calculator and sharing our data, so it made the group work thing a whole lot better. And I really valued the part where we, as groups, we went out and used the overhead projector and we presented our information back to the group. So I just, I really started to see different ways of using it that I hadn't thought of before. So it really enhanced group work, it really showed me that you could do a lot more hands on stuff, the practical activity with the motion detectors. That graphics calculators are good for inspiring all those other good things in teaching, like the hands on, the group work, and really starting to think when we were fitting functions to the data. Really having to think and understand what the intercept and the gradient mean. We weren't just doing, we were really understanding at a higher level. I found that really powerful. Because I had thought that all they do is save you that boring part of maths.
Environmental constraints and affordances (ZFM) seemed to play little part in Lisa's learning, possibly because as Head of her school's Mathematics Department she had considerable autonomy in obtaining desired resources and in managing curriculum and assessment programs. Instead, the re-construction of her identity as a teacher can be understood in terms of the changing relationship between her goals and interests 196 (ZPD) and the ZPAs offered by the professional development and training she experienced. She described previous workshops she had attended as "off-putting", because the emphasis was on procedural aspects of operating the calculators and the mathematics presented was too difficult for participants to engage meaningfully with the technology. She contrasted this with the approach taken in the week-end workshops offered as part of this research project:
I didn't really feel super confident until I went to the workshop. And I think it was then, understanding the bigger concepts, rather than just pushing buttons. Because at the pushing buttons level you never really understand how they operate. And after that I was just so inspired. It was just that whole valuing and that sharing and learning from each other, and just to realise that other people are out there. So that was really the turning point for me to say that this is really exciting stuff.
Lisa seemed to find a professional development ZPA that matched her need to focus on pedagogical, rather than procedural, aspects of using technology, and acknowledged the potential for experienced teachers to learn from each other.
Discussion
This paper has analysed relationships between mathematics teachers' access to technology resources and the ways in which they incorporate these resources into their pedagogical practices. Evidence from research studies carried out in Australian classrooms suggests that simple notions of "access" and "use" are inadequate for understanding the roles that technology plays in mathematics teaching and learning. The case studies of Sandra and Lisa showed that teachers interpret access to technology in relation to what they believe is beneficial for students and feasible in

the light of their own expertise and institutional context.
Teachers' learning can be conceptualised in terms of relationships between Valsiner's (1997) Zones of Proximal Development, Free Movement and Promoted Action, and this provides a useful way of analysing the extent to which teachers adopt innovative practices involving technology. The ZFM can be interpreted as teachers' institutional context, the ZPA represents their experiences in learning about teaching with technology, and the ZPD is influenced by their knowledge of how to integrate technology into their teaching and their pedagogical beliefs. The case study of Lisa illuminated issues facing experienced teachers who are unfamiliar with new technologies such as graphics calculators. While her ZFM presented few constraints, she had to search for professional development (ZPA) that would extend, rather than only accommodate, her existing ideas about teaching with technology (her ZPD). On the other hand, novice teachers like Sandra who are knowledgeable and enthusiastic about using technology may encounter obstacles in their professional environment (ZFM) that hinder implementation of preferred teaching approaches. Thus the theoretical approach outlined in this paper provides a way of interpreting teachers' actions in mathematics classrooms and may generate informed discussion about 197 conditions that support or inhibit teachers' learning and adoption of new technologies.
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The Unrealized Potential of the Internet
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The role of the Internet in teaching and research has been given too little attention in mathematics education, particularly in developing countries and underserved segments of the population of developed countries. Those with inadequate Internet connectivity lack access to the research of others and experience difficulty in achieving recognition for their own work. Exchange of ideas in their formative stage as well as the distribution of completed writing is essential for full participation in the research community. Similarly, use of the Internet to share experience and innovation in teaching and to train teachers in-service and pre-service is a costeffective means of instituting widespread improvements, particularly with respect to increasing access for groups such as girls, adult learners, rural or disadvantaged populations, and the learning disabled. For the learners themselves, the ability to acquire information via the Internet can transform their educational experience. Although there is concern about the hegemony of developed nations in the Internet environment, the solution is inclusiveness, not isolation, as well as sharing within domestic culture. Although investment will be required, new technology and focus on community-based access will reduce the costs of providing adequate communications infrastructure.
In spite of the focus on the use of technology in mathematics education, there has been insufficient attention to the importance of the role of the Internet in teaching and research, particularly with respect to access, equity, and socio-cultural issues. In developing countries with limited resources, to the extent that Internet connections exist, they are generally confined to entrepreneurial enterprises, from local Internet cafes to outposts of multinational corporations.
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Although the Internet first arose in a largely academic context, for the most part it is only in highly developed countries that university faculty and students, much less teachers and students in primary and secondary schools, have extensive access.
Computers are not as rare in developing countries as one might think and even the Internet has experienced rapid superficial growth, far surpassing already Bill Gates' 1997 prediction of 500 million users by 2007. However, educational use of the Internet is very limited outside of developed countries.

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This observation is based in part on the author's extensive experience in the Middle East, including pre- and post-invasion Iraq, and to a lesser extent in Asia and Africa. 199
Extensive surveys have been conducted of the dispersion of the Internet in developing countries, including a number using a six part paradigm: pervasiveness, geographic dispersion, connectivity infrastructure, organizational infrastructure, sectoral absorption, and sophistication of use (Wolcott et al, 2000). Going beyond such simple metrics as numbers of hosts, the researchers measured dispersion of points of presence or toll-free access, domestic and international backbone width, collaborative arrangements and public exchanges, usage rather that just access, and whether the usage is conventional or innovative. By all measures, developing countries have a long way to go. Although they have not focused in much depth on the education sector, the general finding has been that while most universities have some Internet access, a single terminal may serve as many as a thousand students, and while secondary schools may have computers, they seldom have Internet access. As for primary schools, use of computers either for educational or administrative purposes is rare in most developing countries. However, business use, including commercial cybercafes, has expanded greatly, showing the potential for broader application of the technology.
This presents an obvious equity issue as between countries, but also means that there are inequities within the countries themselves--it is the urban, prosperous who acquire the information and skills available via the Internet. We discuss the existing situation and propose remedies.
Research
The obvious disadvantage to those having limited access to the Internet is that they experience difficulty in keeping up with the research of others and in making known their own contributions. Universities and research institutes in developing countries have few print journals available in mathematics or mathematics education (or any other field); whereas a typical US medical school library may subscribe to around 5000 journals, the best university medical school library in many developing countries may have no more than 20, if indeed it has any at all. Textbooks providing up-to-date information are in similarly short supply. There are programs, supplemented by pleas for assistance from particular institutions, providing such journals, but they are not only inadequate, but misguided. Internal and outside resources currently committed to these projects should be redirected to securing online access through JSTOR and other services and databases. The reasons to prefer online access--in addition to the fact that the cost in the long run is likely to be less--include the fact that the electronic journals can be made simultaneously available to many users, storage is not a problem, and searching for particular topics is much easier. Moreover, increasingly journals are published directly online. 200
Not only do those with inadequate Internet connectivity lack access to the research of others, but also they may have difficulty making their own work known. Exchange of ideas in their formative stage as well as the distribution of completed writing is essential to full participation in the research community. In addition, notices of conferences and other opportunities for collaboration come to most researchers via the Internet. Increasing amounts of information about scientific and technological developments are now available only on the Internet. Use of the Internet can improve resource mobilization and make it possible to carry on collaborative research among distant sites.
Teaching
The usefulness of computers in teaching has been well recognized, but too little emphasis has been placed on their enhanced value if there is Internet connectivity. The amount of educational material in all fields readily available free on the Web is huge and ever growing. True, discretion is required in deciding what is worthwhile and what is not, but that there is selectivity needed is not a reason for choosing not to avail oneself of the riches waiting to be discovered.
There is justifiable fear about the hegemony of American and European culture on the Internet and complaints about the necessity of knowing English to acquire much of the information found there. However, one should consider the benefits of having a language that also enables communication among developing as well as highly developed countries throughout the world in order to share knowledge and experiences. Moreover, the Internet can be used to preserve and nurture one's own language and culture through domestic exchanges online.
The benefits of using Internet resources in teaching are extensive and varied. There are voluminous lists of web sites providing, just as an example, up to date maps and statistics to which schools would otherwise be unlikely to have access. No longer must students hunt for stories and pictures of women or minority mathematicians nor must teachers seek in inadequate libraries for the story of the development of the concept of zero. Hieroglyphics, Mayan glyphs, Babylonian cuneiform--they are all there. The history of mathematics--and not just from a European perspective can be vividly incorporated into the teaching of mathematic via this application of technology. The audience can be widened to provide more equity in learning, but also in subject matter learned.
Interesting and relevant applications of mathematics and statistics, tutorial help, and innovative and effective teaching techniques can be found on the Web. In many developing countries resources are not available for in-service training of widely dispersed teachers. Topics such as dealing with learning disabilities are often 201 neglected in pre-service training as well. The ability of teachers to share their own ideas with colleagues in their own culture is another important benefit of Internet access.
Equity
The digital divide can be within a country as well as between countries, but this need not be. The Internet properly used has great potential for reducing this divide, for bringing the information age--and with it mathematics education--to rural areas, to girls and women, and to other underserved populations.
Distance learning applications of the Internet have great potential, especially to reach rural areas and to maximize use of scarce teaching resources. The development of adequate material can be capital intensive, but sharing from country to country as well as within a country can help. Convincing those in charge of the return on initial investment is key to establishing distance learning in terms of both hardware and teaching material.
It should also be understood that "distance" learning need not be over long distances. Particularly for part-time learners who must continue in full-time employment, the ability to get specialized training at convenient urban locations can be crucial to economic and social development.
Girls by no means have equal access to education in many developing countries. For example, in sub-Sahara Africa only six of ten girls attend primary school (compared to eight of ten boys) with the situation becoming even more disparate beyond primary school (LaFraniere, 2005). Long distances, lack of sanitary facilities, and sexual harassment problems can be overcome through distance learning. Setting up Internet access points, particularly in rural areas, can transform girls' prospects for education; the locations can also be used for adult reading and quantitative literacy programs, especially for women with small children. Such local access centers could also be used for teacher training.
But it is not only in developing countries that inequities need to be addressed and where investment in Internet access for the schools could be instrumental in doing so. Although public primary school students in some areas in developed countries may be designing their own web sites, there are others who have no access at all to the benefits of the Internet.
What is needed? 202
Distance learning has great potential, especially to reach rural areas and to maximize use of scarce teaching resources. The development of adequate material can be capital intensive, but sharing from country to country as well as within a country can help. Convincing those in charge of education that the return on initial investment can be massive is key to establishing distance learning support of both

hardware and teaching materials.
The correlation between the number of Internet hosts and the UNDP Human Development Index is high, which suggests that substantial investment will be needed to increase Internet access to the point where it can play an important role in teaching and research. Rather than focusing on individual access, the goal should be more socially beneficial community-based access.
Studies have shown that the major handicap in the broad use of the Internet is deficient telecommunications infrastructure. However, the answer is to leapfrog over, for example, the lack of landline connections. In India a nationwide cellular network was installed without an inch of copper wire, which might later be cut or stolen in any case, at a cost less than one-third of a landline installation. More generally, while there are always problems in the introduction of innovative technology, that the population in developing countries is relatively young is a big advantage.
Instructive is the burgeoning of Internet usage in China (in spite of the restrictive government policies that hamper the openness that should be part of Internet use). Chinese universities decided some years ago to make Internet connectivity a priority, with economic reforms providing the capital for the investment needed; the primary and secondary sectors have not, however, seen similar progress. Universities in developed countries have also made Internet access an important goal, although, at least in the United States, the access is uneven, depending on the institutions' resources. Essential to the Chinese experience was a prior decision to invest in telecommunications infrastructure and human resources.
In discussing Internet-inspired development, Shirin Madon (2000) asserts:
"The establishment of a strategic infrastructure is considered critical for developing countries where the marginal impact of improved network communications can be very high, leading to improved economic productivity, governance, education and quality of life, particularly in rural areas."
However, skepticism regarding the potential of technology includes a fear of increasing dependency on international resources in the form of financing or technical skill. Thus an important component of technological development must be 203
the training of the domestic workforce. In fact the Internet itself can play an important role in this preparation. For example, the Village Internet Programme of the Grameem Bank (Madon, 2000) helped to create technology-related jobs for rural poor and Cuba's school networking project (Press, 1998) stressed grassroots participation of schools in rural areas. In the field of mathematics education, even in developed countries, rural areas or depressed urban areas may have difficulty securing access to qualified teachers and material. Educational authorities must be encouraged to see investment in informational communication as a way to help relieve inequities. Too often education budgets are a source of cuts when savings need to be made in national budgets.
Governments need to understand their role in creating and disseminating knowledge. Because of the worry of the dominance of developed countries in the Internet environment, there is likely to be resistance to the notion of substantial investment. Therefore, "do it yourself" is an important concept for the development of information technology. To limit external dependency, human resource development for network users means concentration on training in data handling, software, monitoring, and management, not just technical matters.
Often nations are willing to commit funds for technology for commercial development purposes, but it makes no sense to invest in information superhighways while cutting down on the prerequisite, solid and adequate education for all. They must be ready to exploit the potential of the Internet for this purpose.
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Learning mathematics in class with online resources
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Internet resources proposing mathematics exercises with an associated environment are frequently used in class, in many countries and for all levels. I address here the question of the theoretical approaches that can be used to describe and understand the way students work and learn mathematics with these resources. I study in particular the possibilities offered by the instrumental approach, complemented by the notion of didactical contract. The resources considered are complex. In order to study their use by students, is thus necessary to consider two activity levels: the resolution of one exercise, where students use the elements of the exercise's environment (feed-back, hint etc.) and the level of a whole session, where students develop working patterns. I describe on examples extracted from various teaching designs how the instrumental approach, and the notion of didactical contract, can help on each level to interpret the students' behavior with the resource. It permits to establish links between the behaviors and the mathematical knowledge involved, and to make a first step towards the consequences on the learning processes.
Introduction
The use of Internet resources in the teaching of mathematics is now widespread in many countries. The resources I consider here propose mathematics exercises with a given environment that can comprise for each exercise tools like a calculator; suggestions, or different kinds of advice; that can send a feed-back, or attribute a mark to the student (see Cazes Bal. (2005) for a grid describing the possible features). I term it an e-exercises resource in what follows. These resources can serve for distance learning, or be inserted in some collaborative learning settings. But they are also frequently used in class with a teacher, at least in France. These classroom uses raise many questions: why do teachers use such resources, that can seem poorer than, for example, microworlds? Which are the consequences of this use on the teaching and learning processes? Like Hoyles and Noss (2003) mention it, these questions have been little studied yet. In fact most of the works that consider internet resources in mathematics are very recent. Engelbrecht and Harding (2005) propose a very precise taxonomy of mathematics web sites, and discuss associated pedagogical issues. Their study is mainly directed towards the design of online courses and of settings using such courses. Bookman and Malone (2003) make precise observations about the way college students learn in an interactive environment (the connected curriculum project). They build a research agenda with three categories of questions: about the technology itself, about the role of the teacher, about the students behavior. But they do not discuss the possible associated theoretical frameworks. 206
In this paper I study the way students work and learn with e-exercises. That subject belongs to the second theme of the ICMI study : "Learning and assessing mathematics with and through digital technologies". I discuss more precisely the question of the theoretical approaches that can help to understand how the use of these resources can favor or on the opposite hinder learning. The instrumental approach is now currently used to study the consequences of the integration in mathematics classrooms of computer algebra systems. Can it also apply to Internet resources? Which phenomena can it illuminate, which other theories can complement it? I try here to contribute to answer to these questions, by analyzing examples stemming from various teaching designs and research projects.
In part two, I present the main principals of the Instrumental approach, and start discussing its potential applications to the use of e-exercises. I also briefly recall the notion of didactical contract, that I use to complement the instrumental approach. In part three, I present on some examples how the instrumental approach can be used to analyze the behavior of a student solving an exercise on the computer. Part four is dedicated to the subject of students paths and working patterns with an Internet resource.
Theoretical frames and e-exercises
Many theoretical frames permit to study mathematical learning in computerized environments. We retain here the instrumental approach, which is grounded in cognitive ergonomy. Rabardel (1995) stresses the difference between an artifact, which is just a given object, and an instrument. The instrument is a psychological construct; constituted of an artifact and of a psychological component defined through the notion of scheme. A scheme is considered here as an invariant organization of behavior for a given class of situations (Vergnaud, 1996). It has several components: goals, rules of action, of information and control, and operative invariants. These operative invariants are implicit knowledge, termed by Vergnaud theorems-in-action: propositions believed to be true by the subject.
Rabardel terms "utilization scheme" of an artifact a scheme organizing the activity with an artifact to realize a given task. He distinguishes between "usage schemes" corresponding to the management of the artifact, and "instrumented action schemes", directed towards the realization of the task. These schemes result from personal construction but also from appropriation of socially pre-existing schemes.
The instrument built by the subject comprises the artifact, and the schemes organizing the activity of the subject. The building process of the instrument is called "the instrumental genesis". Using an artifact influences the students' activity, and the activity influences the way the instrument is built. A detailed presentation of the instrumental approach can be found for example in Artigue (2002) and Trouche (2004). It has been mainly used to understand the impact on the learning processes of the introduction of computer algebra systems, especially through the identification of the schemes constructed by the students (see for example Lagrange 1999). It also 207
helps to understand unexpected uses of a given artifact: the subject constructs unexpected schemes, thus an unexpected instrument.
How can the instrumental approach be used to illuminate how students work and learn in a setting comprising such an e-exercises resource?
The possibilities offered by an e-exercises resource: access to exercises chosen according to their title or to their mathematical theme, help, feed-back etc. are easily understood by the students. For that reason, I will not refer here to "usage schemes", but only to "instrumented action schemes". The first step towards the description of such schemes is the identification of the possible tasks. An e-exercises resource can certainly be considered as an artifact. If it becomes an instrument, which kind of actions can be realized with that instrument? A computer algebra system, for example, can help to solve a mathematical problem. An e-exercises resource comprises tools that can help to solve a problem; but it also proposes problems, which makes a big difference.
Thus I consider here that it is necessary to distinguish between two levels of activity with an e-exercises resource. The first level is the level of one given exercise. The second level is the level of a whole classroom session. For both levels, several kinds of tasks can intervene, according to the scenario in use retained by the teacher, and more generally to the didactical contract (Brousseau & Warfield, 1999). The didactical contract has explicit, but also implicit rules, determining the roles of the teacher, of the student, and here also of the computer. For example, the e-exercises resource can send to the student a feed-back "right" or "wrong", a responsibility which is usually in the teacher's domain.
For the first level, the exercise's level, the task can be to find a numerical answer and type it in the "answer zone" of the computer. But it can also be to write the complete solution on a paper, if the teacher asks for it. Or on the opposite, the student can consider that the task is to obtain a feed-back "right" from the computer, and not to find a proper solution: this can be a consequence of the modification of the didactical

contract induced by the e-exercises resource. According to the characteristics of the resource, several artifacts can be available for these tasks. A calculator; but also the feed-back of the computer, or the advice associated with that exercise. The way these artifacts become instruments naturally depend on the task retained by the student. For the second level, the task can be to solve a given set of exercises. Things start to be really different from the exercise's level when the scenario in use retained by the teacher permits a freedom of paths within a certain range of exercises. In that case students can develop different working patterns, and thus assign themselves different tasks: tackling the maximum number of available exercises, or solving in detail only a few exercises for example. The artifact here is the effective resource made accessible by the teacher, thus it is not always the same for all the students in the same computer lab. These working patterns can certainly be interpreted as schemes. However, the description of these schemes involves rules of the didactical contract. Thus the contract appears as the main explicative tool for that level. 208
These first remarks are further discussed and illustrated by examples in the two following sections.
Instrumented action schemes for the exercise's level
When a student is solving an exercise, proposed by a given e-exercises resource, the artifact is the associated environment. Is it possible to identify instrumented action schemes constructed for that artifact? I try to answer this question on the following examples, stemming from observations realized in class at various levels.
The first example is the case of Alice. Alice is a sixth grade student. She works on an e-exercises resource proposing proportionality word problems. All the expected answers are numerical. The resource allows two attempts, with a feed-back "right" or "wrong" after the first try before attributing a mark. I observed Alice during four one hour sessions. She developed a very stable strategy; I describe it for the following exercise:
A car uses 20 liters of gas for 400 km. How much does it require for 100 km?
Alice knows that computations must be done, that these computations are likely to be division or multiplication, and that the expected result is a whole number. Thus she uses the calculator to make some tries, 100 times 20 for example. However, she controls the size of the result, and only proposes it if it sounds reasonable. Her first proposition is thus 4, obtained by computing 400/100 on the calculator. She naturally gets a feed-back "wrong". Then she goes on computing. Finally, she makes 100/20 on the calculator, finds 5, and proposes that answer. She gets a feed-back "Right".
Alice constructed here an unexpected scheme, relying on a mathematical theorem-inaction: "The answer to a proportionality problem whose text comprises only whole numbers is a whole number obtained by multiplying or dividing the given numbers".
Alice also controls the size of the numerical answer. The artifact helped that scheme to develop, because the calculator made it easier to try the possible computations; and that scheme governed the way the artifact was used, the feed-back was used here to test an answer that sounded acceptable. The pre- and post- test administrated indicate no improvement in her ability to solve such missing value problems. In her case, the unexpected scheme clearly hindered learning.
The use of the feed-back to develop attempts and errors strategies is frequent. These strategies are often grounded on mathematical knowledge; they are in fact schemes associated to the feed-back artifact. Let us give a second example. In an advanced linear algebra course where graduate students work on an e-exercises resource, the following exercise was proposed:
"Let E be a 35-dimensional vector space, and f an endomorphism of E, such that dim(Im(f0f))=13. Determine the minimum value of dim(Kerf)."
The students made several attempts; each time, the resource gives a feed-back with the right answer, and proposes to restart the exercise with new numerical values. 209
After a few tries, all the students were able to guess the general formula (and none of them was able to prove it):
Min (dim(Kerf)) = Integer part (1/2(dim E - dimIm(f0f) + 1)).
The mere feed-back would not lead to the formula with only a few tries. The complete constructed scheme associates the attempts, the observation of the right answer sent by the feed-back, and the following theorem-in-action: "To deduce the dimension of a kernel from the rank of an endomorphism, it is necessary to subtract it somehow from the dimension of the whole space".
That behavior was expected by the teacher, who proposed afterwards in a usual tutorial session a class discussion about the exercise, the formula and its proof. The exercise was finally solved during this session, and writing the complete solution was given as homework. The teacher wanted the students to face a puzzling situation, which motivated them to look for the complex proof.
Can all invariant behaviors of the students solving an e-exercise be explained by schemes? In fact, the central point is more: do all the descriptions in terms of schemes tell us something about mathematical knowledge? It is clear when a mathematical theorem-in-action is involved. But the operative invariant of the schemes are not always mathematical.
Let us consider the general case of the suggestions: different kinds of texts designed to help the students in their solving processes. They can be explanations, methods, hints etc. They are part of the artifact, they are a specific kind of tools that can be termed "cognitive tools" (Rogalski & Samurçay 1993) because they comprise knowledge. In a teaching design about sequences for university first year students using an e-exercises resource, I observed that some students always opened the "suggestions" window just after reading the text of the exercise, and before any personal attempt. It was a very stable, naturally unexpected, behavior. It is expected to make a personal attempt, and have a look at the suggestions if necessary only after a significant personal research. That unexpected behavior can be considered as a scheme; but the description of this scheme leads to an interpretation in terms of didactical contract. These students break the implicit rules of the contract by opening too quickly the "suggestions" window. They leave a part of their mathematical work to the computer.
In the next section further interpretations in terms of didactic contract are developed.
The session's level: working patterns
The possibility for students to work at their own pace, or to follow different paths, is often mentioned as a positive aspect of e-exercises. But do all kinds of paths favor learning? One can easily figure that a student choosing to work during a whole one hour session on the same simple exercise is not likely to learn much. I observed on many occasions that regularities appear in the paths chosen by the students; for that reason I use the term "working patterns". These working patterns naturally depend on 210
the features of the resource and of the associated scenario. For example, some resources send after a given number of exercises a feed-back comprising an advice about the next exercise to try; others even impose the next exercise. Anyway, at least when it is possible, students develop different working patterns.
I present here briefly an example issued from a research with sixth grade students working on proportionality problems with an e-exercises resource. This resource proposed sets of five exercises, identified by a title (the titles' list is displayed on the first screen). It attributed a mark over five for such an exercises set. When the mark is 3 over 5 or less, the resource suggests to restart the exercises set. If students restart, they are proposed the same five exercises with new numerical values. I observed several working patterns of the students working with this resource. The most frequent were: "Making one time each exercise" for students who solve one exercise, and then tackle the following one, independently of their success or failure; and "Following the computer's suggestions": for students who always restart when the resource suggests it. These working patterns remain stable. They can be interpreted in terms of schemes; but the description of these schemes leads anyway to didactical contract effects that organize the students' behavior. For that reason, I focus here only on these contract effects.
"Making one time each exercise" corresponds to students who work with the computer as they would do it with a paper exercises sheet. They consider the Internet resource as a textbook, and place themselves in the usual didactic contract, ignoring the feed-backs. By "following the computer's suggestion" the students clearly attribute to the computer responsibilities that usually belong to the teacher's domain. These students work within a didactical contract modified by the presence of the computer.
The designers of the resource clearly expected the second working pattern. It indicates a belief about the learning processes: a student learns by making several times similar exercises. It is a typical drill and practice choice. In the research mentioned here, we studied the evolution of the students' ability to identify and solve proportionality problems before, during and after their work on the computer. We observed real improvements, but no significant difference between the groups corresponding to each of the two working patterns described here. It means that students learn by drill; but they do not seem to learn more than the one who work like in a usual environment. However, we do not know what the students who practiced drill would have learnt if the scenario, or the resource banned that behavior. Further research is necessary to decide whether leaving the two possibilities has positive consequences for learning.
Conclusion
For students solving a given e-exercise, the instrumental approach provides a framework to describe instrumented action schemes associated to the "environment" artifact. It permits in particular to anticipate behaviors likely to hinder learning. 211
On the more global level of a whole session on an e-exercises resource, stable working patterns can be observed and interpreted in terms of didactic contract. It is then necessary to evaluate the consequences of these working patterns on learning. Stating such results can be useful to design further resources, or to improve the existing ones. But it can also be simply used by teachers to choose appropriate scenarios for a given resource, in order to avoid undesirable behaviors and foster learning. The scenario plays indeed a central role to determine the explicit, and some of the implicit rules of the didactic contract.
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A developing agenda for research into digital technologies and mathematics education: A view from Brazil
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This article aims to introduce the research agenda that is currently guiding activities in the group Tecnologias e Meios de Expressão em Matemática
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(TecMEM) of
PUC-SP. It describes how wide scale attempts to insert digital technologies into Brazil's public schools system have tended to emphasise the computer as a catalyst for pedagogical change, without acknowledging the epistemological and cognitive dimensions associated with such change or the complexity associated with the appropriation of tools into mathematical and teaching practices. To focus research efforts on learning ecologies as complex interacting systems, the paper presents how our group is adopting as a research strategy the involvement of teachers and students in the process of collaborative tool design.
Integrating (?) Technology in the Brazilian Education System
In the late 70s and early 80s, university researchers in Brazil started to develop studies relating to the use of the computer as an instrument in the processes of teaching and learning. Their experiments were based on the use of programming languages inspired by the evolving constructionist perspective of Papert (1980). The introduction of computer technology into the public sector of the Brazilian Education system began later in the 80s, when the Ministry of Education (MEC) funded projects such as EDUCOM and FORMAR. In these projects, the role attributed to the computer was that of catalyst for pedagogical change (Valente & Almeida, 1997). The idea being that the possibilities offered by computer technology would enable innovative approaches to education, helping to form reflective citizens who would use exploit technology in the search, selection and interrelation of information and in the construction of knowledge and hence enable them to better understand and transform their own socio-historical context. A huge challenge given that the dominant pedagogical approach of the time almost exclusively focussed on teaching as transmission of ideas.
Despite, or perhaps because of, the innovative nature of the Brazilian proposal for the insertion of computer in the education system, none of the government-funded programs have yet resulted in the intended transformations in educational approach. The pedagogical aspects implicit in the use of the computer have turned out to be much more complex than originally predicted and, in practice, compounded by the fact that, within the public education system in Brazil, access to technological resources continues to be sporadic and unevenly distributed throughout the country's

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schools. One important result, however, has been the recognition of the critical need
to understand the role of the teacher in every step of the integration process.
It is important to stress that although research in mathematics education has
been a part of the insertion process described above, its role has not been central. In
attempts to prepare teachers to make use of technology, there seems to have been less
emphasis on what to teach – or even what is being taught – and much more on how it
could/should be taught. Epistemological concerns regarding mathematics (or indeed
any other curriculum area) or the meanings for mathematics constructed during
technology-mediated activity seem to have been largely absent from the debates
concerning the integration of computers into the Brazilian education system. Indeed,
in the official guidelines for the Brazilian Mathematics Curriculum published by
MEC (PCN, 1998), again it is principally pedagogical concerns that figure in relation
to the use of digital technologies in the mathematics classroom. The use of
technological resources (such as calculators, video and computers) is one of a set of
three pedagogical approaches emphasised, the other two being the use of historical
resources and games.
Digital tools and mediated agency
It seems that the privileging of the (supposed) impact of digital technologies
on pedagogical approach in the Brazilian context had the result of largely ignoring
the important reciprocal relationships between technology and thinking – that the
resources available for the negotiation of meanings for mathematics, both shape and
are shaped by these developing meanings. According to Borba (2000), the same
could be said of much of the work generated by Brazilian researchers working
specifically in the field of mathematics education. He suggests that, until recently,
investigations of the role of different media in the teaching and learning of
mathematics have been limited to treating technological tools as didactical aids
which might help (or not) the learning of particular material. Drawing from the work
of Levy and Latour, Borba suggests that rather than individual learners we should
focus our attention on the collective "seres-humanos-com-mídias" (people-withmedia), a construct reminiscent of the socio-cultural unit of analysis proposed by
Wertsch and Toma (1994), individual(s)-operating-with-mediational-means or
mediated agency. According to this perspective, the inclusion of the (any) tool in
activity alters the course both of the activity and of all the mental processes that enter
into the instrumental act.
This suggests that in order to understand learning ecologies that include digital
resources, it is important to consider the ways these resources interrelate with both
epistemological and cognitive dimensions. Recent research has also emphasised the
need to theorise more precisely about the ways in which learners come to make use
of the technological resources available in the learning environments in which they
interact or, to put it in the terms of Verillon and Rabardel (1995), to understand the
process of instrumental genesis by which artefacts become transformed into 215
instruments
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. The potential role of digital tools cannot be expected to be transparent
– neither to teachers nor to learners – and if they are to be integrated in a significant
form into mathematics classrooms, an understanding of how to engender the process
of instrumental genesis is crucial. It could be conjectured, then, that one of the
weaknesses of the Brazilian teacher education programs was (and largely still is) that
little attention was given to the process by which the digital artefacts introduced
might be appropriated by teachers and integrated into their practice. In the case of
teacher education, it may even been that the instrumental genesis process is yet more
complex since its ends become twofold: artefacts need to become instruments not
only in the mathematical practices of teachers but also in their didactical practices.
Research into the use of technology in mathematics education within Brazil,
and more specifically the emerging research agenda of the group Tecnologias e
Meios de Expressão em Matemática – TecMEM has begun to take on board these
issues our research is increasingly focused on what meanings for the mathematical
objects concerned are afforded (and constrained) as particular tools are appropriated,
how these meanings evolve as the tools are transformed by their users in practice and
how both tools and the meanings associated with them impact upon the way we think
about the mathematics in question. In considering the constraints and affordances of
particular resources, as we bring epistemological and cognitive questions into play,
and as sociocultural perspectives on tool mediation come to the fore, we believe it is
important not to relegate pedagogical aspects to a second plane. The challenge is to
focus on the learning ecology as a whole, considering the interactions between
different dimensions – epistemological, technological (or perhaps instrumental),
cognitive and pedagogical – concomitantly.
Approaches to tool design
TecMEM is a research group of the programme of post-graduate studies in
mathematics education of the Catholic University of Sao Paulo (PUC-SP).
Geographically, the group is based in the university's Centre of Exact Sciences.
Because of this location, the group counts on participants from a diversity of
backgrounds: researchers in mathematics education and mathematics teachers are
joined by computer scientists and engineers. As a group, we are involved in
understanding the processes by which mathematical knowledge is constructed in the
presence of digital technologies, while building learning environments that support
engagement in these processes. Given the particular background of the group, it is
perhaps not surprising that tool design represents a significant aspect of the group's

24 Verillon and Rabardel (1995) use term artefact to describe a given human-made object. For
any individual person, the artefact becomes an instrument as he or she develops a set of schemes
associated with its use, allowing the artefact to be appropriated and integrated into the
individual's practices. As Trouche (2004; p. 285) puts it "an instrument can be considered as an
extension of the body, a functional organ made up of an artefact component (an artefact, or the
part of an artefact mobilized in the activity) and a psychological component". 216
work. To a certain extent, we have come full circle back to the constructionist
beginnings of computer use in education in Brazil, and underlying our research is a
continued desire to design tools that will provoke changes in the way mathematics is
experienced in Brazilian schools. But the theoretical basis which informs our
research has developed considerably, opening new windows through which to
investigate mediating between learners' personal knowledge of mathematics and the
official mathematical discourse they are supposed to learn.
In terms of tool design, it is possible to distinguish between two contrasting
approaches to supporting connections between conventional and personal
mathematics, usually described using the metaphors top-down and bottom-up.
Software designed from the top-down have their genesis in expert practice, in
"crystallised expert mathematical knowledge" as Gravemeijer (1997) puts it. They
offer to learners a set of tools to enact conventional techniques or methods. Examples
include Computer Algebra Systems, many educational statistical packages (see,
Biehler, 1997, Konold, 2002). Dynamic geometry systems can also be included
within this group: Laborde and Laborde (1995), for example, describe how Cabrigéomètre, was designed to bring students in touch with a model as close as possible
to traditional Euclidean Geometry. On the other hand, are software designed from the
bottom-up, their basis in the learners' practices and reasoning, perhaps the most
classic example of which is Turtle Geometry, which "started with the goal of fitting
children" (Papert, 1980; p. 53) and the hope that students would bring what they
know about their own bodies and their movements to bear as they learn a formal
geometry.
A problem with these metaphors is that they place learners, mathematics
always on a lower plane. The devaluing of individual knowledge in favour of the
conventional can be avoided if instead the terms filling inwards (FI) and filling
outwards (FO) are employed (Healy, 2002). This has the added advantage of
stressing a major difference between constructivist and sociocultural approaches: the
primacy assigned to the individual or the cultural in the learning process. Hence,
these approaches are not limited to tool design considerations and can be interpreted
more generally as two different didactical models. Filling outwards approaches
correspond to constructivist-rooted approaches to mathematical teaching (such as
realistic mathematics education, didactical engineering and the emergent approach of
Cobb et al., 1997, for example), which emphasise an outwards flow whereby
interventions are intended to guide personal understandings gradually towards
institutionalized knowledge, with mathematically significant issues arising out of the
student's own constructive efforts. A reverse filling-inwards flow of instruction
characterises sociocultural accounts of teaching, with interventions aimed at
supporting learners in internalising institutionalized knowledge to construct new
understandings, hence enabling mathematically significant issues to become
appropriated during the learner's constructive efforts. 217
Returning to the issue of tool design, it might be hypothesised the process of
instrumental genesis is rather different according to whether a filling outwards or
filling inwards approach is adopted. For experts, the integration of FI-based tools into
their mathematical practices might be relatively straightforward, but for those less
fluent in the expert practice, the path from artefact to instrument might be more
tortuous. In contrast, FO-based tools might be more easily incorporated into learner's
activities since they have been designed on the basis of what they can be expected to
know. But in this case, facilitating the process of instrumental genesis may not have
the effect of guiding towards conventional mathematical expression. This results in a
somewhat paradoxical situation. When we design tools on the basis of the practices
of learners rather than experts, enabling learners to express their own meanings in
ways that do not necessarily match conventional expressions, teachers are not always
willing to accept the validity of the tools. On the other hand, when our design
decisions are guided by conventional mathematical expressions instead of what we
know about learners' understandings, although the legitimacy of the tools we design
is less likely to be questioned by teachers, the tools themselves do not always enable
learners to interact with the mathematics in question.
Our current research agenda
So the challenge the participants of *TecMEM* are currently confronting is how
to design tools that make sense to learners, connect with their points of view and
resonate with what they already know, while, at the same time enabling them to
express their ideas in ways that are considered mathematically legitimate by their
teachers. Tools that illuminate the mathematical structures and relationships
embedded in them to learners and to teachers alike.
We are choosing to concentrate our design efforts particularly in areas of
mathematics whose current coverage has been highlighted as problematic in the
Brazilian context. This includes, for example, statistics and probability, whose
emphasis in the official mathematics curriculum has significantly increased in recent
years
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, geometry, pinpointed as a domain that mathematics teachers have a tendency
to avoid, and proof, an aspect of mathematics considered rarely, if at all, in the
mathematics classrooms of the public school system
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. Because of the make-up of
our group, it would be possible to divide the research tasks, with researchers and
computer programmers responsible for task and tool design respectively and the
mathematics teachers involved principally in implementing these activities in their

25 A description of some of the mathematically issues that emerged in the design of tools for
the simulation of data distributions, see Healy (in press).
26 We are currently engaged in a CNPq funded project, AProvaME (Argumentação e Prova na
Matemática Escola, No. 478272/2004-9) which is involving 26 mathematics teachers from public
schools in the state of sao Paulo in investigation the strategies of proof and justification of their
students and in building technology-intergrated learning scenarios to support these students in
further developing their competencies in proving.218
classrooms. The problem is, if teachers bypass the design phase, they will not
necessarily come into contact with either the epistemological, didactical or cognitive
considerations that contributed to the conception of the activities. Without this
contact, teachers may not feel ready to come up with new activities of their own or
even to adapt existing activities according to the particular needs of their students.
The strategy we are investigating, then, is to involve all group members not
only in the design of activities to support mathematical reasoning, but also in the
design of the computational environments which form the context in which this
reasoning is to take place. Our aim is not that we create well polished "finished"
softwares, rather that, in line with the constructionist agenda that continues to guide
our work, we create microworlds that represent our tinkering and can be
subsequently tinkered with by others (Papert, 1991). In our strategy, the process of
instrumental genesis begins with the genesis of the artefact itself, that is, it could be
said that, for the designer, the artefact is always an instrument. The conjecture that
drives the projects currently underway is that the design of digital tools for
mathematics learning will necessarily involve participants in making explicit their
own knowledge about the mathematical issues concerned, their beliefs about the
learning trajectories their students follow and their thinking about how best to
mediate between their students' (and perhaps their own) personal knowledge and the
mathematics they are aiming to teach. As such, in relation to the integration of digital
technologies into mathematics classroom, the questions to which we are seeking
evidence are:
• Will participating in design of tools for simulating data sets encourage designers
as learners to reflect upon mathematics concepts incorporated in the tools under
development?
• Will participating in the design process encourage designers as teachers to reflect
upon the kinds of representations that might permit their students to access and
explore these same ideas?
• And what didactical approaches do designers as researchers build into the
learning scenarios in which their developing tools are embedded?
So, one strand of our research work involves mathematics teachers in the
collaborative design of tools for mathematics learning. A second strand involves the
students themselves in this collaborative design process.
In this second strand, we are working with learners whose developmental trajectories
are not yet well represented in the research literature: blind learners and learners with
severe visual impairments. As researchers, we recognise that we still know relatively
little about the mathematical interpretations developed by blind learners and whether
the mathematical meanings they construct and the narratives through which they
make sense of mathematical problems follow significantly different patterns to those
of sighted learners. However, in Brazil, there is currently a strong drive within the
public education systems towards the inclusion of learners with special needs into

mainstream schooling. Because of this policy, a number of the mathematics teachers 219
of TecMEM are already facing the challenge of teaching classes in which one or
several of the forty or so students are blind. One of the particular difficulties is that
not only can the learners not see the inscriptions of mathematics presented to the rest
of the class, the mathematics teachers cannot, in general, read the inscriptions
produced by their blind learners, until they are translated from Braille, making. This
makes it extremely hard to know if and when to intervene, and especially to adopt a
filling-outwards didactical approach. We are currently working on the design of
musical microworlds and microworlds composed of speaking and conversing
mathematical objects, which, combined with tactile tools, aim to allow blind learners
to explore, express and communicate mathematical ideas associated with rational
numbers, function and geometry. Although this work is only beginning, we are
finding that the participation of the learners, right from the conception of the tools is
critically as we simply do not know enough about the stories they construct to make
sense of the mathematics they encounter to proceed without them
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A final word
From the beginning, the attempts to insert digital technologies into Brazilian schools
did not represent a response to an expressed need from the grass-roots of the
classroom. To a greater or lesser extent this continues to be true today. On the other
hand, it might be that digital technologies do have considerable potential to offer in
confronting current challenges at the chalk face. Our research seeks to involve
teachers and their students in the process of collaborative tool design, in order that
they might assess and, where appropriate, begin to harness this potential to resolve
the particular complexities of the classrooms in which they work. By its very nature,
our approach, not unlike the process of instrumental genesis, is complex, timeconsuming and linked to local considerations such as the characteristics of the
learning setting, and the participants' activities, knowledge and former methods of
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Functionalities of technological tools in the learning of basic geometrical
notions and properties
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This contribution presents the comparisons made by two students, each the most
advanced in their classes, of the relative functionality of a variety of technological
tools, the ones they had used to learn basic geometric notions and properties. This
work is part of a line of investigation specializing in the functionality of artifacts
involved in teaching sessions (Verillon and Rabardel, 1995) and the development of
classroom discourse (Sfard, 2001). It revisits the case of "Guillermo" (Hoyos, 2003),
a junior high school student who learns the topic of geometric transformations via
utilization of Cabri-II and pantographs or articulated machines. Also reviewed is the
case of "Marcel", a sixth grade student which takes up problem resolution involving
the notions of angle, turn, and their measurement via Logo and Cabri-II. The paper
argues for complementarity between the tools utilized for the construction of use
schema, which objectified certain mathematical notions involved; and for the
internalization of instruments in use (Mariotti, 2002).
Theoretical Framework
As Mariotti (2002, p. 697) argued, since school use of computers and new
technologies is on the rise, it becomes urgent to identify key points around which to
organize their use in fostering diverse educational processes.
In addition to incorporation of computers in schools, she and other Italian researchers
(Bartolini et al., 1998; Boero et al., 1997; Mariotti et al., 1997) particularly have
recommended using artifacts and contexts of geometric practice which employ
mechanical or jointed models of drawing and tracing machines as a school's way to
generate complex mathematical ideas or notions.
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As an example, in the case of the study of geometrical transformations, each one of the used
machines was made of a set of articulated metal bars mounted on a wooden stage in accordance
with different geometrical configurations representing the elemental geometric principles of
reflection, symmetry, translation, and dilation. The bars are fifteen to twenty-five centimeters long,
the wood base measures about fifty by sixty centimeters. One of the more well-known pantographs
may be the one that enlarges or reduces a drawing by a given ratio. This research employed this
artifact, as well other similarly constructed, which were designed to study symmetry, reflection, and
translation. The materials were originally constructed by the University of Modena (UNIMO), in
Italy. For further detail on the pantographs and other UNIMO instruments, refer to
www.mmlab.unimo.it 222
Actually, current investigations have focused on incorporating use of diverse
technological tools, with the idea of comparing the results from manipulating
different learning environments (cf. Hoyos, Capponi and Genèves, 1998; Vincent et
al., 2002; Hoyos, 2002; Hoyos, 2005) or in the attempt to overcome difficulties
associated with solving mathematical complex tasks, like the proof of geometrical
basic properties. (See Vincent et al., 2002; Hoyos, 2005)
For example, participant observation of sixth grade students (Hoyos, 2005) revealed
that, although Logo's turtle (figure 1) can move forward in any direction, it does so
linearly in such a way that it is not evident for children that the same turn procedure
is being applied both forward and backward.
Fig. 1
Mathematical turn rules are usually referred to clockwise, and using Logo it might
not be evident to children that the turtle procedure of turning to the right (the rt
command) could be associated with clockwise motion, and counterclockwise in the
case of left turns (the lt command). We predict that a broader significance assigned to
turns implemented by this tool would aid students to improve their handling of Logo.
Based on these considerations, and thinking about the advantage of having interactive
tools that could measure the various magnitudes presented in the problems, we used
other materials in addition to Logo, particularly the Cabri-II computer environment.
Briefly, all the technological tools used in that context would provide specific,
functional features to enable students to efficiently leave behind drawing difficulties
related with giving the turtle convenient instructions – a task which requires
anticipating turtle turn direction.
Verillon and Rabardel (1995, p. 77) affirmed the potential for artifacts to address
particular learning difficulties related to the functionalities of the artifacts in play.
Among the most important contributions these authors offer is to point up the
substantial differences in the products of knowledge derived from unlike
manipulation experiences. On the one hand are those (cf. Piaget and Inhelder, 1958)
in which the subject manipulates various sizes and shapes of metal bars or liquids 223
transferred between containers, and on the other are those that involve exploration or
use of technologies, such as activating a robot.
Concerning use of Cabri-II dynamic geometry software in both exploratory studies,
one of the central hypotheses (Hoyos, 2006) was that if from the beginning of the
activities the learners had available a formal language they had not yet used or
experienced
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, it would permit tracking their progress in mathematic discourse, as far
as they construct or assign meaning to these referents.
Sfard (2001) takes as an instance of the development of mathematic discourse a
classroom interaction in grade seven on the learning of negative numbers. She shows
how disorganized the information becomes when these numbers had only recently
been introduced, as well as the students' shallow comprehension. Sfard argues that
this type of basement is insufficient to immediate advancement toward capitalization
in, for example, operational handling of the new terms. For Sfard (2001, p. 28), the
"introduction of new names and new signifiers is the beginning rather than the end
of the story."
Sfard emphasizes that the introduction of new symbols acts as a piston, driving and
creating new semantic spaces which appeal to the needs for new meanings and new
discursive habits (ibid, p. 32).
This author identifies the act of communication with the act of thinking, and thus the
development of discourse in a two phased process. The first phase is characterized by
the use of terms with a template at hand and the second by objectification or
objectified use of the symbols.
Finally, from the approach Sfard proposes to learning as the development of
discourse, it becomes interesting to obtain descriptions of student identifications of
the symbols or terms in use, given that learning some topic creates the capacity to
extend discursive capacities so that in some moment the learner will be able to
communicate on the theme. (Ibid, p. 26)
Aims, Methodology, and Some Results
The two case studies reviewed here (Hoyos, 2003; Hoyos, 2005) were constituted on
the basis of the productions made by the most advanced students in each of the
classes observed. Different technologies were alternated in both cases so that the
learners would be able to strengthen or compare the results from manipulating
distinct learning environments (cf. Vincent et al., 2002; Hoyos, 2002; Hoyos, 2003).

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Note that Cabri-II includes a Help command that displays a legend at the bottom of
the screen. This legend gives the students a formal description of the geometric
construction in turn, just as is the case with the menu for geometric transformations.
In fact, that legend constitutes a formal guide for the student in order to execute the
required construction or action. 224
During the first case study, "Guillermo" (Hoyos, 2003), a very advanced student in a
ninth grade Math class (fifteen years old, approximately), was introduced to explore
Cabri-II and pantographs as a means to approach the theme of basic geometrical
transformations.
With ease and accomplishment he worked all the tasks presented in a long series of
practical activities implemented in the classroom (Hoyos, 2006). Video logs were
obtained of all of Guillermo's movements, especially because he was asked to
describe what they had done at the end of each assigned activity. At the end of the
reflection and translation learning activities, during his description of what he had
done, Guillermo makes reference to the various moments of the activities performed
during the two learning stages that were set.
Here it is going to be briefly mentioned what happened using Cabri-II when he
activated the Reflection command. The student describes it as making the initial and
image objects "move in opposite directions":
Guillermo (hereinafter G): What we saw with this machine [the jointed machine for
reflection or axial symmetry] was the same as for axial symmetry, what we were
seeing, for example ... Like if we took these by the movement points [manipulates
the jointed machine, moving its drawing guide] they go opposite ways. Like [what]
we were seeing in the Cabri software...
What we were proving here [with the jointed machine] was that axial symmetry is
practically a reflection of the original figure, if we move [the guide] to the left ... It's

like when we move the original figure from left to right, the reflection goes … From right to left.
As can be clearly observed from the work report Guillermo wrote after have finished the work sessions with the software, he was able to quite coherently express the properties of dilation. For example, it is presented here his answer to the question "Give a general definition of dilation", which was on the scripted worksheet for the computer exploration:
G: The image [is] on another scale, with proportional movements to what you do [in relation with the initial figure].
The final passage that is going to be presented here concerns with comparison of tools in use. In the last interview with Guillermo he was asked to offer his opinion on the work he carried out in all the sessions.
I: Let's see, Guillermo, tell me how you feel about these work sessions.
G: Yea, well, alright. 225
I: Right. Has what we've done been interesting for you?
G: Yea, because it made us think awful hard.
I: Yes. Did you like the computer work?
G: Yea, just that it's easier with the computer than doing it here, firsthand.
I: But the work with the machines [the pantographs], does it seem productive? I mean, it leaves you with something. Interesting, isn't it?
G: You learn more [there], than with computers, cause you do the work.
I: You learn more with computers, is that what you think?
G: No, you learn more there, [points to the pantographs] cause here you do the work and the tracing, and on the computer the only thing you do is guide it.
Along the second case study (Hoyos, 2005), the hypothesis was advanced that Logo and Cabri-II functions were complementary when applied to the learning of angle and turn. Activities based on manipulating these learning environments and other concrete materials permitted the students to overcome difficulties with the direction of turn and the measurement of angles greater than 180°.
In the previous to the last session it was introduced Cabri-II as a teaching technique, because it was likely to allow the students to overcome the difficulties still demonstrated in the Stage 4 of that study, which had to do with measuring angles greater than 180° and the direction of turns.
The Cabri-II activities may be briefly characterized as simulating both directions of clock movement by placing the hands in various positions and using its Angle command to measure angle sizes, especially those greater than 180°. At this juncture it is worth noting that Cabri's Angle command only works with angles less than or equal to 180°, requiring reflection or planning on how to calculate those of greater dimension. These may be the reasons why comprehension about measuring that kind of angles might finally be achieved.
In sum, what happened during the work session with Cabri-II was that the children were asked to measure an angle like that indicated by an arc in greater than 180°. Nonetheless, the measurement the software displays when one selects three points -- one on the little hand, one in the center of the circumference, and one on the big hand-- (as Angle measurement shows in the Help legend,) is a measurement of 148.9°, which is less than 180°!
Instructor (I): So, what's up with the measurement the machine is giving us [that of 148.9°]? How can I do to know how much the angle labeled by the arc is?
Marcel (M) [The most advanced student in that study case-]: Do you know why it's showing up like that? [He is referring to the software] What the computer is doing is, what I suppose, is it isn't marking on this side, but from here to here [M points to the other side of the angle, which in effect measures less than 180°]. 'Cause it wasn't labeled exactly. [You, sir] put the three points, but the computer can make mistakes, [it doesn't know] if it's to here or to here [M successively indicates each of the two 226 dos angles determined by the clock hands simulated in Cabri-II]. It didn't label right which angle to go to.
I: Marcel is right, since you have to label three points to measure an angle, as the legend in Help says, the angle the software is looking at could be the one marked by the arc or its opposite. Perfect.
I… What should I do to measure the angle labeled by the arc?
M: I see how I can do it. 360°-148.9°, this is the angle labeled by the arc, [because] the 148.9° is the one that the arc didn't label. So, if this is what this angle measures, it's less 360, and so we get the other degrees left over. That's why we know what [that, the result from subtraction] the other angle is.
Briefly, at the conclusion of the work sequence on problem solving with Logo, the instructor asked the children their opinion on working with Logo, or Cabri-II, and whether they had enjoyed working with these programs:
M: Cabri seems easier to me, 'cause there you go to your tools and you get everything. But here [with Logo] you have to calculate 'n' everything. It looks harder to me. It's a bit harder for a kid … Because there you gotta be calculating degrees … But in that one [Cabri] you've got your tools 'n' everything. So for a kid it's easier to get what you need from the tools, not in that one [Logo], you gotta be calculating 'n' everything [gestures to his head].
Towards a Conclusion
One of the lines of research into the question of semiotic mediation (Mariotti, 2002) looks into the cognitive processes of instrumental genesis, with the source of its analysis being the self-same nature and manipulation of the artifacts employed.
Preliminary results obtained in both explorations indicate that, in effect, coordination is possible between all technological instruments deployed. Besides, the working hypothesis that appears to gain credence on the basis the comparison the children made between the instruments used (Cabri-II and pantographs in the first study; Logo and Cabri-II in the second one) is that students might naturally tend to assign greater value to tools and to learning gained once an objectification process has taken place, as they will contrast their current capacities against those they had been able to perform during a previous stage.
In addition, both cases would appear to be evidence of an accomplished internalization process (Mariotti, 2002), because it is probable that idiosyncratic use of contextual terms to evaluate used tools might be evidence of cognitive attainment that probably is only reached through accomplished internalization of instruments in use. 227
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Tools rather than Toys: Fostering mathematical understanding through ICT in primary mathematics classrooms
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The proposed paper addresses the theme "teachers and teaching" by reflecting how far a new methods course on the use of technology in primary mathematics prepares pre-service teachers to critically select and use digital and electronic technologies in classrooms. While the motivational benefit with respect to the use of ICT in classrooms is certainly an important issue, this paper seeks to highlight in how far the careful selection and reflected implementation of technologies in classrooms can help to extend mathematical understanding beyond mathematics teaching and learning with traditional classroom materials. The paper focuses on two learning environments that have been developed in the context of a university methods course and trialed in the form of teaching experiments in grade 4 classrooms by pre-service teachers. The two learning environments go beyond the use of special software designed for (primary) mathematics classrooms and involve a robot and a monitoring device. Both learning environments will be briefly introduced in terms of their technical description. In addition, selected tasks from the teaching experiments, classroom observations and examples of students' work will be presented. More detailed information as well as an evaluation of the pre-service teachers learning processes (currently in progress) would be provided during the conference presentation.
The context: An ICT module within the teacher preparation programme at the University of Oldenburg
The teacher preparation programme focussing on future primary mathematics teachers at the University of Oldenburg
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involves a compulsory 4-hour-module on
the use of digital and electronic technologies in the primary mathematics class-room. The course is based on research findings that students learn more mathematics, more deeply with the use of technology (e.g. see NCTM 2000) and the understanding that technology does not replace the mathematics teacher (e.g. see Way & Beardon 2003).

30 All teacher preparation programmes at the University of Oldenburg have been re-developed in the last two years in order to meet the criteria of the Bologna declaration (UNESCO-CEPES 1999). For future primary and lower secondary teachers the University of Oldenburg offers a three-year bachelor programme (Bachelor of Arts/Science) focussing on two subjects and educational studies followed by a one year master-programme (Master of Education). 229
The more theoretical parts of the course critically reflect the current international literature on the impact of ICT on the teaching and learning of primary mathematics with respect to the following questions:
- How far can ICT improve the learning of mathematics?
- How can ICT support effective mathematics classroom practice?
- How far does the use of ICT effect mathematics contents?

In the more practical oriented parts of the course, pre-service teachers are given the opportunity to explore ICT such as teaching and learning software, construction and logo programmes, small robots and monitoring devices in small groups in order to understand their functions and their role as essential tools for mathematics learning. However, while critical theoretical reflections and practical investigations are seen as crucial elements of the course, they are not regarded as sufficient in terms of teacher preparation.
Guskey's (1985) investigation of teachers' professional development processes suggests that in-service teachers' classroom practice is a critical variable for classroom change. He argues that teachers predominantly define their professional success through the improvement of their students' learning. Hence, the opportunity to explore ICT in mathematics classrooms and to observe and assess students' learning processes is seen additionally as a critical factor for teacher pre-service training. In order to facilitate these classroom experiences, the course also involves the development, classroom implementation and evaluation of appropriate learning environments in primary mathematics.
The development of the learning environments is guided by the paradigm that ICT serve as tools not toys in the classroom (Buchanan 2003) which foster the extension of mathematical understanding and the introduction of new content areas.
Two examples of ICT based learning environments
In the following paragraphs two innovative examples of ICT tools suitable for primary mathematics are introduced. Both have been chosen because they support the extension of mathematical understanding beyond traditional classroom materials.
Introducing Pip
„Pip" is a small robot on wheels which can be programmed to move in centimetres, turn in degrees and wait in tenths of a second. It can be programmed via a keyboard - overall a maximum of 48 commands can be entered in one programme. A pen can be attached through the whole in the centre of the device which allows to record its movements on a 230 sheet of paper for subsequent analysis.
When it is first introduced, many primary students (and pre-service teachers) get the impression that Pip is a great toy and they certainly enjoy "playing" with it. However, pre-service teachers in the methods course have been challenged to develop a learning environment based on Pip that enables fourth graders active discovery of new mathematical contents, such as angles.
Using Pip to teach angles
Primary mathematics curricula in Germany require that students in grade 4 get to know and understand the 90° angle. This is usually taught introducing a protractor. More innovative approaches would also include suitable paper folding activities.

However, Pip allows the introduction of the angle concept in a hands-on and discovery based environment that involves and challenges the exploration of a number of angles other than 90° and their relationships.
Goal of the learning environment developed by the pre-service teachers was the understanding of 90°, 180° und 360° angles as a basis for finding and estimating other angles in between.
For the classroom exploration a drawing of a circle with 10° intervals on a large piece of paper was prepared and Pip was positioned in the centre of the circle with the arrow facing 0°/360°.
The grade 4 students were given the following tasks:
Exploring different sized angles with Pip
1) Program Pip so that it turns 90°/ 10°/ 180°/240°/360° clockwise (anticlockwise).
2) Program Pip so that it turns 40°/60°/20° clockwise. To which degree is the arrow pointing, when it would then turn 60°/30°/180° anti-clockwise?
3) The degrees of the circle are hidden: We want Pip to turn 45°/135°/ 70° clockwise. Show where it will stop! 231
Another task aimed at the understanding of the 90° angle with respect to the concept of perpendicular. A pen was attached so that Pip's movements created a drawing.
Constructing 90° angles
1) Program Pip so that it turns 40° anti-clockwise going all the way to the circle's outline, then turns 180° again going to (the opposite side of) the circle's outline.
2) Then program Pip so that it draws a perpendicular line through the centre of the circle.
3) Finally program Pip so that it draws a parallel line to the perpendicular line.
Apart from understanding and programming different angles and finding and programming the right length, these tasks require reasoning and anticipatory mathematical thinking.
In this context, Pip is either an effective addition or an alternative to the use of the protractor that fosters experimentation and visualisation. Mistakes quickly become obvious and are generally understood in a constructive way. Through empirical approaches and "trial and error" Pip enabled the students to independently control and – if necessary – advance and correct their thinking and modify their actions. They could develop estimation skills with respect to the size of different angles beyond 90° and deepen their understanding of the angle concept.
Overall, Pip provided an enjoyable and stimulating approach to angles for primary students. This frequently lead them to 'invent' and explore their own tasks, e.g. by imagining that the circle line is a clock: Can we program Pip so that the clock shows 5 o'clock (2 o'clock, 9 o'clock …)?
Using Pip to explore lengths 232
Using Pip in the context of length measurement encourages children to estimate distances and to check their estimates. Challenging tasks might also involve the combination of angles and lengths with respect to estimation:
Practising estimation
1) Program Pip so that it goes to the child that is sitting directly opposite to you.
2) Program Pip so that it goes to Anne/Tom/Erkan .
Path diagrams created by Pip are another opportunity to combine the estimation of lengths and angles. Depending on their prior knowledge and individual capability students can be challenged to program Pip so that it follows given path diagrams (either with 90° or other size angles).
Pip has also been used successfully in grade 4 classrooms with respect to the following topics: Axial symmetry, sum of angles in quadrilaterals, features of geometric shapes.
Understanding graphs with the „Ranger"
The „Ranger" is a monitoring device (data logger) that allows the monitoring of movement in a straight line. In combination with the computer programme LogIT it provides the construction of graphs that show the relationship between time and distance.
In using the Ranger primary students can develop basic knowledge about graphs in an experimental environment. They observe and discover how the Ranger reacts to their movements and learn to interpret graphs by developing their understanding of the relationship between time and distance. 233
In traditional approaches to the development and understanding of time – distance graphs without the computer, children would walk forward and backwards between points A and B, timing themselves with stop-watches, then draw a graph of their results. But it would be difficult to understand and correct misconceptions, e.g. wrong distances or a graph that implies that one could go back in time.
With the aid of the Ranger and LogIT the movements can be monitored and recorded as graphs which can then be compared with the graphs predicted by the children.
Introductory activities could involve the following tasks:
Recording time – distance relationships with the Ranger
1) Stand 3 metres in front of the Ranger. Cross your arms and then slowly walk towards it.
2) Stand 3 metres in front of the Ranger. Cross your arms and then quickly walk towards it.
3) Stand 3 metres in front of the Ranger. Cross your arms and then slowly walk a few steps forwards and backwards.
Then the three graphs can be compared by analysing the time-distance relationship that they display.
The computer programme also allows children to overlay. This is useful if one wants to set up a walk beforehand and ask others to copy.
Alternatively, peers can orally describe a given walk that one child has to copy without seeing the graph.
Another activity involved given graphs that had to be interpreted by the students, for example by telling or writing the 'story' belonging to the graph (see picture on the bottom left). The examples chosen by the pre-services teachers deliberately included graphs that do not describe a time-distance relationships and require children's reasoning why these graphs cannot be reproduced by the Ranger. Other groups of children had been given little stories and the students were encouraged to draw the graph that would capture the story. 234

Concluding Remarks
The tasks and classroom examples introduced above refer to very recently conducted teaching experiments during the current winter semester. Pre-service teachers involved in the classroom explorations are in the process of documenting their experiences and reflecting their professional learning. In addition, they have been given a questionnaire that seeks to investigate their knowledge, beliefs and attitudes with respect to the use of ICT in primary mathematics.
In the next couple of months this data will be analysed with respect to the key questions identified in the introductory section of this paper. Findings and artefacts from these analyses would be provided during the presentation at the Study Conference.
With respect to the question in how far ICT might effect mathematics contents, the two learning environments based on "Pip" and "Ranger" illustrated increased mathematical understanding with respect to angles and graphs.
Without the introduction of Pip, the treatment of angles most likely would have been limited to the isolated introduction of the 90°, 180° and 360° angles. The construction of perpendicular lines that form a 90° angle as shown in photos above, certainly would not have been included. While perpendicular lines can easily be drawn by using a protractor, this task rather requires procedural understanding. The construction of perpendicular lines as described above by using Pip, in contrast requires a conceptual understanding of angles. The pre-service teachers involved in the teaching experiments were clearly surprised that grade 4 children were able to demonstrate that level of understanding.
Furthermore, the understanding of time – distance graphs is not a topic that is frequently dealt with in grade 4 classrooms in Germany. Traditional approaches to graphs in grade 7 are often based on text book tasks that do not involve student experiments. But even more innovative approaches so far were limited to hand drawn graphs of movements between two points A and B timed with stop watches. However, while these drawings would reveal students' misconceptions, it would be difficult for students to understand and correct their mistakes due to a lack of empirical experiences. In this context, the Ranger enabled fourth graders to explore, monitor and record time – distance relationships and hence to develop a deep understanding of graphs. 235
While both devices certainly were seen as fun toys by many children, 'playing' with these toys in suitable learning environments turned them into powerful tools for mathematics teaching and learning.
Acknowledgement
We wish to express our gratitude to Audrey Elliot who introduced us to "Pip" and "Ranger" in the context of our joint Sokrates-Comenius 2.1 programme "COSIMA – Communicating Own Strategies in Primary Mathematics". Her classroom ideas and experiences have greatly supported the development of the learning environments described in this paper as well as the contents of the methods course.
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Dynamic Geometry Activity Design for Elementary School Mathematics
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Based on our research in designing mathematics activities for the grade 3-5 classroom, we outline central issues confronting the deployment and integration of Dynamic Geometry software in the elementary school curriculum. We remark in particular on contributions of the elementary setting to specific theoretical and pedagogic issues confronted in Dynamic Geometry use in real classrooms not directly instrumented through researcher interventions, and to specific effects of Dynamic Geometry technology on students' mathematical understanding and practice.
Introduction
In this paper, we report preliminary findings from our work using The Geometer's Sketchpad (Jackiw, 2001) to design mathematics activities for use in grade 3-5 classrooms in the United States. We emphasize how our emerging understanding of Dynamic Geometry's reception and impact differs in this situation (age-group, curriculum, and classroom setting) from the upper-middle-school and secondary mathematics settings more commonly addressed by Dynamic Geometry research (see King and Schattschneider, 1997).
Background and examples
The context of these remarks is Sketchpad for Young Learners of Mathematics (SYL), a research and materials development initiative that grew out of a 2003 conference on uses of visualization software in the early grade levels. The research focus of this project is on technology-supported curriculum design models; the development focus is on a collection of classroom materials using Sketchpad to support, specifically and concretely, several curricular programs in current widespread use in the United States (the National Science Foundation "reform curricula" such as Everyday Mathematics (UCSMP, 1998), and the Connected Mathematics Project (Lappan et al., 1995)). The project aims for impact at national scale, and thus emphasizes the conditions of widespread adoptability of project materials and sustainability of their dissemination and reuse by teachers without direct project support in the way of additional tools, professional development, or

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other resources. Design therefore proceeds as much in reference to practical, cultural, and political exigencies of the intended user community as in reference to theoretical models of cognition, social interaction, or educational formation. Activities from the early phases of the effort are freely available; through the end of 2005, more than three thousand school teachers had requested and received them.
In this section, three representative activities illustrate the project's output and establish a context for our subsequent discussion. Each activity features one or more sketches—interactive Dynamic Geometry microworlds—as well as teacher discussion documents and student activity sheets. Their discussion here suffers from the typical problem of describing dynamic imagery and interactive technology in static print—we make do with a few representative screen shots and verbal remarks, but in doing so mourn the loss of some of the sketches' dynamic character.
Grouping. In this activity, designed for whole-class participation, students encounter a population of wandering bugs—green dots on the screen—that can be marshaled into group formations, and make predictions and observations about the bugs and their groupings (see Figure 1). Initially (a) the screen contains an unsorted collection of insects, two buttons (123 and Redlight), and an editable number statement: group size = 5. Pressing the 123 button (b) causes the insects to wander randomly across the screen, but pressing Redlight causes them to sprint into (c) organized groups of the

same number of bugs.
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As students orient themselves to the environment, the
activity asks them to begin mathematizing it. How many groups of bugs are there?
How many bugs in each group? How many left over? (One group in (c) has only four
members!) How many bugs are there, in total? Either through experience with earlier
activities, or through explicit teacher prompting, students discover they can alter the
value of the group size parameter, and that this value controls the size of the groups
into which bugs cluster. Image (d) shows the bugs frozen by Redlight with a group
size of three. More questions explore this idea of variation: if grouping by five made
four leftovers, what other group sizes will have four leftovers? Students in grade 4
are very concerned with fairness: how can you group them so that no bugs are left
over? Further questions and oriented discussion seek more subjective responses: how
could you group them to best show that there are exactly 24 bugs?

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The names for the buttons here come from a popular children's game—"red light, green light"—
where players move as far and as fast as possible during a count (1, 2, 3…); then stop still on the
cry of "red light!" 238
(a) a "large" collection of bugs… (b) wander aimlessly about the
screen…
(c) until students "collect" them
into groups…
(d) whose size is determined by the
given parameter.
Figure 1. Grouping
Jump Along. This second activity (figure 2) resituates the ideas of Grouping in a
more familiar representation—the number line (a)—governed by two parameters, a
number of jumps and an amount to jump by. Pressing Jump Along causes a point
to hop rightwards (from zero) by the number and size of jumps requested by the
parameters. In (b), six jumps of four units lands on 24. Activity questions seek first
concrete and then generalized descriptions of observed patterns and inferred rules of
the environment. How else might you land on 24? Image (c) shows a related pattern
(four jumps of six) superimposed on the first (six groups of four). Thus the
commutative principle emerges from students' exploration of co-variation of the two
parameters (both land on 24—six fours equal four sixes), while visual imagery
retains the obvious distinction between these two propositions (six fours is not the
same thing as four sixes). Further development in the activity strives to build
additional "visual personalities" for number and number properties through the
jumping metaphor. A destination like 24 ultimately accumulates a denselyoverlapped set of jump factors, while prime numbers emerge with an equally distinct
visual signature (numbers that can be reached only by the smallest jumps). 239
(a) the basic number line… (b) shows n jumps of m
units…
(c) for student choices of n
and m.
Figure 2. Jump-counting along the number line
Flatland. This third activity provides an interactive model of the universe of
Edwin Abbott's 1884 parable of higher-dimensional visualization. Abbott's
protagonist, A. Square, is a two-dimensional being living inside the plane. He thus
perceives other inhabitants of his world always "edge-on:" objects appear with width
but no height, and their depth can only be sensed through relative brightness.
(Ambient gloom in Flatland causes things farther away to appear darker.) Where
Abbott's masterpiece develops these optical principles in analogic pursuit of three-
and higher-dimensional thinking, the grade 5 Sketchpad activity focuses on twodimensional visualization and the development of intrinsic geometry perspectives.
The Sketchpad microworld (Figure 3) is divided into two panels—a "spaceland
view" looking down on the plane from above and a "flatland view" looking across
the plane from within it, specifically from the vantage of the brown square. In (a), the
spaceland view at top shows the square looking (in the direction of its lower left
vertex) at the plane's other inhabitant, a triangle of separately-colored sides. The
flatland view at bottom—the broad horizontal stripe—shows the triangle as the
square sees it: only its red and blue edges are visible, and they occupy roughly the
same amount of the square's field of view. (But they are not equally long! The righthand edge fades to a much darker red at its extremity, indicating that edge recedes
sharply into the distance.) The sketch is also interactive. Students control the position
and orientation of the square either by dragging it (and its "eye") directly with the
mouse, or by navigating indirectly through turtle-like controls. The most important
button, Hide Spaceland, eliminates the top-down view and leaves only the edge-on
view of the triangle. Guided exercises develop students' footing in the flatlandscape,
and then probe their understanding. In the square's view (a) of the triangle, can you
determine with certainty whether the red and blue edges meet at all, or if one simply
eclipses the other? What maneuvers might improve or disprove that conjecture?
What other shapes could masquerade as perspective (a)? Eventually the activity
moves offline: without recourse to trial-and-error experiment, what sort of spaceland
geometries could lead to the four flatland perspectives shown in (b)? 240
(a) Corresponding views of a
triangle in Spaceland (top,
viewed from above the plane)
and Flatland (bottom, viewed
"within the plane" from the
moveable position of A. Square
in the upper-right).
(b) Where was A.
Square and the
triangle when these
photos were taken?
Figure 3. Interactive Flatland
Curriculum: discrete vs. continuous
In contrast to the content domains in which Dynamic Geometry is widely studied at
the middle school and higher levels (notably geometry, but also others), much of the
mathematics at the elementary level is discrete in nature. Students begin with whole
numbers, then move to integers, fractions, and decimals—which themselves appear
not as markers of a real continuum, but rather as another counting system of
somewhat smaller steps. At first blush, these discrete concepts and representations
seem at odds with the continuous motions and visualizations that form the heart of
Dynamic Geometry dragging. To rephrase as a challenge: what relevance has a
technology fundamentally about continuous evolution and dynamic manipulation of
geometric shape, to curricula focused on discrete counting systems and number
representations?
Three possible responses are evident in the SYL activities. We note the most
obvious first: not all of the elementary curriculum in the US concerns number.
Informal geometry (particularly the study of shape properties and hierarchies) is
pursued intermittently through qualitative and relational approaches that—because
non-quantitative—avoid the discrete formulations so common elsewhere. Thus
activities like Flatland, which centrally feature continuous variation and deformation
of shape, remain relevant. While such activities may appear as novel Sketchpad
treatments in terms of their intended age level, their use of Dynamic Geometry is
conventional.
A second observation is that some common discrete mathematical representations
that appear at these grade levels are conventions of, rather than essential to,
conceptual objects of study. This opens up the possibility of substituting, in the 241
dynamic environment, continuous for discrete representations, while remaining
focused on shared underlying mathematical concepts. Within our project, such
substitutions were well-accepted by field-testing teachers provided that the alternate
representations themselves were not identified as "properly belonging" to other
locations within the scope and sequence of their curriculum. In an attempt, for
example, to reinterpret a textbook lesson in data analysis, a draft Sketchpad activity
placed several draggable data markers on a horizontal axis. Each marker displayed its
numeric value, and bright arrows displayed the location and value of the mean and
median of the data set. Where the corresponding textbook activity depicted only
integer-valued data, the sketch displayed values to two decimal places, to provide
suitable numeric feedback to the act of dragging values left and right on the axis. But
these values led to the draft's rejection by field-testing teachers, under the stronglyheld conviction that this was intended as—and that the curriculum called for—an
activity about central tendency, not an activity about decimal numbers. A subsequent
draft removed numeric values from the dynamically-displayed data-markers and
central measures, and relied instead on graphical and positional visualizations of
these quantities. This draft—substituting a continuous, visual representation for the
integerized textbook illustration—met with greater teacher ratification. Thus in many
of our field-test classrooms, the mathematical role of technology is clearly to
innovate within parameters well established by an external curriculum framework—
both in terms of what it may address and what clearly it must avoid.
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Provided
Dynamic Geometry respect that principle, from our teachers' perspectives,
continuous representations though often new were not per se judged inappropriate
(see also Assude and Gelis, 2002). From students' perspective, such substitutions
tended to de-emphasize counting and computation skills, and re-emphasize
qualitative behaviors and comparative relationships.
A final response is that the general-purpose tools such as Sketchpad provide
sufficient capabilities to permit advanced users—in this case, curriculum
developers—to simulate discrete or quasi-discrete representations out of the
continuous ones that come easily in Dynamic Geometry. For example, step-functions
can be used to round or truncate the decimal portion of real quantities, and settings
for the preferred number of visible decimal digits can be adjusted to display these as
whole numbers. In addition, one can take advantage of subtly discrete behaviors that
already exist in the software's mathematics.
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Pursuing "highly authored" activity
design through such techniques of course has its own consequences in terms of the

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Since there is no national curriculum all schools most follow in the US, this "external curriculum
framework" in many cases was defined simply by the classroom textbook, or by school- or districtwide teaching policies and content expectations.
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For example, an intersection of two line segments continuously moves, in Sketchpad, as a result
of motion of the segments—until they no longer intersect, at which moment the intersection point
"blinks out" of existence. Thus on top of the continuous phenomenon of point location is layered a
discrete notion of point existence. Sophisticated users manipulate such behaviors as constructive
building blocks of other discrete and categorical phenomena or effects. 242
software's use and instrumentalization by students. We develop some of these
consequences in the next section, and in our presentation will discuss further the
Sketchpad software functionality we found particularly adaptable to, or relevant in,
young learner activity design.
To close this discussion of the discrete and the continuous, we note similar issues
impact Dynamic Geometry experiences at all levels, not just in the elementary
setting. An example so common it deserves a name of its own—the "defective
triangle defect" problem, perhaps—strikes most users when they first measure the
interior angles of a dynamic triangle, and sum the resulting values. The sum appears
as exactly 180°. But inspection shows that hand-adding the three displayed addends
themselves gives a different answer—180.1° or 179.9°. The issue, of course, is not
one of internal software accuracy, but of decimal representation and measurement.
The three angles are continuously-varying geometric quantities; there may be no
finite number of decimal digits that measures them exactly. Thus the individual
values display rounded to some adjustable, but finite, precision. Their summed result,
on the other hand, is exactly representable in whole-number decimal degrees. The
heart of the issue is how one's assumptions have been fed by prior textbook
illustrations of triangles in which all angles "happen" to have exact whole-degree
magnitudes. In the infinite bestiary of possible triangles one encounters in dynamic
geometry, such triangles are vanishingly rare, and one confronts the realization that
although discrete decimal numbers can usefully measure—that is, estimate—an
arbitrary angle's magnitude, they infinitely rarely describe it exactly. These sorts of
tension between the discrete and the continuous are fundamental to the mathematical
domain itself; but they are brought into particular focus by Dynamic Geometry's
novel and accessible representation of mathematical continuity. At the elementary
school level, occasionally-weaker content knowledge combines with greater
curricular emphasis on whole number operations to make their outbreak more
frequent, unless steps are taken in curriculum design to anticipate and manage their
consequences.
Students: pre-algebraic thinking, motion, and instrumental genesis
While we have yet to study student learning and engagement with these activities
directly, our field-test observations of classrooms and discussions with teacher
research participants identify several distinct features of the young learner
appropriation of Dynamic Geometry tools and activity settings.
The object known within Sketchpad as a "parameter" seems strongly implicated
in students' proto-symbolic thinking. Parameters consist of two attributes—a textual
name and a (possibly dimensioned) numeric value. (Group size = 5, in the Grouping
activity, and number of jumps = 4 and jump by = 6 in the Jump Along activity, are
parameter objects.) When users create a new parameter, they specify initial values for
both halves, which subsequently display together, as in the arrangements of Figures 1
and 2. Users then access a variety of tools to change the halves separately: a textediting tool, for renaming the parameter; a number-editing tool, for typing-in a new 243
numeric value; an adjusting tool, for incrementing or decrementing a present value;
and an animating tool, for varying the value continuously across some preset numeric
domain. The dual name/value appearance of Sketchpad parameters scaffolds
students' conception of variation and, more generally, of "variables." Many SYL
activities involve changing parameter values. Thus both over time within the
activity's unfolding on a single computer, and at any single moment in time across
multiple computers (in a lab or class cluster), the "same" parameter appears with
different values. In communicating with each other and with a teacher, students move
from referring to parameters by value—"let's change the three to a four"—to
referring to them by name: "change jump by to four." In these pre-symbolic uses,
the name clearly acts both a placeholder for a current value and as a summary of all
possible values (as in: "jump by can be as big as you want"—elementary students
like big numbers!—"but you don't necessarily wind up on the same spot").
From a broader pre-algebraic perspective, we note students at this age display
greater degrees of comfort and dexterity negotiating technical formalisms imposed

by constraints of interaction with a computer program than with those imposed by teachers. Where a definition, rule, or requirement stated by a teacher may vex students as arbitrary, insufficiently justified, or otherwise opaque, students appear to expect inflexible responses from machines. The Sketchpad command to measure, for example, requires a student to pre-select two points identifying that distance. This requirement operates as a mathematical formalism that students navigate to proceed in an activity. If a student incorrectly proposes that distance might be measured as a property of a single point, the software simply refuses to cooperate: the command stays disabled. Because this response has none of the overtones of error, fault, and correction that would accrue to similar human responses (by teachers or other authorities), students view it less as a rebuke than as a challenge: "how do I get the computer to do it?" Where students encounter computational representations and operations that parallel mathematical representations and operations, the software milieu offers unique opportunities to support their acquisition.

We turn to consider the role of motion. If motion--particularly, the mathematical motion of geometric objects under dragging or animation--is the most signal ingredient of Dynamic Geometry experiences, it appears motion takes on somewhat different functions in elementary activities than in the standard class of triangle-dragging activities well-studied by the Dynamic Geometry literature. First we note several different categories of motion within the activities described in Section 2. In Jump Along, the motion of jumping particles is strictly determined by given parameters, and exactly replicable for any setting of them. Students choreograph and explore this motion only indirectly, by setting parameter values and then "jumping." The Grouping activity combines this sort of deterministic motion (the grouping formations) with the more organic motion of freely-wandering bugs. Students impose the former on the latter, which they may initiate (by pressing 123) and terminate (Redlight) but not otherwise control. In Flatland, motions are open-ended and entirely under the agency and direction of the student, who may move, position, and 244 spin objects either through direct manipulation of their physical shapes or through a variety of indirect controls.

These diverse types of motion sustain diverse roles and functions of motion within SYL activities. An important purpose of motion in activities like Jump Along and Grouping is to invest some form of continuous dynamics to traditionally or fundamentally discrete mathematical representations and concepts (see Section 3). Continuity in turn endows changing mathematical forms with durable identities, and temporalizes abstract relations and propositions, thus literally animating mathematical representations. The central idea of jumping, in Jump Along, takes place in time and space: motion interprets 24 = 6 x 4 as six jumps (in time) of four units each (in space). Thus student discussion of systems in motion focus on the behavior of mathematical situations rather than on their properties or attributes. Secondarily, we find in the SYL project that motion frequently defines the narrative matrix of an activity's story or context (see Sinclair, 2005). The Grouping activity's bugs "feel like" bugs not because of the precision of their frozen Redlight arrangements but because of their ant-like perambulations when 123 wandering. Such story contexts themselves play multiple roles in an activity. As authors, we hope they motivate and sustain student engagement in problem spaces (but are aware of how often such contexts strike older students as sham devices, to be stripped and discarded as the first step of solving "real" problems). Perhaps more importantly, we see that story contexts provide a mechanism for teachers and students alike to transport discoveries, experiences, and conclusions formed within the technology activity milieu into other contexts and situations. Weeks after encountering the Grouping activity, a student struggling with division may not be able conveniently to return to Sketchpad. His teacher asks "remember what had to happen for the bugs to group up with nobody left out?" and resurrects, through story, the situated activity experience. Finally, we note that where direct motions and manipulations--dragging a vertex, rotating a polygon--provide the most immersive form of free geometric inquiry, activity situations in which motion is initiated and determined indirectly--for example, by pressing buttons--often leads students to patterns of more explicit conjecture formation and evaluation.

Our last observation concerns students' and teachers' instrumentalization of the software. An intellectually exhilarating ingredient of many users' first encounter with the dynamic geometry "behavior" of a simple construction--say, of a triangle and its centroid--is the shortness of the distance between one's perception of what one confronts, and the conviction that you could construct it yourself. (We see this shortness as a central feature of what Hoyles and Noss (2003) call "expressive technologies.") An initial concern in this project, then, was that the authoring techniques required to develop microworlds such as Grouping or Flatland were sufficiently advanced--in terms of both mathematical sophistication and Sketchpad technique--as to be indistinguishable from the advanced techniques required to program a dedicated Flash or Java-based applet of the sort that exist in great number on the web. The internal construction defining the dynamic behavior of Grouping or 245 Jump Along is not trivial; and a desirable degree of transparency--how does this work?--is lost. Happily, though, we find frequent evidence of the dynamic geometry environment's classroom remaining value even when used in conjunction with microworlds whose assembly techniques are ultimately opaque. Even where the mechanisms behind a microworld may appear as a black-box, the available and interesting mathematics emerging from that microworld, as well as the questions and techniques students bring to probing it, are not confined to such boxes. Since over the course of multiple activities, students become exposed to a general stable of Sketchpad techniques (e.g. changing a parameter, measuring a distance, coloring an existing shape or constructing a new one), over time these techniques develop into vertical skills that users re-apply even in contexts not intentionally structured for them. When a student--departing from an activity script--recolors one of the green bugs red, to trace its migration from group to group under different group size values, she expresses her acquisition of the software as an intellectual tool, rather than merely as an activity-delivery environment; and alters her relation to both the activity and its setting. A teacher responding to a spontaneous student question by measuring some quantity within an activity sketch similarly demonstrates vertical appropriation of the software. While novice users certainly remain distinguishable from Sketchpad experts, both exist in continuum and in flux, rather than in fixed camps of activity-producing developers and activity-consuming users. Teachers, students, and curriculum developers alike appear across the usage spectrum.

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On-line professional development for mathematics educators: Overcoming significant barriers to the modelling of reform-oriented pedagogy

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As noted in the ICMI 17 Discussion Document, much has changed since the original ICMI 1985 study. The advent of highspeed Internet and web-based technologies has in many ways revolutionized the educational project, touching all areas of research and practice. For example, on-line course offerings in continuing teacher education are rapidly becoming standard features for faculties of education involved with the profesional development of in-service teachers. However, instructors of mathematics education courses which are offered in a full-distance context must navigate certain formidable obstacles in the planning and delivery of their on-line learning experience. In an era of reform-oriented mathematics education (National Council of Teachers of Mathematics, 2000; Ontario Ministry of Education, 2005), which emphasizes the increased use of manipulatives, technology, groupwork, and communication, the "virtual" instructor must develop creative methods for modelling these important aspects of teaching and learning. Based on three years (eight courses) of instructor/course evaluation feedback and on the author's own observations, the following paper presents four key strategies for bridging this technological gap in the delivery of quality on-line professional development for mathematics educators. In addressing both professional development and distance education, this paper speaks to specific questions found in both the Teachers and Teaching (3) and Connectivity and Virtual Networks for Learning (7) themes, and approaches these in terms of digital technologies and the role of the teacher.

Introductory

On-line learning is quickly becoming a commonplace feature of the postsecondary education landscape (Varnhagen, Wilson, Krupa, Kasprzak, & Hunting, 2005). This reality is clearly, and in an ever-expanding manner, evidenced within pre-service and in-service teacher training programs across North America. Fulldistance education has both benefits and drawbacks for the candidate and for the online instructor. Mathematics education, by the very nature of the its content and delivery, brings with it a unique set of obstacles and opportunities when taught online, particularly from a reform-oriented perspective such as that encouraged by the National Council of Teachers of Mathematics (National Council of Teachers of Mathematics, 2000).

While teaching in an Additional Qualifications (AQ) on-line learning environment for the past three years at the University of Western Ontario in Ontario, Canada, I have attempted to develop several strategies that I believe, based on candidate feedback and my own observations as facilitator, enhance the overall 247 professional learning experience for mathematics educators who are enrolled in such teacher development courses. I would like to share with you four of the most effective strategies in this article.

Building Community with Class Profiles

Like in any regular classroom, the first few instructions given and interactions experienced within the on-line forum are of extreme importance in terms of setting the tone or "creating the ambience" for learning (Kimball, 1995, p. 55). Adult learners enrolled in my Honours Specialist Mathematics courses are first asked to "sign in" on-line within the "Aftermath Café" folder, sharing various details about their professional experiences and personal interests. After several days, I have found it very useful for both myself and the candidates to take the time to collate this data in two forms: a simple table or chart (see Table 1) with columns highlighting their location, school, courses taught, years teaching, and other miscellaneous information; and a geographical map (see Figure 1) upon which each individual is situated according to location and name. These files are shared with candidates on-line, and I ask them to provide me with feedback regarding any possible errors or omissions. As the second and sometimes third drafts are created and posted, I, as instructor, am already modelling collaboration, interest, and direct involvement within the course.

Table 1. Class Profile Chart of Professional and Personal Information

Imran Toronto, Ontario	AASS Mathematics/Business 2a	Worked in Business; MBA;	likes new technologies	
Jean-Jacques Montreal, Quebec	BBSS Mathematics/Computers 18a	Physics/Science background;	enjoys cycling and climbing	
Elizabeth Ottawa, Canada	CCSS Mathematics/Visual Arts 7a	Black belt in karate; loves to	sketch, paint, and integrate	

The table or chart provides the instructor with instant access to important facts which often become helpful throughout the course in terms of mentally "locating" an individual, asking good questions, and mindfully drawing upon candidate expertise as one extends or redirects on-line interaction. The map serves to actually situate learners within a visual context, allowing them to obtain a general "feel" for what the course looks like in terms of geographical representation, and rendering a sense of "place" in an otherwise distant or disconnected context. One obvious benefit of on-line learning is that it allows the learner to not only participate at her/his leisure, but also permits she/he from studying at great distances or while on the move. I've had candidates take my London-based mathematics education courses while teaching in China, working in western Canada, and touring Europe with a backpack and laptop. 248

Figure 1. Class Profile Map of Candidate Locations.

Because the instructor and course participants are often denied the visual element (i.e., as in onsite settings or teleconferencing situations) in much of distance education, this chart/map class profile is very much a part of building community and constructing individual "portraits" of each learner. Mounted near my monitor for the duration of the course, the class profile provides for quick identification regarding who is who, and who is where.

Engaging Minds with Rich Problems

The second strategy which I've found to be very effective is that of posting engaging mathematics problems throughout the course and asking candidates to delay sharing their ideas/solutions concerning these problems until a specified date, thereby allowing all participants more time to grapple with a given problem (see Figure 2 for sample problem and related instructions; this is a classic ratio problem rewritten using Tolkien's Lord of the Rings characters in order to provide a more interesting context).

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Problem: Strider and Boromir are sitting at a back table in a dark, smoky tavern of
Middle Earth known as the Prancing Pony. Strider has three loaves of bread; Boromir
has two. Just before they eat, however, in walks Bilbo Baggins, obviously famished,
and asks if he may share their meal. They agree, and each of the three characters eat
equal amounts of bread. Before departing, Bilbo pulls out five identical gold coins
and says, "Please accept these few coins as thanks." Strider and Boromir watch the
hobbit depart but then become puzzled as to how they should distribute the gift.
Unable to solve the riddle, they take it to Gandalf the Grey and ask him for advice.
As this represented a difficult "two-pipe" problem, they had to wait some time
outside the wizzard's dwelling. Finally, he emerged from the interior, and delivered
his recommendation.

Question: What was Gandalf's conclusion, and why was it the only fair and just way
to divide up the golden coins? Further, think of as many different ways (at least
three) to tackle this mathematical problem as possible. What role would the teacher
play in these? At what grade levels might this problem be appropriate, and where
might it be used in terms of curriculum?
N.B. Again, please wait until Monday July 18th before posting
solutions/ideas/comments, so that all of the hobbits will have a chance to think
through the problem. (N.B. If you are not a Tolkien fan, you can substitute other
names or leave the characters anonymous.)
Figure 2. Sample Engaging Problem, Question, and Response Posting
Instructions.
These problems and subsequent response posts are located in a separate
folder and are thereby kept organized and easily accessible. Four or five such
problems are selected for the course, one being featured for each of the on-line
modules which may last one or several weeks, depending on the course structure. Not
only does this process model the use of rich problems in mathematics classrooms, but
the multiple solutions that are often presented by candidates lead to important
pedagogical discussions surrounding issues of mathematical communication, teacher
questioning, consolidation of learning, and the encouragement of invented algorithms
and novel approaches to problem solving among students.
Creating Anticipation with Live Technology Tutorials
A third on-line teaching strategy is the use of scheduled chatline technology
tutorials in which candidates meet virtually, at scheduled dates/times to share
comments/questions/insights regarding the many software/website resources and
related explorations that I, as instructor, have posted earlier in the course. Like with
the math problems, I have found it beneficial to maintain a separate folder 250
specifically focussed on mathematics technology, in which I can post these various
resources and links. Candidates are asked to have relevant software "up and running"
and/or available during the virtual class sessions wherever possible, and to have
specific questions/comments prepared in advance. During the scheduled tutorials, the
instructor acts as facilitator, sometimes answering specific questions but more often
than not simply guiding the conversation and offering insights where possible.
As technology is now such a vital part of reform-oriented mathematics
education (Gadanidis, Gadanidis, & Schindler, 2003; National Council of Teachers
of Mathematics, 2000, pp. 24-27; Ontario Ministry of Education, 2005, pp. 27-28;
Richards, 2002; Sinclair, 2005), and since on-line learning prohibits the use of
technology software/hardware in an actual classroom, this teaching strategy serves to
address this important area of pedagogy within the distance education forum. I
currently have candidates examine five areas of technology (see Figure 3) with
related introductory activities (e.g., graphing calculators and spreadsheets; dynamic
geometry and data software; on-line learning objects [Interactive Applets] and virtual
manipulatives; Statistics Canada's E-Stat on-line learning resources; and various
websites related to the teaching and learning of mathematics).

Graphing Calculators (Texas Instruments): http://education.ti.com/us/product/graphing.html
Geometer's Sketchpad Dynamic Geometry Software (Key Curriculum Press): http://www.keypress.com/sketchpad/
Fathom Dynamic Data Software (Key Curriculum Press): http://www.keypress.com/fathom/
On-line Learning Objects and Virtual Manipulatives:
NCTM Illuminations (VA, USA): http://illuminations.nctm.org/
CAREO Repository (AB, Canada): http://www.careo.org/
CLOE Repository (ON, Canada): http://cloe.on.ca/
National Library of Virtual Manipulatives (UT, USA): http://matti.usu.edu/nlvm/nav/vlibrary.html
Statistics Canada E-Stat Educational Resource (ON, Canada): http://www.statcan.ca/english/Estat/licence.htm
Mathematics Education Reference Websites:
eWorkshop (ON, Canada): http://www.eworkshop.on.ca/
Leading Math Success (ON, Canada): http://www.edu.gov.on.ca/eng/document/reports/numeracy/index.html
Targeted Implementation and Planning Supports (ON, Canada): http://www.curriculum.org/occ/tips/index.shtml
Figure 3. Sample Web-Based Resources for On-line Technology Explorations.
The web-based resources are obviously widely available, and the Key
Curriculum Press software titles listed above (i.e., Sketchpad and Fathom) are
licensed for Ontario teachers under the Ontario Software Acquisition Program and
are also available to candidates as free downloadable evaluation copies. Therefore,
apart from the graphing calculators--most teachers taking Honour Specialist courses
either have purchased their own machine or can borrow one from their school for the
duration of the on-line course--all of these technology resources are readily
available, and so access is not an issue. 251
To date, I have only made these chatline tutorials optional/voluntary, in
terms of candidate participation, and they have therefore not been assessed as part of
the course evaluation. However, I could see incorporating these sessions as a formal
requirement, thereby encouraging all candidates to explore the posted activities
beforehand and to offer comments/questions during the virtual gatherings.
Depending on the configuration of the distance education system, it may be to the
instructor's advantage to divide up the participants into smaller groups based on
software/hardware topics and/or the most convenient gathering times for candidates.
For those not able to join the group on-line during selected dates/times, transcripts of
the sessions are converted to PDF and posted for everyone following the chatline
sessions. As one teacher shared following the tutorial: "I found the on-line chat the
most fun part of the course. It was quite engaging and I would encourage you to
include more live chat sessions."
Providing Specific and Ongoing Feedback with Candidate Assessment Files
Like all students, adult learners desire meaningful and ongoing feedback
throughout the on-line learning experience. I have found that the best way to respond
to this need in a consistent and appropriate manner is to construct what I've simply
labelled as "Candidate Assessment Files (CAFs)." Sent privately to each participant
via the on-line MailBox, all four instalments of this CAF provide candidates with
both general and individual comments regarding course progress and assignment
achievement, respectively. These CAFs are Word files with each candidate's name
typed on a title page with course information and colourful graphics. Inside, each
course module is briefly synopsised and then individual comments are made
regarding the modular assignments. Particularly effective, in terms of feedback, is the
copying and pasting of specific quotations made within the candidates' assignments,
accompanied by related instructor comments, questions, and/or suggested readings.
Even in the very first instalment of the CAF, all modules and assignments
are represented in blank outline form. Also included are rubrics for major
assignments (i.e., the cells of which are simply shaded to indicate achievement upon
assessment), a "Course in Review" page with all assignments listed and their
respective value (i.e., these left blank for the moment), and a copy of the university
grading scales (e.g., what characteristics constitute a mark of "A" or "B+"). This
complete CAF, although primarily blank at the point of first instalment, provides
each candidate with a clear framework for assessment in the on-line course. They can
be confident that there will not be any surprises in terms of how or when they will be
evaluated. By providing both general (i.e., course reading and on-line discussion
highlights) and individual (i.e., referencing specific assignment quotations and online participation tracking) comments, not only is the instructor more likely to be in
tune with all participants and the course in general, but candidates are reassured that
their instructor is carefully monitoring all that transpires. 252
Towards On-line Teaching Excellence
When I first started teaching on-line courses in the spring of 2003, I assumed
that they would be very limiting in terms of possible interaction with candidates and
my ability to deal, in any meaningful way, with the immensely important area of
technology within mathematics education. In looking back now, however, and
comparing my eight university courses taught on-line with the same number taught
onsite in faculty of education classrooms, I must honestly say that not only have I
enjoyed the on-line immensely, but that in developing and implementing the abovedetailed strategies, I have been much more "connected" to candidates and able to
closely monitor what and how they were thinking as a given course progressed.
Although I could not "see" them, in the end I felt that I actually "knew them" better.
By creating detailed class profiles, posting rich and engaging mathematics
problems, hosting virtual technology tutorials, and providing ongoing and
meaningful assessment files, I have grown as an instructor and feel that I've been
able to provide mathematics educators with quality on-line professional
development. On-line learning does have its inherent limitations, to be sure; yet the
opportunities it affords the "virtual" instructor of mathematics education, and the
participating candidates, are indeed worth investigating.
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Theoretical perspectives on the design
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Designing learning environments entails drawing on theoretical perspectives on
learning while, at the same time, being cognisant of the affordances and constraints
of the technology. This paper reflects on the design process through utilising
evidence from the design stage of the development of a dynamic visualisation
software environment called 3DMath. During the development of 3DMath, a dynamic
three-dimensional geometry microworld aimed at enabling learners to construct, observe
and manipulate geometrical figures in a 3D-like space, the key elements of visualisation
– covering mental images, external representations, and the processes and abilities of
visualisation- were taken into consideration. The aim of this paper is to illustrate how
the design of this particular software was informed by these elements of visualisation, as
well as by theories related to the philosophical basis of mathematical knowledge and by
semiotics. The paper illustrates how the features of software may be designed to take
account of relevant theoretical notions and to satisfy the characteristics of instructional
techniques that are appropriate to theoretical perspectives on learning.
Introduction
A prime aim of the 8
th
World Conference on Computers in Education, held in July
2005, in Stellenbosch, South Africa, was to develop and produce a "global vision for ICT
in education". The vision document produced at the conference, known as the
Stellenbosch Declaration (IFIP, 2005), sets challenges for all stakeholders in ICT in
education - teachers, practitioners, researchers, academics, managers, decision-makers
and policy-makers- all with a view to increasing the access to education for everyone
around the world.
While acknowledging that the development of well-designed ICT-based
educational material is growing, the Stellenbosch Declaration sets out a radical agenda
for research, specifying the need for the research community both to bridge the gap
between technology and pedagogy, and to ensure the development of solid theoretical
frameworks for software design and utilisation. This is because, the Declaration argues,

"in the field of ICT-supported learning, pedagogy and technology have often been treated separately: pedagogy often being based on what the technology appears to permit, rather than fully integrated as a basis for technological design" (emphasis added) and that "the possibility of relying on solid theoretical frameworks is one of the key factors that can 254 enable conception of the many positive experiences already taking place in order to reach the definition of reliable innovative reference models" (ibid, p4, emphasis added). The Declaration also insists that "the output of research should be made widely available, as open source, for improving practice, decision-making, and resources development".
This paper is offered as a modest contribution to meeting the challenges to research set out in the Stellenbosch Declaration. It does this by reporting on the theoretical perspectives underpinning the design of a dynamic visualisation software environment called 3DMath (Christou et al, 2006) aimed at enabling learners to construct, observe and manipulate geometrical figures in a 3D-like space.
Theoretical Perspectives on Design
Traditionally, three-dimensional geometry is taught using static pictures of geometric solids presented in textbooks. Students, however, are known to have difficulty reasoning from two-dimensional representations of three-dimensional objects (Raquel, 2002: Parzysz, 1988). Moreover, developing visualization skills is difficult in a traditional lesson environment using the standard chalkboard because representing 3D objects by a 2D sketch is complicated and time consuming. Such difficulties remain even when commonly available 2D dynamic software is used. For example, Dixon (1995) showed that 2D software can be effective in improving students' two dimensional visualisation but was not effective in improving students' three dimensional visualisation.
In an attempt to overcome these difficulties, and to take account of relevant theory, this paper reports on the design of a 3D software named 3DMath. The main objective of the software development project is to develop a dynamic three dimensional geometry microworld, which enables students to construct, observe and manipulate geometrical figures in 3D-like space on the computer screen. To meet these purposes, the design of the proposed software followed three major fields of educational theory:
(a) the constructivist perspective about learning which argues that learning is personally constructed and is achieved by designing and making artifacts that are personally meaningful (Kafai & Resnick, 1996),
(b) the semiotic perspective about mathematics as a meaning-making endeavour which argues that any single sign (e.g. icon, diagram, symbol) is an incomplete representation of the object or concept, and thus multiple representations of knowledge should be encouraged during learning (Yeh & Nason, 2004), and
(c) the fallibilist nature of mathematics which argues that mathematical knowledge is a construction of human beings and is subject to revision (Ernest, 1994).
In addition, the aim of developing the 3DMath software was to develop abilities and processes in students that are closely associated with the idea of visual imagery as a mental scheme depicting spatial information (Presmeg, 1986). It is generally accepted that learning 3D geometry is strongly associated with spatial and visual ability (Dreyfus, 1991) and that incorporating spatial visualisation and manipulation into learning activity could improve geometry learning (Tso and Liang, 2002). 255
Spatial ability has had many definitions in the literature. For example, Tartre (1990) defines spatial ability as the mental skills concerned with understanding, manipulating, reorganizing, or interpreting relationships visually, while Linn and Petersen (1985) defines it as the process of representing, transforming, generating, and recalling symbolic, non-linguistic information. Lohman (1988) proposes a three factor model for spatial ability, including "spatial visualization", "spatial orientation", and "spatial relations". "Spatial visualization" is the ability to comprehend imaginary movements in a three-dimensional space or the ability to manipulate objects in imagination. "Spatial orientation" is defined as a measure of one's ability to remain unconfused by the changes in the orientation of visual stimuli that requires only a mental rotation of configuration. "Spatial relation" is defined by the speed in manipulating simple visual patterns such as mental rotations and describes the ability to mentally rotate a spatial object fast and correctly.
The core visual abilities that should be taken into account in developing 3D dynamic geometry software could be said to be the following (following Gutiérrez, 1996):
(a) "Perceptual constancy", i.e., the ability to recognize that some properties of an object are independent of size, colour, texture, or position, and to remain unconfused when an object or picture is perceived in different orientations,
(b) "Mental rotation", the ability to produce dynamic mental images and to visualize a configuration in movement,
(c) "Perception of spatial positions", the ability to relate an object, picture, or mental image to oneself,
(d) "Perception of spatial relationships", the ability to relate several objects, pictures, and/or mental images to each other, or simultaneously to oneself, and
(e) "Visual discrimination", the ability to compare several objects, pictures, and/or mental images to identify similarities and differences among them.
In addition, given Yakimanskaya's (1991) claim that the creation of mental images is possible because of the accumulation of representations that serve as the starting point, 3D dynamic geometry software should aim to provide the learner with a variety and richness of spatial images.. The richer and more diverse the store of spatial representations, the easier is to use images in solving problems.
Design Principles for 3DMath
Based on the above theoretical perspectives, and the rich concept of visualisation noted above, the following guided the design and the construction of the 3DMath software:
(a) The software needs to allows students to see a geometric solid represented in several possible ways on the screen and to transform it, helping students to acquire and develop abilities of visualization in the context of 3D geometry.
(b) Given Gutiérrez's (1996) view that when a person handles a real threedimensional solid and rotates it, the rotations made with the hands can be so fast, 256 unconscious, and accurate that the person can hardly reflect on such actions, then the software could usefully place some small limits on the directions and speed of rotation, thus forcing students to devise strategies of movement and to anticipate the result of a given turn.
(c) The interface of the software needs to be intuitive and to provide an open and generative environment that enables learning to learn through making and designing personally meaningful artefacts. It also needs to employ rich semiotic resources that enable multiple perspectives and representations for mathematical meaning-making (for example, students need to be able to represent a solid in 3D, or its correspondence in 2D).
(d) The software needs to be designed to provide the means for students to focus on the mental images they create, and the processes and abilities of visualization they use to solve problems. Given that a mental image is any kind of cognitive representation of a mathematical concept by means of spatial elements, 3DMath needs to makes it straightforward for students to construct different solids and perceive them in a concrete or pictorial form. This is because the repetition of this process helps students to formulate a "picture in their mind's eyes" (Presmeg, 1986). In addition, the software needs enables students to see solids in many positions on the screen and consequently gain a rich experience that allows them to form richer mental images than from textbooks or other static resources.
(e) In terms of external representations, a visual representation means the manipulation of visual images and the transformation of one visual image into another (Bishop, 1980). The form of software developed by the 3DMath project aims to be rich in the ability to manipulate and transform solids - see Christou et al (2006) for an example of how the 3DMath software aids the user in distinguishing between a representation of a pyramid and an octahedron.
(f) Bishop (1980) identified two relevant processes of visualisation: interpreting figural information and the visual processing of abstract information; the translation of abstract relationships and non-figural data into visual terms, the manipulation and extrapolation of visual imagery, and the transformation of one visual image into another. The 3DMath software incorporates Bishop's ideas by focusing on the processes of observation, construction and exploration (see Christou et al, 2006, for examples) in that
- observation allows students to see and understand the third dimension by changing the spatial system of reference (axes), choosing perspective and displaying visual feedback on objects. The 3DMath software is being designed so that students can rotate a geometric figure in reference to the three axes and thence obtain a holistic view of it. The speed and the direction of the rotation are e controlled by the user of the software and the drawing style of the object can be in a solid colour view or in a transparent line view. Students can select, label and colour the edges and faces of the objects.
- construction entails providing users with the facilities to allows a dynamic construction of geometrical figures from elementary objects (points, lines, planes) and 257 construction primitives (intersection, parallel, etc.). Students can also construct geometrical figures by selecting the appropriate 2D figures and then forming the solids by dynamic animations.
- exploration allows students to explore and discover geometrical properties of the figure. This is the main procedure adopted in most of the teaching scenarios being designed to accompany the 3DMath software.
(g) The 3DMath software is being designed in such a way as to accommodate the development of the following visualisation abilities (see Gutiérrez, 1996): (a) the figure-ground perception, (b) perceptual constancy, (c) mental rotation, (d) perception of spatial positions, (e) perception of spatial relationships, and (f) visual discrimination.
The following features are that are thought to contribute to the development of the abilities are being integrates into the software:
- the dragging capability of the software is being designed to enable students to rotate, move and resize 3D objects. Rotation can be executed in all directions by controlling a rotation cursor and determining the speed of the rotation. In addition, students can resize proportionally all the dimensions of the object or resize it only in one dimension, according to the requirements of the problem.
- tracing is a particular instance of the interface where only parts of the figure are displayed. The intended purpose of this feature is to provide the learners with a way to perform a visual filtering of the main construction represented on the screen, i.e., to allow them to extract and observe parts of the construction in an independent view.
- as in 2D dynamic geometry software, students can carry out useful measurements, in the case of 3DMath measures of the length of edges, the area of faces, and the volume of a solid. All measurements are dynamic as solids are resized by dragging. The dynamic characteristic of the measurement facility allows the exploration of properties within and among figures e.g. users can measure the volume of a cone and then double its height and see how its volume is altered.
- a textual feature, which represents the declarative description of the figure, provides the learners with a textual and chronological list of all the geometrical objects involved in the construction of the figure. Additionally, the History file can be used as input to the system. For example, a History file created by one student in one country can be used by another student in another country to reconstruct (or re-use) the same model. Using this feature, it would be possible to construct not only Interactive models, but also Declarative models (by importing History files) and Interactive Programming models.
- a diagrammatic feature provides a representation of the structural dependency graph of the figure.
(h) Other features being developed include:
- the ability to export constructions as images (BMP, JPEG, etc), or in other rendering format (PS, XML, etc). This should help teachers to create supporting educational materials, preparing reports, printed material, etc. 258
- the locking/unlocking of features (primitives), making them hidden from view. For example, the primitive to find the distance from a point to a plane might be initially locked (or hidden). To find that distance, students must solve the problem by making appropriate constructions. Once they do this correctly, the primitive may be unlocked (or made visible) so that it can be used freely in further constructions.
Concluding Comments
As illustrated by this paper, the design of 3DMath is informed by theories based on philosophy of mathematical knowledge, such as constructivism, and by semiotics. The main purpose of 3DMath is to enhance students' understanding of 3D geometry with an emphasis on visualisation. Thus, during the developmental process the key elements of visualization, as defined by Presmeg (1986), Bishop (1980), Clements (1982) and Gutiérrez (1996) (mental images, external representations, processes, and abilities of visualization), are carefully taken into consideration.
In developing the 3DMath software we are seeking to bridge the gap between technology and pedagogy, and develop solid theoretical frameworks that inform the software design. This is so that the pedagogy is fully integrated as a basis for technological design, rather than the pedagogy, as is often the case, being based on what the technology appears to permit a The output of this research, as the IFIP (2005) declaration recommends, is to be made widely available, as open source, for improving practice, decision-making, and resources development in mathematics education.
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Designing for diversity through web-based layered learning:
a prototype space travel games construction kit

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We present a Space Travel Games Construction Kit designed to enable students to learn while building a computer game. Games are built in the context of a metagame that provides motivation, structure, guidance, and background in game making. A key design decision was to encourage layered learning through specially-designed program fragments, which the students could customise and assemble. We present some scenarios and report on the results of testing a version of the kit with students, which suggest that designing levels in the game is an avenue for further development.

Introduction and design principles

In this paper, we discuss the hypothesis that learning of a subject domain can be deep and richly interconnected if it involves exploration within a computer game which combines designing and building as well as game playing. Additionally, we will argue that this approach to learning could potentially widen access to the possibilities of digital technologies, given that it allows diverse layers of engagement and exploits the interconnectivity available on the web.

Our prior research concerning computer games (Noss & Hoyles, in press) has indicated that games designed for learning authored by students tend not to suffer the difficulty noted with educational games more generally around poor production values. In the case of constructed games, even if they are relatively simple and crude, the students become dedicated to their creations. However acquiring the skills to make computer games requires a major investment in time and effort. Here, we present some novel design research that will illustrate an approach that is designed to reap the advantages of learning by programming games without the disadvantages of having to know a priori how to program.

Our starting point derives from the principle that the most powerful way for students to learn mathematics is through their long-term engagement in collaborative projects for which they take responsibility individually and collectively (see, for example, Harel & Papert, 1991). From this basis we have developed the following principles for designing for this learning:

sequenced activities that can engage students at a variety of levels in what we name layered learning;

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We acknowledge the funding of the BBC, and the helpful comments of colleagues in the London Knowledge Lab, notably of Gordon Simpson and Diana Laurillard 261

flexible tools that have adjustable parameters, can be combined in different way and can be also be programmed, and thus allow students to investigate each activity for themselves

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collaborative interactions as part of the activity sequence through which students discuss their emerging ideas in the context of their game design and finally multi-player game playing either at one computer or over the web.

The second design requirement is worthy of particular attention, as it implies that designers can both empower and constrain students by offering a set of components or modules that can be customised to suit diverse students' goals, as well as tuned to the knowledge domain. Programmable modules have the added advantage of providing a consistent way to combine and modify tools, in the control of the learner. There have been numerous attempts to design a programming-based approach to learning over the years. The most successful have achieved tangible learning outcomes across various topics (music, mathematics, language, physics (some of these are discussed in Noss & Hoyles, 1996). They have also provided important pointers to the possibilities of learning that transcends the procedural and superficial by encouraging a playful – yet mindful – spirit of enquiry on the part of learners, aiming to break down the curricular silos that so often characterises traditional schooling. For the most part, what has been missing has been generalised success in tapping into students' own interests on a wide scale and engaging them in debate, investigation and production. To achieve this aim, we have built and tested a space travel construction kit along with a narrative 'metagame' to assist learners in navigating their way through the sequence of activities of design and game playing.

Description of the game environment

The Space Travel Games Construction Kit

We began by considering the classic computer game of Lunar Lander. In the process of trying to land upon the moon, a player engages with the laws of motion, playing in a virtual world where the laws of gravity and momentum are not obscured by friction or atmospheric drag as they are on Earth. We have built and tested a Space Travel Games Construction Kit (using Imagine Logo, Blaho A., Kalas I., 2001) that can be used to build a variety of games similar to Lunar Lander. The construction kit provides small program fragments together with tools for customising and composing them. Thus the software extensions allow layered exploration appropriate for populations of students with differing backgrounds and ages; that is, users can choose how far to delve into the workings of the tools.

The innovative aspects of the software include:

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In the current research aiming to involve groups spread geographically it was important that the tools could be accessed through a web browser - although this is not a 'principle'. 262

1. A child-friendly interface for the composition and parameterisation of pre-built program fragments;
2. An underlying computation model that has been simplified and made composable by building upon multiple independent processes. This is critical for the deeper levels of engagement where students inspect and edit program code;
3. An underlying physics model based upon conservation of momentum that is simpler than the one commonly used based upon first and second derivatives of position and the first derivative of momentum. Yet it is capable of modelling the same phenomena;
4. The concept of a "metagame" where games are made within an overarching narrative game structure.

The narrative metagame

The metagame starts with the player being hired as a game developer at a game company. The player receives help from a team of simulated experts including a programmer, a scientist, a historian, an assistant game designer, and an animator. A player may skip over all this and jump into game making, but the metagame provides structure, background information, guidance, and a gradual introduction of new features and capabilities.

The metagame embodies the design of a learning sequence. The learners are presented with a goal and need to interact with their virtual teammates in order to acquire both the needed components and the knowledge to proceed. The response of each teammate to a visit by the player is scripted but also depends upon both the current state of the game being constructed by the player and the history of the player's interactions with all the teammates. This gives the player freedom to visit the teammates in any order and with any frequency. Furthermore, each game component has an associated help button. When a component's help button is pressed, the player is informed which teammates have something to say about it. For example, the programmer, the scientist, the historian, and the game designer all have a unique perspective when giving an explanation of the component which implements gravity.

The Activity Sequence

Phase 1: exploration of game making

Moving in outer space: An astronaut is adrift and needs to reach her space ship.

Build a game by acquiring program fragments and artwork, which can be accomplished using only components involving horizontal motion that is achieved by 'throwing' rocks (previously collected by the astronaut).

Agreeing some constraints: for example, will the astronaut get back safely? Is she going too fast when she hits the spaceship? 263

Creating a new game: Now invent a new game. For example, the space ship has started to move off in a vertical direction. Can the astronaut reach it now? The final challenge in this sequence involves making the game into a cooperative game between two players over a network.

Phase 2. Building and playing a Lunar Lander game

To proceed to this phase, all the required game-making tasks in Phase 1 need to be performed. Here gravity is introduced.

Landing on the Moon: for example, what speed constitutes a safe landing? At this point we introduce dynamic configurable gauges to measure speed, acceleration, mass etc which monitor and graph these values and can help in the task of landing safely on the moon. Added challenges are introduced: do not use too much fuel.

Using the Autopilot: The settings of any 'manual' landing that is all the variables and how they are changed can be captured by an autopilot. The next challenge is to construct the best autopilot program, by recording what is deemed a good landing and then tweaking the parameters of the recorded landing to produce an optimal landing.

Multiplayer game over the web: Players can compete for example to land with the safest landing speed using the least amount of time or fuel. Students are also challenged to invent new games to play.

The potential layers of learning

Both the metagame and the construction kit were designed to support layered learning. At the first layer, engagement is mainly through reading, watching and making conjectures based on observation and for example describing a motion and reflecting on it. The tools for this layer are basic, perhaps only a handful to control a simulation on video, or the timing of movement. At a second layer, the learner can begin to manipulate motion and predict and test out the effects of different values. At a third layer, the learner might explore further how variables relate to each other, for example position, velocity and acceleration, by reference to the values set by sliders. And finally, a fourth layer that engages with these relationships either by modifying existing programming code or by writing new programs or fragments of programs.

Game construction: the interface

When the player is ready to build the game, a game panel with images of the astronaut and lander is presented (see Figure 1). Beside it is a control panel with a start button. Pushing the start button initially does nothing, since none of the game elements have been given programs. The control panel also has a button that causes the behaviour gadgets panel to appear (Figure 2). It contains behaviour gadgets that consist of one or more code boxes. A picture can be given a behaviour by placing a behaviour gadget on its back. The behaviour can be altered by setting sliders on the gadget's settings page. The code boxes of a behaviour gadget can be removed, whereupon they expand to display the code that implements the behaviour. Portions 264 of the code that can safely be edited without programming expertise are colour highlighted.

Figure 1. The initial game construction page

Figure 2. Behaviour gadgets can be dragged from the Behaviour Gadgets Panel. Initially, the behaviour gadgets panel contains only a horizontal velocity gadget and a horizontal rock throwing gadget. As the metagame progresses, the behaviour gadgets panel acquires more elements.

The total mass of rocks (i.e., total fuel), the largest rock (the maximum rate of fuel

usage), and the rock velocity (the propellant velocity) can all be adjusted by moving sliders. As one does so, the system makes calculations to show derived values such as force and to perform unit conversions. These parameters reflect real engineering tradeoffs. For example, adding more rocks/fuel does increases the duration of manoeuvrability but at the cost of a greater total mass and hence a smaller acceleration from identical rock throws.
Limitations of space prevent us from illustrating a range of further panels, behaviours and other objects that control instrumentation (for example, gauges that monitor any of 13 values in graphical or numerical displays, including velocity, acceleration, remaining fuel, total mass, the application of thrust (by throwing rocks out in the opposite direction), autopilot facility (a recording of all the changes to thrusters made manually), and a two-player version of the Lunar Lander game typically involving a race to be the first to land safely on the moon.
Figure 3 - A snapshot of game play with three active gauges.
Alternative epistemologies underlying design
When designing a toolkit to be used to construct scientific models (and games based upon these models) one needs to determine the underlying ontology of the system. How should time be modelled? How should concurrent processes behave? What are 265 the primitive notions of motion and space and which concepts are to be constructed from those primitives? Are the usual ways of conceptualising the laws of motion that are based upon algebra and calculus optimal for computational modelling?
Here we simply discuss the epistemological question is how to model forces. Should forces define acceleration, which in turn defines velocity, which defines position? A sequential program that models all the processes could be built this way. But it would lack the modularity and composibility of the concurrent processes that we rely upon, and although this provides a coherent epistemological perspective, it may not be optimal for learning. Consider the difficulties that would arise if one process implemented gravity by setting the acceleration to the appropriate value, while a thruster process implemented force by adding to or subtracting from the current acceleration. Clearly, the order in which the gravity process and the thruster processes run will drastically affect the model.
Instead, we chose to build upon the conservation of momentum, rather than F = ma. One reason for this is that it provides a concrete and discrete way to think about forces. It is likely to be more accessible than the alternative, which relies upon a notion of continuous rates of change. It also makes the mechanism underlying rocket thrust transparent. Throwing a one kilogram rock once per second is the same mechanism as real rocket thrusters that "throw" a trillion trillion molecules ("rocks") per second. Force is the derivative of momentum and as such is a more complex and difficult notion than the discrete change in momentum that we build upon.
Some illustrations of learning
We now very briefly outline some of the learning issues that are emerging from the iterative design/test cycle with three groups of students: two drawn from a large, urban comprehensive school (one "Year 7" class aged 11-12; one "Year 8" class, aged 12-13) and a small group of 3 students (aged 12-14) from a second school in an after-school setting.
Developing understandings of Newton's third law: In the first task the Year 7 students began with a relatively low knowledge of the physics concepts involved. For example, R suggested, "you could throw some rocks away and that would make her lighter so she would move." They were unaware that the effect of gravity in space is negligible (in the terms in which the software was devised). Through the course of the activity and experimentation with the horizontal rock thrower they appeared to develop an understanding that throwing a rock would develop an opposite movement proportional in velocity. Indeed, later in the session two of the students worked with the theory that "throwing larger rocks makes her move faster."
Minimising time (of astronaut to spaceship, or lander to moon) proved a motivating task, particularly for the Year 7 students. Most students used an iterative strategy e.g. Tom and Alex were delighted to refine their strategy again and again by optimising the use of the horizontal rock thrower against the speed of reaching the spaceship. 266
Using the gauges: Throughout the sequence much use was made of the gauges and interpretation of their output. Students found this relatively easy to put in place and tended to refer to them constantly as a guide to their use of the rock thrower or thrust. When landing on the moon some students applied far too much thrust causing the lander to move upwards and disappear off the screen. Reading the vertical velocity gauge which they had set up for the lander they predicted how the lander would "keep on getting slower until zero. Then it will fall back again because of gravity." In the 2 player game the ability to attach gauges to the opponent's lander was a particularly successful feature, enabling one group to make a close comparison with the other and to adjust the strategy second by second.
Composing horizontal and vertical velocities: Coming up with the hypothesis that to achieve diagonal movement a combination of horizontal and vertical thrusts would be needed, appeared almost effortless and was tested by, for example, using both horizontal and vertical rock throwers to the astronaut and using both simultaneously.
Gravity: The Year 7 students did not immediately make a connection between the rock throwing astronaut and the rock throwers for the lander, although Year 8 needed no prompting. The Year 7s also only had a more sketchy concept of gravity. Only two - Tess and Alex - volunteered that the lander game would be different from the astronaut in that there would be a gravitational pull (Alex) near the surface of the moon.
Collaboration, competition and motivation: Beating previous best scores proved highly motivating, especially for the team of Year 7 boys. Collaboration centred around agreeing what the two teams should have in common; the total mass of the projectiles, an agreed safe landing action, a value for gravity and the vertical starting position of their landers, for which they sought and found a new gauge, previously not used.
Attempts to minimise fuel use became more sophisticated. The boys realised that with their agreed safe landing speed of 30 metres per second, they needed only to keep just below this figure to ensure a safe landing and minimal fuel consumption. Before this they had been trying to reduce the velocity to the minimum regardless of fuel use.
In summary, the competitive element of the two-player version was an enormous motivation to the students: they loved seeing the opposition's ship on their screen and being able to monitor its progress through gauges. The students became wildly excited during landings.
Reflections on design
Overall the software did allow access to diverse students at many layers of learning. It stimulated huge interest and discussion and students used quite sophisticated ideas in pursuit of their game making and playing. The students evaluated the sessions and were all positive: but came up with many ideas for improvement. For example, they 267 suggested the need to reduce the amount of reading required in the initial stages and simplify some terminology that was too complex in places. The students suggested that a choice should be offered between reading and listening to instructions and that more complex instructions might be communicated through demo buttons or tutorials - for example showing them how to set up a gauge or to use a behaviour gadget.
Perhaps the most interesting suggestion was that the software might be structured into what they described as "levels", such as those commonly found in computer games.
Where the lunar lander software failed to meet the initial expectations was in giving students easy access to the programming code and in creating situations where they would want to analyse and adjust that code. It would seem possible that if the software were remodelled into a series of levels - corresponding in some way to the previously defined expected layered learning model - analysis and use of the relationships inspectable in the programming code or as recorded by the autopilot could become not just a real possibility but an integral part of the game.
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IT in Singapore Mathematics Education
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Introduction
The Singapore's second Masterplan for IT in Education (2002 - 2008), following the first masterplan
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, provides the overall direction on how schools can harness the possibilities offered by IT for learning. For mathematics education, the emphasis is increasingly on using IT to enable and support the teaching and learning. To achieve this would require curriculum and pedagogical changes, the professional development of teachers, and a paradigm shift in teaching and learning.
Integration of IT and Pedagogy
The Singapore school mathematics framework has 5 inter-related components, namely, Concepts, Skills, Processes, Attitudes and Metacognition. (As shown)
The framework was developed in 1990, and has been updated in 2000 and 2005.The new emphases include "Reasoning, communication and connections" and "Application and modelling". IT can play an important role in supporting the development of these.

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The first masterplan (1997 - 2002) has created an IT-enriched learning environment. All schools have an IT infrastructure with a good range of learning resources. Teachers generally have acquired basic proficiency in IT knowledge and skills.
Monitoring of one's own thinking
Self-regulation of learning
Beliefs
Interest
Appreciation
Confidence
Perseverance
Reasoning, communication and connections
Thinking skills and heuristics
Application and modelling
Numerical calculation
Algebraic manipulation
Spatial visualisation
Data analysis
Measurement
Use of mathematical tools
Estimation
Numerical
Algebraic
Geometrical
Statistical
Probabilistic269
A Pedagogy-Driven Model
The development of IT-supported learning resources involves 3 aspects, namely, pedagogy, technology and resources. A successful development would depend much on an effective integration of IT and pedagogy. We would like to propose the following pedagogy-driven model for the development of learning resources for mathematics:
The model involves the development of IT tools to support certain pre-determined pedagogical approaches. Students will use the IT tools to help them understand mathematics concepts and processes. The tools, in turn, will allow students to construct mathematics concepts and models, and to share and discuss their constructions, and the system will provide feedback to facilitate the learning process. Thus students will be actively engaged in an interactive and collaborative learning environment.
Conclusion
We are constantly improving the way we teach mathematics, paying greater attention to the processes of learning mathematics. For example, the Concrete-PictorialAbstract approach
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, grounded in Bruner's theory of constructivism
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and Vygotsky's
theory of zone of proximal development
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, has proven to work in our Primary Mathematics curriculum. This approach is now extended to learning algebra, for the development of conceptual understanding through carefully planned, developmentally- and age-appropriate strategies. With IT support and enhancement, students will develop not only deep understanding of algebra concepts and processes, but also the skills of reasoning and communication, which are required by our school mathematics framework. The aim is to prepare students to meet the challenges of the 21
st
century.

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This approach was developed by MOE in the 1980s, and has become the basis of our mathematics education. It was inspired by Bruner's classical book: The Process of Education (1960) in which he presents three levels of representation of knowledge: enactive, iconic and symbolic representations.
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Bruner defines constructivist learning as an active process in which learners construct new ideas or concepts based upon their current and past knowledge.
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Vygotsky defines the zone of proximal development as the distance between the actual developmental level for independent problem solving and the level of potential development with adult guidance or with more capable peers.
Pedagogy

(Teaching and learning
approaches)
Technology
(IT tools)
Learning Resources
(Online modules)
Development 270
The discrete continuous interplay. Will the last straw break the camel's back?
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We investigate the influence of the technology and in particular the influence of the discrete continuous interplay, which can be demonstrated by the technology, in enhancing students' mathematical thinking. We analyze how students' awareness of the limitation of discrete numerical methods, combined with error analysis, lead to a better understanding of the continuous methods. We identify the new potential offered by the instrumented work, the way students are influenced by their interaction with the Computer Algebra System and the presence of mental images created by this interaction, even when the computer is turned off. We also identify the inability of some students to differentiate between error due to mathematical meanings and error due to meanings specific to the "instrument". Our intention is to employ the possibilities offered by the technology, to elaborate activities based on the discretecontinuous interplay and to investigate their influence on students' thinking processes in relation to the notion of limit in the derivative concept.
Introduction
The cognitive difficulties that accompany the learning of concepts that relate to the continuous such as limit and derivative are well known. Our empirical approach leads us to consider the interplay between the continuous and the discrete, and to examine how to use it to help students enhance their conceptual understanding of these central notions. The discrete continuous interplay is not new. It existed before the computer age. The founders of the mathematical theory developed numerical discrete approaches to better understand dynamic continuous processes. In the last decade, the use of technology, especially the Computer Algebra Systems (CAS), offers a new mean in the effort to overcome some of the conceptual difficulties. We focus on the "instrumentation process" (i.e. how the tool becomes an effective instrument of mathematical thinking for the learner) in analyzing students' reactions in the context of activities based on the complementary aspect of discrete and continuous approaches.
The instrumentation theory
The instrumental approach is a specific approach built upon the instrumentation theory developed by Verillon and Rabardel (1995) in cognitive ergonomics and the

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anthropological theory developed by Chevallard (1992). The term 'instrumentation' is explained in Artigue (2002): The "instrument" is differentiated from the object, material or symbolic, on which it is based and for which the term "artifact" is used. An instrument is a mixed entity, in part an artifact, and in part cognitive schemes which make it an instrument. For a given individual, the artifact becomes an instrument through a process, called instrumental genesis. This process leads to the development or appropriation of schemes of instrumented action that progressively take shape as techniques that permit an effective response to given tasks.
The instrumentation theory focuses on the mathematical needs for instrumentation, on the status of instrumented techniques as well as on the unexpected complexity of instrumental genesis (Artigue (2002), Guin and Trouche (1999), Lagrange (2000)). We can take advantage of the new potentials offered by the instrumented work, for example, by means of discretization processes. Artigue (2002) warns us that the learner needs more specific knowledge about the way the artifact implements these discretization processes. Thus, it is important to be aware of the complexity of the instrumentation process. Working with a CAS introduces the learner to a system of "double reference" (Lagrange, 2000): on the one hand, he is introduced to mathematical meanings; on the other hand, he is introduced to meanings that are specific to the constraints of the instrument. Being aware of the limitation of the instrument might be helpful. Steiner and Dana-Picard (2004) demonstrate how to make advantage of the analysis of the error due to the limitation of the CAS to help students understand the theory of integration.
As a background to the present study, we present some cognitive difficulties that accompany the understanding of the limit concept.
Conceptualization of the continuous
In previous studies concerning the way students conceived real numbers, Kidron & Vinner (1983) observed that the infinite decimal is conceived as one of its finite approximation " three digits after the decimal point are sufficient, otherwise it is not practical" or as a dynamic creature which is in an unending process- a potentially infinite process: in each next stage we improve the precision with one more digit after the decimal point. This is not in accord with the mathematical view as expressed by Courant (1937). Courant wrote that if the concept of limit yielded nothing more than the recognition that certain known numbers can be approximated to as closely as we like by certain sequences of other known numbers, we should have gained very little from it. The fruitfulness of the concept of limit in analysis rests essentially on the fact that limits of sequences of known numbers provide a means of dealing with other numbers which are not directly known or expressible. Thus, the limit concept should lead to a new entity and not just to one more digit after the decimal point.
The derivative function is defined as
h
f x h f x
f x
h
( ) ( )
( ) lim
0
+ !
=
&
. By means of
animation and elementary programming in a Computer Algebra System, students can 272
visualize the process of
h
f(x + h) ! f(x)
approaching f (x) for decreasing h. The
dynamic picture might reinforce the misconception that one can replace the limit by
x
y
'
.
for 'x very small. How small? If we choose 'x =0.016 instead of 0.017, what
will be the difference? There is a belief that gradual causes have gradual effects and
that small changes in a cause should produce small changes in its effect (Stewart,
2001). This belief might explain the misconception that a change of, say, 0.001 in
'x will not produce a big change in its effect.
The discrete continuous interplay
We were interested in a counterexample that will demonstrate that one cannot replace
the limit "
x
y
x'
.
' &0
lim " by
x
y
'
.
.
for 'x very small and that omitting the limit will
change significantly the nature of the concept. The counterexample was found in the
field of dynamical systems. A dynamical system is any process that evolves in time.
The mathematical model is a differential equation dy/dt = y' = f(t,y) and we
encounter again the derivative
t
y
y
t'
.
=
' &0
' lim . In a dynamical process that changes with
time, time is a continuous variable. Using a numerical method to solve the differential
equation, there is a discretization of the variable time and the passage to a discrete
time model might totally change the nature of the solution. In the following
counterexample (the logistic equation), the analytical solution obtained by means of
continuous calculus is totally different from the numerical solution obtained by
means of discrete numerical methods. Moreover, using the analytical solution, the
students use the concept of the derivative
x
y
x'
.
' &0
lim . Using the discrete approximation
by means of the numerical method the students use
x
y
'
.
for small 'x . We will see
that the two solutions, the analytical and the numerical, are totally different.
The design of the learning experiment
The learning experience is described in a pilot study (Kidron, 2003). We report it
here for the convenience of the reader. The students (first year College students in a
differential equations' course) were given the following task: a point (t y,) 0 0
and the
derivative of the function dy/dt=f(t,y) are given. Plot the function y(t). The students
were asked to find the next point (t y,) 1 1
by means of (y y)/(t t) f(t y,) 1
! 0 1
! 0
= 0 0
. As t
increases by the small constant step t t □t 1
! 0
= , the students realized that they are
moving along the tangent line in the direction of the slope f(t y,) 0 0
. The students
generalized and wrote the algorithm: y □t f(t ,) n 1 n n n
y = + y +
for Euler's method. They
were asked how to better approach the solution. They proposed to choose a smaller
step □t . 273
The logistic equation dy/dt = r y(t) (1-y(t)), y(0) = 0
y was introduced as a model for
the dynamics of the growth of a population. An analytical solution exists for all
values of the parameter r. The numerical solution is totally different for different
values of □t as we can see in the graphical representations of the Euler's numerical
solution of the logistic equation with r = 18 and y(0)=1.3.
Fig. 1
In the first plot, the solution tends to 1 and looks like the analytical solution. In the
second, third and fourth plot, the process becomes a periodic oscillation between two,
four and eight levels. In the fourth plot, we did not join the points, in order that this
period doubling will be clearer. In the fifth and sixth plot, the logistic mapping
becomes chaotic. We slightly decrease □t in the seventh plot. For the first 40
iterations, the logistic map appears chaotic. Then, period 3 appears. As we increase
□t very gradually we get, in the eight plot, period 6 and, in the ninth plot period 12
and the belief that gradual causes have gradual effects is false! The fact that a
small change in a parameter causes only a small effect, does not necessarily imply
that a further small change in the parameter will cause only a further small change in
the effect. We knew this long ago. We just did not realize there was a mathematical
consequence. Take, for example, the proverb: the last straw breaks the camel back
(Stewart, 2001)
Findings and Discussion
First year college students in an innovative differential equations' course (N=60),
were the participants in the research. The first author taught the course. The students
interact with the Mathematica software in the exercise lessons which were held in the
PC laboratories. Mathematica was also used during the lectures for demonstrations.
We examined the students' reactions when they realized that the approximate
solution to the logistic equation by means of discrete numerical methods is so
different from the analytical solution. The students were given written questionnaires 274
before and after being exposed to the logistic equation. Some of the students were
also interviewed and invited to explain their answers. We present the analysis of the
students' reactions in the light of the instrumentation theory.
Awareness of the limitations of the numerical method
Before being introduced to the logistic equation, the students were asked if a very
small value for the step size □t in Euler's method will assure a good approximation to

the solution: 80% of the students claimed that a small value for □h might be not small enough. They connect their claim to the limitations of the numerical method. Some students also pointed out the fact that the error in the numerical method accumulates.
The influence of previous experiences in the PC lab on the students thinking processes
The students were influenced by previous experiences in the PC lab even if these experiences took place in other courses.
Irit: We worked on several examples in which we noticed that the more points, the smaller □h and the better the approximation BUT I think that not every function will behave this way. We encountered in the lab, in relation to another subject, a function with a special behavior: In spite of the fact that we added interpolation points, the function did not behave the way we expected. I think it was
( )
2
1
1
( )
x
f x
+
= . Usually,
the more points, the better is the approximation but it is not always the case.
Irit referred to Runge's example. The students worked in the lab looking for polynomial approximations to the function with equidistant interpolation points.
The belief that gradual causes have gradual effects
Before being exposed to the logistic equation, the students were asked to express their opinion about the following statement:" If in Euler's method, using a step size □h = 0.017 we get a solution very far from the real solution, then a step size □h = 0.016 will not produce a big improvement, maybe some digits after the decimal point and no more". The belief that gradual causes have gradual effects was expressed in 53% of the students' answers "it seems to me that if with □h = 0.017 we didn't get a good solution, then □h = 0.016 will not produce a big improvement either". 31% of the students who claimed that gradual causes have no necessarily gradual effects explained their answer by the fact that "There are functions which oscillate very quickly" or by means of the accumulating effect 275
"I think that we use an iterative procedure to find yk+1
, namely, we perform the
algorithm on yk
in order to find yk+1
and so on… After several iterations, we
accumulate differences in the value of yk+1
that might be significant therefore we
might get a big improvement even with a slightly smaller □h ".
The (in)ability to differentiate between error due to mathematical meaning and error due to the instrument
After being exposed to the logistic equation, the students were asked to characterize the source of error in Euler's method. We investigate whether the students realize that the source of error is the fact that in the numerical method the limit has been omitted in the definition of the derivative.
• The students' attention might be distracted by the round off error
The students' attention might be distracted by the round-off error especially if in previous experience with the computer they encountered such kind of round off error. This happened to a student, Hadas, which atributed the error to the round-off effect
Hadas: I remember from an exercise in the Calculus course that the solution with Matlab was 0 but the solution using the symbolic form was 0.5. When we tried to understand why this happened we realized that MatLab computes only 15 digits after the decimal point.
Hadas refered to an episode in the Calculus course in which the students were given the function
6 12
f(x) = -1 (cos(x))/x and they had to explain why some graphs of f might
give false information about lim f(x) xß0
. The limit is " but both Mathematica and
MatLab give the answer 0 when we evaluate the function for x = 0.01. Working the exercise in the PC lab, the students understood that the computer with its limited precision gives the incorrect result that -1 cos(x)
6
is 0 for even moderately small values
of x.
By means of error analysis, we planned to help the students to better understand the continuous methods and the concept of limit. But, working with a CAS, there are other unexpected effects that are directly linked with the "instrument" and the way it influences the students' thinking. In addition to the error due to the discretization process, to the fact that an algorithm that belongs to a numerical method is used to solve the logistic equation in place of the analytical method, there are other sources of error that are directly related to the "tool".
• The students' attention might be distracted by the accumulative effect
22% of the students explained the error by means of the accumulative effect.
• Round off and accumulating effect could not be the only source of error but the students cannot find the exact source of error
This view was expressed in answers like the following: 276
"The software has some limitation concerning the number of digits after the decimal point and it seems that this fact has enough strength (due to the accumulative effect) to influence the solution with Euler's method. But this fact by itself could not cause the crazy behavior of the function" or:
"The round-off is not the crucial part. There is a change in the analytical behavior. We do expect for a change due to the round off, but we expect to a change "in numbers" not in the qualitative behavior".
• The students relate the error to discrete - continuous considerations, but without a mention of the limit or of the formal definition of the derivative.
23% of the answers expressed well developed qualitative approach to differential equations, adequate to explain why there is an error, but inadequate to give a formal account how the discrete method employed the derivative concept.
• The students relate the error to the fact that in the numerical solution the limit is omitted in the definition of the derivative.
This was expressed in 19% of the answers. Some students reached this conclusion by means of mental images created by previous experiences with the instrument. These images were also present when the computer was turned off. This is expressed in the following:
"We have worked this week an exercise that demonstrates that a small change in the initial condition of a differential equation might cause a large change in the solution. Maybe the small error made in the Euler's method induced big changes in the graph of the solution curve also in our case" or:
"A difference of 0.001 might be crucial if it leads to the crossing of an equilibrium solution and therefore to a transition to a zone with totally different slopes like in the example.."
The student related to a figure describing a previous experience in the lab
"In the absence of an equilibrium solution, the error would have increased but there would not have been crucial changes. The error increased but we noticed it only because the equilibrium solution". In connection to the metaphor of the straw and the camel's back, the student added: if there was no equilibrium line, the camel would not have fall down!
Concluding remarks
The emphasis in this research study is laid on the way we take advantage of the discrete continuous interplay to identify the new potentials offered by instrumented work but also on identifying the constraints induced by the instrument. The "instrument" plays a very important role. It enables the students to "see" the significant difference between the discrete and the continuous methods. It also helps 277 the students to analyze qualitatively the behavior of the solution and to connect it with former experiences. Mathematica enables to change very slightly the value of a parameter and to plot the solution. The role played by the instrument is very important and enable the new potentials offered by the instrumented work. However, the percentage of students who did not find the source of error in the numerical method, demonstrated that it might have not been enough to expose the students to a counterexample. Students have to be faced with the necessity of developing schemes that will help them to differentiate between error due to mathematical meanings and error due to meanings specific to the instrument (Artigue, 2002).
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Learning about equivalence, equality, and equation in a CAS environment: the interaction of machine techniques, paper-and-pencil techniques, and theorizing
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The study presented in this report is part of a larger project on the intertwining coemergence of technique and theory within a CAS-based task environment for learning algebra, which also includes paper-and-pencil activity. The theoretical framework consists of the instrumental approach to learning mathematics with technology, in particular Artigue and colleagues' adaptation of Chevallard's anthropological theory. The theme presented herein is that of equivalence, equality, and equation. Two 10
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grade classes were taught by the same mathematics teacher during two successive years, using project materials designed by the research team. Classroom observations, student interviews, student activity sheets, and posttest responses were the main data sources used in the analysis. Findings attest to the intertwining of technique and theory in algebra learning in a CAS environment. In addition, the data analysis revealed that probably the most productive learning took place after the CAS techniques provided some kind of confrontation or conflict with the students' expectations, based on their previous theoretical knowledge. Even if such conflicts in applying CAS techniques may seem to be hindrances to students' progress, in fact our experience suggests that they should be considered occasions for learning rather than as obstacles. However, a precondition for these conflicts to foster learning is their appropriate management in the classroom by the teacher.
Introduction
School algebra has traditionally been an area where technique and theory collide, with technique usually claiming victory. While a parallel with the terms skills/procedures and concepts may suggest itself, both technique and theory are broader in meaning than procedures and concepts (Artigue, 2002). The notion that school algebra can be an arena for the interaction of both theory and technique has not taken hold until recently. The advent of computer algebra systems (CAS) technology in schools, along with the development of theoretical frameworks for interpreting how such technology becomes an instrument of mathematical thought, have both been contributing factors.
The instrumental approach to tool use encompasses elements from both cognitive ergonomics (Vérillon & Rabardel, 1995) and the anthropological theory of didactics (Chevallard, 1999). An essential starting point in the instrumental approach is the distinction between an artifact and an instrument. Whereas the artifact is the object 279 that is used as a tool, the instrument involves also the techniques and schemes that the user develops while using it, and that guide both the way the tool is used and the development of the user's thinking. The process of an artifact becoming an instrument in the hands of a user -- in our case the student -- is called instrumental genesis. The instrumental approach was recognized by French mathematics education researchers (e.g., Artigue, 1997; Lagrange, 2000; Trouche, 2000; Guin & Trouche, 2002) as a potentially powerful framework in the context of using CAS in mathematics education.
As Monaghan (2005) pointed out, one can distinguish two directions within the instrumental approach. In line with the cognitive ergonomic framework, some researchers (e.g., Trouche, 2000; Drijvers, 2003) see the development of schemes as the heart of instrumental genesis. Although these mental schemes develop in social interaction, they are essentially individual. Within the schemes, conceptual and

technical elements are intertwined. More in line with the anthropological approach, other researchers focus on techniques that students develop while using technological tools and in social interaction. The advantage of this focus is that instrumented techniques are visible and can be observed more easily than mental schemes. Still, it is acknowledged that techniques encompass theoretical notions. The focus on techniques is dominant in the work of Artigue (1997, 2002) and Lagrange (2000) in particular.
The study
Theoretical framework: Task-Technique-Theory
Chevallard's anthropological theory of didactics, which incorporates an institutional dimension into the mathematical meaning that students construct, describes four components of practice by which mathematical objects are brought into play within didactic institutions: task, technique, technology, and theory. (By technology, Chevallard means the discourse that is used to explain and justify techniques; he is not referring to the use of computers or other technological tools.) In their adaptation of Chevallard's anthropological theory, Artigue and her colleagues have collapsed technology and theory into the one term, theory, thereby giving the theoretical component a wider interpretation than is usual in the anthropological approach. Furthermore, Artigue notes that technique also has to be given a wider meaning than is usual in educational discourse.
Lagrange (2003, p. 271) has elaborated this latter idea further: "Technique plays an epistemic role by contributing to an understanding of the objects that it handles, particularly during its elaboration. It also serves as an object for a conceptual reflection when compared with other techniques and when discussed with regard to consistency." It is precisely this epistemic role played by techniques that is a focus of our study, that is, the notion that students' mathematical theorizing develops as their techniques evolve. It is noted, as well, that our perspective on the co-emergence of theory and technique is situated within the context of technological tool use, where 280 the nature of the task plays an equally fundamental role. Thus, the triad TaskTechnique-Theory (TTT) served as the framework not only for constructing the tasks of this study, but also for gathering data during the teaching sequences and for analyzing the resulting data.
Aim of the study
The research study, of which this report is a part, is an ongoing one. It has as a central objective the shedding of further light on the co-emergence of technique and theory within the CAS-based algebraic activity of secondary school students. Because of severe space restrictions, this report will highlight the design and findings from one task set, that of equivalence, equality, and equation.
Participants
The research involves six intact classes of 10
th
graders (15-year-olds) in Canada and
Mexico, as well as a class of older students in Oregon. Five of the 10
th
grade classes
were observed during the 2004-05 school year; the sixth class, the following year.
Two of these 10
th
grade classes are featured in this report – one from the 2004 study
and the other from the 2005 study. Both classes were taught by the same teacher, with five years of experience. He is a teacher who, along with encouraging his pupils to talk about their mathematics in class, believes that it is useful for them to struggle a little with mathematical tasks. He elicits students' thinking, rather than quickly giving them answers. The students in this report had already learned a few basic techniques for solving linear and quadratic equations during their 9
th
grade mathematics course
and had used graphing calculators on a regular basis; however, they had not had any experience with the notion of equivalence, one of the theoretical ideas developed in the project materials, nor with symbol-manipulating calculators (i.e., the TI-92 Plus CAS machines used in this project).
Data sources
All project classes were observed and videotaped (12-15 class periods for each of the seven project classes). Students were interviewed, alone or in pairs, at several instances -- before, during, and after class. A posttest involving CAS was administered after the task set on equivalence had been completed. All students were pretested. Thus, data sources for the segment of the study presented in this report include the videotapes of all the classroom lessons, videotaped interviews with students, a videotaped interview with the teacher, the activity sheets of all students (these contained their paper-and-pencil responses, a record of CAS displays, and their interpretations of these displays), written pretest and posttest responses, and researcher field notes.
Task design 281
The research team created several sets of tasks that aimed at supporting the coemergence of technique and theory. Because paper-and-pencil techniques were a fundamental part of the algebra program of studies of the schools where the research was carried out, and because we believe in the importance of combining the two media, they too were included in the teaching sequences. Task sets were planned to take from one to five periods. For the task set described in this paper, one class took three periods, and the other, four. Each task set involved student work, either with CAS or paper-and-pencil or both, reflection questions, and classroom discussion of the main issues raised by the tasks. In designing the tasks, we took seriously both the students' background knowledge and the fact that these tasks were to fit into an existing curriculum; but we also did our best to ensure that they would unfold in a particular classroom culture that reflected a certain priority given to discussion of serious mathematical issues. Tasks that asked students to write about how they were interpreting their work and the related CAS displays aimed to bring mathematical notions to the surface, making them objects of explicit reflection and discourse in the classroom, and clarifying ideas and distinctions, in ways that simply "doing algebra" may not require.
The task set on equivalence, equality, and equation
The underlying motive of this task set is the subtle relationship between arithmetic and algebra: on the one hand, the numerical world is the most important motive and model for the world of algebra, on the other hand algebra goes beyond the numerical world, which is in fact part of its power. This two-sided relationship is reflected in the notion of equivalence of algebraic expressions (see Kieran & Saldanha, 2005; Saldanha & Kieran, 2005). Equivalence of two expressions relates to the numeric as it reflects the idea of 'equal output values for each of an infinite set of input values.' However, equivalence of two expressions also relates to the algebraic in that the expressions can be rewritten in a common algebraic form.
At the start of the teaching sequence, numerical evaluation of expressions by using CAS and comparison of their resultant values are used as the entry points for discussions on equivalence. One of the core tasks here aims at students' noticing that some pairs of expressions seem to always end up with equal results, and thus evokes the notion of equivalence based on numerical equality. The algebraic expressions included in the task were fairly complex so as not to permit the evaluation of equivalence by purely visual means. The task is followed by a reflection question on what would happen if the table of values were extended to include other values of x. The task and the CAS substitution technique lead to the following definition of equivalence of expressions, with deliberate inclusion of the idea of a set of admissible values:
We specify a set of admissible numbers for x (e.g., excluding the numbers where one of the expressions is not defined). If, for any admissible number that replaces x, each 282 of the expressions gives the same value, we say that these expressions are equivalent on the set of admissible values.
The impossibility of testing all possible numerical substitutions to determine equivalence motivates the use of algebraic manipulation and the explicit search for common forms of expressions in the second part of the task set. Different CAS techniques can be used: Factor, Expand, Automatic Simplification. An additional technique is the "Test of Equality," which involves entering an equation, followed by the Enter button. In this test, the CAS checks both sides of the equation for equivalence, by means of automatic simplification and other 'black-box' means. The CAS will come up with 'true' in cases of equivalence (see Figure 1). Restrictions are ignored, just as with Automatic Simplification. The Test of Equality technique has probably the most 'black-box' character, and the output it produces is the most difficult to interpret, especially for cases of non-equivalence. This CAS technique was deliberately introduced in the design of the tasks so as to provoke student questioning of its output.
Fig.1: Illustration of the "Test of Equality" and the way that the
TI-92 neglects restrictions
In the next part of the task set, the relation between two expressions being equivalent or not, and the corresponding equation having many, some, or no solutions is explored in both CAS (Solve now introduced) and paper-and-pencil tasks. For example, students are asked to generate a pair of equivalent expressions, and, in a similar follow-up task, two non-equivalent expressions. The ensuing reflection question concerns the relation between the nature of an equation's solution(s) and the equivalence or non-equivalence of the expressions that form the equation.
Analysis of student activity
Three theoretical elements were found to be intertwined and related to the techniques and tasks of this segment of the data analysis, and thus serve to organize the discussion of results: i) The numeric and algebraic views on equivalence; ii) The issue of restrictions; and iii) Coordination of the notions of equivalence and solution of an equation. 283
The numeric and algebraic views on equivalence
Students seemed to have an intuitive idea of equivalence as having always the same numerical value, even if this was sometimes expressed in an informal way. This notion was clearly supported by the CAS substitution technique, which makes numerical substitutions easy to carry out. The repeated substitution with the CAS presented the students with the phenomenon of equal values, which invited algebraic generalization. Still, the relation between the algebraic and the numeric was somewhat vague. The Factor, Expand and Automatic Simplification techniques are on a more algebraic level, but seem to foster the notion of common form as 'simple' form. That is, some students tended to interpret the simplified forms produced by these commands as ordinary or basic or common, and thought that this was what we meant when we asked them to express a pair of expressions in a common form. The Test of Equality technique is probably the most interesting one from the conceptual point of view, as it seems to act at the borderline between the numeric and the algebraic. This technique provides 'true' in cases of equivalence, but just returns the (sometimes transformed) equation in other cases. The latter was difficult to understand for many students, as they would have expected something like 'false'; whereas returning an equation -- so two expressions with an equal sign in between -- unjustly suggested equivalence to them.
The verbatim in Figure 2 illustrates that Suzanne found the output of this CAS test surprising. She tried to interpret it by means of her existing theoretical thinking, but was unable to do so satisfactorily. In spite of the confusion that she expresses in the last line, we appreciate that she takes it as an incentive to rethink about her conceptions. In fact, that is what the tasks and techniques, if dealt with properly by the teacher, can provoke: a rethinking of the theoretical knowledge. The classroom discussion that followed did help to move Suzanne's thinking forward.
Suzanne Uhm, I entered the problem (20)(3 2)1 3( )(1)(2)5
2 2 2
x + x! x + x! = x! x ! x! x+ and it
gave me pretty much the same problem back, but rearranged, it's the same answer. When you think that the other one said "true," it is kind of puzzling. … The answer that it gave me. I figure that that's this statement, like the first expression equals the second expression is true. … When I see an equal sign, I figure they are equivalent, the same.
[…]
Interviewer How would you now interpret such a display when you enter in two expressions like that?
Suzanne Uhm, that it can be right sometimes, but isn't always right. With specific numbers, it is correct.
Interviewer So, when you mean correct?
Suzanne That you would get the same number in the end on both sides. But only sometimes. 284
Interviewer Only for some numbers.
Suzanne Yah.
Interviewer So how do you feel about that?
Suzanne I'm still confused. With the "true"s and the "="s, to me it all has sort of the same meaning. I guess I just have to change my way of thinking.
Fig. 2. Confusion about the CAS returning the equation for the case of nonequivalence
A second issue that is related to the role of techniques in the evolution of the students' thinking about equivalence concerned the coordination of different techniques as a means to check consistency. In several cases, students used different techniques, both paper-and-pencil and CAS, to verify the consistency of their theorizing. Surprising CAS results in some cases gave rise to conflicts that invited reasoning. For example, at first Andrew was puzzled when the CAS simplified 2(! x)(1! 2x) as (x ! 2)(2x !1). After some thinking about this, he found a justification that involved the technique of substitution:
"I think since it's switching them both that it works out. Let's just say x was represented by 6, -4 times -11, which is 44. And the other one it's 6 2, which is 4 □ times 11, which is also 44. It's just two negatives, since it's switching both of them it's OK."
By the way, this verbatim shows the student's returning to the numerical to check algebraic relations -- not a bad habit of course. Still, when so asked, Andrew indicated that he had several means to check algebraically the equivalence of 2(! x)(1! 2x) and (x !)(2 2x !)1 , such as entering the corresponding equation or expanding them both. The other students also used these CAS techniques to check their consistency with by-hand results.
The issue of restrictions
The question of how to deal with restrictions, both with CAS and paper-and-pencil techniques, played a role in the algebraic view on equivalence. It also disclosed limitations in certain students' thinking about zero in fractions and dividing by zero. This was the case for Andrew when dealing with the equivalence of two particular

expressions: (3x !1)(x
2
! x ! 2)(x + 5) and
(x
2
+ 3x !10)(3x !1)(x
2
+ 3x + 2)
x + 2
. At first, he had
difficulties with identifying the restriction of x = !2 . The question to consider the denominator revealed a misconception: "If x were -2 then the denominator would be - 2 plus 2, which is zero and anything over zero is equal to zero. One over zero equals to zero," he said. After some intervention, Andrew concluded that the result of division by zero is undefined, but it remained unclear about whether another zero might appear somewhere. To check this out, he substituted x = !2 into the first expression and got -84 as a result, clearly not zero. So, he concluded: "Basically, it 285 will work with everything except the !2." Then he substituted x = !2 into the expanded form of the first expression, which of course gave !84 once more. This seemed to be a check for consistency, although he was not completely sure about what to expect. Then he wondered about the value of the second expression when x = !2 would be substituted. He expected -84, but the calculator displayed 'undefined.' He explained this as follows:
"That's what I figured out that it should be, undefined, but I didn't think the calculator would show it. Just based on all the other results, just based on the fact that this came out to -84, and this came out to -84. … Well like it substitutes it and then it fills everything in and anything divided by zero is undefined, no matter what the equation is on top, it's still divided by -2 plus 2, so it's undefined."
Andrew's fuzzy thinking about rational expressions, division by zero, and substitution of inadmissible values into the numerator interacted with his expectation of a certain CAS output. He was, however, eventually able to provide a technical interpretation that made sense to him about how the CAS produced 'undef' – as disclosed by his last comment.
Coordination of the notions of equivalence and solution of an equation
Results concerning the coordination of solutions and equivalent/non-equivalent expressions were mixed. For example, after students had generated a pair of equivalent expressions and were asked what they could say about the solutions of the equation formed from this pair -- but without actually solving the equation -- half the students' responses included true, equal, equivalent, and did not refer explicitly to solutions of the equation. Furthermore, the word solution itself seemed problematic. While some students were able to relate the set of equation solutions to the equivalence of the two expressions involved, this remained unclear for others. The Solve technique in itself was not a problem for the students; but its coordination with the other techniques on equivalence required a change of perspective, which was not easy. Evidence suggests that a language issue is involved here as well: students use the word 'solve' for any operation leading to a result, the result being called the 'solution.'
Concluding remarks on equivalence, equality, and equation
If we consider our findings on the task set of equivalence, equality, and equation in retrospect, two main issues come to the fore: the relation between students' theoretical thinking and the techniques they use for solving the proposed tasks, and the specific role of the confrontation of CAS output with students' expectations. To elaborate on the first point, our findings suggest that the relation between Theory and Technique, as it is established while working on appropriate Tasks, can hardly be underestimated. On the one hand, the development of students' theoretical thinking was guided by the techniques that the tasks invited; on the other hand, students' 286 conceptions influenced the development of these techniques. More specifically, students' numerical view on equivalence of expressions was found to be related to three techniques: the numerical substitution technique, and, to a lesser extent, the Test of Equality and the Solve technique. The fact that students seemed to give priority to a numerical view on equivalence was tied to their use of the numerical substitution technique to check equality. In the emergence of an algebraic view on equivalence, the CAS techniques Factor, Expand, Automatic Simplification, and Test of Equality played important roles, even to the extent that the factored and expanded forms were considered as common forms. Finally, with regard to relating the numeric and the algebraic views on equivalence, students had difficulties with the coordination of the Solve technique and the techniques on equivalence; nevertheless, classroom discussion of these techniques turned out to be quite productive.
To elaborate on the second point, the data analysis revealed that probably the most productive learning took place after the CAS techniques provided some kind of confrontation or conflict with the students' expectations, based on their previous theoretical knowledge. The students' seeking for consistency evoked theoretical thinking and further experimentation. Also, the fact that the CAS Automatic Simplification technique and the Test of Equality both neglect restrictions led to an increasing awareness of the importance of these 'exceptions.' Finally, the CAS just returning an equation in cases of non-equivalence struck the students, and gave rise to interesting discussions on the interpretation of the output, as did the interpretation of 'true' and 'false' in cases of numeric or algebraic application of the Test of Equality. Even if such complications in applying CAS techniques may seem to be hindrances (see Drijvers, 2002) to students' progress, in fact our experience suggests that they should be considered occasions for learning rather than as obstacles. However, a precondition for these complications to foster learning is their appropriate management in the classroom by the teacher.
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The impact of the use of graphics calculator on the learning of statistics: a malaysian experience
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Educational tool such as the graphics calculator (GC) is increasingly being used in school and college mathematics worldwide. In Malaysia, the Ministry of Education together with a couple of GC distributing companies (such as CASIO and TEXAS INSTRUMENTS) has conducted a series of GC workshops to train the secondary school mathematics and science teachers. To date, undergraduate mathematics integrating the use of GC has already been offered in the local universities. This paper emphasizes on the impact of GC in the learning of statistics. Even though many research findings have reported favourable use of GC in the teaching and learning of mathematics, the effect and the impact of GC however, could be different in a diverse learning environment. In the study conducted, students' thinking and feeling towards engaging GC in their learning process were explored through observation, students' written self-reflection and interviews. Analysis of the data highlighted three major changes in the students' learning process that engaged the use of GC. These three changes are students' perceived value of GC, changes in the norm classroom practices and changes in the perceived peer status.
Background
Educational tool such as the graphics calculator (GC) is increasingly being used in school and college mathematics worldwide. New ideas for graphing technologies are continuously being featured at international conferences as well as published in mathematics educational journals. The integration of such innovative technology in the mathematics classroom is anticipated to bring about changes in the approach in completing a mathematical task. Studies over the past (Arcavi & Hadas, 2000; Selinger & Pratt, 1997) have shown that GC can be used in ways that can promote and enhanced learning of mathematical concepts through visualization, symbolic, algebraic and graphic presentations. In addition, there are studies (Cedillo, 2001; Ruthven, 1990) that shown using of GC might enhance students' thinking skills in mathematics such as analyzing, reasoning and translation to algebraic or symbolic form.
In Malaysia, the integration of GC in the teaching and learning of mathematics has received much attention from the Ministry of Education (MOE), though the pick up is still slow. Since mid-2003, the MOE has been sponsoring educational officers and mathematics teachers to attend short-term GC training courses or workshops as part of their professional development program. With the support of the MOE and the initiative given by a couple of GC distributing companies (such as CASIO and 289 TEXAS INSTRUMENTS), a series of GC workshops was conducted to train the secondary mathematics and science teachers in 388 schools throughout Malaysia. Besides these training, each participating school was given a classroom set (40 units) of GC from the MOE. This program is planned to continue until the majority of the mathematics teachers are familiar to the practice of using the GC technology. In addition, the teachers training colleges have also taken drastic measure to gain access to the portable mathematical tool in order to equip more trainee teachers with the GC knowledge.
To date, undergraduate mathematics applying the GC knowledge has been offered in certain local universities (Rosihan et. al, 2002; Rosihan & Kor, 2004). However, the use of GC in mathematics at school level is still very limited. Nevertheless, more and more schools and higher institutions throughout Malaysia are encouraged to use GC in the teaching and learning of mathematics. For example, the latest curriculum specification of the Secondary Mathematics Syllabus 2005 has explicitly recommended the use of GC and GSP in the teaching and learning activities. Consequently, new mathematics textbooks that incorporate graphing technologies as part of the solution steps inevitably help to provide impetus to intensify the use and integration of graphing technologies in the present mathematics classroom.
Rationale of the study
Statistical literacy is vital to all aspects in our lives that deal with data based information. The importance of statistical literacy can be seen by the recent move taken by all colleges and universities in Malaysia to require their undergraduates to pursue compulsorily a first course in statistics regardless of their future profession. Even though many research findings (Kissane, 2004; Rosihan & Kor, 2004) reported favourable use of GC in the teaching and learning of mathematics, the effect and the impact of GC however, could be different in a diverse learning environment. Apparently, the Malaysian mathematics classroom that comprises of Asian students with a different learning culture is a worthwhile case to study. A study of this kind complies with Koc's (2005) assertion that awareness of cultural differences in technology could help instructional designers and trainers to build more culturally and socially sensitive materials for educational purposes. With these in mind, this study was conducted to examine how students' engagement with GC changes the

norm practice of their learning of statistics.
Method
The respondents in the study comprised of 76 second year Diploma in Business students. All were non-mathematics majors who had no prior classroom experience in using the GC. The statistics course in this study was conducted using the TI-83 Plus. 290 The topics taught were Measures of Central Tendency, and Correlation & Simple Linear Regression. The researcher conducted the whole course in twelve lessons. One lesson was two hours long and there were two lessons a week.
Students were required to write down their reflection corresponding to five openended questions given after each lesson. Interviews were conducted twice during the twelve lessons to explore students' thinking and feeling towards engaging GC in their learning process. Four independent facilitators were also employed to observe the classroom climate. Respondents' written reflection, the audio taped interviews and observation notes were collected, transcribed and analyzed.
Results and discussions
Analysis of the data highlighted three major changes in the students' learning process that engaged the use of GC. These three changes are students' perceived value of GC, changes in the norm classroom practices and changes in the perceived peer status.
(A) Changes in students' perceived value of GC
(i) GC as a technological tool
During the interview, the respondents expressed that they were not too convinced of the capability of the GC at the beginning of the lesson. They thought it would be just another calculator with more advance features and is solely used to assist them in doing more complicated computation. After a few lessons, many respondents were very impressed with the many functions a GC can perform. They were especially attracted to the graphs that appeared on the screen at the press of a button. The respondents responded by saying,
The first time we see a GC, we did not expect it to be able to draw graphs like what we do now… you just need to key in the right data and press the right button and there is the graph.
Another group found that the GC enriched their understanding on a set of data by presenting the data in different forms because "we saw that the same set of data can be represented in many different forms". These students' responses highlighted the significance of GC as a tool for learning. They learned that data is not just about a set of numbers but can be presented in many different forms within the switch of the GC screens.
However, there were some who dislike the use of GC voiced different view about GC such that, "learning to use GC is very difficult. There are so many steps to follow… 291 so many functions to remember…very confusing … cannot tell us if we have keyed in the wrong data or have pressed the wrong button!"
(ii) Use of GC provides a new approach in learning statistics
An appropriate use of technological tool such as GC has the potential to bring about students' cognitive activities rather than just to amplify their human capabilities. There was description about using GC to "work backwards" with the existing data when there is an uncertainty in solving a particular problem. One average achiever who favours GC presented a refreshing perspective by saying that
Sometimes I used GC to find the answer to a problem that I am not sure of doing … use the answer to work backwards to get the solution… like a box plot looks… can use GC to plot the box plot with the data I had and then use the "TRACE" key to locate the points and transfer these points to my graph paper to complete a box plot.
One group mentioned that they worked out the solutions with GC repeatedly until the final correct answer was achieved. They elaborated further, "To locate a point on a regression line for example, we can zoom repeatedly and trace the point that is not in the range until we get it."
(iii) Perceived personal gains
Interestingly, respondents' views about the new GC-enhanced statistics lessons were a mixture of great expectation and cynicism. Some respondents expressed that GC enhanced their learning, they said, "… there is reinforcement when we learn to solve a problem in two different methods: the calculator and the manual methods. By doing it twice we can remember better."
In addition, some related their bad experience when using GC such as, "It is very difficult to remember the functions in GC. We cannot see anything meaningful on the GC screen as compared to learning with manual method." One facilitator observed that, "The students seemed to show disappointment when they failed to follow the instructions on how to use the GC to plot a regression line."
(B) Changes in the norm classroom practice
(i) Classroom learning atmosphere
Before the GC engagement, the study sample was observed to be serious and followed the lessons passively. They were seen taking notes from the white board and occasionally exchanging words in a whisper with their neighbours. The learning atmosphere was solemn. However, the inclusion of GC seemed to change the classroom norm. It was observed that the classroom atmosphere was lively and 292 cheerful. Time passed by without anyone noticing as most of the respondents were busy sharing their findings with each other. One facilitator strongly believed that the invigorating classroom climate reflects that learning is in progress as she commented, "The class was noisy but it is a good sign of learning because they were busy asking questions. Discussions were seen among the students and between students and lecturer."
(ii) Students' learning attitudes
In general, the engagement of GC in the statistics lessons had motivated students' interest in statistics as well as improved their confidence in the learning process. A typical remark that explains the transformation was,
I find that I work harder now in this subject… always rush to finish off the given exercises in the classroom. I would try out again and again if I fail to get the answer… want to get it done because nobody can offer you help on GC other than the lecturer …all of us are still new to this thing.
Another respondent reported the excitement when using GC as, "I can draw frequency polygon faster than the traditional method. I am excited …using GC to learn is really interesting."
However, some expressed difficulties in using GC to learn because they could not concentrate on doing two things at the same time.
… need to listen to the instruction given by the lecturer and at the same time we need to understand the concepts…it's very difficult…we could not catch up with the others. We were always left behind in the lesson. It creates tension and phobia in us.
(iii) Classroom participation
Compare to the normal class, more respondents were seen offering peer tutoring among themselves. It was recorded by the facilitators that, "When they (respondents) were confused in receiving instruction from the lecturer, they overcome this by encouraging and helping each other."
The learning process was dynamic. Group discussions and didactic conversation between lecturer and students and among students were a common sight. One group explained the reason why they asked more questions than the normal class. They said, "We are forced to ask many questions because there are many new things that we do not know about GC." 293
Conversely, weak students who dislike using the GC revealed their problems that kept them from active participation. This is because, "We were hardly active in the group discussion… we were not confident of contributing anything about GC. We were afraid that we might say the wrong thing… we just kept quiet and let the others do the talking."
(c) Changes in perceived peer status
Some respondents seemed to associate GC with the state-of-the-art modern electronic device. They were optimistic that GC as a new technological gadget can create an appeal to the future statistics classroom. They developed a sense of pride from the knowledge gained from GC. There were two groups of respondents who used the analogy of "hand phone" while another two groups used "computer" to describe the GC. There were groups who felt that learning statistics with GC is compatible with the present computer age. These respondents considered themselves as technology savvy and more knowledgeable after learning how to use GC in statistics. They said, "GC is like a hand phone … when we compare ourselves to those without the GC knowledge, we are more professional and proficient in doing statistics."
Conclusion
The findings of this study suggest that GC can bring changes to the statistics classroom culture of learning. These changes include the cognitive changes in the pattern of learning and doing statistics, the norm classroom practice and changes in the students' perception and feelings about the new tool and the new GC knowledge. Although there were reports that the respondents were confused in the process of handling GC, the number was relatively small in the study sample. Apparently not all respondents in the study recommended highly the engagement of GC in doing and learning statistics. Nevertheless, majority admitted that GC enhanced their understanding through visualization and multiple representations. One of the most common complaints among the respondents was the frustration they faced in mastering the GC skills. Inevitably, respondents who were incompetent in GC skills often encountered with technological problems. As a result, the weak students often find using GC annoying.
In fact, the success of GC implementation depends much on tool competency of the users. Poor mastery of GC could lead to high frustration and subsequently reduce learners' motivation to learn the subject. Consequently, this study advocates strongly that the more experiences the learners have with GC prior to instruction, the more likely they are to perform better through the course. This study also shows that many weak students did not possess the necessary GC skills and sound understanding of 294 the statistics content that allowed them to work collaboratively in a group. Therefore, it is pertinent that special attention must be taken when including a GC into a mixedability classroom. This study supports Lindsay's (2003) attestation that studentcentred activities, inquiry-based approaches of mathematics teaching might be more suitable for average and high-achievers only. For weak students, it is important that their skills in handling GC should always be evaluated and monitored throughout the learning period. Furthermore, the facilitator must provide ample guidance and gives immediate feedback to reinforce the learners' short-term memory.
In sum, we observed that the Malaysian students' experience in engaging GC to learn statistics was not far different from the global norm. However, it remains our concern on how to exploit the use of educational tools such as GC so as to make the learning of mathematics more enticing and motivating to students of all abilities.
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Analysis of effects of tablet PC technology in mathematical education of future teachers
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This paper presents description of authors' current and proposed work using Tablet PCs mobile computer lab in future teachers' preparation classes. Faculty from the Colleges of Education and Science at the University of Texas at El Paso worked together to study the effects of incorporating Tablet PC technology in pre-service teachers' math education. We assessed the significance of the technology by evaluating and comparing students' final project and course grades. We did a statistical comparison of two groups: the treatment group where students extensively used Tablet PCs to work on mathematical investigations and lesson plans and the control group where students worked on identical math investigations and created lesson plans without utilizing any technology. The outcome shows a greater improvement in the treatment group's mathematical content knowledge versus that of the control group's. Current and future work involves evaluation of the change in acquiring mathematical pedagogical knowledge by pre-service teachers. Future teachers (in both groups) are asked to create original math lessons using unique manipulatives and hands-on activities. Students in the treatment group are required to use Tablet PCs to create hands-on activities. Groups' pedagogical knowledge will be compared using pre/post tests, questionnaires and knowledge and attitude surveys.
Introduction
The concept of a "digital divide" separating those with access to computers and communications technology from those without is simplistic.
Research (Peslak, 2005) shows that computers per students and total number of computers in a school significantly effects student learning, but surprisingly there is a negative impact of this metric on standardized reading and math scores.
Another study (Warschauer, 2005) shows that kindergarten through 12th grade students from a higher socioeconomic status are more likely to use computers for experimentation, research and critical inquiry; students from a lower socioeconomic status usually engage in less challenging drills and exercises that do not fully utilize the advantages of computer technology.
To benefit from computers teachers should have access to good educational software and be familiar with the available software. Ideally teachers should be able to use appropriate software to create math activities that guide students to higher order

thinking. Pre-service (future) teachers should be confident and knowledgeable about effective instructional strategies that incorporate variety of digital technologies. 296
Located on the Rio Grande River in the far western edge of Texas along the borders of Mexico, New Mexico and Texas, El Paso is a bustling urban area of 700,000 people, more than 74% of whom are Mexican in origin. Widely known as a major passageway for land travel through the mountains from Mexico to the U.S., El Paso sits in close proximity to the Mexican City of Juarez, with a population of more than 1.2 million; together El Paso and Juarez represent the largest metropolitan along the 2,000- mile U. S. /Mexico border. Generations of Mexicans and Mexican-Americans view El Paso as a place where they can pursue their hopes, aspirations, and dreams. Almost a quarter of El Paso's population is foreign born, and more than 50% of El Paso's households speak Spanish as the language of preference. The University of Texas at El Paso prepares a large number of bilingual educators to work with the growing Hispanic population.
In the beginning of 2004 the team of researchers from the College of Education and College of Engineering received a grant from Hewlett Packard that allowed UTEP to organize a mobile Tablet PC lab. This lab was readily available for use in the math and math methods classes taught in a Field-based environment. It was this HP grant which provided us the necessary technology for this study.
In this paper we present the study on the effects of Tablet PC technology on mathematical content knowledge of pre-service teachers. We also in the process of collecting data and evaluating students' math pedagogy, attitudes toward digital wireless technologies, and effects of this technology on collaborative learning of mathematics. Thus, the scope of this paper addresses the following research questions (from Discussion Document http://www.math.msu.edu/~nathsinc/ICMI/):
roles of different technologies in teaching and learning mathematics, assessing learning of mathematics using digital technologies, and how can technologyintegrated environments be design so as to capture significant moments of learning.
The significance of Tablet PCs: recently, at Technology Review's Emerging Technologies Conference held at MIT, Nicholas Negroponte, a founder of MIT's Media Lab, showed off a laptop design he hopes can be sold for just $100. These small laptops work similarly to Tablet PCs, using "digital ink" thus providing students the opportunity to write on the screen using a specially designed stylus pen. With the affordability of these computers, schools will be able to provide every child with a computer.
The Study
In spring semester of 2005, 38 pre-service elementary teachers were enrolled in math content and math methods courses. These students were also enrolled in internships at local elementary schools. The students' internships and class placement was random.
Our study focused on two groups of students. The treatment group consisted of 15 students that regularly met in a professional development school where they were provided with 14 Tablet PCs for use in their math and math method classes. The control group consisted of 23 students who were enrolled in the same courses with 297 the same instructor but met at different times and location. In this control group Tablet PCs were not used.
All math and math methods classes were team taught using a series of rich math investigations, in-depth discussions on topics in the methods textbook (Van de Walle, 2004) and group lesson preparations and implementations. Specifically, math and math methods course were designed to foster conceptual understanding of mathematics and pedagogy in the following major strands: rational numbers, geometry, algebra, number theory, and functions. All projects were designed to include open-ended problems that required thorough investigations to achieve successful solutions. Instructors utilized collaborative learning and inquiry based methodology while students worked in small groups.
The treatment group used Tablet PCs to explore the investigations or projects. This technology allowed students to explore each activity fully without being limited to paper and pencil drawings. The real time feedback in the various programs and websites used gave students a good evaluation tool of their problem solving methods. In addition, the ability to iterate calculations allowed students to focus on the "big" picture without getting bogged down with repetitive calculations.
Besides the rich investigations, students from both groups planned individual/group lessons and implemented them in local elementary schools. Students created these lesson plans by applying the concepts investigated in their math and math methods courses. Students in the treatment group successfully implemented Tablet PCs in their individual and group teachings allowing pupils to engage in meaningful technology based activities.
By the end of the four month period both groups had completed and presented several investigations. The students' final exam was an oral presentation covering the functions and algebra investigation. The students' final grade was a cumulative grade indicative of the students' performances in each investigation, presentation, teaching and methods review.
Implementing the Tablet PCs in a technology-enhanced classroom
Tablet PC's are fully functional PC's running an enhanced version of Windows XP Professional. One of their most interesting feature is the "digital ink" that allows a user to write on the screen using a stylus pen. The same pen is also used as a mouse. The handwriting recognition software allows the written text to be converted to digital text in Microsoft Word. Traditional keyboard and mouse are also available. Tablet PC's also have built-in wireless connectivity, so they can communicate with each other even if there is no internet connection present (ad hoc mode). They can also wirelessly connect to the internet. In newer Tablet PCs battery life can be up to six hours, however in our case our Tablets only have a battery life of three hours.
Students had the opportunity to utilize the Tablet PC during their presentations by connecting the Tablets to a data projector. In addition since we had 14 Tablets and 15 students, then students were able to work with the Tablet PCs individually or in small groups. Faculty used special software to communicate with the students. First, using 298 VNC software (http://www.realvnc.com/what.html) and internal wireless capability, faculty could connect to any of the students' Tablet PCs and check their work, help them and project their successful solution on the screen. Second, Discourse software (www.ets.org/discourse) was used for assessment. Using this software, faculty is able to ask various types of questions including open-ended or multiple-choice questions. Each question can be accompanied by an image guiding students to explore this particular question. In addition to images, faculty can guide students to a specific internet site relevant to the question. Students can answer their questions concurrently or on their own pace.
The faculty is also able to simultaneously monitor each student's answer in real time. Some of the students prefer typing the answers; others can open a writing pad, and record their answer on the pad. This writing pad automatically converts it to digital text. Also, students can have discussions using the "chat" option (that can be disabled by faculty).
Pre-service teachers employed the Tablet PCs in their university courses and during their internship teaching in a number of ways. The most popular program was a program called Microsoft Journal, which comes free on Tablets and is used as an electronic whiteboard. The Power Point program was extensively used for presentations and for the creation of animated virtual manipulatives. The students were very successful in creating virtual manipulatives for solving word problems, place value tasks, fractions, and geometrical designs. Similarly, Kidspiration and Inspiration software was used for the creation of math activities and games. Students also used this software to create concept or mind maps.
Variety of specialized free software was also used. One of the most interesting software is Java Bars
(http://tt.uga.edu/tt/jwilson.coe.uga.edu/olive/welcome.html) that provides a creative workspace to explore fractions. Another software used extensively was the Tangram editor that will no longer be available for free. Excel was used in mathematical projects by utilizing the different spreadsheet capabilities such as graphing and formula calculator. Another interactive site used for fractions was Cynthia Lainus' website from Rice University where students could explore fractions with pattern blocks. Students also used a variety of internet sites such as the National Library of Virtual Manipulatives, NCTM Illuminations, Math Playground and InterMath Investigations. Preservice teachers used a WebCT portal provided to all students by the university to submit their work for grading and discussion as well as for submitting questions outside of class time.
Results, Conclusions and Future Work
We statistically compared the effectiveness of our technology enhanced method for mathematics, using the Tablet PC, to the standard inquiry based method. This comparison is based on the results of two distinct items. The first is the students' Final Exam given at the end of a 4 month learning period. The second is the students' Final Grades which is a cumulative grade based on all the investigations throughout 299 the semester. Of a random sample of 38 students, 15 were taught by the technology enhanced method and were considered our treatment group. The other 23 students were taught by the standard inquiry-based method and were considered out control group. All 38 students were taught by the same qualified instructors under similar conditions.
We computed students' grade point average (GPA) using only their previous mathematics courses. The average math GPA for the treatment group is 2.99 while the average math GPA for the control group is 2.98. This indicates that both groups had approximately the same mathematical background and content knowledge prior to the study. We developed descriptive statistics for both samples and both teaching methods see (Table 1).

Final Exam Final Grade Group	n	Mean X	Var 2 (	St.Dev.	n	Mean X	Var 2 (	St.Dev.
Treatment	15	90.8	14.31	3.78	15	93.04	8.95	2.99
Control	23	84.9	77.55	8.81	23	87.70	42.49	6.52

Table 1: Sample's Descriptive Statistics

Based on our observations during the previous semesters, we expected that the new teaching method is more effective. If we can show a statistically significant difference between these groups, we can conclude this observation to be valid. To do this, we set up a null hypothesis that the two sample means are the same (or the two groups come from the same population), and attempt to reject that hypothesis.
The standard approach for this situation is to use the small-sample t-test. The first theoretical assumption about the independency of observations is satisfied, based on the information provided about the sampling procedure. The second assumption is about Normality (the histograms of all four distributions are bell-shaped and approximately symmetric). To verify the Normality in more rigorous way, we used the
2
- test for the Final Exam and Final Grades for both groups. The results of four tests are represented in (Table 2).
The test shows that for all four distributions the calculated value chi-square is less than the critical value for the indicated degrees of freedom and . There is insufficient evidence to reject the null-hypothesis about the Normal distribution for given samples. Consequently, each sample data set appears to come from a 300 population that is approximately Normal. Moreover, as we know, the t-test is robust to moderate departures from Normality.
The third and weaker assumption is that the two samples come from distributions with approximately the same variance.
For our samples we can see that the ratio of the larger to smaller standard deviation is greater than two, so the unequal variance test should be used. In the case where the sample sizes and variances are different (n1 n2 and 1
2
2
), an approximate smallsample test can be obtained by modifying the standard deviation and the degrees of freedom associated with the t-distribution.

Final Exam Group	n	2	df	critical
Treatment	15	3.53	1	0.05 3.84
Control	23	4.37	1	0.025 5.02

Final Grade Group	n	2	df	critical
Treatment	15	1.52	1	0.05 3.84
Control	23	0.78	2	0.05 5.99

Table 2: The Results for
2
- test for Final Exam and Final Grades
We used the Approximate Small-Size Procedure (McClave, Sincich, 2003, p.390, J.Devore, 2004, p.373) when 1
2

□ □2
2
. We let □1 and □ 2 represent the population
mean for the Final Grade of the treatment and control groups, respectively. The null
hypothesis H0: (□1 - □ 2) = 0 signifies that samples do not differ significantly. The
alternative hypothesis is H1: (□1 - □ 2) > 0.
Test statistic 1 2
2 2
1 1 2 2
X X
t
s n s n
!
=
+
where t is based on degrees of freedom equal to
( )
( ) ( )
2
2 2
1 1 2 2
1 1 2 2
1 2
1 1
s n s n
df
s n s n
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Table 3 (test results) shows that the t-values for both tests are much bigger than the
critical value of the t-test with the 5% Type I Error. The table of t-distribution
(J.Devore, 2004, p.747) shows that the area under the 32 degrees of freedom t-curve 301
to the right of 2.851 for the Final Exam test is 0.0035. So the p-value for the uppertailed test is also 0.0035, that corresponds to the probability 0.35%. For the Final
Grade test the p-value is only 0.001 or 0.1%.
t df □t Confidence Interval critical
Lower limit Upper
limit

Final Exam
2.851 32 0.05 1.694 2.40 9.45
Final Grade
3.418 33 0.05 1.693 2.70 8.00
Table 3: The results of the Small-Size t-test
With a confidence level 0.95, we estimate the difference in the mean of the Final
Exam scores between treatment group and the control group to fall in the interval
(2.40, 9.45). This means that with 95% confidence we estimate the mean Final Exam
score for the new method to be anywhere from 2.40 to 9.45 points more than the
mean of the Final Exam score for the standard inquiry-based method.
The 95% confidence interval for the Final Grade is (2.70, 8.00), so we estimate the
mean of the Final Grade for the new method to be anywhere from 2.70 to 8.00 points
more than the mean of the Final Grade for the standard inquiry-based method. In
other words, the new method is associated with higher mean scores.
Therefore there is enough evidence to indicate that (□1 - □ 2) differs from zero and that
(□1 - □ 2) > 0. Using a significance level of 0.05, we can reject the null hypothesis that
the two sample means are the same in a favor of the alternative hypothesis which
states that the treatment group mean is significantly bigger then the control group
mean for the both the Final Exam and the Final Grade.
Consequently, the statistical analysis of the data collected shows that the technology
enhanced group achieved significantly higher mean scores than the control group.
These higher mean scores obtained by the treatment group translates into the
treatment group having a greater understanding of math content when compared to
the control group. This greater understanding can be directly contributed to the
effective implementation of the Tablet PC technology in the math and math methods
courses. Thus, we simultaneously achieved two goals. We increased students' math
education software literacy and math content knowledge.
This study was conducted in math and math methods classes. The natural extension
of our work is to evaluate the change in acquiring mathematical pedagogical
knowledge by pre-service teachers. Future teachers (in both groups) are asked to
create original math lessons using manipulatives and hands-on activities. Students in
the treatment group are required to use Tablet PCs to create hands-on activities and
virtual manipulatives. Groups' pedagogical knowledge will be compared using
pre/post tests, questionnaires and knowledge and attitude surveys. Preliminary 302
observations show that even students who did not improve significantly their math
content knowledge benefit from using Tablet PCs in terms of mathematical
pedagogy. They become more creative in developing hands-on activities, become
very confident in searching the Internet and develop good skills to critically analyze
existing mathematical activities posted on the Internet.
Another important aspect that will be studied is how well the collaboration helps preservice teachers to learn mathematics and pedagogy. Collaboration is recognized as
an important forum for learning (Bransford, J. D., Brown, A. L., Cocking, R. R.,
Eds., 2000.), and research has demonstrated its potential for improving students'
problem-solving and learning (Slavin, R. E, pp. 145-173, 1992, Johnson, D. W. and
Johnson, R. T., pp. 23-37, 1990). In both treatment and control groups students are
working in teams. However in the treatment group Discourse software is extensively
used for teaching, learning and assessment by both faculty and students. Important
feature for collaboration evaluation is "chat" option. Students are encouraged to chat
only on the topics discussed in the class session. These chats can be saved in the
archive file and then be analyzed to evaluate the level of cooperation, team members'
participation in collaborative work on the project as well as level and complexities of
content questions asked.
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What can be learned from metacognitive guidance in mathematical online
discussion?
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This study compares two mathematical online learning environments: Online learning supported either with
explicit metagognitive guidance (MG) or with no metagognitive guidance (NG). The metacognitive guidance
was based on three aspects: Using the IMPROVE self-metacognitive questioning method for problem solving
(Kramarski & Mevarech, 2003), discussing features of mathematical explanations, and practicing ways of
providing online feedback. The effects were compared between mathematical online problem solving of a
real life task and students' mathematical and metacognitive discourse. Participants were 79 ninth-grade
students in Israeli junior high schools. Results showed that MG students significantly outperformed the NG
students in online problem solving with regard to mathematical explanations. We also found that the MG
students outperformed their counterparts in various criteria of mathematical and metacognitive discourse.
The practical and theoretical implications of supporting online learning with metacognitive guidance will be
discussed at the conference.
Introduction
New advances in technology have brought challenges and opportunities to
mathematical education and instruction methods. Online environments provide
students with dynamic, interactive nonlinear access to a wide range of information
represented as text, graphics, and animation as well as to self-directed learning in
online discussion (e.g., Jacobson & Archodidou, 2000).
Discussion mediates shared meaning. Through critically examining others' reasoning
and participating in the resolution of disagreements, students learn to monitor their
thinking in the service of reasoning about important mathematical concepts (e.g.,
McClain & Cobb, 2001). Online discussion allows asynchronous exchanges and
permits both one-to-one as well as one-to-many interactions. The students exhibit
motivation, learn independently, and transfer and apply the knowledge to real-life
situations. Learning in such an environment requires students to self regulate their
learning; that is, to make decisions about what to learn, how to learn it, when to
abandon and modify plans and strategies, and to increase effort. Specifically, students
need to analyze the learning situation, set meaningful learning goals, determine which
strategies to use, assess whether the strategies are effective in meeting the learning
goal, evaluate their emerging understanding of the topic, and determine whether the
learning strategy is effective for a given learning goal. Students' need to monitor their
understanding and modify their plans, goals, and strategies (e.g., Zimmerman &
Schunk, 2001). However, research indicated that too few students are skilled at
regulating their learning to optimize self-directed learning. For the most part, studies
have found that students learn little in online environments and they do not deploy
key self-regulatory processes and mechanisms such as effective cognitive and
metacognitive strategies during learning (e.g., Azevedo & Cromley, 2004). 304
Recent research has begun to examine the role of students' ability to regulate several
aspects of their cognition, motivation, and behavior during learning environments
with explicit metacognitive guidance as self-questioning and providing feedback
(e.g., Azevedo & Cromley, 2004; Kramarski & Mizrachi, in press; Kramarski &
Mevarech, 2003).
There is little research in the field of mathematics to accurately determine the benefits
and pitfalls of new technology such as online discussion particularly when compared
to a learning environment embedded with metacognitive guidance. Gaining
knowledge about process and outcomes of online discussion with and without
metacognitive guidance, help educators and researchers to gain insight on students'
problem solving of real-life task and online discourse.
The purpose of the study is two-fold: (a) To investigate the ability to solve online
real-life tasks of students' who were exposed either to metacognitive guidance (MG)
or with no such guidance NG; and (b) to examine the online discourse of students'
who were exposed to these instructional guidance with regard to mathematical and
metacognitive aspects.
Method
Participants
Participants were 79 (boys and girls) ninth-grade students who studied in two classes
within one junior high school in central Israel. Each instructional approach was
assigned randomly to one of the classes. No statistical differenced on a mathematical
pre test were found between the two groups (M=83.30; SD=16. 80; M=80;
SD=15.70; t(78)= 2.01; p>.05).
Measurements
The study utilized two measures: (a) an online real life task; and (b) online discourse.
(a) An online real life task
A real-life task was administrated in online discussion adapted from PISA (2003).
The task describes an orchard planted by a farmer. The students are asked to find
patterns in change and relationships by comparing the growth of apple trees planted
in a square pattern and conifer trees planted around the orchard and to explain their
reasoning. Relationships are manipulated in a variety of representations, including
graphical, tabular, and symbolic.
Scoring: Each item on the task was scored from 0 (not responding, or wrong
response) to 1 point (correct answer/explanation). The scores were translated into
percents. The quality of explanations were analyzed based on two criteria of
arguments: Mathematical arguments (e.g., formal or daily arguments); and
Procedural arguments (e.g., calculation example).
(b) Online discourse 305
Students' online discourse during the solution of the real-life task was analyzed in
two aspects: Mathematical discourse and metacognitive discourse. Mathematical
discourse refers to four criteria: Number of statements, mathematical terms,
mathematical representations and final solution. Metacognitive discourse refers to
three criteria: Errors identification, process description and mathematical
explanations.
Scoring: Sum of references provided to each category during the online discussion
was calculated.
Instructions
General online discussion instructions: Students from both groups (MG & NG)
participated in an asynchronous forum based on problem solving discussion of reallife tasks. Students practiced problem solving in pairs for a four-week period once a
week in the computer lab (45 min). Three stages were implemented in the online
discussion: First, each pair were asked to solve the task and to send the solution to
another pair online as a text in the forum. Second, each pair were asked to provide
and receive feedback to the solution from a counterpart pair. During the third stage
each pair corrected the solution and sent it as a text to the forum and as an attachment
file to the teacher. The teacher encouraged students to be engaged in the whole forum

discussion by providing mathematical explanations and feedback, and resent their corrected solutions.
Metacognitive online discussion guidance: The metacognitive guidance was based on two parts: The first part was based on the IMPROVE metacognitive questioning method (Mevarech & Kramarski, 1997 and Kramarski & Mevarech, 2003) and the second part was based on practicing explicit strategies for providing mathematical explanations and feedback. The IMPROVE method utilizes a series of four selfaddressed metacognitive questions during problem solving: Comprehension, connection, strategic, and reflection.
Comprehension questions prompted students to reflect on the problem/task before solving it (e.g., "What is the problem/task all about?"); Connection questions prompted students to focus on similarities and differences between the problem/task they worked on and the problem/task or set of problems/tasks that they had already solved (e.g., "How is this problem/task different from/similar to what you have already solved? Explain why"). Strategic questions prompted students to consider which strategies were appropriate for solving the given problem/task and for what reasons (e.g., What strategy/tactic/principle can be used in order to solve the problem/task?" and WHY). Reflection questions prompted students to reflect on their understanding during the solution process (e.g., "What am I doing?"; "Does the solution make sense?").
In the second stage students were engaged in a discussion about the question "what does it mean to provide a good mathematical explanation". They were exposed to features of mathematical explanation such as, mathematical expressions, 306 representations, conclusions and clarity. In addition, students practiced how to provide feedback technique by reflecting and discussing examples. The metacognitive questions and instructions on how to provide explanations and feedback were printed on an index card and students referred to this guidance in the following circumstances: During their turn to solve the task, during the discussion about the solution, and their providing explanations and feedback regarding peers' solutions.
Results
The primary purpose of our study was to investigate students' online real-life problem solving with regard to mathematical explanations.
Table 1: Means
1
, and standard deviations on online real-life task by method of guidance
MG
N=43
NG
N=36
F(1,77)
Problem Means 86.68 74.46
Solving S.D 19.90 26.87
7.98*
Mathematical Means 39.93 27.90
explanations S.D 21.09 21.17
9.94*
Note:
1
range 0-100; *p<.001
We performed a one-way ANOVA on real-life task scores. Results indicated that the online MG students significantly outperformed their counterparts (NG) on mathematical problem solving and providing mathematical explanations. In addition, we found that at the end of the study more MG students provided mathematical arguments than the NG students (72%; 50%, t(77) = 3.97, p<.05 respectively), whereas no significant differences were found between the two environments on procedural arguments (17%; 20%, t(77) = 0.57, p>.05 respectively).
The second purpose of our study was to examine students' online discourse with regard to mathematical and metacognitive aspects.
Table 2: Means
1
, and standard deviations on online mathematical discourse by method of guidance
MG
N=43
NG
N=36
F(1,78)
Number of Mean 5.44 .94
statements SD 3.38 1.28
29.80* 307
Mathematical Mean 1.39 .44
terms SD .90 .50
17.82*
Mathematical Mean 11.04 .15
representations SD 15.71 .02
11.15*
Mathematical Mean 18.22 12.03
solution SD 19.72 26.31
2.05
Note:
1
range: Sum of references provided to each category. *p<.001
We performed a MANOVA and an ANOVA on each criteria of mathematical and metacognitive discourse. Table 2 indicated that the MG students significantly outperformed their counterparts (NG) on mathematical discourse regarding three criteria of providing feedback: Using number of statements, mathematical terms, and reference to mathematical representations. No significant differences were found on providing feedback regarding the final solution.
Table 3: Means
1
, and standard deviations on online metacognitive discourse by method of guidance
MG
N=43
NG
N=36
F(1,78)
Errors Mean 2.40 .13
identification SD 6.11 .04
5.39*
Process Mean 7.84 4.17
description SD 14.15 12.67
2.8
Mathematical Mean 16.64 4.17
explanations SD 19.18 12.68
5.53*
Note.
1
range: Sum of references provided to each category. *p<.05
Similarly, Table 3 indicated that the MG students significantly outperformed their counterparts (NG) on metacognitive discourse regarding two criteria of providing feedback: Errors identification, and mathematical explanations. No significant differences were found on providing feedback regarding process description.
Discussion and conclusions
Our findings indicated that metacognitive guidance in an online learning environment might be a vehicle for students' mathematical problem solving and discourse. There are possible reasons for the beneficial effect of the metacognitive guidance. It seems, that IMPROVE guidance integrating with discussion about mathematical 308 explanations and providing feedback might help students access and interact with the content functionality, think about the deeper concepts and structure of disciplinary relations, and avoid superficial details. Our findings extend other findings in nontechnology environments which indicated that the IMPROVE method had a cognitive effect on students' mathematical reasoning (e.g., Kramarski & Mevarech, 2003).
We recognize the need to understand more about how mathematical problem solving and students' discourse emerge in different advanced technology environments. Our study focuses on questions such as how do students learn mathematics with technology and what mathematics do they learn. The metacognitive framework, findings and implications will contribute to the discussion of theme 3 (contribution to learning mathematics).
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HALF-BAKED MICROWORLDS IN CONSTRUCTIONIST TASKS
CHALLENGING TEACHER EDUCATORS' KNOWLEDGE
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This paper illustrates how five teacher educators in training were challenged with respect to their epistemology and perceptions of teaching and learning mathematics through their interactions with expressive digital media during a professional development course. The research focused on their experience of communally constructing artifacts and their reflections on the nature of mathematics and mathematics teaching and learning with digital media. We discussed three different ways in which this media was used by the teachers; firstly, as a means to engage in techical-applied mathematics to engineer mathematical models; secondly, as a means to construct models for students to engage in experimental-constructivist activity; thirdly, as a means to engage in a discussion of a challenging mathematical problem.
This paper addresses theme 3 and in particular the kind of professional development programs which might be appropriate to prepare teachers for using mathematical microworlds in their classrooms. It discusses 'half-baked microworlds' as tools in tasks designed to generate teacher educators' reflection on pedagogy and mathematical ideas based on the experience of jointly constructing and reconstructing such microworlds. Half-baked microworlds are programmable pieces of software designed so that teachers would want to build on them, change them or decompose parts of them in order to carry out some mathematics for themselves or to build microworlds for students. They are meant to operate as starting points, as idea generators and as resources for building or de-composing pieces of software. They are not built and presented as ready made environments to be understood by the teachers and then used by students. Instead, the point is to change and customize them and thus to gain ownership of the techniques and the ideas behind microworld construction. A crucial characteristic of half- baked microworlds is that they invite construction and re-construction of structural parts of the software, i.e. they are not just 'containers' of content which teachers can put in or take out. They are thus based on technological platforms which allow for deep structural access (diSessa, 1997). It is the very nature of the structure and the tools for structural change which are an integral part of the mathematics at hand. The platform used in the study described in this paper is 'E-slate' (Kynigos 2004, 2002), a programmable construction kit similar to diSessa's 'Boxer', but different in that the elements for construction are generic pieces of software themselves (diSessa, 2004).
In this paper, I describe the process by which one of these E-slate half-baked microworlds was developed and changed within respective tasks in a course for 310 teacher educators, focusing on its potential for generating mathematical, pedagogical and professional reflections amongst them. The paper is a synopsis of alarger paper describing this research and including different half baked microworlds.
The study was carried out within the context of an innovative systemic initiative to employ three university sites to prepare experienced teachers to take on the role of teacher educators in the use of ICT in educational practice. At the Educational Technology Lab, , these teachers participated in courses, constructed digital artefacts and corresponding materials for student learning and engaged in teacher education courses as part of their training. A particular aspect of the pedagogical design of the course was to support these teachers to acquire some experience not only with the technology but most importantly with pedagogical knowledge about constuctionist (Papert et al, 1991) experiential mathematics and at the same time with the process of joint reflection on the practice of this kind of mathematical activity. Korthagen and Kessels (1999) discuss Aristotle's notion of episteme versus phronesis, i.e. theoretical de-contextualized knowledge applicable to a wide spectrum of situations versus situation – specific knowledge derived directly from experience within that situation and aimed at meeting a problem within the situation itself. In arguing for a realistic teacher education pedagogy, they suggest that building on teachers' phronesis is critical to their understanding a theory and most importantly in building a constructively reflective habit of mind in the teachers,

helping them perceive their profession as a developing one. Our particular focus was on the ways in which half-baked microworlds designed as tasks for doing mathematics with, were used in phronesis-generating situations, i.e. within the process of action, discussion and reflection during the course. My agenda for supporting these kind of situations was formed mainly because of the context of the Greek Education system where a frontal encyclopedic and revelatory educational paradigm coupled with minimal teacher education mechanisms makes it very hard for such mathematical activity to be understood, valued and practiced.

Context

Describing the context in which a course like this is situated is a complex thing to do, since one can select a large variety of contextual characteristics, from the macro-level of the educational system and the cultural - historical and political time in which the course took place, to the specifics of the institutional dynamics and the roles of the actors engaged in the course. The course described here, for instance, was in itself an innovation within the Greek education system.

The context of this system is characterized by a centralized nation-wide administration coupled with a single national curriculum. In this sense, the teacher is placed in the role of the technical implementer of this curriculum rather than in the role of a professional implementing a developing personal pedagogy. With such constraints, it is very hard to distinguish innovation from systemic reform, and it is difficult to imagine individual teachers involved in curriculum design and in trying out alternative teaching methods, both of which were central to the aims of the 311 course. Furthermore, at the secondary level there is almost non-existent teacher education at both pre-service and in-service levels, only some optional courses are provided in undergraduate programs, while at in-service level there is mainly a range of Masters' level university based courses. It would be thus fair to suggest that from a systemic point of view, the teacher profession is considered as that of a nondeveloping practitioner. In this kind of context, it was unlikely that teachers would start constructing things with a piece of educational technology unless this was done through starting up a program for teacher education that aimed to institutionalize not only the use of technology in schools, but also the idea of teacher professional development supported by the system.

The course was a constituent element of a middle-scale initiative from the Ministry of Education involving the installation and use of digital technologies in 10% of secondary schools ('Odysseia' project

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, Maritsas et. al., 1992). The objective of the teacher education course described here was to train experienced teachers, selected by the Ministry of Education, to become teacher educators in the use of digital technologies for teaching and learning in their respective subject. It was one out of three such courses funded by the Odysseia project and was carried out by the (Author's site). During and after completion of the course, these teacher educators were relieved from their school duties and given the task to engage in in-service teacher education programs in 3-5 schools neighboring their own.

The course at the (Author's site) was thus designed on the premise that, from an educational point of view, digital technologies can at best provide the educational community with sources of information and media for communication and expression (Author, 2001). With respect to mathematics education, if this technology is put to educational use it may support richer learning activity based on symbolic expression, construction, experimentation, investigation, data handling. It may also support the generation of social learning modes where authentic questioning, research, use of human and artificial feedback, argumentation become recognized and valued in school. A course to train teacher educators would thus need to perceive teacher education as a systematic, life-long professional development activity addressing teachers' epistemologies, practices, pedagogies and subject-related knowledge. It would entail teachers supported to adopt the role of reflective practitioners personally engaged in co-implementing the innovation. This role for teacher educators cannot be prescribed and handed to them by their administration when it constitutes such an innovation. Rather, it needs to be actively claimed and shaped through practice.

The project (code name 'E42') involved a year - long course

2

to train 15 selected secondary teachers (five from each mathematics, science and humanities disciplines). The aim of the course was a) to provide the teachers with methods; knowledge and experience in in-service school based teacher education, and b) to educate them in the pedagogical characteristics and uses of exploratory software and communication technologies. The E42 project involved a "sandwich" course where 312 teachers would alternate between full time presence at the University and practice at in-service teacher education in three schools to which they were each assigned. The content of the course ranged from teacher education methods to teaching and learning with subject specific exploratory software. In this paper, the focus is on work with the five mathematics teachers.

Technology

During the course, the five mathematics teachers worked with four kinds of digital media for learning mathematics:

Data handling software (they used 'Tabletop', see Hanckock, 1995)

CAS software (they used Function Probe, see Confrey, 1993)

DGS software (they used Geometer's Sketchpad, Jakiw, 1991)

Programmable microworlds software (they used Turtleworlds and E-slate software, see Author, in press)

In this study, the focus is on the teachers' work towards the end of the course, when they were learning to use the fourth category. During the whole of the course, a central feature of the method used to encourage the teachers' engagement with and appropriation of the technology, was to start each time with what we called 'a halfbaked microworld'. Half-baked microworlds are pieces of software designed so that the teachers would want to build on them, change them or de-compose parts of them in order to carry out some mathematics for themselves or to build microworlds for students. They are meant to operate as starting points, as idea generators and as resources for building or de-composing pieces of software. In a sense, they operate like diSessa's toolsets (diSessa, 1997) in that they are not built and presented as ready made environments to be understood by the teachers and then used by students. Instead, the point is to change and customize them and thus to gain ownership of the techniques and the ideas behind microworld construction as outlined earlier. In this paper, I describe the process by which two of these half-baked microworlds were developed and changed through the experience of the course, focusing on its potential for generating mathematical, pedagogical and professional reflections amongst the teachers.

Method

A design research method was selected as appropriate for the study, the researcher taking on the role of participant interventionist. The data collected included research journals, the to-be educators' reports from their pilot courses with colleagues, the instructor's reports on observations of a sample of these courses, a semi-structured interview on their perceptions of mathematics and mathematics teaching with digital media and the various versions of the software they constructed. The analysis of the data is presented in the form of a story addressing a key aspect of the research findings. The research focused on the trainee teacher educators' experience of communally constructing artifacts and their reflections on the nature of mathematics and mathematics teaching and learning with digital media. The story respectively discusses the way in which this media was used by the teachers; firstly, 313 as a means to engage in technical-applied mathematics to engineer mathematical models; secondly, as a means to construct models for students to engage in experimental-constructivist activity; thirdly, as a means to engage in a discussion of a challenging mathematical problem.

A half-baked microworld: creating functional relations between number lines

The 'connected number lines' microworld consists of a series of sliders connected through Logo scripts so that moving the pointer on the first would result in a mathematically related corresponding movement on the others. The version the teachers started with had an additive relationship between first and second slider (+100), a multiplicative one between first and third (x2) and an exponential one between first and fourth (power of 2). In that sense, the first slider 'x' is the independent variable in three distinct cases. In each of these, sliders y, z and w are respectively the dependent variables in the three corresponding functional relationships. The interesting aspect of this microworld is the representation of function in one dimension rather than two in an ortho-canonical system. Since the number lines are aligned, dragging the cursor on the x slider causes simultaneous movement on the others. The additive function resembles a movement where x and y have the same velocity, the multiplicative function resembles y moving at greater but constant speed than x and the exponential one resembles acceleration. The microworld thus provides a sense of the 'effect' the independent variable has on the dependent one in each case.

So, in this teacher educator course, the strategy was not to provide the participants with exemplary tools to master, but rather to engage them in situations where phronesis (in the sense of Korthagen and Kessels, 1999) would be built, as a result of their own experiences with designing courses with the technology. The halfbaked microworld was thus designed as phronesis - generating tool, in the sense that it was not imposed on the participants as exemplary or complete case. What they would do with it was left open at the beginning, so that discussion and challenge related to constructivist learning theory would emerge from pragmatic situations.

Findings: Teachers' theoretical mathematics

The teachers were shown the half-baked connected number lines microworld. They were shown by the instructor how the sliders-number lines were connected through the E-slate authoring mechanism of writing Logo scripts defining the connections between component pieces of software, which in this case were replicates of one simple such component, the slider. The commands of the script were shown and explained and subsequently, the behavior of the software and the didactical design behind it was discussed. The idea of what the slider representations 'this is a framework which aims to engage the students in the activities through problem solving, the problem being within the framework of the curriculum... to engage students in generalization processes'. 314 show seemed interesting and teacher B referred to the idea of rate of change which he had read in a math education paper by Kaput. That is, that the slider connected with a multiplicative function seemed to go steadily faster than the independent one, whereas the one connected with an exponential function seemed to behave very differently. The teachers were then asked to think up microworlds of their own, based on the connected slider idea.

Not surprisingly, at first, focus was on re-creating a connection between two sliders so that the syntax and the mechanism would be understood. Soon however, the discussion centered on the mathematical relations between the sliders. What kind of representation do the sliders provide? How can we make something interesting happen? A teacher then had the idea of composite functions, i.e. making one slider dependent on the next and so forth. They inserted a functional relationship of x-1 between first and second slider and another of 3x between second and third and observed what happened when moving the first. In mathematical notation terms, we have: f(x) = x-1, g(x) = 3x, f(g(x)) = 3(x-1). In slider notation terms we have: sa, sb=sa-1, sc=3sb (sa, sb, sc are the slider names respectively). The discussion centered for a while on what kind of tasks we could give to students. For instance, if slider sa has a range of 30, what range should slider sb have so that the pointer covers it when we drag the pointer of slider sa the whole distance? How can I make a currency conversion machine (drachmas to euros)?

At some point, however, this style of designing exercises for their students suddenly changed. Teacher B suggested trying to create a model of an interesting mathematical investigation he himself was engaged in a short while back involving fractal geometry. Teacher B was the ultimate "bricoleur" (do it yourself) personality. He enjoyed using technology and spent a lot of time developing elaborate constructions especially with Geometer's Sketchpad. He perceived computers to be "a tool to play with ideas" as he said during the interview. Also that 'mathematics is a means for solving any problem, it's creative. It's an interplay between theory and practice'. However, he was quite overwhelmingly proficient in relation to the other teachers. This influenced his teacher education pedagogy, at least at the beginning as shown by one of the instructors' reports.

The other teachers agreed to build the model suggested by teacher B. He explained the problem, which involved the process of repeatedly forming the composition of a function with itself. He represented that by starting with an independent slider and then consecutively linking sliders, each taking double the value of the immediately previous one. What is formed in this way, is a series of

Teacher B (instructor's report): He himself organized what the topic and the precise use of the software would be. This was good because it helped focus on specific functionalities of the software. However, it did not allow the trainees to act personally and to make decisions on what functionalities they needed for the activity. 315

compositions starting from the function f(x)=2x, then onto f1(x)=2(f(x))=2

2

x, then

onto f2(x)=2(f1(f(x)))=2

3

x and so on. By placing these sliders in vertical alignment, the slider cursors represent the values of x. The formation of these cursors is equivalent to the graphical representation of plotting the number of iterations with the value of fn(x). In teacher B's representation, the variation tool was inserted (the emergent connecting of pieces of software as a microworld construction progresses is characteristic of E-slate authoring, see Kynigos, 2004) to signify the value of the parameter, in a generalization of the problem where the initial function is f(x)=ax. What was interesting and was part of teacher B's reading and investigating, is that for a given value x, changing the value of the parameter created different cursor patterns. The teachers discussed the mathematical issues in theoretical terms, away from the medium, occasionally sketching on paper. However, this whole discussion was initiated from selecting a problem which was interesting and challenging to them as mathematicians (from a mathematical, not an educational/teaching point of view), representing it with a digital medium so that it could be dynamically manipulated. The manipulation ignited discussion on the behavior of the representation which quickly went on to a discussion of the mathematics behind it.

Being immersed in an activity where the mathematics itself was a challenge may have allowed the teachers to feel what doing mathematics is like. Even though teacher D had stated that 'new knowledge comes form the teacher and the book', in

another part of the interview, he allowed for students' generation of mathematical meaning through experimentation with the tool.
In discussing his own teacher education pedagogy, he also referred to the value of teachers' experiences in using computers and in mathematical pedagogy based on experiential learning with this technology.
So, the use of this story was not to point out that such experience will in itself change teachers' beliefs in a predictable and reproducible way. Rather, stories like that constitute situations rich in potential for phronesis-style experiences to take place. This is because they are based on construction and representation activity with
Teacher D (interview): There are times when the student discovers things from trying them out on the computer and observing what happens. I've seen students, because of the tool, find solutions which were unexpected, yet they were correct. Once, the teacher I was observing scolded a student for doing his own thing. That teacher was greatly surprised when he realized that the solution was correct, after I had pointed this out to him.
Teacher D: For teacher education to take effect, the teacher needs hands-on experience with the tool.
Teacher D: I believe computers will be widely used when teachers understand that this tool differs from the classical method when everything had to do with the teacher's brain. 316
the software and on discussion related to the constructs and their designed use. An integral and crucial part of the activity was the interchange of roles played by the teachers which in this case was model constructors, microworld designers and mathematicians. Bringing these changes into awareness and turning them into material for reflection operated as a means for associations with different aspects of pedagogical knowledge. The pedagogical milieu orchestrated by the instructor is of course crucial to the emergence of clear associations with episteme style knowledge and the potential for that knowledge to be put to use in subsequent teaching.
The research shows how the emergence of these uses was a process which challenged teachers' knowing with respect to teaching and learning mathematics, but also regarding their view of the nature of mathematics itself. In a sense, it was the process of experience with the half-baked microworld which seemed to play a critical role in bringing genuine mathematical discussion and activity into the context of professional practice. In parallel, the same process encouraged reflection on mathematical teaching and learning issues. In that sense, I argue that constructionist mathematics may not necessarily be an agenda only for student tasks. There is a place for constructionist mathematics in task design in contexts where the pedagogical agenda for teacher education is the generation of knowledge based on phronesis. What this study suggests is that it might be worthwhile to consider this perspective as a design factor for tasks in teacher educator development contexts.
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Study of a teacher professional problem: how to take into account the instrumental dimension when using Cabri-geometry?
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The question addressed by this contribution is how teachers organize the conditions for an instrumental genesis and to what extent they foster mathematics learning through an instrumental genesis, in the case of geometry learning and teaching based on the use of a dynamic geometry software. The contribution is based on a French project aimed at studying the integration processes of dynamic geometry at primary school and the beginning of middle school.
Integrating ICT into teaching and especially the teaching of mathematics is supported by official institutions for education in several countries. It is now well known that such a political will is not a sufficient condition for a real integration of digital technology into the everyday teaching practice. The present paper addresses the problems teachers are faced with in a context of institutional support of use of technology when they make practical decisions for organizing the use of technology in their teaching. This issue is discussed on the basis of a research project developed in France. There is currently in France an institutional injunction for using ICT in the teaching. However there is still a long way until teachers view technology as a tool for fostering learning. In our approach, we consider that teachers are faced with a professional problem. They must turn institutional demands into everyday decisions by making choices within a system of constraints.
The aim of our contribution to ICMI study 17 is to make explicit the constraints and conditions of integration of digital technology into the teaching practice and by means of theoretical tools to propose indicators allowing researchers to evaluate the degree of this integration in a teaching practice. The focus will be on the instrumental dimension (Artigue, 2002). We refer here to the theoretical approach developed by Rabardel and Vérillon (1995) according to which the individual must learn how to use a tool (or an artifact) for carrying out a task by means of the tool. When the tool is complex and offers the possibility of performing operations referring to theoretical domains, this process of instrumental genesis may be long and may need the help or intervention of a more expert person. What we would like to stress here is that, as

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We are grateful to the French Ministry of Education supporting this project with the status of a Technological Research Team in Education called MAGI (Mieux Apprendre la Géométrie avec l'Informatique) 318
technology involved in mathematics education embodies mathematics, the technical and the conceptual parts are intrinsically intertwined (Artigue Ibid.): the use of technology shapes the knowledge constructed by students (Hoyles et al. 2004). The question addressed by this contribution is how the teachers organize the conditions for an instrumental genesis and to what extent they foster mathematics learning through instrumental genesis in the case of geometry learning and teaching.
Our contribution comes within the theme « Teachers and teaching » by addressing the two following questions: "What theoretical frameworks and methodologies illuminate the teacher's role in technology-integrated environments for mathematics teaching?" and "What kinds of pedagogical approaches and classroom organisations can be employed in technology-integrated environments and how can they be evaluated?"
Context of the research work
It is carried out in the frame of a national project (Technological Research Team in Education) entitled MAGI ("Mieux Apprendre la Géométrie avec l'Informatique", in English "Better Learning of Geometry with Computers"). The project is a development and research project involving twenty researchers, teacher educators and teachers divided into groups located in different places in France: It is aimed to study the integration processes of dynamic geometry software, namely Cabrigeometry, into ordinary teaching contexts at primary school and beginning of secondary school. The project consists of two parts:
designing and implementing scenarios of use of Cabri in several classes of primary and secondary schools (in about ten classes)
study of the impact of teacher preparation sessions to the use of Cabri onto their classroom practice.
Dynamic geometry is conceived in this project as a tool for helping students to move from a purely visual conception to the construction of geometrical theoretical concepts such as collinearity, perpendicular, parallel, congruence... Therefore the school levels, primary school and the beginning of secondary school, were chosen.
The entry to a « theoretical geometry » must be prepared in France at primary school and achieved in the first years of secondary school. In addition, geometry is not a favoured subject for primary school teachers who view the teaching of geometry as essentially the teaching of a vocabulary and not as the construction of a coherent model of spatial phenomena and objects. Dynamic geometry can change their view of geometry and motivate them for changing the teaching geometry. This is why it seemed particularly interesting to investigate the integration of dynamic geometry at this school level.
Theoretical framework: mode and degree of instrumental integration
Our analysis of technology integration into teaching is based on a multidimensional approach (Assude & Gélis 2002, Artigue 2001, Guin & Trouche 2004, Lagrange 319 2001, Trouche 2005) that takes into account several dimensions: epistemological, cognitive as well as instrumental, institutional and anthropological.
What do we mean by "degree of instrumental integration"? It measures to what extent mathematical knowledge and knowledge of handling a tool are intertwined in the organization of the instrumental genesis by the teacher. We could consider that it can take any value in a continuum from zero to one, zero meaning no integration at all and one the highest level of integration. However it is difficult to assign such a precise degree to an observed teacher practice and we prefer to define modes of integration, characterizing a teacher practice on a period of time, a session or part of a session, or sequence of teaching sessions.
To determine these different modes, we use several indicators:
• focus of the tasks given to pupils: is the focus on the tool or on mathematics?
• techniques for solving these tasks: do they come mainly within mathematics or technology or within both, or in other words does the task require mainly instrumental abilities (IA) or mainly mathematical knowledge (MK), or both are equally important?
• extent of the intertwining of instrumental abilities and mathematical knowledge involved in the task (IA/MK).
Until now we propose to distinguish four modes of integration: instrumental initiation, instrumental exploration, instrumental reinforcement, instrumental symbiosis. We distinguish between two cases of instrumental abilities: either pupils are beginners or novice in using the artifact (low IA or no IA) or they already have got knowledge of the artifact but they have not yet a good knowledge how handle all the facilities (average IA).
Pupils are beginners:
In the instrumental initiation, the teacher's aim is mainly that the pupils learn how to use the technology (pupils must learn some IA). Pupils are given tasks focusing on the way to use Cabri. The relation between IA and MK is minimal.
In the instrumental exploration, the teacher aims at improving both some IA and MK. Pupils explore the technology through mathematical tasks. The relationship between IA and MK can vary according to the mathematical task and to the content teacher interventions: the teacher may just give information items of how to use a specific facility of the artifact or (s)he may express links between IA and MK.
Pupils are already introduced to handle the artifact:
In the instrumental reinforcement, pupils are faced with instrumental difficulties when solving a mathematical task. The teacher gives them elements of information about how to use a specific item of the artifact to allow them to overcome the technical difficulties. The teacher's aim is improving mathematical knowledge. The relationship between IA and MK may vary according to the way the teacher formulates his/her help for using the artifact (see below § 4.1 the case of Robert on regular polygon). 320
In the instrumental symbiosis, pupils are faced with mathematical tasks that allow them to improve both their IA and MK because these ones are connected. The relation between IA and MK is maximal: each one allows the other to increase and the connection between paper-pencil work and Cabri work is strong. An example of such symbiosis is the construction of a square with given side AB. The difficulty of the task essentially lies in using a circle as a tool for transferring length and thus conceiving a circle as a set of points equidistant of a given point.
These different modes of instrumental integration led us to classify the integration practices from the lowest level (where the only instrumental integration taken into account is the instrumental initiation) to the highest level (where the different modes are taken into account according to the adequacy of the moment).
Characterizing the design of a sequence by means of degrees of instrumental integration
Part of the work carried out by the Grenoble subgroup of our team was to build a teaching sequence aimed to initiate 11 year-old pupils to deductive reasoning and to introduce them to the use of Cabri. Actually, the drag mode in Cabri-geometry provides a means for distinguishing between the properties that belong to the geometrical figure (hypotheses and their consequences) and the ones that belong are valid only for a specific diagram (a particular case). The first ones are preserved by dragging and not the second ones. Thus, Cabri-geometry seems to be a relevant tool to be used by the pupils at the very beginning of the learning of deductive reasoning.
Within this purpose, the mathematical learning objectives are strongly intertwined with the ability to use the software. In fact, the pupils give a mathematical sense to the effects of the dragging as far as they understand the geometrical properties of a figure. To reach its objectives related to deductive reasoning, the sequence has to support the instrumental genesis that turns the dragging into a pupil's instrument to state about the validity of a geometrical property. For the design of the sequence, we paid attention to create a strong interaction between the discovery of Cabri, the development of instrumental abilities and the learning of mathematics.
An example of the design of a task:
Four diagrams were displayed in Cabri. They have not been obtained by the same construction process but they were all looking like a triangle and an inscribed

quadrilateral in the triangle (Fig.1). The pupils were asked to answer questions about parallel or perpendicular lines: are lines BC and GF perpendicular? Are lines GF and DE parallel? Are lines EG and DF parallel? To answer, the pupils were supposed to move every free point of the diagram and to observe whether the property was preserved or not (Figs 2 & 3). This kind of task is possible only after a significant work about moving objects and interpreting mathematically what happens on the screen. With pupils already introduced to Cabri, this task can be considered as coming within instrumental symbiosis. The task is of instrumental and mathematical nature: pupils need to decide 1) to drag elements 2) to drag enough points elements in order to decide about the validity of mathematical properties. 321

A
B
C
D F
E G

Fig. 1 The initial state of all four diagrams Fig.2 Dragging D Fig.3 Dragging A, B or C in one of the diagrams

Dragging D is not enough to state about the parallelism of lines ED and GF. Because these lines look always parallel when A, B or C are dragged, pupils must find a mathematical reason in the construction program (obtained in Cabri with the facility Replay construction): ED and GF were both constructed as perpendicular to line BC and dragging D has no influence on the quadrilateral because F is the reflected point of D with respect to the midpoint of segment BC. However the relationship between dragging and mathematics may strongly vary according to the prior knowledge of pupils. Pupils may be attracted by the only fact that "it moves", once they drag points but do not pay attention in a more precise way to what happens while dragging or misinterpret the phenomena because they do not relate it to geometrical properties. In this case, the expected instrumental symbiosis turns into instrumental exploration since the pupils may be considered as beginners with respect to the interpretation of drag mode and the aim of the teacher becomes to reintroduce dragging as a tool for checking properties. In the same vein, the activity may turn into instrumental reinforcement, if pupils do not know how to get information on the construction program of the diagram, the teacher may give information about the existence and use of the tool "Replay construction" and extend the instrumental abilities of the pupils.

The observation of the teacher during two years showed us that the instrumentation of the drag mode takes a very long time. In the first year, it took seven weeks until pupils decide to drag points on their own with a mathematical intention.

When the gap between the planned mode of integration and its actualization in classroom is important, it reveals the incomplete instrumental genesis of the pupils and thus may be a research tool for analysing the integration of technology.

Characterizing teachers' practices with degrees of instrumental integration

Two subgroups of our project, one in Toulon and one in Amiens, worked with primary school teachers who had no knowledge of dynamic geometry before the project. They were introduced to dynamic geometry in a short training session (half a day) for The Toulon subgroup and in a longer session (one week) for the Amiens subgroup. Each group worked with three teachers, in charge of 10 year-old pupils in Toulon, and of 9 and 10 year-old pupils in Amiens. The agreement was made with the teachers that they would integrate the use of the software in relation to the whole 322 work of their class, that they were completely free to choose activities with the software and that the role of researchers was restricted to observing teachers without intervening on the choice and the design of the activities as well as on the management of the class. We wanted to bring out the conditions and constraints of integration of the software by ordinary teachers who had to construct – everything – including a relationship with Cabri. Analyses are done by means of observation notes, videos, students' notebooks, and interviews with teachers. The teachers of Toulon were observed during one year, those of Amiens during two successive years. It is interesting to note that in both subgroups, the number of sessions devoted to the use of the software varied according to the teachers: from 5 to 15 sessions in Toulon, from 3 to 11 sessions in Amiens.

Two contrasted cases in the observation of teachers

These cases were observed by the Toulon subgroup.

A low level of instrumental integration: the example of Ingrid

Teacher Ingrid

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proposed two initiation sessions to the pupils. During these sessions pupils created and dragged successively a point, a straight line, a circle, a segment and they named points. They did it from reading a form indicating all the actions they were required to do and there was no collective institutionalization

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of these

Instrumental Abilities. The teacher did not point at the status of points, and though pupils dragged, the teacher did not insist on the interest of dragging to check the constructions.

In these sessions, the type of task was a Cabri task whose aim was to build and drag some basic mathematical objects. The relation between IA and MK is minimal. We consider that these sessions come within an instrumental initiation. In addition, these sessions did not insist enough on the changes in the didactical contract: the function of dragging the constructions and the contribution of Cabri to an experimental approach of geometry were not expressed at all. Such an initiation is aimed at creating mathematical objects on the computer instead of emphasizing the kind of work the software allows to do with those objects. In the following sessions the teacher was reluctant to use instrumental reinforcement (although it sometimes occurred) and instrumental symbiosis.

The mode of instrumental integration was limited to an instrumental initiation and that fact leads us to make the hypothesis that during the first year during which teacher Ingrid tried to integrate Cabri software, she did not pay enough attention to the instrumental dimension (although there were initiation sessions). From this point of view the degree of instrumental integration can be estimated as low.

A medium level of instrumental integration: the example of Robert

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Teachers are given fictitious names

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By institutionalization we mean that the teacher is synthesizing knowledge used in the task, which is part of the official knowledge and must be learnt. 323

Teacher Robert did not propose initiation sessions to his pupils. They discovered Cabri through exploring the software tools to achieve some mathematical tasks. In such a case the IA constitute a tool for mathematical tasks (instrumental exploration) but the relation IA/MK is minimal because the link between them was not made explicit. The mathematical task proposed by the teacher was to construct the perpendicular bisector of a segment. This kind of task as well as the notion itself of perpendicular bisector was new for the pupils and the focus of the teacher was on solving problems; pupils were expected to make hypothesis on what is a perpendicular bisector of a segment by handling Cabri. They also discovered how to create a segment, a midpoint of a segment and the perpendicular bisector of a segment by exploring the menu items. As the teacher asked them to justify their answers, pupils were also faced with other IA, such as measuring a segment or checking whether a line and a segment were perpendicular. This instrumental exploration can reach its limits if the teacher doesn't institutionalize some IA at one moment or another, and if they remain under the private control of the pupils.

Later we could observe what we call instrumental reinforcement when teacher Robert brought additional information on how to measure segments when they were not created as such. The pupils had to construct a square and to check that they obtained a square: they used the Cabri tool "regular polygon and wanted to measure the sides of a square. In that case, Cabri provides the measure of the perimeter, but not the measure of the side of the square. As long as the teacher didn't inform the pupils that they needed to create segments to be able to measure them (IA), they could not perform the mathematical task.

We consider that teacher Robert carried out an instrumental integration of medium degree: he took into account several instrumental integration modes according to the pupils' mathematical needs. However he did not institutionalize the instrumental abilities needed to perform the mathematical work and he did not take into account relations between IA and MK, and relations between Cabri tasks and paper-pencil tasks.

Observation of the instrumental dimension on a teacher long term practice

In the Amiens group, teachers were observed during two years. This long term observation allowed us to observe evolution in the modes of instrumental integration.

An increase in the level of instrumental integration: the example of Sara

The first year, the objective of teacher Sara was to come back to geometrical concepts studied at the beginning of year in paper/pencil environment while using dynamic geometry and construction programs. Sara organised three types of session: an initiation session: pupils created and dragged a point, created a segment or a line from two points, created and dragged a point on a segment, created the intersecting point of two lines. Each action to perform on the software was described on a worksheet. Then pupils analyzed the constructed objects to study their properties and answer the questions. Finally teacher Sara institutionalized the three kinds of points, by linking instrumental abilities and geometrical knowledge. 324

two strongly guided sessions: pupils performed given construction programs (rightangled triangle, square) in the environment and analysed the obtained constructions;

eight less guided sessions: pupils had to use these programs to construct or reproduce simple diagrams drawn on paper. Pupils had time for searching and teacher Sara brought information only when pupils had instrumental difficulties. But Sara must bring a lot of instrumental assistance in an individual way. In a collective synthesis, Sara established the link between properties of a figure and the used functionalities of the software.

In the first year, the initiation session aimed at both exploring the software instructions and improving mathematical knowledge in interaction with instrumental abilities. It can be viewed as an instrumental exploration. Later during tasks about reflection, teacher Sara gave information to build symmetric points (instrumental reinforcement). But there was no relationship between software tasks and paperpencil tasks. Furthermore, Sara proposed only construction tasks in which the aim was just to obtain a diagram and the software did not contribute to an experimental approach of geometry. From this point of view, we consider that the degree of instrumental integration is low.

During the second year, Sara alternated software tasks and paper-pencil tasks. Unlike the first year, Sara divided the initiation phase into two parts and in the first part gave more written instructions about the use of the software. She didn't organize any collective institutionalization of these instrumental abilities. But then she articulated geometrical tasks and software tasks to approach the concepts of circle, reflection and parallelogram. The aim was to analyze figures and to make conjectures about geometrical properties. On the other hand, Sara proposed less construction tasks than the previous year.

The first part of the initiation phase can be considered as an instrumental exploration in which the ratio between mathematical knowledge and instrumental abilities became minimal. But in the following sessions, Sara changed the didactic contract: the function of dragging the constructions and the contribution of the software to support an experimental approach of geometry have been developed. Sara also organized instrumental reinforcement when it was necessary to construct new geometrical knowledge by means of the software. Even if the same modes were observed in the first year and in the second year, Sara changed the organisation of the instrumental genesis by increasing the relationship between instrumental abilities and mathematical knowledge.

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The examination of Computer Algebra Systems (CAS) integration into university-level mathematics teaching

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Although the first ICMI study was almost exclusively concerned with the integration of technology into university-level mathematics, there has been little focus on this phase of education as technology-related research has become dominated by schoollevel studies. Computer Algebra Systems have quietly become an integral component of university-level mathematics, but little is known about the extent of CAS use and the factors influencing its integration into university curricula. School-level studies suggest that beyond the availability of technology, teachers' conceptions and cultural

elements are key factors in technology integration into mathematics teaching and learning. In this proposal I report on an ongoing project and summarize results of the first phase of this study, which is based on interviews and observations of 22 mathematicians in three countries, Hungary, UK, and US. In addition, I outline the development of the second-phase in which a questionnaire will be sent to a sample of 3500 mathematicians in the participating countries to investigate the extent of current CAS use and to examine factors influencing CAS integration into universitylevel mathematics education. My research contributes to the ICMI-17 by considering cultural diversity, reflecting on actual uses of technology and addressing potential impact of CAS upon mathematics teaching and learning in universities.

The first ICMI study in 1985 reviewed the history, the potentials, the constraints, and the impact of computers on mathematics and its teaching and learning (Churchhouse et al., 1986). Despite difficulties articulated by several of its authors, the study presented an optimistic future for technology integration into mathematics education. Some years later, due to increasing accessibility to both computers and calculators, Kaput (1992) predicted that technology would become rapidly integrated into all levels of education. However, the accumulated evidence of the last fifteen years indicates that this prediction has not been realized with technology still playing a marginal role in mathematics teaching and learning (Cuban, Kirkpatrick, & Peck, 2001; Ruthven & Hennessy, 2002).

The first ICMI study was almost exclusively concerned with the integration of technology into university-level mathematics (Holton, 2001). More recently, despite a small number of studies reporting on innovative technology-assisted teaching practices and examination of university students' learning, technology mediated mathematics education research has been dominated by school-level studies (Lagrange, Artigue, Laborde, & Trouche, 2003). 327

Papers in the ICMI-11 study discuss, inter alia, the role of technology in a variety of mathematics courses taught in universities, accounts of the ways in which technology can be used to enhance students' learning, and the impact of technology on classroom communication (King, Hillel, & Artigue, 2001). However, the study provides neither an overview of the extent of technology use in universities nor discusses the reasons for the slowness of technology integration, preferring to offer examples of particular practices in particular universities in particular countries. The totality of the report suggests that technology use remains 'cosmetic' (Hillel, 2001). Even though little is known about the state of technology use in universities, recent surveys tell us much about its use in school mathematics and the factors influencing its classroom integration at both national (Becker, 2000; Ofsted, 2004) and international levels (Gonzales et al., 2004; OECD, 2004). These studies suggest that investment in technology can enhance, but not guarantee, increased use of ICT in education, although the TIMSS 2003 study implies that funding for educational technology may not increase the actual use of ICT

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in classrooms (Gonzales et al.,

2004). Other studies, which have investigated the cause of slow technology integration, suggest that, beyond the accessibility of technology and policy pressures, teachers' beliefs and attitudes as well as cultural aspects are vital factors influencing technology integration (Hennessy, Ruthven, & Brindley, 2005; Ruthven & Hennessy, 2002). In addition, international comparative studies have reinforced the importance of cultural aspects by demonstrating that teachers' didactical beliefs and conceptions of the subject, as well as the characteristics of their classrooms and their relation to technology, are heavily affected by teaching traditions and geographic locations (Andrews & Hatch, 2000). Results from such school-based studies may be applicable to the university setting, but a systematic investigation essential.

Aims of the study

Due to the paucity of university-level research outlined above, I designed a study to investigate the current use of technology together with the factors that influence its integration into university mathematics education. In my study, I focus on a specific technology application, Computer Algebra Systems (CAS), because this type of software package is the most widely used in university mathematics education. CAS is explicitly designed to carry out mathematical operations (not a general technology application such as a web-based homework system); and CAS has the potential to become a mathematical tool in students' future studies and career (Artigue, 2005). Specifically, my study aims to examine:

- the extent and manner of CAS use in university mathematics departments;

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Information and Communication Technologies (ICT). Frequently used as a reference to technology (-ies) in the UK. 328

- the pedagogical and mathematical conceptions of university mathematicians regarding CAS, including the factors influencing their professional use of CAS; and
- the extent to which nationally situated teaching traditions, frequently based on unarticulated assumptions, influence mathematicians' conceptions of and motivation for using CAS.

I decided to adopt an international comparative approach in order to understand more completely different teaching traditions and subject-related conceptions (Andrews & Hatch, 2000) at the university level. The participating countries, Hungary, the United Kingdom (UK), and the United States (US), pose a variety of cultural and economic considerations. Obviously, my selection has also been influenced by my personal and professional background, as well as by my familiarity with the higher education systems of these countries. However, international comparative literature advocates the comparison of considerably dissimilar (Hungary vs. UK, US) and similar (UK, US) teaching traditions to elicit similarities and differences (Kaiser, 1999).

Methodology, methods, and preliminary results

To investigate the outlined aims I designed a two-phase study following a mixed method methodology (Johnson & Onwuegbuzie, 2004). The first, qualitative, phase of the study explored those issues that influenced university mathematics lecturers' CAS-assisted teaching. In this phase, I interviewed 22 mathematicians, observed classes, and collected course materials in Hungary, the UK, and the US. Data were analysed by means of a grounded theory approach (Glaser & Strauss, 1967; Strauss & Corbin, 1998). Building on the results of the first phase, I am designing a largescale quantitative study to further examine the issues that emerged from the analysis of the first phase data, to gauge the extent of CAS use in universities, and to uncover additional issues that did not surface in the initial phase of the study.

The analysis of the first phase data identified three clusters of issues:

1. Personal characteristics
2. External factors (institutional and technology issues)
3. Mathematicians' conceptions (of mathematics, mathematics teaching/learning, CAS, CAS teaching/learning)

Many of these issues will be further investigated in the second phase of my study, but space does not permit a detailed discussion here. (A more complete list of the subcategories of these issues can be found in Lavicza (In review). However, I highlight three of the more interesting findings of this phase below.

Firstly, similarly to results of school-level studies, academics' conceptions, proved to be a crucial factor in technology integration into mathematics teaching. Moreover, 329 their conceptions appear more important an influence than for schoolteachers because, due to the academic freedoms of university life, they are less prone to policy pressures. Also, mathematicians are less constrained than schoolteachers by prescribed curricula and uniform examinations. Therefore, mathematicians have better opportunities than schoolteachers to experiment with technology integration in their teaching. However, academics are frequently more concerned with research than teaching and so experiments with technology in their teaching may be seen as counterproductive.

Secondly, mathematicians' primary use of CAS in their teaching is to enhance the transmission of mathematical concepts. Many described using CAS to illustrate mathematical concepts and I did not encounter any instance when they referred to CAS use as a motivational tool. In contrast, school-level studies report that teachers often emphasize the use of technology as motivational and classroom management tools (Hennessy et al., 2005; Ruthven & Hennessy, 2002). This result challenges the applicability of school-level findings to university settings while presenting new possibilities for collaboration that may enhance the integration of technology at all levels of mathematics education.

Thirdly, school-level studies demonstrated noteworthy differences in teachers' conceptions of mathematics and its teaching owing to nationally situated teaching traditions (Andrews & Hatch, 2000). In my study, no distinctive teaching traditions of technology use at the university-level were identified. This may be due to the fact that the participants of my study constituted an internationally mobile group with many experienced in or aware of international university-level teaching practices and research. In addition, the use of technology in university teaching is a fairly recent endeavour. This result accords with Atweh, Clarkson, and Nebres' (2003) idea that mathematics research and mathematics education have become an international enterprise, particularly at the university level.

For the quantitative phase of the study, I am designing a web-based questionnaire for sending to 3500

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mathematicians in Hungary, the UK, and the US. In part this will draw on the results of the first phase but will also attempt to uncover additional issues relating to the aims of the study. Therefore, sections of the questionnaire will enquire about:

1. the current use of CAS by selected participants;
2. participants' personal characteristics and institutional settings;

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This preliminary estimate is based on the estimated population of 35,000 mathematicians in the selected three counties and the desired 20% response rate for the web questionnaire. Detailed sampling strategy is available upon request. 330

3. participants' variety of conceptions of CAS, CAS-assisted teaching, role of CAS in the field of mathematics and mathematics teaching.

It is my expectation that the analysis of the questionnaire data will expose relationships between participants' personal characteristics and institutional settings, their CAS use in teaching and research, their conceptions of mathematics, and their CAS-related conceptions. Furthermore, I plan to exploit factor analytic and structural equation modelling techniques to uncover additional factors that influence CAS integration at universities.

My study will contribute to our knowledge of CAS and its use at the university level by

- providing an overview of CAS use at a large number of universities;
- identifying factors that influence CAS integration at universities and highlight similarities and differences between university- and school-level results;
- allowing insight into mathematicians' understanding of and thinking about CAS and the impact of their teaching/cultural traditions;
- pinpointing some effects of nationally based teaching traditions of CAS use at the university level and mathematicians' conceptions of CAS-assisted teaching.

Results of the study will enable researchers and practitioners to

- pinpoint directions for improvements and show limits of CAS applicability at universities;
- align research into local practices with international trends;
- assist in the possible development of CAS training workshops;
- improve the mathematical preparation of university students.

Once this second aspect of the study has been completed a number of possible research directions present themselves. The study may reveal issues for examination by means of a qualitative study. If, as I hope, the questionnaire proves effective, the study could be replicated in a larger set of countries. In addition, a similar study could be conducted in schools to uncover similarities and differences in university- and school-level use of technology. Furthermore, it would be possible to collaborate with mathematicians as well as school teachers to develop curricula, supporting materials, and a variety of workshops to enhance the use of technology in mathematics education.

If invited to the ICMI-17 conference, I would be able to report on the data collection of both phases of my study and outline the preliminary results of the entire research project. 331

Contributions to the ICMI-17 study

The research reported in this proposal supports the aims of the ICMI-17 study, as set out in the discussion document, in the following ways.

My work aims to identify and analyze aspects of technology integration, primarily in universities (diverse curricular organizations), but also consider connections with pre-university level mathematics education. In addition, my study incorporates and investigates cultural diversity as it takes an international comparative approach to compare the use of technology in a less developed (Hungary) with more developed countries (UK, US) in terms of ICT resources and investment in educational technology . My study addresses well the following aim of the ICMI-17 study:

ICMI Study 17 will also seek to take account of cultural diversity and how issues of culture alongside those related to teacher beliefs and practice all shape the way digital technologies are used and their impact on mathematics and its teaching and learning. (Hoyles & Lagrange, 2005, p.4)

My work is also in line with the following two ICMI-17 study aims:

1) to reflect on actual uses of technology in mathematics education, avoiding mere speculations on hypothetical prospects

2) to address the range of hardware and software with a potential to impact upon or contribute to mathematics teaching and learning. (Hoyles & Lagrange, 2005, p.4)

Although my study concentrates on a particular software package its results should be applicable to a wider range of applications. The examination of CAS is significant for technology-related research because such packages explicitly focus on mathematical activities and do not only reorganize communication in classrooms.

I believe that my research best fits Theme 3 – Teachers and teaching – as it contributes to many of the questions raised in the Study's discussion document. Particularly, it addresses the three questions:

- How are teachers' beliefs, attitudes, mathematical and pedagogical knowledge shaped and shaped by their use of digital technologies in mathematics teaching and how are these issues influenced by access to resources and by differences in culture?
- What can we learn from teachers who use, or who have tried to use, digital

technologies for mathematics teaching?
How can teachers be supported in deciding why, when and how to implement technological resources into their teaching practices? (Hoyles & Lagrange, 2005, p.8) 332
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Developing Learning and Assessment Tasks in a Dynamic Geometry Environment
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This paper presents the development of manipulative tasks in a dynamic geometry environment as a tool for learning and assessment in geometry. In a research study, groups of junior secondary students were asked to manipulate dynamic geometry figures, in the form of Java applets, by dragging movable points to create particular configurations satisfying specified conditions. The tasks are designed in such a way that students are expected to easily make use of learnt knowledge in school geometry to produce the required results. They had to consider real-time measurements provided and constraints on the variation of the figures. As part of design of the tool, students' results of dragging can be recorded as an image and in terms of numerical parameters for later analysis. Students' responses to the tasks were analyzed by first examining quantitatively the variety of configurations produced, followed by clinical interviews probing into the process of students' working on selected tasks. The analysis reveals the complexity of students' interaction with and interpretation of dynamic figures. Based on these results, some major questions are suggested to further explore the nature of dynamic geometry figures and implication of uses of these manipulative tasks in the context of classroom learning and assessment.
Introduction
Dynamic geometry environment is a powerful milieu in which the teaching and learning of plane geometry could be re-conceptualized due to the drag-mode (see for example the discussion in Lopez-Real & Leung, 2004). Indisputably, dragging in dynamic geometry plays a crucial role in the formation of geometrical concepts and conjectures (see for example, Arzarello et al, 2002; Leung and Chan, 2005). In particular, the epistemological values of robust and soft constructions with respect to dragging have been the foci of many studies in dynamic geometry (see for example, Laborde, 2005). Though fruitful insights have been obtained, there is yet a deep understanding on how students perceive the relationship between the spatio-graphical representations (what they actually see on the computer screen) and the theoretical knowledge (the traditional school geometry they learnt). The variation visualized from the drag-mode in dynamic geometry may give students a new reasoning pattern that diverges from the traditional deductive thinking. In particular, the (pseudo) accurate representation of geometrical objects and measurements under dragging offers a confluence of simultaneities that could bring about discernments that might be different from a static paper-pencil environment. Sinclair (2004) concluded in her study on Grade 12 students working on pre-constructed dynamic geometry sketches that "students who focused on visual information and drew conclusions based on the appearance of the onscreen image did not have the tools to develop and communicate a proof based on visual concepts" and this was due to students' unawareness of the 335 accurate representation in dynamic geometry sketches. It would be interesting to probe into the impact of accurate representation and variation under dragging on students' spatio-graphical reasoning that leads to explanations of phenomena in dynamic geometry environments. In this respect, we initiated an experimental instrument to assess students' geometrical understanding in the context of IT activity in a mathematics competition for regional primary schools in Hong Kong (Lee, Wong & Tang, 2004; Wong, Lee & Tang, 2005). The instrument was in the form of dynamic geometry manipulative tasks in which students can vary a point (a dimension of variation) in a geometrical configuration via dragging. Students were asked to drag the point to a position that would satisfy certain required condition. A coded and varying (as the point was dragged) numerical value was associated to the point for the purpose of recording students' answers. This opened up a new arena for quantitative analysis in dynamic geometry research that might yield interesting collective information on students' different ways of interpreting spatio-graphical data in dynamic geometry environment. In this proposal, we continue the quantitative experiment with groups of secondary 1 and 2 (Grade 7 and Grade 8) students in Hong Kong. After patterns are observed in the quantitative analysis, selected students are invited for one-to-one clinical interviews during which they will work on a specific manipulative task while the interviewer will probe them with questions on geometrical understanding of the task.
Design of dynamic geometry manipulative tasks
Each task requires students to manipulate geometric objects on screen by using the mouse to drag around movable points. The dynamic figures are created with the software C.a.R. ("Compass and Ruler" by R. Grothmann, http://mathsrv.kueichstaett.de/MGF/homes/grothmann/java/zirkel/doc_en/) and presented as Java applets so that students can access them through common browsers. These are basically pre-constructed geometric figures in which certain geometric properties are preserved during dragging. Students need not know the hidden construction nor perform additional constructions on the figure. What they observe will be the constrained movement of dependent parts and some displayed real-time measurements (of length, angle size, etc.) as a result of their dragging of movable points. By making use of such observation or information, the students try to turn the figure into a required configuration.
We begin by considering situations in which students may make use of their knowledge and understanding of geometric properties to complete a task. For illustration, let us consider parallelism and angles. In some of the tasks, students are required to make a line parallel to another or to make two lines intersect at a particular angle. In a given figure, students can drag movable points (while the rest are fixed or dependent on others) to re-position lines or segments while seeing measurements of angles continuously updated. Specific solutions, whether there is only one or more, are expected. However, we are not simply interested in the 336 correctness or accuracy of students' solutions, but the variety of results of students' manipulation initiated from the task.
Two of the tasks are explained here to illustrate our considerations.
Task 1
Figure 1a shows the initial configuration of a dynamic figure. The task is to move the point P to make angle a equal to 120°. All the lines are fixed except the one containing P, which will be turned about a fixed point outside the viewing window. Figure 1b shows the result of dragging P to the right and the only position of the variable line that gives the required angle.
Figure 1a Figure 1b Figure 2a Figure 2b
Task 2
This task starts with the figure 2a, in which A, B and C are fixed points while D can be moved freely in any direction. The task is to make a quadrilateral with at least one pair of parallel sides. Figure 2b shows only one among many possible ways fulfilling the requirement.
Collecting and analyzing students' responses
The open source software C.a.R. is modified to allow students to submit electronically the final configuration of their dynamic figures in the browser when a task is finished. We are therefore able to record and retrieve students' responses as an image and numerical values specifying final state of key variable parts. In the examples above, the coordinates of point P (in task 1) and point D (in task 2) will be recorded when students submit their results.
The test was conducted in 9 junior secondary classes from 4 schools with different academic backgrounds in Hong Kong around June 2005. These students had learnt about properties among angles related to parallel lines and transversals, in the same or previous academic year. We first examined distribution of students' responses to individual tasks. By reducing students' submitted configurations into one or two numerical descriptors, we were able to capture the variety of students' results and identify interesting patterns, which provided a useful picture before we probed into individual students' attempts to a task.
For the task 1, we examined students' positioning of the movable point along a horizontal track, which corresponded to different angles formed between the variable transversal and the fixed lines. Nearly half of the students could accurately generate 337 the required angle. Meanwhile, we noticed an unexpected outcome from about one eighth of the students, giving 3 angles in progression, as shown in Figure 3b.
Distribution of x-coordinates of point P in task 1 from 169 students.
Values at about 7 (red column) correspond to result shown in figure 3b.
Figure 3a Figure 3b
Scatter plot of position of point D in task 2 from 162 students.
Positions along the diagonal line (dots in red) give a pair of equal opposite angles, as shown in Figure 4b.
Figure 4a Figure 4b
For the task 2, we considered the scatter plot of all final positions of the vertex D of the quadrilateral. Most students could put the vertex along a line parallel to either fixed side of the quadrilateral, in particular the intersection where the figure resulted in a parallelogram. Among the rest seemingly without pattern, we noticed a special diagonal line attracting some points (about 28 students). Putting the free vertex along this line (near but except the intersection) produced a pair of equal opposite angles

but no parallel sides.
After this preliminary quantitative analysis, we proceeded to interview individual students and observe their processes of working on selected tasks from the test. We 338 interviewed 24 students from 4 schools with similar background but had not done the test in the first stage. Even from such a small sample, we realized the complexity of students' processes of working with the dynamic figures despite overall fairly good results from the test. Even though some students did eventually come to a correct result, the process could be far from a straightforward application of their knowledge of school geometry on the manipulation of the dynamic figures. Students' explanation of their work reflected their incomplete or conflicting conceptions regarding basic geometric notions such as parallel lines and angle relation. This may be due to their inexperience in articulating their geometric thoughts, or unfamiliarity with behavior of dynamic figures and interpretation of real measurements.
A student working on task 1 started with a slow and careful dragging of point P across the screen. He stopped at the position in fig. 3b and declared that it was the required angle. He explicitly pointed out that the "angles differ by 2˚ and then 2˚", leading to his belief that angle a was 120˚ at that position. (Note that the given angles made by the transversal on the right kept differing by 2˚ as P varied.) Yet he also noticed by himself the given angles on the left (84˚, 86˚, 84˚) were not "considered". He started to wonder if the lines were parallel, partly based on their appearance and partly based on their angles between the transversal. He tried to relate the "imbalance" of the lines to the difference in the angles. He finally chose to ignore the middle line, which "deviates from the other two". He proceeded to adjust his answer to that of fig. 1b, where the angles on the upper and lower lines "look symmetric". At the end he was sure that the upper and lower lines were parallel, following his previous inspection of the angles and inclination of the lines. Yet this was not a simple deduction of the angle properties from relevant geometrical theorems, which we may assume from students giving a correct result. On the other hand, we also noticed how the student tried hard to make sense of the real representation and varying measurement during the process.
The results of Task 2 as mentioned above reveal an interesting fact that students attend to a pair of equal opposite angles rather than a pair of angles associated directly with one pair of parallel lines. Some students in our interviews on this task made explicit attempts at making BAD equal to BCD (Figure 4b). We find this point even more intriguing when we look at students' responses to another similar task but with different given conditions. On the one hand, similar responses along a diagonal (as in Figure 4a) are observed. On the other hand, interviews on this task show clearly this undue attention to the opposite angles. Suffice to mention one telling example. A few students first came to a parallelogram (Figure 5a) but provided with a strong reason that each opposite angle gives the same sum of 99˚. When a student was prompted by the interviewer to produce another answer – i.e. a quadrilateral with at least one pair of parallel lines, she managed to produce one with an equal pair of alternate angles (Figure 5b). But she failed to recognize it as an answer and explained that the sum of angles at one corner (33˚ + 66˚ = 99˚
) was not equal to the other
(66˚ + 36˚ =102˚
). 339
Figure 5a Figure 5b
Questions for Investigation
Through the design of the DG manipulative tasks and analysis of students' responses in both quantitative and qualitative manner, we attempt to develop new perspectives on the role of DG in the classroom learning and assessment. We start with a fundamental question.
• How do students understand the dynamic figures presented in this kind of manipulative tasks? In particular, when they drag a figure, how do they interpret the given measurements together with the varying shapes? And, when they want to achieve a certain target diagram, how do they decide on the pertinence of various geometric properties?
We go on to elaborate this question in the context of teaching and learning and then assessment.
• How will this kind of DG manipulative tasks be useful in learning and teaching?
Usually, when setting a mathematics question for students, a teacher has a good idea of what knowledge (e.g. a geometric property in its precise expression) is to be applied. This kind of conception leaves little room for exploration in the student's problem-solving process and thus does not appear to be consonant with the potential benefits granted by the DG environment. Nevertheless, a manipulative task of this kind has a very precise goal but leaves a reasonable scope of exploration (by very limited number of movable points or constrained movements). On a task with such reduced complexity, we seek to understand how students perceive the spatiographical representations presented and what geometric knowledge (from the formal curriculum) is to be enacted. The collective quantitative information from a large number of students and the qualitative data from the interviews should provide us with evidence.
Inspired by the distinction between soft and robust construction and also the benefits of the former in teaching (Laborde, 2005), we consider this kind of manipulative tasks more like the "soft" ones in the sense that a manipulative task of this kind 340 requires a student to attain a figure that meets a certain condition by dragging a preconstructed figure. The important point is that the pre-constructed figure does not have all the necessary invariant properties that lead to the target condition – otherwise it is already a robust construction. By a careful design of what is allowed to vary and what is kept invariant in the background, the student would be helped to visualize how a particular property emerges from a varying figure and to realize the interdependency between the geometric properties. Collection of all students' responses on the other hand can help teachers or researchers to visualize the 'locus' of the points resulted from students' examples chosen to satisfy some particular conditions. Interview data in the current research project would help us understand better how students perceive a geometric figure and, in particular, what geometric features of it are taken into consideration when they are trying to make it satisfy a certain geometric condition. Understanding on this aspect will guide us on another path of developing the tasks for classroom use in the process of learning and teaching.
Can students' performance in this kind of DG manipulative tasks be assessed with the aid of the quantitative measures? If yes, what are we assessing and what can we tell out of the quantitative results?
Traditional paper-and-pencil assessment used to test whether a student can make use of basic geometric properties to find an unknown geometric measure or to produce a written proof of an unfamiliar property. This accounts at least partly for the concern of most teachers with what students know (a product-oriented curriculum) rather than how they come to know (a process-oriented curriculum). Just as DG and the research thereof seem to have suggested more on the latter but not the former, many teachers are hesitant to bring DG into the curriculum.
This new type of DG manipulative tasks suggests a potential use for assessment (Lee, Wong & Tang, 2004). With a parameter assigned to describe the relevant geometric feature(s) of the DG figure, a numerical value as a result of a student's manipulation is obtainable. The value in turn reflects, to a certain extent, how well a student has fulfilled the requirement of a certain manipulative task. Despite various difficulties pertaining to the design of the task as well as the designation of an appropriate parameter, this quantification of student performance in the dynamic geometry environment offers a possibility for assessment. Apart from a numerical score given to individual students, collective information from a large group of students may also reveal common mistakes and help serving the diagnostic purpose of the assessment (Lee, Wong & Tang, 2004). Numerical data collected also allows various quantitative analysis. Wong, Lee & Tang (2005) has attempted Factor Analysis and, in their interpretation and characterization of the factors, the authors start to be convinced of a distinctive element of exploratory interaction as opposed to working with formulas and computation in accounting for student performance. If successful in this integration of DG in assessment, we would be in a better position to materialize such new learning objective (e.g. in the curricular context of Hong Kong) as "to explore 341 and visualize geometric properties." Analysis of the data in the current research project will shed more light on this aspect.
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Using technology in the teaching of mathematics at Lesotho National University
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Introduction
In 1993 while I was studying for my Masters degree in the UK, I came across the book "Computers in the Mathematics Curriculum" by the Mathematical Association. After reading through this book, I felt an uplifting in spirits which had not happened in a long time. How wonderful it would be to get into my Pre-entry programme class back in Lesotho and draw all these different graphs (linear, quadratic) using excel spreadsheet instead of pencil and a graph sheet. I recalled how it used to be: The pencil should have a very sharp tip, and should actually be "HP" so that it does not a make a mess on the graph sheet when one rubs it off. To draw the quadratic graph, one should not move their hand as the graph might have bumps.
This happened around the time when I was about to finish my dissertation "Discontinuities in Mathematics Education between High School and University in Lesotho: Bridging the Gap". I had learned most profoundly how technologies could aid in the teaching and learning of mathematics, and had experimented with "Derive" and "Cabri-Geometre".
My view was that, the usage of computers and other technologies was quite a breakthrough in my teaching of mathematics, and for my students as, whom the biggest concern at the time for them was their poor and seemingly deteriorating performance in mathematics at o' level and beyond.
I returned home in 1993 and was about to experiment with my new-found knowledge on using technology to teach the Pre-Entry Science class just about to enter into first year of their BSc degree. Then reality hit home.
Socio-Economic factors in Lesotho
Lesotho is a country of approximately 30000 km
2
in area, with a population of just
over two million people. According to the Human Development Report 2005, Lesotho is ranked 149
th
out of 177 countries on the Human Development Index with a HDI value of 0.497. It is one of the low human development countries with GNP per capita (PPP US$) of 2561, and 49.2% of the population living below the income poverty line, therefore, quite a poor country.
In terms of technology and technology development, Lesotho has 16 telephone mainlines per 1000 people, 47 cellular subscribers per 1000 people, 14 internet users per 1000 people, and 42 researchers per 1000 000 people (HDR, 2005). Therefore, for most of the population of Lesotho acquiring a computer would be quite impossible, so that the usage of computers is done mostly from the work-place by adults instead of school children.
Political factors
At the National University of Lesotho (NUL), in 1993 when I got back from my studies, there were about one hundred and twenty students joining the faculty of 343 science and technology at first year level. The computer laboratories available in the university were stocked with about fifty (50) machines which were used exclusively by students at second year level and above, who measured in computer science.
In recent years, the university has introduced a computer literacy course for all students of the university making it compulsory for each student to have done at least this computer literacy course by the time they graduate. To this effect, the university introduced a few computer laboratories stocked with computers to serve this purpose. All first year students in all faculties of the university excluding the faculty of science do the computer literacy course.
The faculty of science and technology has two streams of first year students. The first stream, the "specialized programme" has about one hundred and twenty students majoring in Computer Science, Information Technology and Statistics. The second stream is the BSc general stream of some four hundred students following common first year programme. Both streams follow a similar curriculum for mathematics in the first year, namely, "M1501 – Algebra, Trigonometry and Analytical Geometry", and "M1502 – Calculus I". The implications here are that for the specialized programme, one can introduce the usage of computers in the teaching, but not for the general stream, as this stream would only have access to computers in their second year.
The National University of Lesotho is mostly funded by government for its activities. At present, top on government agenda are issues of HIV and AIDS which is 30% prevalent in Lesotho, and issues of poverty reduction. Hence, the need for acquisition of more computers would have to be prioritised with all these other national programmes. Since the acquisition would be just for supplementing the teaching of

mathematics there would have to be a major political support on the side of the university to place its demand on computer acquisition to government.
Lesotho and the Republic of South Africa
Lesotho has an unusually distinct feature of being completely landlocked by another country, the Republic of South Africa. This makes South Africa Lesotho's immediate neighbour at all points. This is one rich neighbour ranking 120
th
on HDI scale, with
GNP per capita (PPP US$) of 10346 and 192 researchers per 1million people,
features which cannot be compared at all with those of Lesotho.
Due to this proximity and South Africa's international position academically, most of the students from Lesotho attend school in South Africa, from primary level up to university level. It is obvious that those students who start schooling in South Africa and are able to continue throughout university there would have better access to computers.
On the other hand, most of the students only manage to get to South Africa at tertiary level. These students would have to work extremely hard to compete with the best of South African students who have had access to computers from early schooling.
Mathematics education in Lesotho 344
It is my opinion that mathematics education in Lesotho has not transformed in any major way in the last two decades or so. Factors affecting student performance in mathematics at high school and university were grouped into three categories way back in 1993 (Makoele: M.Sc Dissertation), namely:
Teaching Methods used in the schools
There was evident lack of teaching approaches such as investigation and problemsolving. It was believed by both students and lecturers interviewed at the time that the teaching was not geared to understanding due to pressures of examinations at the end of three years and five years respectively for the Lesotho schools, so that the teaching was seen to encourage rote learning.
I have not found reason to believe that things have changed for the better in recent years as I teach more new students at university coming directly from the schools.
Mathematics at University
On this particular issue, important factors were:
i. The method of lecturing, which is traditionally followed at university requires some adjustment in the learning habits of students.
Big classes at university of about 400 students taught together as opposed to classes of 40 to 50 students at school. The syllabus which in other countries would be covered over a period of two years at A'level, is done over nine months at the NUL.
ii. Textbooks.
Due to foreign exchange rates, the cost of textbooks is very high, hence there would be not much variety in the books for reference purposes. It has been the case that at NUL for some years there had not been prescribed books and students depend entirely on lecture notes.
iii. Research.
In my opinion, research in mathematics education is virtually non-existent. Where there has been some research done, it would be uncoordinated, and hence unavailable to the public. In particular, at NUL, the departments of mathematics education and mathematics fall under two different faculties, their only interaction being of students in mathematics education taking mathematics courses in the department of mathematics. The department of mathematics education main focus is on the teaching of mathematics at school level, so that it does not work on research issues in undergraduate teaching of mathematics.
On the other hand, there is a feeling among the members of the department of mathematics that issues of undergraduate teaching of mathematics fall under education and therefore have no place really in the department of mathematics. This has caused a major neglect of this important sector in the teaching of mathematics, so that at NUL there is no study or research whatsoever about the teaching of mathematics at undergraduate level, let alone the usage of digital technologies in the teaching of undergraduate mathematics.
Mathematics Education in the World 345
The Lecturers at NUL are expected to be conversant with the stuff that they teach to students like everyone else. In the light of unavailability of textbooks as mentioned earlier, they depend largely on the wide world web for reference. Here one finds lecture notes of other lecturers teaching the same subject in their own universities, research on current issues on specific subject matters, and innovations in the development of mathematics teaching. In as much as the stuff from the internet is fascinating, it is also very intimidating to some extent. For example, in teaching a topic in complex functions, I was distraught when introducing the concept of "complex maps", until I discovered a web page where someone did the very mapping using computer technology. This was so helpful to me that I conducted the next lecture to some 30 students in my office showing them how the functions map. And the next thought was, if only I could do that for all the courses and to all students that I teach. The possibilities created by technology in the teaching of undergraduate mathematics cannot be overemphasized, but the challenges posed by inaccessibility to these technologies and to be part of the "global village" that researches first hand on issues of using these technologies in the teaching of mathematics are similarly devastating to those interested in the technologies.
Conclusion
I believe that for many of us in the developing countries, the possibility of using digital technologies in the teaching of mathematics would be a great advantage. It is true that these technologies do not come cheap, but if they could be available, then a lot of work done in specialised programmes such as "bridging and remedial" programmes instituted for the science students entering tertiary education would be minimised, and there would be better results from these remedial and bridging programmes observed than without the technologies.
Despite all these, I agree with Harold Wenglinsky who says that "Computers can raise student achievement and even improve a school's climate. But they have to be placed in the right hands and used in the right ways".(Education Week).
What is needed is research on the aspects of the teaching of mathematics at all sectors of education, that is, from primary throughout to tertiary level so that individual efforts made could be adequately monitored and publicised. The research would help in giving guidance for educators to know exactly where and how to use the digital technologies.
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Instrumental Genesis in Dynamic Geometry Environments
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Over the past decade, much research has been done on dynamic geometry software investigating how this virtual environment can change our perception of mathematics and doing mathematics (in particular geometry), in the hope of enriching the pedagogical practices of mathematics teaching/learning. This proposal attempts to employ Vérillon and Rabardel's theoretical construct – instrumental genesis – to study the evolution of utilization schemes by persons engaging in dynamic geometry explorative tasks; a first step in a programme to probe deeper into how geometry is conceptualized and learnt in dynamic geometry environments (DGE). In this study, the theory of variation in the phenomenographic research approach is used as an interpretive tool. In particular, the drag-mode in dynamic geometry is perceived as an artifact, hence an instrument. Consequently, instrumentation/instrumentalization of dragging via functions of variation (contrast, separation, generalization, fusion) and dragging modalities will be a main focus of DGE instrumental genesis. In the proposal, a pair of Hong Kong pre-service mathematics student-teachers' DGE exploration episode is presented and briefly analysed. A possible variational dragging scheme is then proposed for their process of discovery. The studying of the conversation between the two student-teachers during their collaboration in the DGE task further identified a few DGE utterances which illuminate ways to conceptualize discourses in DGE.
The proposal
Over the past decade, much research has been done on dynamic geometry software investigating how this virtual environment can change our perception of mathematics and doing mathematics (in particular geometry), in the hope of enriching the pedagogical practices of mathematics teaching/learning (see for examples, Educational Studies in Mathematics 44:1-161, 2000; International Journal of Computers for Mathematical Learning 6:229-333, 2001; Math ZDM 34(3), 2002; Leung and Lopez-Real, 2002; Lopez-Real and Leung, 2004). Dynamic Geometry Environments (DGEs) opened up a milieu for the integration of experimental mathematics into classroom didactic and consequently brought forth the role of ICT as a key contributor to mathematical discourse. A key feature of DGE is its ability to visually represent geometrical invariants amidst simultaneous variations induced by dragging activities. This dynamic tool – dragging – induces potential dialectic between the conceptual realm (abstraction) of mathematical entities and the world of virtual empirical objects. Because of this possibility, dragging has been a major focus of research in DGE resulting in fruitful discussions on promising dragging modalities and strategies that seem to be conducive to knowledge construction (see for 347
examples, Hölzl, 1996; Arzarello et al, 2002). Indeed, when the drag mode "acts" on a virtual empirical object in DGE, the object undergoes transformations in a domain in which the dual nature of mathematical object (Sfard, 1991) can be "lived out". This domain is a complex network that covers various aspects of DGE and human behaviours (physical, psychological and cognitive) where artifacts/tools could be turned into instruments/psychological tools. It thus provides an integrated environment (DGE plus human) where situated (mathematical) abstraction (Noss and Hoyles, 1996) could be constructed and "behaviour of mathematics" could be studied. The transition from concrete to abstract (epistemic transformation), or vice versa (pragmatic transformation), in this integrated environment spans a zone where creativity resides. This zone is the object of study for most of the DGE research. It is where experiences and "imagination" in DGE meet, merging to shape geometrical concepts. The idea of instrumental genesis proposed by Vérillon and Rabardel (1995) is a suitable framework to begin the exploration in this transition zone.
Instrumental genesis differentiates an artifact (a man-made object/tool) from an instrument (a psychological construct) by defining the latter as formed by an artifact together with one or more associated utilization schemes that emerge from SIA (Situated Instrumented Activity). SIA (Vérillon, 2000) is an inter-activities web formed by a triad that consists of a subject (as user), an instrument (as tool) and an object (as epistemic transformation, e.g. geometrical knowledge). Tools are artifacts that can amplify or modify our abilities to transform the world around us. They are shaped and fashioned in ways that contain the potential to reify human imagination. Hence, the value of a tool is inseparable from the one who uses it, in particular, how one uses it. We learn mathematics with tools. A pair of compasses gives us a vision of the ideal circle, a ruler is a representation of straightness, a calculator enables us to see patterns behind the complexity of routine calculations, and the list goes on. A user turns a tool into an instrument for a specific mathematical task by associating with it a scheme of use. A scheme is a systematic procedure on how to use a certain tool to achieve a certain purpose. Thus, an instrument is a psychological construct in the cognitive ergonomic domain (Vérillon and Rabardel, 1995). This is the useroriented micro-genetic process of instrumentation in instrumental genesis. At the same time, specific functionalities, even purposes, are attributed by the user to the tool (not necessarily intended by the designer of the tool) in the instrumental genesis process. Rabardel (1995) called this instrumentalization of the artifact. Hence, an instrument is a dual entity – artifactual and psychological.
A user-oriented utilization scheme is somewhat reminiscent of Kant's schema – "representation of a universal procedure of imagination in providing an image of a concept" (Kant, as cited in Tasi, 2001, p.11). In a sense, a utilization scheme in instrumentation can be thought of as a local realization of a schema; however, this raises the important question of whether different utilization schemes would converge to the same concept object. Mariotti contended 348
"The artifact, although incorporating mathematical knowledge and integrated by appropriate utilization schemes, might not function in generating mathematical meaning; its user might not access the meaning incorporated in the artifact."
(Mariotti, 2002, p.704)
Hence a utilization scheme can be just a routine procedure for a particular task and remains a situated abstraction that might not explicitly harvest the universal mathematical meaning behind it. Instead of artifact and instrument, Mariotti favoured the Vygotskian distinction between the mediating functions of technical tools and psychological tools (or signs: a mental construct) and preferred using semiotic mediation as an umbrella over instrumental genesis. Instead of focusing on the evolution of a utilization scheme in instrumental genesis, meaning construction becomes the core activity in a process of internalization where technical tools are being transformed into psychological tools for the purpose of shaping new meanings. In semiotic mediation, the focus is not on distinction between artifact and instrument; rather it is on the externally or internally oriented usage of the tool that could bring about the construal of mathematical meanings.
How do we gain geometrical knowledge in DGE? This is the ultimate question that we seek to answer. Instrumental genesis and semiotic mediation are promising theoretical frameworks that could pave a path to start the journey. Any path to knowledge construction should be an experiential one. How a person experiences the world of DGE will determine the kind of knowledge that he/she gains in DGE. The utilization schemes in instrumental genesis are schemes on how to experience the potentiality of a tool/artifact. This somewhat echoes to Vergnaud's notions of scheme and operational invariant (Vergnaud, 1999). In semiotic mediation, the process of internalization concerns the transformation of experiences; a change in the way of seeing something via a change in the nature (pragmatic, epistemic) of the tools.
Experiences, discernment, variation and simultaneity are the central concepts in the phenomenographic research approach in which learning and awareness are interpreted under a theoretical framework of variation.
"The unit of phenomenographic research is a way of experiencing something, ..., and the object of the research is the variation in ways of experiencing phenomena."

(Marton and Booth, 1997, p.111).

Phenomenography literally can mean the act of representing an object of study as qualitatively distinct phenomena. In particular, it concerns the second-order nondualistic (neither internal/mental nor external/physical) categories of description 349 of the variation in ways of experiencing something (phenomena), and is about categorizing the limited number of qualitatively different ways of seeing, or experiencing, a phenomenon in a hierarchical fashion. DGE is rooted in variation in its design. It is a milieu where mathematical concepts can be given visual dynamic forms subject to our actions, powerful or not and mathematical concepts can be developed through experiencing invariance under different dimensions of variation mediated by the tool/artifact in DGE. DGE is a natural experimental ground to experience variation since it has the built-in mechanism that enables the generation (via intelligent construction and dragging by us) of various qualitatively different ways of literally seeing a geometrical phenomenon in action. In this respect, Leung (2003) and Leung and Chan (2005) made initial attempts to discuss how functions of variation (specifically contrast, separation, generalization and fusion) in the phenomenographic research approach could be realized under different DGE dragging strategies in problem-solving and conjecturing episodes. It seems to be a worthwhile research to consider the functions of variation as functions of dragging and investigate how these functions can be realized through different dragging modalities in different DGE contexts (for examples, construction, problem solving, conjecturing, proving), or vice versa. That is, how to use which dragging modalities for different functions of dragging in different DGE contexts; in other words, an instrumentation of dragging in DGE via functions of variation and dragging modalities. Hence, the overarching question is how to integrate the theory of variation in the phenomenographic research approach into instrumental genesis and semiotic mediation in the study of how geometry is learnt (experienced) in DGE, which might lead to sound pedagogical content knowledge in DGE. In particular, here are a few possible (inter-related) research questions:

1. Explore how the functions of variation interact with features (e.g. dragging modalities/strategies) in DGE and investigate how this interaction contributes to the formation of utilization schemes in the process of instrumental genesis.
2. Can ways of experiencing geometry in DGE be categorized in some sort of hierarchical schematic fashion? If yes, how does this hierarchy contribute to the conceptualization of geometrical knowledge?
3. DGE is somewhere between physical and psychological, hence may be regarded as a kind of non-dualistic field of experience, a natural niche for the phenomenographic assumption. Does the idea of non-dualistic tool for semiotic mediation in DGE make any sense? If so, in what ways?

These are questions that need in-depth research to reach some sort of answers.

Extensive research has been done on instrumental genesis in Computer Algebra System (see for examples, Artigue, 2002; Guin and Trouche, 2002, Trouche, 2004; Guin, Ruthven and Trouche, 2005); however, there seems to be a gap in the literature on instrumental genesis in DGE. In the following, we describe a portion of an episode 350 in which two pre-service mathematics student-teachers studying at the University of Hong Kong for the Postgraduate Diploma in Education (PGDE) were taking part in a conjecture making activity in DGE. The DGE they used was C.a.R. (Compass and Ruler, a DGS developed by R. Grothmann, http://mathsrv.ku-eichstaett.de/MGF/homes/grothmann/java/zirkel/doc_en/). During their exploration, an implicit dragging
scheme was evolved for the problem they were
working on that eventually led them to the
discovery of a (correct) conjecture. This
dragging scheme composed of (a) key
functions of variation mentioned in the
discussion above and (b) dragging modalities
that have been identified in research and
practices (the student-teachers were not aware
of these). The problem they were working on was finding a relation that was essentially the necessary condition in Ceva's Theorem. In particular, they were given a C.a.R file (see figure on the right) in which A, B, C and P are independent draggable points while X, Y and Z are points dependent on P. The task was to find a relation connecting the lengths of the segments BX, XC, CY, YA, AZ and ZB. These two student-teachers had undergraduate degrees in mathematics-related subject areas but they didn't know Ceva's Theorem. They were introduced to C.a.R. in a two-hour session in the PGDE programme and they had no knowledge of the above mentioned functions of variation. At the beginning of the exploration, they spent a few minutes familiarizing themselves with the C.a.R environment and they measured the length of all the segments in question. The following is a brief outline and description of a dragging scheme that seemed to have evolved out of their exploration.

A Variational Dragging Scheme in DGE

1. Create contrasting experiences by wandering dragging until a dimension of variation is identified.
P was dragged to different positions inside triangle ABC in a wandering fashion while focus was put on the numerical values of the length measurements. Side BC (in particular, the point X) was chosen as a controlling variable (a dimension of variation) and while X varies as P was dragged, focus was given to the length measurements of CY, YA, AZ and ZB.
2. Fix a value (usually a position) for the chosen dimension of variation.
The midpoint,X) , of BC was constructed.
3. Employ different dragging
modalities/strategies to separate out
critical feature(s) under the fixed value
(i.e. a special case for the configuration) 351
P was dragged to keep X andX) as close as possible (see figure on the right) and attention was given to the length measurements of CY, YA, AZ and ZB. This was a guided dragging/drag-to-fit strategy. A numerical pattern was observed: the product of YA and ZB "appeared to be somewhat equal" to the product of CY and AZ (a calculator was used). Consequently, YA, ZB and CY, AZ were separated out as two related pairs. During this exploration, some DGE utterances were developed between the two student-teachers: "very difficult to control", "try best to keep on the line", "try to squeeze it", etc.
4. Simultaneously focusing, hence "reasoning", on co-varying aspects during dragging. A preliminary conjecture is fused together.
Further refinement of the dragging techniques (as reflected by the DGE utterances) confirmed the speculation in 3. This is a fusion experience in which co-varying aspects (numerical values, position of P and X, the changing line segments) were simultaneously experienced together. A preliminary conjecture was proposed: when X is the midpoint of BC, the product of YA and ZB equals the product of CY and AZ.
5. Attempt to generalize by a change to a different value for the chosen dimension of variation.
P was dragging in a wandering fashion, however, with random patterns. For example, it was dragged horizontally for a while. After a period of fruitless exploration, the student-teachers decided to place X at a different special position: CX : CB = 1 : 3.
6. Repeat steps 3 and 4 to find compromises or modifications (if necessary) to the conjecture proposed in step 4.
The dragging and reasoning strategies developed in steps 3 and 4 were employed to this modified situation (a new value for the chosen dimension of variation). The student-teachers at this moment were more experienced with the "utilization scheme" that they have developed and used it again to tackle a new situation. More DGE utterances were developed: "the point moves, all will be changed", "it feels like the property has to do with length", "put it on top", etc. They discovered that the product of YA and ZB was not equal to the product of CY and AZ in this case; rather, they found that the product of YA and ZB is 3 times the product of CY and AZ. A modified conjecture was then proposed: the product of YA, CX and ZB is equal to the product of CY, XB and AZ.
7. Generalization by varying (via different dragging modalities) other dimensions of variation
The vertices (instead of P) A, B and C were being dragged in a wandering fashion and different values were assigned to the position of X; literally trying to see whether the modified conjecture still holds. In particular, P was dragged to move X continuously along BC. By seeing that the conjecture remained invariant under variation (by dragging) in all dimension of variation (expect when P was outside the 352 triangle which could be seen as a design fault for the moment), the student-teachers were convinced that they had a generalized conjecture.

The 7-step variational dragging scheme described above may serve as a good starting point to investigate research question 1 and it is the intention of this proposal to undertake such an attempt. To begin with, more in-depth analysis will be done on the above episode (only the first portion of the episode was analyzed here; for the rest of the episode, the student-teachers continued the exploration in DGE and eventually proved the conjecture successfully). As mentioned above, DGE utterances were developed by the student-teachers during their exploration, it would be interesting to note the evolution of a DGE discourse as another strand of study. Further exploration tasks will be designed and persons with good mathematical background (students or teachers from secondary or tertiary) will be invited to participate in the research. It is hoped that by researching into how instrumental genesis in DGE could take place under the framework of variation, research questions 2 and 3 could become more well-defined and tangible.

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Pappus in a Modern Dynamic Geometry: An Honest Way for Deductive Proof

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This study shows that dynamic geometry using the "analysis" method systemized by the Greek mathematician Pappus in the 3rd century AD can provide a good learning environment to teach deductive proof for secondary students. Traditionally, in teaching deductive proof the axiomatic or synthesis method to deduce a new result from assumptions has been far more emphasized at the expense of the mathematical discovery process. The method systemized in Euclid's Elements is not an honest way for teaching deductive proof in that it shows only final results by mathematicians and does not help students to appreciate why and how to prove. To improve deductive

proof abilities through the analysis method, a dynamic environment in which geometric figures can be easily manipulated are required for an "active justification" to find the heuristics for proof. This paper suggests four phases to solve construction problems in dynamic geometry: First is the understanding phase to recognize problem conditions and goals. Second is the analysis phase to assume what to be solved is done and to find the proof ideas by the analysis method. Third is the synthesis phase to construct a deductive proof as a reversed process of the analysis and finally, the reflection phase to reflect on the problem solving process as a whole.
Deductive proof is a process used to deduce a new result from assumptions existed in the problem, axioms, what was previously proven, etc. Since the 6th century BC, it has been the flower of mathematics and marks a distinction between mathematics and other science. Euclid's Elements written in the 3rd century BC has been used as a textbook to develop students' deductive proof over 2000 years. However, the axiomatic method used in Euclid's Elements has been criticized in the sense of showing none of mathematical activities as imagination, intuition, experiment, thoughtful guessing, trial and error, making mistakes, etc. (Clairaut, 1741; Lakatos, 1976; Vincent, 2005). Similarly the proof process that appears in secondary mathematics textbooks shows only the final result produced by some mathematicians and does not help students to see why and how to prove. Students have few meaning in the proof explained by teachers based on textbooks and lose confidence in constructing proof. eventually.
This paper argues that the axiomatic method of Euclid's Elements and current mathematics textbooks are not honest ways for teaching deductive proof, and in order to improve students' proving abilities, an "active justification" to find the heuristics for proving by students themselves should be required rather than a "passive justification" that occurs through teacher's explanation or persuasion about the process. This paper introduces one strategy for active justification to improve 355 students' deductive proof: 'Analysis' with dynamic geometry software. Dynamic geometry provides a good environment for students to develop deductive proof abilities when it is combined with the 'analysis' method systemized by Pappus, who criticized Euclid's 'synthesis' method which shares the same order as the proof process that appears in secondary geometry textbooks.
Analysis and Synthesis Mathematical heuristics related with proof goes back to the Greek era. In the 3rd century, the Greek mathematician Pappus systemized in his book "Collection" the "analysis" which was also emphasized by Euclid but did not appear in his book "Elements". The analysis method, which is the oldest mathematics heuristics in the history of mathematics, assumes "what is sought as if it were already done and inquire what it is from which this results and again what it is the antecedent cause of the latter and so on, until by so retracing the steps coming up something already know or belonging to the class of the first (Hearth, 1981, p.400)." The synthesis as the reverse of the analysis take as already done that which was last arrived at in the analysis and arrives finally at the construction of what was sought by arranging in their natural order as consequences what before were antecedents and successively connecting them one with another.
Greek mathematicians thought the dialectic integration of analysis and synthesis to be a substance of mathematical thought. However, Euclid's Elements considered only synthesis to reduce theorems from the foundation as a way to guarantee the truth of mathematics. As same as Euclid's Elements, current secondary geometry textbooks introduce only the synthesis. The analysis also should be introduced in order to develop students' proof abilities.
Design of an instructional scheme to improve deductive proof ability
Dynamic geometry One problem is that the analysis method is very difficult to be applied in the paper and pencil environment because various dynamic operations such as manipulating geometric figures are required for the method. Particularly, the paper and pencil environment is worse for normal level students than for high achievement studentsIt might be because of the lack of proper dynamic tools that the analysis known well by such Greek mathematicians as Plato and Euclid had not emphasized in schools since the Greek era. Dynamic geometry, which has been developed since the late 1980s, can provide a circumstance for the revived use of the analysis in that it allows students to drag and transform geometric figures.
Four phases of problem solving In this paper four phases of problem solving is suggested as an instructional scheme to improve deductive proof ability: First is the understanding phase to clearly recognize problem conditions and goals. Second is the analysis phase to assume what solving is to be done and to find the construction ideas by using the analysis. Third is the synthesis phase to construct a deductive proof as a reversed process of the analysis and finally, the reflection phase to reflect on the whole problem solving process. 356
Problem situation There are two kinds of geometry problems: Proof problems or construction problems. There are two kinds of analysis. The first kind of analysis is to find proof process by getting a series of previous sufficient conditions of the conclusion to be proven under the assumption that what is required to be proven is already proven. The second kind of analysis is a problem solving strategy for construction problems. This is a strategy to find the construction process by extracting a series of necessary conditions from the assumption that what is required to be constructed has already constructed. In this paper, geometric problems are limited to problems of construction although the analysis method can be applied to both construction and proof
Analysis & synthesis phase for a proof problem
There is a triangle ABC. Make three equilateral triangles ABD, BCE, AFC by using each side of the given triangle ABC. Then prove that quadrilateral BEFD is a parallelogram
The following axiomatic or synthesis proof process usually appears in secondary school textbooks:
1. ABD, 'BCE, 'AFC are equilateral triangles
2. FCA and ECB are 60˚ and FCB is common (DAB and FAC are 60˚ and FAB is common)
3. ACB = FCE and AC = FC, BC = EC (BAC = DAF and AC = AF, AB = AD)
4. ABC □ FEC (ABC □ ADF)
5. AB = FE (DF=BC)
6. BD = FE (DF=BE)
7. Quadrilateral BEFD is a parallelogram
Here, thought flow 1 2 3 4 5 6 7 is a series of sufficient conditions in that 1 is a sufficient condition of 2 and 2 is a sufficient condition of 3 and … and 6 is a sufficient condition of 7. However, this synthesis process does not explain to students why they have to consider "FCA and ECB are 60˚ and FCB is common" from 'ABD, 'BCE, 'AFC are equilateral triangles. Actually it is for showing 'ACB = 'FCE. Therefore students should consider 'ACB = 'FCE first, then they have to find the reason why 'ACB = 'FCE. The reason is "ECB is 60˚ and 'FCB is common". Similarly, students should consider 'ABC □ 'FEC before considering "'ACB = 'FCE, AC = FC, BC = EC" and AB = FE before considering 357 'ABC □ 'FEC, BD = FE before AB=FE and finally "quadrilateral BEFD is a parallelogram" before considering BD=FE.
Thus, the most natural thought in order to prove this theorem is to assume first that quadrilateral BEFD is a parallelogram. Then to find a series of sufficient conditions 6, 5, 4, 3, 2, 1. That is, thought flow 7□ 6□ 5□ 4□ 3□ 2□ 1 is more natural than 1□ 2□ 3□ 4□ 5□ 6□ 7. Here, Pappus called this natural thought flow "analysis" and the reversed thought flow "synthesis."
Analysis & synthesis phase of a construction problem
There is a circle O and there are two lines m and n which are perpendicular to each other. Construct a circle whose center is located on the line n and to which the circle O and the line m are tangent

According to the textbook, the synthesis proof process is as follows (See Fig 1):
1. Draw a circle H with radius OG.. Then let F be the intersection point
2. Draw a perpendicular bisector of OF. Let P be an intersection of the perpendicular bisector and line n. Then 'OPF is an isosceles triangle.
3. Then, TP = PH
4. Then a circle can be drawn with the center P and the radius PH. Draw a circle P with radius PH. That is the circle to find.
Fig. 1 358
Here, thought flow 1 2 3 4 is a series of necessary conditions in that 1 is a necessary condition of 2 and 2 is a necessary condition of 3 and 3 is a necessary condition of 4. However, in this synthesis process students cannot understand why they have to draw the circle H with radius OG in order to obtain the circle P tangent to the given circle O and the given line m. What is important is how to find point F such that OG = HF. Students have to find the way to get point F by themselves. Similarly, they cannot appreciate why they have to draw a perpendicular bisector of 'OF. Students also cannot appreciate the relation between point F and Point P.. Through only the analysis method, they can understand the relation between the circle H and the tangent circle P.
The analysis process of the problem is as follows (See Fig. 2 and Fig. 3):
1. Assume that the circle P is constructed satisfying the given conditions. Let T be a tangent point to the circle O and H be a tangent point to line m.
2. Then TP=PH
3. Draw a circle P with radius OP and Let F be an intersection point of the line n and the circle. Then, 'OPF is an isosceles.4. Then OT=HF. Draw a line perpendicular to OF passing through P. Then E is a mid point of OF
Of course, this analysis process is not an easy route. Students have to find a series of necessary conditions starting from line 1 by using operational activities in dynamic geometry. In this phase, students need special help from their peers and teacher through discussions or dialogues with them. And, after finishing the analysis phase, students have to go to the synthesis phase as the reversed process of the analysis to improve their deductive proof abilities.

Fig 2 359
Fig 3
Reflection Phase in a construction problem
In the reflection phase, students look for another proof. The analysis process is as follows (See Fig 4):
1. Assume that the circle P is constructed satisfying the given conditions. Let L be a tangent point of two circles O and P.
2. Draw OP which passes through L. And, draw OQ which is perpendicular to m. And, make two isosceles triangles OQL and PHL
3. Then segments QL and LH makes one segment QH i.e. Angle QLH is 180˚
The synthesis proof is as follows (See also Fig 4):
1. Draw a perpendicular line to m passing through O and let Q be an intersection point of the given circle O and the line. Then draw QH and let L be an intersection point of the line and the circle O.
2. Draw a line OL and let P is the intersection point of the line OL and line n. Then 'LPH is an isosceles triangle. That is, LP=PH.
3. Draw a circle P with the radius PH
In the reflection phase, students can check whether the construction process by the synthesis is right or wrong. If the relation among components of the figure constructed by synthesis is preserved while dragging it, the construction process can be considered as a right procedure.
360
Fig 4 Fig 5
The functions of dynamic geometry for the analysis method
Three functions of dynamic geometry for the analysis method are as follows: First, dynamic geometry can help students draw a precise figure. To find more easily a series of necessary conditions to a final conclusion, students have to show relations among components in the problem situation. In a figure roughly drawn on the paper it is very difficult for students to find the relation. Second, dynamic geometry has a measure function. It makes students determine a good starting point for analysis by continuously measuring length or angle while dragging the point continuously. In a paper circumstance, it is very difficult for students to make a precise enough measurement to find a good starting point for analysis. Third, dynamic geometry is dynamic. It can make students perform various experiments to find necessary conditions by drawing, erasing and manipulating figures easily as well as dynamically. In a paper and pencil circumstance, it is almost impossible to perform analysis because the figure drawn on the paper cannot be manipulated. Finally, dynamic geometry is a reflective tool. If the relation among components of the picture constructed by synthesis is preserved while dragging it, the construction process can be considered as a right procedure. In a paper and pencil circumstance, there is no way to check whether the construction process is right or not.
Conclusion
In the late 1980s, Cabri and GSP were designed as dynamic tools for students to investigate the properties and relations within and between figures through operating figures on the computer screen directly. It is a very powerful environment in which much more activities than in traditional construction using normal compasses and rulers made possible: Construct, erase, drag and transform figures, measure segments and angles etc. A dynamic method for Euclidean geometry was proposed by Clairaut, a French mathematician of the 17th century (Laborde, 1999). However, then neither did he have the proper tool for the dynamic method. In dynamic geometry, students can make a conjecture to geometric properties and confirm them informally and feel the need to prove the conjectured and informally confirmed geometric facts. In dynamic geometry, students can improve their proof abilities by using the analysis method. Dynamic geometry is a very excellent tool for the analysis method which is a good mathematical strategy proposed by Greek mathematicians but that has been forgotten for a long time, perhaps due to the lack of a proper tool. In mathematics textbooks, the conclusion is a conclusion. In mathematics education, the conclusion should be a starting point rather than a conclusion. Dynamic geometry can provide a better and safe route from the starting point to the development of deductive proof by normal students.361
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Cabri 3D: potential, problems and a web-based approach to instrumental genesis

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Cabri 3D is a relatively new software which has great potential in the teaching and learning of both 2D and 3D geometry, in enhancing student ability to visualize, in modeling physical structures and motion and in developing new mathematics. In order to facilitate the instrumental genesis of this software an approach based on a web-based integration of text, hypertext, both static and dynamic "pictures" and interactive demos is being developed. This approach may well be useful with other applications.

Introduction

When I first saw a prototype of Cabri 3D demonstrated in 2001, my reaction was "so what?" Oldknow (2005) suggests that this might be a common response "It is possible to view…Cabri 3D to be just for fun and of no curricular relevance!" After a year of experimentation with Cabri 3D I am now convinced that it is an important software with the potential to radically alter our perceptions of geometry, open up new mathematics and provide engaging mathematical environments for students. This paper will discuss a number of areas in which Cabri 3D could be important together with some issues that may arise and will also discuss a particular approach to its instrumental genesis.

Three-Dimensional Geometry

Laborde (2005) sees mathematics as a science dealing with variable objects. Theorems in geometry, which state that certain geometrical properties remain invariant as a figure varies are comparable to algebraic identities, which remain true as the variable changes. Interactive geometry software is hence of particular importance in that students, in being enabled to manipulate tangible variable objects are introduced to this essential feature of mathematics.

Cabri 3D shares many basic features with 2D interactive geometry softwares such as Cabri 2+ and The Geometer's Sketchpad (Jackiw, 2001). In particular, Cabri 3D is about such tangible variable objects, objects which are constructed in geometrical relationships which are preserved when the position of initial objects is changed. Hence much of the research evidence concerning 2D interactive geometries is likely to generalize. For example, in a summary of research findings on proof, Mariotti (2006, p. 193) states that interactive geometry environments have been successful in enabling students to link informal explanations with formal proof. There is as yet little research specific to Cabri 3D: however, Laborde (2005) has shown how the idea of soft construction, an important aspect of generalization in which certain properties are deliberately constructed by eye in order to empirically explore the locus of possible figures, can apply to Cabri 3D in exploring the altitudes of a tetrahedron. 363

Cabri 3D has some important differences with 2D interactive geometry, which can act as both affordances and constraints. One difference is that many of the features of Sketchpad or Cabri 2+ are still in the process of being developed. The most significant feature lacking in dealing with three-dimensional geometry, particularly in the context of the school curriculum, is measurement, although, as will be seen in the section on pedagogical models, this does not rule out the use of Cabri 3D in this area. Other features not yet implemented are macros and loci. However, some of the limitations are also enabling. For example, the absence of number made the construction of the volume and cross-product pedagogical models an interesting mathematical challenge, rather than just a useful task.

Cabri 3D also has features not shared with 2D interactive geometries, such as the ability to create polyhedra, cut these polyhedra to form new polyhedra, and to create the nets of polyhedra.

A more fundamental difference is that, while a 2D interactive geometry represents 2D space with no loss of information, in the 2D representation of 3D space provided by Cabri 3D there is an inevitable loss of information (Parzysz, 1988) as any point on the screen represents a line in space. This is overcome to some extent by the ability to change the viewing angle, but has the consequence that creating or dragging any unconstrained object may have unexpected results: what looks like a small difference from one viewing angle may create a large change . Hence free "play" with object creation may be less satisfying than in 2D. On the other hand, geometric properties have greater importance: the student is forced to continually consider the geometric relationship between objects, as any apparent spatial relationship is unlikely to hold true.

It is also not possible to easily infer the properties of objects on the basis of spatio-visual information. Accascina and Rogora (2005) found that students using Cabri 3D found it difficult to infer the properties of objects on the basis of spatiovisual information. They were unable to decide whether all parallelogram crosssections of a cube were rectangles. On the other hand, the consequent uncertainty may act as incentive for proof (Hadas, Hershkowitz &Schwarz, 2000).

Cabri 3D is a good 2D representation of 3D, with a choice of perspectives and objects rendered to give an impression of distance, but inevitably has the limitation that the verisimilitude of the representation is dependent on the precise position of the viewer with respect to the screen, which will vary. Hence, experience with the software is needed in order to perceive objects as realistically three-dimensional.

Cabri 3D in 2D geometry

Constructions in 2D geometry may be performed on any plane, and hence can be easily viewed from different directions, which may enable students to more readily distinguish between incidental spatio-graphical properties such as the position or orientation of an object and the necessary links between its spatio-graphical properties which arise from the way the object was constructed. This distinction is critical for understanding the nature of these links (Laborde, 2005). This is illustrated by the two views of the construction of an equilateral triangle below. 364

Looking directly down on the base plane. The orientation of this figure is readily changed.

A view from an oblique angle: an accurate construction which has given rise to an "inaccurate" diagram, necessitating separation of the visual and the geometrical for its interpretation.

Cabri 3D also provides a link between 2D and 3D geometry. Familiar 2D objects need to be constructed in new ways, or change substantially in 3D, as illustrated below:

A circle is defined by a point and an axis.

The perpendicular bisector of a segment is a plane.

Rather than 3D space being an extension of 2D space, 2D objects may hence be seen as embedded in 3D space, which may consist of a "cartoon" world, with links to the world of experience. Here is an example which uses a character called Claude to mediate the mathematical meaning of reflection:

The Claude "behind" the mirror is a reflection.

Claude, reflection, and mirror seen from above.

Visualisation

Visualization is an important aspect of doing mathematics, the ability to visualize is not developmental and hence a major issue is the development of "effective pedagogy that can enhance the use and power of visualization in mathematics education" (Presmeg, 2006, p. 227). Research evidence shows that dynamic computer software facilitates visualization processes (Presmeg, 2006, p. 220). Cabri 3D may facilitate 3D visualization through allowing students to manipulate structures which are not limited by gravity or solidity and which can be given varying degrees of visibility. In a computer environment, such manipulation 365 involves an explicit awareness of the nature of actions such as rotation upon objects (Gutiérrez, 1996). Here are some questions using Cabri 3D which are designed to require student visualization:

What shape is this cross-section?

What happens to the cross-section as the plane moves towards the bottom vertex?

What characterizes this hexagon? Is it possible to get a regular hexagon as a crosssection?

Physical Modelling

One of the reasons for the early importance of Logo was its connection with body motion: the turtle represented an object turning in the physical world. However, the lack of correspondence between the geometry of Logo and the Euclidean geometry of the school curriculum has been problematic (Laborde, Kynigos, Hollebrands & Strässer, 2006).

Cabri 3D can also be used both to embody motion and to create models of real-world structures, and depends upon the Euclidean geometry of the school curriculum.

Little House by Jean-Jacques Dahan

Tower Bridge by Adrian Oldknow (file available at www.counton.org/cabri/index.htm)

Unlike real-world model building, the creation of these structures requires the use of geometrically based tools and hence an explicit awareness of the geometric relationships embodied in the structure.

This is also true when modeling motion: to create Claude on the swing with a shadow requires the use of the rotation and parallel line tools. To make Claude row requires a specific awareness of bodily motion and the means by which this may be modeled geometrically. 366

A Microworld?

The objects above show that in Cabri 3D there is certainly scope for students to design and create using mathematical tools, in which mathematical problem-solving and experimentation is required. However, with no textual interface or ability to create macros, it cannot be called a microworld in the sense defined by Hoyles and Noss (2003). Is the distinction important? It may be, if we find that the integration of symbolic and visual learning is universally important.

Pedagogical Models

These models, which cannot be entirely reproduced in the physical world, are designed to illustrate mathematical dependencies:

Effect of changing length, width or height on volume and surface area.

The cross product of two unit vectors as the angle between them varies.

A cone of variable height and radius can be transformed into a sector.

New Mathematics

As well as enabling new results in research mathematics (e.g. Oldknow (2005) generalized the Soddy line of a triangle to a tetrahedron), objects may be created which give rise to areas of new mathematical exploration which are both attractive and accessible to students. An example is the "net" of a truncated icosahedron in which the dihedral angle between connected polygons is progressively being decreased (Mackrell, 2005a): 367

The unexpected emergence of the final three objects, with high degrees of symmetry, gives rise to the question "why", which leads to further investigation and proof.

Problem:

Given the potential of Cabri 3D, how do we engage learners? In the early years of the use of technology, it was assumed that learning would emerge simply from the interactions between the student and the machine. It is now recognized, however, that the choice of task and learning environment is crucial (Laborde, Kynigos, Hollebrands, & Strässer, 2006, p. 279). An initial problem is that of instrumental genesis: how can we provide a learning environment in which Cabri 3D progresses from being an artefact to becoming an instrument (Rabardel, 2002) which can be used in problem-solving or design where appropriate and laid aside in favour of other instruments when inappropriate?

Laborde, Kynigos, Hollebrands, & Strässer (2006, p. 280) recognize that there is an intrinsic link between mathematical knowledge and knowledge about how to use a tool and that hence developing the ability to use a tool may also involve developing mathematical knowledge. This accords with my experience with teacher education students engaged in learning how to use Cabri or GSP in Ontario and the

UK. Many learners have very little geometrical awareness and hence cannot easily distinguish between a geometry problem and a problem of not knowing how to use a particular tool. For example, a student, unable to create a sequence of equally spaced points on a line commented that this was due to not knowing how to use the "equidistant" tool.
Hence, for many learners, teachers as well as students, the process of instrumental genesis must also include enhancing geometrical understanding. This is an important justification for using interactive geometry software in a classroom: with relatively straightforward interfaces, learning how to use the software is mainly about coming to grips with the underlying geometry. In the process the software will become the geometry in some sense for the student (Mariotti, 2002), and its particular features may have a profound effect on geometrical thinking. A concept image of "perpendicular bisector" based on the use of a straight edge and compass will differ from that of students creating a concept image through using the Cabri 3D "perpendicular bisector" tool.
Possibilities:
An early effort with multi-leveled text was a resource consisting of three documents for learning how to construct a kaleidoscope using GSP or Cabri. The first document 368 gave a framework for the task and asked specific questions and suggested extensions. The other two documents showed the process in detail, using screenshots, one with Cabri and the other with GSP. Learners were encouraged to use the documents as they felt comfortable: the confident could use the framework as a set of challenges to meet whereas those lacking in confidence could follow the process in detail. This has had a positive response in a number of workshops for teachers. It allows learners to work at their own pace, choose their level of challenge and create a mathematically and aesthetically pleasing object.
I correspondingly designed an introduction to Cabri 3D, consisting of activities now posed at three levels of challenge with details communicated by screenshots, followed by extension questions (Mackrell, 2005b). One issue, however, is that all hints and details are visible and it is hence hard to ignore the details. Another is that younger learners may not feel encouraged to continue exploration. Using similar materials showing how to create Claude on the swing, a mathematically able sixteen year old felt challenged, but was able to create the figure. A 12 year old with extensive experience with Cabri was able to follow the materials and create the figure, but showed no interest in further work with Claude.
I have recently been designing interactive demos using the software TurboDemo. These show brief movies of the steps involved in solving a problem, but also ask the viewer to anticipate what might happen next, to consider mathematical questions and to engage in extension activities. The viewer is in control, choosing whether to proceed at the end of a step or to view the step again. Combining this with the idea of multi-layered text, I am now in the process of writing an HTML resource to introduce fold-up polygons such as the truncated dodecahedron shown above. This includes hypertext (with links to a glossary), popups for hints, pictures, embedded Cabri 3D files and interactive demos. Part of this resource may be found at http://educ.queensu.ca/~mackrell .
In trying out this resource, it has become clear that 12 year olds can easily follow a demo and create a fold-up dodecahedron. A major question, however, is whether students are learning a sequence of moves to follow or are learning about the rotation which is the purpose of these moves, particularly as little interaction with text is required when following a demo. This will be the subject of further research.
In conclusion, Cabri 3D has great potential, but, as with any interactive geometry software, instrumental genesis is problematic due to lack of geometric understanding on the part of both teachers and students. New approaches are needed to overcome these difficulties and a web-based approach is being developed.
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Dynamic statistical software: How are learners using it to conduct data-based investigations?

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Bakker (2002) identified two categories of learning software in mathematics: landscape-type software and route-type software. Route-type software was designed to guide learners through a hypothetical learning trajectory with a fairly fixed destination. Alternatively, landscape-type software is designed to support learners in conducting open-ended investigations. The use of these dynamic software tools for the learning of mathematics and statistics has gained increasing prominence in schools because of its ability to support multiple purposes defined by the user rather than the software. Little is known, however, about the diversity of approaches in which learners use these software packages to conduct investigations. This paper reports on a study of eighteen prospective secondary mathematics and science teachers' approaches to conducting a statistical investigation using the dynamic data analysis software Fathom™ (Finzer, 2001). Three distinct approaches were identified by the research—Wonderers, Wanderers, and Answerers—each with measurable differences in their approach. This paper describes qualitative and quantitative differences in these approaches as well as their potential epistemological roots.

Introduction

Bakker (2002) identified two categories of learning software in mathematics: routetype software and landscape-type software. Route-type software was designed to support a predetermined learning trajectory with a fairly fixed destination. Alternatively, landscape-type software is identified by its ability to support learners in conducting open-ended investigations by providing them with dynamic tools—for purposes defined by the user rather than the software. The use of these dynamic software tools for the learning of mathematics and statistics has gained increasing prominence in schools because of its ability to support multiple learning routes with varied destinations. Little is known, however, about the diversity of approaches in which learners use these software packages to conduct investigations. This paper reports on a study of eighteen prospective secondary mathematics and science teachers epistemological approaches to conducting a statistical investigation using the dynamic data analysis software Fathom™ (Finzer, 2001).

Dynamic statistical learning software

Two significant shifts have been made in the teaching and learning of statistics, largely due to the influence of technology and improved theories of learning in 371 mathematics. The first shift, beginning around the time of the first ICMI study on technology (late 1980s), was largely due the availability of computing software such as Excel, SPSS, and SAS which freed the user from tedious calculations. This change affected primarily upper secondary and tertiary instruction in formal statistics coursework (e.g., hypothesis testing, regression) as statistics was deemed too challenging for younger students. The shift provided an opportunity for instruction to evolve from a focus on learning formulas and practicing calculations towards a greater emphasis on conceptual development. These newer approaches to teaching statistics relied more heavily on technology for calculations, putting new emphasis on activity-based learning, development of conjectures and data collection, and more complex analysis of larger data sets. Like in mathematics, this opened opportunities in statistics for learners with a variety of backgrounds and learning styles, including access to powerful statistical tools for secondary students.
The drawback of employment of software to calculate statistical measures and hypothesis testing, however, was its use as a black box. That is, students would input data into the software, apply a few keystrokes to request a particular graph, calculation, or statistical test, and the results would be "spit out" by the computer. This generated much complaint by researchers that learners were developing a blackand-white view of statistics (Abelson, 1995; Gardner & Hudson, 1999), a deterministic mindset of data as a means for providing "answers" to complex questions. An opposing epistemological stance was of statistics as a tool for inquiry and analysis which acknowledges a complex world.
Statistical software packages developed in the past few years counter this "black box" approach by taking a more visual approach to statistical analysis. Software packages such as Fathom™ (Finzer, 2005) and Tinkerplots™ (Konold & Miller, 2004) were developed as learning software for doing statistics, by encouraging learners to visualize statistical relationships and develop skills in informal inference. Software packages designed to support statistical learning through open-ended investigations with data have the potential to provide users with multiple opportunities to become data wonderers, those who explore a series of "I wonder" questions as they explore data, generate and test hunches, seek insight into the phenomenon being investigated, and communicate statistical evidence for discussion and debate. Whether these software packages actually support learners in this way, however, has not been investigated. This is the goal of this paper.

Study context and method

A study conducted at a large university in the United States examined preservice teachers' development of statistical reasoning through exploration of assessment data (Makar, 2004; Makar & Confrey, in press). The participants in the study were eighteen prospective mathematics and science teachers enrolled in an innovative course on assessment developed and taught by the authors. The course themes examined standardized and classroom testing, analysis of data using technology, and focused on developing analytic tools to highlight issues of equity in test data. 372 Structured investigations were used throughout the course to introduce the teachers to ways in which interpreting data distributions could uncover hidden issues in highstakes testing (Confrey & Makar, 2005).
The primary purpose of the study was to examine the interaction between the prospective teachers' understanding of variation and distribution and their use of data as statistical evidence in an open-ended data investigation of equity in testing. This paper will focus on the epistemological approaches that the preservice teachers made use of the software Fathom™ to conduct a semi-structured data investigation.
Individual interviews were videotaped as each teacher conducted an investigation in Fathom; their work on the computer was also captured. Videotape data, linked to their computer capture, were transcribed and analyzed qualitatively using Grounded Theory (Strauss & Corbin, 1998). One analysis focused on individual actions performed (e.g., observations, evaluations, and conclusions) and a second macrolevel analysis documented patterns of inquiry. The action-analysis was coded by two researchers independently, with 95% agreement.
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mean ( ) = 79.0476
Hispanic - Urban vs. Rural Dot Plot
Figure 1: Representation created by nearly all participants
A random sample of data on 273 students' scores on a state exam was provided to the prospective teachers for the investigation. The dataset contained fourteen variables

(demographic information, current and previous test scores, economic level, Englishlanguage background, etc.) on sixteen-year old students of Hispanic descent from rural and urban communities. Before seeing the dataset, the teachers were asked to state a conjecture about the relative performance of Hispanic students in urban and rural schools. After stating their conjecture, they were told to investigate their conjecture in Fathom until they felt they were ready to reach a conclusion. They were given the data set but no representations (tables or graphs). Most of the participants began by creating a graph similar to Figure 1. The upper distribution is of urban student test scores and the lower distribution is of rural student test scores.

Results

All of the preservice teachers demonstrated facility with the software in conducting their analysis and did not find the process difficult, even though the dataset was 373 relatively large. An analysis of the patterns of inquiry resulted in three categories of epistemological approach to conducting their investigation: Wonderers, Wanderers, and Answerers. Briefly, Wonderers are those who took an inquiry approach through cycles of "I wonder" questions: investigating the phenomenon by following a hunch, then seeking further explanations in the data through refined conjectures. Rather than using the data to investigate hunches, Wanderers, used the data to create hunches. They systematically examined graphs or summary statistics on each variable until they found a pattern that looked "interesting". Finally, Answerers developed an initial hunch, used the data to test their hunch, and quickly ended their investigation. Further analysis is provided below for each category.

Wonderers

Six of the eighteen participants (33%) were classified as Wonderers. This group exhibited the kind of behaviour for which dynamic software was likely designed. They developed a conjecture of what they expected to find, then were lead through an investigation by cycles of "what if" questions, before finally settling on a conclusion (Figure 2). These questions refined their theories of understanding about the phenomenon being investigated. Their investigations in the data were guided by these theories.

Figure 2: Model of Wonderer behaviour

Wonderers were distinct from the other participants in two measurable ways. First, they spent a significantly longer period of time conducting their investigation (p = 0.02), spending on average 26.2 minutes (s = 12.1) compared to an average of 10.3 minutes (s = 5.1) by the other participants. Wonderers used their time differently as well, making significantly fewer unfocused observations (statements about the data not connected to the context or conjecture) than other participants (p = 0.03), making on average 2.6 observations per 5 minutes (s = 1.2) compared to a mean of 4.4 (s = 2.2) by other participants. Although their investigations were longer, they were focused in their exploration into the situation. Their use of the software appeared to be as a tool for inquiry - to seek insight into the context of their investigation.

Wanderers

The largest category of investigative behaviours was by those identified as Wanderers, with eight of the eighteen participants (44%) being categorized as such. Wanderers used the software to go through the variables available in the data, often systematically, until a relationship "popped out" at them. They did not enter their investigation with any particular theory in mind (Figure 3) nor did they conduct their inquiry in a purposeful way. They appeared to possess a belief that the data would "speak for itself" and that any relationships to be found would emerge on their own. Once a particular interesting relationship was found, they used their understanding of Theor
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the context (rather than the data) to try and explain the phenomenon they had found. To them, relationships pre-existed and their job was to discover these relationships.

Figure 3: Model of Wanderer behaviour

The Wanderers used their time primarily making observations (statements not connected to the context or conjecture, such as comparing sample sizes) or drawing preliminary conclusions; few of their statements were evaluative in nature (interpreting the data with respect to the context). One of the participants summed it up well when she said during her investigation "Well, I always like to look at everything". Their use of the technology was as a filter to "catch" potential insights. The Wanderers (mean = 1.7, s = 0.44) posted a significantly higher rate of conclusion statements per five minutes (p = 0.02) compared to the Wonderers (mean = 1.0, s = 0.46) and posted an average of 70% more observations per five minutes (mean = 4.4, s = 2.4) than the Wonderers (mean = 2.6, s = 1.2), although the difference was not significant (p = 0.09). In general, the Wanderers did not appear to seek insight into the phenomenon being investigated, but looked for significant results in the data devoid of context.

Answerers

The third group of behavior types recorded, the Answerers, used the software as a tool to locate a particular piece of evidence in the data to test a conjecture (theory) and then were quickly ready to draw a conclusion (Figure 4). Four of the eighteen participants (22%) were identified as Answerers. To this group, the computer was an efficiency tool that they could use to answer a question they had. This group was identified by their decision process: they looked for a particular, single piece of evidence and once they found it were satisfied that they had "answered" the question put to them. Their approach was somewhat similar to the use of computers to support statistical analysis using traditional statistical software. Rather than be a tool to explore, it was used as a way to test and decide upon the validity of a hunch.

Figure 4: Model of Answerer behaviour

Answerers were distinct from the other two approaches in two measurable ways. First, their investigations were significantly shorter (p < 0.01) than the other two groups, with a mean time of 5.7 minutes (s = 2.6 minutes) compared to the other Theor
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participants (mean = 18.5 minutes, s = 10.8 minutes). Second, they spent significantly greater proportion of their time (p = 0.03) drawing conclusions (mean = 2.4 conclusions per 5 minutes, sd = 0.55) compared to the other participants (mean = 1.4, sd = 0.55). Their investigations were very efficient and appeared to use the software to test their conjecture rather than seek insight into the phenomenon under investigation.

Discussion

The three approaches to conducting data-based investigations with dynamic software revealed three different epistemological approaches by learners to computer supported inquiry. Wonderers approached their investigation by developing a hunch or theory about the phenomenon in question. They then used the data to both test their hunch and refine their conjecture. Wonderers possess curiosity in a way that Dewey (1910/1997) argues is vital for reflective thinking and inquiry. "Such curiosity is the only sure guarantee of the acquisition of the primary facts upon which inference must base itself" (p. 31). It is likely that this was the model of a learner that the software developers had in mind when the software was designed, although this category emerged in only one-third of the participants. This approach reveals a learner who acknowledges that their role as an inquirer is to construct a deeper understanding of the phenomenon by creating more and more refined conjectures and demonstrates a deeper epistemological grounding.

Two different approaches, however, also emerged in the analysis. Wanderers were identified by the lack of purpose in their investigatory approach. Rather than use the data to test a hunch, they are more opportunistic in their approach, sifting through the data until something interesting caught their eye. From their findings, they try and develop a theory which they believe the data are telling them, however they explain their theory not with data but by anecdotal evidence. This epistemological approach is one in which the learner believes the data "speak for themselves" and their role as investigators is to seek out predetermined messages hidden in the data. Unlike the Wonderers who take charge of the investigation, Wanderers are led through the investigation by the data. Dewey again describes this group well, saying that their conclusions are "generated by a modicum of fact merely because the suggestions are vivid and interesting" (p. 20), and that they "find it difficult to reach any definite conclusion and wander more or less helplessly among them" (p. 36). Because the software was designed to make graph construction easy through its 'drag-and-drop' technology, it is possible that it may encourage this kind of behaviour in some learners. This can be of concern when the goal is to develop curious investigators, not those who wander aimlessly through graph after graph until an interesting results "pops out" at them, then explaining it post-hoc. If the number of relationships examined by the Wanderer is high, it is possible that the investigator will happen upon a significant result just by chance. 376

Finally, Answerers use the software as an efficiency tool by developing and testing a hunch to find an "answer" to their question. Rather than see the data as an opportunity to seek further meaning of a phenomenon, Answerers see data as having the potential to reveal answers to questions under investigation. This is consistent with Dewey's description of learners hardened by routine: "A conclusion reached after consideration of a few alternatives may be formally correct, but it will not possess the fullness and richness of meaning of one arrived at after comparison of a greater variety of alternative suggestions" (p. 36). Dewey highlights the importance of taking one's time in inquiry: "Time is required to digest impressions, and translate them into substantial ideas. Failure to afford time for leisure conduce to habits of speedy, but snapshot and superficial, judgment. The depth to which a sense of the problem, of the difficulty, sinks, determines the quality of the thinking that follows" (p. 38). The Answerers' approach is similar to one taken by many learners who use traditional software packages; they allow the user to input data, select a particular measure, graph, estimate, or model, and then present the user with the result. It requires a highly creative user with a strong conceptual understanding of both statistics and the context under investigation to use traditional software packages as something more than a black box, often bemoaned by teachers who recognize students' black-and-white approach to statistical analysis.

The epistemological approach taken by the user is of critical importance if the goal of innovative software is to support the learner in the construction of knowledge through inquiry-based approaches. The desire is not just to guide the learner through a set of facts and procedures as it may have been previously. Rather, these software packages are developed to enable mathematical thinking that goes beyond facts and procedures. The purpose is to develop skills in inquiry as well as a mindset of mathematics as a human construction developed through the refinement of proofs and refutations (Lakatos, 1976).

The findings of this study allow us to now investigate how we can support each of these categories of user. For example, how can one encourage a Wanderer to envision a goal of structured inquiry? What kinds of tasks must be developed in order to encourage Answerers to seek further insight into their initial findings? These are important questions for the mathematics community.

The use of new dynamic software packages has provided learners with powerful tools to support a more visual approach to informal statistical inference. A critical next step is to investigate ways that learners are using these tools. This includes both acknowledging the diversity of approaches to learning that they support as well as ways in which tasks can be developed which encourage productive inquiry skills.

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New artefacts and the mediation of mathematical meanings

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The paper proposes a theoretical reflection, based on a long-term research project. The aim is that of presenting a Vygotskian perspective for interpreting the functioning of new technical tools within the theoretical framework of social construction of mathematical knowledge. Our study has developed the relationship between a general hypothesis, concerning the teaching and learning process mediated by artefacts, and specific hypotheses concerning the semiotic potential of specific computational tools. The original notion of semiotic mediation has been

elaborated in order to become a theoretical construct both inspiring the design of the teaching experiments and guiding the analysis of the collected data, according to the methodology of research for innovation. The discussion proposed aims to situate the notion of semiotic mediation in relation to other theoretical constructs, in particular, to the notion of instrument and instrumental genesis, as introduced by Rabardel and now developed in the field of mathematics education.

The impact of new technology

In spite of the great expectation expressed by many educators more than twenty years ago, it is hard to say that new technologies have found a real integration in school practice. For this reason it becomes more and more urgent to identify key points around which to organise the discussion on the way to exploit the potentialities that new technologies (computers and all the new technologies related to them) offer to education: how and why new technologies are going to influence education and in particular mathematics education in the future.

This paper aims to contribute to the debate from a theoretical perspective. Drawn from a number of long term teaching experiments, developed within different research projects in the last years, the proposal of a theoretical perspective will be discussed to explain the functioning of different tools in the construction of mathematical knowledge. Such a perspective has the both modern computational tools and old artefacts, but mainly seems suitable to express this functioning in terms of educational goals.

Historic and social perspective

Human history is accompanied and punctuated by technological innovations. Generally speaking, artefacts and their use can be considered characteristic of human activity and their contribution at the cognitive level (Norman, 1993) is largely acknowledged. Besides this general statement, it seems that there is something in computer related technology which makes it a very peculiar artefact in respect to education and mathematics education in particular. Since the very beginning, even before the appearance of the personal computer, computational technology has been 379 considered as a powerful tool to be used for educational purposes. Papert (1981) certainly had the merit to firstly point on this aspect: interacting with a computer offers many different opportunities of meaningful activities, often neglected in school practice, but involving ways of thinking recognisable as typical of mathematics.

The appearance of new technologies and in particular of new tools related to mathematical activity, led mathematics educator to foresee a deep transformation in the relationship between problems and knowledge, both in respect to the type of problems to be proposed to pupils and the solution processes that new resources may affect.

That is obviously the case of all the environments which have a direct relationship with mathematical knowledge: for instance, symbolic manipulators, such as DERIVE, or more sophisticated products such as Maple or Mathematica; but it can be also the case for other applications which have no direct relation to mathematical knowledge, but never the less incorporate it, for instance professional software such as EXCEL (Sutherland & Rojano, 1993) .

Much work has been done in order to study the effects of acting in computer environments on intellectual processes involved in problem solving and concept formation. As clearly pointed out by Noss and Hoyles (1996, p. 44), availability of computational objects has drawn a deep transformation, in particular, concerning the classic distinction, historically rooted in the western culture, between abstract and concrete. Reification of mathematical objects and relations (Dörfler, 1993) has challenged longstanding assumptions about what mathematics is to be taught and claims for a radical change of perspective (Noss & Hoyles, 1996; Lagrange, et al. 2003; Trouche, 2005).

The cognitive-oriented research, carried out for many years has been principally centred on the learner, but it has also highlighted the need to enlarge the investigation in order to study the effect that activities in computer environments may have on the mathematical classroom as a whole.

In fact, although strictly related and deeply affected by practice, and in particular by practice mediated by tools, the process of construction of mathematical meanings is not directly and simply related to practice. To go back to the origin consider the use of the compass and the meaning of circle

The definition of the geometric figure is certainly related to the use of compass, which on the other hand realises graphic representation of circles. But the passage from the use of the compass to trace round shapes to the conception of the circle as "the locus of the points equidistant for the centre" is not immediate (Bartolini Bussi et al., 1996) neither is it the nature of the definition which can be formulated and used in the solution of geometrical problems, for instance construction problems (Mariotti, 2000).

In conclusion, mathematical meanings, rooted in the use of artefacts, might remain inaccessible to pupils: they may remain "in the eyes of the observer".

This phenomenon can be evaluated differently, according to what possible learning achievements are expected, however the discussion on the results of the interaction 380 with a computer seems to become more and more complex, I would relate it to a general issue that can be condensed in the following crucial question (Mariotti, 2002):

Is it possible to co-ordinate the autonomy of the student to construct his/her own knowledge, and the authority of mathematical knowledge, as a cultural domain of knowledge?

The relationship between artefacts and mathematical meanings asks a particular analysis, taking into account the fundamental role played by the emergence of signs, related to the instrumented activity, and their evolution within social interaction.

The instrumental approach

An instrumental approach, developed in the domain of mathematics education in a number of research studies and rooted on the work of Rabardel (1995), provides a very powerful frame for describing situations where the use of tools is involved, in particular, to account the differences that might appear in students' use of a tool and the relationship they may construct between the use of a tool and the mathematics knowledge .

In the analysis, carried out by Rabardel (1995), the main point consists in separating the bare object from the ways of using it in relation to accomplishing a task. In other terms, he proposes the following distinction between:

• the artefact, i.e. the particular object with its intrinsic characteristics, designed and realised for purpose of accomplishing a particular task:

• the instrument, that is the artefact and the modalities of its use, as are elaborated by an individual in accomplishing different tasks.

The introduction of an instrumental approach makes it possible to analyse and to interpret the cognitive complexity of an instrumented action and in particular, to see its potential in terms of socially shared meanings related to this action. The cognitive development (instrumental genesis), related to the acquisition and evolution of utilization schema of an artefact, carefully described by the author and subsequently elaborated in different studies (Artigue, Lagrange, Trouche, Drijvers), may contribute to explain some of the difficulties encountered by students, and at the same time fruitfully inspire the design of classroom activities.

Semiotic mediation at school: the role of the teacher

In the process of instrumental genesis the evolution of the artefact into an 'instrument', may support the emergence of sign, and consequently of meanings, related to those utilisation schemes. As clearly explained by Radford:

"In other words, to arrive at the goal, the individuals relay on the use and articulation of several artefacts and semiotic systems through which they organize their actions across space and time. [...] these artefacts and varied systems of signs that individual use in social meaning making process to make apparent their intention and carry out 381 their actions in order to attain the goal of their activities I call means of semiotic objectification." (Radford, 2001, pag. 4)

Meanings are expressed through signs, words, gestures, drawings ... and in so doing they may become socially shared and evolve. Imagine this process in the school environment, in particular in the community of the mathematics class, where social interaction means not only general interchange between human beings, but also interchange oriented by the common goal of teaching/learning mathematics.

Signs sprouting from activities with the artefact are socially elaborated: in particular, they can be intentionally used by the teacher to exploit semiotic processes, aiming at guiding the evolution of meanings consistent with mathematical meanings that are objectives of the educational activity. In this case one can say that the artefact, used by the teacher for her/his educational objectives, is functioning as a "tool of semiotic mediation".

In summary, the process of semiotic mediation develops on two different levels:

- The pupil uses the artefact, according to certain utilisation schemes, in order to accomplish the goal given by the task. In so doing the artefact may function as a semiotic mediator, i.e. meanings emerge from subject's involvement in the activity in relation to particular utilization schema and to particular emerging sings.

- The teacher uses the artefact and the signs derived from its use in specific activities, according to a specific educational motive, that is the evolution towards mathematically consistent meanings.

The mathematical meaning, related to the artefact, become accessible to the learner by its use, but the construction of meanings is supported by the guidance of the teacher, as long as specific activities are organised, the motive of which is the evolution/construction of meanings recognisable and acceptable mathematically. According to that perspective the following hypothesis can be stated.

Meanings are rooted in the phenomenological experience (actions of the user and feedback of the environment, of which the artefact is a component) but their evolution is achieved by means of social construction in the classroom, under the guidance of the teacher.

In the dialectics between these two levels the social construction of mathematical meanings occurs, as the product of a process of internalisation guided by the teacher.

The organization of the activity

The process aimed to guide the development of the dialectics between the two levels (the individual and the social) has different components, which can be separately studied although has to be considered as a whole. Consider, for instance, the notion of instrumental orchestration (orchestration instrumentale) introduced by Trouche, as the author explains:

"Orchestration instrumentale est exactement l'agencement systématique par un agent intentionnel des éléments (artefacts et humaines) d'un environnement en vue de mettre en oeuvre une situation donnée et, plus généralement, de guider les apprenants 382 dans les genèses instrumentales et dans l'évolution et l'équilibrage de leurs systèmes d'instruments".

(Trouche, 2005, p. 126)

In same paper, the author carefully examines the aspects of the orchestration centred on what he calls specific configurations didactiques, related to different types of organization of classroom work with the artefact; the author proposes some examples of configurations where the teacher guides the students through the process of evolution of the instrumental genesis.

Consistently with the general hypothesis formulated above, in the last years a long term research projects have been carried out, with the aim of testing the effectiveness of this theoretical hypothesis, in the case of particular artefacts and particular mathematical meanings. According to a methodology of "research for innovation" (Arzarello & Bartolini Bussi, 1998) teaching experiments have a dialectic relation with specific theoretical framework, on the one hand the theoretical framework shape the design of the particular teaching experiment, on the other hand results coming from the analysis of the collected data lead the researchers to develop their assumptions, and generally speaking to develop the theoretical framework.

The long term teaching experiments followed a common methodology, consistent with the hypothesis concerning the functioning of specific tools of semiotic mediation and aimed to further elaborate the theoretical perspective. The methodology is based on the following basic points.

- Selection of the artefact. Analysis of the artefact and of its use from an epistemological and cognitive point of view; the objective is that of identifying its semiotic potentialities, in rapport to a mathematical ideas, processes, ..., that can be generally referred as meanings.

- Design of the teaching intervention. Design of a sequence of activities, including different types of tasks; some of the tasks are centred on the use of the artefact, other centred on semiotic activities related to the use of the artefact, and aimed to generate germ signs (for instance writing reports on lab experiences, or writing the mathematics notebook). Among these semiotic activities a special role has been played by collective discussions (Bartolini Bussi, 1998), where germ signs have to be developed under the guidance of the teacher.

- Analysis of the experimental data, in particular the analysis of the collective activities, recognising in social interaction that interpersonal context where the evolution of signs may occur. Results showed, both within a single collective discussion and along a sequence of successive discussions, the evolution of what can be called a semiotic net, i.e. a system of related meanings progressively merging towards the mathematical meaning. A complex and delicate semiotic process is accomplished, rooted in the activities with the artefact, but put into place only thanks to the active direction of the teacher (Mariotti, 2000, 2001; Mariottiß Cerulli, 2001; Cerulli & Mariotti, 2003; Mariotti et al. 2003; Falcade et al., 2004). 383

Since the beginning, the role of the teacher has been considered crucial, and the fine grain analysis of the semiotic process, taking place during the collective discussions has shown both its effectiveness and its complexity. In particular, specific actions have been described, that the teacher intentionally, but also automatically put in place, with the aim of guiding and prompting the emergence and the evolution of meanings, consistent with the mathematical meaning which is the goal of its teaching endeavour .

The analysis of the teacher's actions highlights that, besides the general categories of actions, directed to manage the didactic contract in the class, the teacher has to put in place specific actions directed to promote the semiotic process (Bartolini Bussi & Mariotti, 1998). This has opened a new and promising field of investigation and a specific research project is now in progress aimed to develop this direction of study.

Conclusions

Taking an instrumental perspective and exploiting the process of semiotic mediation that artefacts make possible permit to consider new technologies in a broader problematique context: computer technologies, as "older" technologies, have a potential which comes from their being products of human culture, embedding knowledge and expertise (Bartolini Bussi et al., 2005; Bartolini Bussi & Mariotti, 1999), but overall being able to evoke mathematical knowledge consistent with

possible educational goals.
Moreover, the instrumental approach articulated in the semiotic perspective, presented in this paper, has the great advantage of proposing a common perspective for any kind of tools, as a consequence the discussion on new technologies may profit from the comparison with the results coming form research studies concerning 'old' technologies (Bartolini Bussi & Mariotti, 1999; Bartolini Bussi et al. 2005), and more generally any kind of artefact that presents semiotic potential consistent with our educational goals.
A conscious use of available technologies in terms of 'semiotic mediation' requires an attentive and careful planning of classroom activities, taking into account the double use of the artefact in play, both form the point of view of students' actions with the tool and from the point of view of teacher's actions with the tool as an instrument of semiotic mediation. Besides a deep knowledge about the artefact and the evolution of the instrumental genesis, which overcomes the simple familiarity with the use of the artefact itself, it is necessary to analyse the artefact and its potentialities in terms of instrument of semiotic mediation in order to organise and carry out classroom activities according to this function.
All that represents a great challenge in the field of math education research, in order to provide teachers the knowledge and the support that they need. Further studies and investigations are requested, mainly concerning the identification of patterns of actions that teachers may accomplish to exploit semiotic mediation.
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Theoretical Approaches to Learning with Digital Technologies
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Introduction
There is a growing body of research oriented towards theoretical approaches to the use of digital technologies in mathematics education. One thinks of the work of the "French school" in Computer Algebra Systems (CAS) (Artigue, 2001; Lagrange, 1999), Kaput (2001) in visualization and dynamic relationships, and the likes of Stroup and Wilensky (2000) in networked classrooms. There remains a considerable amount of work to be done showing how certain theoretical approaches to the understanding of digital technology, and its place in the classroom, inform how students learn in a technology rich environment and how we can harness that understanding to improve student learning. In this paper I will offer a theoretical framework, based on my research (Meagher, 2004) and inspired by the work of Brian Rotman (1993, 1995), for understanding the place of digital technology in student learning. The research was based on the use of CAS in students learning calculus but I believe the framework applies well to digital technology generally.
Digital technologies and theory
Perhaps one reason that theoretical frameworks are running behind use of digital technologies is that calculators/computers are, in many ways, a stealth tool in the classroom. Use of digital technologies in the classroom demands innovation (at least inasmuch as any new agency on a classroom demands innovation) but doesn't make clear the operationalisation of that innovation. Indeed it is impossible to judge those demands in advance as they are manifestations of the difference between what Moursund (2002) has called first-order use of technology (amplification) and secondorder use. Moursund argues that, in the first instance, we use technology to do what we can do already, but to do these things faster and more accurately. The classic example of this is the car which was first thought of as a "horseless carriage." Therefore, the car was first conceived of as a technology which would move people from A to B just as a carriage does, only faster. The second-order effects of cars are now well-known to us: complete transformation of the very conception of the city, dismantling of public transport, rampant pollution, raise in the age for legal drinking, etc., but were certainly not part of the design and conception of cars. The following example shows how digital technology introduced into the classroom setting, without attention to theory-informed use of technology, can bring second-order effects to the fore which corrupt and subvert the work of the classroom.
An example of technology use 387
In typical high school algebra classes students learn to factor quadratics in order to solve quadratic equations and are subsequently introduced to the relationship between the solutions of such equations and the roots of the graphs of quadratic functions. I have seen students being taught to use the functionality of the graphing calculator to find the roots of a function from a graph and, using those roots, work backwards, to factor quadratic expressions. I offer this example not so much to criticise this approach but, rather, to point out the contradictions that can become manifest in the coherence of the standard mathematics curriculum when digital technology is introduced into the classroom. Students often think of mathematics as a school-based game which they can play well or not. When digital technology is used in this way to subvert a curriculum is it any wonder that students perceive mathematics as such a "bag of tricks"? (A perhaps more enlightened approach is to recast the approach to teaching of this subject to concentrate on functions, as opposed to equations with equations becoming particular instantiations of functions.)
I discuss this example at some length because it serves to show the nature of the change that the introduction of digital technologies in the classroom can impose on the classroom environment. Furthermore it begs the vital question that needs to be asked when digital technology is introduced: In what way is my classroom now different? This is the key question toward the path to theoretical approaches that illuminate learning in a technology rich environment. As we see in the example above the classroom is not the same: the goals have changed, the paths to those goals are new, the dynamics of interaction are new. This is an example of Moursund's (2002) second order effects.
The benefits of graphing calculators are many and obvious: an entire strand of dynamic visualisation is added to the classroom and student's ability to experiment with parameters of functions and their effects on graphs is clear. We see, however, in this example a second-order effect: suddenly the learned procedure of factoring quadratic equations is brought into question. If the goal is simply to solve the equation then, surely, using the digital technology is more effective. A second order effect has occurred whereby the goal of factoring shifts perhaps to an understanding of the relationship between the algebraic form and the coordinate geometry being the driver of the concept rather than a noticed consequence once the student has gone through the conceptual path from algebra to coordinate geometry. Another possible shift is that what matters in terms of factoring quadratics is an awareness of algebraic structure: some quadratic forms can be written as products of linear forms. In this case the second order effect of the technology is to force the broader, structural mathematical issues to the forefront of the understanding of the concept. In either case, and there are many other cases, the introduction of the digital technology does more than simply "soup up" the existing status and dynamic of the class. 388
Approaching theoretical approaches
As suggested above, as a first step to exploring theoretical approaches to the use of digital technology it behooves us to draw on experiences of using technology in teaching and ask what is different about the classroom when digital technology is introduced.
Another example: When I first taught with CAS it was in an Algebra I class where the students had hand-held devices (TI-92s) with inbuilt CAS capability. The beginning of one of the worksheets was as follows:
Type 3a + 2a. Press enter and record the result
Type 4b + 5b. Press enter and record the result
Type 3x + 2y. Press enter and record the result
What do you think will be the result of 3m + 2m? Record your guess and check.
What do you think will be the result of 3r + 2q? Record your guess and check.
Do you observe a pattern or rule? State the pattern or rule.
The students worked through this sheet with great success and their attempts to articulate the rule provided a good opportunity for discussion of definitions, vocabulary, and concision in mathematical definitions and descriptions. What struck me as curious about all of this is when I thought to myself: would this be different if I, as the teacher, went to the blackboard and wrote "3a + 2a = 5a; 4b + 5b = 9b" etc. and asked the students to do the examples and tell me the pattern/rule. Having taken both approaches I can attest that each approach plays out very differently in the classroom. Students regard the mathematical authority of a calculator/computer differently to how they regard the authority of a teacher. There is a clear sense in which students felt they were discovering something when getting feedback from technology rather than responding to my probing. This difference begs many questions about how students relate to a calculator/computer (as opposed to a person) and, in particular, how students relate to the authority of a calculator/computer. The way that I found to characterise, and theorise about, the interaction that was going on was to regard it as a "trialogue" between the students, the technology and the mathematics.
In looking for theoretical approaches to help illuminate students' learning of mathematics in technology-integrated environments this tripartite relationship led me to the Rotman Model of Mathematical Reasoning (1993, 1995).
The Rotman Model of Mathematical Reasoning 389
Drawing on the work of C. S. Pierce, Rotman's (1993, 1995) semiotic reading of mathematics discourse constitutes mathematical reasoning as the unison of three agencies: the Subject, the Person, and the Agent (Fig. 1.1).
Figure 1.1: The Rotman Model of Mathematical Reasoning
Rotman's (1993, 1995) theoretical model is concerned with ontological questions about mathematical objects and processes. Its principal value for my purposes is its tripartite nature with the student, the mathematics, and the technology at the vertices of the triangle. Nevertheless, it is worth looking in some detail at what Rotman has to say because of his understanding of agency in mathematics and where that agency resides. In his model the Subject is "the agency [which] reads/writes mathematical texts and has access to all and only those linguistic means allowed by the Code" (p.396), the Code being, essentially, mathematics as sanctioned by the mathematical

community. The Person is the agency who works within the Metacode "the entire matrix of unrigorous mathematical procedures" (p.396), such as stories, ideograms and pictures. Finally the Agent is the agency who "acts on mathematical objects in a purely formal way" (p.397), i.e. with a Virtual Code. Mathematical reasoning in this formulation is considered to arise from the interplay between a learner, mathematics as a subject, and a disembodied agent who carries out mathematical procedures accurately and without prejudice. For my purposes the disembodied agent is the digital technology. For Rotman, mathematical reasoning is "an irreducibly tripartite activity in which the Person ... observes the Subject ... imagining a proxy - the Agent ... - of him/herself" (p.397), and is persuaded by the closeness of the Subject and the Agent of the validity of the mathematical activity. My argument is that digital technology can be interpreted as the Agent of mathematics with which students interact to gain access to calculus. This adaptation of the Rotman model is seen in
CODE
Subject
VIRTUAL CODE
Agent
METACODE
Person 390
Figure 1.2:
Figure 1.2: The Adapted Rotman Model of Mathematical Reasoning
Thus one avenue for the development of mathematical reasoning could be engaged in creating or facilitating intellectual space where the learner can experiment and play within the realm of the Metacode i.e. the "stories, motives, pictures, diagrams, and other so-called heuristics" (p.396), through which the learner gains access to the Code but with reference to the Virtual Code i.e. the acting on of mathematical objects in a formal way. My research (Meagher, 2004) shows, in the realm of CAS, that technology can furnish just such intellectual space in mathematics education. My research found that CAS allows students to take experimental steps in working with mathematical objects. Which is to say that CAS provides a Virtual Code (Mathematica) allowing students to act experimentally and observe the consequences of those actions within the Virtual Code (Mathematica) to examine, or negotiate, a relationship with the Code (Calculus) itself. More importantly my research shows that the Model, as I am using it, provides a way of understanding the place of technology in the mathematical activity of the learner. (Digital technology can be thought of, perhaps, as providing various microworlds in the Papertian sense, CAS as an algebra microworld, Cabri as a geometry microworld etc.) The important part of the model is its triangular nature which affords separate agency to three entities: the learner, the technology, and the mathematics. What I mean by agency here is that none of these entities is neutral but that each has a relationship with the other two, effecting and being effected by the other two in a manner such that each contributes continually to the students mathematical reasoning. Each of these entities acts on the other two and thus has agency.
Application of the Model
My research (Meagher, 2004) explored each of the three sides of the triangular model above with significant conclusions found on each side. The temptation to describe the digital technology to mathematics relationship simply as "the machine has been
MATHEMATICS
Code
TECHNOLOGY
Virtual Code
STUDENT
Metacode 391
programmed to perform accepted algorithms" is belied by closer examination. For example, if one looks at the different inputs necessary to "ask" different CASs the same question, and one looks at the different outputs from different CAS systems in response to those questions, it becomes quickly apparent that, while the mathematics and algorithms might be agreed upon by the different programmers, the interfaces of the programs and the presentation of results are radically different.
Mathematica, a program originally written when memory was at more of a premium than it is nowadays, is a program which depends on the user learning a very particular syntax. This syntax is not reflective of mathematics as it is written in textbooks or mathematical journals and cleaves more closely to high level programming languages. Another memory saving device of Mathematica's is the rendering of graphs. Graphs are regularly displayed in a non-standard format (for example the axes do not intersect at (0, 0) in a way that is efficient but may not be easy for the user to interpret. DERIVE, by way of contrast, is a program which was designed, particular in later versions, which considerable attention paid to the fact that the user may be trying to learn mathematics using DERIVE. The program is essentially menu driven, looking more like graphing calculator interfaces and word processing programs than Mathematica. In addition, graphs in DERIVE always include the origin and error messages tend not be as intimidating as those in Mathematica.
In the realm of graphing calculators we see privileging of certain forms such as the "y =" form of equations of lines in order to facilitate graphing. On an even more basic level students need to learn particular syntax orders to perform, for example, exponentiation. This order is different from what they see in text books or typically write in the class.
Understanding digital technology as a separate agency is crucial to exploring students response to and use of technology. My research showed, in particular, that the transition to a new form of digital technology (graphing calculator to CAS) is a delicate moment and the students' relationship to the new technology was heavily dependent on both their history with the old technology and how the transition to the new technology was facilitated. This is further evidence that the introduction of digital technology is not merely a passive add on to the current situation but dynamically impacts students behaviour and achievement in a technology rich environment.
Students' relationship to mathematics itself is also highly influenced by their use of technology in learning. My research showed that students whose idea of success in mathematics is oriented towards successful implementation of algorithms and a 392 strong sense of "getting the right answer" can feel somewhat cheated by digital technology. This is particularly true in the use of CAS where the machine is doing the work which students often take as their domain of success and an emphasis is placed on explanations and reasoning which some students consider to be less authentic mathematical activity.
Conclusion
The theoretical approach outlined above is important in providing a step away from reductive thinking of technology as a simply a mediator between the student and mathematics. The Adapted Rotman model provides a more sophisticated and authentic picture of the learning relationship between students, digital technologies and mathematics. It can also be thought of as residing in a larger network structure which would include, in a more explicit sense, the role of the teacher and teaching in technology-rich environments.
The model is, in many respects, consonant with constructivist theories of learning but emphasises the extent to which the students' knowledge is constructed by the technology: digital technologies become, very explicitly, part of the sociocultural framework of the classroom. However, as explained above, digital technologies have a different agency from other students or a teacher in the classroom environment. The model should also be an important consideration in curriculum design since it places great emphasis on students understanding of the role of technology in their learning and places a non-traditional emphasis on what is considered authentic mathematical activity.
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Comments for ICMI STUDY 17: "Digital technologies and mathematics teaching and learning: Rethinking the terrain"
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The paper will reflect the process of learning and understanding mathematics when working with calculators or computers. How do we succeed in developing powerful mental representations, which in German we call "Vorstellungen"? We distinguish two types of Vorstellungen, but the traditional kind of teaching mathematics gives strong emphasis only to one of them, to a reflective, logical and analytical thinking. Most teachers or students or even researchers in mathematics often are unaware of their spontaneous and intuitive Vorstellungen.
But only the interaction of both types, the interference of "reflective" Vorstellungen with "intuitive" Vorstellungen, develops powerful mental concepts, procepts, frames, micro worlds, ... The use of calculators or computers seems to further this development. Working with a computer we often see a typical guess and test behavior or trials to discover properties or repeating similar key stroke sequences just to make sure ...
We regard this mainly unconscious behavior as a vehicle to further the development of "intuitive" Vorstellungen and we designed a special teaching method which we called "One-Way-Principle" (abbr. OWP). The OWP is an intermediate step to discover in a set of examples intuitively common properties to move on then to generalize these observations algebraically. Examples will be given.
Cognitive Aspects
Studying the Discussion Document for the ICMI STUDY 17, I appreciate very much the broad, deep and balanced discussion of the topic field. In this paper I will concentrate on cognitive aspects related to the use of digital technologies. The comments basically will discuss aspects of theme 2 "Learning and assessing mathematics with and through digital technologies". But also aspects related to the themes "Mathematical practices in the class room" ,"Teachers and teaching" and "Design of learning environments" will be touched.
In our research group in Muenster we think the final version of the ICMI STUDY 17 document should reflect in some detail also psychological aspects related to the use of teaching and learning mathematics when using digital technologies. What do we know about advanced mathematical thinking (TALL, VINNER)? Does the use of technology further the development of procepts (GRAY, TALL e.a.) or enhance processes of encapsulation (DUBINSKI)? Being confronted with challenging software, which is the role of reification (SFARD)? 395
The document also should include investigations on the interaction processes between the external representations of a problem, which we call Darstellungen
46
and the related internal mental images or cognitive structures of the problem solvers, which we call Vorstellungen
47
(MEISSNER 2002). The following picture gives a
survey. For a possible scenario of interactions between Darstellungen and
Vorstellungen see page 3 (example decimal grid).
Two types of Vorstellungen
"Vorstellungen" are like "Subjective Domains of Experiences" (in German "Subjektive Erfahrungsbereiche", BAUERSFELD 1983), they are personal and individual. The goal of mathematics education is to develop mathematical

46
We call external representations of mathematical ideas Darstellungen. Darstellungen we can read, or see, or hear, or feel, or manipulate, ... These external representations or Darstellungen can be objects, manipulatives, activities, videos, pictures on the screen, graphs, figures, symbols, tags, words, written or spoken language, gestures, ... In such a Darstellung the mathematical idea or example or concept or structure is hidden or encoded. There is no one-to-one-correspondence between a mathematical idea, concept, etc. and a Darstellung.
47
Human beings are able to "associate" with these objects, activities, pictures, graphs, or symbols a meaning. That means each Darstellung evokes a personal internal image, a Vorstellung (cf. concept image, TALL & VINNER). Thus Vorstellung is a personal internal representation, which can be modified. Or the learner develops a new Vorstellung. A Vorstellung in this sense is similar to a cognitive net, a frame, a script or a micro world. That means the same Darstellung may be associated with many individual different internal representations, images. Each learner has his/her own Vorstellung. And again here, there is no one-to-one-correspondence between a Darstellung and a Vorstellung.
mathematical
concepts

Darstellungen
(external

Vorstellungen
(internal
a b
a b
2 2
+
!
„a fraction is 396
"Vorstellungen" which are extensive and effective, which are rich and flexible. We
distinguish two kinds of "Vorstellungen", which we will call intuitive Vorstellungen
and reflective Vorstellungen. Thus we refer to a polarity in thinking which already
was discussed before by many other authors
48
.
"Reflective Vorstellungen" may be regarded as an internal mental copy of a net
of knowledge, abilities, and skills, a net of facts, relations, properties, etc. where we
have a conscious access to. Reflective Vorstellungen mainly are the result of a
teaching. The development of "reflective Vorstellungen" certainly is in the center of
mathematics education. Here a formal, logical, deterministic, and analytical thinking
is the goal. To reflect and to make conscious are the important activities. We more or
less do not realize or even ignore or suppress intuitive or spontaneous ideas. A
traditional mathematics education does not emphasize unconsciously produced
feelings or reactions. In mathematics education there is not much space for informal
pre-reflections, for an only "general" or "global" or "overall" view, or for
uncontrolled spontaneous activities. Guess and test or trial and error are not
considered to be a valuable mathematical behavior. But all these components are
necessary to develop spontaneous Vorstellungen. And these spontaneous
Vorstellungen mainly develop unconsciously or intuitively.
Both types of Vorstellungen together form individual "Subjective Domains of
Experiences" (SDE). For a well developed and powerful SDE both is essential, a
sound and mainly intuitive "common-sense" and a conscious knowledge of rules and
facts. Both aspects belong together like the two sides of a coin. And whenever
necessary the individual must be able, often unconsciously, to jump from the one side
to the other. Along the following example the reader may realize this situation.
Example Decimal Grid
Select a path from A to B (see grid on the next page). Change the direction at
each crossing. Multiply (with a calculator) the numbers of each step you go. Mark
each path in one of the small grids. Find the path with the smallest product. You have
4 trials. (Worksheet for each child (age > 10), first individual work, then discussion
of the results)
At a first glance, the problem is easy, children start immediately ("reflectively"):
(1) Find the shortest path (minimum of steps) or
(2) Select at each corner the smallest factor
But then ("intuitively")
(3) Perhaps there are better rules than (1) and (2)

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BRUNER discusses analytic vs. intuitive thinking. VYGOTZKI talks about spontaneous and
scientific concepts. GINSBURG compares informal work and written work. STRAUSS discusses a
common sense knowledge vs. a cultural knowledge. He especially has pointed out that these two
types of knowledge are quite different by nature, that they develop quite differently, and that
sometimes they interfere and conflict ("U-shaped" behavior). 397
(4) Let us try just another path to find out
And after some trials suddenly cognitive jumps (SDEs get changed):
• Multiplication not always makes bigger
• More factors may give a smaller product
• Running in a circle forth and forth (i.e. ... 0.3 x 0.8 x 0.6 x ...)
• infinite path (ß intuitive concept of limit)
And finally at the end: The smallest number will be ZERO! And even one step
more: ZERO on the number line? Or ZERO on the calculator display?
Summarizing, we distinguish two types of internalizing our experiences from
interacting with "Darstellungen". On the one hand we develop conscious reflective
Vorstellungen and on the other hand we create (mainly intuitively) "spontaneous"
Vorstellungen. Both types together create or modify an individual "Subjective
Domain of Experiences" in which this situation is imbedded then.

Reflecting the Role of Digital Technologies
As already mentioned, the emphasis of the traditional mathematics teaching
obviously lies on the development of powerful and conscious "reflective"
Vorstellungen. Overemphasizing the algorithmic and procedural approach we must
face the danger that our children get trained in skills but not in getting enough
insight. This is true for the four basic operations as well as for using sophisticated
CAS programs or others. "The importance of the ability to serve as a poor imitation
of a $4.95 calculator is rapidly declining" (KAPUT). 398
Despite the existence of digital technologies we still observe a lack of chances for the
learner to create their own ideas, to develop or to verify or falsify their own
assumptions, to invent theories and to apply them, or to explore facts or properties or
relations in given situations. The chance to develop intuitive Vorstellungen is limited.
The One-Way-Principle
In this situation we designed a special teaching method that we called "OneWay-Principle". To explain the method we will start with a few examples. In the first
and second we train number sense, in the third percentage feeling and, in the last two
examples, function sense.
Hit the Target
Hit the Target is a calculator game which furthers an
intuitive understanding of multiplicative structures: An
interval [A,B] is given and a number n. Find a second
number x so that the product of n and x is within the
=n
x [A,B]
interval. Our more than 1000 guess-and-test protocols show that the students after a
certain training develop excellent estimation skills (guessing the starting number) and
a very good proportional feeling (very often less than three guesses to find a correct
solution).
Big Zero and Big One
In the calculator game "Big Zero" we hide a subtraction operator and ask "Which is
the input for getting 0 in the display? In the game "Big One" we hide a division
operator and ask "Which is the input for getting 1 in the display? Discovering these
hidden operators by guess-and-test develops an intuitive understanding of additive
respectively multiplicative structures. Playing these games we observe after some
training excellent approximation skills.
Teaching Percentages
There are calculators that work syntactically like we speak in our daily life.
635 + 13 % = ... needs the following key stroke sequence:
6 3 5 + 1 3 % =
We taught percentages with the percent key, without using formulae or reverse
functions or algebraic transformations of formulae. If necessary the missing values
had to be guessed and verified by pressing always the same key stroke sequence from
above. The students became excellent in guessing each
value and they developed an astonishing "%-feeling".
We administered the same test
with 6 problems in our experimental group (white
bars, N $ 250) and in a control group with the
traditional "reflective" approach (N $ 500, dark bars).
The results are shown in the graph below. 399
Functions
Use your plotter software to find via guess and test an algebraic term to plot the
following graph. (Graphs given on a work sheet)
Trigonometric Functions
Use the calculator to find via guess and test at least three different values for x in sin
x = 0.2, tan x = -0.2, cos x = 0.5, sin x = 1.5, tan x = 2.5, cos x = 1, tan x = 4.
The idea of the One-Way-Principle
(OWP) is to solve with calculators or
computers a package of related problems
always with the same key stroke
sequence, independent of which variables
are given and which are wanted. That
means there is only one way to solve all
problems. Either we work syntactically
just pressing the buttons along the once
given sequence or, instead of applying
algebraic transformations or using
different formulae, we work semantically
by guessing and testing, still using the
same sequence of buttons.
Via the semantic guess and test the
user usually develops unconsciously a
feeling for the global "entity of input,
function and output", see table on the
right. The shadowed "?" may be an
operator (examples 4.1 or 4.2) or a
function (examples 4.3 or 4.5) or a
software (example 4.4).
input display

And of course we must work with intervals because the quality of a good estimation
depends on the size of related intervals.
Working along the OWP method is similar to the working with simulation
software (to learn car driving or flying an airplane, etc.). In mathematics education
the OWP is an intermediate step between simple examples and algebraic
generalizations. The OWP is a method to develop intuitive and spontaneous Vorstellungen about the relations between and about the order of magnitude of the many
variables of a mathematical concept before we start with introducing "reflective"
400
Vorstellungen with formulae or functions and reverse functions and algebraic
transformations.
Research Results
Our research group, TIM ("Taschenrechner Im Mathematikunterricht"), have
worked with calculators for more than 25 years (and with "computers" more than 18
years). Our goal is to develop ideas on how to integrate the new technologies into the
existing mathematics education on the base of empirical findings. A main
investigation was done by LANGE (1984). She trained mental arithmetic and number
sense in a primary school project (age – 9) where the calculators were on the table of
the students all the time. In the post-test she found that the subjects did significantly
better in mental arithmetic than the students of a control group.
Almost 20 years later we repeated an international inquiry about the use of
calculators in primary schools. The feed back from about 25 countries was
disappointing. In almost all of the countries the use of calculators in primary schools
just was not allowed. Thus we started a new project by the help of our teacher preservice students and many teachers (8 schools, 186 students, age – 9). Title and goal
of the project: "Use the calculator to become independent from it". The results were
similar those in LANGE's project, details see MEISSNER (2006).
A systematic application of the OWP also was administered in our percentage
projects (see 4.3) and in the dissertation from MUELLER-PHILIPP (1994). She
concentrated on linear and quadratic functions. Her students succeeded impressively
in building up intuitive Vorstellungen between the gestalt of a graph and the related
algebraic term.
In case studies we also used the OWP to teach the topics "Interest", "Compound
Interest", "Growth and Decay", and others. Our observations showed that the
students were working very concentrated with quite different strategies. Very often
they got an unconscious feeling about the new concept before they could explain
their discoveries or their good guesses. Thus we urged them to write down protocols
from their guess and test work because these protocols are excellent Darstellungen
from their Vorstellungen. By reflecting the protocols also unconscious Vorstellungen
may become conscious.
When guessing and testing became boring the students themselves started
asking for more efficient solution procedures. Then we could introduce reverse
functions and algebraic transformations. And when they got lost in the algebraic
approach, they could go back to their guess-and-test procedures and to their intuitive
Vorstellungen.
Important, in all our investigations the students got a conscious or unconscious
feeling for the mathematical relations and properties which also were available when
they just had to guess or to estimate, especially also in situations where a calculator
or computer was not available. In this sense the use of machines had furthered a
certain independence from these machines. 401
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Evolution for a revolution: professional development for mathematics teachers using interactive whiteboard technology

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The interactive whiteboard (IAW) is a presentation technology currently being used extensively in mathematics lessons in England. A growing research base shows that although use of the IAW initially improves pupil motivation it need not necessarily improve teaching and learning. It is suggested that to maximise impact teachers need to move through 3 stages to that called 'enhanced interactive' where thinking and pedagogy change. At this stage lessons have become more interactive and involve more discussion and pupil activity. However professional development is required to move teachers most rapidly to this stage. Based on observations of over 100 mathematics lessons, discussions with teachers and pupil surveys we believe that there is value to be gained by considering the role of gesture as mathematics teachers use IAWs. We also note that where the IAW has been fully exploited there appears to be a dynamic between activities at the IAW, on the pupil's desk and, we contend, 'in the pupil's head'. Finally we suggest that in order for a revolution in teaching and learning using an IAW there needs to be an investment in professional development to enhance reflective practice and to support pedagogic change.

Background

An interactive whiteboard system consists of data projector linked to a computer which in turn is linked to a large 'touch sensitive' e-screen - the actual interactive whiteboard (IAW) itself. The size of the IAW can be up to 2 metres across by 1 metre high. Usually the IAW is fixed to a wall, the data projector to the ceiling (though both can be 'mobile') and the computer is away from the IAW. Images from the computer are displayed onto the e-screen and can be 'touched', usually by a finger or a special pen, in order to control the computer from the IAW in the same way that a mouse can be used with a standard computer. Figure 1 shows one schematic arrangement.

Figure 1: An interactive whiteboard system with its components

The IAW usually comes with its own software that allows it to be used in one of three ways: with standard or subject specialist software; overwriting other application software (as you would on a transparent sheet placed over a normal computer screen); and with the software that accompanies the IAW. The key point is 403 that you stand at the IAW and use a finger or pen. The cost of the equipment usually depends on the make of the interactive whiteboard and the software provided with it.

During the past decade the interactive whiteboard (IAW) has passed from being a novelty to being part of the equipment of many mathematics teaching rooms within the UK (especially England), and to a lesser extent within parts of Western Europe, North America, S.E .Asia and Australasia. Widespread national introduction, like in England, has followed largely as part of government policy aimed at learning for the globalised digital age by the provision of additional finance. In England special funding for IAWs has followed but purchase of IAWs is also a reflection of self-government within schools and their intention to support individualised pupil motivation and learning through more appropriate pedagogy. Other countries where IAW use is increasing include Brazil, China, Mexico, Singapore and South Africa with some involving government initiatives (e.g. Mexico and Singapore).

Early evidence into the use of IAWs in mathematics classrooms suggests that practitioners pass through stages of developing both technology and pedagogy (Glover et al, 2003), but that the availability of equipment alone is no guarantee of enhanced teaching and learning (Miller et al, 2004). Recent government reports in 2005 by the Office for Standards in Education and the Qualifications and Curriculum Authority in the UK point to the need for teachers to become more aware of the inherent value of interactivity at the heart of a changed pedagogy.

Our contention is that as the technology becomes more widely available it is essential that teachers should be offered professional development that fosters a rapid move to technological competence and pedagogic flexibility. To this end we draw on evidence from on-going research at Keele University, UK, involving analysis of over 100 video-recorded lessons filmed with mathematics teachers, mostly with pupils aged 11-14, in 25 schools from varying socio-economic contexts. All videos have been analysed using agreed criteria for assessment of lesson structure, content, approach, teaching and learning strategies, and conceptual and cognitive development. Teacher and pupil use of the IAW has also been analysed according to techniques, tactical deployment and pedagogic interactivity. Associated survey work, discussion with teachers and pupils, and peer group problem solving has led us to offer a schematic outline for professional development that could encourage competent and flexible use of technology so that pupil engagement and attainment can be enhanced.

There are stages of development that can be considered in the drive to introduce and use IAWs, so some teachers have just a data projector and computer in their classroom, others include the IAW, while others extend this by adding a tablet PC. In some classrooms the tablet PC is used with just a data projector, this option is usually cheaper than purchasing an IAW and computer. To a large extent the more expensive the system the more that it appears to offer for teaching and learning, however without adequate support and training teachers have been observed to use 404 the more expensive systems in the same that they could use a computer and data projector (Glover et al., 2004).

The introduction of any of these systems into a mathematics classroom brings with it the opportunity for teachers to use mathematical and generic software as part of their teaching, but surprisingly this is initially overlooked by many teachers. Our evidence suggests that for maximum impact teachers need to have available a geometry program (such as Geometer's SketchPad, Cabri-géomètre), a graphing and statistical package (such as Autograph), a spreadsheet and interactive software designed to be used with an IAW (such as EXP Maths). In addition access to the internet provides further opportunities to use appropriate resources. The importance of using specially designed IAW programs, also under-estimated, allows teachers to see how the potential of the IAW might be used to benefit teaching and learning – often, though not always, bringing with them new ways with which to work with pupils and encouraging interactive teaching. The aim of such is to develop pupils' knowledge and understanding of mathematics as well as their competence and 'technical' skills. However some such programs consist of PowerPoint presentations, or equivalent, that provide colourful material but lack an interactive pedagogy.

Research into the use of IAWs generally and in mathematics has been so far emanating from those countries where there are many systems in place primarily the UK, to a lesser extent the USA (mostly small scale at a teacher level), though work is now being undertaken in, amongst others, Mexico, Singapore and South Africa.

Developmental framework

Our earlier work (Glover et al., 2004) led to the proposition that teachers pass through a 3 stage process in developing personal confidence in the use of IAWs.These are:

supported didactic: the teacher makes some use of the IAW but only as a visual support to the lesson and not as an integral tool to conceptual development; there is little interactivity, pupil involvement or discussion

interactive: the teacher makes some use of the potential of the IAW to stimulate pupils' responses from time to time in the lesson and to demonstrate some concepts; elements of lessons challenge pupils to think by using a variety of verbal, visual and aesthetic stimuli

enhanced interactive: this approach is a progression from the previous stage marked by a change of thinking on the part of the teacher who seeks to use the technology as an integral part of most teaching in most lessons and who looks to integrate concept and cognitive development in a way that exploits the interactive capacity of the technology; teachers are aware of the techniques that are available, are fluent in their use and structure the lesson so that there is considerable opportunity for pupils to respond to IAW stimuli either as individuals, pairs or groups, with enhanced active learning; the IAW is used as 405 a means of prompting discussion, explaining processes, developing hypotheses or structures and then testing these by varied application; a wide variety of material is used - 'home-grown', Internet, IAW specific, specialist and commercial software designed for use on the IAW.

Evidence from our research projects has shown that 'missioner' teachers (Glover and Miller, 2002) recognise the value of rapidly moving to the enhanced interactive stage. Continuing work with teachers in 7 schools has shown that teachers initially cling to didactic approaches as they gain fluency in the use of the IAW and that the rate of progress towards enhanced interactivity depends on the extent to which they have received support, shared software and experience, and the opportunity for mentoring or coaching to overcome specific problems that are usually related to general information technology skills and understanding how to use the IAW in a way different from non-IAW teaching (i.e. the new pedagogy of the IAW).

Enhancing motivation - understanding techniques

Pupil comment throughout the work has shown that the IAW brings benefits.

Our findings lead us to think that enhanced motivation is secured initially through the use of a range of techniques that attract pupils to what is presented, and then 'once caught' prompt learning through the use of visual and kinaesthetic impact. At its simplest it may be that computer generated numbers and text are easier to read but the use of colouring, highlighting and shading for emphasis add interest to conventional material. It was noticed that the opportunity for over writing material on the IAW was seen as an advantage for early users but this can cause them to cling to didactic approaches and that early encouragement to use more dynamic techniques, known as manipulations, leads to teaching which is more readily appreciated by pupils. These include dragging and dropping elements around the IAW (for example in showing balance in equations); hiding items and revealing them later (answers to calculations); colour and highlighting for emphasis (to show similar items); matching items to show some relationship (equivalent fractions); movement or animation (to show the stages in a proof); and immediate response to one of these (by a computer response or a comment from the teacher or another pupil). However, there is strong evidence that teachers seek to have even more options available. These are collectively summarised as virtual manipulatives and usually are software miniprogrammes that allow immediacy of response when teacher or pupil working at the IAW use them to illustrate some aspect of mathematics.

For example using a word processor a teacher can create with pupils a fraction wall and while doing this discuss the concept of fractions. A fraction wall virtual manipulative, such as one shown in Figure 2, can then be used on the IAW where a moveable rectangle can be used to discuss with pupils equivalent fractions and ideas of addition and subtraction of fractions. Pupils can then work in pairs with their own fraction wall (a manipulative) and use tracing paper to create and move rectangles in the same way that they have seen the teacher do this on the IAW. 406

Figure 2: A fraction wall – a virtual manipulative on the IAW

Applying techniques

As teachers develop facility in using the techniques marking the move from didactic to interactive approaches they seek to use them in a range of mathematical processes. Observation of teachers during their personal learning period suggests that these fall into three groups. The first show the use of verbal-visual approaches including annotating of data on the IAW, connecting elements of a process or argument with coloured lines, framing similar stages or responses in processes, grouping with the use of identification symbols, and emphasising either with highlighting or underlining. The next stage involves the use of the emergent data by discriminating as in the sorting of a range of integers or formulae; sequencing as in using successive IAW 'screens' to build up an understanding of the relationship between sides and the angles of a polygon; matching where differing representations of data can be analysed, and, as pupils gain competence and confidence in the use of the IAW as a medium for them to explain their own mathematical thinking. At this point the observed teachers appear to undergo a change in their pedagogic approaches. One expressed this as 'realising that we have a tool of potentially shattering importance, that it depends upon developing interactivity as a way forward, and recognising that our own lesson planning needs to be more precise and totally different in structure'.

Further discussion with a group of similarly 'convinced' teachers shows that at the final stage they then seek to maximise the potential of the technology through the use of the IAW as a means of group and individual assessment either with pupils showing their responses on individual mini-boards or, as in one or two schools, through the use of electronic tablets. Assessment is but a means to an end and good practice included: rapid feedback through the use of recalled screens which had supported the teaching stages; the development of differentiated materials which were readily recalled to support individual learning; and recapitulation with differing explanations or materials so that primarily visual, numeric or kinaesthetic learners can be offered suitable materials. It was noticed that fluent users of IAW technology and interactive pedagogy become adept at building up a catalogue of curriculum 407 related screens that can be used in successive lessons, revisited in successive years, and shared with other departmental colleagues when satisfactorily developed.

Changing approaches

Enhanced understanding of the ways in which the use of the IAW can be a positive support for teachers and hence learners, thus leads to changes in the way in which they think. Three elements indicate this. Initially teachers recognise the need for improvement in their own personal understanding of the processes involved in explaining, illustrating and developing mathematical understanding … often

achieved through 'having the time to reflect not just on what is to be taught but how this can be broken up into sufficiently small learning chunks often with their own assessment criteria so that we have concept maps and know what we need to get them across'. Consideration of all the evidence from 9 of the most successful practitioners suggests that this then becomes reflexive so that pedagogic reassessment becomes an ingrained behaviour both during the lesson and in planning future learning strategies.

From this emerges a second element – that of changing lesson structure. There are many schools in England already following lesson structure based upon the use of a brief and stimulating starter, an extended main section of the lesson, and a whole class plenary. Inspection reports in the UK indicate that such structures are more successful because they use different thinking skills, promote pace in the lesson, and overcome potentially boring 'chalk and talk' lessons. Effective IAW users favour such approaches because they 'draw on the full range of what the IAW has to offer – the fun of a dynamic and mind challenging starter, the material to support staged learning, and the opportunity to reflect upon what has been learnt and how it fits into the bigger scheme of things'. Our evidence is that early users who have not already developed the use of such a structure tend to revert to explanation, illustration and worked examples without using potentially stimulating software or the opportunity for pupils to develop their own reflective skills. They can rely on presentation software and reproduced series of screens that offer little interaction.

Thirdly, where concepts have been analysed, understood and developed and where practice and reflection become part of the normal lesson routine pupils develop greater confidence in offering explanations, using mathematical language and reasoning. Survey evidence shows that 13% of pupils feel that one of the positive gains from IAW use is that there is now more group work both at the IAW and in preparing materials and arguments for presentation to other pupils. This suggests that the IAW may be prompting a shift in the focus of the mathematics classroom away from the teacher who becomes learning facilitator towards the pupils who are becoming keen to use IAW resources in response to challenge. As one 14 year old expressed this 'lessons with Mr. X are ace because he really makes it seem understandable by using the IAW to show us different explanations, and makes it fun too ... and we rise to the challenge'. It is significant that the class concerned had not408 dispensed with their teacher but they had discovered some joy in learning mathematics.

Evolution to revolution?

An opportunity was provided by the observation of over 100 lessons to ascertain how the changed focus from teacher to IAW might need a reassessment of both mathematics learning and teacher development. Although not restricted to those lessons taught with an IAW it became clear that teachers who had developed an enthusiasm for the use of the technology also communicated their enthusiasm for learning mathematics – shown in their use of gestures. This is beyond the scope of the current position report but our findings echo the work undertaken by Goldin-Meadow (2003) and Rotman (2005) and show that as the focus shifts to the IAW both teachers and pupils develop gestures that enhance personal understanding and group communication, but that after each learning period gesture falls back indicating that concepts and processes have been internalised. If there is to be any value in training teachers to analyse their gestures and those of pupils it must lie in the transferability of gesture patterns. Rasmussen et al (2004) suggest that gesture and argumentation (visual and verbal explanations) together support the establishment of 'taken as shared' ideas and that once established these can then be used in similar but different situations – a vocabulary of gesture develops as a result. Our evidence is that IAW use may lead to patterns of teaching that may enhance gesture development and do so with some consistency in the same schools.

This shared secondary language is a reflection of the dynamic nature of mathematical learning where the IAW has been exploited to the full but our observations indicate that teachers are moving to an enhanced understanding of the learning process so that they can offer learning mediation. This leads to a greater understanding of a three way dynamic of learning which we believe can help the teacher enhance pupils' understanding:

At the IAW – teachers are recognising that the learning process is changing and there is evidence that they are constantly developing a new vocabulary of explanations linking concept and explanation for a variety of learning styles.

Their thinking is however not directed towards alternative verbal or verbalvisual explanations but to the incorporation of kinaesthetic momentum. They link their IAW activities and use of virtual manipulatives, like the fraction wall, to pupil activities.

On the pupil's desk – teachers note a changed practice away from the use of the IAW as the sole source of explanation towards tasks that emulate similarly lively techniques – so manipulatives, like the fraction wall, are used in a kinaesthetic way by the pupils on their desks. There is a continued use of conventional textbooks and worksheets on the desk, but to a lesser extent.

In the pupil's head – our subsidiary investigation of pupil attitudes to learning mathematics has shown that motivation and attainment are fostered where the teacher provides frequent opportunities for pupil discussion and offering of 409 ideas and theories followed by assessment of what has been understood. We believe that linking activities at the IAW with those on the pupil's desk can help focus learning in the pupil's head on understanding rather than calculation capability.

Our early contention is that teachers who are offered access to the technology but who have to take time to go through the stages outlined above may fail to ensure that the IAW is a worthwhile investment unless they are offered access to the exploration of the link between interactivity and mathematical development, coaching to support the use of the range of presentational and pedagogic opportunities and continuing group support to enhance reflective practice … and then the revolution could come! It could be that this evolution begins with the development of a bank of materials for distance learning supported by mentors within schools.

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Teachers, technology and cultural diversity

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This paper addresses secondary teachers using technology. It considers cultural matters and offers a cultural account. This account is not a model or a theory but a consideration of cultural factors in teachers' use of technology. Different cultures need not be 'far way' and cultural differences within a locality are recognised. Artefacts are considered as fundamental constituents of culture. Artefacts include hand tools and modes of action of using such tools such as beliefs and classroom norms. Pedagogy is viewed as a cultural undertaking. A review of literature suggests that many papers on using technology are 'acultural'. A set of papers is introduced that are considered relevant to building a cultural account. Constructs from these papers include teachers' routines, establishing a dialogue with teachers, teacher privileging, the software of the teacher, motives and goals, emergent goals, ergonomic and anthropological approaches (including orchestration and epistemic and pragmatic values ascribed to techniques) and situated abstractions. These constructs are used to suggest both a cultural account of teachers using technology and ways of working with teachers from other cultures. Mutual respect for teachers from other cultures is emphasised. An end-note considers differences between primary, secondary and university teachers.

Introduction

The essence of this paper is that (secondary) teaching is a complex undertaking that varies across and within cultures and that introducing technology into teachers' class-rooms adds to this complexity. After this introduction I consider artefacts and culture. I then provide a selected review of research on teachers using technology in which I pay particular attention to studies I regard as useful for formulating a cultural account of teachers using technology. I then present this cultural account and look at the implications of it for working with teachers who intend to use technology in their classrooms.

Artefacts and culture

Culture, in a short paper, must remain a tacit construct but "artifacts are the fundamental constituents of culture" (Cole, 1996, p.144) Cole's argument, very briefly, is that the things people do in their everyday settings involve a multitude of coordinated artefacts which mediate their social interactions and their actions on the nonhuman world. Technology is a subset of artefacts, but what are artefacts? Wartofsky's (1973) three levels of artefacts is a widely respected account. Primary artefacts, e.g. hand tools. Secondary artefacts, representations of, and modes of action using, primary artefacts. Tertiary artefacts are imagined worlds, arenas of 'free' play. In considering teachers, technology and culture, attention should be given to 411 secondary artefacts as these encompass beliefs, conventions and norms which are important aspects of culture and pedagogy.

Pedagogy is a central consideration to teachers using technology. The important thing to note about pedagogies is that there are many pedagogies. Pedagogy concerns why and how teachers do things with artefacts around them (and how these artefacts shape and are shaped by what teachers do). There are many things a teacher can do and many things a teacher can use, but what they do and use is intrinsically bound up with their culture – the culture of their country or community and the culture of their school. Pedagogy is a cultural construct and many people not working in education have clear opinions on suitable and unsuitable pedagogies. Daniels (2001, p.69) calls pedagogy "the fundamental social context through which cultural reproduction-production takes place".

It appears reasonable to say that pedagogies from affluent countries have been, implicitly and explicitly, offered to (or adopted by) educators from developing countries. Nkhoma (2002) investigates attempts to shift teachers' practices from being teacher-centred to learner-centred in Black South African schools. He found that the poor material conditions of the environment and the culture of Black township schools were not condusive to Western style learner-centred classrooms but found that rich learning experiences can be provided by committed teachers who teach from the front. He concludes:

It is not beneficial to stereotype classrooms practices into, simply, teacher-centred therefore bad, and learner-centred therefore good … rich experiences can be provided in practices that appear teacher-centred. (Ibid, p.112)

Both the material conditions of life and the secondary artefacts which weave culture are important in considerations of teachers using technology. Noss & Hoyles (1996, p.185) consider doctoral studies which examined Portuguese and Turkish teachers using technology and ascribe differences in attitude to and use of technology (Logo) to different educational contexts and beliefs.

Teachers using technology, a review of literature

I consider the literature in two parts: that which I do not regard as particularly informative for accounting for cultural differences and that which may be informative for this purpose.

Part 1

The majority of mathematics education academic papers, especially older papers, on teachers using technology are, in my opinion, written 'aculturally', i.e. they do not describe the culture of the situations they examine and sometimes make recommendations which do not pay regard to the coordinated artefacts of other cultures. I do not have space for an independent review in this paper and so I call on two refereed reviews to make my points.

Lagrange et al. (2001) is a meta study of literature on technology in mathematics education, 1994-99. The approach has many dimensions, not all of which are possible to describe here. The study involved quantitative and qualitative 412 analyses. I summarise their qualitative analysis which was restricted to the educational use of computer algebra systems (CAS). They found that papers considered could be classified into five types of problématique (I consider the first and last type here). The first type is technical descriptions of possibilities (53%), optimistic accounts which stress capabilities that the authors consider educationally relevant. As a personal note I usually find such papers, e.g. Using CAS to introduce exponential functions, most enjoyable. My point with regard to the focus of this paper, however, is that expecting the teacher or student behaviours described to 'jump' contexts/ cultures is expecting too much. The fifth type is papers focusing on the integration of the technology, the conditions of use (7%). It is only these kind of reports, I believe, that can inform the focus of this paper; only by focusing on the conditions of primary and secondary artefact coordination can we provide a cultural account of teachers' use of technology. Monaghan (2004) is neither a meta study nor a comprehensive literature review but it does attend in some detail to the division in the literature between papers which are 'prescriptive', e.g. encourage teachers to relinquish didactic roles and papers that recognise the complexities of practice. He notes a gradual increase in papers of the latter type over time.

Part 2
The following summarises studies which I consider contain elements that have the potential to contribute to a cultural account of teachers using technology.
Olson (1992) presents a way of viewing teaching and teachers. This book is not explicitly concerned with the use of technology although an important study that contributed to this book was on Canadian primary teachers using Logo. Routines are central to Olson's account of teaching. Routines are not 'thoughtless' but they are based on 'tacit knowledge' which precedes articulated knowledge. "Routines express culture" (ibid., p.24). Teaching is essentially a moral undertaking and teachers have a vision of what is good for their students. Innovations, e.g. exhor-tations to incorporate technology into teaching, must match teachers' visions. "Classroom routines are not what computers will replace, they are where computers must fit if they are to be useful to teachers." (ibid., p.26). With regard to working with teachers for change "A dialogue needs to be established instead of compliance." (ibid., p.90).
Although a background study to this work involved primary teachers I believe that comments on routines, morality and visions apply equally to secondary teachers. The approach values teachers as people and this, I strongly believe, is important in any cultural approach; any approach which treats teachers as 'subjects' in a scientific study does not deserve consideration here.
There is no contradiction in treating teachers as individuals and as representatives of their cultures, the oft used phrase 'shaped and shaped by' applies to individuals and their cultures. Two PhD studies which focused on individual teachers' use of technology are Kendal (2001) and Lins (2002). Kendal worked with three Australian teachers who designed and taught an introductory calculus programme using symbolic calculators. She found great differences in the emphases 413 these teachers placed which was manifested in the differential performance of their students in specific domains (graphic skills, procedural competence and conceptual understanding) although overall test results for the classes were similar. Her analysis used the construct of 'privileg-ing' (elevating one form of mental functioning over another, e.g. algebraic over graphical reasoning), a term she took from Wertsch (1991). Teacher privileging, over different cultures and within specific cultures, is, I believe is a useful construct to incorporate into a cultural account of teaching with technology. Lins worked with four British teachers, two using Cabri and two using Excel. Her theoretical framework was 'anti-essentialist' - technology as a 'text' that teachers 'read'. The meanings produced by the teachers for Cabri and Excel were different. Lins introduces the construct of 'the Excel and the Cabri of the teacher'. This idea, the software or the tool of the teacher, is potentially important for a cultural account of teaching with technology. The fact that a tool was not a neutral 'given' in one culture suggests great variation in the appropriation of technology over cultures.
Another aspect of the non-neutrality of technology is why one uses it, the goal and the motive. These are essential components of activity theory, e.g. Leont'ev (1978). With regard to mathematics teachers globally and any specific mathematics software it would, I believe, be absurd to assume that the goals/ motives are the same. Ostensibly 'objective' issues here (what will I do?) are intricately tied up with affective matters (why do I want to do this?). In the construct frame of Wartofsky primary and secondary artefacts are interrelated, e.g. teachers attitudes to technology use in mathematics classes shape and are shaped by teachers' cultural identity.
Monaghan (2004) introduces another dimen-sion of teachers' goals in using technology, 'emergent goals' as described by Saxe (1991). Emergent goals come into being and fade away according to the situation. Saxe examines emergent goals with regard to four parameters: activity structures; social interactions; conventions and artefacts; and prior understandings. Monaghan used data from a project on teachers using technology to examine how these parameters interact in the development and resolution of emergent goals. His analysis presents some uncomfortable 'home truths' for advocates of tech-nology, e.g. how the emergent goal of a teacher, whose ostensive goal was to provide a rich spreadsheet activity for her students, was managing the printer queue and managing the behaviour of students waiting for work to be printed.
Work following the ergonomic and the anthropological approaches contains many important constructs relevant for a cultural approach. It comprises a large body of academic papers, not possible to list in this paper, so I refer to a recent critique of this work, Monaghan (2005). Work in the ergonomic (instrumental) approach addresses how a tool, a material object, becomes an instrument, a psychological construct. It emphasises the subject-tool dialectic (the tool shapes the actions of the subject and the subject shapes the tool). It examines schemes, the structure of actions, which have pragmatic, heuristic and epistemic functions. 'Gestures' are the bits of schemes we can see and techniques are sets of gestures. A role of the teacher is to 414 guide students' 'instrumental genesis', the evolution of a tool-scheme dialectic. How a teacher does this with a class is termed 'instrumental orchestration'. Guin and Trouche (1999) is a study of a particular technological classroom environment that includes students with TI-92s and exercise books, a rotating (amongst the class) 'sherpa-student' who operates the viewscreen, a viewscreen, a blackboard, specific tasks and a teacher. In commenting on how to support instrumental genesis they argue for strong teacher involvement. Sherpa-student orchestration is culturally specific but what is important for a cultural account of teachers using technology is that, whatever one calls it, some form of instrumental orchestration, even if it is minimal, will take place and that forms of orchestration are unlikely to transfer between cultures.
Papers by Artigue and Lagrange tend to follow the anthropological approach of Chevallard where practices are described in terms of: tasks; techniques (used to solve tasks); technology (discourse used to explain techniques),); and theory. These four Ts are essentially cultural constructs and it would be a mistake of any cultural account of technology to ignore them. An important aspect of techniques is that they have pragmatic (efficiency, breadth of application) and epistemic values. The interplay between different values in traditional and technology classes introduces many complexities.
Noss & Hoyles (1996) discuss, amongst other things, two constructs, 'webbing' and 'situated abstraction' which are ostensibly about student learning but which have cultural implications for teachers using technology. Webbing is about making connections and concerns building a structure a student or teacher can use to (re)create meaning. Situated abstractions concern how people construct mathematical ideas in a specific setting, with a specific tool or set of tools. We come to a mathematical task with knowledge artefacts (mathematical objects and relationships between them) and tool using capabilities. We must see something prior to new learning or we could not approach the task. Bit by bit, moving between established and emerging knowledge, we focus on new knowledge artefacts. The two constructs are interrelated and complementary. They are consistent with a view of using technology celebrating cultural diversity – we build knowledge from that which we know.
Towards a cultural account of teachers & technology
The need for a cultural account of teachers using technology
Speaking of the marginalisation of technology in mathematics education around the world Hoyles et al. (2004, p.311) point to "... a failure to theorise adequately the complexity of supporting learners to develop a fluent and effective relationship with technology in the classroom". There is an equal need to provide a theoretical account of mathematics teachers using technology, if for nothing else, to avoid advocating ways of working that do not help real teachers develop and assist their students. Such an account must, I hold, be a cultural account and must address the history, curricula opportunities and constraints, the routines and the motives of 415 groups of teachers of different cultures. Focusing on the use of digital technologies of teachers from developing countries, as the ICMI 17 Discussion Document suggests, is important, both in principle as a mark of respect and for the insights these contributions may provide. But a cultural account which provided a framework for these individual accounts would be a step forward.
My considerations are an 'account' rather than a theory or model because it is far from fully developed. I am, of course, interested in the practices of teachers from different countries but I feel strongly that such an account should also address differences within countries. This is grounded in my experience of working with a large number of teachers in England on using technology in their classrooms: their motives for using technology, the affordances and constraints offered by their school/departmental practices and the resources that are available vary tremendously. These, at a local level, are cultural differences.
Aspects of a cultural account of teachers using technology
I could present a 'ready made' cultural account of practice and simply tailor it for the case of mathematics teachers. This would have the benefits of being authoritative and more likely to be consistent than a specially developed account. If I did this, then I would look to Cole's (1996) brand of cultural-historical activity theory as he, more than anyone, puts culture at the forefront. But mathematics is a special case and the considerations of mathematics educ-ators, noted in 'Part 2' above, provide insights not addressed by Cole because he is not a mathematics educator. Of activity theory, however, I believe a cultural account must retain artefact (of various levels) mediation and goal-directed actions. One need not accept all the tenets of activity theory to accept this.
The special case of mathematics is due, in part, to its history; mathematics is the oldest subject in the school curriculum and we cannot, even if we wanted to, ignore this history when we address the use of technology in its teaching and learning. This is, in effect, one of the points of Artigue and Lagrange, when they speak of the epistemic and pragmatic values of techniques. Beyond the 'breadth of application' pragmatic value and the 'facilitating understanding' epistemic value of techniques, however, there are the values accorded to this 'breadth' and these 'forms of understanding' over hundreds of years (we cannot ignore history in any attempt to answer why it is "good" to be able to factorise a
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). Lagrange (2000, p.3, my
imperfect translation) states "the impossibility to conceive or to analyse teaching with these symbolic systems without taking into account the new and the usual techniques that interact in the mathematical activity of the students". This 'impossibility' is, in large part, due to the history of our subject and its historically privileged techniques (I assume that there will be some commonality and some variation in 'historically privileged techniques' in different cultures). The work of Artigue, Lagrange and other on values is in its infancy. To date this work has been largely restricted to work with CAS. Apart from considering other software, 416 however, it is important to locate epistemic and pragmatic values in other cultures' school mathematics.
Instrumental genesis and a form of orchestration will be enacted, more or less successfully, by all teachers using technology but how it is enacted and the criteria for success will, I believe, vary across cultures. Recognising the diversity of ways in which people communicate with and through technology, Hoyles et al. (2004) put forward 'situated abstraction' as a complement to instrumental genesis through the construct of a boundary object, a shared, between different communities of practice, knowledge artefact. Boundary objects take form though do not bring meaning with them when they cross communities of practice. Meaning, however, can be negotiated over communities of practice. Gestures in a software system can generate situated abstractions that others (re)construct, "orchestration becomes a mutual act, rather than something one community does to another" (ibid., p.321). These are general claims and putting them into practice is a significant undertaking. My belief is that many unforeseen problems will arise and that these will be due, to an extent, to emergent goals which arise over and within cultures in different ways. Monaghan's (2004) application of Saxe's model may help us to understand these problems.
Mutual acts are, I believe, an important factor in any consideration of one group of teachers learning from another (especially so if one group represents a dominant culture and the other does not) and are consistent with Olson's (1992) call for a dialogue in place of compliance. Mutual acts are founded on mutual respect of the kind Olson puts forward: respect for visions and routines that may not be the same as ours, respect for teachers of other cultures whose coordinated systems of primary and secondary artefacts have a different historical foundation. Mutual acts are also founded on individual differences between and within cultures. The work of Lins (2002) and Kendal (2001) highlight, respectively, how individual teachers within similar institutions in a single country appropriate software differently and privilege different aspects of the same software with their students. We should expect differential appropriation and privileging to increase in teachers from others countries and cultures. Clearly research in this area would be useful but Lins' and Kendals' work provides a basis for appreciating, and working with, these expected differences.
Endnote on primary, secondary and university teachers
I have focused on secondary teachers. Like Noss & Hoyles (1996) I think it unlikely that primary and secondary teachers will have the same roles or beliefs; their routines are different and one group is exclusively focused on mathematics. I further think that, in general, a difference between how secondary teachers and university teachers use technology is likely. I suspect that university lecturers' focus on mathematics over pedagogy creates different goals and different roles.
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Developing Interactive Learning Environments that can be used by all the
classes having access to computers. The case of Aplusix for algebra.
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Our research team has developed and experimented software for the learning of
algebra, named Aplusix, with the idea of being usable and useful for all the classes
having access to computers, and of helping teachers to teach the curriculum. In this
paper, we list 19 principles that we consider relevant to this goal and we briefly
describe the Aplusix system. This system is distributed in France since early 2005
and will be distributed in many countries from 2006. It has proven to be efficient
(students learn) and to facilitate the teacher's work.
Introduction
Interactive Learning Environments (ILEs) is a term used in the AI-ED community
(Artificial Intelligence in Education) for computer systems designed to help students
learn a domain in an interactive way
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. ILEs can be produced by research teams or
companies. Research teams generally develop prototypes
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in order to implement and
experiment ideas; generally, they do not develop products
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that can be used in
ordinary classes by teachers who just want to teach the curriculum. Companies
develop products to be used in ordinary classes. Most of these products do not
include advanced functionalities and have a poor interactivity, which is sometimes
limited to multiple choice questions.
Many mathematics teachers complain they do not have enough time to teach the
curriculum. This produces a strong limitation to the use of ILEs because these
systems consume a lot of teachers time (study of the system, preparation of learning
situations) and time of their students (it replaces well known activities by activities in
the computer room having an efficiency which is not obvious).
This paper is devoted to ILEs in mathematics (more particularly in algebra) to be
used in the usual framework of classes that can access to computers, both in
developed and developing countries. We particularly focus on ILEs that can help

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ILEs do not include all the pieces of software that can be used at school, for example CASs and
spreadsheets are not ILEs, they are professional software. The fact that they can be used in classes,
at some level, in well-prepared situations, does not make them ILEs.
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A prototype is an application which is not well finished and debugged and cannot be distributed
as a professional system.
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A product is a well finished and debugged application which can be distributed as a professional
system. 419
students learn the curriculum with little modification of the class functioning and
without the goal of changing what is learnt. We consider that this is the first kind of
ILE that most of the teachers in the world need. However, we do not mean that this is
the only interesting kind of ILE.
In section 2, we list principles of ILEs for the usual framework of the class and a
few algebra ILEs are situated with regards to these principles. In section 3, we
present the Aplusix ILE project in algebra. After a first period where prototypes were
built and experimented, we adopted the goal of building an algebra ILE for the usual
framework of the class and we redesigned our system according to this goal. At the
present time, experiments have shown that the system verifies these principles. We
are now entering in a deployment phase.
Principles of ILEs for the usual framework of the class
The principles we consider are oriented towards the students or the teacher.
1. The tasks proposed by the ILE must be part of the curriculum.
2. The activities must be close to the usual activities of the curriculum. This
includes a place for errors and a general interaction mode with little
scaffolding (there is scaffolding when the system executes a part of the
task).
3. The main representations at the interface must be close to the usual ones.
Other representations must be added only for didactical reasons.
4. There must be transfer on paper. After an adequate amount of activities
with the computer, students must have better scores on paper tests.
5. Teacher must be able to use the ILE in the theoretical framework, either
explicit or implicit, they are used to.
6. The domain of the ILE must be large.
7. The manipulation of the representations must be natural (not involving
intermediary representations) and easy.
8. The ILE must be in the natural language used in the school.
9. The familiarization with the system must be easy. When the overall
interaction is complex, there must be different stages so that the
familiarization of each stage is easy.
10.When important features of the system depend on human choice,
parameters must allow teacher's decision on the behavior of the system.
11.The ILE must present some added value compared to traditional
environment.
12.The ratio between the time for familiarization and the duration of learning
activities must be low (maximum 20%).
13.The interaction modes and the feedbacks must allow a good level of
autonomy of the students when they use the ILE, so that the load for the
teachers is not heavy.
14.The time the teachers need for preparing learning situations must be short. 420
15.Information concerning the student activities must be accessible to the
teachers.
16.The teachers must be involved in the global learning process.
17.The price of the system must be adapted to schools. When the system
requires the use of another system, like a CAS, this system must not be
expensive, and must be easy to obtain or included in the installation
package.
18.The installation of the system must not be complex, because it is often done
by teachers who have no advanced knowledge in computer science.
19.The organization developing the ILE must be durable (10 years or more),
because an ILE must evolve (correction of bugs, development of new
functionalities, etc.).
Principles 1 to 6 are adequacy and utility principles. Principles 7 to 10 are
usability. Principles 11 to 14 are economical principles at a cognitive level.
Principles 15 and 16 exclude AI systems that would take in charge the entire learning
process and leave no place to the teacher. Principles 17 to 19 are general economical
principles.
Cognitive tutor for algebra [Koedinger et al., 1997] is an example of ILE for the
usual framework of the class. Its domain is elementary algebra: word problems,
linear equations and systems. It has been designed by the Carnegie Mellon University
of Pittsburgh and is distributed by the Carnegie Learning company
[CarnegieLearning], which is a spin-off of this University. It is oriented to the US
curriculum. The Website of Carnegie Learning announces "325 000 students using
the system over 750 school districts across the United States" and better scores of
students using this system with regard to other students.
MathXpert [Beeson, 1996] is an ILE for algebra and calculus designed at the
University of San José and distributed by the MathWare company [MathWare].
Experiments have proven that students benefit from the use of MathXpert. However,
MathXpert does not follow the principle 2: it provides strong scaffolding (at any
time, the student selects a sub-expression and the system provides a menu with the
rules that can be applied to this sub-expression) and makes no place for errors (the
student chooses a rule in the menu and the system applies this rule). This non respect
of the principle 2 may be a reason of a weak success of the system.
The Aplusix ILE
Brief history of Aplusix
During the nineties, we developed an ILE for algebra [Nicaud et al., 1990]. It was
a command-based ILE like MathXpert with rules in menus and calculations done by
the system. The main differences with MathXpert were: (1) more abstract rules; (2)
Presentation of all the rules (not only the applicable ones); (3) Demand of the values
of the rules variables; (4) A small domain (factorization of polynomials). Several
experiments were conducted and provided good results. 421
Figure 1. The Aplusix ILE.
As researchers in computer science and developers of an ILE, we were not
satisfied with the usage of the system: it was used only in small experiments driven
by researchers. So, we decided to redesign the systems with the goal of being
attractive and usable by many teachers and students. At this moment, we did not have
the ILEfor the usual framework of the class idea but a will to have a system that can
be widely used like dynamic geometry software [Laborde, 1989]. So we built the
Aplusix system [Nicaud et al., 2004], see figure 1, with the following main features:
(1) To allow the student to freely build and transform algebraic expressions and solve
algebra exercises by producing his/her own steps like on paper; (2) To produce the
first fundamental feedback, the indication of the correctness of the steps, in a non
intrusive way, see figure 2. This follows principles 2, 3, and 7. According to the
principle 17, we decided to develop ourselves the entire system (without using a CAS
or some other piece of software). For feature (1), we developed an advanced editor of
algebraic expressions and for feature (2), we developed a module of formal
calculations including the calculations of the equivalence of expressions.
Figure 2. A correct step in the left, with an equivalence sign; an incorrect step in
the middle, with a red crossed equivalence sign; a step without feedback, in the
right.
The teachers and the students who used the first version of this new Aplusix were
very interested and the first tests showed that the students learned well. However, the
teachers had soon demands of new functionalities: (1) A second fundamental
feedback, the indication of the correct end of the exercises; (2) A mode without 422
feedback; (3) Exercises ready to be used. At this moment, we entered in the ILEfor
the usual framework of the class idea.
Short description of Aplusix
First versions of the new Aplusix were principally pieces of software devoted to
students, based on the concept of microworld, with a rich replay system usable by
students, teachers and researchers. In last versions, we continued to improve the
microworld aspect of our environment, but we worked to nest it into some kind of
exerciser where the work done by students could be automatically analyzed and
scored, fundamental feedbacks can be hidden to permit use of Aplusix for tests,
exercises could be automatically generated, solutions could be automatically found
out. Both students and teacher were targeted by these new components, and their
works have been facilitated. But we have worked for teachers specifically too, trying
to provide them with tools for the administration of their classes and of their students,
tools for the edition of specific exercises, list of exercises, or richer exercises (for
word problem, or problems with many linked sections), and last, tools for statistical
analysis of student's results, see figure 3.
Figure 3. Statistical analysis of student's results. Fist the total of
well-solved exercises, second, the total of attempted exercises.
As a consequence, the main student's activity envisaged with Aplusix is no more
an exploration activity with the microworld but a training activity on a list of given
exercises under the control of fundamental feedbacks or a test activity when these
feedbacks are hidden, these new activities are closer to the one practiced in class. We
have added also a new activity, we call self-correction, where students, after a test
(i.e., without fundamental feedbacks), can benefit from fundamental feedbacks to
correct their errors. The last activities are visualization of past activities, either
globally (the final form of the exercise) or action by action.
The organization of work with Aplusix according to activities has been a solution
to reduce the use of the parameters. A set of parameters allows customizing the

system for each class and situation. Parameters continue to exist and can define for examples: the mode and scope of the verification of the equivalence; how the system must manage an incorrect or ill-formed step; the access to the solutions and to the CAS-like commands, the order of exercises obtained from a list (randomized or not); 423 the introduction of strong invitation to students to comment their steps. See figure 4. Activities do not set all the parameters but the most important ones and reduce the number of those which still need to be set. Because of the complexity of the use of parameters (set of parameters can be assigned to each class, for each session), there was a real need to find a way to have customized version of the system without big effort. Activities have been our solution to make the system easier to use.
Figure 4. Teacher's panel for choosing the values of the parameters.
The domain available for explorations and exercises concern algebra: numerical calculations (from integers to square roots), expansion of polynomial expressions of several variables, factorizations of polynomial expressions of one variable and maximum degree 4, polynomial equations and inequations of one unknown and maximum degree 4, rational equation leading to polynomial equations of one unknown and maximum degree 4, system of linear equations up to 10 equations and 10 unknowns.
For more precise information consult documentations at http://aplusix.imag.f
Current state of the global project
Aplusix has been developed in French. We have developed tools allowing to easily translate the texts and the help file of the system. Aplusix has now been translated in English, Portuguese, Italian, Vietnamese, and Arabic. Translations are ongoing in Spanish and Japanese.
Until the end of 2004, several experiments have been carried out [Nicaud, 2005a] in different countries (France, Brazil, Italy, Vietnam, and India) and contexts (a few sessions, regular use during the entire school year, one student per computer, two or four students per computer) for a total of about 15 000 students?hours. At the present time, principles 1 to 18 are verified. In particular: (1) students gain autonomy and improve their knowledge; (2) Aplusix facilitates the teachers' work (because of students' autonomy and of already-made lists of exercises). Furthermore, enquiries showed that all the students worked with pleasure with Aplusix.
Early 2005, a contract was signed between our University and a French publisher, and Aplusix began to be marketed in France. This type of contract appears to be 424 unsatisfactory with regards to duration (principle 19), because, according to French law, the royalties received by the University cannot be used to pay engineers for maintaining and developing the system. So we decided to move the development structure. First, it will go for 4 years to a company, which is an affiliate of the University; then a spin-off company will be created. We have contacts, for several months, with publishers who are willing to sell Aplusix out of France. We will be able to sign contracts with them in March or April 2006. We will adapt the price to the gross domestic product of the countries.
Discussion
The above list of principles has been built from our view of ILEs for the usual framework of the class. As Aplusix has this goal for several years, it is not by chance if it follows these principles. However, we may have forgotten some principles in the elaboration of this list and ideas are welcome.
This list may also be used to estimate the distance between an ILE (an existing ILE or an ILE to be developed) and the for the usual framework of the class concept. An important distance does not mean a poor ILE (the ILE may be very good), but we think that an important distance means a limited possible use. In that case, the benefit of the development of the ILE is only at a research level (doing experiments, producing results of these experiments, publishing papers), not at a usage level.
In the case of Aplusix, we are developing new functionalities that will help students and teachers (through students' autonomy): (1) a companion as an ideal student of a given level who can provide suggestions, explanations and calculation steps; (2) a tutored mode in which the students' calculation steps will be analysed and when a misconception is diagnosed [Nicaud at al. 2005b], an adequate feedback is provided, for example: When you move an additive expression from one side to the other side of an equation, you must change its sign.
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Using Computing Technology to Enhance Mathematical Learning in a Developing Society
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Computing technology can offer teachers a means to incorporate more open-ended tasks and problem solving into their repertoire of pedagogical strategies. But can this be done within the parameters set by traditional curricula, textbooks and classroom cultural contexts? What are the pedagogical approaches that will facilitate this task? How can teachers be helped in making decisions regarding the use of such technology in their classrooms? What are the theoretical frameworks that will support the teacher's work? What are the implications of the use of such technology for teacher development programs? In this study, we use the case study of a teacher-researcher collaboration in a classroom to offer some suggestions on these aspects. The study was conducted in a secondary school classroom in India and offers some insight into the use of a tool that is still in its nascent stages in developing societies.
Historically, society has had two important goals in mind while establishing schools for children. One, schools were a useful means of conveying to children aspects of learning that a society had accumulated over a period of time. Two, schools provided children a means of honing and achieving their potential in different spheres of life. Thus, schools were expected to fulfill societal and individual aspirations in a compatible fashion. As institutions that are geared towards the goals of individuals in particular, and society in general, it is expected that schools will change with the changing needs of a society and help young learners to emerge from them with a broad understanding of the world that they inhabit and the potential to contribute to it in a meaningful manner. Thus, the skills and knowledge that learners acquire in a school system change over a period of time so as to keep pace with changing societal goals and aspirations.
However, in developing countries such as India, traditional approaches to teaching and learning have dominated the school scene for a very long time. The educational scenario in India, in particular, has remained remarkably impervious to changes not only in the global society but also to local, fast paced economic and social changes. School curricula and syllabi, textbooks, teaching strategies as well as classroom contexts have changed little in the past several decades. Naturally, teacher education programs – both preparation programs as well as professional development ones – have reflected this strong resistance to change both in their content and approach to education. It is only in recent years that some of the winds of change sweeping across other spheres of life in Indian society have permeated across the school scene, and even though such changes remain confined to few areas, in terms of both – content and geography, they have begun to play an important role in bringing about changes in the wider educational arena. 427
Mathematical teaching and learning in particular has suffered much as a result of the static modes of curriculum design and transaction that have prevailed in the classroom. Students in schools and colleges learn mathematics by and large by the chalk-and-talk method (both instruments – chalk and talk – being pretty much the exclusive domain of the teacher!) and remain largely mute spectators to the entire process of teaching that goes on in the classroom.
It is into this dismal and depressing scenario that some brave teachers are now venturing forth to disturb the prevailing status quo. Various influences have worked to create this situation. Individual educators who have a broader vision, pressure from some (limited) sections of society who want children to be part of the larger global society, educational researchers who are geared towards helping learners become independent thinkers rather than machines that regurgitate memorized facts, a multitude of private business houses who are marketing their wares in India -- either textbooks (very few) or tools such as hand held technological implements or computing software. Whether intentionally or not, all these factors are beginning to create a demand for change in Indian mathematics classrooms and slowly but surely, the old guard is coming face to face with a host of novel challenges.
We may need to offer some information about the Indian schooling system at this point. Broadly speaking, there are two kinds of schools in India – the state administered schools, also called "government schools" and schools run by private bodies such as individuals or trusts, also called "private" or "public" (actually a misnomer since these are managed by private bodies) schools. Private schools usually charge far higher amounts as fees as compared to the government schools where the fee is very nominal. The quality of education offered in the former institutions usually ensures that all parents who can afford to will send their children to private schools. Another important distinction between these two kinds of schools is that almost all private schools across the country use English as the medium of instruction (regarded as an international language) whereas in government schools, regional languages are used for instruction.
Regardless of whether they are government or private schools, the majority of schools in India work within a curriculum prescribed by a national autonomous body called the National Council of Educational Research and Training (NCERT). This body is responsible for framing syllabi and course guidelines, and creating textbooks that are largely followed as such by most schools in India.
At the end of first ten and then twelve years of schooling, students are expected to take examinations that are conducted by national or state bodies that are responsible for holding them. The outcomes of these examinations are extremely important for most students as much of their future depend upon these results. Thus, to a large extent, much of the teaching in the final three years of schooling is geared towards these so called "board examinations" (as they are conducted by central or state level boards) and preparing students to take them. 428
As in many other countries, reforms in the educational system are driven by changes in the social and economic spheres, but for any such changes to become an integral part of the entire school system requires changes in the mindset of the people who are part of the institutions that look after the administration of the schooling system – the NCERT, the analogous organizations at the state levels called the SCERT's, and the boards that conduct the board examinations. However, at the initial stages, these changes are made at the individual level by people and institutions willing to take the initiative to bring about such reforms. Through such individuals (people or institutions), the reforms may spread to others so as to become more widespread.
The use of technological tools in Indian classrooms has been one such reform that has been very slow in coming, and having made its initial appearance has been even slower to spread. Attitudes and beliefs about mathematics and its learning as well as the cultural heritage of mathematics in this country where people have prided themselves on developing mental skills to perform rapidly various operations that have long and complicated standard algorithms have been partly responsible for the slowness of such changes. "Mathematics is about thinking and how can any technology help us to think?" has been the traditionalists' response to the use of digital technology in mathematics classrooms.
In such a culture, this then is the central question: "How can we use technology to help students to think better in mathematics?" We must also remember that this question must be answered within the context of the prevailing educational system – the curricula, the syllabi, the mathematics textbooks and the traditionally prepared teachers. It is therefore heartening that there are some teachers within this system who are willing to take the risk – to make some efforts towards answering this question. Obviously, they cannot be expected to face this challenge alone. Thus, for educational researchers, the crucial question is: How best can we help teachers in using available technology in their classrooms? Even more importantly, we would like to know whether the use of such technology would help in learning or teaching mathematics in this traditional set up.
It is in seeking answers to such questions that we have set out to do this study. We would like to know: (a) how can we help teachers to decide if, why, when, and how can technology be integrated within the prevailing contexts to provide a better mathematical experience to students? (b) what kind of pedagogical approaches can help us to use technology in the mathematics classrooms?, and (c) if technology can be integrated into the classroom, then what are the lessons for teacher preparation and professional development programs that must prepare teachers for future classrooms? Thus, in this paper, we have chosen to work on the theme "Teachers and Teaching" (with reference to the Discussion Document of ICMI 17) and the broad approach that will define our work is the role of the teacher. Naturally, this approach will be complemented by discussions on the kind of mathematics learning that takes place with the use of a specific technology in the classroom. We would also make 429 suggestions on how curricula may be modified to take into account the mathematics learning that occurs in technology-integrated classrooms.
The interventionist design experiment that is the basis of this study took place

in a local private school in New Delhi, India, that is well established and has been known to take the lead in placing different kinds of reforms on its agenda. A supportive principal and governing body of the school make it possible for teachers to seek and implement a variety of innovative teaching strategies. One of the senior school teachers of mathematics in this school, Dr. Jayanti Gopal, has been responsible for bringing about a lot of change not only in the kind of mathematics teaching that happens in this school but also for taking the lead in spreading awareness and information about such changes to other schools in the city.
Jayanti set up a mathematics laboratory in her school at a time when the concepts of mathematics and laboratory did not go together in this country. She used the mathematics laboratory as a means of having students work on simulated real life problems that could be connected to their regular school curriculum. It goes to the credit of the school that in a packed school curriculum, Jayanti was able to find the time and space for an activity -- the mathematics laboratory -- that was nowhere on the regular mathematics curriculum. Jayanti invited mathematicians to help her set up the laboratory and devise activities that would introduce students to varied and fascinating aspects of mathematics. Gradually, motivated students helped her to extend the frontiers of her laboratory. Jayanti then invited students and teachers from other schools to visit the laboratory. She also organized an annual mathematics day at which teams of students could present mathematical projects designed by them. A variety of other related mathematical activities such as talks by research mathematicians, different kinds of mathematical quizzes and contests, and professional development programmes for teachers helped to set up Jayanti's mathematical laboratory as an important landmark in the mathematical firmament of school mathematics within the city.
My own association with Jayanti helped me to discuss ideas related to school mathematics with her on a regular basis. I visited her laboratory on numerous occasions and I have been involved in different capacities in the mathematical activities that she organizes in her school. Jayanti has also shown her keenness to understand different theoretical perspectives that prevail in mathematics education, particularly those related to the teaching and learning of school mathematics. For me, the opportunity to work with her on different projects has given me an insight into different aspects of the field. Thus, over time, we have developed a working relationship that has helped to foster and feed into each of our individual interests.
The introduction of the Geometer's sketchpad (hereafter called the sketchpad) in India provided Jayanti and me the opportunity to work on a project that is of interest to both of us. I had used the sketchpad while working on developing some activities for pre-service elementary school teachers in the United States. Thus, I was somewhat familiar with the software and believed that it would provide a useful 430 learning environment in geometry, a neglected area of school mathematics. Jayanti felt that the software was interesting enough that it would capture the imagination of her students. Both of us believed that it would challenge students to look at different aspects of geometry from a problem solving perspective. Together, we hoped to utilize the software in Jayanti's laboratory with middle school students in order to answer a variety of questions that challenged us regarding the use of technology in the mathematics classrooms.
Besides stimulating our academic interests, the use of different kinds of tools and implements in the learning of mathematics has acquired additional interest in the Indian context because of some other developments that are currently taking place in the school education scene in this country. A year ago, the government made the mathematics laboratory a mandatory component of middle school mathematics. Thus, all schools are now required to create a mathematics laboratory and have their middle school students visit them at least twice a week. The government's directive in this regard has left many teachers quite confused. Many of them have very little idea of how to go about setting up a mathematics laboratory, and even if a room is designated as such by the school, teachers do not how to put them to use. They are seeking some guidance in this area. A second important development has been the revision of syllabi and textbooks in all subjects at the school level. As curricula in mathematics undergo changes there is need to look at the research in areas related to mathematics education and to utilize the results of this research in order to help learners adjust to the changing needs of a global society. Thirdly, there are a number of schools that are willing to invest in computing technology and want their teachers to find ways to make use of this tool in their classrooms. Parents are also keen to see their children become adept users of computing tools in their everyday lives. Thus, teachers and educators are under pressure to find appropriate spaces whereby learners can use digital technology as a means of learning different subject areas, particularly mathematics, as it is seen as a subject that is in close affinity to this kind of technology.
The collaborative model of learning that Jayanti and I worked with is based to some extent on the teacher development model described by Cobb and McClain (2001) where the researcher and the teacher are part of a professional team that situates its learning within the classroom and the development that evolves is grounded in the interactions that take place in the classroom.
The model was particularly useful for both of us as the learning environment had novel features and allowed us to complement each other. For the first time, we were looking at the kind of mathematical learning that evolves amongst secondary school students through the use of digital technology. Both of us believed that this research study would provide us with ideas on how to integrate such technology better in the classroom. For me, additional lessons were to be learnt in the context of profesional development of pre- and in-service school teachers. Besides, our areas of knowledge and expertise complemented each other – Jayanti knew her students 431 quite well as she had been working with them for several months and so she was aware of their strengths and the places where they would need help. Her laboratory and the use of different kinds of technological tools in it bore testimony to the fact that she believed that school students could be helped to learn mathematics through the use of tools and emerging technology. My own expertise lay in the theoretical perspectives that would help us to contextualise and frame the study as also to design the working model within the classroom with the students, and between Jayanti and I. Together, we developed the tasks that we would have our students use in the laboratory.
Thus, in this case, we believe that the first condition for bringing about any change in classroom practice was fulfilled – the teacher supported the developments and was assured of its usefulness in helping her students to learn mathematics. As Ruthven (1999) states, "A more promising source of guidance on improving the effectiveness of teaching is research into classroom processes and their effects. Here in particular, some researchers have been able to identify core features of effective teaching, and then to test the robustness of their findings through intervention studies, leading to the identification of what can be broadly characterized as active teaching." (p. 208). While we are still some distance away from being able to identify the exact aspects of the usefulness of technology that we will benefit from in the classroom, we hope that we will able to use it to foster a sense of exploration among our students. Through such explorations, we hope to have our students draw conjectures and verify them. We also hope that the visualization strategies that students will be using during the mathematical activities with the sketchpad will encourage them to use these in other situations.
Moreover, the students themselves were keen to use the tool. Early results from the study showed that students were able to manipulate the sketchpad tools and tried to use it in contexts that had not been stipulated by us, showing both their ability to handle such tools and their eagerness to explore ideas using this technology. As the study progresses, we look forward to asking our students more specific questions regarding their own interest in this tool, and other ideas that they would like to explore through it.
Discussions between Jayanti and me together with the work done by the students will help us to evaluate the effectiveness of the tool and our strategies. We will also conduct interviews with students as they work on the tasks in small groups. Thus, our own observations, interviews with students, records of their work, our discussions and regular diaries of our reflections will help us to document and develop strategies for effective teaching within our contexts.
As we develop this case study, we hope that there will be aspects of it that can be extrapolated and used in other secondary school classrooms. Primarily, our interest is in pointing to specific attributes of technology that can be useful in the mathematics classroom. While we are doing this study within the confines of the current classroom contexts, we hope to identify some elements of the curriculum that 432 need to be modified given the availability of technology. In particular, we now have visual strategies possible that help students to see the connections between algebraic and graphical representations of different concepts. Finally, we hope that we will be able to make some suggestions for changes in pre- and in-service professional development programs for teachers. These suggestions will not be merely geared towards skill based knowledge that teachers may require in the future. In fact, we hope to address the attitudes and beliefs regarding mathematics and its teaching and learning that hinder the incorporation of digital technologies in traditional classrooms in societies where they have been conspicuous by their absence. We believe that this is essential if mathematical learning has to keep up with the changing needs of our society, and we are to fulfill the aspirations of our learners.

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Students' constructions of dynamic geometry

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This paper draws on a study investigating the development of the proving process in a dynamic geometry environment in the context of open geometry problems at secondary school level. Starting with a paradigmatic example, the paper will explore how the modalities of interaction with Cabri that students show, influence the construction of different 'dynamic geometry instruments' and direct the proving process in different directions. The modalities of interaction with the software will be interpreted within the instrumental framework (Verillion & Rabardel, 1995) and illustrated by students' protocols.

A Paradigmatic example: the Cabri of Bartolomeo and the Cabri of Tiziana

The paper starts with a paradigmatic example, taken from a study (Olivero, 2002), that shows how interactions with a dynamic geometry software in the context of the same problem differ for two students working in pairs, and impact on the shaping of the proving process. Bartolomeo and Tiziana are 15-year-old Italian students who have used Cabri a couple of times before they were given the problem 'Perpendicular bisectors of a quadrilateral'
52
to tackle in Cabri in pairs
53
. They both have an average mathematical background and considerable Cabri experience.
Tiziana has the mouse. After constructing the figure, the students start exploring the situation and quite soon they get to this extract that leads to the formulation of the conjecture 'If ABCD is a parallelogram then MKLM is a parallelogram too', through an episode of dragging.
69
Bartolomeo: what have you done, a rectangle? (Figure 1)
70
Tiziana: yes, well…
71
Bartolomeo: so… it is a point… try to make it bigger…
75
Tiziana drags D up and stops to observe and think (Figure 2)

52 You are given a quadrilateral ABCD. Construct the perpendicular bisectors of its sides: a of AB, b of BC, c of CD, d of DA. H is the intersection point of a and b, K of a and d, L of c and d, M of c and b. Investigate how MKLM changes in relation to ABCD. Prove your conjectures.
53 For more information about the methodology of the project this example is taken from see Olivero (2002). 434
76
Tiziana: excuse me! This (she points at LM) follows what this (AB) does, this (LK) follows this (AD) … (she laughs)
77
Bartolomeo: let's examine some more cases
78
Tiziana drags A up and gets Figure 3
79
Bartolomeo: ah, when it's a rectangle it's always a point… (he writes down the second conjecture) […]
80
Tiziana: No, because now it's a point too. Tiziana drags B so that ABCD is no longer a rectangle but inside there is still a point (Figure 4).
Tiziana
Figure 1
Bartolomeo
Figure 2 Figure 3
D
C
B
A
MHKL
Figure 4
The students start from the same figure (a rectangle) but use Cabri in two different ways, that open a window on their aims, and potentially direct the proving process in different directions: by the end of the process Tiziana discovers the most general conjecture for this problem, while Bartolomeo goes on with a systematic exploration

of particular cases, as the one explored in this extract.
The Cabri of Bartolomeo
In 72, 77 and 79 Bartolomeo shows how he wants to use Cabri: to produce and check conjectures in a very systematic way ("let's examine some more cases" – 77). During this episode of dragging, he pays attention only to the initial and final figure (Figure 1 and Figure 3), as two snapshots, as his aim is clear: checking if HKLM is always a 435 point when ABCD is a rectangle
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. And, as soon as Tiziana stops in Figure 3 he formulates a conjecture (79).
This episode represents well the overall interaction with Cabri shown by Bartolomeo throughout the proving process. He shows a quite 'controlled' use of the software, and seems not to be absorbed by it. He does with Cabri something that he could have done on paper too. He has a precise strategy, which is to examine particular cases and to use dragging as a tool for validating conjectures, as is made clear at the beginning of the process ("let's see what happens in every case, shall we?"): the software is used to carry out this plan. Photo-dragging
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(Olivero, 2002) characterises his behaviour
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because dragging itself is only used to transform a figure into another one, and the attention is focused only on the initial and final state of dragging.
This 'controlled' use of Cabri, in which the software is incorporated in a predetermined solution process, is limited to the exploration of particular cases/ conjectures and may hinder the discovery of new properties/conjectures.
The Cabri of Tiziana
Tiziana's use of Cabri in this episode is different from Bartolomeo's. Tiziana is observing the figure over the dragging which takes her from Figure 1 to Figure 3, through Figure 2, 'reading' what the figure suggests her. She stops in 75 and reads a relationship between elements of the configuration (the sides of ABCD and the sides of HKLM), which opens up a new thread in the proving process and will be transformed into a general conjecture later on in the process
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. At the end of this episode, Tiziana does not stop on the rectangle configuration but moves to another 'unknown" configuration in which there is still a point inside (80), opening up another new thread for exploration
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This episode represents Tiziana's prevalent type of interaction with Cabri. She does not show a pre-specified plan of action and she shows a more open use of Cabri: she uses Cabri in order to experiment, explore the situation, get ideas and discover new properties. With a metaphor we can say she is 'dragged by dragging', in that she reads what is happening in Cabri while she is dragging. Her modality is film-dragging
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54 For Bartolomeo "always" means in two cases only.
55 Photo-dragging incorporates "modalities which suggest a discrete sequence of images over time: the subject looks at the initial and final state of the figure, without paying attention to the intermediate instances. The aim is to get a particular figure" (Olivero, 2002, p.141).
56 Sometimes it is 'indirect' dragging when it is Tiziana who is in fact using the mouse.
57 The general conjecture is: ABCD and HKLM are similar.
58 Which will lead to another general conjecture: if ABCD is cyclic then HKLM is a point.
59 Film-dragging incorporates "modalities which suggest a film: the subject looks at the variation of the figure while moving and the relationships among the elements of the figure. The aim of dragging is the variation of the figure itself" (Olivero, 2002, p.141) 436
(Olivero, 2002): she focuses on the intermediate state of the figure while she is dragging and stops whenever she sees something interesting. Cabri does for her something that she would not be able to do on paper and is an integral part of her actions.
This more open use of the software transforms Cabri (and dragging in particular) into a tool for discovering new relationships and facts, leading to general conjectures. The overall process is determined by what emerges from observing what happens in Cabri.
The research problem
The paradigmatic example has shown different ways in which the students exploit and incorporate the software in their solution processes and how this affects in different ways the development of the proving process. What is the research problem suggested by this example?
Educational innovations tend to take on an objectified character in popular thinking. Innovators advocate and administrators endorse the educational use of this new technology or that, as if the instrument were invariant and its use determinate (Ruthven, 2005). This shows how the risk of the "fingertip effect" (Perkins, 1985), that is simply making a support system available and expecting that people will more or less automatically take advantage of the opportunities that it affords, is always there. However, research has shown that technologies do not work by themselves and people do not automatically take on board the technology or software: "The computer is an expressive medium that different people can make their own in their own way" (Turkle & Papert, 1990). This leads to the exploration of students' constructions of dynamic geometry, to be interpreted within the instrumental framework, as developed by Verillion & Rabardel (1995) and elaborated by Mariotti (2002), that may help us understand why this is the case. The research questions we may ask are:
• What are the instruments-Cabri constructed by the students starting from the artefact-Cabri?
• How do students make Cabri their own instrument-Cabri? What elements play a role in the process?
The instrumental framework
The instrumental approach elaborated by Verillion & Rabardel (1995) provides a new perspective on the effect of technical devices
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on learning processes.

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This approach considers the use of tools in generals, not necessarily new technologies.437
According to the instrumental approach, any technical device has a double interpretation: on the one hand it has been constructed according to a specific knowledge which assures the accomplishment of specific goals, and on the other hand, there is a user who makes his/her own use of the device. In other terms, in this perspective it is important to highlight the distinction between artefact, which is "the particular object with its intrinsic characteristics, designed and realised for the purpose of accomplishing a particular task" and instrument, that is "the artefact and the modalities of its use, as elaborated by a particular user" (Mariotti, 2002, p.702) within a given activity. "For a given individual, the artefact at the outset, does not have an instrumental value. It becomes an instrument through a process, or genesis, by the construction of personal schemes" (Artigue, 2002, p. 248), or schemes of use. As different and co-ordinated schemes of use are successively elaborated, the relationship between user and artefact evolves, in a long-term process called instrumental genesis, which is linked to: the characteristics of the artefact (its potentials and constraints) and those of the subject (its knowledge and former work habits) (Verillion & Rabardel, 1995). Therefore the instrument does not exist in itself, an object becomes an instrument when the subject has been able to appropriate the artefact for himself/herself and has integrated it with his/her activity. At different moments different instruments can exist even if the artefact used is the same and it may happen that an artefact is never transformed into an instrument.
The Cabri of Carla: A conflict is generated
This section shows how the instrument-Cabri a student constructs is not appropriate for the situation at stake. Carla and Alessandra are 15-year-old students solving the Varignon's problem
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. They have a weak mathematical background and only used Cabri twice before this problem. After having formulated the conjecture 'if ABCD is a square then HKLM is a square', they prove it correctly on paper. Afterwards, they go back to Cabri to 'check' their proof, but the Cabri figure does not show what they have just proven. So their conclusion is that "it's all wrong".
Figure 5 (on paper) Figure 6

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Varignon's problem: Draw any quadrilateral ABCD. Draw the midpoints L, M, N, P of the four sides. Which properties does the quadrilateral LMNP have? Which particular configurations does LMNP assume? Which hypotheses on the quadrilateral ABCD are needed in order for LMNP to assume those particular configurations?438
215 Carla: all this stuff…these…they are congruent (the halves of the sides of ABCD - Erreur ! Source du renvoi introuvable.). [Then Alessandra writes down the thesis: LM equals MN, equals NP, equals PL. Meanwhile Carla uses a ruler to measure the sides of LMNP]. so PL equals MN. The same for …PDN triangle and LBM triangle [illegible] PN equals LM…Should I do a cross comparison? PDN triangle and PAL triangle [illegible] PN equals PL. What's missing? These two are done, these two are done…. […] They all have equal angles. So it is a square! Ok! […]
225 Carla: the problem …is that this is not a square (ABCD) […] look… no…(Erreur ! Source du renvoi introuvable.) Because if you say that this equals this (PD and DN) and you say they have an equal angle (D) and then this equals this (PN and LM) and this and this (PL and MN)…then this becomes a square (LMNP), but we've just seen that it is not a square. So it's all wrong!
228 Teacher: why? What puzzles you?
227 Carla: because…if this is the midpoint (she points at P) then it divides this side in two equal parts (she points at AD and AP and PD) so it should be: if it is a square, the quadrilateral inside is a square too. Why the figure doesn't show that?
228 Teacher: What do you trust more, the figure or your proof?
In this episode, a conflict between a theoretical result (proof) and the empirical answer given by Cabri (the figure does not look like a square) arises. This happens because the students try to validate their proof in the spatio-graphical field (Laborde, 2004; Olivero, 2002). This would require looking at the figure from another point of view, not only empirical, as it may happen at the beginning of the exploration process, but also theoretical. When validating the proof the pupils 'read' the figure at an empirical level, they 'read' the properties of LMNP from the measurements: it has not equal sides therefore it cannot be a square. The students do not consider that their hypothesis is ABCD square while the Cabri figure is not a square because the angles are not right angles. Instead of 'reading' the Cabri figures, they should have looked at them from a theoretical point of view, according to which ABCD and LMNP are both 'approximations' of squares. The proof would have then been validated.
If we look at this episode in terms of the research problem highlighted above and in the context of the instrumental framework, we can see that the Cabri instrument they 439 construct has the following characteristics: the Cabri feedback is interpreted in a visual-perceptual-numerical way and the students do not show a theoretical control over Cabri62. The students take on board the software to the point that the answer they see on the screen is believed to be true even if it contradicts what they found without the software, by proving and using geometric properties of the figure. This particular instrument in the context of this problem provokes a conflict.
From this example we can see how the interaction with Cabri in the context of open geometry problems needs to be mediated by the use of the theory that allows a control over the Cabri spatio-graphical field. Also the intervention of the teacher becomes crucial to solve possible conflicts between what the software does/shows and the mathematical theory and to mediate the construction of an appropriate instrument (228).
Students' constructions of Cabri: the role of the cabri/ mathematics experience
The two examples discussed in this paper illustrate how the process of construction of a particular Cabri instrument affects the proving process. From the analysis of the case studies that formed the research Olivero (2002), a pattern emerged in relation to what sort of Cabri instrument was constructed in the context of the problems used in the study
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. An instrument is constructed in order to solve the given task, which involves the construction of conjectures and proofs. Can we characterise the type of instrument that the students construct and identify what this depends on? The research showed that what play a role in the process of construction of the instrument is the mathematical theory and the use of Cabri, which evolve together throughout the proving process.
It was observed that students with considerable Cabri experience and average mathematical background seem to manage better the interaction with Cabri, showing a wide range of dragging modalities and a successful proving process. They generally do not use paper. They are able to link the spatio-graphical and theoretical field in a productive way, which leads to the production of many conjectures and proofs. In this case, the artefact Cabri is transformed by the students into an instrument that is appropriate to deal with the situation they are presented with. There is a theoretical control over Cabri, which is used as a discovery tool within processes of conjecturing, and is then re-interpreted and used as a validation tool or support for thinking within processes related to the actual construction of proofs. This is the result of both mathematical and Cabri long term experience, which allow the students to transform Cabri into an internally oriented tool (in the sense of Vygotsky (1978)). Some students falling in this category often talk about dragging and the way they are

62 For a detailed analysis of this episode see Olivero (2002).
63 Open problems (Arsac, Germain, & Mante, 1988) requiring conjecturing and proving in geometry. 440

using it, there is a control over what they are doing in Cabri and they understand well what Cabri can do and show.
A second case is when students have with very little Cabri experience but have a strong mathematical background: in this case they do not fully exploit the possibilities offered by the software. In general they show more controlled exploration in Cabri. It seems that the artefact Cabri is never transformed into an instrument for these students. It remains an artefact which is used occasionally but is not really taken on board by the students. The students prefer to use other tools (as for example paper and pencil) they are more used to and show a successful production of conjectures and proofs. Given their mathematical strength, it seems that these students are less eager to experiment with new tools they are not familiar with. This behaviour can be observed with 'experts' at different levels; Cabri offers possibilities of exploring and opening up spaces that the 'expert' does not necessarily need.
Finally, students with very little Cabri experience and weak mathematical background, like Carla and Alessandra, usually experiment a lot with Cabri but do not always use it successfully. Conflicts may arise between results produced in the spatio-graphical field and possible theoretical explanations, and the focusing process may take a wrong direction, as shown in the case of Carla above. In this case, the process of instrumental genesis develops through different steps and the intervention of the teacher is needed in order to direct students towards the construction of the appropriate instrument which allows the evolution of the focusing process in the construction of conjectures and proofs. The artefact Cabri is first turned into an instrument based on a scheme of use that relies on a visual-perceptual-numerical interpretation of the software's feedback. This is not the instrument which serves to accomplish the goal of the problem situations. A new instrument needs to be constructed by the students, based on a theoretical way of 'reading' the Cabri figures. The role of the teacher is crucial in developing this new scheme of use and provoking students to see the same figure from a different point of view which leads them to conjecturing and finally proving.
Conclusions and Implications
To conclude, this paper shows how, given the same tool (dynamic geometry software) and the same activity (proving open problems), students develop different proving processes, both in terms of the way they interact with the software and in terms of the conjectures and proofs they produce. The instrumental approach explains this through the fact that students are constructing different instruments by transforming the same artefact (Cabri). The mathematics and Cabri experience affect and influence the instrumental genesis. Understanding the different instruments and how they are constructed is important because the construction of the instruments affects the development of the proving process in terms of production of conjectures and proofs. Further research will focus on a detailed analysis of the development of 441 the schemes of use related to the particular elements of dynamic geometry software, as for example dragging and measurements.
The fact that students construct different instruments shows that the integration of dynamic geometry in the classroom practice is not a straightforward process but requires a careful analysis. A key challenge for the integration of technology into classrooms and curricula is to understand and to devise ways to foster the process of instrumental genesis towards the construction of the appropriate instrument for a given task. The role of the teacher emerges as important, showing that dynamic geometry per se does not guarantee a successful proving process that manages well the key relationship between the spatio-graphical field and the theoretical field. The teacher constructs different instruments too (Lins, 2003; Ruthven, 2005), which influence the instrumental genesis the students develop and their appropriation of the software. The teacher should act in 'transforming' the tool used by the students into a "semiotic mediator" (Mariotti, 2002) in the proving process so that a process of internalisation of the tool itself takes place and the artefact is then transformed into an appropriate instrument for the situation at stake.
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Curricular innovation: an example of a learning environment integrated with technology
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An important question in considering the introduction of new technologies in mathematics curricula is that of their effectiveness in enhancing (or damaging) the real capabilities of students. To answer this question the paper sketches a theoretical framework, which frames the new technologies for mathematics as representational infrastructures: as such, they are analysed both as cultural semiotic systems and as cognitive energizers. The two concepts allow defining suitable adequacy criteria for testing the new technologies in the classroom. A teaching-learning environment integrated with technology is described as a concrete realisation of a technologicaloriented Italian curriculum. An example of how learning can happen in this environment is described and a few final comments are drawn with respect to some questions asked in the Discussion Document of ICMI Study 17.
Introduction
The focus of this paper is on the theme 4 of the Discussion Document. To do that we introduce a theoretical framework suitable to test how the environments integrating digital technologies are adequate for the effectiveness of maths learning (§1). Then we exemplify this definition illustrating a concrete learning-teaching environment where the learning experiences take advantage of the affordances supported by the new technologies (§§ 3, 4). The environment is developed within a concrete technological integrated Italian curriculum, which is sketched in advance (§ 2). In the end (§ 5) we discuss how our theoretical framework and the methodologies illustrated in the examples are useful in understanding the impact upon the teaching/learning of mathematics.
Theoretical Framework
The introduction of Information and Communication Technologies (ICT in brief) in mathematics curricula has been stressed and encouraged in these last years. However the benefits from such an introduction are neither necessary nor automatic; the matter must be considered carefully, looking at its many aspects and possible negative effects. In fact, as shown in Artigue et al. (2001), much of the theoretical and empirical research dealing with the use of ICT in mathematics education is prevalently concerned with the added-value component provided by the technology and rarely faces critically an approach to ICT based on an ecologically sustainable use. A major disadvantage consists in the fact that papers on ICT are more concerned

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in "learning how to use technical tools incorporated in the computer ... than [in] understanding the theory behind those tools" (ibid.), namely in what can be called a blind use of technology. This often corresponds to a-critical practices in the classrooms, which are limited to control the functioning of the tool, which is "insufficient for a successful mathematics outcome" (Thomas & Hong, 2004).
What is needed is an aware use of technology, which means to understand if, how and when the technological artefacts can mediate/support/carve the construction of the student's mathematical knowledge in the classroom. To do that, one must consider ICT's in a wider setting, namely to analyse them from a multiple perspectives: that is, from a didactic point of view (e.g. considering the role of the teacher, of social interactions induced by the used technology, and so on); from a cognitive point of view (e.g. considering how technology changes the mental structures of the learners);from a cultural point of view (e.g. considering the framework of rationality towards which the use of technologies may push the student). An interesting analysis in this direction has been developed by Kaput et al. (2002), who introduced the notion of representational infrastructures (= RI):
The appearance of new computational forms and literacies is pervading the social and economic lives of individuals and nations alike....The real changes are not technical, they are cultural. Understanding them... is a question of the social relations among people, not among things. The notational systems we use to present and re-present our thoughts to ourselves and to others, to create and communicate records across space and time, and to support reasoning and computation constitute a central part of any civilization's infrastructure. As with infrastructure in general, it functions best when it is taken for granted, invisible, when it simply "works".
The challenge today is how to design learnable systems within suitable and up-todate representational infrastructures. Roughly speaking, the major point consists in seeing how the ICT involved in a system fit within RI according to the following two criteria: a) as Cultural Semiotic Systemsb) as Intrinsic Cognitive Energizers. Cultural semiotic systems (Radford, 2003) are those systems which make available various sources for meaning-making through specific social meaningful practices; such practices are not (only) to be considered within the strictly school environment but within the larger environment of the whole society, embedded in the stream of its history. For example, Kaput & Schorr (2002) claim that the development of algebra in the history of mathematics was made possible by an entirely new mode of thought "characterized by the use of an operant symbolism, that is, a symbolism that not only abbreviates words but represents the workings of the combinatory operations, or, in other words, a symbolism with which one operates".
Intrinsic Cognitive Energizers are here defined as systems which make available varied sources for meaning-making through specific cognitive meaningful practices; such practices do enter intrinsically in positive cognitive resonance with the subjects, because of cultural and biological reasons. Examples are given by the quick and mainly visual interaction with the information provided on the computer screen in 445 computer games and in access to all kinds of information provided by the internet system.
In the next two paragraphs we shall illustrate this framework introducing a concrete teaching-learning environment developed within a technological integrated curriculum of mathematics. We'll sketch the latter in next paragraph and discuss the former in the successive § 3.
An example of technological integrated curriculum: the Umi65 proposal and the mathematics laboratory
Modern society requires massive use of mathematical knowledge and skills. A significant act with this respect is the UNESCO resolution in 1997, that underlines how "mathematics education has a key role, in particular at the level of primary and secondary school, in the comprehension of mathematical concepts and in the development of rational thinking" (http://www.unesco.org/science/physics.htm).
The UMI proposal of mathematical curricular innovation is inspired by the UNESCO resolution and, in particular, take into account both the instrumental and the cultural functions of mathematics. The essential core of UMI curriculum provides a foundation for pupils' mathematical competencies through 4 content areas (numbers and algorithms; space and shapes; relations and functions; data handling and previsions) and 3 processes areas (argumentation, conjecturing, proving; measuring; solving and posing problems). The 7 areas are essentially the same for the whole pre-university school from 6 to 19 years. The teacher is supposed to tackle these themes in an integrated manner, trying to connect them to other topics and to other subject disciplines. The UMI curriculum (
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) contains also some reflections on
the Mathematics Laboratory that are very important for the scope of this paper. In fact, the Mathematics Laboratory is conceived as a teaching-learning environment based on the use of instruments and aimed at the construction of mathematical meanings. It is not intended as a physical place other than the classroom, but as a structured set of activities with the goal of building the meaning of mathematical objects. As such, it involves people (students and teachers), structures (classroom, instruments, organization of space and time) and ideas (plans for didactical activities, experiments). In this sense, it passes the two adequacy criteria, stated in § 1. This is shown also by a metaphor introduced in the UMI document, where the Mathematics Laboratory is compared to what happened in the Bottega d'Arte of Renaissance artists, where the novices learned through a cognitive apprenticeship, namely by doing and watching what was done by experts, communicating with one another and with the experts, who pointed out the cognitive difficulties that the newcomers would encounter. The construction of meanings within the Mathematics Laboratory is strictly linked to the instruments used when carrying out the given activities and to the interactions among the participants in the activities (Bartolini et al., 2004). It is

important to remember that tools are the result of a cultural evolution: they have been

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Unione Matematica Italiana (Association of mathematicians and mathematics Italian teachers)
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The curriculum is available at: http://www.dm.unibo.it/umi/italiano/Didattica/ICME10.pdf446
produced with specific aims and, as RI, represent ideas in a culturally shared and
technologically up to date way. This has important didactical implications: first of
all, meanings do not live only in the tools and cannot emerge purely from the
interaction of the pupils with the tools. Meanings are rooted in the aims for which the
tools are used as culturally shared artefacts, and in the strategies related to the use of
those tools that are elaborated in the course of the activities, which make palpable
their cultural content. Moreover, the appropriation of the meanings requires
individual reflection on the objects of study and the proposed activities. The
construction of meaning is strictly linked to the communication and sharing of
knowledge in the classroom, through collaborative or cooperative group work and
through the mathematical discussion orchestrated by the teacher.
An example of a technological integrated teaching-learning environment
The UMI document has also many examples of teaching activities; however it does
not contain indications on specific didactical paths that suggest how to develop its
contents. How its curricular indications can be transformed into structured and
concrete didactical paths in the classroom is an affair that has been left to the schools
and to the teachers, in accordance with the Italian curricular tradition. For this reason
a group of upper secondary school teachers has designed a didactical plan from the
curricular indications of the UMI proposals and according to the educational
framework illustrated in §1. The result has been a teaching-learning environment,
where the ICT are integrated as RI, called Mathematics in the Web (indicated as
MW-project in the following).
The main features of this environment are sketched in the following points:
a) using ICT's to design, to build and to realize mathematical activities, which
make sense (attività sensate in Italian); the Italian word is used with three different
meanings: reasonable (that is attentive to the specific possibilities and constraints of
the class); linked to the natural abilities and particularly to perception; ruled by the
intellect and specifically by a theory (the last two meanings were used by G. Galilei
to feature what he called sensate esperienze);
b) using a long-term didactics (contrasted to short term educational projects),
with particular attention to the construction of the meaning of the mathematical
concepts and to the development of a critical thought in the students, considered
essential for citizens' awareness in nowadays society;
c) engaging students in explorations, building, scaffolding, communicating
activities, so that that the teacher can have information not only about their products
but also about their thinking processes;
d) gradually introducing the students to theoretical knowledge (different from the
factual one), where such questions like "why does it work?" make sense and where
the answers give reason not only for links among related facts but also in terms of
logical consequence between the phenomena that one is trying to explain and the
statements that are the foundations of the theory within which the facts are framed.
As far as the specific goals of the project are concerned, there are three main
conceptual areas, which are developed according to the UMI curriculum: 447
(i) beginning probabilistic thinking; (ii) beginning the study of changing
quantities: specifically modelling activities of phenomena that evolve and change in
time, in order to describe them and to foresee their evolution; (iii) fostering students
spatial intuition in all activities where they use exploring, observing, finding
abilities.
We shall make some comments only on point (ii), which is particularly akin to the
topics discussed in this paper. These activities concern the context of change and of
movement, within the core area Relations and functions in UMI curriculum and
approach some of the basic concepts of Calculus: from functions as modelling tools
for phenomena of changing quantities to derivatives and integrals. The project uses
systematically a variety of different ICT devices which incorporate suitable different
aspects of change & motion phenomena (e.g. motion detectors connected to
computers). As widely discussed in Kaput et al. (2002) such devices are genuine new
representational infrastructures, which can produce a positive cognitive resonance in
pupils and support their learning. There are many projects in the world that are
developed according to this philosophy, e.g. Simcalc, Playground, Data Capture,
WebLabsTproject. The use of ITC for the mathematics of change is a good example
of a teaching design which fits very well the adequacy criteria a) and b) stated in §1.
Essentially the concept of function is approached within a very powerful RI,
where cultural and cognitive aspects are in deep resonance.
Specifically, the MW-project is based on some ideas by D. Tall on the approach
to Calculus, which are widely sympathetic to the general philosophy discussed
previously. In fact the approach to the main concepts of Calculus is built up starting
from the three following fundamental cognitive roots (Tall, 2000): the notion of local
straightness as a cognitive root for differentiation; the idea of a graph that "pulls flat"
when it is stretched more horizontally than vertically for the mathematical concept of
punctual continuity; the notion of area under a graph of a continuous function and the
graph that "pulls flat", for the relationships between integration and differentiation.
The project develops such concepts using the software TI-InterActive!, which
works as a generic organizer, in the sense of Tall (Tall, 2000). Such a software is an
interesting example of an ICT, which fulfils the adequacy criteria of §1. In fact, it
collects the functions of different products into a unique environment: a numerical,
graphical and symbolic calculator; a word processor; a spreadsheet; the possibility of
importing data from different environments, specifically from graphic-symbolic
calculators and from probe devices that get measures of physical quantities; a
browser to navigate in the web and to interface with other environments and software
(in § 4 we shall show an example where the students use the software Graphic
Calculus). The project uses TI-InterActive! to design work-sheets in its teachinglearning environment for exploring activities that foster the production of conjectures
by students, as well as their validation. In all this work the activity of the teacher is
essential to coach all the different processes. Successively, the teacher supports
students in structuring and scaffolding the learnt concepts within a theory. In the end 448
the work-sheets can mediate part of the theory, which organizes and systematizes the
learnt concepts and the algorithms in a coherent framework.
An example of a teaching experiment
To give an idea of the type of learning that happens in such an environment, we shall
present some excerpts from the protocols of students who are approaching the
concept of the derivative dy/dx as the limit of the rate of change □y/□x. The excerpts
illustrate in an emblematic way how the interaction with Graphic Calculus, suitably
coached by the teacher, can support conceptualisation processes in pupils. In the
work-sheet, which is built with TI-InterActive!, the students are required to explore
Gradient, a modulus of Graphic Calculus and, successively to write their results in
the TI-InterActive! page. The space does not allow to enter into details and we shall
limit ourselves to some spots. The students are in the 11
th
grade of a scientifically
oriented school, participate to the MW-project and are introduced to the fundamental
concepts of Calculus since the beginning of high school (9
th
grade). They are used to
work in small groups and to participate to collective discussions orchestrated by the
teacher. As said above, they are also accustomed to use technological devices, e.g.
sensors to investigate motion experiments. In the example, they must study the graph
of the function f(x) = 0,5 x3 - 5x2 + 3. As shown in , the software generates: the
graph of the function; its tangent, which moves dynamically along the graph, while a
point traces it; the graph of the slope value of the tangent, while it is moving.
The figure shows some emblematic steps in the genesis of the rate of change
concept. It is precisely the interaction with
the software to generate in the students the
first germs of the relationships between the
rate of change and the slope of the tangent.
In fact they have produced the graph of a
secant to the graph of the function, which
joins couples of points on the graph whose abscissas differ for a constant value □x,
determined by the students themselves. They fix successively □x equals to 0.1, 0.01,
0.001, and so on, so that the secant becomes a quasi-tangent (Fig. 1). It is interesting
to observe in the pictures of Fig.2 how the gestures of the student (more than his
words) show the way he is acquiring the rate of change notion.
Stud1: This straight line must join [Fig 1a], ok, the X interval...it is [always?] the same [Fig 2b],
Teach.: The X interval is the same; delta X [□x] is fixed
Stud1: Delta...eh, indeed, however there are some points where to explain it, one can say that this
straight line must join two points on the Y axis, which are farther each other hence it is steeper
towards... [Fig 2c]
Stud1: Let us say from here, when, here, when however it must join two points, which are farther,
hence there is less distance [Fig 2d] [...]
Teach.: is it decreasing? [Fig 2e] [...]
Stud1: They are less and less far; in fact we can say that the slope is going towards zero degrees.
Teach.: Uh, uh
Stud1: Let us say so
Stud2: Ok, □y over □x at a certain point here it reaches points, ...
Stud1: The points are less and less far [Fig 2f] [...] 449
Teach.: What does this object represent, when h represents this distance, this small interval? ...
Stud3: No, It is neither a tangent, it is a ... secant...the more I make this small, when I have h very
small, then it represents the slope of the tangent in that point, exactly in that point and hence... [...]
Stud2: It is something, which is useful to determine the slope in X; since we cannot do it directly;
that is we need the X + h, which when multiplied by a certain number of things , you see it through
computations, then it goes to zero; so we can eliminate it and we have ...
It is a dynamical idea that contains: the limit process with x that becomes smaller and smaller
(Fig. 2 a, b), the relationship between the slope and the rate of change (Fig. 2c); the relationship
between the □x (here constant) and the□y (Fig. 2 d, f).
The data of our teaching experiment, with all their limits, confirm that this
approach to ICT is an useful research tool to understand the ways in which
technological artefacts can support the construction of the student's mathematical
knowledge. In fact, the analysis of the cultural and cognitive ingredients of the ICT
used in the classroom allows to consider the added-value component provided by the
technology not limited to its purely technical features. In particular, the analysis of
ICT according the two dimensions (cultural and cognitive) stresses students' learning
that happens in such an environment: a rich interplay between the perceptuo-motor
and the symbolic-reconstructive learning, as discussed in Arzarello et al. (2005).
Some conclusions
Our example and the whole MW-project illustrate how the mathematical content and
the methodological issues of a curriculum must take into account suitably the rich
resources made available by RI. It is the same concept of mathematical literacy to
change within such new technologically integrated environments. A major goal of
today math education consists in contributing, together with all other subject
disciplines, to the cultural development of citizens, in order to enable them to take
part in the social life with awareness and a critical eye. The competencies required
for a citizen, to which mathematics education can contribute, include for example:
communicating information appropriately, perceiving and imagining, solving and
posing problems, planning and constructing models of real situations, making
choices in conditions of uncertainty, These goals are much more ambitious than the
old ones, like making computations and remembering a corpus of knowledge, which
was supposed essential for a mathematical culture. The new literacy requires a strong
mathematical experience and the habit of working with mathematical objects and
within mathematical environments integrated with the new ICT. As such, ICT are
specific for mathematics but share a lot of common features with other RI.
The main design principles according to which the new ICT are to be articulated,
should satisfy a twofold principle. From the one side, they should make available rich
and multimodal experiences to approach mathematical conceptualisation within a
rich and meaningful RI. From the other side, the experiences made in such
environments should be 'easier' and more motivating for the students because of
cultural and cognitive reasons. In fact, concepts should be tackled starting from their
cognitive roots and the environment should represent them in a powerful dynamic
and interactive way, so that it can support and enhance a perceptuo-motor approach
to them, namely an approach that involves massively action and perception and 450
produces learning based on doing, touching, moving and seeing (like in the excerpt
of our example). But the ultimate rationale of any ICT remains the teacher: it
depends heavily upon her/his teaching project if the students will be satisfied,
provided they know simply what to do or how to do something, or on the contrary if
they will ask themselves also why things they have found or they are using are
effectively so.
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The challenge of teaching and learning math online
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As the call for papers of this study expressed, since 1992 there have been
substantive developments in digital technologies, both in terms of hardware and
software. However, the networked use of the Internet, empowering the interactions
among people and contents, states the most significant difference between the first
ICMI study and this one.
The Web is only a medium of delivering learning and instruction that has the
potential to support the creation of significant learning environments providing
opportunities for communication, collaboration, and learner-centered learning.
However it is necessary to change the instructional paradigm to match the
requirements of the information age. A shift is required from a focus on presenting
materials to a focus on making sure that learners' needs are met, from passive to
active learning, and from instructor-directed to learner-directed (or jointly directed)
learning. Teaching online, faculty have to deal with learners who interact in a
completely different way and all have to reflect on their learning assumptions in
order to understand their new roles.
The change in the instructional paradigm that teaching online requires is a
challenge for all disciples, although math has additional difficulties due to the
intensive interactions with content that it necessary to establish in order to produce
the learning process.
Online math teaching requires the design of interactive material that could be
very expensive in time and money if each teacher would design completely a course.
However this can be faced chunking the course content in learning objects and
sharing them. A learning/knowledge object is the smallest 'chunk' of instruction or
information that can stand alone and still have meaning to a learner.
Currently the Web is plenty of math knowledge objects: The Mathlets. They
are the equivalent of a good example that can be explored by learners in an
interactive way. Creating object libraries, different learning programs can share the
same objects using them repeatedly, even for different purposes, reducing
redundancy, lowering costs, and enabling the customization of learning because the
configuration of the objects can be dependent on the needs of learners. E-learning
solutions based in objects can be quickly reconfigured to meet changes in user's
needs. The same knowledge object can be used (and reused) to build different
learning solutions associated to different learning goals and instructional strategies. 452
A Case Study Of Developing Students' Ability to Design Algorithm in Logo
Environment
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The algorithmic idea has been a kind of necessary mathematics quality for modern
people in this information society. In China the algorithm was represented fully as
one of the new mathematics contents in the secondary level for the first time when
The Standards of Mathematics Curriculum for the Senior High School was
promulgated in 2003, so the research about the teaching of algorithm undoubtedly
has its practical implications for mathematics education. In this paper, with the
conceptual framework of The Mathematics Task Framework as the research tool, an
algorithmic teaching case based on LOGO software was introduced in detail, and
data including observations, interviews and worksheets were collected, then the case
was analyzed, and the results showed that the teaching of algorithm is feasible and
effective in the LOGO environment. In the last, some beneficial implications about
the instructional design of algorithm were discussed.
Introduction
An algorithm refers to the step-by-step systematic procedure used to accomplish an
operation, which characterized as finiteness, definiteness, input, output, effectiveness
and named by the ninth-century Arabian mathematician Mohammed al-Khowarizmi.
Due to the way of mechanical operation, the algorithm hasn't been emphasized much
in the long history of mathematics education (Peng, 2004).
With the rapid development of modern information technology, the algorithm begins
to play a fundamental role in the development of science, technology and society, and
even penetrates into every aspects of life. Being considered as a kind of necessary
mathematics quality for modern people, the algorithmic idea is gradually emphasized
in educational circles, which arouses the interests of the related research from
mathematics education. The initial work can trace back to 1978, in which Engel
outlined the comprehensive topics of the mathematics curriculum at school from an
algorithmic standpoint (pp.255, 274). Similarly, Ziegenbalg argued that the concept
of algorithm belongs to one of those fundamental concepts of mathematics (pp. 239,
241).
At present, as the process of problem solving and mathematical application in the
authentic life are highlighted in modern mathematics teaching reform, the learning
and understanding of the algorithmic process is especially emphasized. It's promoted
about mathematics teaching reform in almost every country that the traditional way
of teaching algorithm should be changed to teach students to design their own 453
algorithms and to solve realistic problems through using the algorithmic ideas,
furthermore more, students should decide their own approaches and steps (Xu, 2001).
Although algorithmization is a distinctive feature of mathematics in ancient China
(Ma et al, 1991), the term algorithm hasn't been represented in the mathematical
textbooks for schools until 2003, when The Standards of Mathematics Curriculum for
the Senior High School was promulgated in China and the algorithm was represented
fully as one of the new contents for the first time. From it we can understand that the
research about algorithm is a new issue and the reason why there is not much work in
this field in China, except the research from the angles of cognitive psychology (Xu,
2003), curricular value (Li, 2004; Liu, 2003) and the significance of learning
algorithm (Li, 2004). In particular, the problems about which undoubtedly should
have the practical implications for mathematics education, such as how to design the
mathematics teaching according to students' real level, especially, how to integrate
the information technology into the algorithmic teaching, are topics worthy of
research while still are scarce.
In the following, an algorithmic teaching case in which the main classroom task is to
draw a pentagram based on LOGO software will be introduced in detail, and data
including observations, interviews and worksheets will also be collected, then indepth analysis will be shown with the conceptual framework of The Mathematics
Task Framework as the research tool. In the last, some beneficial implications about
the instructional design of algorithm will be discussed.
Conceptual framework
The framework that guided much of this study stems from work done in the
QUASAR project
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. Based partly in the ideas and research of Stein and her
colleagues at QUASAR developed a framework that focuses on the cognitive demand
of mathematical tasks and the various phases tasks pass through in their instructional
use (Stein, Grover, & Henningsen, 1996; Stein and Smith, 1998). This framework is
depicted in Figure 1.

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QUASAR (Quantitative Understanding: Amplifying Student Achievement and
Reasoning) was a multi-year teacher capacity-building initiative sponsored by the
Ford Foundation to change mathematics teaching practices at six middle schools
ended in 1995. 454
Figure 1: Mathematical Tasks Framework
In describing the framework, Stein, Grover, and Henningsen (1996) write:
"Instructional tasks are seen as passing through three phases: first, as curricular
materials; second, as set up by the teacher in the classroom; and third, as
implemented by students during the lesson" (p. 460). Certain cognitive demands are
inherent in the way that a mathematical task is written. For example, tasks that ask
students to memorize a fact or to perform an algorithm rotely encourage a certain
type of mathematical thinking. Tasks that ask students to look for patterns,
generalize, make connections, or think conceptually encourage a different kind of
thinking (Stein & Smith, 1998).
In my research, the Mathematical Tasks Framework guides of my data collection,
analysis, and reporting. It made sense to use this framework as a research tool
because I was particularly interested in understanding how the students perform their
classroom tasks pertaining to the ideas found in the Mathematical Tasks Framework.
While organizing the teaching case, I was guided by my use of the Mathematical
Tasks Framework in making decisions about what data to collect. These data could
have been analyzed in a way that I was guided by my use of the Mathematical Tasks
Framework in choosing "a coherent way of thinking about how to organize and
interpret the data" (Eisenhart, 1991, p. 204).
The mathematics teaching case
Task: drawing a pentagram
The content of drawing a pentagram is chosen from the colorful LOGO-figures
world, a subsection of a learning textbook, LOGO experiment, as a help of new
textbooks compiled according to The Standards of Mathematics Curriculum during
Compulsory Education. What should be mentioned is that although the case is taken
from junior high school, the results showed that it can realize the notion of The
Standards of Mathematics Curriculum for the Senior High School, so it's helpful not
only for the instructional design of algorithm in senior high school, but also for how
to develop algorithmic idea in junior high school.
Teaching condition
TASKS
as they
appear in
curricular/
instructional
materials
TASKS
as set up
by
teachers
TASKS
as
implemented
by students
Student
Learning 455
The students have learned some basic geometry knowledge, and they can use some
basic and simple LOGO orders to operate. Everyone has one computer in the wellfurnished computer laboratory, which is linked by local area network, through which
students and teacher can communicate freely.
Teaching process
Stage 1 Review (about 3 minutes)
The teacher guides students to review the LOGO orders, FD (FORWARD), BK
(BACKWARD), RT (RIGHT), LT (LEFT), which will be used in this lesson, through
the strategy of asking-answer way.
Stage 2 Learning the new LOGO order (about 10 minutes)
The students begin to learn the new order REPEAT, with the help of the teacher,
through drawing the triangle, quadrangle and pentagon. In this stage, students can
describe the complex recycling process and use the format:
REPEAT 3 [FD 60 RT 120]
REPEAT 4 [FD 60 RT 90]
REPEAT 5 [FD 60 RT 72]
Stage 3 Exploration of the experiment (about 27 minutes)
The teacher shows a model of a pentagram and encourages students to draw it
through computer.
As we all know that pentagram is a very popular geometry figure, it can be seen
everywhere, such as the Chinese National Flag.
Many students are very excited when the teacher asks them to draw a pentagram,
because they are familiar with it, and most of them have the experience of drawing a
pentagram in a paper-pencil way. But in the face of computer based on LOGO, they
feel out of place, for it's difficult for them without considerable mathematics
knowledge and the thinking way of precise expression.
The angle is the core of solving the algorithmic problem. Based on the knowledge of
measure of angle in the seventh grade, the teacher guides students to review the
conception of supplementary angles and adjacent angles, then tells them the
characteristics of the pentagram(for example, every angle is 36°)
Students devote themselves to draw right now, and some who are good at computer
can color the pentagrams.
The students find the following approaches to draw pentagrams: 456
Figure 2 Regular pentagram
Figure 3 Hollow pentagram
After discussion with students about the meaning of the data 5, 60, 144, 25, 72, the
teacher gives another approach of drawing.
Figure 4 Solid pentagram
Stage 4 Exploration and innovation (about 5 minutes)
Fill in the blanks
REPEAT 5 [RT FD 100 RT]
RT 90
REPEAT 5 [FD 100* LT 72]
LT 90
END
Figure 5 Pentagon and pentagram
This practice provides an approach through connecting the diagonal of the pentagon,
especial and meaningful, which is a bridge of the pentagram and pentagon. It
motivates students to further understand the algorithm of drawing a pentagram, and

do prepare for the continuing learning of the pentagon in the future.
Discussion and conclusion
This lesson is designed from the aspect of how to explore algorithm and how to design various algorithm, aimed at developing and deepening students' algorithmic
Solid pentagram added by the teacher
REPEAT 5 [FD 25 LT 18 BK 25 * 1.9
FD 25 * 1.9 RT 162 FD 25 LT 72]
Hollow pentagram four algorithms
REPEAT 5 [FD 25 RT 144 FD 25 LT 72]
REPEAT 5 [FD 25 LT 144 FD 25 RT 72]
REPEAT 5 [FD 25 RT 72 FD 25 LT 144]
REPEAT 5 [FD 25 LT 72 FD 25 RT 144]
Regular pentagram two algorithms
REPEAT 5 [FD 60 RT 144]
REPEAT 5 [FD 60 LT 144]457
idea through the experience of the algorithmic diversity. We'd like to analysis the obtained goal from the exploration of algorithm, algorithmic diversity and mistakes during the study of algorithm.
Students' exploration of algorithm
In this case, taking advantage of the characteristics of LOGO language, such as intuitive and easy-operate, the teacher guided students to explore the way of solving problem during the operation, to develop their own algorithmic ideas step by step and foster their ability to solve realistic problems (drawing a pentagram from many aspects) by using the algorithmic ideas.
Students can experience the algorithmic characteristics of finiteness, definiteness, input, output and effectiveness during drawing the pentagram, and deeply impressed with the input language of FD … and RT …. recycling language of REPEAT …
The following are their feeling and thoughts about exploration of the algorithm, which we got from the observation and interview in the classroom.
It's easier to understand the approaches of drawing a pentagram through computer, when we can think carefully as well as looking at the turtle moving (the order is operated by a turtle).
It's interesting to draw different colors and sizes of pentagrams (by using the order of color, students color red, yellow and blue for the pentagrams).
Programming is not a difficult thing (students can summarize the step of drawing a pentagram as a program, and then put different numerical data into the parameter, thus making the pentagram variable in size. In fact, it has realized the transition from mathematics language to procedure language).
The turtle can finish my demands well, for which I'm highly required that no mistakes would happen during the operation.
What the above mentioned show that with the guiding of the teacher in the teaching of algorithm, it's effective to arouse students' enthusiasm and develop their algorithmic ideas, while letting students take active part in exploring algorithm by themselves.
Students' various algorithms
There are six algorithms of drawing a pentagram developed by the students, which can be divided into two types of algorithms. According to the moving way of the turtle, the first type is based on the drawing through connecting the diagonal of the pentagram (there are two approaches, see figure 2), and the second type is based on the drawing through moving along the sides of the pentagram (there are four approaches, see figure 3).
Other two students used this approach: [FD 60 RT 144 FD 60 RT 144 FD 60 RT 144 FD 60 RT 144 FD 60 RT 144], which we call regular algorithm, for it don't need the order of REPEAT, only but FD and RT. It seems like the ordinary paper-pencil way, which can be finished step by step. 458
From 55 students' procedure records, we get 49.There are 18 students who used the first or second type of algorithm, and there are 4 students who used both (see Table 1).
Types of algorithm First Second First & Second Regular Wong
Numbers of student 18 18 4 2 7
Table 1: The types of the algorithms and the corresponding number of students
According to our interview, the students who used the first or the second were affected by the order of REPEAT taught by the teacher, because it can simple the repeated process. There are only 2 students who used the regular algorithm, and we know that they didn't catch the meaning of REPEAT, so they chose the way of step by step.
These results indicate that:
Teacher's action has highly active or negative effects on students' learning of algorithm;
The algorithmic structure of pentagram in students' mind and LOGO language can help students to express the algorithmic model;
Students' past mathematics knowledge and skills have effects on the subsequent learning of algorithm. The first type of algorithm is simpler in expression than the second and more difficult to get. While the results show the number in any of these two types are the same. It is recorded by the students that because of the transition of the ordinary paper-pencil drawing, they form the habit of connecting the vertex of pentagram to draw. As for the second, it is intuitional, although relatively complex in expression and easier to find for them.
Students' mistakes
There are 7 students whose algorithms are wrong, of course, they didn't get the pentagram on computer. There are some ordinary mistakes in their algorithms, for example, procedure of REPEAT 5 [FD 25 LT 144 FD 25 RT 72] is replaced by the procedure of REPEAT 5 [FD 25 RT 144 FD 25 LT 36]. The mistakes result from the failing to understand the mathematics basic knowledge. Some students know that every angle of the pentagram is 36
0
with the help of the teacher, so 144
0
of the
parameter of LT, rather than 36
0
, which can be learned only though computing
according to the theorem of the total of internal angles of a triangle. It shows that students haven't deeply understood the relationship between the angles. Further more, the angle of LF or RT is also important, which easy to be mistaken without a logic and precise thinking way. 459
From the above analysis, we can conclude that the students have obtained the expected goals and the teaching of algorithm is feasible and effective in the LOGO environment.
Implications for the instructional design
In this section, we'll look back some characteristics of the instructional design of algorithm in this case.
Roles of the teacher and students
The learning of algorithm is a kind of uncreative learning in the psychology. In this case, the teacher didn't indoctrinate the existing algorithm, but let students try themselves to draw a pentagram to experience the initiative of the algorithm through their constructive learning in the LOGO environment, thus transiting the learning of algorithm into a kind of creative learning. It's helpful to eliminating their fear and hate to mathematics and algorithm, also promote students to understand the constructive process of algorithm (P. Dowling & R. Noss, 1990). It's more interesting if students can create an algorithm, then it means that he or she has not only understood the algorithm, but also has known how to apply the algorithm to ordinary life.
So the role of the teacher is to create actively environment for students and to help them take part in exploring and constructing their own algorithm, rather than to teach the existing algorithm. Also, from the time of students' participation, total to 32 minutes, we can see that students are the dominant of the learning.
Task-directed to arouse the motivation
The motivation can arouse students' enthusiasm for learning, provide the direction and goal of learning. In this case, an interesting and familiar problem (drawing a pentagram) was given, catching students right now, and then resulted in requirement of recognition in the algorithm learning during the operation, thus the mathematics knowledge need to be taught and learned naturally. In the LOGO environment, the teacher can choose much more realistic problems, by using the strategies of taskdirected to guide the students.
Taking advantage of the LOGO network
LOGO is such a good mathematical environment that it can help students toward more intuitive mathematical strategies rather than avoid analytic and it together, also it's something of scaffold for the learning (C.Hoyles & R.Noss, 1992). Its good characteristics of friendly face, convenient language and simple operation, is easy to be learned and well liked by students. The most important is its open system that can allow students to create new orders, namely creating new algorithms. In the teaching of algorithm, we should make use of it. 460
On the other hand, everyone have a computer, which allows them to devote to the exploration of algorithm freely. Further more, through the monitoring system, the teacher can see everyone, and every student can ask for help from the teacher. Students can communicate their algorithms and cooperate with one another to share resources and make progress together.
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Building up the notion of Dependence Relationship between Variables: A case study with 10 to 12-year old students working with Math Worlds
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This paper reports the results from a study with 10 to 12-year old students working on activities involving various functional representations (graphs, tables, and numerical relationships) in a motion phenomena simulation environment such Math Worlds. Results from the study suggest that pupils that have not been received formal instruction in algebra symbolism are able to evolve towards a better understanding of functional relationships, when working with a variety of representation systems. Duval's registers theory was used for activity design and data analysis. This study is part of a broader project entitled Enseñanza de las Matemáticas con Tecnología (EMAT) (Teaching Mathematics with Technology), which was developed by the Mexican ministry of education at the end of the 90s (Rojano, T., 2003).
Introduction
At the end of the 90s, Mexico's Secretariat of Public Education (SEP) together with the Latin American Institute of Educational Communication (ILCE) undertook the initiative of implementing an educational innovation and development project known as Teaching Physics and Mathematics with Technology (EFIT-EMAT). The principles listed below were the underpinnings of the project conception: a) Didactic principle through which classroom activities are designed following a phenomenological treatment of the concepts taught. b) Specialization principle through which tools and pieces of content software are chosen. The selection criteria were derived from the specific didactics of each subject (Physics and Mathematics). c) Cognitive principle through which tools are chosen that enable direct manipulation of mathematical objects and phenomenon models through executable representations. d) Empirical principle according to which tools proven in some educational system are chosen. e) Pedagogic principle through which ICT usage activities are designed in order for them to promote collaborative learning and interaction among students, as well as among teachers and students. f) Equity principle with which tools are chosen that enable secondary school students to have early access to powerful scientific and mathematical ideas.
Specifically EMAT (Teaching Mathematics with Technology) is a model that

contemplates use of a variety of technological pieces (specialized software and graphic calculators) each of which is very closely related to the specific didactics. In concrete terms usage of dynamic geometry software was included for topics of geometry; spreadsheets were included for teaching of algebra, arithmetic-algebraic

68 Part of the contents of this study was presented at the Twenty-sixth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA) held in Toronto, Ontario, Canada. 463
problem solving, and probabilities topics; graphic calculators were included for introduction of algebraic syntax and problem solving; simulation software and software to represent movement phenomena were included to teach mathematics of variation. This article reports on the findings of a pilot study on the activities designed for work with mathematics of variation.
Background
There has been recently a growing interest on children's potential to learn algebra at early stages of their development. In some of these early algebra studies the possibility of teaching algebra to young students (7 to 8 year old) has been explored through problem solving activities that elicit the algebraic nature of arithmetic competency (Carraher et al, 1999 and 2000). Other approaches emphasize the role of young children drawings and representations in word problem solving processes as a basis to develop algebraic ideas (Dougherty and Zolliox, 2003). Smith and Davis (2001) say that the history of algebra may be used as a source of information about the possible difficulties faced by young students when they are introduced to algebraic thinking. L. Lee stresses out the idea of considering the relevant aspects of different methods used to teach algebra (as a language, a way of thinking, a tool, or generalized arithmetic) to encourage the learning of algebra (Lee, 2001). Every study, however, has reported the feasibility of introducing young students to the algebra domain either by using algebra symbolism or through other representations. The purpose of this study is to investigate the possibility of introducing fundamental algebraic ideas to students from elementary schools through the use of representation systems such as Cartesian graphics and numerical tables generated by a learning tool that includes a motion phenomena simulation environment.
Theoretical Framework
Theoretical references on representation devices are based on R. Duval (1999), who describes how semiotic registers provide an effective way to materialize knowledge and deal with mathematical objects. In this regard it is necessary to promote a kind of learning, where several representation devices are integrated and coordinated in such a way that the student does not mix up the mathematical object and its semiotic representation, and relates the mathematical object to several representations. R. Duval claimed that it is necessary to encourage three cognitive activities: 1) formation (create a representation to describe an object); 2) treatment (transform the representation into the device); and 3) conversion (transform the representation of a device into another).
Our research takes these elements as a basis for the development of a didactic strategy to design learning activities, which enable students to approach up algebraic concepts such as functional variation through the cognitive activities of treatment, conversion, and formation.
We must highlight the fact, however, that in certain situations students may or may not be aware of such cognitive activities; for example, when they create a functional table from position graphics (corresponding to a phenomenon of constant and 464 positive speed). This was possible because a software application (a simulator) was used as a mediation tool to create links between the students' algebraic knowledge and cognitive processes. The idea was to introduce 10 to 12-year old students, who had never received formal education on algebra, into the notion of functional relationships through the use of the SimCalc Math Worlds computing environment. Math Worlds provides animated worlds, where animations move according to changes in graphics. Graphics are represented through rectangles meaning speed: The height of a rectangle means "how fast", and the width means "how long". Position, speed, and acceleration graphs are dynamically linked. If there is a change in speed, the corresponding changes in the position or acceleration graphs are instantly displayed.
As for environmental usage, theoretical references are taken from J. Roschelle (1998) and J. Kaput (1998). Ideas such as functions, equations, and variables are used to promote the development of skills in the initiation of algebra.
In this paper we discuss the results from the students' analysis of position and speed graphics that help them gain a better understanding of dependence relationship between two variables, as well as to provide concepts such as "it goes faster" or "this is quicker" with a mathematical meaning.
As other studies based on a functional approach to algebra (Kieran et al, 1996; Nemirovsky, 1966; Heid, 1966), our study used the computing environment to design modeling activities that allowed students to explore quantitative changes in variables, and analyze how these changes modify functional representations.
Methodology
• 12 activities were designed to encourage the three cognitive operations of formation, treatment and conversion among registers (Duval, R. 1998, 1999).
• Such activities are used to promote i) the use of more than one representation register: a) interpreting position graphs, b) building up tables and position graphs, and c) interpreting speed graphs; ii) the notion of constant speed (functional relationship); and iii) problem solving in a motion phenomena simulation environment. This paper is focused on aspects i) and ii).
• A database consisting of: a) a diagnosis questionnaire on basic notions of physics (speed), pre-algebraic operations, arithmetic, and reading and gathering data from tables and graphs; b) structured guided interviews (A. Brown et al, 1998); c) students production from learning activities with Math Worlds (student records); d) videos from interviews; e) learning activities aimed to encourage the formation, treatment and conversion of registers to gain evidence on the students understanding of notions such as variable, functional relationship, speed, and others.
• The participants were 10 to 12-year old children from elementary schools, who had never received formal education on algebraic symbolic language, and were 465 selected from their answers to the diagnostic questionnaire. This paper reports on the results from the six-student group working with Math Worlds.
Diagnosis Questionnaire Results
Based on a diagnosis questionnaire, subjects were grouped in three levels, according to their arithmetic competency, register-handling abilities (data gathering from tables and graphs), and notion of constant speed. Levels I, II, and III vary from a substantial proficiency on arithmetic (Level I) to a limited competency (Level III).
Based on this questionnaire, six students (3 from 5
th
grade and 3 from 6
th
grade) were
chosen to participate in the study. They took part in 12 sessions, working with the computing environment on: a) the use of more than one representation system to interpret position graphs; build up position graphs and charts; interpret speed graphs; b) the notion of constant speed as a functional relationship; c) relating problems to motion phenomena.
In the middle and at the end of each session sequence individual interviews were carried out with all six students to analyze the strategies they use to interpret different representations as they went on solving the proposed activities. This paper discusses the results related to a) and b).
10-year old Students working with SimCalc.
All three students required the use of simulations along the learning activities to verify their answers, complete tables or build up graphs. Only the Level I student (Erick) built up continuous graphs from the first activity, compared to the other two students, who did it in a discrete way, by running the simulation step-by-step or making operations to identify the next section on the graph. In addition, Level III student (Rodrigo) found out in the last two activities that he could build up the position graph continuously by knowing the total distance and duration of a path, as shown in the following table:
The two students from levels II and III seemed to focus their attention on the use of graphs to calculate speed, while Erick showed at first some reluctance to use tables to identify speed when requested to do so, but once convinced, he worked on the table to identify the required information and calculate the answer.
Toward the notion of speed (10-year old children)
Level I (Erick).
Math Worlds incorporates motion of various characters, such as frogs or lifts (according to the World chosen). From the first sessions he shows to deal with a notion of speed involving both variables, and providing explanations such as "the frog moves forward four meters every second, and the clown two meters." After becoming familiar with the simulator, his explanations were "the frog moves forward twice as fast as the clown", or "it moves three times faster than the green lift." As his identification of the variables improves, his answers to the notion of speed include elements such as "Frog 2 moves forward three meters per second, and frog 1 one and a half meter per second." 466
In the middle of the activity sequence, Erick is requested to give a definition of speed. His answer, "speed is what it runs in one second," could be considered as focusing his attention on one of the variables. At the end of the sequence he was requested again to provide a new explanation of speed. This time he uses a specific example, but after a few questions he comes out with a more general notion pointing out both variables:
Er: It is… it moves forward one point five meters (the character's speed in the simulation);
E: And speed involves… what?;
Er: Seconds and distance.
Level II (Ana Karen)
At the beginning of the session she provided explanations such as "The frog's steps are four meters long," or "The clown's steps are two meters long." After comparing the speed of some characters she expressed the following: "The truck goes faster, and the car slower." After moving on through the activities, Ana managed to include a little more information: "The slower clown moves five meters, and the faster clown moves eight meters."
In the middle of the sequence her definition of speed is "the number of kilometers a car or anything else moves forward." At this point she also perceived the possibility of using distance and time to calculate the speed from the information contained in the position graph. By the end of the sequence she employed a particular situation to explain the notion of speed, taking into consideration the time and distance variables, but on a particular example:
E: How would you explain speed?;
Ak: Speed is the distance and time a car travels;
E: How do you read speed?;
Ak: If we take meters and seconds, then it will be 81 meters per second.
Level III (Rodrigo)
His first explanations about motion took into account the physical features of characters, with definitions such as "The clown is small, and that is why his legs go slower," or "The tires of the truck are bigger, and that is why it goes faster." After completing the first activities he included in his explanations elements related to the characters movement, such as "the red one moves slow, and the green one moves fast," until he finally takes into consideration both variables: "It advances one third every second," or "the red one goes up two floors every second."
When first asked about his notion of speed, his answer was "every lift goes up a number of floors per second," making use of an example to generate an explanation. Once the activities sequence was concluded, his notion of speed evolved to include both variables, distance and time, making it easier for him to calculate speeds from a position graph, but avoiding any oral explanation:
Ro: Then we looked at the graph; 467
E: And what did we notice there?;
Ro: The hours (he writes his answer: "We looked at the graphic, and paid attention to the hours and kms");
E: Ok. Now, how would you explain speed to your schoolmates?;
Ro: I don't remember (he could not provide an explanation).
11 to 12-year old Students working with SimCalc.
On both ends of the table, Level I student (Eduardo), and Level III student (Rafael) draw graphs step by step, and it was not until the last activities that they started on building up continuous graphs, using as reference the total distance and period. With this information they didn't have to make any further calculations to identify every portion of the graph. On the other hand, Level II student (Clara) drew continuous graphs from the beginning, when she was asked to build up a register.
As for the verification process, all three levels repeatedly used simulation to get data and match their results. In those situations where the environment didn't provide directly the required information, they made calculations using pencil and paper, an electronic calculator or mentally. For the final activities the use of the simulator in all three cases was restricted to the analysis of situations rather than to getting information to give an answer.
As for the preference on the kind of representations, only Clara (Level II) showed a clear disposition to use graphs rather than tables to get information. Eduardo and Rafael used indistinctively data from graphs and tables to make calculations, solve problems, and answer questions.
Toward the notion of speed (11 and 12-year old children)
Level I (Eduardo)
From the beginning he took into consideration the two variables to build up the notion of speed, with answers such as "Because the frog went over more meters in a second than the clown," or "Clown 1 goes six meters in one second, and clown 2 goes eight meters." These answers were consistent along the sequence. When he was asked to give a definition of speed, he said:
E: How would you explain to your schoolmates what speed is?;
Ed: Through distance and time. You can say that it represents how fast two people or two objects go.
Together with the comments above, he recognizes the need to divide distance by time to calculate speed either from a table or a graph: "We can see how far did he go, and the time, and all that."
Level II (Clara)

Her first explanations about the movement of characters indicate her focus on the physical conditions of the phenomena: "It runs more and gets first", or "It walks slower". After the third activity, she includes other elements such as distance and time: "The second frog, because it runs more meters per second." 468
For her first explanation of speed, she uses a specific example: "The number of floors per second it climbs, the distance it travels per hour." Once the sequence is complete, she takes into consideration the variables involved in her notion of speed, even if she requires a specific example:
E: How would you explain to your schoolmates the concept of speed?;
Cl: The distance traveled every second.
We may consider her definition of speed after completing the sequence as an evolution from a very intuitive and inconsistent notion of speed at the beginning of the study.
Level III (Rafael)
At the beginning of the sessions, his explanations were: "it goes farther than the frog", or "it goes two by two and is slower." His answers indicate that he takes into consideration one variable. From the third activity, however, he includes in his explanations both variables as follows: "it goes only a few meters in many seconds," and "the red one, because every second it climbs three floors."
His first definition of speed indicates the relationship between the distance and time variables: "the distance traveled in a period of time." This may suggest that he is adjusting his notion of speed, since he also recognizes the need to identify distance and time in order to calculate speed:
E: What elements should be taken into consideration to calculate speed?;
Ra: Floors and seconds.
In spite of recognizing both variables in the notion of speed, his defective use of the division produces false explanations, as the following:
E: And now what did you do to calculate speed?;
Ra: A division;
E: What did you divide?;
Ra: Hours and meters.
This was a continuous obstacle throughout Rafael's work, and it could not be overcome.
Results
After working with the activities designed for Math Worlds, the participants showed an improvement on their understanding of motion phenomena and on the dependence relationship between variables. There was, however, a differentiated use of the representational means available in the computing environment, according to children's level of arithmetic competency (identified in the pre-questionnaire). In addition, it was observed that whereas some children during all the sessions required the simulation to analyze the dependence between variables, others were able to focus their analyses on mathematical representations (Cartesian graphs and function tables). These children in turn succeeded in building up a quantitative notion of constant speed. For example, in the final session Erick (11 years old) described his notion of 469 speed as "the frog moves forward four meters every second, and the clown two meters" (referring to the characters in the simulation environment). When he was asked about his conception of "speed", he said: "Speed is what it runs in one second."
Final Comment
Results from this study suggest that students at a pre-symbolic stage (pupils that have not received formal instruction in algebra symbolism) are able to evolve towards a better understanding of functional relationships, making use of a variety of representation systems (including simulations) to analyze different aspects of motion phenomena. Duval´s registers theory was not only used at the activity design stage, but also helped us to define an analysis framework that enables feasible explanations of the way children make use of representational systems when solving problems in a computing environment.
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Mathematics Revisited and Reinvigorated
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Despite enormous changes in technology over the past twenty years, and numerous changes in the UK curriculum, and elsewhere, the content of school mathematics remains very recognisably the same, notwithstanding dramatic changes in the world's modus operandi and the need for citizens to deploy a far wider range of mathematical skills than before..
We identify areas where changes in the curriculum could usefully reflect changing uses of mathematics and address some of the negative perceptions of mathematics as boring, irrelevant and inaccessible; and develop this in the context of our current work on reasoning from evidence.
Reasoning from multivariate evidence is pervasive in political speeches and in the media, but is largely absent in UK schools. Currently, we do not prepare young people adequately to understand important social debates, decision-making under uncertainty in a business environment, nor to make informed decisions about their personal well-being.
Two strands of work will be described. The first presents evidence that students can work effectively with multivariate data if they are supported appropriately with good computer interfaces. Second, current work with teachers of mathematics, citizenship, and geography on curriculum materials to develop skills in reasoning from evidence will be reported.
Introduction
There are various perspectives from which the rationale for studying maths in the 21 st
Century may be considered: there is a utilitarian aspect, but there are also cultural and aesthetic aspects. The report Mathematical Skills in the Workplace (Hoyles, Wolf, Molyneaux-Hodgson and Kent, 2002) offered many insights into what mathematics is used routinely. Some of the aspects listed on page 5 of the report would be recognisable components of the current curriculum but there are substantial areas which would not: what is used nowadays in the world of work is quite different from 'traditional maths' and the curriculum should shift in response to this. Much of this shift relates to the pervasive nature of the use of ICT in the workplace - many activities which employ mathematical skills involve the use of some form of technology, and could often be described generically as modelling, but the researchers found that often those involved in these activities would not describe what they did as mathematical in any sense. This relates strongly to the cultural aspects of the curriculum: we need to break the cycle of "I can't do Maths" and "I hated Maths at school" which seems to exist across much of society - where it is seen as acceptable to admit to being innumerate: indeed, it is almost a badge of honour to many people who take pride in other achievements. Many fewer people would admit 472 to similar sentiments about illiteracy. Understanding the process of modelling is a prerequisite to critical engagement with many aspects of modern society.
For the report by Hoyles et al. a number of case studies were undertaken, and the type of activities with any mathematical component were recorded. A high proportion of these were related in some way to handling data: these could be at very different levels of sophistication - recording, summarising, reporting, interpreting, and analysing data are an integral part of many processes, and making decisions based on the data and its interpretation can be critical to the success of any business. More generally, the skills related to handling data can be viewed as a part of modelling, and it is in this area that we see the main imperatives for change, with ICT playing an integral role.
Modelling, and fitting parameters in models, is conceptually accessible at school level, and can now be done reasonably [and in realistic contexts] using ICT. Technology allows multiple variables to be investigated in complex data: if students leave school without ever working with more realistic levels of complexity they will find it very hard to make the transition later in their cognitive development. Contexts such as biomathematics, psychology, business studies, geography and citizenship offer a vast array of examples where mathematics plays an integral part which have been largely ignored by the subject itself. Technology offers the opportunity to develop short digital video descriptions of applications of mathematics in such areas which could be used by teachers to motivate the study of certain topics.
It is generally accepted that mathematics has a poor 'image' with the majority of students: it is viewed as hard in many cases, but also as irrelevant and boring. We argue that we should consider the inclusion of areas which are made accessible now by the use of technology to handle the complex processing required, and which are of widespread use in technology today, and that such changes would improve the perception of mathematics, and increase its accessibility at the same time. Areas include: matrices, vectors, calculus and geometry in 3D, curve sketching, large samples in statistics and multivariate data, of which we will say more later.
Mathematics recently has concentrated too much on small scale activities: largely this has been driven by the structure of assessment, and the justification seems to be in terms of ensuring reliability. However, the effect of this is that the whole is less than the sum of the parts, and the validity of the assessment of mathematics is questionable: the abilities to solve problems, to apply simple techniques in new contexts, to combine reasoning and techniques from different areas of the curriculum, and to critically evaluate arguments and data presented in context are parts of mathematics which have been devalued.
Statistics within the mathematics curriculum - the place of reasoning from evidence
Many branches of mathematics have been invented in response to real-world problems. Probability theory and statistics provide vivid examples. In the case of statistics, the early development of theory and technique were constrained by the 473 absence of computer power. For calculations to be tractable, strong assumptions were made, and a somewhat bizarre process of 'hypothesis testing' was invented. This has had a regrettable legacy: if a curriculum were to be devised from scratch the light of current statistical knowledge, it would not be dominated by univariate parameterized distributions and linear bivariate models. It would include bootstrapping methods, nonlinear regression and multiple regression models, all of which are computationally intensive but conceptually accessible. It might include time series modelling with simple autocorrelations, to introduce the concept of feedback loops.
Very large data sets are now available to schools for analysis. Here, almost any way of partitioning the data set is likely to reveal statistically significant differences, which will not necessarily have any practical or conceptual importance: that is, analysis will often show very small 'effect sizes' with no real world 'significance'. Schield (2005) observes that while chance (random error) dominates in small sized, well-designed experiments, bias (systematic error) can dominate in poorly designed studies regardless of size, and confounding (the influence of a lurking variable) dominates in populations or large-scale, well-designed observational studies.
Many other subjects depend implicitly, if not always explicitly at the school level, on reasoning from evidence. While we believe that the core responsibility for teaching and learning statistics lies within the mathematics curriculum, it is vitally important that statistics is not seen as a set of abstract mathematical techniques (Nicholson, Ridgway &McCusker, 2006). There is already pressure on curriculum time, and there are demands for greater fluency in other important areas of mathematics such as algebra, so something has to give. We believe there is scope for substantially reducing the amount of time spent on repetitive, routine tasks such as calculations of summary statistics and graph drawing, which are now automated in virtually every working environment, and replacing it in mathematics by the core skills of reasoning with complex data, supported by the co-ordinated use of those skills in other subjects to encourage transferability. This should make the mathematics curriculum more

relevant and also create some extra time within mathematics to improve other key activities such as algebra, while not reducing the total amount of time students spend on working with data in the curriculum.
In UK schools, we teach the statistical techniques of the 1920s, and (on rather rare occasions where the curriculum extends beyond the practice of technique) choose contexts to which this restricted set of methods can be applied. Reasoning from evidence is a central theme that runs through all empirically based subjects, yet is dealt with very badly in mathematics. It is not simply the case that students are exposed to an impoverished curriculum; we will argue that it is actually a pedagogy that trivialises mathematics, and disempowers students, with unfortunate consequences for public debate on important social issues. Students are not equipped to deal with the world outside the classroom, either to engage meaningfully in political debates on topics such as climate change, poverty, health, and crime, or to make informed decisions about their own lives that depend on understanding 474 multidimensional data - such as their health, career choices, or finances. The second unfortunate consequence is more parochial, and concerns the locus and influence of mathematics within the curriculum. Mathematical thinking should be at the heart of the educational process, not confined within a curriculum box.
Here, we present evidence that young students can reason effectively with multivariate data if supported appropriately with ICT. This offers the prospect of a radical, mathematical based, reform of the whole curriculum. Reasoning from evidence has a number of features that generalise across domains and mathematics, and the community of mathematics educators are ideally positioned to provide an integrated framework to develop generalisable skills associated with reasoning from evidence in students. There are at least three major challenges to be faced. First is the nature of the evidence itself. Much of the data in the public domain is presented in indigestible ways - for example, extensive tables of data in a printed form. Second is the current mathematics curriculum. Statistics, when it is taught within mathematics, focuses on the mastery of a narrow range of statistical techniques that are ill-suited to understanding complex problems. Third is our understanding of the nature of the development of reasoning from evidence. Here, we explore some key ideas that underpin the modelling of complex situations, and identify some components which are accessible to young students.
If reasoning from evidence is to become a key focus in the new curriculum, we need to be convinced that students can actually engage in reasoning from evidence, and we need accounts of the qualitative stages in the development of reasoning from evidence. The latter will be an essential guide for the design of curriculum, pedagogy, and assessment.
Paradoxically, a generic approach that emphasises universal principles in reasoning from evidence is likely to provide a firm foundation for modelling global and regional data, to the benefit of local, regional, and national communities. If educational systems are to help students reason effectively from evidence, then accounts of stages, progress, and pedagogy are essential.
Ridgway, Nicholson, and McCusker (in press) present an analysis of a speech by David Blunkett, then Home Secretary, on the social problems apparent in multiethnic communities, and the actions that politicians might take to alleviate these problems. The paper set out to establish some of the theories about modelling complex phenomena implicit in the speech of a thoughtful politician. A number of principles were identified, including the following ones:
• Every complex problem has a number of components, which are influenced by a variety of factors. Effects occur over a range of time scales, and at different magnitudes;
• Models of change need feedback loops;
• Correlation is not the same as causality. Possible moderator variables need to be considered. 475
Citizens need to understand these principles if they are to engage effectively in political debates and if they are to understand arguments and evidence in the media. If students are to become responsible citizens, they need to be exposed to these ideas in the curriculum. This implies (at least) that their experiences should include:
• Exposure to multivariate problems:
- Exposure to non-linear relationships between variables;
- Working with time lags between causes and effects;
- Seeing different effect sizes;
• Analysis of possible causal links between variables, and an understanding of moderator variables;
• Linking data analysis to the situations from which the data were derived;
• Speculation about possible courses of actions beyond the ones considered.
Such changes may well be uncomfortable for many teachers of mathematics, who are attracted by the certainties that pure mathematics can provide (because of the tautological nature of results that derive from reasoning from premises, and manipulating systems of rules). In mathematics past attempts to use ICT to support mathematical modelling at the school level have been seriously flawed. They have emphasised the teaching of mathematical formalisms that have been applied successfully in particular domains (such as Newtonian mechanics), but have failed to address the process of modelling seriously - either in terms of its philosophical underpinnings, or in terms of appropriate pedagogy.
Digital technologies can make a number of important contributions to mathematics, to pedagogic practice and (we believe) to cognitive development, student empowerment, and richer public debate on important social issues. Here, we will:
• Demonstrate the paucity of the statistics curriculum in the UK;
• Describe some of the key ideas that should be core elements in any attempt to model non-trivial phenomena;
• Show some of our interfaces for presenting multivariate data (extending to 5 variables);
• Provide empirical evidence that students can, indeed, reason effectively about multivariate data where relationships are non-linear;
• Show our early explorations of a hierarchy of 'reasoning from evidence'.
Two strands of work will be described. The first presents evidence that students can work effectively with multivariate data when they are supported appropriately with good computer interfaces. A study will be described where 195 students aged 12 to 15 years were presented with computer based tasks that require reasoning with multivariate data, together with paper based tasks from a well established scale of statistical literacy. All the tasks fitted well onto a single Rasch scale; computer tasks were cognitively more complex, but were ranked as being only slightly more difficult than paper tasks on the Rasch scale. Current work developing measures of students' 476 ability to 'reason from evidence' will be described. Several distinct levels of reasoning are evident in student responses, associated with comprehending; manipulating; and drawing conclusions. Competence ranges from working with single values, one step computation, and elementary reasoning, through to fluency using a variety of representations, fluency with number, and in synthesising evidence and communicating results clearly. We will show examples of computer-based tasks, student work, the Rasch scale, and will describe the development of a short unidimensional scale.
Second, current work with teachers of mathematics, citizenship, and geography will be reported. Here, we are creating curriculum materials designed to develop an understanding of complex issues in the curriculum, and reasoning from evidence in general. Examples of the use of complex survey data on sexually transmitted diseases, which students explore using powerful interfaces, will be shown to illustrate our approach.
Implications for assessment, the curriculum, and public presentations of complex evidence will be discussed.
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On the role and aim of digital technologies for mathematical learning: experiences and reflections derived from the implementation of computational technologies in Mexican mathematics classrooms
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In this paper we reflect, based on the Mexican experience of massive implementation of digital technologies in "real-world" mathematics classrooms, on the role and aim of these tools for mathematical learning. The experience in our country has yielded inconsistent results and the main aim of improved mathematical learning appears to not have been achieved. There have been some positive results (e.g. students' better attitudes and increase of enthusiasm, of motivation, of class participation; the possibility of formulating and proving conjectures and of analysing particular cases that can lead to generalizations) but many factors not present in laboratory settings come into play (from teachers' abilities to administrative difficulties), when attempting a massive implementation such as this one, "out in the real world". Furthermore, the experience has led us to readdress certain questions: a) What is it that students are learning when using new technological tools?; b) What kind of mathematics skills are they actually developing; c) What mathematics do we actually want students to learn with these technologies?; d) Can we put together the learning that does or can take place with the use of these tools, with the learning of what we usually consider as formal basic mathematical knowledge?
For decades, many research studies have investigated the various possibilities that new technologies could offer for improving the teaching and learning of mathematics at different levels (e.g. Balacheff & Kaput, 1996; Hoyles & Sutherland, 1989; Dettori et al., 2001; Mariotti, 2005). These have implied that there are certain ways of using technology that can help students in their learning of mathematics. For example, (although it is not our purpose in this paper to synthesize all the available findings) among the possible results are that:
• Technological tools may offer students a means: to learn to formulate and test hypotheses; to create and experiment with mathematic models; to work within different representational registers; to develop problem-solving abilities (all of which can lead to a better understanding of mathematical concepts).
• Technology can also provide immediate (non-personal) feedback that allows students to discover their mistakes, analyse them and correct them; in this way, errors become a means to assist learning.
• The tendency to work on routine problems in an algorithmic way decreases, while students can focus more on problem-solving activities. Thus, with technology, school mathematics can cease to be a simple mechanisation of procedures and instead become a space for reflection and development of concepts. 478
These encouraging results, derived from serious research in different parts of the world, in experimental settings of various carefully designed computational environments, seemed to be enough guarantee to insure that an adequate use of technology for assisting the teaching of mathematics, could produce satisfactory results and help improve students learning, in a large-scale implementation. As we will discuss in this paper, the step from laboratory setting to a large-scale implementation is far from being straightforward.
On the other hand, the positive results from the use of technological tools in experimental settings have led, first, to an over generalisation of their possible benefits that has spawn campaigns

promoting the use of digital technologies in classrooms without consideration of the pedagogical design, akin to what happened to the Logo programming language when Papert's (1980) Mindstorms book came out and many schools took up Logo without taking into account what to do with it.
Second, and perhaps more worryingly, now there seems to be a tendency of replacing formal mathematics with tool-based approaches that seek to make mathematics more accessible, but where the real mathematics is hidden from the user and thus may only develop a superficial mathematics understanding. In the following sections, we begin by recapitulating the Mexican experience of large-scale implementations of digital technologies in classroom, and end with a reflection on the results, difficulties and a more general concern of the tendencies for the use of those technologies in schools.
The Mexican experience
In Mexico, since the 1980's, the Mathematics Education community has been developing research on the use of computational and other new technologies in education and the Mexican government has tried to address this issue. In 1989, a government initiative called MicroSep shipped specially-built computers to schools, pre-loaded with different tutorials, Basic, Logo, etc. The problem was that it was a very naïve initiative: no training was given to teachers and it was an era when teachers feared computers enormously, both in how to use them but also in that the machine would eventually replace the human teacher. The outcome of the MicroSep experience was that most computers remained unused and the project was a failure (with the consequence that some of the software that came with the machines was also considered a failure, such as Logo). In the subsequent years, there were smaller initiatives to introduce computers to schools, all without much success.
In 1997, the Mexican Ministry of Education began sponsoring a, still-ongoing today, national project called EMAT (Teaching Mathematics with Technology), that has been led by a group of researchers in Mathematics Education, and which aimed at incorporating computational technologies to the mathematical curriculum of secondary schools (children aged 12 to 15 years old). Specifically, the EMAT project aims at promoting the use of new technologies using a constructionist approach to

Currently, in Mexico, there is a large government-sponsored campaign, that includes many advertisements in radio and television, claiming that computers (without any reference to the way they are used) improve children's learning. 479
enrich and improve the current teaching and learning of mathematics. A study carried out in Mexico and England (Rojano et al., 1996) involving mathematical practices in the Science classes, revealed that in Mexico few students are able to close the gap between the formal treatment of the curricular topics and their possible applications. This suggested that it is necessary to replace or complement the traditional formal approach, with a "bottom-up" approach capable of fostering the students' explorative,

manipulative and communication skills. Some of the fundamental ideas characterising the project, as described in the official documents (Ursini & Rojano, 2000), are the following:
• A use of computer software or technological tools (e.g. calculators) that
a) makes it possible to deal with mathematical concepts in a phenomenological way;
b) provides objects or representations of (mathematical) objects that can be directly manipulated;
c) is related with a specific area of school mathematics (arithmetic, algebra, geometry, probability, etc.)
• Specializes the users of the technology (teachers and students) in one or more pieces of software and/or tools so they become proficient in its use and are able to apply it for the teaching and learning of specific curricular topics.
• Puts into practice a collaborative model of learning: students work in pairs with one computer, thus promoting discussions and the exchange of ideas.
• Incorporates a pedagogical model where the teacher's role is that of promoting the exchange of ideas and collective discussion; at the same time, acting as mediator between the students and the technological tools (the computational environment), aiding the students in their work with the class activities and sharing with them the same expressive medium (tool).
The first phase of the EMAT project began in 1997 with a pilot phase during which the technology-based educational models were put to trial in secondary schools using relevant results from previous computer-based educational studies carried out in different countries. The project was designed with utmost care drawing from the expertise of the international and national scientific advisors, and was set up in stages so that adequate assessment and corrections in the implementation could be made before massive expansion. The stages of the project were also meant to allow for an adequate training of human resources, which is perhaps where the most emphasis in the development of the project should be placed. Nearly 90 teachers and 10000 students at the secondary school level participated in the project in the first three years. Since 2000, hundreds of schools across the country have incorporated the EMAT materials into their programmes, and we have discovered that many more are doing so in an unofficial way. Recently, a curriculum reform aims to officially incorporate technological tools into math and science education. 480
A main criterion for the choice of the software and tools used in the project (Ursini & Rojano, 2000) was to have open tools; that is, where the user can be in control and has the power of deciding how to use the software. This allows for the construction of learning environments where students are able to decide on how to proceed, as opposed to other types of computer software that direct the student and the activity. These open tools have to be flexible enough so that they can be used with different didactical objectives, such as those of the activities designed for the project. The technological tools and software used during the pilot phase were Spreadsheets (Excel), Cabri-Géomètre, SimcCalc-MathWorlds, Stella, the TI-92 algebraic calculator, and the Logo programming language, all aiming at covering curricular topics such as arithmetic, pre-algebra, algebra, geometry, variation and modelling. For each tool, activities and worksheets were developed by national experts, in collaboration with external international advisors. The calculators and the spreadsheets, and later the Logo programming activities, were easily incorporated. Cabri-Géomètre has also been well received, despite some difficulties due in part to the lack of preparation of teachers in the area of Geometry as well as licence problems. These tools have continued to be incorporated in the expansion phase of the project, but Simcalc and Stella were dropped because it was hard to fit these tools into the curriculum without more teacher-training that was hard to achieve due to administrative reasons.
Some results from the EMAT project and the problem of large-scale implementations
The pilot phase of EMAT, despite some difficulties had a positive impact (see Sacristán & Ursini, 2001). That phase was groundbreaking in changing the role of the teacher and the traditional passive attitude of children. It created an irreversible change allowing for technologies to be incorporated into the Mexican school culture, hopefully in an adequate way.
On the other hand, what became apparent since that pilot phase is that factors not present in laboratory settings come into play, when implementing a project such as this one, "out in the real world". The more outstanding issues have been: lack of adequate mathematical preparation on the part of the teachers; lack of experience working with technology by both teachers and students; difficulties in adapting to the proposed pedagogical model; teachers' lack of free time to prepare anything outside the established curriculum (all of these factors have contributed in making the activities much more directed than originally planned); lack of adequate follow-up teacher training and support because of administrative issues; many other bureaucratic difficulties; and lack of communication between the different levels of authorities.
Since the year 2001, we have tried to assess through different studies whether there have been improvements in the mathematical learning of the children using the technological tools. We have not been too surprised to find inconsistent results. One study (Sacristán, 2005), has correlated students' results in multiple-choice 481 mathematics tests, with teachers' performance (e.g. understanding and employ of the proposed pedagogical model and of the materials) as well as teachers' attitudes towards the technological tools. Putting it bluntly, "good teachers" achieve good results: they are able to take advantage of the technological tools and their students benefit from those experiences; but less experienced, poorly trained teachers, or simply teachers who dislike the technological tools, do not do so well. In the EMAT proposed pedagogical model, the role of the teacher is considered very important, as it is she/he who has the job of making students aware of the mathematics they are exploring with the tools (otherwise the knowledge constructed remains "situated" within the technological context); as Clements (2002, p.165) put it: "children do not appreciate the mathematics in Logo [or technology-based] work unless teachers help them see the work mathematically"; but one thing is the theory and another the practice…
By and large – from several studies that have used mathematics tests to compare EMAT students (i.e. students who are using computational tools) with non-EMAT students— the tested large populations of students who have been using the technology-based tools have NOT shown much benefit, if any, from that use, some groups even do slightly worse (Ursini et al., 2005).
We are aware that many factors come into play: not only the teacher's role, but also the amount of use of the tools (which we have detected is also very inconsistent), and most likely, many other factors; on the other hand, the benefit of the use of the tools is perhaps not so much in the development of specific knowledge-content but in the development of mathematical abilities that the instruments used in the studies described above do not measure.
But the crux of the problem is this: the EMAT programme was designed to improve the learning of curriculum-based mathematics. Yet, in curriculum mathematics tests, the use of the tools doesn't show benefits when the populations tested are part of large-scale implementations were many factors are beyond the control of the Project designers or main instructors. This leads us to ask ourselves the following questions, some of which we discuss in the next section: a) What is it that children are learning when using these tools?; b) What kind of mathematics skills are they actually developing; c) What mathematics do we actually want children to learn with these technologies?; d) Can we put together the learning that does or can take place with the use of the technological tools, with the learning of what we usually consider as classic basic mathematics?
We must, however, also give the positive results of the EMAT experience. In general, the use of the computational tools has had a very strong positive impact on children's attitudes towards mathematics. A study developed with 24 teachers and 1113 students (Ursini et al., 2004) shows that there is a clear increase in their enthusiasm and motivation; and although the impact is different for girls and for boys, the behavioral changes observed seem to lead to more gender equity. 482
Another positive results is the changes in classroom dynamics that have modified the traditionally passive attitudes of children and empowered them, giving them a status almost equal (and sometimes even higher than the teacher) when involved in mathematical explorations with the tools.
What is it all about? Some reflections and words of warning
We would like here to go a bit beyond the EMAT experience and reflect on some of the questions we posed before, as well as raise others:
First, what mathematics do we actually want children to learn with these technologies? If we want children to learn classical school mathematics, it is not so straightforward because, as some evidence shows, it doesn't always work. The failure could be attributed to an approach of simply adding technology to teach the same mathematics; but projects such as EMAT have tried NOT to do that, but to actually introduce technology as a means to explore mathematical ideas and concepts and enrich the current curriculum. But they still don't work. If we want something else from the use of the technological tools, then that is what needs to be made explicit and evaluated. Introducing technologies to enrich an actual curriculum "out in the real world", no matter how well thought out the implementation may be, is perhaps a contradiction. The use of new technologies seems to require making fundamental changes in the theoretical and pedagogical conception of the curricular structure and contents.
What is it that children are actually learning when using these new technological tools?
Outside experimental approaches, there are hardly any large-scale implementation studies or even theoretical research approaches for doing so (in great part due to the difficulties in evaluating such massive results of the kind of new knowledge that is being generated) that can give us data on what students actually learn in technologyassisted environments. In Mexico, we are attempting to do this kind of large-scale research for the EMAT programme, as discussed earlier in this paper. Some of the initial results that we can report can be useful but also depend on many factors. First, we can say that what is learned is extremely dependent on the specific technological tool being used, how the implementation is conceived, the mathematical knowledge of the teacher and his/her ability to link and make explicit the knowledge developed and situated within the technological environment, and formal mathematics. Nevertheless, we do know that the use of technological tools does develop motivation, a more positive attitude towards mathematics, an increase in student participation, in student abilities to defend their ideas; that the technology-based environments allow students to generate and test conjectures and to go from the particular to the general.
Therefore, if what students are learning is to develop certain abilities, then the use and implementation of the technological tools should not be conceived as an aid to improve current mathematics learning, but simply to develop those abilities that 483 underlie mathematical knowledge. In fact, that is what a tool like the (classic) Logo programming language does.
But, where are we going?
We would like to address several tendencies in the use and implementations of digital technologies for mathematics education that worry us. First, there is the idea that with technological tools, mathematics can be made more accessible through "new infrastructures" for the great majority. For example, there is a tendency to present students to representations of (often advanced) mathematical ideas, sometimes called by many "microworlds" – though contradicting Papert's vision of that term (since in this case the user doesn't create: s/he only explores)— but that we would simply call 'models' or 'interactive software structures' (such as applets or other closed structures over more open platforms). This tendency is further encouraged as officials and educators try to accommodate to the fast changing pace of technologies. The aim of these structures is for students to explore ideas that may be too difficult for them if presented in purely mathematical terms; the problem is that more and more often the actual mathematical concepts that create those models are not transparent or open for the user to see. While these tools may be useful for building intuitions, it is very questionable as to what actual mathematical knowledge students can derive or understand. Thus, while we may be thinking that we are making hard mathematics accessible to all, what we may really be doing is training people to use tools blindly while the mathematical design is not accessible. Even in EMAT, that had as theoretical guideline the use of open tools for students to create and explore, we find many activities that follow the above tendency. The problem with the above tendency is: who will finally benefit from this? It seems to us that deeper formal mathematics knowledge is in jeopardy of being reserved for a small elite. Some would argue that a new mathematical paradigm, based on the new infrastructures, may emerge; but even in that case we are in danger of creating elites of those with access to the technological tools and leaving out large sectors of the population that do not have access to those tools.
Related to the use of these models or structures, we are also worried about the tendency to rely only on the "point-and-click" method and the use of icons. Fifteen years ago, many in the mathematics education community called for the incorporation of more visual elements into mathematics teaching to complement the dominating algebraic/symbolic approach of mathematics, and researched how the use of digital tools could help in this endeavour. Today, some of us are worried that the use of symbolic and algebraic expressions is taking a back step in favour of easier, but perhaps less formal, forms of expression. The main question rises again: what will be the consequences of doing this and who will finally benefit from it?
Finally, another tendency is to want to use "state-of-the-art" technological tools: always the latest, fastest, most advanced, etc… This doesn't allow for creating a stable use of a technology; furthermore research of any particular technology thus soon becomes obsolete, so with this tendency, whatever is being used is relatively 484 untested (except perhaps for general implementation philosophies). True, we should acknowledge and research new tools, but this shouldn't imply abandoning technologies that have proven to have the potential to be beneficial.
So, why are we actually teaching mathematics and what happens to formal mathematics?
New technologies should always serve the learning and teaching of mathematics and not the other way around. Let us not forget that there is some basic mathematical knowledge that children should still learn. After all, mathematics is part of culture and not only a tool for solving problems. We thus would warn against loosing sight that what we want is for students to learn mathematics and not just that kind of implicit mathematics that may remain situated within a technological context. What, of course would be ideal is to make a use of the technologies that could allow to make the formal mathematics (e.g. algebra) accessible for all. Is this possible? How

should we use the technology to reach this goal?
What Seymour Papert and his colleagues had in mind when they developed the Logo programming language, had this potential as there was an emphasis on symbolic descriptions that were truly mathematical. We would like to end by reflecting on this valuable tool.
Logo has probably been researched perhaps more than any other and proven its benefits under appropriate conditions. Yet, we believe the potential of this tool was never adequately developed and now, sadly, in many places, particularly in the Western world, Logo has been abandoned because it is considered old or even obsolete, and has been replaced by other tools. But what Logo provides is not as easily found in other tools: to begin with it provides an invaluable tool for symbolic expression and for symbolically describing geometric figures. The benefits of Logo are well-known and it is not our intent to review them here. What we do want to claim, is that Logo – even when we refer to what we call "classic Logo" and not second generation Logo-based environments – is far from obsolete and is still an invaluable tool. Classic Logo activities were incorporated in the fourth year of the EMAT project. Most students who have used Logo in EMAT, claim it is their favourite tool; furthermore, there are instances where the use of the other tools has been enriched by the Logo experience: children demand a different use of Spreadsheets or Cabri, where they feel they can program these tools, like they do in Logo; they begin using the tools more according to their own needs and projects, rather than simply following preset activities. All of this from a tool that many consider outdated. The purpose of this reflection on Logo is a call to attention to look back on the valuable pedagogical lessons that we learned from that tool and that we can still learn, in particular in terms of a use of technologies that fosters the development of true mathematical knowledge.
Summary
We have attempted, in this paper, to reflect, based on the Mexican experience of massive implementation of digital technologies in "real-world" mathematics 485 classrooms, on the role and aim of technological tools for mathematical learning. The experience in our country has yielded inconsistent results leading to the necessity of making a deeper reflection of some questions. We have considered some of these here, but our main aim is to provoke reflection for further discussion during the study conference.
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Assessment within computer algebra rich learning environments
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This paper considers the design of learning environments and curricular, with a particular focus on automatic assessment. Mainstream computer algebra systems (CAS) are currently being used to support assessment, particularly in higher education. For example, the CAS can establish the algebraic equivalence of the student's and teacher's answers. This application of CAS is quite different from the traditional use, which is to model in an exploratory manner rather advanced mathematical ideas. While mainstream CAS have been used successfully for this application for the last five years, this paper examines the affordances and constraints of using CAS in this way. By using CAS-supported assessment it is possible to use open-ended questions which are traditionally difficult to assess, but which the educational literature suggests can be pedagogically valuable. In assessing an answer to such a question the CAS is used to establish various mathematical properties of the student's response. To focus this paper we concentrate exclusively on the assessment function, while bearing in mind the place of assessment in learning cycles and online learning environments.
Introduction
Mathematically rich computer based learning environments are increasingly being used at all levels of mathematics learning to support various functions in the learning cycle. It is the design of learning environments and curricular, with a particular focus on learning and assessing mathematics with and through technologies which is the issue addressed here. This paper is a discussion of some affordances and constraints of using computer algebra systems (CAS) to underpin such assessment. This a relatively novel use of CAS, an understanding of which is important given the crucial place of assessment in any learning cycle, and its role as the primary driver of many student's learning.
As an illustration, consider the situation in which a student enters his or her response to a mathematical question into a computer aided assessment (CAA) sub-system of a learning environment. The CAA system then uses a CAS to subtract the student's response from the teacher's response and to simplify the resulting expression algebraically. If the result is zero an algebraic equivalence between the student's answer and the teacher's answer has been established. Note the important pedagogic principle being implemented: the CAA system evaluates the student's answer which contains mathematical content, rather a selection from a list of teacher provided answers, such as in multiple choice or multiple response questions. A mathematical property of the student's answer has been established, with algebraic equivalence as 488 the prototype. Appropriate action, such as providing feedback, assigning a mark and storing these outcomes in a database, can then be taken.
To focus this paper we concentrate exclusively on the assessment function, but keep firmly in mind the place of assessment in the learning cycle and as an integral part of a coherently designed online learning environment.
Approaches
Roles of different digital technology
Until around 2000, the use to which CAS had been put in the learning and teaching of mathematics was "almost exclusively [...] to model in an exploratory manner rather advanced mathematical ideas", (Hoyles and Lagrange, 2005). However, during the last five years a community of practice has developed for automatic assessment of mathematics which makes significant use of CAS for the following.
1. Random generation of structured problems, (see Section 2.2)
2. generation of feedback,
3. to establish mathematical properties of the student's answer.
One of the most commonly cited advantages of CAA is the ability to generate feedback almost instantly. A CAS can be used to manipulate the answer of the student, and calculations based upon this can be incorporated into this feedback. For example, if a student makes an incorrect attempt at an integration problem, feedback of the following type might be given.
The derivative of your answer should be equal to the function which you were asked to integrate. However, the derivative of your answer with respect to x is ..., so you must have done something wrong!
Here, the ... is automatically replaced by the derivative of the student's answer, and such feedback encourages the student to check the result for themselves by differentiating. The ability to do this requires CAS tools within the assessment system.
The third function CAS enables within a CAA system is establishing a range of mathematical properties of a student's response. Algebraic equivalence with a correct answer is the prototype property, however algebraic equivalence with expressions arising from a range of mistakes common to the particular question can also be established. In this case feedback can be given, all of which may take place in the context of random question versions.
There are many other properties which we might look for in a student's answer, including whether an expression is factored, "simplified" or perhaps a solution to a given equation. In establishing these properties a CAS manipulates a student's expression, and as a further illustration, we consider how a CAS might establish if a student has found the general solution to a differential equation such as
y''(t) - 9y'(t) + 18y(t) = 0. (1) 489
First the CAS substitutes the response of the student into the left hand side of the equation and simplifies, which includes performing the differentiation of the student's expression where necessary. If the result of this calculation is zero, the property of primary importance has been established: the student's answer satisfies the differential equation. Other tests can be devised to ensure the expression is a general solution. In particular, that the answer consists of the superposition of two linearly independent solutions, and the presence of general constants, can be established. However, the choice of which letters used to express the general constants can be at the discretion of the student. The CAA system does not use a CAS to simply establish the algebraic equivalence of the student's answer with an expression such as
Ae
3t
+Be
6t
.
While solving (1) is a relatively standard problem, the use of a CAS allows the teacher to set and assess questions which would require significant computation to establish the required properties, or have non-unique or complex solutions. We examine such questions in more detail in Section 2.2 below.
The first system to make CAS a central feature was the AIM system (Klai et al. 2000). This system operates using Maple, as does the Wallis system of Mavrikis and Maciocia, (2003), and Maple's own proprietary MapleTA. Other systems have access to a different CAS, such as CalMath which uses Mathematica, CABLE, (Naismith and Sangwin, 2004), which uses Axiom and the STACK system (Sangwin and Grove, 2006), of the author, which uses the CAS Maxima, see www.stack.bham.ac.uk. Private correspondence indicates that Derive is being used in a similar way.
A common feature of these systems is their use of an existing CAS. There are significant differences between CAS implementations, which have been discussed elsewhere, for example Grabmeir et al., (2003), or Wester, (1999). These and other comparisons are from the point of view of the research mathematician or student essentially using CAS as a "super calculator". The functionality required for the application of CAA is quite different. We give concrete mathematical examples to illustrate some of the issues involved.
The problem of recognizing that an expression entered by a student is factorized (over some field), is significantly more subtle than comparing the student's expression with the result of applying the CAS's "factor" command to the teacher's answer. For example, a CAA system may have to respond to any of the following expressions
(x-3)
2
, (3-x)
2
, (x-3)(x-3), (3-x)(3-x), 9(1-x/3)
2
.
Only the first of these is returned by the "factor" command, while the others could all be argued to be correct factored forms. Similar problems occur with other syntactic forms, such as partial fractions. To be useful as part of a CAA system, functions which establish properties, either syntactic or semantic, are needed. 490
The reader might consider all the different senses in which the word "simplify" is used in an average textbook on elementary algebra. Often "simplify" seems to be a synonym for "do what I have just shown you". Two different examples occurs with what Nicaud et al., (2004) terms sorted and reduced form, when a polynomial is represented as x
2
+2x+1 rather than x+1+x+x
2
. If a CAA system is to provide useful

feedback to students it must be capable of distinguishing between expressions which are not fully simplified in various senses, and respond. However, such functions are usually not present in a CAS designed for computation and subsequent automatic simplification to canonical forms. Indeed, most CAS perform automatic simplifications which cause technical problems for CAA, and do not currently provide the level of fine grained control necessary for this application at the most elementary levels. Despite the fact that CAS-supported CAA is now commonly used in higher education, specifying the characteristics required of a CAS for CAA is a substantial project which has yet to be undertaken. It will require close collaboration between mathematics educators, research mathematicians and computer scientists. This is an important issue in the design of learning environments.
It should be noted that a CAS is not required for the three functions described in this section. Indeed, there are very many examples of highly mathematical CAA and computer based learning systems which accept and respond to student's answers without using a CAS. However, the authors of such CAA systems often replicate libraries of CAS-like functions, which represent and manipulate mathematical expressions. Hence, while they do not make use of a recognized mainstream CAS we would argue that they are in fact implementing computer algebra in its broadest sense. Examples of such systems are the CALM system of Ashton et al., (2005), the Aplusix system of Nicaud et al., (2004) and the Metric system of Ramsden, (2004). The issues raised here are just as relevant to these, and similar, systems.
Contribution to learning
Mathematically rich CAS functions are ideally suited to generating random versions of a particular problem within carefully structured question spaces. Worked solutions, with various steps, can similarly be constructed from templates. Such problems can be used for repeated practice or to reduce plagiarism. Indeed, so far CAS supported CAA has predominantly been used to provide traditional practice of routine techniques. Since the systems cited above originated in higher education they have also seen application to questions from linear algebra, vector calculus and differential equations.
It might be argued that since the CAS can perform simplifications and other calculations, the student should not be required to do so fluently themselves. Even if fluency in the actual calculations is not necessary, basic competence will always be and so practicing routine manipulations will remain a valid application.
However, by harnessing CAA within a learning cycle group work can be encouraged to aid understanding of a topic, with each student evidencing their own learning by completing their unique set of tasks. Often no one cares about the actual answer 491 itself, and the numbers used in a typical mathematics questions are themselves unimportant. Hence, the ability to randomly generate questions within constrained variation may be used to help students perceive the structure of the problem underlying that which their version represents. This embeds the assessment into the learning cycle, ensuring it is integral to the experience of the student.
Given that CAS enabled CAA establishes properties, rather then simply checks for "the correct answer", more open-ended questions can be set and assessed. As an example, consider the following question. "Give an example of a function with a stationary point at x=1." To assess this, the CAS differentiates the student's answer with respect to x, substitutes x=1 simplifies. Hence there is an infinite family of correct responses and as one student commented, in an anonymous feedback questionnaire to a course taught by the author, after answering this question using the AIM CAA system:
Recognising [...] the functions produced in question 2 was impressive, as there are a lot of functions [...] and it would be difficult to simply input all possibilities to be recognised as answers.
More than one property can be requested, such as the following.
Give an example of a function with a stationary point at x=1 and which is continuous but not differentiable at x=0.
The CAS functions are used to establish whether the student's answer (eg |x|(x-2)) has each of the required properties.
In questions such as this the student must decide what properties are required, and then construct a mathematical object, such as a function, which satisfies them. The cognitive processes required are quite different from following or repeating a routine procedure given by the teacher. The pedagogic potential for this style of question is well documented in the educational literature, for example Watson and Mason, (2002) or Michener, (1978). The work of Dahlberg and Housman, (1997), suggests that it "might be beneficial to introduce students to new concepts by having them generate their own examples or having them decide whether teacher-provided candidates are examples or non-examples, before providing students examples and explanations".
Such questions are usually absent from contemporary teaching, probably because of the practical constraints of time under which teachers operate. Using CAS-enabled CAA to assess such questions is considered in, for example, Sangwin, (2005). Considering how CAS-enabled CAA supports learning in this way is an important issue of curriculum design, and of learning environment design to support this.
Role of the teacher
Using CAA forces the teacher to consider the properties required of an answer, and describe carefully what outcomes should result in advance of the assessment being set. For example, how should the system respond if a student enters 0.5 as part of an 492 expression? While this is five tenths, ie 1/2, the use of floating point numbers is often discouraged. Conversely, if a student approximates a rational number using a floating point number, such as 0.3333 for 1/3 the CAA system will not establish algebraic equivalence even if the student is essentially correct. Hence, it is no longer sufficient for a teacher to be able to recognize a correct or worthy response when a student provides one. They have to consider carefully the various possibilities and be able articulate what they require in CAA terms. The issues of how teachers author CAA materials in this new context is addressed in Sangwin and Grove, (2006), which argues that teachers are themselves "neglected learners".
The open ended problems described in Section 2.2 require a different classroom approach from the teacher. Work of the Standards Unit in the United Kingdom has piloted very similar activities with students using personal white boards, and sets of cards containing mathematical expressions. The freedom of expression which very open ended questions affords students requires the teacher to think on their feet when responding to students. While the CAA system can help with computations to establish properties, the explanations for various outcomes will still be given by the teacher.
When using CAA a teacher risks becoming detached from their students. Marking written work by hand, while immensely time consuming, gives an insight to a particular group of students' learning which CAA cannot currently replicate.
Impact on mathematics
Novel forms of assessment should be subject to close scrutiny from the mathematics community. However, this should be a comparison of the new with the strengths and weaknesses of traditional methods of assessment. Assessment drives learning, and any novel assessments will hence affect what is learned and how this learning takes place. Given the central role of assessment, the perception of what constitutes mathematics is likely to be altered by new assessment regimes. This is not the place to engage in epistemic argument about the nature of mathematics, other to acknowledge that we send out strong messages about what we, as a community, value by what we assess.
Detailed inspection by the author of high stakes public examinations in the United Kingdom, indicates that much of what is currently actually assessed are routine tasks rather than open-ended problems and these can be automated by CAS supported CAA. However, it is hard to envisage how such systems would ever be used to assess method or the quality of the type of argument one might expect in a proof. Hence, it is imperative that the community establish what is really valued in the mathematical work of students before adopting CAA to simply replace existing paper-based work.
In CAS supported CAA the student interacts with the system, both to enter a response which contains mathematical content, and to read mathematical expressions on the computer screen. Hence, one specific area where CAA is likely to have an impact on mathematics is in the area of notation and symbolism. When used as a sophisticated 493 calculator, all existing CAS allow a linear syntax for expression entry. Here a user has the ability to edit their input, look up the syntax in the online help and possibly experiment. In computer aided assessment the stakes are higher: the user is being evaluated on their input. By implication this evaluation mixes their ability to express themselves using the correct linear syntax and their ability to actually solve the problem in hand. The work of Ramsden and Sangwin, (2005) details significant variations between CAS as to what symbols mean. The inconsistencies between linear syntaxes have implications for CAA, particularly if used in high stakes assessments. Other methods, such as an "equation editor" or pen-based character recognition, are also in use for CAA, however they also suffer from many of the same fundamental problems.
Conclusion
It appears likely that automatic assessment will become increasingly important at various points in learning and teaching. To make such assessment mathematically rich it is necessary that these include a CAS, or CAS-like functionality. In doing this we can automate the assessment of pedagogically valid and rich questions which are traditionally impractical. This application is rather different from the traditional use of CAS as a "super calculator". This is an existing and rapidly expanding field but one in which many challenges, both pedagogic and technical, remain.
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Students' Development of Mathematical Practices Based on the Use of Computational Technologies
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We document and discuss themes and aspects of mathematical practice that appear relevant during the development and implementation of activities associated with a research project whose aim is to analyze students ways of reasoning that emerge in problem solving classes that promote the use of diverse computational tools. In particular, we focus on identifying (i) the types of questions and conjectures, including arguments to support them, that students exhibit as a result of using dynamic software, (ii) the students' construction of mathematical relationships that come out from examining dynamic geometric configurations that are formed by simple mathematical objects, and (iii) the curriculum changes that we need to think of in order to validate and promote mathematical practices that favor the use of digital technologies.
The availability of computational tools to represent and explore mathematical ideas has influenced notably the ways to generate or develop mathematical knowledge. In this context, the formulation of questions or problems, the problematique, and mathematical practices, including methods, used to investigate those questions are shaped and mediated by the use of those tools (Moreno & Santos, submitted). In school mathematics, the use of Computer Algebra Systems (CAS) or dynamic geometry software has increased the awareness for students to develop numeric, symbolic and geometric sense during the study of the discipline. Thus, it becomes important to document and contrast features and mathematical processes shown by students that emerge in problem solving environments that enhance the use of technology.
Professional worlds as well as society at large have a pragmatic relationships with computational tools: their legitimacy is mainly linked to their efficiency…The educational legitimacy of tools for mathematical work has thus both epistemic and pragmatic sources: tools must be helpful for producing results but their use must also support and promote mathematical learning and understanding (Artigue, 2005, p. 232).
What type of reasoning do students develop as a result of using distinct digital tools in their mathematical learning experiences? What type of mathematical representations of problems or mathematical objects do students construct during the

process of understanding mathematical ideas or solving mathematical problems?
How do students and teachers participate or interact during the learning and the development of instructional activities in educational scenarios that promote the use of technology? These are relevant questions that guide the development of an ongoing research project that aims at documenting the type of mathematical competences that students develop when they systematically use dynamic software, 496 Excel and algebraic calculators during their problem solving approaches. This project also aims at understanding how mathematical concepts and objects are transformed as they are given a new life through the use of digital tools.
The project began two years ago and we have worked directly with high school teachers who have been implementing series of mathematical tasks in their regular mathematical classes. Fundamental principles that helped structure and frame the project involve: (i) the recognition that students need to think of their mathematical learning as a problem solving activity in which contents, problems or phenomena are seen as dilemmas that need to be examined, explained, and solved in terms of formulating and pursuing questions or inquiry methods (Schoenfeld, 1994); (ii) the importance for students to think of distinct ways to represent, explore or solve mathematics problems. That is students need to develop distinct modes of thought and habits that reflect the proper use of mathematical resources and strategies (Cuoco, 1998); (iii) The relevance for students to use various computational tools to represent mathematical objects in order to identify, explore, and support mathematical conjectures or relationships. Here, we argue that different tools offer students different opportunities and ways to think of a problem or situation and as a consequence, it is important for them to use several tools to develop diverse problem solving approaches to solving those problems (Santos, et. al, in press).
At this stage, we have collected data regarding the process of designing the tasks that help student think and reason mathematically, ways in which the tasks were implemented within instructional scenarios favoring students' development of mathematical ideas, all with the mediation of digital technologies. Based on the analysis of the gathered data, some preliminary results have begun to emerge and we would like to focus on relevant aspects related to:
(i)The type of conjectures and arguments that students propose as a result of representing the problems dynamically using dynamic geometry software.
(ii)The process of using simple objects (segments, lines, triangles, perpendicular bisector, etc.) to ensemble a geometric configuration that becomes a platform to pose and pursue question that lead students to reconstruct or develop series of mathematical results or relationships.
(iii)The contents and curriculum transformations that seem relevant to make explicit when the use of the new mathematical instruments become relevant in mathematical instruction. As Hoyles and Noss (2003, p. 325), stated "…exploiting the real power of technology requires such innovative approaches, that comparison to a traditional class is inappropriate".
To elaborate on each of the above themes, we have chosen examples that illustrate students' ways of reasoning or thinking of mathematical problems that have emerged while using digital tools during their mathematical problem-solving experiences.

1. Formulation of Conjectures and their validation. 497
With the use of dynamic software, students can construct executable representations of mathematical objects and problems that reflect changes or invariants in different contexts. Thus, in general, students tend to construct dynamic representations of problems that lead them to quantify attributes (lengths of segments, perimeters, angles, areas, etc.) and observe their behaviors or loci as a result of dragging particular elements within that representation. Students also tend to examine the viability and pertinence of a particular conjecture in terms of using the software to (i) identify the conjecture visually and dynamically, (ii) examine whether the conjecture includes a structural relationship (dragging test), (iii) construct a macro that crystallizes the construction and verify whether the conjecture is held in objects generated by the macro, (iv) quantify and verify properties of mathematical objects to detect patterns, and (v) present formal arguments to prove the emerging conjecture.
An example illustrates the way students deal with a conjecture.
The problem: Cross's Theorem:
Squares are drawn on the three sides
of a triangle. Show that the areas of
the four shaded triangles are the
same (figure 1) (Faux, 2004). Fig.1: Are the triangles' shaded
areas the same?
Students represented the problem dynamically and provided distinct arguments to support it. They also had opportunity to examine a related conjecture: What about if we draw rectangles instead of squares on each side of the given triangle, how are the shaded areas? While representing dynamically figure in this question, students noticed that they could draw many rectangles taken as the base one side on the given triangle and as a consequence it was difficult to trace the area behavior of the shaded triangles. How can we relate the construction of all rectangles? Discussing this question led them to assume that the corresponding sides of the rectangles should share the same proportion. They decided for
EC
CB
=
GA
AC
=
IB
BA
=
1
2
.
They drew the rectangles holding
this condition and verified (using the
software tools at their disposal) that
the areas of triangles CEF, AGH and
BDI were the same (figure 2).
0.37 cm
2
0.37 cm
2
0.37 cm
2
1.49 cm
2
C
A B
E
D
H
I
G
F
Fig. 2: Drawing rectangles with
proportional sides
The software became a powerful tool to explore both the plausibility of the conjecture and the search for ways to validate it. We illustrate the ways students dealt with this conjecture: 498
Visuo-perceptual Recognition.
An important feature while using dynamic software is that mathematical figures can be drawn accurately. Students drew triangle ABC and the corresponding rectangles (with the same constant of proportionality among their sides) and concluded that the area of triangles CEF, AGH, and BDI were the same. The initial (perceptual) conjecture is supported with the corresponding areas calculation (figure 3).
Fig. 3: Visual recognition of a relationship in triangles CEF, AGH, and BDI
The Dragging Test.
Here students explored the validity of the conjecture for a family of triangles. They dragged the position of the vertices of the given triangle ABC to generate a family of triangles with the same construction, perceiving a structural relationship. They observed that when one vertex is moved, the family of triangles generated held that the area of triangles CEF, AGH, and BDI was the same (figure 4).
0.50 cm
2
0.50 cm
2
0.50 cm
2
2.01 cm
2
C
A B
E
D
H
I
G
F
0.61 cm
2
0.61 cm
2
0.61 cm
2
2.44 cm
2
C
A
B
E
D
H
I
G
F
Fig. 4: Verifying the conjecture for different positions of triangle ABC
Constructing a Macro.
Another way to explore and eventually verify the conjecture was that students built a macro to reproduce the construction for any given triangle. That is, students identified initial objects (triangle ABC and ratio R of rectangle sides) and final objects triangles CEF, AGH, and BDI. This procedure adds a formal status to the involved figures while trying to validate the conjecture. Applying the macro to different triangles, students confirmed the conjecture, that is, they verified that triangles CEF, AGH, and BDI all have the same area.
Quantifying attributes and Patterns.
In addition to observing the behavior of the triangles areas, students focused on comparing (ratios of) areas of triangle CEF and triangle ABC for distinct values of
0.54 cm
2
0.54 cm
2
0.54 cm
2
2.15 cm
2
C
A B
E
D
H
I
G
F499
the proportionality coefficient r of the sides of the rectangles (figure 6). Based on this information, they notice that
area of 'CEF
area of 'ABC
= r
2
for different values of r.
Formal Analytic Proof. To prove both conjectures, students followed different approaches that were discussed within the whole class. Here, we present an analytic proof that was constructed during the class discussion. The idea is to construct a triangle with one vertex on the origin of the Cartesian system and other on the X-axis and the third a point on the first quadrant. That is, the vertices of the triangle will be A (0, 0), B (c, 0) and C (a, b) as is shown in figure 5.
Fig. 5: Identifying coordinates of vertices of the given triangle
What is the area of triangles AGH, BID, CEF and ABC? Given the coordinates of the vertices of those triangles, students recalled that the area of each triangle will be:
2
2
1
0 1
1
0 0 1
2
1
( ) bcr

cr
Area AGH br ar =
!
' = ! (1);
2
2
1
1
1
0 1
2
1
( ) bcr
c br cr ar
c cr
c
Area BID =
+ !
' = ! (2) and
2
2
1
1
( ) 1
1
2
1
( ) bcr
a br b ar
a br b c a r
a b
Area CEF =
! +
' = + + ! (3), in
addition, Area ABC bc
2
1
( ) = (4).
Based on the information given in (1), (2), (3) and (4) they concluded that:
Area('AGH) = Area('BDI) = Area('CEF) =
2
Area('ABC ") r
Comments
There is evidence that the use of dynamic representations of the problems and the treatment of mathematical objects that students showed with the mediation of the software became an experience-enhancing platform to identify and explore mathematical relationships. The easiness to quantify mathematical attributes and the exploration of visual representations of problems were fundamental activities that permeated the students' process of formulation of conjectures. To assess the validity 500 of the conjecture, students relied on using the tool to explore particular cases visually, to examine family of cases by dragging particular elements within the representation (perceptually discovering structural relationships), to construct a macro to also examine a family of cases (a higher level of formalization), and to observe patterns that emerge as a result of exploring invariance in the behavior of particular data (recognition of a mathematical object). In this context, the use of an analytical method to prove the conjecture came out only as a way to confirm the validity of the conjecture.
A core idea is that the processes of formalization and abstraction are not best served - from a didactical viewpoint- when we conceive of abstraction as an extraction process. It might be better to view it as an additive process (Noss&Hoyles, 1996) in which it is important to connect, in terms of mathematical properties and meaning, distinct representations of the problem, including those generated with the use of the software. Thus, these representations are seemed as abstraction domains from which to build mathematical objects and relationships. Here is where the full power of the "dragging exploration," for instance, is revealed. In this process, students think deeper: situating mathematical results within a digital environment and generating new mathematical explorations. As a result of this activity, new conceptualizations of mathematical objects will emerge and will be important to frame mathematical practices in classrooms.
Construction of Relationships
Another type of activities that students explored with the use of dynamic software is the construction of geometric configuration by using simple mathematical objects. For example, in one of the problem-solving session, a student constructed a dynamic configuration that involves perpendicular lines. That is, line L2 is perpendicular to line L1; line L3 is perpendicular to L2 and line L4 is perpendicular to line L1 (figure 6).
L1
L2
L
3
L
4
P
R
Q
S
Fig. 6: Constructing an initial geometric configuration
L1
L2
L3
L4
L5
P
R
Q
T
U
S
S
Fig. 7: Drawing the perpendicular bisector of segment QR
Later, he drew the perpendicular bisector of segment QR that intersects lines L1, L4 and L3 at S, T, and U respectively (Figure 7). 501
Here, the student moved objects within the representation and observed path or invariants left by other objects. When he moved point P along line L1, the path left by point S appears in figure 8
L1
L2
L3
L4
L5
P
R
Q
T
U
S
S
Fig. 8: What is the locus of point S when point P is moved along line L1?
At this stage, the student's goal was to provide an argument to show that the locus actually represents a parabola. It is easy to show that point Q is the focus and L3 is the directrix of this parabola. This became an interesting scenario to discuss the classical definition of conics. Thus the dynamic environment mediates the construction of the definitions and concepts.
Comments
An important feature in constructing geometric configurations is that students do not have initially a well-established problem to be solved; rather they rely on simple objects (line, points, segments, perpendiculars) to build a platform for identifying and exploring mathematical relationships. In this case, it is observed that by moving a point along a line, a particular path left by other point can be visually identified as a parabola. However, it becomes important to provide an argument to actually show that the locus corresponds to that conic. Students used distinct arguments to explain and "prove" that, in this case, the locus was a parabola. They include verifying empirically, using the software, the definition of parabola and other geometric arguments. Dragging objects, observing behaviors, formulating conjectures, presenting arguments and communicating results seem to be important activities that students can practice systematically with the help of dynamic software in order to detect and explore mathematical properties. Mathematical objects are, now, evolving objects that students constantly explore in terms of using more refined and robust resources and strategies.
Curriculum Transformations
The use of digital tools seems to offer students the possibility of formulating and exploring mathematical conjectures from diverse perspectives. In this context, rather than thinking of a sequence of themes or contents to be included in a curriculum proposal, what it seems relevant is to structure the curriculum in terms of fundamental mathematical ideas and concepts dynamically conceived for students to solve problems and use in further studies. Thus, students need to develop mathematical resources, strategies and ways of thinking that are necessary to comprehend and apply mathematical ideas. It is evident and clear that as part of the scaffolding process to conceive of a technology oriented-curriculum, a new way to conceptualize mathematical objects and their study is needed. For instance, as formalization is relative to the medium in which it takes place, there is a need to 502 reflect on the new ways students have to prove mathematical assertions in the classroom. Take for instance, the Hilbert space-filling curve:

To prove it along classical lines can be an intricate task, but what happens when one turns the result into a digital one? For instance, to argue in favor of the validity of the theorem, we can translate it: given a (screen) resolution, there is a step in the recursive process that generates the curve, that fills that screen. This is a way to empower students in order for them to have access to deep results properly translated. As Schoenfeld (1994, p.76) stated, "proof is not a thing separable from mathematics…is an essential component of doing, communicating and recording mathematics", nevertheless the use of new digital media requires a fresh approach to this important mathematical and curricular topic. It is central for a new and needed epistemology.
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Using visual microworlds in teaching mathematics
of computation in primary and secondary schools
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The most rapidly growing branch of mathematics is mathematics of computation. Today it deals with modern information processing as well as with human reasoning and formal acting. Comparatively slow it appears in secondary school curricula. It is not evident that computers can effectively support learning in this field, but it happened that many key topics of it have been visualised (structural programming, parallel processing, interpretation of logical formulas, etc). The visualisation is of two kinds: 'one-to-one', where all objects and processes of mathematical reality are represented on a computer screen, with the only limitations being the object's size and process time, and 'specialized', where specific objects and processes are

represented, but this is enough to form general skills of students applicable for all problems of the topic. The major challenge here can be summarized in one word: 'integration'.
Changes in mathematics and changes in civilisation
Over the last century (starting in 1870s and even before) the content and methods of mathematics changed dramatically. One of the most important areas of these changes was connected with mathematical investigations of human thinking and acting. The first summit of the events was achieved in 1930-s in Gödel's completeness and incompleteness theorems and Church's thesis. Naturally, the primary results were on mathematical thinking (formal mathematical reasoning) and mathematical acting (execution of formal mathematical algorithms). They constituted, respectively interconnected branches of mathematical logic and theory of algorithms.
In the following 30 years many results of mathematical logic and theory of algorithms were implemented in hardware and software of the rapidly developing information industry. In 1930s it was hard to predict development of integration circuits, but the "coincidence" in time of ICT and the preceding mathematical events are astonishing. The precedence of mathematics appeared further. For example, the major concepts of structural programming (constructions, invariants, inductive proofs) were fully developed and used for rigorous proof of correctness of a compiler by Andrei Markov (Jr.) in the framework of his theory of algorithms based on string processing.
In the next years on the one hand the field of mathematics of computation based on mentioned mathematical logic and theory of algorithms became perhaps the most massive and rapid development. On the other hand computers became a powerful tool of visualisation and became personal.
We can say that machines of material and energy processing were implementations of continuous mathematics, as well as many processes in nature. Computers are 504 today implementations of discrete mathematics, as well as processes of human thinking and formal behaviour. We believe that these dramatic changes in mathematics and in the world should be resembled in education. As was indicated by many people and considered in details by Don Knuth, rigorous thinking on algorithms is mathematical and concerns abstract mathematical objects and methods of reasoning constitute a special type of mathematical thinking - the algorithmic one. There should be a shift from training how to execute routine algorithms that are much better executed by machines now to learning, on the one hand how to use these machines, and on the other hand, the mathematical basement for these machines' construction, operation and using.
The content of mathematics of computation on the secondary level
We believe that the major achievements of mathematics of computation constitute today an important part of the general culture. As an example we quote here the Russian standards for general school education (1-11) approved in 2004 with minor changes for standards that are in the process of development now.
General notions
Systems constituted by acting elements, states of elements, signals. Control, feedback, stability. Information, information objects of different types (symbols, numbers, sounds, images). Universality of discrete (digital, in particular, binary) presentation of information, accuracy of presentation. Quantity of information. Compression of information.
Process of information transmission, information transmitter and receiver, coding and decoding, distortion of information in the process of transmission, transmission speed.
Storing, transmitting, processing of information in social, biological, and technical systems. Information perception, memorising, processing and transmitting by living organisms and humans. Value of information.
Language as medium of presentation for information: natural and artificial languages, semantics. Formalisation of description and simulation of real objects and processes. Formalisation of problem. Computer simulation.
Mathematical concepts and their applications
Transforming of information by formal rules. Algorithms. Different ways to describe algorithms, flow-charts. Logical values, operations and expressions. Constructing algorithms using names, branching, cycles. Top-down analysis of a problem, using sub-algorithms. Objects of algorithmic processes: strings of symbols, binary numbers, lists, trees, graphs. Algorithms (the following list is flexible): Euclid's, conversion from binary to digital and vice versa, examples of sorting, search (for a winning strategy in the tree of a game).
Computable functions, formalization of the notion of computable function, its completeness. Complexity of computation. Complexity of information object. 505
Non-existence of algorithms. The problem of exhaustive search (P-NPproblem).
The quoted approach was a cornerstone in building up Russian curriculum on computer science and technology in the mid-1980s. The mathematics of computation was originally presented there as a non-computer activity. So, the course was introduced in all schools of the Soviet Union and was criticised for "teaching to ride a bicycle without a bicycle". That was evidently correct in the aspect of computer technology but was not so true in the aspect of development of algorithmic thinking. A little bit later a set of microworlds (see 'Visualisation' below) was developed for 8-bit, 32K or 64K memory computers.
Can computers help?
The content given previously can be changed in different traditions cultural and technological surroundings. At the same time, the core essence of this content is invariant and associated with the concepts of discrete objects and discrete processes. The very natural idea of computer application in learning and teaching the field of mathematics is that computers can represent any discrete object and discrete process. So, if a specific process deals, for example with numbers, the numbers can be stored and transformed by a computer. So, we can compare the result of implementation of our algorithm in the computer with our intention and verify our design. But the most important reason for using computers in learning mathematics of computation is visualisation.
Visualisation
Visualisation is considered as one of the most powerful processes in mathematical discovery and mathematical education. There is a long discussion on different styles of mathematical thinking, but in any style there is a space for visual presentation of objects and processes.
So, the background idea of visualisation in learning mathematics of computation is to find such specific environments, or micro-worlds, rich enough to present important features of algorithmic processes and algorithmic constructions.
The Turtle
One of the first and the most famous microworlds is the microworld of Turtle. The Turtle lives on a plane - a finite part of an infinite plane, or the same assembled as a torus, or a potentially infinite plane. The Turtle can act - move forward to a given distance and turn on a given angle. This environment today is extended to different directions and has different applications in primary and secondary education:
• A creative environment for children's self-expression and development through making texts, pictures, and animations;
• An environment for learning concepts of mathematics such as angle, polygon, approximation of a circle by a polygon, probability, etc. 506
• An environment, where a child can control the Turtle (primarily in its moves in the plane on the computer screen). The control can be direct and immediate: 'command - action' or, on the later stages, programming it.
The programming is done in Logo language. The language was originally developed by mathematicians and AI researchers from a military oriented company, BBN, and Seymour Papert, who joined them. Seymour's inspiration made the Logo community the strongest one in the field of advanced approaches of using computers in schools. Logo became associated with the Piagetian constructivist and Papert constructionist philosophy of education.
Turtle microworlds as well as programmed Drawer were used in Russian schools among other microworlds (the Robot, discussed later was the most important one).
Over the last decade a version of Logo (called Icon Logo, or LogoFirst) was developed in Russia. The main feature of it is that programming starts there without and before textual literacy. Programming itself (not execution of programs only) became visual: the primitives of the language are iconic.
Through visualisation Logo helps students to understand better such important essentials of algorithmic thinking as iteration, recursion, top-down analysis.
The robot and structural programming
The natural behaviour of the Turtle does not assume conditional branching. Of course, we can add to it some additional features like seeing a colour of plane underneath. But there is something more natural and more intriguing that involves conditionals. This is the microworld of Robot in the maze.
It needs some additional investigation to find out all origins of this creature. Some of these are Slovakia, formerly - part of Czechoslovakia - the native land of robots - fantastic creations of great Czech writer of the XX century Karel Capek (pronounced Cha;pek), Cornell University where Karel the Robot was used to teach students structural programming starting from the beginning of 80s, and Moscow state university, where the same happened approximately at the same time, and then was used in mid-80s to teach algorithmic thinking to 2.5 million Soviet high-school students.
Let us go to the essence. A maze is a rectangle surrounded by walls. It can also contain walls inside. The only limitation is: all walls are vertical or horizontal segments with integer ends. The Robot is a creature (or a machine) with four options of moving: by one unit up, down, right, left, and four senses: to see a wall immediately up, down, right, left. (This definition can be changed, for example we can expect the robot to turn to four directions and to see a wall in front of it.) A problem is usually formulated as: "program the Robot to move from one position to another". For example, the first position can be unknown a priory, and the second one can be upper-left corner. Even more important is that the maze can be also unknown beforehand, but some restriction on the wall's position can be given, for example: there arbitrary number of wall segments, but all should go in the vertical direction. These unknown parameters represent the major feature of algorithms and 507 algorithmic thinking: to invent a general instruction of behaviour that is valid for a possible infinite number of situations.
In the practical school context it is important that the programming, on the one hand is presented as text construction, like in Logo and 'adult' computer languages. On the other hand, the construction is done by using 'building blocks' in a structural way, not letter by letter. The result is also presented in a structured, visualised way. In the process of execution the executed command can be indicated on the screen.
Investigating different kinds of restrictions we can obtain a big variety of problems constituting a natural space of tasks of increasing difficulty up to the problems of proving impossibility (non-existing of algorithms). In this case the non-existence is not caused by the general limits of computability but by specific limitations of the computational model. One more dimension of variety here is the different primitives of programming, for example, numeric variables can be permitted or not.
Lacking the creative power of Logo, as an instrument of doing something relevant for the world outside mathematical education, the described microworld is a very important environment to learn the basic methodology of structural programming.
The builder and parallel programming
One more visual environment used in Russia for teaching algorithmic thinking is a microworld of building construction. The process of construction (to be programmed by a student) consists of putting together building blocks, and can done in parallel by different brigades. A problem is an assignment to build a construction in the shortest time with a given number of brigades.
This microworld is interesting enough and provides some space for variety of problems for secondary school. Its major limitation is connected with the fact that one program is being constructed for one building only, it is not assumed to work in multiple situations.
An IT extension: Controlling real moving objects
The natural extension of the visualisation idea can be called "materialisation". This means controlling and programming, not events inside a computer, not visible on a computer screen only, but happening in physical reality. For example, a "real" electro-mechanical turtle is a popular device in British primary and secondary schools.
An additional dimension comes with an opportunity of assembling the moving models from blocks of a construction set. Such construction sets are produced by LEGO and other companies.
Turing Machine
In the mathematics of computation traditionally more simple devices than abstract computers are considered such as Turing machines. We think that considering them could be helpful in the development of algorithmic thinking in connection with hardware - software relations, understanding more about complexity, specific syntactic algorithms, etc. At the same time, Turing machines in their standard 508 presentation are very far from being intuitive. A group in Stanford University developed a visualization of the machines. As they write in [1]:
'In Turing's World, a collection of graphical tools lets you design Turing machines by directly drawing their state diagrams. When you run a Turing machine in Turing's World, the operation of the machine is displayed graphically, both on the tape and in the state diagram window. On the tape, the read/write head moves, making the changes required by the machine you've designed. In the state diagram, the nodes and arcs highlight to show the changing state of the computation. Turing's World also allows students to display the text-based "4-tuple" description of their machines, though we have found that they rarely do.'
The Turing machine looks out of the scope of the secondary education. At the same time finite automata presented in a visual way on computer screen can naturally fit into mathematics of computation for secondary and even primary school. (There were several attempts to do this even without computer.) So, first - finite automata can be presented similarly to Turing machines, second - in the case of finite automata are in a course, it is easier for Turing machines to appear.
The Life Game
Several attempts were made to implement the microworld of the Life Game of Conway. It looks like this microworld can be used not only in mathematics of computation study, but also as an environment for student's investigation of a 'natural science' phenomena (in this case - phenomena of artificial nature). These phenomena resemble different phenomena of biology and can be productively investigated even on the primary school level. As in other cases it is helpful to start with pen and paper exercises and then pass to the computer environment.
Pre-computer experiences in algorithmic thinking

Algorithmic thinking and other elements of basic mathematics of computation can be developed a non-ICT context. Formally it is outside the subject of the study but we believe that some of the non-computer activities should be integrated into the learning process, for example:
• Students can count seeds in a jar and learn divide-and-conquer principles
• Students can sort LEGO bricks and design sorting algorithms to be implemented on computer
• Students can play games with complete information, construct winning strategies, possibly using tree of the game, etc.
Static microworlds of logic and mathematics of computation
Some problems concerning objects of mathematics of computation can be pretty sophisticated and importantly not being immediately associated with any process. For example, it can be a problem of finding the truth value of a formula, or constructing an object satisfying a given condition, etc. The objects can be visualised on the computer screen.
'Tarski's (Micro)World' was designed by the mentioned team from Stanford University. In this microworld as in some others (Robot etc.) a virtual interpretation 509 of some general concepts is given. Here the general notions of relation is interpreted via a handful of (quasi) spatial relations: 'to be a cube, 'to be a pyramid', 'to be small' 'solid (or 'block'), 'A is back of solid B', etc. It looks very limited, but the microworld happens to be rich enough to develop in its visual context the major skills and heuristics, associated with first-order logics. Visualisation here gives a model for a formula and used also in formula interactive analysis.
Let us mention also another product of Patric Suppes group at Stanford – it is Hyperproof. Being beyond secondary level it gives an interesting example of visualising syntactical structures and inferences.
A different example is given by the Informatica microworld, used in teaching basics of discrete mathematics including mathematics of computation in thousands of Russian schools for last decades. In this case the microworld contains and graphically presents on a computer screen such objects as strings, bags, trees and tables (of beads, symbols, digits, numbers etc.) as well as provides graphical tools to construct objects and to operate with them: add bags, or unite them (maximum operation), represent bag as a table, multiply to bags (as algebraic expression), concatenate strings, etc. This microworld is supported (or, in many cases, substituted) by 'penand-paper microworld' of graphical objects. In this case it is interesting that, as it happens in algebra with graphics or in dynamic geometry, the objects of a child's work can be always presented on the screen with the size of them being the only limitation. Next to this there are Tabletop and, especially, Tabletop Jr. of TERC – instrument for data and operations on data visualising.
Games
It would be interesting to construct microworlds for specific (possibly, to be invented) games (of two persons with perfect information), where, on the one hand dynamic strategy thinking is developing, and on the other hand, understanding of quantifiers as opponents' moves appears. A known example is the Nim game. Here is one more brilliant case of such a game from the field of recreational mathematics:
Let us have a round table and two players with infinite stock of congruent coins. They put coins in alternative moves, one by one, non-overlapping. The player that has made the last move wins.
The known winning strategy (to be given in the conference talk) proof needs discussion of termination of computational processes, invariants, and inductive reasoning as well as general mathematical concepts of symmetry.
In this field of recreational mathematics and Olympiads there are more problems in the field (not games only) that can be visualised and studied with computer.
Not covered yet
Naturally, not all topics of mathematics of computation are visualised or can be visualised at our level of understanding today. At the same time, we remember, that, for example, Euclid's algorithm was developed as a geometric one. So, experience in 510 visualisation of actions in dynamic geometries (Geometer's Sketchpad, etc.) can be used here.
Conclusions
As we can see, there are several virtual environments visualizing objects and processes of mathematics of computation including logic and algorithms on the potentially secondary and, even primary, level. Some of these environments represent some part of mathematical reality in a full form, others are of generic nature, where you see and work with some specialization of general notions (e.g. structural programming) but can develop general skills, needed in a general context.
The important interconnected tasks for the future work are:
• To redesign primitives and interfaces of the microworlds
• To implement the microworlds in the modern operating systems' coding, possibly, as open source Java, or Flash, etc. code, to cover mobile devices as well
• To provide some kind of unification over the microworld (in the interface details, terminology, etc.)
• To design specific topics and whole mathematics of computation part of the modern mathematical curriculum
• To integrate math of computation into mathematical curricula and into general education
• To reconsider the goals and content of mathematical education on the basis of ICT technology, mathematics of computation and modern society needs.
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Teaching with technology: complexity theory as a lens for reflecting on practice
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The main thesis of an article by Davis and Simmt (2003) is that "mathematics classes are adaptive and self-organizing complex systems". Several years ago I examined three technology experiences in light of that thesis, to determine how software, organization, and task impact the blossoming of a complex learning system in the lab-classroom. Since that time I have continued to think about the implications of these ideas in the broader context of implementing, and helping preservice teachers implement, technology in teaching mathematics. Complexity theory, as applied to education, is a relatively new theoretical framework; however, I believe that it is very appropriate. Based originally on biological models, it offers a perspective from which to examine multiple intertwining relationships, emergence of new ideas from seemingly insignificant events, the creation of unexpected connections, and the development of student understandings that are more than the sum of the parts. In this paper, in light of some ideas from complexity theory, I reflect on my own teaching experiences with technology, and the consequences for my research.
For us, perhaps the most important conclusion of complexity science is that, in any learning system, complex co-adaptive activity is always happening across several levels simultaneously. It is impossible to affect a part of the complex unity without affecting its global character, and vice versa." (Davis & Simmt, in press)
In addition to familiar examples such as anthills, brains, and cities (Johnson, 2001), Davis and Simmt (2003) argue that mathematics classrooms are instances of complex systems – self-organizing, self-maintaining, adaptive phenomena. Like living organisms they grow and develop, responding on many levels to even minor changes. Davis and Simmt identify five necessary (but insufficient) conditions for such a system to be able to learn – internal diversity, internal redundancy, distributed control, organized randomness (in recent work (Davis & Simmt, in press) "enabling constraints"), and neighbour interactions. Complexity theory says nothing about the quality of this learning; for example, a class may learn to be apathetic; but it does provide a perspective for analysing the complex entity that is the classroom.
I suggest that, in a similar way, complexity theory offers a powerful framework for analysing teaching with technology. I believe that it can help us explain some of the problems we face, particularly with regard to implementation. Research has shown that appropriate opportunities to use technology help students develop a deeper understanding of mathematical concepts (cf., Clements & 512 Sarama, 2002; Jones, Autumn 2002; Ruthven, 1999). As a result, school districts are providing support; for example, the Ministry of Education in Ontario, Canada has licensed Geometer's Sketchpad (GSP) and Fathom for student use, and has released new curriculum documents that mandate the use of technology in elementary and secondary mathematics. School boards in Ontario have responded with funds for inservice and the purchase of hardware and graphing calculators. Yet in many (perhaps most) Ontario math classes, technology is used infrequently or not at all. The case in other provinces in Canada – and in other countries is similar. For instance, Artigue (2000) notes that despite the active support of digital technologies by the Ministry of Education in France, integration at the secondary level has been minimal. She proposes that this is because: 1) computer technologies, though having strong scientific and social legitimacy, have poor educational legitimacy; 2) issues around the "computerisation" of mathematical knowledge have been underestimated; 3) opposition between technical and conceptual dimensions of mathematical activity (which is not new) has been affected by the introduction of technologies that make the technical aspects easier; and 4) and the "complexity of instrumentation processes", i.e., dealing with the impact (both mathematical and technical) of technological tools, has been underestimated. (p. 9) She says:
Various characteristics of the mathematical culture and various constraints act as obstacles to integration and the strategies spontaneously developed by the educational system are not necessarily the most adequate. A better understanding of the way these characteristics and constraints shape teaching and learning processes in technological environments and the way they mutually intertwine, is today more than ever a necessity for research. (Artigue, 2000, p.9)
Artigue's analysis indicates that introducing technology in (secondary) mathematics has been difficult because of the "intertwined" interactions of numerous factors – both mathematical and technical. That is, the situation challenges our ability to use a straightforward cause and effect analysis; I suggest therefore that complexity theory may be an appropriate theoretical framework within which to investigate issues around implementation.
In a small way, I have used complexity theory to examine some experiences in my own practice of teaching with technology.
Several years ago I wrote about three technology activities I had used with my high school students: a linear transformations project using a spreadsheet, a set of proof tasks with JavaSketchpad, and an independent study that made use of a variety of technological applications – Maple, AsEasyAs, and GSP. I used complexity theory as a framework to consider the question: Can we – or how can we – nurture the development of a "learning system" in a technology-supported environment? The paper examined the three mathematics activities with respect to the five conditions mentioned earlier. My analysis suggested that these conditions were present to 513 varying degrees but were affected in the lab environment by: a) lab configuration; b) program ease of use and/or depth of options; c) task design; and d) student opportunities to share knowledge with peers, and to communicate with the teacher. While these practical concerns are still important, I want to focus on other aspects of one activity.
I found that the independent study project environment was the only one that satisfied all the conditions for the development of a learning system. The broad range of student knowledge (mathematical and technical) guaranteed sufficient internal diversity for students to move ahead; shared understandings of mathematical topics, common terminology and familiar algorithms provided enough redundancy for meaningful communication; the out of class timeslots automatically allowed for distributed control, and the individual topics provided the matter over which to exercise that control. Although the class did not focus on a single goal, shared understandings emerged with regard to the use of technology, and the nature of mathematics; for the first time many students saw mathematics as a creative endeavour. With regard to "enabling constraints", I saw in the independent study project many examples of "serious play" (Reiber, Smith, & Noah, 1998), that is, absorbed, intense, self-directed action. I attributed this to the fact that the tasks were proscriptive, rather than prescriptive. [Davis and Simmt (2003) note that tasks that are proscriptive (i.e., that tell you what you can't do), allow more options for responding than tasks that are prescriptive (i.e., that tell you what you must do).] Students in the classes were free to interact with one another and with other pairs; their interactions gave rise to new ideas and explorations. In addition, students interacted with the software, the hardware, the reference materials, and the mathematical topics; all of these interactions provided fruitful opportunities for emergent understandings. Learning flourished as the classes 'gelled'. I wrote:
Observations of the Independent Study Project groups suggest that a structure allowing for sharing, play, and individual choice, that involves activities based on broadly applicable and adaptable software can result in the emergence of a beneficial complex learning system in a technological environment. (Sinclair, 2004)
A second critical experience involved a group of students in a masters' geometry course.The students in the course were novices to the investigative study of geometric concepts on a sphere. Some had gone beyond the Euclidean geometry studied in high school; however, even those students viewed geometry as formal, abstract, and logic-based. In-class activities focused on the manipulation and analysis of concrete models, but early in the course students were briefly introduced to two programs that permit the dynamic exploration of spherical geometry concepts – Cinderella, and Spherical Easel. There was neither subsequent use of the programs in class, nor any requirement that they be used; yet program screenshots appeared in the 514 assignments of some students. Assignment comments revealed that use of the tools had contributed to students' sense of confidence about what they knew and about how they could continue to learn. The programs had provided extended opportunities for investigation and allowed the students, outside of class, to deepen their understanding of spherical relationships by developing stronger mental images.
This particular group of graduate students had embraced technology for their own purposes, and had used it in a natural way – whenever it seemed appropriate and in conjunction with other methods. How different from the behaviour of my preservice and undergraduate students. [The undergraduate course is designed for teacher candidates or those considering teaching, who want to strengthen their ability to "think mathematically". It is not open to those taking other university math.] Each year in my courses I encouraged students to use technology. I uploaded activities and resource links to the class conferences, gave sessions in the lab on GSP and EXCEL, provided tutorial files, and developed assignments that required technology use. Despite this support and encouragement, few student projects even hinted at

technology use, and I heard many versions of, "It's nice, but..."
Nevertheless, my own delight at seeing 3D graphs in Maple, my secondary
students' excitement over playing with mathematical ideas, and the graduate
students' unexpected and natural use of software to explore spherical geometry, led
me to look for a new approach to teaching with technology. I pulled back from my
focus on technological tasks to consider the broader realities.
We know that technology is important to students. They surf the internet to
find information; upload, download and edit digital photos and graphics; use word
processing software to prepare assignments; and talk to one another via email and
chat. However, it isn't always part of their mathematical experience. Aside from
using scientific calculators, many students have never used technology to do math.
But my students had used technology for mathematics. In the short time
allotted for a pedagogy course (36 hours) I had provided opportunities for them use
technology to make connections between numerical, visual, and symbolic
representations, to work with multiple examples and to examine the behaviour of
mathematical objects. They had responded enthusiastically to activities in the lab -
but I recognized that they still did not use technology in their teaching or in their own
mathematical work.
What about the teachers at their placement schools? We know that teachers use
technology in their own lives. They use the internet, use word processing software to
prepare assignments and tests, and many regularly use email and chat. But teachers
often see technology as an add-on in teaching mathematics. They may do specific
technology activities, but deep down they 'know' that they can teach math without it.
After all, if they couldn't - they wouldn't have learned mathematics. My students
admitted that technology use was not being modeled in their host classrooms.
My research has focused on the design and use of technological tasks for
mathematics -- and I want to emphasize that these are important areas of research and 515
development work -- however, on reflection, I recognized that in my teaching I was
not using technology in a natural way, reaching for it at any appropriate time as I did
in my own mathematical work. Instead, I was teaching applications of technology.
This year I set out to make technology an integral part of my teaching. In
addition to scheduling lab sessions I decided to use technology whenever
appropriate, throughout the course. To ensure that I would follow through, I ordered
a computer and LCD projector for each session, and planned a very brief use of
technology for each class (e.g., showing pictures of tilings, examining an applet,
using a GSP sketch or an EXCEL graph to start a discussion.)
Over the past four months I have noticed changes in my own attitude and
practice. Although I only plan a brief use of technology for each class, I have found
myself using other technology examples, accessing the Internet to explore an idea
that has been raised, or creating a sketch or graph "on-the-fly" to illustrate a point or
to provide a shared image for a class discussion. Similarly, while I have not collected
data in a systematic way, I have noted differences in students' attitudes towards, and
use of, technology. For instance, four out of twenty students in one class handed in
an assignment that contained GSP images (they had obtained Geometer's Sketchpad
on their own through the Faculty library); another student asked if she could submit a
GSP file with an assignment. Many have requested that I upload examples to the
class conferences, and two have asked for my advice on preparing lessons that made
use of technology for their math practice teaching.
It is also clear that the students, through observing, have gained an
understanding of how to use the software, and how technology can be used to explore
mathematics. For example, during class students have asked me to create a GSP
sketch that has particular features (e.g., a quadrilateral with a right angle), to alter a
sketch or graph for them to analyse the result, and to surf to a particular website they
have discovered. Thus far, two of the three classes have had lab sessions in which
they used GSP. Students who had already obtained a copy of Sketchpad were well
beyond the beginner stage, but even those who hadn't used the program were aware
of its capabilities and the contents of the menus. I was impressed that most students,
after a paper-folding experience and a discussion of the properties of a rhombus,
were able to use Sketchpad to construct a rhombus with very little help.
Reflection
Complexity theory played a large part in my thinking as I examined my own
practice. It enabled me to step back and see the broader picture and to consider
whether the conditions that Davis and Simmt identified were met. Thinking about my
classes this way led me to conjecture that these "learning systems" were not ready to
embrace the use of technology - they lacked a shared, foundational language and
experience in the area. As a result I concluded that there would be few opportunities
for neighbouring ideas to bounce off one another, and even fewer opportunities for
new conceptions of what it means to learn mathematics, to emerge. My decision to 516
build foundational technological experiences for the students by embedding
technology use in my own teaching grew out of this analysis.
The experiences of this year have caused me to rethink a project that I, along
with several other researchers, will be starting soon; it involves evaluating the use of
a linking system for graphing calculators with grade 9 applied students [the applied
stream leads to apprenticeship and some college programs]. Originally the plan was
to inservice a number of teachers on the use of the linking system, encourage them to
use the system in their classroom, and then to use surveys, interviews and
observations to evaluate the impact on student achievement. However, Davis and
Simmt note that "it is impossible to affect a part of the complex unity without
affecting its global character and vice versa" (in press). If math classes are complex
learning systems, changes will impact the environment at many levels. An
implication for research is that any intervention must be designed to take into account
the possible effects on the whole system, especially in respect to the conditions that
have been identified as necessary for a system to learn. In the revised proposal
students will be expected to take their calculators home; the linking system and
projection equipment will be available and switched on for every meeting of every
class in the experimental group; and parents will be interviewed and surveyed.
Teachers will receive professional development sessions on using the technology but
also on developing mathematical ideas with applied students. The additions address
the need to create an environment in which technology will be embedded. The
broadened analysis recognizes that the changes will impact many aspects of the
learning environment.
Conclusion
Though these comments on my teaching practice are subjective and limited in scope,
I believe that they illustrate that the concept of a complex learning system is a useful
construct for thinking about teaching with technology. While such topics as task
design, student interactions, and developing understanding of particular mathematical
concepts are critical foci for research, complexity theory challenges us to see the
whole system in a new way. Based originally on biological models, it offers a
perspective from which to examine multiple intertwining relationships, effects that
are magnified by feedback, emergence of new ideas from seemingly insignificant
events, the creation of unexpected connections, and the development of student
understandings that are more than the sum of the parts.
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Distance Learning: Mathematical Learning Opportunities
for Rural Schools in the United States
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Much has been written about technology in the mathematics classroom, in the
computer lab, and in students' homes. Such technology includes calculators,
SmartBoards™, computers, Internet access, VCR or DVD recorders or players,
digital cameras, computer projection apparatus, and other technological solutions to
particular teaching and learning needs. Many schools, especially those in remote,
often impoverished locales, cannot provide such a cornucopia of goods and services,
but there is one technological resource that can provide students in almost any
location with the very best learning opportunities available anywhere in the world.
This technology is known as Distance Learning, and even at its most basic level, any
school with Internet access can open new doors of opportunity for its students. When
access to interactive video systems can be achieved, Distance Learning can enable
those students to become acquainted with world-class mathematicians and scientists.
All schools, including the rural schools which seek to educate almost one third
of the children in the United States (Beeson & Strange, 2003), make decisions about
the use of technology on a more or less continual basis, and all schools have similar
factors to consider -- the population they serve, the resources at their disposal, and
the technological expertise of their staff. According to a recent NCES report, rural
schools are leaders in U.S. education in at least one area of computer based
instruction: distance learning (Setzer & Lewis, 2005). In many ways, schools in the
rural United States are similar to schools in developing countries that have limited
fiscal resources for education and may be remote geographically. The intent of this
paper is to provide information and insight into some of the ways that distance
learning can benefit mathematical learning opportunities in such schools and how
this technology might be procured and utilized.
In the NCES report, distance learning courses are defined as those offered in a
district with the teacher and the students in different locations. Approximately 46%
of rural districts in the United States have students enrolled in distance education
courses, and two-way interactive video, the crème de la crème of distance learning, is
the technology most often used as the delivery method (Setzer & Lewis, 2005). With
two-way interactive television, or I-TV, distance learning is instructor-led, classbased, and synchronous -- providing real time instruction and communication
(Yaunches, 2004).
As typically implemented, I-TV is the technology that most closely resembles
a traditional classroom, and in thousands of rural schools, educators are discovering
the instructional benefits and cost effectiveness of this technology. According to a 519
report about distance learning opportunities for elementary and secondary students,
of the 15,040 school districts in the United States, 5480 (44% of secondary schools
and 36% of all schools) had students enrolled in distance learning classes.
Administrators were so pleased with the results that 72% of participating schools
were planning to expand their distance learning programs. Fifteen percent of students
in distance learning courses were taking mathematics. Nationwide, 14% of students
in distance learning classes were taking Advanced Placement or college level
courses, but in more than half (53%) of the rural schools offering distance learning
classes, students were taking such accelerated offerings. Similarly, for 56% of the
schools in areas with high poverty, where distance learning was offered, students
were taking advanced courses (National Center for Education Statistics, 2002). This
suggests that for students in rural and economically stagnant areas, distance learning
is vitally important for gaining access to the highest levels of academic work.
Research comparing distance learning of mathematics to traditional learning
has been limited, usually at the college level, and often indicates there is no
significant difference in achievement; but most mathematics educators in rural
schools would quickly point out that for rural schools, the question is often not a
choice between traditional and distance instruction but whether an advanced
mathematics course can be offered at all.
A driving force behind rural schools' embrace of distance learning, beyond the
ability for small rural schools to provide an expanded mathematics curriculum for
their students and continuing professional development for their teachers, is that
distance learning is often an effective response to the ever-looming threat of
consolidation (Yaunches, 2004). Nearly 54% of rural and small town secondary
schools in the U.S. have enrollments of 400 or fewer students (Hoffman, 2003), and
state legislatures often turn to consolidation of schools or entire districts in an attempt
to cut costs and provide better educational opportunities. Research has shown,
however, that small schools can be effective, with lower dropout rates and higher
percentages of students graduating (Hobbs, 2004); and, as it turns out, attempts to
lower costs through consolidation almost always fail (Rural School and Community
Trust, 2003). The preservation or demise of small schools is a vitally important issue
for many communities, and providing distance learning opportunities for students, as
opposed to pursuing consolidation strategies, certainly seems to be a more effective
solution to the problems spawned by isolation, teacher shortages, and fiscal
pressures.
Having limited numbers of students interested in taking an elective
mathematics course presents challenges for school administrators who have to
balance demands for classroom space, teachers, and other resources -- and without a
distance learning alternative, such requests often must be denied. Distance learning
programs can provide access to such coursework, even for just one or two students.
Students in need of remedial work, as well as those ready for advanced mathematics
courses, can benefit from these offerings. At most schools, even those with limited 520
technology, students can be directed to computer resources on an as-needed basis,
before, during, and after school. Because many colleges and universities in the
United States offer distance learning courses, high schools with this technology can
offer dual enrollment for students who are ready for college-level mathematics but
have not graduated high school.

Some schools have found that distance learning can provide a virtual schooling alternative across the curriculum. Alternative school programs are commonplace throughout much of the United States, many of them relying on computerized instruction of some sort; and the high school in Cairo, Georgia, has a novel approach to the problem of students who are unable to meet traditional requirements. These students spend the school day in a computer lab at Cairo High School taking on-line courses offered by Griggs University in Maryland; but this is not a remedial or tutorial program. As soon as they have enough credits to complete graduation requirements, the participating students receive a high school diploma from the state of Maryland (Williams, 2005).
Web-based instruction generally refers to distance learning as well as instructional programs that include on-line sites for access to data sets, software applications, or other instructional materials. Unlike the fixed content in conventional computer-based instruction (CBI), Web-based instruction can be modified to meet specific needs, is accessible from almost any location, and can be linked to related sources of information – factors that establish what has become known as "anytime, anywhere" learning.
Ultimately, successful integration of any technology, including web-based instruction, into the mathematics curriculum is dependent on teachers and administrators who embrace new technologies and are willing to participate in professional development programs designed to help them take advantage of the benefits and opportunities embedded in computer-based learning (Huffaker, 2003). Much of the research about technology integration assumes that once technological tools are in place, everyone will enthusiastically support technology-based instructional methods; however, this does not usually occur without a conscientious effort by school officials to address a multitude of issues. Schools must explore issues dealing with professional development, equitable access to appropriate hardware and software, Internet access and controls, and out-of-class availability of computers (Alexiou-Ray, Wilson, Wright, & Peirano, 2003).
Providing support to teachers may involve extensive and on-going professional development. Many distance learning providers expect that a teacher will be available to assist students with the course content as well as the technology, and those teachers often need additional training – often through web-based professional development programs. The InterMath
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Project, a collaborative effort of The

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InterMath web page: http://www.intermath-uga.gatech.edu/homepg.htm521
University of Georgia, CEISMC - Georgia Institute of Technology, and regional technology centers in the state of Georgia, funded in part by the US National Science Foundation, is one functioning example of a web-based professional development resource for middle school mathematics teachers. InterMath focuses on building teachers' mathematical content knowledge through mathematical investigations that are supported by technology. The Project includes a workshop component as well as an ongoing, web-based, support community that includes a lesson plan database and a discussion board.
Clearly, all of this technology costs money, a commodity often hard to come by in rural areas. Initiatives such as the e-rate program, which is a telecommunication, Internet access, and internal networking discount program administered by the Federal Communications Commission, and technology challenge grants from a wide range of sources can alleviate some fiscal difficulties. Many school districts band together to develop distance learning consortiums, establishing partnerships with other schools, higher education partners, and/or other outside vendors to provide cost effective options for distance learning by sharing teachers, maximizing the benefits of the I-TV network investment, sharing costs of operations, combining classes across multiple schools, providing access to professional groups, and maximizing the use of technology by addressing objectives for athletic, administrative, health, and other educational staff (Hobbs, 2004).
After receiving a grant in 1992, the North Carolina School of Science and Mathematics (NCSSM) developed a distance learning program with the goal of distance learning becoming a school outreach program. They began with the delivery of three courses, AP American History, Precalculus, and Science of the Mind, utilizing one-way video and two-way audio. In 1994, NCSSM became a state-funded provider of educational programming to teachers and students, utilizing the North Carolina Information Highway. Now, with two-way video and audio, their system supports full interactivity between school sites and teachers and among school sites with each other, sharing resources and curriculum. Through distance education NCSSM can provide rural and isolated areas of the state with courses, enrichment programs, paired teaching collaborations, workshops, and graduate level courses (The North Carolina School of Science and Mathematics, 2002).
In Florida, the Florida Virtual School (FLVS) was funded through a $1.3 billion dollar initiative to ensure that the project did not threaten general education funding. Now, while many other states struggle to maintain a virtual-schooling option for their students, FLVS has a substantial revenue stream, generating approximately $500,000 in profits during 2004 (Wood, 2005). The advance governmental support enabled FLVS to provide educational opportunities not only for students in Florida but also for students around the world as FLVS has joined the

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ranks of courseware entrepreneurs, selling to other schools its curriculum and offering franchises with hardware, software, curriculum, and professional development for teachers (Wood, 2005).
Rural schools are not alone in their effort to take advantage of grants as a means of financing technology to meet the special needs and challenges facing rural educators. In a press release from Merrill Lynch in December 2004, it was announced that students in Pennsylvania, Ohio, and Georgia will benefit from a new $5 million grant from the U.S. Department of Education's Office of Innovation and Improvement to fund the development of a national model for increasing educational opportunities for students attending small and rural schools.
Over the next five years, the Association of Education Services Agencies (AESA) and Catapult Learning, LLC will establish a streamlined contracting and purchasing system so that small and rural school districts will have greater access to high quality supplemental educational services (PRNewswire, 2004). These supplemental educational programs are an integral part of many districts' school improvement plans, especially in regards to the provisions of No Child Left Behind (NCLB). During the pilot program, 2,300 Pennsylvania, Ohio, and Georgia students will receive live, individualized, direct instruction delivered through Catapult Learning's online tutoring system (PRNewswire, 2004).
Distance learning is not a fad, nor is it a panacea for all of the shortcomings of the traditional mathematics classroom, but all evidence indicates that its widespread adoption is likely to continue into the future, with students accessing the technology at school, public libraries, and their homes – perhaps even via their cell phones. For their very survival, rural schools have to provide this technology. Issues such as curriculum controls, accreditation, course evaluations, teacher certification, accountability, academic integrity, Internet filters, per-student funding, preservice teacher training, and professional development need to be addressed as distance learning becomes a more substantial part of students' educational experience.
There is no benefit to be derived from ignoring the fact that technology is essential to today's education environment. In the United States there are ways to fund it and ways to train staff to utilize it; but schools, rural and otherwise, need to spend their money wisely, critically examine research reports, and realistically assess the ways in which various technologies can attend to the needs of the students in their schools. In an upcoming research effort, the first author of this paper will be examining in depth the mathematics program in a rural school in the southeastern United States, and of particular interest will be the extent to which that school, located in an area of extreme poverty, has been able to provide access to Web-based instruction, including distance learning, to enhance learning opportunities for its students. 523
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Geometry Through the Lens of Digital Technology: Design and the case of the non-euclidean turtle
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This paper addresses the theme of designing learning environments and curricula with a focus on geometry. It approaches the theme by first analysing role(s) that digital technology plays in learning by considering the multiple meanings of the microworld concept. Highlighting the fundamental role that pedagogy plays in a microworld, the paper proposes a descriptive theoretical framework derived from Activity Theory to capture those meanings. After discussing the close epistemological relationship between technology and the logical structure of geometries, the paper compares the differences between Euclidean dynamic geometry environments and turtle geometry, and their implications for school curricula. Finally two principles for designing digitally-based environments for geometry are illustrated in the case of non-euclidean turtle geometry: learner-centred development of tools and activities that mediate understanding in specific geometries, and the use of an iterative design process.
Introduction
What is the relationship between mathematics and digital technologies? Central to this paper is the theme of designing digital environments for learning geometry, and the dynamics of the relationship between technology and epistemology. The paper approaches the theme by considering how designing learning environment using a microworld paradigm relates to pedagogy- conceptualised using aspects of Activity Theory-and the relationship between of digital technology, geometry and learning. Key aspects of designing learning environments for geometry are illustrated by the case of a turtle-based microworld for non-euclidean geometries.
Microworlds and pedagogy
Papert's seminal notion of a microworld provided learners with computational tools to engage actively in building their own meanings, and control the trajectory of their learning. (1980: 122). His turtle geometry microworld enabled young children to use powerful computers, with the turtle seen as a transitional object for individual learners within a Piagetian inspired framework. More recently he has identified six dimensions that draw out the sense in which a microworld contains a model of a knowledge domain, a set of tools for learning and a theory of cognitive development. (Papert, 2002).
Other conceptions of a microworld locate its meaning in the relationship between learners, technology, knowledge domain and pedagogic setting. Hoyles, Noss and Sutherland's experience in developing a microworld for ratio and proportion led them to extend its definition to take account of the pedagogical context. This includes "carefully sequenced set of activities on and off the computer, organised in pairs, 525 groups or whole classes each with specified learning objectives." (Hoyles, Noss and Sutherland, 1991:3). More recently, Jackiw and Sinclair (2002) have developed a social definition of a microworld which makes whole-class activities the central focus of microworld design. Pedagogy takes centre stage in this design process which, although present in earlier conceptions, now involves groups of learners and teachers.
Papert's early notion of a microworld aimed at tapping into the sense of intrinsic motivation and play which can drive reflective abstraction. However the implied "adidactic" pedagogy led to the "play paradox"-by intervening learners are denied the opportunity for pursuing their own interests, yet, learners do not learn what is required by the designer. (Noss and Hoyles, 1996). Non-intervention can lead to learners not becoming aware of the epistemological base of the microworld, making unreflective use of tools, and avoidance of using certain tools or analysis. (Noss and Hoyles, 1992). With the emergence of social-cultural views of mathematics and their emphasis on the role of context, (Lerman, 2000; Steffe and Thompson, 2000) pedagogy has become a central concern in discussion of mathematics education. From this perspective, microworlds can be seen as the site of structured social interactions that develop over time, involving teachers, learners and technology.
Modelling Microworlds

Two implications for designing microworlds follow from these descriptions. First, microworlds are systems which can include "capable others" as well as technology and learners. A second implication is that a broad range of approaches to pedagogy and learning need to be described in connection with microworlds. These span the "classic" approach with its focus on individual or pairs of learners using technology through to groups working together with a range of resources, including digital technologies.
Edwards (1995) identifies both the structure and function of a microworld in the ways that the knowledge domain is made available to learners through the tools and activities that they undertake. Drawing on notions of an Activity Structure (Leonte'v, 1977) and System (Engeström, 1987) microworlds can be modelled to take account of their structure, function and variety. As an Activity System, microworlds make tools and resources available to both teachers and learners, and structures their pedagogical roles and the organisation of their context. As an Activity Structure, microworlds have three interdependent levels which connect their purpose and outcomes to their structure and sequencing within a context that is constrained by a range of factors. Interactions between tools, the roles adopted by the participants and their organisation condition the sequence and structure of a microworld's evolution. Figure 1 shows how the outcomes and the structure mutually determine one another, and the constraints condition what kinds of structure and outcomes might be possible. These could include the type and configuration of hardware available ranging from single machines to rooms of more than 12, and types of mathematics software DGE, CAS, Graph plotters, browsers, calculators and topic specific applications. 526
Figure 1 Structure of a Microworld. The outcomes, structure, and constraints mutually condition one another.
Table 2 shows spectrum of possibilities for the configuration of tools, roles and organisation, drawn from and validated by OECD/CERI (1999) and Hoyles and Sutherland (1989), which enables this framework to be used for describing microworlds. Combinations of statements from each column can describe a particular pedagogical structure, which can be sequenced to give a full account of a microworld.
Organisation Roles Tools
Teachers working with whole group
Teachers teamwork
Learners teamwork
Learners working individually
Learners working with whole group
Teachers giving information
Teachers directing questions and answers to reproduce facts
Teachers directing conversation
Teachers stimulating reflections or other critical analysis
Learners directing conversation with peers or teacher.
Teacher using ICT
Learners using ICT initiated by the teacher
Learners using ICT initiated by themselves
Learners interacting via ICT, initiated by the teacher
Learners interacting via ICT, initiated by themselves
Learners creating using ICT (visual arts, music, mathematics)
Table 2 Spectrum of descriptions for the central section
A classic microworld might be described by a single phase so that "Learners working individually" (Organisation), with "Learners directing conversation with peers or teacher." (Roles) and "Learners using ICT initiated by themselves" (Tools). Outcomes for this approach are self-defined by the learners, and they use a single machine with appropriate software. Alternatively, there maybe a more complex structure with several stages such as Jacklw and Sinclair (ibid.) describe. A teacher may start a set of activities that make use of technology in different ways, with learners subsequently taking control as they develop their understanding.
Digital Technologies and Geometries
Arguably the oldest and most common form of mathematics found in cultures across history is geometry. Practical techniques for measuring land, making buildings and laying out public and private spaces can be found in writings and sites dating back to the dawn of time. (Gray, 1989). Geometry emerged alongside the growth of advanced cultures to resolve disputes, and as a means for transforming the natural
Purpose
Structure & Sequence
Constraints
Outcomes
Tools Roles
Organisation
Curriculum, Resource Distribution, Policy
Tools Roles
Organisation527
world into niches suitable for humans. In this sense geometry reverses the Darwinian notion of natural selection: humans construct niches in which they thrive rather than being selected by those niches. (Berger and Luckman, 1966).
With the emergence of Euclid's Elements in the third century BCE, geometry became a systematic and organised body of knowledge that provided the paradigm for mathematical reasoning. However, the practical origins of the geometry still play a significant role: euclidean geometry is synthetic. It relies on the use of a straightedge and compass in the process of establishing its results. (Bkouche, 1989). Only with Hilbert's treatment of Euclid at the start of the twentieth century did the constructional assumptions and practices implicit in the process of proving results become apparent. (Stillwell, 1992). By the same token these technologies also constrained what was possible to achieve with the geometry. All problems have to be interpreted in a way that the geometry and its technology can be applied.
This raises the question about what kinds of geometry should be taught and learnt. History indicates that euclidean geometry is not unique, and represents a normative rather than a natural interpretation of our spatial experience. (Riechenbach, 1957). It is arguable that learners need to be able to move between geometries that are appropriate to their experience in different situations. Euclidean geometry is intimately related to the technology of straight edges and compasses, and has been used effectively over "local" distances for millennia. Spherical geometry has been used for over four centuries for practical navigation, and is accompanied by a range of instruments and mappings for practical uses. (Kreyzig, 1991). Hyperbolic geometry and its euclidean models, by contrast, have emerged recently as a working tool with the development of computational representations to navigate the web. (Lamping et al., 1995). What these examples have in common is the interdependence between geometry as a theoretical discipline and the technologies which enable it.
4. Dynamic geometry-what is it?
A new generation of computational environments to support learners of euclidean geometry combine tools for creating on-screen objects with their direct manipulation. Known as dynamic geometry environments,(DGE), they enable learners to construct euclidean diagrams, and, through selecting points on the figures, move the diagrams around to examine the logical dependences. (Jacklw, 1995; Laborde and Laborde, 1995; Richter-Gebert, J. and Kortenkamp,1998). DGEs rely for their semantic power on learners intuitively identifying DGEs' functionalities with their own experiences of paper and pencil constructions. Most DGEs make use of the technical vocabulary of euclidean geometry in menus and icons to represent a range of operations. However, there is not a one-to-one correspondence between DGEs and "paper and pencil", since design decisions make DGEs different to euclidean geometry in significant ways. (Goldenberg and Cuoco, 1998). It remains to be seen what the implications of this semantic disparity are for learners.
Although DGEs do have tools to animate diagrams and produce loci of points, their principle source of dynamism comes from learners who move diagrams physically around the screen. Such "external" dynamism contrasts with the internal 528 dynamism of turtle geometry in which movement is an integral part of the commands that control the turtle: forward, backward, left, and right. Turtle geometry differs from DGEs in two other respects. Although the turtle produces euclidean objects, it is conceptualised as syntonic or body geometry. (Papert, 1980). Objects are created by learners as they walk the turtle over the shape rather like they might do in a physical setting, and express that process in terms of its four commands and other programming structures, such as procedures, control structures, and loops. Turtle geometry, although it can produce euclidean objects, is essentially differential. It is defined locally and intrinsically without reference to an external and global coordinate system. (Abelson and DiSessa, 1980). Turtle geometry differs from DGE in a second key respect; it's a space-time geometry. Motion is an integral part of defining a turtle's state, which refers to events rather than spatial objects.
The "failure" of turtle geometry to become a central element in mathematics classrooms may be viewed not as a lack of will or pedagogic ingenuity. (Noss and Hoyles, 1996). The contrast with other more successful dynamic geometry environments suggests that the problem lies in the type of geometry that turtle graphics enables, and what counts as geometry within the school curriculum. Current applications that are normally referred to as DGE (e.g. Cabri, SketchPad, Cinderella) are tools for exploring the euclidean model of spatial experience. By contrast turtle geometry, which is also dynamic, mediates other kinds of geometry that model our spatial experience but do not form part of the school curriculum. It remains an open question as to whether the DGE categorisation of software should include just the euclidean-based applications.
Euclidean DGEs success lies in the fact that they have been created principally to support the learning of school geometry, since they reproduce and automate many of the functions that are routinely undertaken using paper and pencil. Unlike turtle geometry DGEs are apparently designed around what their creators perceive as being the necessary functionality derived from epistemology of Euclidean geometry, rather than what learners might need. (Squires and McDougal, 1994). However given the tensions that Hoyles, Noss, and Adamson (2002) identify in relation to what they call "superstructure"-what is available to the learner in a microworld, and "platform"-the environment in which the superstructure is articulated, focussing on learners is a complex and time consuming process. The next section explores the process of learner-centred design and its connection to "superstructure" through a specific example.
Learner-centred design: spherical and hyperbolic turtle geometry
Two principles derived from the construction of a turtle-based microworld for noneuclidean geometry illustrate the importance of the interplay between design and learning. (Stevenson, 1996; Stevenson & Noss, 1999; Stevenson, 2001)
Learner-centred development of tools and activities that mediate understanding in specific geometries. Papert's turtle was chosen originally for its syntonic link to cognitive development, and finding cognitive links for non-euclidean contexts was, therefore, a central design issue. The links emerged by working with learners to find 529 what engaged them with the structures of the new geometries. The turtle's screen behaviour was governed by euclidean models obtained from projecting spherical and hyperbolic geometry onto the screen "plane". Three types of links were needed to help learners connect with non-euclidean turtle geometry because of the complexity of the screen images: physical surfaces, metaphors, and on-screen structures. Tracing paths on the physical surfaces with their fingers enabled learners to make sense of what they saw on screen by metaphorically linking their action with the screen turtle. Using the metaphor "turtles walk straight paths" helped learners identify "straight lines" on curved surfaces with straight lines left by the turtle. (Abelson and Disessa, 1980). A second metaphor used to understand the turtle's behaviour was the screen's temperature. (Gray, 1989). Spherical geometry projects a surface that increases in temperature as one moves towards the screen's edge, while hyperbolic geometries gets cooler towards the edge. Dashing the turtle's path so that the dashes grew longer or shorter according to the geometry indicated that the turtle's steps were expanded or contracted by the "temperature" of the screen. A corresponding speeding up or slowing down of the turtle's movement as it left dashes, coupled with a tool that drew the large-scale path which a turtle might take given a particular position and heading provided a dynamic structure for learners to build up their understanding.
The key point is that the microworld's physical, conceptual, and virtual resources emerged through paying careful and systematic attention to learners' needs in these specific geometrical context. As Harel (1991) points out, learning and designing are intimately connected both for "learners" and "designers".
Iterative design process. Paying careful and systematic attention to learners' needs took the form of a thorough evaluation of the pedagogical, technical, and cognitive aspects of the microworld. The microworld emerged through analysis of a series of structured activities and observations based on the relationship between the roles, tools and organisation of resources in Table 2. Over three cycles of development a combination of didactic intervention, reflective discussions, task-based interviews and non-participatory observation of learners were used. Each of these roles was applied explicitly in designing activities to achieve particular research objectives. In terms of Table 2, for example, "didactic intervention" consisted of "teacher working whole group", with "teachers directing conversation" and "teacher using ICT". Reflective discussions, however, consisted of learners working individually with teachers (researcher) stimulating reflections and learners using ICT initiated by themselves. Interspersed between working with learners were periods of reflection to redesign tools and resources, activities, and pedagogic approaches based on analysis of video tapes, field notes and dribble files. Each new section of work with the learners was carefully sequenced according the pedagogic structure described by use

of tools, roles, and organisation.
Conclusion
Microworlds are essentially pedagogic. The descriptive framework proposed in this paper is intended to articulate the range of meanings that microworlds can have, and provides a means for designing learning environments. The example of designing a 530 turtle-based environment for non-euclidean geometry illustrates a number of key issues. Geometries other than Euclid are available and they form part of our everyday experience. They can be mediated by digital technology, but they are constrained by the standard euclidean representations of the "screen". For that reason designing learning environments must pay careful and systematic attention to what mediates understanding in specific geometries. This implies an iterative approach to design that places learners at the centre of the process, and the framework derived from Activity Theory provides the means to undertake this systematically.
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Informed by experiments and experience with Dynamical Geometry Environments ("DGE"), the paper elaborates on changes, which are linked to the use of DGE. DGEs allow breaking out of the narrow confines of Euclidean Geometry – best illustrated by the use of the dragmode to introduce movement into static Euclidean Geometry. DGEs offer ways of teaching and learning Geometry, which are not available in a traditional paper-and-pencil environment. The macro-functionality of DGEs appears to be an excellent possibility of structuring the material and cognitive representation of a construction process. On the other hand, the explorative potential of DGE often implies a "de-goaling" from the initial task. Worksheets and the intervention of the teacher are suggested to cope with these difficulties of learning with the help of DGEs. An excursion additionally shows that some of the design decisions within (Geometry) software are constrained by unavoidable mathematical necessities – with implications for pedagogical and cognitive (dis)advantages. Following an'instrumental genesis' approach, design decisions should be made after close inspection of the ways the users 'instrumentalize' the software. Some of the findings on DGE can obviously be generalised to other types of mathematical software available or forthcoming in the near future.
Euclidean Geometry left with DGE
Comparing traditional, especially Euclidean Geometry and its teaching and learning with a more recent Geometry education using Dynamical Geometry Environments (in the following: "DGE" - like Cabri-géomètre, Geometry Inventor, Geometer's Sketchpad or The Supposer – to name but a few of a whole variety of DGEs, for an early description see Sträßer 1992), the most obvious difference is the dragmode, i.e. the possibility of moving initial points of a drawing with the mouse while the construction is "updated" according to the construction commands used. It is obvious that DGEs have left the narrow confines of static Euclidean Geometry (and philosophers and historians of Mathematics do name the avoidance of motion as one of the characteristic features of Euclidean Geometry). According to the designers of DGEs like N. Jackiw (for Sketchpad) or J.-M. Laborde (for Cabri), this was not intended when creating these pieces of software. As a consequence of the software design and only after some time, the designers, mathematicians, educationalists and users realised what DGEs do to Euclidean Geometry (see for instance Laborde 2001). They also had not expected the problems linked to this potential of DGE (see section 3). 533
Looking more closely into DGE use, another major change becomes evident, which is not as well researched (and not so widely used in Mathematics education) – namely the use of geometrical transformations like reflections, translations and rotations. Even if again closely related to movement (especially with translation and rotation), DGE nowadays have a functionality to transform geometrical elements as a whole, showing the image of a reflection, rotation or translation after one click of the mouse.
Example 1:
With a DGE like Cabri, it is easy to study the transformation, which takes place when a quadrilateral is successively reflected at two different straight lines. We can even drag the lines and study the consequences of different respective positions of these lines (e.g.: parallel or intersecting). Such an exploration would offer excellent handson experience to study the composition of line reflections, more general: congruence transformations in plane Euclidean Geometry.
The "locus-of-point" feature of Cabri can also be used to go for complete images of certain elements of a drawing (for a detailed analysis of this feature see Jahn 2002 and the underlying PhD-thesis of Jahn). In all, DGEs offer a potential for transformational Geometry, which is only seldom exploited – even if disciplinary Geometry indicates this possibility (see Felix Klein's "Erlangen program").
In addition to this, the dragmode offers a more fundamental potential within DGEs: If appropriately introduced and interpreted, the dragmode can facilitate the understanding of the difference und relations between a material "drawing" and the underlying geometrical "figure" (for the introduction of these concepts see Parzysz 1988, related to DGE: SträßerßCapponi 1991; for a slightly different conceptualization see Mariotti 1992). Offering a huge variety of drawings belonging to the same figure at a click and drag of the mouse, DGEs can help to distinguish between the material representation of a geometrical configuration, its drawing(s), and the underlying logico-geometrical relations, which -after the cutting of the ontological binding (see Freudenthal)- are most relevant for the mathematician.
Example 2: The'Varignon'- parallelogram in a quadrilateral
To end this section, it should be mentioned that models of non-Euclidean Geometry are also easily available within this type of software (see the set of macros for hyperbolic Geometry created for Cabri by Lister 1998 or the possibilities in CINDERELLA described in Richter-GebertßKortenkamp 2000). 534
Teaching Geometry with DGE
If we look more closely into the teaching and learning of Geometry with the help of DGEs, there is more to DGEs than the dragmode. transformation Geometry and the distinction of 'drawing' from 'figure': For the constructive part of Geometry and its analysis, it is most helpful to structure this activity not according to the functionalities and menu of the individual software, but to be able to think in units defined within Geometry. DGEs normally offer a most helpful feature to meet this demand: the possibility of defining macros. After having done a specific construction and under certain restrictions, DGEs can be told to repeat a defined part of a construction with an input defined by the user and a necessarily unique output (no 'if-then'-decisions allowed at present – for a discussion see Kadunz 2002 or Sträßer 2003).
Example 3: Square-Macro
In contrast to a traditional construction of a square ("construct a segment, perpendiculars in the endpoints, a circle through one endpoint with the other endpoint as centre, ..."), a well-defined DGE-macro can draw a square just by indicating a segment or two points as input by using a macro pre-defined by the user and stored in her/his computer. To appreciate the consequences, imagine you want to draw a configuration illustrating Pythagoras' Theorem!
A macro definitely reduces the cognitive burden of a construction and gives a chance to cognitively structure the construction in a way appropriate for the geometrical problem under study – provided a well filled set of pre-defined macros is available for the user. Sometimes, the absence of a macro-feature in CINDERELLA is given as a reason not to take this piece of software as a full-fledged DGE.
Even in the early days of DGE, one major problem implied by the use of DGEs to help teaching and especially learning with the help of DGEs became obvious: The intuitive interface and easily understood handling of the software really facilitated the exploration of geometrical configurations, but – when handed over into the full responsibility of the learner – DGEs most often lead to a "de-goaling" in terms of the initial purpose of the construction. The user often played around with the DGE and – being intrigued by the nice drawings – forgot about the task set by the teacher (for an early description of this phenomenon see Hölzl 1994).
From the very beginning, DGE-use for teaching and learning Geometry has to cope with a dilemma not only linked to DGE-use: How to handle the non-trivial relation between the personal constructions of a learner and the often very precisely defined concepts and definitions from disciplinary Mathematics? This dilemma seems to be especially difficult to handle in a constructivist approach to learning Geometry, more general: Mathematics.
A suggestion brought forward by practitioners of teaching in the 1990ies not only tries to avoid this de-goaling: "Worksheets", prepared in advance and with clear tasks and questions to be answered, try to control the trajectory of the learner's activities in a way to avoid not only the de-goaling problem, but also try to save time by offering 535 prefabricated DGE-files. These ready-made files should directly offer the configurations to be explored – saving time and avoiding the construction mistakes often made by un-experienced users of DGEs (for this approach see Elschenbroich 2001 and Heintz 2001).
A different way to cope with this is a suggestion brought forward by Italian colleagues and grounded in a more fundamental approach: As a consequence of a social constructive understanding of teaching and learning and inspired by Vygotsky, the importance of collective discussions in the classroom are stressed. In order to guide the learners in a way to help them arrive at socially, also mathematically accepted solutions, the interventions of the teacher are of fundamental importance and cannot be left aside (see for instance Bartolini-Bussi 1996).
Excursion: Mathematics strikes back!
The end of the last section already illustrated: Using DGEs to teach and learn Geometry is not a simple success story. On closer inspection of the dragmode for

instance, the story gets even more complicated. Two demands on the dragmode seem obvious: Using the dragmode should not imply a discontinuity, the drawing should change continuously if the dragmode is used ('continuity' of the dragmode).
Example 4:
Try the angle bisector of an angle bisector and move the point defining the initial angle to make the initial angle greater than 90°. Construct only one second angle bisector to see the jump (e.g. in Cabri).
On the other hand, the user expects that the drawing should be the same after a drag of a point when this point is brought back into the initial position ('reversibility' of the dragmode, producing 'deterministic' drawings; see counter-example next page!). Unfortunately, these two somehow natural demands cannot be met together. The software designers have to make (and indeed: make) a choice between 'continuity' and 'reversibility (as shown by Gawlick 2001). The designers of CINDERELLA for instance are proud of having produced a 'continuous' piece of DGE (and there are even limit cases where the software also produces unwanted jumps), while most of the other DGEs go for 'reversibility' (for instance Cabri-géomètre), but have to cope with discontinuous changes of the drawing produced with the DGE.
When a straight line g intersects with a circle in E and F (see drawing 1 above), in a way, that g can be moved not to intersect g (drawing 2), then moved back to intersect g (drawing 3), the intersection points E and F change position.
The interesting point is that algebraic, differential Geometry can prove that this choice is inescapable, a DGE is either 'continuous' or 'deterministic'. As a consequence, the software designer has to make a (hopefully deliberate, at best pedagogically motivated) choice. 536
Example 5: Non-reversability in CINDERELLA (see Laborde 2001)
I want to add another example of the 'same' making: Numbers can be represented in software within a finite number of bits - as long as we stay with the usual discrete and binary code in digital (!) computers. The different ways to represent and store numbers in widely used pieces of software like spreadsheets (as EXCEL: at best floating point numbers with clear restrictions about numbers accurately represented, numerical analysis provides the details) and Computer Algebra Systems (CAS, especially DERIVE with its representation of numbers as quotients of natural numbers as long as storage is available, that is: rational numbers accurately represented "in principle", hence irrational numbers approximated) also show that simple Mathematics does constrain the use of software in doing, teaching and learning Mathematics. This implicitly was already an issue in the first ICMI-study, where the role of "Discrete Mathematics" (especially for college Mathematics in the USA) was fiercely discussed - without reaching a consensus. The question of discrete Mathematics seems to be open and still has to be solved for the upper secondary school mathematics curriculum worldwide.
From the two examples above (continuity versus determinism and representation of numbers), we can conclude that sometimes it is not a simple, deliberate design decision how a piece of software is constructed. Heavy, sometimes unavoidable mathematical constraints force software development into decisions, which may not be desirable from an educational point of view, but inevitable because of unavoidable mathematical constraints.
From artefacts to instruments
If we go back to the most obvious characteristic feature of DGEs, namely the dragmode, we can illustrate another major lesson to be learnt about teaching and learning with the help digital technologies: For mathematics education, (one of) the most important insights into DGE-based teaching and learning came from detailed user studies undertaken by colleagues from Italy (for a comprehensive overview see Arzarello et al. 2002). With a meticulous analysis of the different ways how students use the dragmode in a DGE, they offered not only an inventory of the utilization schemes of the users, but gave excellent hints how to profit from this most prominent feature of DGEs for teaching and learning Geometry. From the analysis published in different places, one can develop a position on the most intriguing question related to (traditional) Geometry teaching - namely whether DGE-use hinders or enhances the teaching and learning of proof in Geometry. In fact, it seems to depend on the 537 occasions and challenges offered by the teacher (or the teaching material), if DGEuse destroys or necessitates the need for mathematical proofs.
In my view, the results of this study very well illustrate how a detailed analysis of the use of software can enhance the teaching and learning of Mathematics. It is not enough to study the artefact as such, to study the features and potentials of a piece of software. Only a detailed analysis of the actual software use offers a sound and pedagogically useful understanding of its use. To put it into the words of the 'instrumental genesis' approach: the artefact together with its utilisation schemes converts digital technology into an instrument of learning! For a short description of this approach see RabardelßSamurçay 2001. Artigue 2002 shows how the approach can be helpful for better understanding the use of Computer Algebra Systems (CAS).
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Solving Equations in a Spreadsheet Environment
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The use of spreadsheets in a beginning algebra course is considered traditionally with regard to their potential to promote generational activities. However, much less is known about their possible use to construct and solve equations. The overall purpose of this paper is to consider the potential of spreadsheets to enhance conceptual understanding of equations and their solutions. For this purpose, we analyzed the work of beginning algebra students with an activity that required algebraic modeling, solving equations, and interpreting results that were obtained as an Excel output. As a result, we recommend expanding the traditional use of spreadsheets from mathematical investigations of variations and patterns to include a conceptual understanding of algebraic relations and transformations as well.
This paper is related to Themes 1 and 2:
1. Mathematics and mathematical practices
- What new types of mathematical knowledge and practices emerge as a result of access to digital technologies?
- How are new types of technology-mediated mathematical knowledge and practices related to current classroom curricula and values?
- What role can the "mathematics laboratory" play in different educational contexts, including secondary education?
2. Learning and assessing mathematics with and through digital technologies
- How can the benefits of existing technologies be maximized for the benefit of mathematics teaching and learning?
Background
The use of spreadsheets in teaching algebra - both at a beginning stage and at more advanced levels, has been reported in various research reports. Usually the potential to produce ample numerical tables, the need to use general expressions to create these tables, and the possibility of obtaining a wide variety of corresponding graphs allows the use of spreadsheets as a tool for promoting a functional approach in teaching and learning algebra (Filloy & Sutherland, 1996; Heid, 1995; Wilson et al., 2005). According to Kieran (2004), most investigated spreadsheet activities are conducted at a generational or at a meta-global level, and according to Thomas & Tall's (2001), categorization of algebraic activities, these activities belong to the domains of generalized arithmetic or evaluation algebra. 540
However, the usefulness of spreadsheets for investigating relationships, such as solving equations or inequalities (according to Thomas & Tall, manipulation algebra) or performing transformational operations (according to Kieran) is less clear. Moreover, some studies show that attempts to use spreadsheets as a means of strengthening a conceptual understanding of algebraic techniques encountered considerable difficulties. Dettori and her colleagues (2001) investigated 13 to 14-year-old students' work on algebraic problems using spreadsheets and concluded that "spreadsheets can start the journey of learning algebra, but do not have the tools to complete it. Being able to write down parts of the relations among the considered objects, but not to synthesize and manipulate the complete relations, is like knowing the words and phrases of a language, but being unable to compose them into complete sentences" (p. 206). Friedlander and Stein (2001) investigated the abilities and preferences of 14-year-old students in solving linear and quadratic equations and systems of two equations with both paper-and-pencil and various technological tools (Excel, Derive, and graph plotter). The interviewed students demonstrated a high ability to choose, employ, and integrate technological tools into their work, but on the other hand, the students' use of spreadsheets for this purpose was quite ineffective. The purpose of this presentation is to analyze the potential of spreadsheets in promoting understanding the concept of equations, intended for beginning algebra students.
The activity
The activity Elections was administered in two 7th grade classes that attended a beginning algebra course based on a functional approach, and on the use of Excel as a technological tool (Hershkowitz et al., 2002). For several lessons before beginning the activity, the students encountered for the first time the concept of an algebraic equation, and solved linear equations of the form ax + b = c by trial and error or by a method of "undoing" the operations involved. At this time, the students had not encountered any conventional methods of solving linear equations.
The Elections activity presented the following problem-situation involving a campaign for a class committee:
Avi, Ben, and Gil are candidates for the election.
Avi received 24 votes more than Ben, and
Gil received 1.5 times more than Ben.
Next, the students were required to consider several situations of equalities between the number of votes received by the candidates, and to model them as equations of the form ax + b = cx + d. For these equations, the method of "undoing" or trial and error cannot be applied easily. In the activity the students were allowed to use Excel, and were provided with appropriate instructions. 541
Excel is not an algebraic manipulator, and it does not have the ability to provide directly the solution of a given equation. As a result, the activity presented the following procedure:
• Write in one column several values for the independent variable (i.e. the number of votes received by one of the candidates).
• Use the independent variable to write an equation at the beginning of another column.
• Copy (drag) the equation down the column.
• The resulting output is a sequence of TRUE or FALSE values - indicating the equality or inequality obtained by substituting the corresponding value of the independent variable in the equation.
• A TRUE value indicates that the substituted number is a solution for the equation (but not necessarily the only one).
• A sequence of only FALSE values can be the result of either an equation that has no solution or an equation whose solution was not included in the set of values selected for the independent variable.
As a result of the possible situations described in the activity, the students constructed four types of equations:
1. An equation with a unique solution that is also meaningful for the given problem-situation.
2. An equation with a unique solution that has no meaning in the given problemsituation.

3. An identity, i.e. an equation with all real numbers as a solution set.
4. An equation with an empty solution set.
Because of the space limitation, we will consider the rationale and analyze the students' work only for equations of the first two types.
1. An equation with a unique and meaningful solution. Item 2a referred to the possibility that "Avi and Gil receive the same number of votes", and the expected algebraic equation is x + 24 = 1.5□x (where x represents the number of votes received by Ben). The recommended Excel procedure was to insert in column A whole numbers from 1 to 50 as possible values for Ben's votes, and to write and drag down the equation
=A2 + 24 = 1.5*A2 in column B. In this particular case, one cell in column B will contain a TRUE output (indicating the solution), and all the other cells in the output column will contain FALSE values (see columns A and B in Table 1). Next, students are expected to interpret the output, and conclude that the situation described in this item can occur if Ben received 48 votes, whereas Avi and Gil received 72 votes each. Note that the expected solution process includes formulating an equation, (in both algebraic and Excel, or only in an Excel format), "dragging" it down, and finally interpreting the obtained output. 542
2. An equation with a unique solution that does not apply to the given situation (e.g., a fraction or a negative number as a solution). Item 2d referred to the possibility that "the number of votes received by Ben and Avi together is the same as the number of votes received by Gil". In our particular case, the expected algebraic equation is x + x + 24 = 1.5□x (where x represents the number of votes received by Ben), and its solution (x = -48) cannot represent a number of votes. Copying down the Excel format of this equation (=A2 + A2 + 24 = 1.5*A2) for positive values of A produces a column of only FALSE values, since the equation has no solution for the selected substitution set (see columns A and C in Table 1). Next, the students were expected to interpret the output, and to conclude that the situation described in this item cannot occur.
Data collection
Thirty-seven students worked on the activity, mainly in pairs, for 45 minutes, and saved their work in 20 files. In addition, the work of ten pairs was audio recorded. The recorded pairs were chosen at random, so we considered their work as representative of the whole class. The following analysis is based mainly on the work of the ten audio-recorded pairs of students.
Table 1. Excel output for items 2a and 2d.

	A	B	C
1	Number of votes received by Ben	2a: Avi and Gil receive the same number of votes	2d: Ben and Avi's votes together equal Gil's votes
2	1	FALSE	
3	2	FALSE	FALSE
4	3	FALSE	FALSE
5	4	FALSE	FALSE
6	5	FALSE	FALSE
.			
.			
47	46	FALSE	FALSE
48	47	FALSE	FALSE
49	48	TRUE	FALSE
50	49	FALSE	FALSE
51	50	FALSE	FALSE

1
This output is the result of entering the equation =A2 + 24 = 1.5*A2
2
This output is the result of entering the equation =A2 + A2 + 24 = 1.5*A2 543
Analysis
In this section we will present students' answers to items 2a and 2d as representatives of their work on a wider, similar collection of items in this activity.
Item 2a: Avi and Gil receive the same number of votes.
We identified four types of reactions to this item.
- Six pairs of students completed independently the solution process (i.e. formulated the algebraic and the Excel equation, copied it down the column, and interpreted the output).
- One pair formulated the algebraic equation correctly, entered in cell B1 its Excel version, and received FALSE as an output. At this stage, they became confused and asked the teacher to help. The teacher encouraged them to continue, and as a result, they were pleased to obtain a sequence of one TRUE and otherwise FALSE values, and were able to identify the first as the desired solution.
- Two pairs read the item, but did not follow the expected solution path. Instead, they used columns A, B, and C to collect their data: similarly to others, column A was used to indicate possible values for the independent variable (Ben's votes); column B indicated Avi's votes (obtained by copying down =A1 + 24); column C contained Gil's votes (=A1*1.5). Next, they searched in columns B and C and found a row with the same number of votes for Avi and Gil. The teacher accepted their strategy, but asked them to apply the new procedure as well. They followed the activity's instructions without any difficulties, and were so pleased with the results, that one student's reaction to the teacher was "Why didn't you teach us this method earlier this year?"
- One pair of students did not understand the situation, and needed the teacher's help to obtain the equation, to write it on the spreadsheet, and to interpret the results. They could work independently only after receiving the teacher's help on two additional examples.
Item 2d: The number of votes received by Ben and Avi together is the same as the number of votes received by Gil.
All pairs wrote a correct equation (=A2 + A2 + 24 = 1.5*A2), and copied it down the column. Since the solution of this equation (-48) was not included in the first column, they received a sequence of only FALSE values. The students attempted to interpret this output in various ways.
- Two pairs concluded that there is no solution for this item since "Avi and Ben together received 2x + 24, whereas Gil received only 1.5x, and these two cannot be equal."
- Four pairs did not provide an explanation, besides noting that this situation "does not make sense". 544
- One pair looked at their neighbors' computer screen and concluded that since they received a similar output, apparently their solution was correct.
- Three pairs concluded that they must be wrong, and asked the teacher for help. The teacher encouraged them to think again about the given situation, and only then they reached the conclusion that there is no solution to the given item.
The interpretation of the output for this equation was also considered during the summarizing class discussion.
Discussion
The main purposes of the activity were
- to enhance the understanding of the concept of algebraic equations,
- to provide an opportunity for algebraic modeling, and finally
- to present a situation that requires interpreting and understanding the meaning of the solution of an equation.
The Excel procedure for solving an equation (substituting a set of numbers in the equation and searching for cases of equality) is not efficient as a routine solution tool. However, the findings indicate that from a technical aspect, the procedure can be learned and applied easily.
More importantly, this solution process provides opportunities for achieving conceptual understandings of important algebraic ideas. The need to interpret the logical TRUE/FALSE values encourages students to consider several equationrelated issues.
- Not all equations can be solved by trial and error or by undoing operations.
- As shown by the many FALSE values in the Excel output, most equations have a large set of values that do not satisfy the given equality.
- Receiving a sequence of only FALSE values as output indicates an empty solution set or that the required solution is not among the values selected and substituted for the independent variable.
Our findings indicate that these insights occurred spontaneously, in the context of the activity, and were the result of a natural need to interpret the Excel output, and furthermore to solve the problem. Some of these conceptual understandings were acquired at the stage where students worked on the activity or during the following class discussion.
Analyses that concluded that spreadsheets are not appropriate to serve as a tool for algebraic transformational activities may be based on a tendency to separate or even contrast conceptual versus procedural knowledge. An analysis of the Elections activity and of student work reveals some interesting possibilities of integrating conceptual and transformational approaches into teaching and learning algebra. Similar recommendations and conclusions were reached by Kieran (2004) and by Star (2005). 545
We consider this approach of a "conceptual solution" of equations particularly suitable and important at the early stages of learning algebra. We believe that a deep understanding of the involved concepts can provide a basis for learning the routine solution procedures employed at a more advanced stage. This notion needs of course further validation by additional research.
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Teachers using computers in mathematics: A longitudinal study
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The computer has been in mathematics classrooms for over 20 years now, but with widely varying implementation in mathematics teaching and learning. This paper describes a ten-year longitudinal research study that has investigated the changing nature of how secondary school teachers use computers in their mathematics classroom, and their perception of constraints or obstacles to improving, or extending, such use. The results show that while there are now many more computers available in schools, access remains a key obstacle to their increased use as mathematical learning tools. There is also a change in the kinds of software used, away from content-specific programs and towards generic software, especially the spreadsheet. Teacher attitude remains a key factor in progress.
Introduction
While many mathematics educators, including the author have been positive about the possible role of computers in the learning of mathematics (see e.g., Thomas & Holton, 2003), there have been doubts raised about a) whether computers have any real value in learning (Cuban, 2001) and b) whether current teacher use is qualitatively and quantitatively sufficient to promote any benefits that might exist. Around 10 years ago Askew and William (1995) reported on a review of research in mathematics education in the 5-16 year old age range, and found that "Although computers have been in use in mathematics education in this country [UK] for well over twenty-five years, the pattern of usage is still very varied and very sparse." (p. 34). A UK Department of Education report (DFE, 1995) also noted a low level of usage of computers in mathematics, with an average of 15.6 minutes of lesson time per week spent using the computer, and in the United States the position was very similar (Ely, 1993). While some might hope that this position has changed in more recent years, a survey by Ruthven and Hennessey (2002) on school computer use concluded that "Typically then, computer use remains low, and its growth slow." (p. 48).
There are a number of possible reasons for a low level of computer use in mathematics teaching and learning, including teacher inability to focus on the

mathematics and its implications rather than the computer and many teachers not believing that the computer has real value in student learning. Certainly, Veen (1993) has argued that teacher factors outweigh school factors in the promotion of computer use. More recently Becker (2000a) reported on a national US survey of over 4000 teachers and concluded that "...in a certain sense Cuban is correct–computers have not transformed the teaching practices of a majority of teachers." (p. 29). However, he noted that for certain teachers, namely those with a more student-centred 547 philosophy, who had sufficient resources in their classroom (5 or more computers), and who had a reasonable background experience of using computers, a majority of them made 'active and regular use of computers' in teaching. Becker (2000b) has added a description of some characteristics of such an 'exemplary' computer-using teacher, but concludes that extending these to other teachers would be expensive. This paper reports on a ten-year longitudinal study describing the changing pattern of computer use in the mathematics classroom in New Zealand. Both the level and kinds of use were recorded, together with some of the obstacles teachers perceive to increased use.

Method

Genuine longitudinal studies, where at least two sets of data are acquired from the same population over an extended time span, are relatively rare in mathematics education research. This longitudinal study, which has as its population all secondary mathematics teachers in New Zealand, began in 1995, when a postal questionnaire on computer use was sent to every secondary school in New Zealand. Replies were received from 90 of the 336 schools (26.8%), a reasonable response rate for a postal survey. Apart from information about the mathematics department in the school we received information from a total of 339 teachers in these 90 schools.

Some of the results of this survey were published at the time (Thomas, 1996). This original survey was followed by a second in 2005 in order to gain longitudinal data on how the situation might have changed over this period. In the years since 1995 teaching has become an even more stressful profession in many ways, particularly in terms of demands on time. Hence, teachers are more reluctant than ever to spend their valuable time filling in forms or research questionnaires. However, we had learned some lessons from 1995 and this time stamped, addressed envelopes were enclosed for all the schools and it was followed up several weeks later with a faxed copy. Using this approach we achieved a response from 193 of the 336 secondary schools in the country, an excellent 57.4% response. Completed questionnaires were received from a total of 465 teachers in these 193 schools, as well as the school information. In both years we are confident, due to the sample size, that the responses form a representative sample of the population of secondary school mathematics teachers, especially since we received a good proportion of responses from noncomputer users (over 30% in each case). Of the respondents, in 1995 51.5% were male and 48.5% female, with a mean age of 41.5 years, whilst in 2005, 52.6% were male and 47.4% female, with a mean age of 44.8 years; the teachers are getting slightly older. While the questionnaires sent out in the two years were not identical, for example questions on the use of the internet were added in 2005, they had a considerable number of questions in common. On both occasions they used both closed and open questions to provide valuable data on issues such as: the number of computers in each school; the level of access to the computers; available software; the pattern of use in mathematics teaching; and teachers' perceived obstacles to computer use. This data enables us to come to some conclusions about the changing 548 nature of computer use in the learning of mathematics in New Zealand secondary schools.

Q1 Do you ever use computers in your mathematics lessons?
If you answered 'No' please go straight to Q14
Yes
No
1
2

Q2 How often do you use computers in your mathematics lessons?
At least once a week
At least once a month
At least once a term
At least once a year
Never
1
2
3
4
5

Q5 Where are the computers you use usually situated? In the computer room
In the mathematics room
1
2

Q6 If the computers are in the mathematics room, how many do you usually have?
One
Two
Three
Four
Other_____
1
2
3
4
5

Q10 Please rank these areas of mathematics in the order in which you most often use the computer in your mathematics lessons
i.e. 1 for most often, 2 for next etc. Leave blank any you do not use the computer for.
Graphical Work
Algebra
Trigonometry
Geometry
Statistics
Calculus
Other_____
___
___
___
___
___
___

Q13 Would you like to use computers more often in your mathematics lessons?
Yes
No
1
2

Q14 If you answered yes to question 13, what do you see as obstacles to your use of them? Please rank in order any of these that apply (ie 1 for biggest obstacle, 2 for the next, etc.).
Other_____
Lack of confidence
Lack of training
Computer availability
Availability of software
School policy
Other_____
___
___
___
___
___
___

Q22 Please give the main advantage or benefit you have found, or feel to be true, of using technology in mathematics lessons._____

Figure 1: Sample questions from the 2005 survey (some formatting changed).

Results

In 1995 67.2% of the teachers said that they used computers in their teaching, and this remained steady at 68.4% in 2005. Looking at how often the teachers are using the computer in teaching, in 1995 5.9% said they used them at least once a week, but 549 in 2005 this had risen to 13.3%. In 1995 the schools reported a mean of 40.0 computers per school, with 1.7 computers in the mathematics department. By 2005 there had been a jump in these numbers, with a mean of 74.4 computers per school (excluding an outlier school with 1800 laptops), 21.9 of which are laptops, and 26.9% of the schools now have over 100 computers. Mathematics departments have 6.5 computers on average (4.2 laptops). One change has been the increase in the number of ICT rooms, up from 71% of schools in 1995 to 96%, with a mean of 2.46 per school, up from 1.79 in 1995. However, while in 1995 89.1% of mathematics teachers usually used computers in labs this dropped to 59.1% in 2005, with 10.7% using them mostly in their classroom. The question arises though as to whether these increased numbers of computers have changed the pattern of use in the teaching of mathematics.

Computer use in mathematics teaching

The mathematics curriculum in New Zealand schools is divided up into Number, Statistics, Geometry, Algebra and Measurement strands, along with a Processes strand. Number and Measurement are principally primary and intermediate school activities (secondary school usually starts at age 13 years) so those using the computer were asked in which of the remaining curriculum areas (along with specific topics of graphs, trigonometry and calculus) they used them (see Table 1).

Area of Use % of 1995 teachers (n=229) % of 2005 teachers (n=318)
Some Use Most Often Used
Some Use Most Often Used
Geometry 34.1 4.8 28.2 3.9
Statistics 75.1 38.0 85.4 59.5
Graphical work 74.2 35.4 75.5 28.0
Algebra 32.3 4.8 33.4 3.5
Trigonometry 22.7 3.1 22.5 2.3
Calculus 24.0 3.9 22.5 2.6

Table 1: Curriculum areas where secondary teachers are using computers.

These figures show a significant increase in the use of computers for the learning of statistics, both as first choice curriculum area (,2 =24.5, p<0.001), and for some use (,2 =9.47, p<0.01). This not surprising since there is a strong emphasis on Statistics in New Zealand schools, and it lends itself readily to an approach where the computer can be used to perform routine calculations, as well as graphical and investigational work. It is rather surprising in view of the excellent packages Cabri Géomètre and Geometers SketchPad, that there has been a fall (although not a significant one; ,2 =2.07) in the use of geometry packages. Cost may well be a factor in this. Of the 193 schools in the 2005 survey only 20 mathematics departments had a technology budget. The amount of money available ranged from NZ$200 to $NZ15000, with a 550 mean of NZ$2762.50 (NZ$1=US$0.68), and one head of department commented that "Annual [software] fees also take up a lot of the allocated budgets ".

To gain some idea of the variety of uses that computers are being put to in schools each survey asked the teachers to rank in order of regularity of use the types of software they employed in teaching mathematics (see Table 2). It appears that there has been a significant change in the kinds of software used in mathematics classrooms over the period, away from specific content-oriented graphical (,2 =5.59, p<0.05), mathematical (,2 =38.7, p<0.001), and statistical packages (,2 =12.3, p<0.001), and towards generic software, especially the spreadsheet (,2 =28.0, p>0.001), which may handle statistical work well enough for secondary schools.

Area of Use % of 1995 teachers (n=229) % of 2005 teachers (n=318)
Some Use Most Often Used
Some Use Most Often Used
Spreadsheet 67.2 31.9 86.2 62.6
Mathematical Programs 61.1 25.8 34.3 5.0
Graph Drawing Package 61.1 22.3 50.9 17.7
Statistics Package 44.1 11.8 29.6 5.0
Internet --- --- 46.1 6.6

Table 2: Types of software used with computers.

The trend away from specific graphical packages is a little surprising since there are now some excellent programs, such as Autograph, available. Possibly the graphic calculator has made inroads into the use of the computer for graphing functions.

Questions on the use of the internet were new in 2005, and 46.1% of the teachers reported some use of it to teach mathematics. 61.1% of the teachers have access in their classroom (and 68.4% in a staff room). For the students, only 26.4% have classroom access, although 95.6% of schools have ICT rooms connected for them. The question of how teachers organise their lessons around computer use arises.
Since 1995 a number of student-centred constructivist perspectives on teaching very have been widely encouraged in mathematics education circles (e.g., von Glasersfeld, 1991; Ernest, 1997). Has this influenced how computers are used, as one might predict?
Method % of 1995 teachers (n=229) % of 2005 teachers (n=318)
Some Use Most Often
Used
Some Use Most Often Used
Skill Development 67.7 37.6 58.5 24.5
Free Use 34.9 3.1 18.9 2.8
Investigations/PS 68.6 38.4 58.8 27.4 551
Demonstrations 40.6 10.9 59.6 29.7
Programming 8.7 1.3 6.9 1.6
Table 3: Teaching methods used with computers.
We can get some idea of what has happened in the classroom by looking at Table 3, which describes the methods that teachers employ when using the computer. The constructivist approach broadly encourages student-centred investigation and problem solving, rather than teacher-led instruction and enforcing of skills; so one might expect teachers to use the computer to do one or the other, but not both. However, in both 1995 and 2005 it appeared that a substantial proportion of teachers used both methods and did not see themselves on one side of a dichotomous ideological fence. This was shown by around 60% reporting computer use for skill development and demonstrations, as well as investigations. There was, however, a significant decline in the proportion of teachers using the computer for skill development (,2
=4.79, p<0.05), and in those allowing free use of the computer
(,2
=18.0, p<0.001). However, the use of demonstrations significantly increased
(,2
=19.5, p<0.001), and so the data implies that while directed use and demonstration is more common in 2005, it is not as often skill-directed. Again this is not entirely what one might expect from a constructivist perspective. We note that the percentage of teachers who value programming sufficiently to spend some time on it has remained reasonably constant, if somewhat low. It may be that those who are convinced that programming may encourage the formation of mathematical thinking have strong convictions. More recent ideas related to the value of programming suggest that allowing students to interact with games where they are in control, programming attributes and functions in microworld-like games software (Noss & Hoyles, 2000) may be beneficial for learning.
Obstacles to computer use
In the original 1995 survey 93.5% of the teachers responded that they would like to use computers more in their mathematics teaching, however, in the latest survey those agreeing with this sentiment had dropped to 75.1%. While this is a highly significant decrease (,2
=47.0, p<0.001), one must take into account the increased rate
of use of computers, and hence some teachers may feel that they have reached their optimum usage level. In any case there is still a sizeable proportion of the teachers who would like to use them more, and so we are led to ask 'what factors do they perceive as preventing them from making greater use, or using them at all?' The results from the two surveys on this aspect are shown in Table 4.
Obstacle % of 1995 teachers (n=339) % of 2005 teachers (n=452)
First
Mentioned
Mentioned First
Mentioned
Mentioned
Available
Software
17.4 52.5 10.8 39.4 552
Available
Computers
43.7 67.8 42.7 58.0
Lack of Training 17.4 45.4 7.5 31.9
Lack of
Confidence
12.7 34.8 5.3
22.4
Government
Policy
4.1 12.4 --- ---
School Policy 0.6 8.0 0.4 9.3
Table 4: Obstacles teachers mention as preventing computer use in teaching.
In 1995 there were two areas where the teachers wanted to see improvement in order to reach their goal of using computers more. They were the provision of resources, in terms of available hardware and software and the increasing of their confidence through satisfactory training. In 2005 we see that the lack of training has been better addressed, with significantly fewer teachers mentioning it (,2
=15.2, p<0.001),
although only 39.6% of the teachers had recently been on any kind of professional development covering use of technology to teach mathematics. Clearly there is still a need for training, since when department heads were asked how many of their mathematics teachers would not feel confident using technology in their teaching, the mean response was 3.1, compared with a total of 7.2 full time and 3.1 part time mathematics teachers. In addition, significantly fewer feel that they lack confidence in computer use (,2
=15.0, p<0.001), possibly due to greater penetration of computers
in homes over the period. Further, the need for software may have been covered by the greater use of the spreadsheet, which is now provided with virtually all computers. However, the problem of the availability of computers remains the major issue. Although the number of computers in schools is increasing, since they are primarily located in large ICT rooms access to them by mathematics teachers is still the primary problem preventing greater use. The 2005 survey asked teachers if they seldom used the computer room what was the reason, and 38.7% said that it was because of the difficulty with booking the room, and a few said that it was too difficult to organize. There were very few other reasons of note given. Typical teacher comment were "Access to computers at required time (of year and within school timetable blocks)" was difficult, there is a problem "...getting into overused computer suites" and "Due to the increased demand for IT classes it is very difficult to book a computer room for a class of 20-30 students". In addition, in 1995 13% of teachers mentioned some other obstacle, and in 2005 the figure was 18.4%. These included the time and effort needed by both students and teachers in order to become familiar with the technology. It appears that some teachers are concerned that this instrumentation phase would impact on time available for learning mathematics.
Conclusion
What does this research tell us about the changing face of computer use mathematics teaching in New Zealand secondary schools? The percentage of secondary 553 mathematics teachers never using them has remained constant, at around 30%. While there are many more computers in the schools and an increased frequency of use, access to them is still the major obstacle to use in mathematics. They are usually in ICT rooms, and 89.6% of mathematics departments do not have their own technology budget. The primary uses of the computer are for graphical and statistical work, with the spreadsheet and a graph-drawing package the two most common pieces of software. There has been a significant decrease in the use of mathematical programs and statistical packages, and an expected increase in the use of the internet. While teachers are using computers less for skill development, its use is still high, and they have increased the use of demonstrations. Use of the computer is directed over 80% of the time. This pattern of changing use could not really be described as teachers warmly adopting the computer, and there are two important factors worth mentioning here. Only 20.7% of the schools had a technology policy in place, and when they did it usually comprised general statements such as "Technology should be used wherever possible as an aid to learning", "All teachers are expected to integrate ICT into their teaching and learning practices", "Access for all students to internet" or it specified what technology would be used by which year groups, or set rules for internet access and computer room use. Only rarely did it include the acquisition and replacement of software and hardware or the professional development of staff. Such an important omission has been noted previously (Andrews, 1999). It is not surprising that without such a policy the use of computers in schools will tend to lack clear focus and direction. The second issue arose when the 2005 teachers were asked what they thought were the advantages and disadvantages of using computers (technology) in mathematics. While just 8% believed that it aided understanding (compared with 32% who thought it made working quicker or more efficient), 16.8% claimed that it impeded learning or understanding. As Manouchehri (1999, p. 37) reported many "...teachers are not convinced of usefulness of computers in their instruction...", they still feel, like Cuban (2001), that benefits are small or exaggerated, and students rely on technology too much. As several teachers in this research put it "I feel technology in lessons is over-rated. I don't feel learning is significantly enhanced...I feel claims of computer benefits in education are often over-stated.", "Reliance on technology rather than understanding content. ", and "Sometimes some students rely too heavily on [technology] without really understanding basic concepts and unable to calculate by hand." Clearly teachers have a crucial role to play, and their beliefs and attitudes are major elements in the progress in computer use. This is an area for further research.
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Developing resources for teaching and learning mathematics with digital technologies in Enciclomedia, a national project
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Enciclomedia is a Mexican national project that intends to complement already existing materials for primary school classrooms with computer programmes and teaching resources that are to be used with an interactive whiteboard. In this paper we report on the process of development of resources for teaching and learning mathematics with Enciclomedia. Our approach is guided by an enactivist theoretical perspective and methodology (Maturana and Varela 1992; Varela, 1999; Reid, 1996) which invites us to consider our work as a learning process in which we continuously refine the resources we develop. Our work includes analysing existing teaching materials, having conversations with teachers, reading the literature and doing research with the purpose of identifying the kinds of activities the new programmes foster in the classrooms. Throughout this work, we discuss the different

aspects of this process of development of resources. We illustrate our way of working through three different mathematical themes: fractions, probability and area. We have found that working with the multiple perspectives which are prevalent in our group has enriched the production of resources. Finally, the circular nature of the process helps us in refining our methods and in questioning our assumptions; that is, in developing our own learning and making our educational initiatives more effective.

Introduction

Enciclomedia is a large-scale Mexican project that has been devised with the purpose of enriching primary school teaching and learning by working with a computer and an interactive whiteboard in the classrooms. An electronic version of the textbooks that are used in Years 5 and 6 in all primary schools is being enhanced with links to computer tools designed to help teachers with the teaching of all subjects. Our task, as part of the Mathematics group, consists in creating resources which complement the textbooks and can help teachers in their teaching of mathematical concepts.

In this paper we focus on the way of working that has emerged in the practice of creating resources and investigating their use in the classrooms. The process includes the analysis of the textbooks used by the students; the identification of some of the difficulties students and teachers have with the mathematical concepts we are addressing, found both in the literature and in conversations with teachers; the production of resources and the investigation of the way the materials are used in the classroom. We begin by considering some ideas related to enactivism, a theory about learning which guides us both theoretically and methodologically. Later, we discuss the different aspects of the process we are immersed in and which we outlined above. 557 Finally, we talk about the way in which this process has helped us in refining both our ideas and the resources we are creating so that they are more effective in the teaching and learning of mathematics.

Theoretical Framework

Our ideas about learning are based on enactivism, a theory of knowing which considers learning as effective or adequate action (Maturana and Varela, 1992). In enactivism, our minds are seen as 'embodied' and cognition as 'embodied action'. These ideas of 'embodiment' entail two fundamental senses: on the one hand cognition is seen as 'dependent upon the kinds of experience that come from having a body with various sensorimotor capacities' and on the other, individual sensorimotor capacities are considered to be 'themselves embedded in a more encompassing biological and cultural context' (Varela, 1999, p. 12). The first meaning of embodiment locates cognition in our bodies, and prevents us from thinking about it as an abstract notion that is detached from our everyday experience. The second situates our learning in a wider social and cultural context.

In enactivism learning occurs when individuals interact with each other, changing their behaviour, that is, their actions, in a similar way. In a particular context or location, the participants create together the conditions that will allow actions to be adequate. Learning outcomes cannot be predetermined or predicted, but the criteria for the adequateness of actions are, at least in part, specified by teachers and students. With this in mind is that we are interested in developing initiatives that can help teachers to create contexts in which certain actions, related to the learning of different mathematical concepts, can be fostered.

Learning mathematics with computer tools

From an enactivist perspective, the use of computer tools is part of human living experience since 'such technologies are entwined in the practices used by humans to represent and negotiate cultural experience' (Davis et. al., 2000, p. 170). Tools, as material devices and/or symbolic systems, are considered to be mediators of human activity. They constitute an important part of learning, because their use shapes the processes of knowledge construction and of conceptualization (Rabardel, 1999).

When tools are incorporated into students' activities they become instruments, which are mixed entities that include both tools and the ways these are used. They are not merely auxiliary components in the teaching of mathematics; they shape students' actions and therefore they are important components of learning (ibid, 1999).

Every tool generates a space for action, and at the same time it poses on users certain restrictions. This makes possible the emergence of new kinds of actions. The influence that tools exercise on learning is not immediate. Actions are shaped gradually, in a complex process of interaction. In the classrooms, students can construct meanings through the use computer tools, in a process of social interaction and with the guide of the teacher (Mariotti, 2001). The purpose in, Enciclomedia, is to develop programmes which can broaden students' and teachers' experiences with mathematics. 558

Some ideas about methodology

'Enactivism, as a methodology [is] a theory for learning about learning' (Reid, 1996, p. 205). Research is considered to be a way of learning, and therefore researchers are seen as individuals developing their learning in a particular context. From an enactivist perspective, researchers interpret the world in a particular way, influenced by their previous experiences. In addition, in the process of doing research researchers influence and shape the context in which they are immersed (Reid, 1996, p. 206). The interdependence of context and researchers makes the research process a flexible and dynamic one. Research does not occur in a linear fashion; rather, it is seen as a recursive process of asking questions. The enactivist ideas apply not only to research we carry out in the classrooms, but to the whole process of development of resources in Enciclomedia. We think of our educational initiatives as dynamic suggestions which are under constant modification. Teachers are intended to work with our proposals in their practice which is inevitably an ever-changing process. The interactive materials related to specific concepts in the curriculum are used by students in a process which is also dynamic. Researchers' and developers' ideas will be modified as they interact with textbooks, teachers, students, and with each other.

The process of development of resources

Analysing the textbooks

When we work with a particular mathematical concept or process, we usually start by looking at those chapters in the textbooks which are related to it. We try to identify areas where a multimedia resource could be of help. The approach taken by the National Curriculum and the textbooks consists in 'taking to the classrooms those activities which elicit students' interest [in mathematics] and invites them to reflect, to find different ways of solving problems and to formulate arguments that validate their answers' (SEP, 2003, p. 7). We find most problems and activities posed in the textbooks challenging and useful; however, we have also noticed limitations in the presentation of concepts. The problems in the textbooks require of mathematical knowledge which students usually do not have. Complex ideas are often presented without providing the students with any strategies for dealing with them. For example, we have found that the idea of probability is introduced before the students have had enough experiences with random events. Also, sophisticated problems involving rational numbers are presented without providing students (and teachers) with activities that could assist them in their learning. Our purpose is to complement the textbooks providing teachers and students with resources that can help them in working with the problems posed in them.

Conversations with teachers

As we have already mentioned, we work with teachers through the whole process of development of resources. Three teachers who use the resources in their classrooms work part-time in Enciclomedia. They carefully go through the materials we produce and make comments about them. We visit schools regularly, and we work, in 559 particular, with four teachers from two different schools. We have also worked in several workshops including 200 teachers in total.

Teachers find many of the problems in the textbooks difficult. They often say that the books do not provide students with enough exercises for them to learn the concepts. Most teachers also say they do not understand the goal of the chapters and that they do not know how to solve the more complex activities in the textbooks. For example, they often avoid teaching lessons related to probability since they consider it too difficult and not relevant. Regarding area and perimeter, even when teachers say to have a good understanding of the concepts, they prefer to use conventional formulae instead of working with non-standard units as is suggested in the textbooks. We have found that teachers' strategies often differ from the approach taken by the National Curriculum and the textbooks. During the lessons they give definitions for concepts and work on repetitive exercises. Students are frequently left to work on the textbooks' problems on their own and they verify answers in the group without discussing them. Collaborative work is seldom used and students often get distracted.

Looking at the literature

Finding out what is said in the literature about the teaching and learning of the mathematical concepts we are addressing in our project provides us with valuable ideas for our work. We are familiar with different areas of the mathematics education literature; some members of our group have experience in doing research on the teaching of mathematics with the use of digital technologies and each of us has looked at research studies on different mathematical concepts.

For example, regarding the learning of fractions we have found that many studies show that it is a complex process where plenty of difficulties arise (e.g. Cramer et al., 2002). Studies often suggest activities that might help children in developing meanings for rational numbers, such as the use of multiple concrete models. Concerning probability we have found that identifying a random event is difficult as students often confuse a random experiment with a sequence of results (e.g. Schlottmann, 2001). Children often use non-conventional strategies to find the probability of winning a game and many students believe that in all random experiments the events are equally likely. Researchers give suggestions such as the use of simulations to help students observe the temporal and spatial features of many random phenomena and facilitate the development of probabilistic reasoning (Drier, 2000). As to the learning of the concept of area, research shows that many students do not see it as a measure of the spread of a surface, even when they can use formulae to calculate it (e.g. Baturo and Nason, 1996). Students often confuse perimeter with area, and researchers advise to work with this distinction from the beginning. Early introduction of formulae is seen as a contributor to this problem and therefore the use of non-standard units and of ways of obtaining areas and perimeters is suggested.

From the analysis of the textbooks, the conversations we have with teachers and the findings from the literature, many questions arise. We ask ourselves what kinds of experiences can help teachers and students in their learning of the 560 mathematical concepts we are thinking about and whether digital technologies can be of particular help. For example, we start thinking about how multiple representations and computer simulations can help teachers and students reconsider their ideas about these concepts. We then start the production phase, in which we develop interactive programmes and teaching guides.

Producing the materials

Computer programmes. We have developed different types of programmes. They vary, for example, in the kinds of interactivity they promote and in the types of problems they pose to the users. The programmes are closely related to the activities in the students' textbooks, but they are mostly thought of as spaces for mathematical exploration. They usually provide users with something they would not get if they used the textbook only. Programmes give the students immediate feedback on their actions on the computer and they often simulate situations that are difficult to recreate or experience in the classroom. For example, the interactive programme Dados, which is related to probability, simulates large number of occurrences of random events by recreating different games with dice. The Balance, a programme that reproduces a problem situation from the Year 6 textbook in which scales need to be balanced using fractions, provides the users with feedback that helps them in identifying which parts are balanced and which are not. Finally, the programme Perimarea invites students to work with non-standard units and with different ways of calculating the area and perimeter of different shapes.

Teaching guides. Our work includes the development of teaching guides with suggestions on how to use the computer programmes with the interactive whiteboards. We believe learning occurs through interactions, and therefore we recommend teachers to promote collaborative work in their classrooms. The activities we propose include discussions and small-group work. We also advise teachers to promote the manipulation of concrete objects before getting students to work with abstract ideas. We acknowledge the complexity of teaching and learning and the uniqueness of every classroom. Our intention is to provide teachers with guidelines, and to work with them in the development of their practice of the teaching of each particular concept.

Doing research in the classrooms: Back to the beginning

After a first version of the programme related to a particular topic is completed, we start doing research in the classrooms. We make observations and ask questions to students and teachers in order to identify mathematical actions they undertake and to obtain more information on what their difficulties and strategies are. We investigate students' behaviour and, in particular, their mathematical actions. We look for patterns in effective behaviour, which for us define a classroom culture (see Maturana and Varela, 1992). We use audio and video recording as well as fieldnotes to collect data.

The data show, for example, that the use of The Balance invites students to act mathematically in a number of ways. They work with mathematical concepts such 561 equivalent fractions and they operate on mathematical symbols. They use different representations when dealing with fractions and they often ask mathematical questions such as "why is ¨ heavier than #?". We have also observed students giving explanations to justify their answers to the group. Justifications are usually incomplete although we have also recorded sophisticated explanations, such as using graphic representations (of pizzas) to show how a fraction with odd numerator can be divided into two equal parts. Classroom observations have also shown that, using the program Dados students are able to interpret graphs of frequencies, to identify equally likely events and to make predictions taking into account the probability of the events. All this is only possible with teachers' support and intervention. Even when programmes prove to be useful as a means of enriching children's experiences with mathematics, teachers' organisation and strategies are found to be crucial.

We have also observed students as they use the programme Perimarea, where they need to calculate the area for different shapes by counting the squares on a grid. We noticed, both during the lessons and on the video from those sessions, that students often give the answers by trial and error. They get feedback from the programme, showing them graphically whether they are missing or they give too many square units in their answers. Many students write random numbers, while others count squares. In both cases, they do not relate these actions to the concept of area or the formulae they have used beforehand. The original purpose for which we developed Perimarea was not accomplished. For this reason, we decided to modify

the programme so that students were not given automatic feedback and so that they could establish relationships between their counting actions and the concepts of area and perimeter. We have already done some changes to the programme and others are in process. Figure 1 shows the way in which the interface has changed. For example, it new button 'Listo', which teachers can use whenever they want the programme to give feedback to the students, was incorporated.
Figure 1: Changes in Perimarea
Research on how this version of the programme works in the classroom is currently being carried out. We are also wondering whether we can develop more activities that can help students deepen their understanding of area. Revisiting the textbooks and the literature are places to start. Classroom observations often make us reconsider our ideas about the concepts we are working with and about the 562 programmes we are developing; therefore taking us back to the beginning of the process (see Figure 2 for a diagram that shows the recursive process of development of resources).

Figure 2: The circular process of production of resources in Enciclomedia
Conclusions
Research done shows that, when the materials we are developing in Enciclomedia are used in the classroom, students look interested in solving mathematical activities. They reflect on problems and they give explanations to justify their solutions, which are key elements in the approach taken by the National Curriculum. The culture in the classrooms includes interaction and collaborative work. We believe that the open nature of most of our interactive programmes promotes this kind of work.
Difficulties have been encountered in the process of incorporating the new tools in classroom activities. Teachers are often afraid of using Enciclomedia at the beginning and they usually need support and encouragement. Not all the programmes have worked in the way we expected them to and therefore, as in the case of Perimarea, modifications have been needed. We have also changed teachers' guides when, after having been in the classrooms, we find that teachers need particular suggestions. The circular process in which we are immersed has been extremely important in the refinement of our materials thus making them more efficient.
Collaborative work amongst ourselves, the members of the mathematics group in Enciclomedia, has been of prime importance in the process of development of our initiatives. Throughout the process, we use multiple perspectives, which is another feature of the enactivist methodology (Reid, 1996, p. 207). This refers to the exchange of ideas with other researchers and also to the examination of different kinds of data. Through the comparison of different events we are able to explain more. The analysis of the books is done by each member of the team separately; we
Textbooks
Themes
Analysis selection
Conversations with teachers Reading the literature
Learning
Teaching
Production of resources
Computer programmes Teaching guides
Research in schools
Students
Teachers
Weekly meetings
with the academic
group
Analysis of results 563
then discuss our ideas with each other and with teachers. The development of the programmes and the teaching guides involves multiple views, as a number of programmers, designers, mathematicians, educators and students collaborate with us. Doing research in the classrooms also involves the use of multiple views as we collect many types of data and we analyse them from different perspectives.
In the future we will carry on developing our resources as we continue exploring classroom cultures in which the digital technologies provided by Enciclomedia are used. Our recursive way of working helps us in enriching teachers' practices and in developing our own learning about the teaching and learning of mathematics.
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Gender and socio-economic issues in the use of digital technologies in mathematics education
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This paper has been prepared to address the issues and questions of the theme 'access, equity and socio-cultural issues'. Findings from two studies are reported. In the first study gender issues in mathematical learning environments when computers were used were investigated. In the second effective practices for teaching disadvantaged or marginalised students with digital technology are canvassed. Teaching for equity and social justice in the digital age is complex. Teachers need to be aware that their beliefs and classroom practices may exacerbate gender and cultural inequalities in mathematics learning. Approaches that are consistent with social-constructivist and democratic theories need further investigation.
In this paper a synopsis of previous research of equity issues in computer-based secondary mathematics, especially gender issues, and findings from current work that is focussing on responding to diversity and disadvantage of students in mathematics when using digital technologies, are described. The issues raised in this paper concern students' engagement in and attitudes to mathematics learning with digital technologies, teachers' practices and pedagogical approaches that erode or enhance equity and social justice in these environments and the theoretical frameworks that inform and arise from the studies.
Equity involves equal access, equal treatment and equal outcomes in mathematics learning, participation and attitudes (Fennema, 1995). Teaching for social justice requires a commitment to 'closing the gap' and involves fairness, respect, inclusivity and redressing power imbalances (Boaler, 2002; Skovsmose & Valero, 2002).
Impact of digital technology on gender equity
In an earlier ICMI Study 'Gender and Mathematics Education' it was reported that the use of technology in mathematics might erode the advances made toward gender equity in mathematics (Hanna and Nyhof-Young, 1995). Only a few people have investigated gender issues with respect to the use of technology in mathematics. In the context of a narrowing gap in gender differences in achievement in Australia but persistent differences in senior secondary mathematics participation (Vale, Forgasz & Horne, 2004) I began to explore the proposition of threats to equity.
The research involved a grade 8, grade 9 and grade 10 mathematics class and their teachers. The classes were located in two schools where the students came from socio-economic backgrounds that were in the mid-range in Victoria, Australia. The research focussed on classroom practices and culture when using computers in mathematics as previous studies had shown the relationship between classroom practices and differentiated learning outcomes (Fennema, 1995). It was naturalistic in 565 the sense that it sought to study what was actually happening in classrooms rather than to invoke change or innovation using ethnographic methods. Mathematics lessons were observed and video-taped, teachers and students were interviewed, and students completed a questionnaire. Teachers determined the content and learning approaches used in the classrooms and we negotiated the timing of the study. Qualitative and quantitative methods were used to analyse the data collected.
The grade 8 class was timetabled in a computer laboratory for one lesson each week. Students in the grade 9 class owned (or leased) laptop computers that they brought to school each day. They were used for some lessons, when appropriate to the content as determined by the teacher. Students in the grade 10 class used computers for only one topic in the year. They accessed a computer laboratory for three consecutive lessons for this topic. These learning settings are typical of the range of contexts in which teachers can access computers for mathematics lessons. The content of the lessons observed included algebra, number and geometry. Grade 8 students used PowerPoint to present and explain the solution of multi-step linear equations and a spreadsheet to solve applied problems about percentage change. Grade 9 students learned to use dynamic geometry software, completed two guided investigations on geometric properties and a project on the construction of various geometric shapes. Grade 10 students learned to use Graphmatica and used it to investigate the family of quadratic functions and to solve problems about the paths of bouncing balls.
The classroom cultures and students' attitudes and the factors influencing these findings are presented here. Data are to support these findings are published elsewhere (Vale, 1998; Vale, 2002; Vale, 2003; Vale & Leder, 2004). The students in these classrooms were motivated to complete the tasks and with a few exceptions they worked individually on their computers. While the behaviours and attitudes of girls and boys were similar in many respects, the classrooms were masculine domains since the behaviours and interests of the boys defined the cultural norms of the classroom. The boys were louder and took up more space (and in the year 9 class they outnumbered girls 2:1); they were more demonstrative and public about their computer knowledge and competitive about their achievements in mathematics and with computers, a finding consistent with previous studies of gender (Boaler, 1997; Forgasz & Leder, 1996; Schofeld, 1995). Boys benefited in these computer-based mathematics classrooms because they took control of their own learning to learn more about computers. They did this through their off-task activities such as loading software and searching the Internet. Girls and their needs and interests were on the periphery in these classrooms; they did not participate in general classroom discussions, male students denigrated their achievement and the teachers were generally ignorant of their computer skills, especially girls with lower mathematics achievement. Some high achieving girls worked individually as 'silent' participants. Students were positive about the use of computers in mathematics and considered it a natural learning environment for the 21
st
century. However, girls viewed the use
computers in mathematics less favourably than boys. Boys believed that computers 566 were a male domain and that they provided pleasure, relevance and success in mathematics. Girls more often commented on whether computers aided their learning or enabled success in mathematics and high achieving girls in particular were concerned that the use of computers may lead to deterioration in their mathematics skills. More positive attitudes to computers by males have been commonly observed in studies of computing in education. Forgasz (2002) found that the socio-economic status of students mediated gender differences in attitudes to computers in mathematics. Students of high and low socio-economic status were more likely to gender-stereotype the use of computers in mathematics.
In these classrooms the students used the computers as a tool for doing mathematics. There were relatively few interactions between students about the mathematical concepts that they were exploring. When these did occur they were between high achieving boys, who were more likely to comment that computers aided learning of mathematics. High achieving girls also displayed efficient strategies when using the computers to solve problems in the different classrooms. Attitude to the use of computers for learning mathematics was more strongly correlated with attitudes to computers than to mathematics, and this was more strongly the case for boys than girls (Vale & Leder, 2004). Galbraith, Haines and Pemberton (1999) also observed this phenomenon among tertiary students of mathematics but they did report any findings by gender.
Teachers' practices
Teachers' practices, beliefs about computers and mathematics learning, expectations of students and lack of experience with computers and software in junior secondary mathematics, contributed to the culture of these classrooms. The data showed that the approaches and views of the teachers were more strongly in accord with the learning preferences and views of boys.
Many of the tasks observed in these classrooms were consistent with a constructivist approach to learning mathematics, and they had the potential of promoting collaboration in the classroom and engagement in mathematical thinking but this rarely occurred and not for lower achieving students. Furthermore, the teachers in this study perceived computers to be a tool and an opportunity for student enjoyment in mathematics and the grade 8 teacher believed he had a responsibility to teach and use generic software in his mathematics program and sought ways to do this. The grade 9 teacher believed that high achieving mathematics students would benefit the most from using computers.

Teachers differentiated their interactions between girls and boys in the classrooms and according to the mathematical achievement level of students. They were more likely to interact with high achieving students about the mathematical concepts. The grade 8 teacher spent long periods of time individually instructing students with fewer computer literacy skills or confidence. They held gender stereotyped views of students and assumed boys' to be the computer experts in these classrooms and called on them to solve problems. They did not acknowledge the computer skills of 567 lower achieving girls who took on different roles in these classroom settings as successful tutors. Opportunities to engage these students in mathematical thinking while using technology were missed. If mathematics teachers believe boys "know" about computers and girls "learn" computers, then teachers will have different expectations of students in computer-based mathematics lessons.
These teachers needed to be more explicit about the mathematical learning objectives of these tasks, to facilitate collaboration among students and to discuss the processes and findings of their investigations and problem solving in the public forum, for the benefit of all students. They could have provided opportunities for students to generate their own questions, draw on their own ideas, use other software or mathematics knowledge and to work in groups. According to the data gathered from student interviews these approaches would have appealed to the girls and boys in this study. The findings from this study indicate that teachers need to reflect on their own practice and beliefs and the way that these impact on the attitudes and performance of the different groups of girls and boys in their classrooms.
Type of digital technology
A range of software accessed through computers, either desktops in a laboratory or laptops were used. Did this make a difference to gender equity? One might argue that the cultural norms would be similar in normal classroom settings for these teachers and students. The "control" that students exercised with computers, especially laptops, resulted in gendered patterns of activity. Boys used these lessons as an opportunity to learn more about computers and to have fun. Girls were less likely to have computers at home and had less experience of computers. Girls with fewer opportunities to use computers relied on learning their computer skills in classrooms. Without adequate support from their peers or the teacher, students who were not computer literate were excluded from the mathematical learning. Fennema (1995) argued that there has been little progress toward gender equity for lower-achieving girls and findings from this study suggest that this phenomenon is evident in computer-based classrooms. Would hand-held digital technologies be any different? Shaoff-Grubbs (1995) reported positive achievement and attitude outcomes for girls using graphics calculators, but there have been few gender-based studies. The propensity of girls to use and perform better than boys with by hand methods for algebra items in graphics calculator and CAS environments (Forster & Mueller, 2001; Tynan & Asp, 1998) suggests that further research is warranted.
RESPONDING TO DIVERSITY AND DISADVANTAGE
In the current research and work with teachers I have begun to document teaching practices that will support the learning of disadvantaged and marginalised students in technology-based mathematics. Teachers who regularly used digital technologies in their junior secondary classrooms and who gave priority to enabling all students to experience success when using digital technologies were selected. Eight junior secondary mathematics teachers have been involved in the first stage of this project. They are teachers in some of the most disadvantaged schools in Melbourne. Their 568 schools are located in communities with below average socio-economic status, high proportions of students from non-English speaking backgrounds where the most disadvantaged are recent refugees or students living in poverty. I interviewed each of the teachers and we spent one whole day together presenting and sharing teaching materials and strategies.
Each teacher defined equity in terms of equal treatment and fairness, and developing mutual respect. Two teachers were also committed to 'closing the gap' by improving the outcomes of their students relative to students from more advantaged socioeconomic backgrounds. They talked about empowering their students.
The framework for beginning to document and analyse teaching practices is drawn from a number of studies on inclusive practices and social justice (for example, Boaler, 2002; Hayes, Lingard & Mills, 2000; Skovsmose & Valero, 2002). It is summarised using six main characteristics of teaching for equity and social justice: equal access to learning and the use of digital technologies; connected learning; collaborative methods; supportive environments; intellectual quality and respect for difference. Brief descriptions of these ideas and a few examples gathered through this project are presented below.
Equal access is non-trivial. Ensuring that students in schools that are poorly resourced with digital technologies or from poor family backgrounds means that this concept extends beyond merely ensuring that students in a class have equal time hands on with materials and digital technologies. Furthermore as observed in the previous study, the cultural norms of the classroom are critical if students are to be included in mathematical practice and thinking. Teachers in the current study talked about the strategies that they are using to get access to digital technologies for their students and also to find ways of 'closing the gap' for their students. These included school initiatives to provide computers for some families and the provision of additional hands on time in class and homework sessions for students who do not have computers or the Internet at home.
The teachers used some approaches related to constructivist, inclusive and democratic theories of connectedness and empowerment. They described learning tasks that enabled students to build on their prior knowledge, in particular their skills with technology. This was especially the case for two of the teachers who used integrated projects that were socially and culturally relevant to their students. In these projects students explored mathematical concepts or applications and presented their findings using a range of digital media or conducted other inquiries using mathematics and statistics with technology to communicate their findings. These projects were open-ended and aspects of the tasks were negotiated with students. Making mathematics relevant was clearly a goal for teachers. But what are the empowering mathematical concepts, skills and teacher practices in the context of digital technologies? Four of the teachers believed that a focus on the language of mathematics was particularly important for their empowering their students. 569 Selecting or designing tasks based on what students knew and understood about mathematics was less apparent in their practise. One teacher described an investigation of the relationship between the diameter and circumference of a circle after discovering that her grade 9 students held some misconceptions about pi. Each student in the class entered data into a spreadsheet on one laptop connected to a data projector. This teacher described a strong sense of community inquiry as students discussed and asked questions in response to the immediate feedback available by the technology being used in this way.
Collaborative practices recognise the importance of discussion and social interaction for the learning of mathematics and students are encouraged to share their knowledge and skills and to explain their thinking. Presentation and discussion of findings from integrated projects and problem solving tasks was important practice for two of these teachers, but group work with technology was not a common practice. Perhaps this was because all but one of the teachers used computers rather than graphic calculators with their students and they wanted to ensure hands on access for all students. Three teachers used particular seating plans in computer laboratories in order to facilitate peer tutoring and assistance, but effective practices for group work with computers needs to be documented.
In supportive learning environments students feel safe, free from abuse, and respected. Expectations for mathematical thinking and practices are made explicit for students and teachers model and scaffold mathematical thinking in the classroom. The concept of fairness, equal treatment and respectful relationships with students were common meanings of equity and social justice given by the teachers. One teacher deliberately used the grouping of students and seating plans in the computer laboratory to develop more understanding, respect and harmony among his students of diverse cultural backgrounds and educational talents. Each of these teachers demonstrated the technical skills to model mathematical practices with technology and they used guided investigations. Three of them believed that detailed step-bystep instructions were important. In three of the schools students accessed learning tasks through the school network. One of these teachers included voice-overs and another imbedded hints as comments in the instructions and examples that she provided her students for problem solving tasks. Another teacher talked about the need to design questions in written instructions that would support students to interpret the dynamic visual feedback afforded in digital environments.
High expectations and engagement of all students in meaningful mathematical thinking are central to social justice. One way in which teachers in the current study conveyed their expectations was through the dissemination of criteria for assessment tasks, especially the integrated projects. One teacher showed her students examples of similar projects. She gave particular importance to thinking creatively and providing the opportunity to display high-level skills with technology. Four of these teachers regularly used non-routine problems in digital environments to engage students in higher-order thinking and provide challenge. They commented that the 570 instant visual feedback of the digital environment afforded students the freedom to experiment without the fear of failure and public disclosure.
Two teachers in particular talked about the need to understand students' cultural background and create tolerance and respect within their classrooms. While teachers generally recognised gender differences and used real data and applications of mathematics related to the interests of their students, both boys and girls, only one was concerned that they were using mono-cultural contexts for their real life applications of mathematics.
Conclusion
Australia is one of the few countries that have consistently shown no significant gender differences in achievement in the large international studies over the last decade (TIMMS and PISA studies) but socio-economic differences in achievement are more dramatic in Australia than for the OECD average. The research into gender issues summarised in this paper reveals practices that threaten advances toward gender equity. Paying attention to gender issues when using digital technology in mathematics is necessary if further progress is to be made in achieving gender equity in achievement and participation in mathematics around the world.
Further there is reasonable concern that the use of technology in mathematics may focus on the learning and needs of the most successful and socially advantaged mathematics students (Hoyles, 1998). Indeed, a report of a recent global survey to gather cases of exemplary innovative practises in the use of digital technology in education included very few cases that focussed on disadvantaged or marginalised groups (Kozma, 2003). I have attempted to shed some light on the practice of teachers working with disadvantaged students. We need to continue to work toward empowering disadvantaged and marginalised students in the digital age.
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Online homework, quizzes and tests enhanced with an equation editor and tools for collaboration
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An effective use of the Internet for teaching and learning mathematics cannot be accomplished without a method of displaying and manipulating mathematical expressions on the web. There are currently several equations editors used for this purpose, but none of them can be regarded as a standard method yet.
This session contains an overview of currently available equation editors, including: i) MathType, ii) TeX and related tools such as LaTeX2HTML, TtH, JsMath and mimeTeX, iii) MathML and WebEQ, with a discussion their advantages and disadvantages. This overview of available methods of mathematics publishing on the web demonstrates that at present there is no one single solution which is superior to all others in all aspects.
Motivated by the principle that users have various needs and therefore should be given options to choose between different platforms in the most convenient way, we propose another online equation editor, in which we combine ingredients of ITeX, mimeTex and some ideas behind TtH and MathML to offer a solution that can meet today's needs as well as take advantages of possible future improvements in MathML. We discuss features of the new online equation editor that make it a possibly more useful tool for online teaching and learning than other existing equation editors, and we demonstrate the use of this equation editor in online class activities such as discussion boards, chats, online office hours and tutorials.
As an example of the potential application of the equation editor, we describe a project under way at Ohio University on a centralized system of homework, quizzes and tests for undergraduate introductory calculus courses. The central idea of the project is collaboration between instructors and students to build a database of problems with solutions, for which the online equation editor serves as a working instrument. We use the well known WebWork for part of our online homework system, but we also use resources created by instructors, especially for those tests which require detailed written answers. We discuss methods for facilitating the fast and efficient entry of mathematics problems into the databases, and for using these databases in teaching, learning and students knowledge assessment. Finally, we discuss possible collaboration of mathematics teachers between institutions in the U.S. and other countries in the creation of online learning content in mathematics. 573
Use of Graphing Calculators in Pre-university Further Mathematics Curriculum
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In Singapore, the use of graphing calculators in the pre-university level Further Mathematics examinations was first permitted in the year 2001. This study examines Further Mathematics students' performance and usage of GC in solving problems on Curve Sketching, Linear Spaces and Statistics. A total of 190 students enrolled in the second year of the two-year pre-university course in Singapore who took Further Mathematics were surveyed on 3 occasions at intervals from March to September 2003. The results obtained seem to suggest that graphing calculator users perform better academically than non-graphing calculator users. Temporal benefits of graphing calculator usage in a timed pencil-and-paper assessment are also alluded to. Further in-depth studies need to be performed to ascertain the factors surrounding graphing calculator usage, such as teacher proficiency and instruction, and a formal teaching scheme that incorporates the use of the graphing calculator on a regular basis needs to be developed and systematically carried out to ascertain the many facets of graphing calculator usage and its potential as a learning and teaching tool at the pre-university level in Singapore.
Introduction
Ever since its birth more than 40 years ago, the electronic calculator has undergone a number of changes as a result of progress in technology: from models that perform the four basic operations of addition, subtraction, multiplication and division, to complex machines that are able to perform extremely specialised algebraic and symbolic computations instantaneously and to do so with precision (Pomerantz, 1997). Consistent improvements have been made on older models of electronic calculators, the products of which are devices with increased speed and more sophisticated capabilities. Apart from being used frequently in a large variety of vocations, electronic calculators have also gained the attention of mathematics educators and researchers in mathematics education as a tool in the teaching and learning of mathematics. In the local context of Singapore, scientific calculators have been used in the teaching and learning of secondary school mathematics from Secondary One through to Secondary Four and in junior colleges (JC; pre-university level) for a number of years. However, the use of the graphing calculator (GC) in the teaching of pre-university level mathematics is a fairly recent endeavour locally. In fact, students were only allowed to use a GC while taking the GCE Advanced Level Further Mathematics examinations since the year of 2001. As there is a dearth of research on the impact of GC use and JC student achievement in Singapore, the 574 purpose of this study therefore, is to determine whether the use of graphing calculators by Year 2 JC students in Further Mathematics enhance academic achievement and overall performance in problem solving.
Since its inception in the learning and teaching of mathematics, a number of researchers have studied the use of calculators in the classroom. Campbell and Stewart (1993) have found that calculators stimulate problem solving, extend number sense and reinforce comprehension of arithmetic operations. Research conducted by Hembree and Dessart (1986) also show that students who utilise calculators regularly obtain higher achievement scores and improved paper-and-pencil skills in basic operations and problem-solving. This directly addresses the concerns that paper-andpencil skills will be rendered obsolete with the frequent use of the calculator.
Suydam (1987) surmised, through scrutiny of over 100 studies, that the use of calculators encourages achievement, improves skills in problem solving and heightens understanding of mathematical concepts. Some researchers propose that the use of calculators make available precious time that was used previously for manual computations that allow teachers to introduce more mathematically investigative work and therefore let students have the opportunity to access previously limited concepts and experiences, specifically in exploration and experimentation in mathematics (Fey & Hirsch, 1992; Pomerantz, 1997).
Despite the rise in popularity of technology, the use of calculators in classroom teaching and learning has been an area of contention since its introduction. Pomerantz (1997) addresses what she calls "myths" in classroom calculator usage that have been raised by various individuals involved in children's education such as mathematics educators and researchers themselves as well as parents. These "myths" include notions that calculators are a crutch for lazy students, that they reduce the level of challenge of mathematical tasks and prevent students from learning the basic mathematics required in many vocations. As a result of frequent calculator usage, individuals might also develop overdependence on these instruments which will therefore cause helplessness in manual computations should the need for paper-andpencil approaches arise. Boers & Jones (1994) found that students were particularly concerned about the possible outcomes of over-dependency on graphing calculators and Dunham (1991) noted that this was particularly true of female students. Some parents have also expressed reservation about the use of calculators in school on the basis that acquisition of mathematical concepts and skills requires paper-and-pencil computations, algorithms and drill work encompassed in traditional mathematics education, which they have been part of. While these are practical concerns that could have stemmed from the misuse of calculators in education, research has shown that the use of calculators in the teaching and learning of mathematics does have its merits. As with all teaching and learning tools, the calculator needs to be utilised appropriately in order for students and teachers alike to reap its benefits.
The value of calculators not only lies in the improvement of students' mathematical cognition and achievement as discussed above. Other studies have shown that 575 calculator usage increases students' self-concept and fosters a better attitude toward mathematics (Dunham, 1995; Hembree & Dessart, 1986). Strong evidence has been found that the use of graphing calculators has significant influence on students' mathematical achievement and on the ways they approach the solution to a problem (Ruthven, 1995). Later work by van Streun et al. (2000) supports Ruthven's findings but goes further in noting that the calculator does not confer any additional advantage in data interpretation. These findings align themselves with research that calculators in general are a merely aids in problem solving and do not think on the behalf of the student. Calculators remove the tedious computation steps that frequently deter students from taking mathematics education to a higher level, which means accessibility to more complex mathematics and allows students to focus on developing mathematical understanding and their repertoire of problem solving techniques (Pomerantz, 1997).
With the progress of technology in mathematics, there have also been questions raised by countries such as Britain regarding students' use of calculators while they are being assessed in mathematical knowledge and skill (Qualifications & Curriculum Authority, 1999). Ellis and Brown (1997) propose that assessment modes should complement the curriculum since students who are expected to utilise a calculator throughout the course of their study need to be assessed fairly. Many examination boards in the world have started incorporating technology in the teaching and learning of Mathematics curriculum of late. For example, Principles and Standards for School Mathematics published by the NTCM in year 2000, advocates the use of electronic technologies - calculators and computers as essential tools for teaching, learning and doing Mathematics. Also, the various tests administered by the College Board (SAT, SAT II, Advanced Placement) either permit or require the use of a graphing calculator. From May 2000, the use of graphing calculators (GCs) was compulsory for three out of the four Diploma Programme Maths course in the International Baccalaureate (IB) curriculum. Victoria state in Australia has been approving the use of a GC in many of its Mathematics examinations since 1999 and it is looking further into a CAS calculatorbased Mathematics module. In Britain, the Qualification and Curriculum Authority (QCA) has listed the use of contemporary calculator technology" as one of the five assessment objectives within its "Subject Criteria for Mathematics" (QCA, 1999). Other countries, such as Denmark, Norway and Portugal which have a national curriculum for 16-19 age group are making the use of GCs compulsory (Edward, 2000).
In Singapore, students taking the advanced level Further Mathematics examinations at the end of their second year of junior college have been allowed the use of a GC since 2001, though the questions set will be GC neutral. This has tremendous implications on the learning and teaching of Further Mathematics at the JC level in Singapore. As yet, students appear to be underutilising the GC that might be due to the lack of familiarity with GC use on the part of teachers and as a result, students themselves. The aim of this study is to ascertain whether students' utilisation of GCs 576 throughout curriculum time have an impact, either positive or negative, on the subsequent assessments of local JC students in Further Mathematics.
Design of Study
The research participants comprised of all 190 Year 2 students (in Year 2003) who took Further Mathematics at a junior college in Singapore. Of the 190 students, approximately 60% possessed either a TI83 or TI83 Plus GC. The research was carried out by conducting surveys of students' performance on 3 different tests covering 3 different topics over a period of 6 months, from March to September 2003. The surveys were conducted after each test was marked and returned to students.
Survey 1 involved students' usage of the GC while attempting Question 6 of the June Common Test on Curve Sketching, where students were requested to state whether they used a GC in obtaining or checking the graph. Survey 2 on Linear Space which was part of a lecture test and required students to indicate the length of time spent on the question posed and whether they utilised a GC in obtaining the solution or for checking their solutions. The final survey, Survey 3 on Paper 2 of the Preliminary Examination was obtained based on students' academic overall academic score and GC usage patterns.
Table 2: Surveys and their related test items
Survey Question
1
Curve Sketching (June Common Test Question 6)
(a) The curve C has equation x
y x
*
= + *+
where
* - 0, 0 x -
.
(i) Find the set of values of
*
for C to cut the x-axis at
two distinct points. [2]
(ii) If 1 <
*
< 2, draw a sketch of C, labelling the
asymptotes, stationary points and any intersections with the
coordinate axes, if they exist. [5]
[Please indicate on the cover page whether you made use of the graphical calculator.]
2
Linear Spaces (Lecture Test)
If x is an eigenvector of each of the square matrices A and B with the corresponding eigenvalues . and p respectively. Show that x is an eigenvector of
(i) kA,
(ii) A + B,
and find their corresponding eigenvalues. 577
[4]
Find the eigenvalues and the corresponding eigenvectors of the matrix A, where

A =
!
!
!
!
!
!
!
!
8 2 7
8 6 3
1 3 3
.
[6]
Hence, find a matrix P and a diagonal matrix D in terms of n such
that SP = PD where
S = A + 2A + 3A + ...+ nA, n/Z
+
.
[5]
3
Preliminary Examination Paper 2
One of the questions that GC could be useful is Question 7 as
follows.
A certain local authority was looking into the length-of-service
characteristics of its employees. Jobs were classified as being
manual, technical or administrative. Records were available
showing how long each employee had served with the authority.
150 employees, chosen at random from all the employees of the
authority, were investigated and the results were as follows:
Type of Jobs Length of service
Manual < 6 months 6 months to 2 years > 2 years
Technical 30 11 19
Administrative 13 13 20
9 8 27
Examine whether the data provide evidence of an association
between the types of job and the length of service of the employee
at the 1% significance level. [6]
RESULTS AND DISCUSSION
The results of study are divided into 3 sections, each based on the corresponding
survey that was carried out. The outcome of the first survey that is presented in Table
2 shows that out of the 33 students who responded that they used a GC in either
checking or obtaining the graph, 20 of them (60.6%) received 4 or 5 marks out of a 578
possible 5. Conversely, 23 students out of the 55 who responded that they did not use
the GC in attempting the question received 4 or 5 marks (that is, 41.8%). The
difference between the percentages of students obtaining 4 or 5 marks for with and
without GC is 18.8%, which seems to suggest that students who used the GC
obtained better results. Similarly, a comparison between students who obtained 0 or 1
mark shows that those who utilised a GC are less likely to perform poorly. This
appears to be in keeping with Dunham's review of research that students who utilise
graphing technology have greater overall achievement on questions that require
graphing solutions (Dunham, 1993).
Table 3: Results of Survey 1 showing students' use of the GC and their
results
Marks Obtained (Total = 5)
0 or 1 2 or 3 4 or 5
Graphing calculator used
(Total of 33 students)
1
(3.0%)
12
(36.4%)
20
(60.6%)
No graphing calculator used
(Total of 55 students)
13
(23.6%)
19
(34.5%)
23
(41.8%)
Table 3 shows a breakdown of survey responses obtained from students during the
lecture test. As the results show, students generally take less time in obtaining the
solution to the computationally intensive Linear Spaces question when they utilised a
GC during the problem solving process. 22 of the 34 students in class 06 used a GC
in getting answers and found that they did not spend a lot of time on the question in
general. On the other hand, classes where majority of students did not use a GC at all
show that a considerable number of them spent a significant amount of time on the
problem (e.g. classes 01 and 09). However, as the duration of "a lot of time" is
subjective, the results are certainly not definitive, but rather indicative in nature and
hints at the possibility of the temporal benefits of GC utilisation.
Table 4: A comparison of the number of students and time spent on a
Linear Spaces question based on their usage of GC
Class
Used a GC in
getting
answers
Spent a
lot of
time
Did not use
a GC at all
Spent a lot
of time
Used a GC
for checking
only
01 2 0 17 5 0
02 8 0 14 4 7
03 3 1 16 1 2
04 6 1 7 2 9
05 22 0 2 1 11
06 10 1 10 4 0 579
07 0 0 17 11 3
08 6 0 15 7 7
09 0 0 21 6 1
10 1 0 13 0 3
In the final survey examines the number of students who utilised a GC and their
corresponding marks in Paper 2 of the Preliminary Examination. A total of 182
students took the test conducted in late September 2003. 115 students (63%) used a
GC in either solving or checking solutions and 67 students (37%) did not use a GC.
The number of students who utilised the GC increased to 63%, presumably due to
their improved familiarity with the GC towards the end of the year through increased
usage or else they might have found that the GC's capabilities were more suited to
solving questions on Statistics.
Table 5: Comparison of student achievement in the Preliminary
Examination based on usage of GCs
Used a GC Did not use a GC
Total marks < 45 45 to 70 0 70 < 45 45 to 70 0 70
No of
students
30 70 15 21 40 6
% of total 16.5% 38.5% 8.2% 11.5% 22.0% 3.3%
% of
category*
26.1% 60.9% 13.0% 31.3% 59.7% 9.0%
*Category refers to those who used a GC and those who did not
The percentage of students who obtained less than 45 marks in total was slightly
higher for non-users of GC at 31.3% than for GC users at 26.1%. While the
percentages of students who acquired a total mark between 45 and 70 are comparable
(60.9% for GC users and 59.7% for non-users of GC), there was a higher percentage
of students who obtained greater or equal to 70 marks for GC users (13.0%).
Although a detailed breakdown of marks for Question 7 (see Table 1) was not
acquired, the student distribution of marks obtained for that particular question might
be similar to the pattern that is observed through the total marks for the paper. This
suggests that academic achievement of Year 2 JC students of Further Mathematics
who use GCs is likely to be in keeping with observations made by researchers on the
general improvement of student achievement evidenced through the use of
calculators (Dunham, 1993; Hembree & Dessart, 1986; Kitchen, 1998; Suydam,
1987).
Conclusion
The 3 surveys reveal that the GC is likely to be a useful tool for most of the Year 2
JC Further Mathematics students surveyed in terms of improved academic
achievement and might confer temporal advantage during test and examination 580
situations where time is limited. The advantages in using the GC, however, are also
contingent upon appropriate usage as Burrill et al. (2002) have determined. As
evidenced from the number of students surveyed who have yet to utilise the GC and
the results of some GC users, the instrument's strengths have not been fully
capitalised in terms of its capabilities in the learning and teaching of Further
Mathematics.
The results of GC users in the surveys, while better in general than those who did not
use GC users were marginal in some cases and there were individuals who utilised
GCs but yet took a longer time in obtaining the solutions to questions. The reason for
this is yet unknown in the local JC context. However, this result does indicate that
there is a possibility that GCs might not contribute to academic achievement and
improved problem solving skills or that students are not utilising the GC to full
advantage. From the general trend of the findings in this study and those of others,
such as Dunham and Dick (1994) and Ruthven (1995), the former is not likely the
case. Students' competency in GC usage is liable to be influenced by a number of
factors, such as accessibility to the GC, familiarity with its functions and the extent to
which students are exposed to its use, often predetermined by teachers. As such, this
study is lacking in terms of qualitative data that surround the learning and teaching of
GC skills. Hence the results obtained, while not definitively conclusive, do intimate
the potential and pitfalls in the application of GC in the learning and teaching of
Further Mathematics.
Further refinement of the current mathematics curriculum needs to be considered as a
GC-neutral mathematics syllabus does not promote the usage of handheld
technology. Moreover, maintaining a GC-neutral stance in set problems is difficult to
achieve as evidenced by the question identified in the Preliminary Examination
(Table 1). Proficient GC users will therefore gain an added advantage due to their
skill in the instrument, resulting in what might be construed as an unfair advantage to
those who choose not to utilise the GC for a variety of reasons, based on the
knowledge that the syllabus is GC-neutral. Future research needs to look specifically
into some of the factors outlined above, such as teachers' competency in GC usage
and the impartation of GC skills as well as conflicts in local curriculum and
assessment.
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Challenging Known Transitions: Research of technology supported long-term learning

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In this paper I address questions related to the implementation of curricula. I offer a framework for research that build on the assumption that although the uses of digital technologies offer ways to redesign curricula with an attempt to create as smooth as possible sequence there is no a single design that can achieve this challenge. Thus, it is important for research to study the stability of known transitions and to explore new critical transitions. A transition is a learning situation that is found to involve a noticeable change of point of view. This change could become apparent as an epistemological obstacle, as a cognitive discontinuity or a didactical gap. The implementation of new curricula and practices in the classroom depends on the ways research would help teachers and designers anticipate transitions. Examples of studies that sought rational for students' progress throughout analysis of curricular decisions will be given below.

Long term learning with technology: A role for research

Technologists and educators speculate about the degree to which new technologies will lead to replacement of current curricula with new content (Papert, 1996; Schwartz,1999; Noss,2001). How does the use of a new curriculum that is based upon new epistemological assumptions change our capability to anticipate students' difficulties and strengths? To start answer the question I would introduce the term: transition. A transition is a learning situation that is found to involve a noticeable change of point of view. This change could become apparent as an epistemological obstacle, as a cognitive discontinuity or a didactical gap. Transition would be identified as a necessity in entering into a different type of discourse (in terms of the language, symbols, tools and representations involved) or more broadly as changing "lenses" one uses to view the concept at hand.

Obviously there are transitions in any sequence of learning. In attempting to design a smooth as possible sequences Tall (2002) defines cognitive roots to be the kernel of continuous cognitive sequences and argues that while cognitive root would not always work for all students as creating cognitive continuity it would in many cases offer a solution to some critical transitions. Technology, when appropriately designed and used, can help to design learning environments that may change to various degrees the assumption about previous knowledge and the order new concepts are introduced. In designing new technology supported sequences we would expect that some known discontinuities disappear but others would not and very probably new gaps will appear. Noss (2001) who is elaborating on the implications of rethinking the mathematics learned with new technological environments is expecting that the epistemology of the mathematics learned with technology would change our ideas about cognitive hierarchies and the didactical attempts to construct them. Thus, an interesting challenge for research is to question stability of known transitions when new computational environments are 583 introduced to the learning and teaching, to go beyond known transitions to distinguish types of transitions and to illuminate the nature of critical transitions in technological based curricula. The implementation of new curricula and practices in the classroom depends on the ways research would help teachers and designers anticipate transitions that might be reflected in students' difficulties. In fact I believe that the degree to which technology is likely to be essential productive part of new curricula depend on the availability of studies that sought rational for students' progress throughout analysis of curricular decisions.

Visual Math: An example of long-term learning sequence

The Visual Math curriculum (1995) is an algebra, pre-calculus and calculus curriculum where technology is being used to help learners develop knowledge from their perceptions of the world and to develop conceptual understanding of symbols. The growing research in the field of embodied cognition suggest the idea that bodily activities are centrally involved in conceptualization of mathematics and that important parts of Algebra and Calculus are understood via conceptual metaphors in term of more concrete concepts (Lakoff and Nunez, 2000). In particular the notion of continuous functions and directed graphs are viewed as mathematical concepts developed through human motion experience. The learning of algebra in VisualMath is preceded by semi-qualitative modeling and an environment that allows users to construct and model motion generated by the movement of the computer's mouse supports the introduction of modeling of motion. Another challenge of the innovative development is rooted in the symbolic world where technology that supports multi representations of functions allows students to develop symbolic understanding using the feedback from graphs or table of values, generating and viewing a rich repertoire of non-prototypic examples. In general, an important goal of this curriculum is to help students develop strong symbolic skills and to learn to do a variety of standard algebraic manipulations. But, the curriculum is aimed at helping students learn to do such manipulations with an understanding of the graphical and tabular meanings of these manipulations, as well as a sense of the purposes for which such manipulations are useful. Such proficiency involves moving across the various views of symbols, graphs, equations and functions and to help students learn to shift their point of view.

We will look at how such new epistemological structures afforded by digital technology impact the cognitive hierarchies; resolve or change the nature of known transitions or mark new critical discontinuities in the curricula.

Known critical moments demanding new transitions

The lion's share of the early parts of the Visual Math Algebra curriculum focuses on functions of one variable and equations of one variable conceptualized as the comparison of two functions of one variable. With this way of thinking about equations that is now taken by a few new technological based developments, students acquire alongside the algebraic procedures alternative methods to solve equations. Studies report that algebra beginners viewing an equation as a comparison of two functions, students who had not learned procedures beyond the linear equation can solve problems for which they have not 584 yet been taught an algorithmic solution method with and without use of graphic software (see for example: Huntley et al. 2000, Hershkowitz et al. 2002).

In typical algebra instruction, solving an equation in two variables and then a system of equations requires to shift from a non explicit form (x+y=2) to an explicit function form: y=2-x. One then substitute and use similar solution techniques as in a single variable case. While technique does not require dramatic change the shift from an equation in a single variable to two variables requires a shift in understanding the nature of the solution: from a single definite solution to a set of solutions. For the Visual Math function approach students the equal sign of the equation represents a symmetric comparison sign and the function equal sign represents an a symmetric assigning sign. Thus, the fact that simple manipulations techniques can help move from one form to the other does not seem to be useful. Thus many would not choose this option and would keep viewing equations as comparisons of two functions f(x,y)=g(x,y). Taking this view the nature of the solution does not change brutally - it remains the intersection values of the intersection of the two functions. But then they have to overcome another transition: The graphical and tabular representations of functions of two variables must be developed in order to help students see their connections to the ways of representing functions of one variable with which they are familiar. Thus the transition remained critical; either one has to develop new ideas about presentations of function in two variables or one has to rewrite the equation in a way that violets the distinction between function and equation. It is required to acknowledge that simple algebraic technique can change the mathematical objects in hand: from equation in two variables to a function in a single variable. Thus, in looking at critical known transitions we would have to study the probable different nature of the transition in this new domain.

The second example offers another familiar critical moment that technology might not smooth but rather introduce a reformed transition that develops in a different direction than in the traditional sequence.

Using technology such as simulations' software, MBL or other modeling tools that includes dynamic forms of representations of computational processes, it is now possible to construct graphical models without first writing symbolic expressions with x's and y's. Several studies suggest that such emphasize on modeling offers students means and tools to reason about differences and variations (rate of change). Apparently, throughout the curriculum's focus on qualitative modeling, the students we will describe had developed ways of using tools to solve complex problems that concern non-constant rate of change. Graphs and what we will call staircases (a graphical depiction of differences in y value for a set change in x Schwartz and Yerushalmy 1995) emerged as models of situations, and also as models for reasoning about mathematical concepts. Using the grammar of objects and the operations on them, young students constructed complex mathematical models, based on qualitative analysis of variation. Technology, like the one implemented in Visual Math or a spreadsheet may suggest that a closed rule is no longer a more natural way to describe a function (for related thoughts based on student performance see Stacey and MacGregor 2000). Thus, if students become familiar first with ideas of continuous change 585 and finite differences, explicit closed forms may become less natural aspect of expressing phenomenon. For example: in Yerushalmy (2005) I describe students whose earlier experience with ideas of differences complicated their use of explicit forms. They were seeking help in understanding why two numerical phenomena they identified as appropriate: the linearly increasing differences and the squaring describe the same quadratic phenomenon. In other words: why does the solution of the difference equation f(x+1) -f(x)= ax+b is of the type: f(x)=x^2. This complication might not have arisen if they had not had earlier support for recursive reasoning. However, the other option, the one most sequences follow of emphasizing explicit rules (closed forms) and then learn to describe it as analysis of differences is problematic as well. Thus the affordances of technology that made the recursive thinking the natural way to think about a phenomenon and to symbolize it in a model is viewed as strength however, it challenges thinking about teaching that can support the transition to algebraic close rules.

A similar state of a known critical difficulty that introduce a transition although the technological based curricular sequence has been redesigned is described by Tall (2002) and by Schnepp (unpublished). Tall describes the strength of a sequence that is based on visual notions of the function to explain the chain rule (the derivative of a composite function) coming from the derivative of addition of functions. However, Tall argues, the teaching of the product rule (derivative of multiplication) is yet to be supported by this sequence. Schnepp points on similar difficulty and argues that the discontinuity can be fixed if coming to the multiplication from the composition. Thus known critical transition remains a challenge but the use of technology turns the order of the sequence in which this transition occurs and reforms the cognitive hierarchies developed and thus the difficulties might arise.

Identifying new critical transition

The third example challenge what the examples above suggest as stability or partial stability and it attempt to support the claim that such critical moments could be mutable. I will argue that a new epistemology introduces new critical moments rather than just changing the cognitive hierarchies as demonstrated in the first two examples.

Research suggests that mature problem solvers of word problems in algebra devote a substantial portion of their work to representation of the problem at the situational level. Forming the situational model is a necessary stage in understanding the story of the problem and is a major component of model-based reasoning. Recently Gilead (Gilead 2002) studied Visual Math students and equations' based algebra students solving word problems. 87.3% of the 196 solutions given by the Visual Math students for problems in context included graphical description of the situation that formed a situation model either using a sketch (73%) or an accurate graph (14.3%). A comparative investigation of the Visual Math students with comparable algebra students suggest that the students who were the more successful students of a traditional algebra sequence which focus on unknowns and stress manipulations of equations were substantially less capable to solve the same problems that were part of their curricular sequence as well. While 90.8% of the solutions 586 of the Visual Math were correct solutions only 57.2% correct solutions were given by the equations' based approach students.

This same study that was designed to analyze the index of difficulty of algebra word problems as related to the approach students learn algebra included a set of problems that were hard to the Visual Math students. They succeeded to provide 41.8% correct solutions on these problems (while the traditional algebra students provided 51.5% correct solutions.) The harder problems were problems that we categorized in another study (Yerushalmy & Gilead 1999) as non canonic problems or problems that have Sketchable Situation Structure. We define canonic problems to be Graphable: those for which the functions in the situational structure can be uniquely described symbolically and graphically as functions of time. In fact Gilead found that a main problem in solving Sketchable problems is the complexity involved in formulating an equation that represents rules describing behavior of unknowns. The following summary of results strengthen this view: While in 70.9% of the correct solutions of canonic problems the equations were based on coherent graphic model only in a single correct solution of a non-canonic problem a situational model was coherent to the equation used to solve the problem.

Function's approach to algebra that we took stressed the algebra signs and symbols and the expressions and equations using these symbols to be a meaningful language to express ideas. One of the ways to observe meaning is to view equations as describing situations out of the mathematics and as graphical models. This habit has been proved to support students learning when solving constant rate problems. However, situations that were difficult to be described by graphical models that can be easily mirrored in an equation were less natural for students. Thus, while known complexities disappeared and students were successful in solving new problems as mature problem solvers, we identified a range of problems that were found to be complex to the function's approach students and were not at all harder to the equation's approach students.

The study of this transition sprang of a learning experiment and in a way was incidental. It led us to a systematical investigation of the structures of algebra word problems that suggest new insights on epistemology of constant rate models (Yerushalmy & Gilead 1999). It also raised a more general question regarding strategies that could guide profound studies of curricular decisions to support teaching and learning with new technologies.

Concluding remark

I have demonstrated the necessity to study changes of cognitive hierarchies that involve learning with technology. Studying these changes is appropriate when one has a chance to follow learning and teaching for a substantial period of time, observing students' strengths, identifying the resources for these strengths, watching how the students get involved in a transition and analyzing the reasons for the discontinuity. Although, I believe that obstacles should be reconsidered when new tools are involved, I suggest that often transitions between mathematical views of concepts remain complex and suspect them to be found independent of the technology used. Computational technologies allow us to improve the design of mathematical learning environments. In order for research to be helpful for 587 teaching and learning in the new context it is important to devise and use strategies that

would support systematic analysis of critical transitions.
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Introduction
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The study aims to produce a state-of-the-art summary of relevant
research and to go beyond that summary to develop new insights and new areas of
knowledge and study about task design. In particular, we aim to develop more
explicit understanding of the difficulties involved in designing and implementing
tasks, and of the interfaces between the teaching, researching, and designing roles
- recognising that these might be undertaken by the same person, or by
completely separate teams.
Background
In her plenary address to the International Group for Psychology of Mathematics
Education (PME) Sierpinska (2003) identified task design and use as a core issue in research
reports and in mathematics education research more generally. She commented that research
reports rarely give sufficient detail about tasks for them to be used by someone else in the
same way. Few studies justify task choice or identify what features of a task are essential and
what features are irrelevant to the study. In some studies using intervention/treatment Introduction
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comparisons to investigate cognitive development, the intervention tasks are often vague, as if
the reader can infer what the learning environment was like from a few brief indications. A
similar view had been expressed by Schoenfeld (1980). Yet we learn from applications of
variation theory to learning study (e.g., Runesson, 2005), from studies of learning from
worked examples (e.g., Renkl, 2005), and from the Adaptive Control of Thought model
(ACT-R) (e.g., Anderson & Schunn, 2000) that seemingly minor differences in tasks can have
significant effects on learning.
At the same time Burkhardt has drawn attention to the importance of design, with the
founding of an international society and a journal, Educational Designer
(www.educationaldesigner.org). (Schoenfeld, 2009) makes a plea for more communication
between designers and researchers, making the point, among others, that many designers are
not articulate about their design principles, and may not be informed by research. In 2008, the
International Congress on Mathematics Education (ICME) hosted a topic study group (TSG),
Research and development in task design and analysis, which provided a forum for that kind
of interaction (http://tsg.icme11.org/tsg/show/35). Designers had to be explicit about their
principles and demonstrate how they used them. Participants were given the opportunity to
experience various tasks, and compare and critique design principles. Drawing from a wide
international field, an overview of the papers makes it apparent that:
(1) it is necessary to have theories about learners' intellectual engagement to have
successful design; and
(2) most design principles included the use of several representations, several kinds
of sensory engagement, and several question types.
The TSG increased its membership during the conference, indicating that a serious,
organised look at task design was of growing interest. A further TSG is due to take place at
ICME 12 in July 2012 in Seoul, Korea. Working groups on task design using digital
technologies, and design of digital learning environments, proliferate, but we are not aware of
a similar level of activity in other environments.
Mathematics educators have focused to a great extent on the social cultures of
classrooms and designed learning environments, on patterns of argumentation, on emotional
aspects of engagement, and on measures of learning. A distinct mathematical contribution can
be made in understanding whether and how doing tasks, of whatever kind, enables conceptual
learning. For example, Lagrange (2002) suggests that applying routine techniques can achieve
results, and also provide the basis for conceptual understanding and new theorising; (Watson
& Mason, 2006) have shown how a set of procedural exercises, seen as one object, can
provide raw material for conceptualisation; Realistic Mathematics Education (RME) from the
Netherlands and Mathematics in Context materials (from the United States) show how
carefully designed situational sequences can turn a learners' attention to abstract similarities.
Our statement that task design is core to effective teaching is well-illustrated by the
success of theoretically-based long term design-research projects resulting in publications
such as those from Shell Centre (Swan, 1985), Realistic Mathematics Education (de Lange,
1996) and Connected Mathematics (Lappan & Phillips, 2009). In these, design and research
over time have combined to develop materials and approaches that have appealed to teachers.
In addition, research related to the QUASAR project (Quantitative Understanding:
Amplifying Student Achievement and Reasoning) found that the cognitive demand of
designed tasks was often reduced during implementation (Henningsen & Stein, 1997). A
research forum at PME in Mexico (Tzur, Sullivan, & Zaslavsky, 2008) offered cogent
explanations for the inevitability and even desirability of teachers' alteration of the cognitive
demand of tasks. Further, Choppin (2011) suggests how adaptation differs among teachers.
Thus, a possible area of investigation is how published tasks are appropriated by teachers for
complex purposes. In variation theory, a distinction is made between the intended, enacted, Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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and lived objects of learning. The Documentational Approach of Didactics (Gueudet &
Trouche, 2009, 2011) also refers to the practitioner perspective in terms of the resources on
which teachers draw. Didactic engineering was the topic of the 15
th
summer school in
mathematics didactics in 2009 (Margolinas, Abboud-Blanchard, Bueno-Ravel, Douek,
Fluckiger, Gibel, Vandebrouck, & Wozniak, 2011). The discussion focused not only on
various principles of task design (see the contributions of Bessot, Chevallard, Boero, and
Schneider) but also on the problem of the influence of task design on the development of
actual mathematics teaching (see contributions of Perrin-Glorian, René de Cotret and Robert).
The tasks in these references are all complex, multi-stage tasks which address complex
purposes, such as those usefully summarised in Kilpatrick, Swafford, & Findell (2011),
namely the development of conceptual understanding; procedural fluency; strategic
competence; adaptive reasoning; and productive disposition.
We would like to encourage an interest in tasks that have more limited but valid
intentions, such as tasks that have a change in conceptual understanding as an aim, or tasks
that focus only on fluency and accuracy. Research can investigate how students perceive and
conceptualise from the examples they are given, or on which they work. Most mathematics
learners world-wide learn procedures and possibly concepts through 'practice', regardless of
the de-emphasis on procedures held by reform enthusiasts. Thus, the design of sequences of
near-similar tasks deserves attention. For reasons of global reality and equity, the study
conference shall also focus on textbook design partly because textbooks are often informed by
tradition or by an examination syllabus rather than through research and development
(Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002), but also because in some countries
textbooks are the major force for change. Textbooks are not the only medium in which
sequences of tasks, designed to afford progressive understanding or shifts to other levels of
perception, can be presented, and we expect that study conference participants will look also
at the design of online task banks.
Work from Sullivan indicates the need to educate new teachers in the use of complex
tasks (Sullivan, 1999) and it is inevitable that teacher education will cross several of our
suggested areas. A volume of the Handbook of Mathematics Teacher Education was devoted
to the tasks and processes of teacher education (Tirosh & Wood, 2009). A particular
relationship between teacher education and task design is the design of tasks for teacher
education purposes. Mathematics teacher education, as a subfield of mathematics education,
has paid significant recent attention to the nature, role and use of tasks with a triple special
issue of the Journal of Mathematics Teacher Education (volume 10, 4-6) edited by Mason,
Watson and Zaslavsky, and a book edited by Zaslavsky & Sullivan (2011)
The meaning of 'task'
The word 'task' is used in different ways. In activity theory (Leont'ev, 1975) task
means an operation undertaken within certain constraints and conditions (that is in a
determinate situation, see Brousseau (1997)). Some writers (Christiansen & Walter, 1986;
Mason & Johnston-Wilder, 2006) express 'task' as being what students are asked to do. Then
'activity' means the subsequent mathematical (and other) motives that emerge from
interaction between student, teacher, resources, environment, and so on around the task. By
contrast, in some professional traditions, 'activity' means a situation set up by the teacher in
which a student has to engage in a certain way. Other traditions (e.g.Chevallard, 1999)
distinguish between tasks, techniques, technology and theories, as a way to acknowledge the
various aspects of a praxeology. We also aware that 'task' sometimes denotes designed
materials or environments which are intended to promote complex mathematical activity (e.g.
Becker & Shimada, 1997), sometimes called 'rich tasks'. In this study, we use 'task' to mean
a wider range of 'things to do' than this, and include repetitive exercises, constructing objects, Introduction
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exemplifying definitions, solving single-stage and multi-stage problems, deciding between
two possibilities, or carrying out an experiment or investigation. Indeed, a task is anything
that a teacher uses to demonstrate mathematics, to pursue interactively with students, or to ask
students to do something. Task can also be anything that students decide to do for themselves
in a particular situation. Tasks, therefore, are the mediating tools for teaching and learning
mathematics and the central issues are how tasks relate to learning, and how tasks are used
pedagogically.
Task design
The design and use of tasks for pedagogic purposes is at the core of mathematics
education (Artigue & Perrin-Glorian, 1991). Tasks generate activity which affords
opportunity to encounter mathematical concepts, ideas, strategies, and also to use and develop
mathematical thinking and modes of enquiry. Teaching includes the selection, modification,
design, sequencing, installation, observation and evaluation of tasks. This work is often

undertaken by using a textbook and/or other resources designed by outsiders.
The extent and detail of design varies widely among those who work on task design.
For some (e.g., Shell Centre) design includes full necessary materials, task sequences and
advice about effective choices, and detailed pedagogic advice about ways of working, verbal
interventions, likely misconceptions and possibly extensions. For others (Ainley, Bills, &
Wilson, 2004, 2005) there may be provision of a question, or a microworld, or some physical
material, with no written object to describe 'the complete task', but rather a series of things
that the teacher might say, perhaps supported by some written prompts. During the resulting
activity, learners may ask questions or make comments to which the teacher needs to respond,
and part of the design is trying to anticipate these and have a general picture of the shape of
responses which would complement the task design. Another form of design is to refine a
question or problem-situation until it is most likely to promote intriguing mathematical
reactions (e.g., (ATM, various dates)). Sullivan, Zevenbergen, & Mousley (2006) have
identified a need to design whole lesson sequences around certain types of tasks. All of these
approaches have implications for implementation, with some relying on teachers' existing
skills, some providing advice to extend teachers' skills, and others dependent on teachers
maintaining or adapting the original task intentions (see, e.g., Kieran, Tanguay, & Solares,
2011).
Tasks also arise spontaneously in educational contexts, with teachers and/or learners
raising questions or providing prompts for action by drawing on a repertoire of past
experience. We are interested in how these are underpinned with implicit design principles.
Task sequences
This discussion of tasks may lead readers to assume that we are focused only on tasks
as single events, but it is important to address also the question of sequences of tasks. There
are different aspects embedded in the design of sequences and, while this is an obvious
consideration when designing textbooks, it also stretches across the whole field of task design.
To achieve the goal of teaching a whole conceptual field (e.g., rational numbers), we
have to describe the different aspects of this knowledge and the way the aspects are linked
(for interesting examples see Brousseau, Brousseau, & Warfield, 2004a, 2004b, 2007, 2008,
2009). In Brousseau's Theory of Didactic Situations (Brousseau, 1997), particular situations
(or single tasks) are generated from more general situations. The earlier tasks in a sequence
should provide experiences that scaffold the student in the solution of later tasks, allowing
them to engage in more sophisticated mathematics than would otherwise have been the case.
In some published sequences, the earlier tasks might be technical components to be used and Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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combined later; in others, the earlier tasks might provide images or experiences which enable
later tasks to be undertaken with situational understanding.
To understand how tasks are linked in order to support teaching, it is important to
understand the nature of the transformation of knowledge from implicit knowledge-in-action
(see Vergnaud, 1982) to knowledge which is formulated, formalized, memorized, related to
cultural knowledge, and so on.
However, there are different ways to create sequences of tasks, some of them are
more commonly known by teachers themselves. One of these types of task sequences is that
in which the problem formulation remains constant but the numbers used increase the
complexity of the task, say moving from small positive integers (for which answers might be
easy to guess) to other ranges of numbers for which a method might be needed. Another type
of sequence is one in which the problem is progressively made more complex by the addition
of steps or variables, such as in a network task where additional nodes are added. A third type
of sequence may be one where the concept itself becomes more complex, such as in a
sequence of finding areas or progressively more complex shapes from rectangles, to
composite shapes, to irregular shapes. These different types of sequences, and their relation to
the teaching unit as a whole, are often the focus of lesson study cycles, such as those reported
in for example Corey, Peterson, Lewis, & Bukarau (2010); Huang & Bao (2006) Yoshida,
(1999).
The importance of sequencing is explicit in Realistic Mathematics Education. In that
tradition, a task sequence starts with situated problems (Gravemeijer, 1999), like dividing
large numbers of people into smaller groups (quotative division problems) to evoke informal
strategies and representations, and continues by changing the focus to formalizing and
generalizing solution procedures, i.e. in this case a general algorithm that can be used for
various division problems. In this type of task sequence the idea of 'guidance with didactical
models' from informal to formal is important as an alternative strategy for the increasing
mathematical complexity of problems students encounter (Van den Heuvel-Panhuizen, 2003).
The situated problems are often already rather complex and can be solved before you know
'the' mathematical solution procedure, and therefore can be good starting points for
problematizing a concept.
Design communities and methods
Of course, teachers also design tasks explicitly and deliberately. Whereas some
authors think it desirable that designing and teaching are separate acts carried out by separate
groups of people (e.g. Wittman, 1995), the experience of the authors of this discussion
document indicates that the communities involved in task design are naturally overlapping
and diverse. Design can involve designers, professional mathematicians, teacher educators,
teachers, researchers, learners, authors, publishers and manufacturers, or combinations of
these, and individuals acting in several of these roles. In the study, we wish to illuminate the
diverse communities and methods that lead to the development and use of tasks. In all
methods, the central consideration is the interaction between teachers and learners through the
designed artefacts and/or the design process. A major focus in the study will therefore be on
learning how design impacts on learners and learning, rather than research which focuses
solely on the design process. For example, research which identifies implicit design principles
would be of interest if connections are made between these principles and the impact on
learning; research about identities of different players in the design process would be of
interest if it contrasted 'teacher-as-task-designer' and 'teacher-as-task-user'. Introduction
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Theme A
Tools and Representations
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In the mathematics classroom, concrete tools (for example, compasses and ruler, unit
blocks, interactive ICT platforms) are usually used as resources to enhance the teachinglearning activity (see for example, Bartolini Bussi & Maschietto, 2008; Maschietto &
Trouche, 2010; Radford, 2011). In this context, tools are broadly interpreted as physical or
virtual artefacts that have potential to mediate between mathematical experience and
mathematical understanding. This theme concerns designing teaching-learning tasks that
involve the use of tools in the mathematics classroom and consequently how, under such
design, tools can represent mathematical knowledge. A task here is a teacher designed
purposeful 'thing to do' using tools for students in order to activate an interactive tool-based
environment where teacher, students and resources mutually enhance each other in producing
mathematical experiences. On a meta-level, it is about possible tool-driven relationships
within the design, teaching and learning triad. In this connection, this type of task design rests

heavily on the complex relationship between artefacts and mathematical knowledge.
There are a few theoretical grounds on which to build and expand this discussion.
Instrumental genesis explicates how the usage of a tool can be turned into a cognitive
instrumentation process for knowledge acquisition. A Vygotskian approach examines how an
artefact can be turned into a psychological tool in the context of social and cultural interaction
developed through the zone of proximal development and internalization processes. Semiotic
mediation can be used as an integrated approach to explore the mathematics classroom under
which a tool takes on multiple pedagogical functions (Bartolini Bussi & Mariotti, 2008).
Embodiment theory proposes that there are strong relationships among sensory activities and
cultural artefacts in the appropriation of mathematical practices, and in particular, their
application to inclusive mathematics education (Healy & Fernandes, 2011). The guided reinvention principle of RME (Realistic Mathematics Education) practiced by the Freudenthal
school can be used to direct the design of tool-based mathematical tasks. These theoretical
orientations, and/or others, may serve to facilitate discussion on tool-based task design and
representation in the mathematics classroom.
An important question to address in this theme is: How to design tasks that can bring
about situated discourses (hence representations) for the mathematical knowledge mediated
by tools in the mathematics classroom and how these discourses relate to mathematics
knowledge? This in turn comprises several additional questions.
Possible questions about tools and representation:
• What mathematics epistemological considerations are taken into account when
designing tasks using tools? Theme A - Introduction
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• How do we create a tool environment for the mathematics classroom to support the
design of teaching and learning tasks for specific mathematic topics?
• How do different types of tools afford different mathematical activities/tasks,
different representations and/or discourses, and different interactions between
representations?
• How do different task designs using tools impact on students' learning and
understanding of mathematics?
• How do we design mathematical tasks that can transform an artefact into a
pedagogical instrument?
• Are there models (theoretical or pragmatic) of tool-based task design for the
teaching and learning of mathematics?
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Using crises, feedback and fading for online task design
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A recent discussion involves the elaboration on possible design principles
for sequences of tasks. This paper builds on three principles, as described by
Bokhove and Drijvers (2012b). A model with ingredients of crises, feedback and
fading of sequences with near-similar tasks can be used to address both procedural
fluency and conceptual understanding in an online environment. Apart from
theoretical underpinnings, this is demonstrated by analysing a case example from
a study conducted in nine schools in the Netherlands. Together with quantitative
results of the underlying study, it shows that the model described, could be a
fruitful addition to the task design repertoire.
Keywords: task, design, sequence, near-similar, crisis, feedback, fading
Introduction
In recent years more and more attention has been paid to task design. In the call for
papers for the 22nd ICMI study on task design the reasons for this are clearly described. One
problem is that tasks are often only described vaguely. Furthermore, Schoenfeld (2009)
advises on having more communication between designers and researchers. In this way
educational research and design can be bridged, as the communities involving task design are
naturally overlapping and diverse. This paper was triggered by some of the remarks that were
made in the call for papers:
The topic of understanding whether and how doing tasks, of whatever kind, enable
conceptual learning. The study reported in this paper supports Lagrange (2002) who
suggested that applying routine techniques can achieve results, and also provide the basis for
conceptual understanding and new theorizing.
To not only address tasks as single events, but also address the question of sequences
of tasks.
It is suggested that the design of sequences of near-similar tasks deserves attention.
Therefore, the paper makes a point of defining a 'task' to mean a wider range of 'things to do'
than just one task, and include repetitive exercises.
Several types of task sequences are mentioned. "One of these types of task sequences
is that in which the problem formulation remains constant but the numbers used increase the
complexity of the task, say moving from small positive integers (for which answers might be
easy to guess) to other ranges of numbers for which a method might be needed.". Building on
an earlier article (Bokhove & Drijvers, 2012b) an extra type of task sequences is proposed,
whereby the complexity of tasks first increases, and then -with the help of feedback-
decreases.
In one sense this can be seen as an adaptation of the 'variation' Watson and Mason
(2006) coined: "From a modelling perspective the term micromodelling' may be helpful to
describe learners' response to exercises in which dimensions of variation have been carefully Theme A - C. Bokhove
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controlled, because the aim is to promote generalization of the dimensions being varied in the
exercise, and thence to focus on mathematical relationships between dimensions." (p.104)
Summarizing these points we would want to formulate design principles for: (i)
sequences of tasks; (ii) near similar and/or repetitive tasks; and (iii) addressing both
conceptual understanding and procedural fluency. This paper synthesizes, elaborates on and
illustrates design principles of a study first described by Bokhove and Drijvers (2012a, b).
The principles regarding crises, feedback and fading are applied to a sequence of digital
tasks. This paper sets out to describe the three principles in an additional cohesive model, and
describes one case example of student work. Bearing the aforementioned goals in mind it
should be a model that could prove to be fruitful while designing tasks. The model bears
elements of both my roles as a designer and researcher when doing my PhD as a teacher at a
secondary school in the Netherlands. The study called 'Algebra met Inzicht' [Algebra with
Insight] was designed in the Digital Mathematical Environment
(http://www.fi.uu.nl/dwo/en)
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. The intervention consists of a pen-and-paper pre-test, four
digital modules, a digital diagnostic test, and a final digital test and, finally, a pen-and-paper
post-test. It was deployed in fifteen 12th grade classes from nine Dutch secondary schools
(N=324), involving eleven mathematics teachers. The schools were spread across the country
and showed a variation in school size and pedagogical and religious backgrounds. The
participating classes consisted of pre-university level 'wiskunde B' students (comparable to
grade 12 in Anglo-Saxon countries). As this article is about design principles, I refer to
different articles for more details of the set-up of the study and the actual effects of the digital
intervention (Bokhove & Drijvers, 2012a, b).
Figure 1 shows the proposed model for sequences of (near-similar) tasks.
figure 1: proposed model for crises, feedback and fading
I contend that it is okay to use near-similar tasks and repetitive exercises, but suggest
that the sequence is interspersed by intentional crises i.e. tasks that are hard or impossible to
solve with skills and knowledge that are available. In other words, the "load" of the task is too
high. I will not go into the word "load" in detail. There is a vast body of knowledge connected
to the term Cognitive Load Theory (Sweller, 1988). There also is, rightly so, criticism (De

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An English translation of part of the module can be found at http://www.fi.uu.nl/dwo/soton. Log in
as guest, and choose 'Demo for 22nd ICMI study'. Java is needed. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.

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Jong, 2010). For the purpose of this paper, we will only assume that knowledge that isn't
known potentially will bear a larger load than unknown knowledge. Then, let students
overcome crises by providing feedback. To avoid a dependency on feedback for the
summative assessments fade the feedback during the course of the sequence of tasks.
A model for sequences of near-similar tasks
I will first elaborate on the three main ingredients of the model: crises, feedback and
fading.
With a crisis we refer to a principle that the poet John Keats so eloquently described
in the early 19th century 'failure is the highway to success'. This principle corresponds to
similar concepts that have been described during the years. Piaget (1964) used the concept of
equilibrium and disequilibrium. Essentially, whenever the child's experience/interaction with
the environment yielded results that confirmed her mental model, she could easily assimilate
the experience. When the experience resulted in something new and unexpected, the result
was disequilibrium, and a child may experience this as confusion or frustration. Eventually,
the child changes his or her cognitive structures to accommodate the new experience and
moves back into equilibrium. Piaget studied the individual case, on a societal level Kuhn
(1962) referred to a paradigm shift, arguing that scientific advancement is not evolutionary,
but rather a "series of peaceful interludes punctuated by intellectually violent revolutions",
and that in those revolutions "one conceptual world view is replaced by another". Tall (1977)
refers to cognitive conflicts: "one of the distinguishing factors in catastrophe theory is the
existence of discontinuities, or sudden jumps in behaviour when certain paths are taken.". In
his 'levels of thinking' van Hiele (1985) discerns structure and insight. There can be a 'crisis
of thinking', which has a link to the vygotskian zone of proximal development. The common
ground between the two is that there is a need for challenge. More recently, Kapur (2010)
uses the term productive failure and cites Clifford (1984): "However, allowing for the
concomitant possibility that under certain conditions letting learners persist, struggle, and
even fail at tasks that are complex and beyond their skills and abilities may in fact be a
productive exercise in failure requiring a paradigm shift". Kapur explains this by stating that it
is reasonable to reinterpret their central findings collectively as an argument for a delay of
structure in learning and problem-solving situations, be it in the form of, feedback and
explanations, coherence in texts, or direct instruction. The difference with my own work
(Bokhove & Drijvers, 2012b) seems to be whether crisis are an inherent part of learning when
solving open problems, or actually embedding tasks that could intentionally cause a crisis. It
is proposed that intentional crisis tasks are added to sequences of near-similar tasks, for
example in the way depicted in table, which illustrates the way in which crisis items are
integrated within the digital tool. The general structure of a sequence is: pre-crisis items, crisis
item, post-crisis items. But then the question becomes: how can students address this crisis?
Can we add another principle which enables students to use assessment for learning. One way
would be to make use of formative assessment. Black and Wiliam (1998) define assessment as
being 'formative' only when feedback from learning activities is actually used to modify
teaching to meet the learner's needs. From this it is clear that feedback plays a pivotal role in
the process of formative assessment. Hattie and Timperley (2007) conducted a meta-review of
the effectiveness of different types of feedback. The feedback effects of hints and corrective
feedback are deemed best. However, in my personal experience as a teacher I have seen there
can be an over-reliance on feedback that is provided. Assuming that students finally have to
pass an exam themselves, it makes sense to address this over-reliance on feedback. In a
follow-up paper Kapur (2011) notes that scaffolding implies help to overcome failure (Pea
2004). It turns out that when dealing with previously stored information in the long term
memory, these limits tend to disappear. Theme A - C. Bokhove
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1.1 Tasks: "Solve the following equation:" Pre-crisis items
In the initial items students are confronted with equations
they have experience with. Students may choose their own
strategy. Many students choose to expand brackets as that is
the strategy that they have used often: work towards the form

0 and use the Quadratic Formula. There is
some limited feedback on the task.
1.2
1.3
1.4
1.5
1.6 Crisis item
Students are then confronted with an intentional
crisis: if a student uses his/her conventional strategy
of expanding the expression. The yellow tick at the
bottom of the screen denotes that the equation is
algebraically equivalent to the initial one, but that it
is not the final answer. This is accompanied by a
partial score for an item and some feedback in Dutch:
'You are rewriting correctly'. Although these
students showed good rearranging skills, in the end
they were not able to continue, as they did not master
the skill to solve a third order equation. There is
some limited feedback on the task.
1.7
1.8 Post-crisis items
After the crisis item students are offered
help by providing a 'voorbeeldfilm', an
instructional screencast, and buttons to get
hints ('tip'), the next step in the solution
('stap') or a worked solution ('loop').
These features have in common that they
provide feedforward information at the
task level and self-regulation.

1.9
1.10
1.11
1.12
1.13
table1: Sequence of items illustrating crises and feedback
As Kirschner et al (2006) argued "any instructional theory that ignores the limits of working memory when dealing with novel information or ignores the disappearance of those limits when dealing with familiar information is unlikely to be effective." As a design principle it is therefore proposed that initially a lot of feedback is provided to foster learning, but the amount is decreased towards the end, to facilitate transfer. Using scaffolding this way is based on the concept of fading (Renkl, Atkinson, & Große, 2004). Formative scenarios Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.

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(Bokhove, 2008) are a variation of this concept, starting off with much feedback, and providing a gradually decreasing level of feedback. Figure 2 shows how this principle was implemented in the intervention. At the start feedback is provided for all intermediate steps of a solution. The subsequent part of the intervention concerns self-assessment and diagnostics: the student performs the steps without any feedback and chooses when to check his or her solution by clicking a "check" button. Feedback is then given for the whole of the exercise.
figure 2: Outline of fading feedback in formative scenarios
Finally, students get a final exam with no means to see how they performed. Just as is the case with a paper test, the teacher will be able to check and grade the exam (in this case automatically) and give students feedback on their performance. A student needs to be able to accomplish tasks independently, without the help of a computer. An implicit advantage of implementing feedback in a sequence of tasks is that teachers and designers have to think upfront about possible student responses.
Principles at work: a case example
Let's look at one student named Paula during the course of this module. The student starts off with a pre-test. Apart from the calculation error on the right hand side of the equation, figure 3 clearly shows that Paula's strategy here is to expand the expressions, similar to students in earlier phases of the study (Bokhove & Drijvers, 2010).
Figure 3: Example of Paula's pre-test pen-and-paper work Theme A – C. Bokhove
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Not surprisingly this strategy fails in the case of this equation. Paula only scores 14 out of 100 for the whole pre-test. With regard to symbol sense, Paula scores a -4 for (for details on the calculation I refer to Bokhove & Drijvers, 2012a). Paula then starts with the sequence of digital tasks. In the first task the student has to get acquainted with the digital environment. The pre-crisis items pose no problem for most students, including Paula. On arriving at the crisis item students roughly exhibit three behaviours, roughly corresponding with the ones already observed in the pre-test: (i) students solve the equation correctly, (ii) students recognize the pattern of the equation but subsequently make mistakes (for example by losing solutions in the process), and (iii) students expand the expressions and get stuck with an equation with a third power. Figure shows that our case student Paula again shows the third type of behaviour. At this moment feedback is still restricted to correct/incorrect. In addition, students are allowed to choose their own strategies, even when they aren't efficient or would lead to problems. In the post-crisis items, as well as feedback correct/incorrect, Paula is provided with buttons for hints, and a movie clip demonstrating the solution. From the log-files of the online environment it becomes clear that Paula fails at the crisis-item (0 out of 10), but succeeds at the post-crisis item with feedback (10 out of 10). When looking at attempts made, Paula attempts the crisis-item 73 times, and the post-crisis item, being aided by feedback, only 3 times.
Figure 4: Paula's digital work. Left: crisis item left. Right: post-crisis item
Finally, in the post-test Paula shows a significant increase in the total score (70 out of 100) and symbol sense behaviour (+1, an increase of 5). Even though mistakes are made they were not caused by a lack of symbol sense any more but errors in calculations. Focusing only on similar types of equations it becomes clear that Paula manages to solve these equations correctly. Paula is not a unique case in this school. Overall, students in participating schools improved on their scores and symbol sense behaviour.
Conclusion
Illustrated by the theoretical underpinnings, the overall results in the study, and case example, it is concluded it would be a good idea to study design principles that can be used to design sequences of near-similar tasks in more detail. By combining three principles from an initial study -crises, feedback and fading- in one model for sequences of tasks, three important aspects of this ICMI study are addressed: (i) sequences of tasks; (ii) near similar and/or repetitive tasks; and (iii) addressing both conceptual understanding and procedural fluency. We propose that educators, teachers, designers and researchers alike can adopt these principles when designing and implementing sequences of (near-similar) tasks. It is, however, important to note some points of discussion. The difficulty of every task or sequence of tasks depends on the context. What can be a simple task for one year eight student can prove to be difficult for another student, even when at first sight they seem fairly similar. Also, the way in Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.

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which a crisis is overcome can differ: some students learn from repeating near-similar tasks, others seem to recognize 'a pattern' immediately and apply this to new tasks. Given this diversity, it is important to field-test and evaluate sequences of tasks, again combining the power of teaching, researching and designing. I think it would be unfair to indefinitely classify certain tasks as 'more creative' and other tasks as 'less creative'. This, too depends on the background and context of the learner: a wonderful, new and creative task can become a repetitive task the second time around. Looking back on this paper I wonder whether the predicate 'near-similar' does not actually apply to all tasks, if a student has seen a task before, even the elaborate, creative ones. It is my wish that we look at the total picture, and integrate all these tasks in one clear picture for the learning student. One way would be to not so much study the nature of solitary tasks but place them in sequences and their corresponding contexts. Hopefully, this paper provides general design principles that can be used, and task design can be taken forward.
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A Holistic Approach for Designing Tasks that Capture and Enhance Mathematical Understanding of a Particular Topic: The Case of the Interplay between Examples and Proof
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This study
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illustrates an approach to task design that aims at revealing and enhancing students' understandings of a particular mathematical topic: the roles of examples in proving and refuting mathematical statements. As a preliminary stage we conducted a content analysis that led to the construction of a comprehensive mathematical framework that describes the various aspects of understanding this topic. Based on this framework, we constructed six prototypes of tasks that complement each other in assessing and enhancing students' understanding of the logical connections between examples and statements. In this paper we describe the design principles that guided the construction of the collection of tasks that can be considered a special kind of tool and provide illustrations of how this tool was used.
Keywords: task design, examples, reasoning and proof
Introduction and Background
With the current curricular emphasis on students' engagement in mathematical investigations and proving, it becomes increasingly important that students develop the understanding of the logical connections between empirical and formal aspects of mathematics. Such understanding manifests itself in understanding of the roles of examples proving and refuting conjectures. Despite the fact that several studies indicate that students encounter difficulties in this area (Fishbein, 1987, Antonini et al., 2011, Bills & Watson, 2008), such connections are not explicitly addressed in the school curriculum and are left to students to develop indirectly mostly on their own. Moreover, there is only scarce work in conceptualizing what constitutes an understanding of the roles of examples in proving (e.g., Zaslavsky & Ron, 1998, Barkai et.al., 2008, Buchbinder & Zaslavsky, 2009, Tabach et. al., 2010). The goal of our study was to develop a tool that could be used both to assess students' overall understanding of the above topic and to enhance these understandings. Moreover, we were looking for a tool that would help capture the

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This study was conducted at the Technion – Israel Institute of Technology with support of Israeli Ministry of Education. Theme A – O. Buchbinder & O. Zaslavsky
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nature of this understanding. The first question we posed was: What does it mean to understand the roles of examples in determining the validity of mathematical statements?
In order to answer this question, we developed a general framework
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that describes four types of examples (confirming, non-confirming, contradicting, and irrelevant) and their status in proving or refuting the two types of mathematical statements (universal and existential). We describe the framework in the next section.
The framework
Every mathematical statement can be characterised by the domain (D) of objects (x) to which it refers to and a proposition (P(x)) that specifies some property. A Universal statement states that a proposition is true for all the objects in the domain: . An Existential statement asserts that there exists an object in the domain for which the proposition is true: . For example, if the domain D is 'all integers for which the sum of their digits is divisible by 6' and the proposition P is 'multiple of 6', we can formulate two types of statements: 1. A Universal statement, "All integers, for which the sum of their digits is divisible by 6 are multiples of 6"; and 2. An Existential statement, "There exists an integer for which the sum of its digits is divisible by 6 that is a multiple of 6".
With respect to a given domain D and a property P, four types of examples can be defined, depending on whether (a) the object x is an element of D or not (or), and (b) the proposition P(x) is true for it or not (or). The logical status of these examples depends on the type of statement (Figure 1). The first type of object x, which we term a confirming example, is an element of D for which the proposition P(x) is true ( ). In the above case, 24 is a confirming example. While 24 is insufficient for proving the universal statement, it proves the corresponding existential statement. The second type of object, which is termed a counterexample, is an element of D that does not satisfy the proposition P(x) ( ). For example, 33 is a counterexample, thus, it contradicts the false universal statement; we refer to such an example as a contradicting example (of the universal statement). However, with respect to the corresponding existential statement 33 is insufficient for refuting it. It neither contradicts nor confirms it; thus we term such an example a non-confirming example (of the existential statement). All of the above types of examples are relevant for examining the validity of a given statement. The other two types of examples are objects that do not belong to the domain D: and . From a logical stand both types are equally irrelevant for determining the validity of either type of statement. However, example of type (e.g., 36) is potentially misleading.

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For a detailed description and discussion of the framework see Buchbinder & Zaslavsky,
2009.
∀x ∈ D, P x)(
∃x ∈ D xP)(,
x∈D x∉D xP)(¬P x)(
x ∈ D P x)(,
x ∈ D,¬ xP)(
x ∉ D, xP)(x ∉ D,¬ xP)(
x ∉ D, ¬P x)(Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Type of Statement
Universal statement Existential Statement
Goal
Type of Example
To prove To disprove To prove To disprove
Confirming Insufficient
Non
applicable
Sufficient
Non
applicable
Contradicting (the universal statement)
Non-confirming (the existential
statement)
Non
applicable
Sufficient
Non
applicable
Insufficient
Irrelevant
Non
applicable
Non
applicable
Non
applicable
Non
applicable
Irrelevant
Non
applicable
Non
applicable
Non
applicable
Non
applicable
Figure 1: A framework for examining the logical status of examples in determining the
validity of mathematical statements
We take "understanding of the logical status of examples in determining the
validity of mathematical statements" as becoming fluent with the types of inferences
that can and cannot be made based on the four types of examples with respect to the
two types of statements. We consider the following four types of cognitive activities
as eliciting evidence of such understanding: construction of examples (spontaneously
or on demand); recognition of and differentiation between types of statements and
types of examples (without necessarily using our terms); making logical inferences
based on examples, and correctly justifying then. Such operational approach to
conceptualizing understanding is consistent with Borgen & Manu (2002), Schoenfeld
(1989) and Zaslavsky (1997) and sets the grounds for task design.
The design principles behind the collection of tasks
Based on our framework, we constructed 6 prototypes of tasks, each of
which addresses a particular aspect of understanding. The set of tasks as a whole
covers all the aspects of the framework and proved effective as a diagnostic and
facilitating tool for 10
th
grade students in Israel.
The overarching design principle was to create a balanced representation of
all of the following variables: 1. Type of statement (universal or existential); 2. Type
of example (confirming, contradicting, non-confirming, and [two kinds of]
irrelevant); 3. The truth-value of the statement (true/false); 4. The content (secondary
level algebra or geometry, all of which students are supposed to be sufficiently
knowledgeable); 5. Number of relevant examples that exist for each statement (e.g.,
no example, a single example or a finite number, an infinite number of examples of a
particular type).
From a mathematical point of view, the specific number of examples that
exist for each statement is irrelevant
4
. A single contradicting example is sufficient for
refuting a statement, even if there exist an infinite number of confirming examples.
However, the number of examples may affect students' inferences, their perseverance
in evaluating the truth-value of the statement, and the methods they use.

4
If a domain of a universal statement has a finite number of elements, it is possible to prove
that the statement is true by showing that the proposition holds for each one of them. However,
showing that the domain is indeed finite is an imperative part of the proof.
∀x∈ D, P x)(∃x∈ D, P x)(
x ∈ D, xP)(
x ∈ D,¬P x)(
x ∉ D, xP)(
x ∉ D, ¬ xP)(Theme A – O. Buchbinder & O. Zaslavsky
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Thus, the collection of tasks included statements for which all existing
examples are supporting ones, statements that have both supporting and contradicting
(or non-confirming examples), and statements which have only contradicting or nonconfirming examples. The rationale for such design was to create a rich scope of
situations for examining students' understanding of the roles of examples in proving.
While we wanted to ensure that students have the necessary knowledge to
approach the tasks, we also wanted to confront the students with statements that were
unfamiliar to them of which truth-values would not be immediately evident. Such
statements have the potential to evoke uncertainty and doubt, which are widely
recognized as powerful diagnostic tools as well as vehicles for creating situations in
which the need to prove arises intrinsically (e.g. Zaslavsky, 2005; Buchbinder &
Zaslavsky, 2011; Hadas, Hershkowitz & Schwarz, 2000). We expected that the
uncertainty evoked by the tasks would trigger students' discussion and attempts to
convince each other through argumentation, which would allow for various aspects of
their understanding to be revealed.
The design of our tool (i.e., the collection of tasks) was inspired to some
extent by Harel's (2007) DNR principles for learning environments. In particular, the
necessity principle that emphasizes the importance of evoking students' intellectual
need to prove; and the repeated reasoning principle that states that students need to
practice reasoning in various settings. Thus, the design of our tool aimed at providing
multiple opportunities for students to practice reasoning, to reflect on their knowledge
of the logical connections between examples and proving, and to enhance their
understanding of these connections through resolving uncertainty.
The collection of tasks
The collection includes 6 types of tasks: (1) What kind of example is this?
(2) True or false? (3) Always, sometimes, or never? (4) Who is right? (5) Is this a
coincidence? (6) Does it exist? For space constraints we cannot go into equal length
of detail for each task.
Task 1:What kind of example is this?
This task has two parts, each involving different statements: a false universal
or a true existential. Each statement was accompanied by 7 examples of various types.
For each part, the students were asked to determine the types of the 7 examples; to
construct two additional examples (one confirming and one non-confirming); to
decide whether the statement is true or false and to justify the decision.
Figure 2 shows one of the false universal statements and the analysis of the
examples used for the task.
This type of task was the only one that explicitly calls for examining
irrelevant examples and determining their logical status, and that explicitly requires
construction of examples that satisfy a given criteria. Note, that the students did not
receive any explanation of the framework including its vocabulary prior to engaging
in the task.
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The Given Statement: The product of any two numbers a and b, for which the
sum is positive, is also positive.
Design-Analysis of
the examples
is a confirming / contradicting / irrelevant example of the
statement?

confirming
is a confirming / contradicting / irrelevant example of the
statement? contradicting
is a confirming / contradicting / irrelevant example of the
statement?

irrelevant
is a confirming / contradicting / irrelevant example of the
statement? irrelevant
is a confirming / contradicting / irrelevant example of the
statement? contradicting
is a confirming / contradicting / irrelevant example of the
statement?

confirming
is a confirming / contradicting / irrelevant example of the
statement?

irrelevant
Figure 2: Examples and design-analysis of the categorization part of Task 1 for the given statement
Task 2: True or False?
Tasks of this structure are rather common. We adapted this familiar model to
fit our mathematical framework. In this task a set of 6 statements were given: 3
universal and 3 existential. Within each type of statement, we included one statement
for which all relevant examples are confirming, one statement which has both
confirming and non-confirming examples, and one statement for which all relevant
examples are contradicting or non-confirming.
Figure 3 presents the structure of the algebraic version of this task. Note that
constructing a case for each row is a non-trivial task for the designer.
This task addresses most of the aspects of understanding of the roles of
examples with respect to the framework. In order to solve the task correctly, students
need to distinguish between the different types of statements and the corresponding
types of examples; they also need to construct counterexamples to disprove false
universal statements (1 & 3), and confirming examples to prove true existential
statements (4 & 5). The task calls for demonstrating the understanding that examples
could be insufficient to prove or refute statements (2 & 6), and that in such cases a
general logical argument is required. Students may arrive at this conclusion either by
constructing and checking examples or by applying algebraic procedures.

a = ,2 b = 8
x ∈ D P x)(,
a = – ,3 b =10
x ∈ D,¬ xP)(
a = ,4 b = –11
x ∉ D,¬ xP)(
a = – ,6 b = –5
x ∉ D P x)(,
4
3
,
2
1
a –= b =
x ∈ D,¬ xP)(
, 7
3
1
a = b =
x ∈ D P x)(,
6
4
,
5
2
a = b –=
x ∉ D,¬ xP)(Theme A – O. Buchbinder & O. Zaslavsky
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The Statement
Type of
statement:
U / E
Truth
value:
T / F
Possible types of
examples for the
statement
Type of
justification
required
Only confirming
nonConfirming and
confirming /
contradicting
Only noncontradicting
confirming /
Every three numbers a, b, c satisfy the
equation: .
U F
Refutation by
contradicting
example
The (positive) difference between the
squares of two consecutive natural
numbers is equal to their sum.
U T General proof
Every two numbers n, m satisfy the
equation:
.
U F
Refutation by
contradicting
example
There exist four numbers a, b, c, d that
satisfy: .
E T
Proof by a
confirming
example
There exists a number that
satisfies the equation:
E T
Proof by a
confirming
example
There exist three distinct positive
integers a, b, c that satisfy:
.
E F
General
refutation
Figure 3: Design structure of the algebraic version of Task 2
Note: E stands for Existential, U for Universal, T for True, and F for False
Task 3: Always, Sometimes, Never
This task is related to the "True or False?" task, with a shift of focus. Given
the propositions from the statements in Task 2, students are required to determine
whether the proposition is always, sometimes or never true (Figure 4). From a logical
point of view, a statement involving quantifiers is either true or false. Thus, a
universal statement cannot be "sometimes" true, even if both confirming and
contradicting examples exist. However, a proposition P(x) can be true for some values
of x and false for others. For example the proposition "x is greater than 5" is true for
x=3, but false for x=13. Thus, a proposition P(x) can be always, sometimes, or never
true (Rosen, 2003).
Figure 4 presents one of the items that were included in Task 3. It is a
different take on item 1 that appears in Figure 3.
Choose the correct answer and explain your reasoning.
[Note that the letters a, b, c represent numbers].
Is the equation: always / sometimes / never true?
Figure 4: An example of Task 3
c
b
a
b c
a
= +
+
m n n + m
+ =
1 1 1
d
c
b
a
b d
a c
= +
+
+
a = 1
□
□
□
□
□
□
□
□
□
□
□
□
□
□
□
–
= - +
–
+ +
1
1
1
1
1
1
a
a
a
a
b
a
b c
a c
=
+
+
c
b
a
b c
a
= +
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Task 4: Who is right?
This task has two parts involving one false universal and one true existential
statement. Each statement is followed by utterances of five hypothetical students
stating their opinion on the truth-value of the statement. The task requires, for each
utterance, to determine whether it is correct or not and to justify the decision.
Figure 5 presents the design structure of the Task 4.
Hypothetical
student
Given: A False Universal
Statement.
Given: A True Existential
Statement.
Correctness
Student A Uses multiple confirming
examples to "prove" the
statement.
Uses multiple non-confirming
examples to "refute" the statement. ×
Student B Refutes the statement using a
counterexample.
Proves the statement using a
confirming example.
Student C Requires multiple
counterexamples.
Requires multiple confirming
examples.
×
Student D Maintains that the statement is
false but does not accept
counterexamples as sufficient.
Requires a general argument.
Maintains that the statement is true
but does not accept confirming
examples as sufficient. Requires a
general argument.
×
Student E Maintains that it is impossible to
determine whether the statement
is true or false since there are
both confirming and
contradicting examples.
Maintains that it is impossible to
determine whether the statement is
true or false since there are both
confirming and contradicting
examples.
×
Figure 5: Design structure of Task 4
Task 5: Is this a coincidence?
This task presents a hypothetical student's actions and his/her observation
based on a single example. The question "is this a coincidence?" invites the students
to evaluate the generality of the observed phenomenon in order to determine whether
it holds for every relevant case or just for some specific cases, one of which the
student examined. Successful completion of the task involves either proving that the
described phenomenon is a general one or constructing a counterexample.
A student chose two fractions: and . He generated another fraction by adding
the two nominators and the two denominators in the following way:
. The student observed that the resulting fraction is between
the two original ones: . Is this a coincidence?
Figure 6: An algebraic example of Task 5
Note, that there is no explicit requirement in the task to prove any claim. The
intention was to draw students' attention to the extent of generality of the
phenomenon, and to study the ways in which they deal with the task, focusing on their
need to form an assertion and justify it by means of proof or refutation.
2
1
4
3
3
2
6
4
2 4
1 3
= =
+
+
3

2
4
3
3
2
2
1
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Task 6: Does it exist?
This task focuses on existential statements and the roles of examples in
determining their validity. The task included four statements, two in algebra and two
in geometry worded in the form "Does an object with a certain property exist?"
(figure 7). In some ways this task resembles the "True or false?" one, however, in the
pilot study we found that the wording of the statements as questions appeared to be
more appealing to the students.
Does there exist a triangle with two heights that are perpendicular to each other?
Does there exist a triangle with two angle bisectors that are perpendicular to each
other?
Does there exist a pair of natural numbers a, c that satisfy: ?
Does there exist a pair of natural numbers a, b that satisfy:
a·b[a·b(a·b)] = (a·b)(a·b)(a·b) ?
Figure 7: Examples of Task 6
Concluding remarks
In this paper we offer a holistic approach to task design. Figure 8 illustrates
how the conceptual framework that we developed guided the design of a whole
collection of tasks that formed a diagnostic tool.
The collection of tasks constituted the research instrument for a study of 10
th
grade students' understanding of the roles of examples in determining the validity of
mathematical statements. This tool was also used for task-based interviews with pairs
of students who worked aloud on these tasks. Overall, the rich scope of types of tasks,
as well as the specific statements and examples that were chosen for each type of task,
proved useful in tapping into students' thinking and enhancing their understandings.
Space does not permit to include examples of such manifestations in this paper but
they will be included in a presentation.
Type of Statement
Universal statement
1, 2, 3, 4-U, 5
Existential Statement
1, 2, 3, 4-E, 6
Goal
Type of Example
To prove To disprove To prove To disprove
Confirming
1, 2, 3, 4, 5, 6
1, 2, 3, 4-U, 5 1, 2, 3, 5 1, 2, 3, 4-E, 6 1, 2, 3, 6
Contradicting (the universal
statement) Non-confirming
(the existential statement)
1, 2, 3, 4, 5, 6
1, 2, 3, 5 1, 2, 3, 4-U, 5 1, 2, 3, 6 1, 2, 3, 4-E, 6
Irrelevant
1, 2, 3, 5, 6
1, 2, 3, 5, 6 1, 2, 3, 5, 6 1, 2, 3, 5, 6 1, 2, 3, 5, 6
Figure 8: Connections between the overall collection of tasks and the conceptual framework.
[Note that the numbers refer to the numbers of the task-types]
a c c
a 1
=
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Rotational symmetry: semiotic potential of a transparency
toolkit
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In this paper, excerpts of lessons on using tool-based tasks to
teach the concept of 'rotational symmetry' were analyzed. Theory of
semiotic mediation was adopted as theoretical framework. The analysis
focused on how the tasks could (or could not) bring out the semiotic
potential of the tool used. We hope that this analysis could provide some
insights on how the design of tool-based tasks may enhance the bringing
about of semiotic potential of the tools.
Keywords: semiotic potential, rotational symmetry, tool-based task
Introduction
It has been a long history to use concrete manipulative objects to teach
mathematics (Dienes, 1960, 1971). Designing tool-based learning tasks becomes
popular in recent decades because of the new development of virtual manipulative
software such as Geometer's Sketchpad or Carbri. By a task we mean a teacher
designed 'thing-to-do' using the tool, either concrete or virtual manipulatives, for
students to experience potential mathematical meanings carried by this tool. Empirical
studies suggest that teachers can promote the evolution of mathematics knowledge
through "orchestrating" tool-based learning tasks and post-task mathematics
discussions. (See for example, Jones, 2000; Mariotti, 2002; Falcade, Laborde, &
Mariotti, 2007.) We follow Mariotti & Maracci (2012) to use "orchestration" as a
metaphor for classroom discussions with the aim "of developing shared meanings,
having an explicit formulation, de-contextualized from the artifact [tool] use,
recognizable and acceptable by the mathematicians' community" (p.60). In this paper,
excerpts of lessons on using tool-based tasks to teach the concept of 'rotational
symmetry' were analyzed under the framework of semiotic mediation. The analysis
focused on how teacher's orchestration of tool-based learning tasks and post-task
mathematics discussions could (or could not) bring out the semiotic potential of the
tool. It is hoped that this analysis could provide some insights on how the design of
tool-based tasks may enhance the bringing out of semiotic potential of tools.
Theoretical Perspective
The framework of "semiotic mediation" (Bartolini Bussi & Mariotti, 2008)
which is rooted from Vygotskian perspective on "social construction of knowledge" is Theme A – Y.-C. Chan & A. Leung
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adopted as theoretical framework. A tool (artefact)
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which carries mathematical
meanings can become a "tool of semiotic mediation" by which students can
experience the development of mathematical concepts. In particular, a tool of semiotic
mediation provides a means to express mathematical ideas. Bartolini Bussi & Mariotti
(2008) points out that there is "double semiotic link" between a tool, a task and
mathematical knowledge when the tool is used to accomplish a specific task. They
further point out that:
"The main point is that of exploiting the system of relationships among artefact,
task and mathematical knowledge. On the one hand, an artefact is related to a
specific task … that seeks to provide a suitable solution. On the other hand, the
same artefact is related to a specific mathematical knowledge." (Bartolini Bussi &
Mariotti, 2008, p.753)
This double semiotic relationship is called the "semiotic potential" of the tool
(artefact) (Bartolini Bussi & Mariotti, 2008, p. 754). In the mathematics classroom,
teacher plays a crucial role in the process of semiotic mediation. As "the voice of
mathematics culture", the teacher guides mathematical discussions which aim at
bringing out the semiotic potential of the tool: a progression from students'
production of mathematical discourse to mathematical knowledge. Building on this
perspective, in Azrarello, Bartolini-Bussi, Leung, Mariotti & Stevenson (2012, pp.
107-108), a model was used to describe the process of semiotic mediation which
highlights the classroom dynamic relationship among the tool (artefact), the task,
students' productions, the teacher, and mathematical knowledge. The dynamism
consists of a network of interactions wherein the teacher uses a tool (artefact) as a
mediator between mathematical knowledge and tasks performed by the students. The
model highlighted two responsibilities of a teacher: 1. choose suitable tasks based on
the tool used; and 2. monitor and manage the process of progression from students'
production to mathematical knowledge to be taught.
The Context
The excerpts of lesson episodes chosen for discussion in this paper were
taken from a research lesson study carried out in Grade 5 classroom at a Hong Kong
primary school
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based on the Japanese's Lesson Study (Fernandez & Yoshida, 2004)
and the Learning Study (Lo, Pong & Chik, 2005) models. The research lesson aimed
at improving mathematics teaching through tool-based tasks. Five Grade 5
mathematics teachers worked together over period of five months to design lessons
for a selected topic. The topic selected for the research lesson was rotational
symmetry. One of the teachers implemented the lesson whereas other teachers
observed and evaluated the lesson. A modified lesson was agreed by the teachers and
was implemented by another teacher (at another class). This cycle was repeated until
all the five teachers have taught the lesson to their own classes. A researcher in charge
(one of the authors) acted as a participant observer and gave advice from theoretical

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Some theoretical perspectives such as instrumental approach make distinction between
artefact and instrument in order to highlight the differences between the potentialities and the actual
uses of the tool respectively. In this paper tool and artefact are regarded as synonyms which refer to
both the potentialities and the actual uses.
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The authors would like to express their gratitude to the team of mathematics teachers from
St Edward Catholic Primary who designed and implemented this Lesson Study. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22
consideration. All the teacher preparation meetings, lessons and the post
discussion were video-taped.
The Tool and the Task
The objective of the lesson was to introduce to students the idea of a
rotational symmetric figure; that is, a figure 'overlaps itself' at least two times during
one cycle of rotation. A toolkit was specially designed for this lesson. The toolkit,
which we will call the
overhead transparency, pen and a Styrofoam board (
purpose to verify whether a given/constructed figure has rotational symmetry.
Students were given a figure (or co
the figure was copied on the overhead transparency. The copied figure acted as an
identical copy of the original figure and was placed on top of the original figure.
Students used the push pin to locate an
While rotating the transparency, the original figure and the rotated copied figure could
be seen at the same time. This synchronic simultaneity enables the concept of

rotational symmetry to become visible an
potential for the transparency toolkit.
Figure 1: Transparency Toolkit
The lesson structure was basically the same among all the five lessons
whereas there were slightly differences in details. The teacher started the lesson by
giving a brief review on line symmetry through a whole
transition the idea of rotational symmetry was introduced. Then a figure was given to
students and they were asked to verify/explore whether the figure has rotational
symmetry using the transparency toolkit. The last part was the main part of the lesson.
Four identical plastic square
(worked in pair) were asked to design rotational symmetric figures using these four
plastic squares and used the transparency toolkit to verify their works. Afterwards, a
whole-class discussion was led by the teachers along with student presentations.
There were variations among different classes. In one class, a sheet of square
grids (figure 2) was used instead of plastic squares. In another class, a 'virtual toolkit'
using computer software that was designed to be equivalent to the transparency
toolkit was used instead of the plastic squares and the transparency toolkit.
In the following, we will analyze the implementations of the last part (i.e.
designing rotational symmetric figures) in two of the lessons (
B) where the transparency toolkit wa
Lesson A and plastics squares were used in
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consideration. All the teacher preparation meetings, lessons and the post
taped.
The Tool and the Task
f the lesson was to introduce to students the idea of a
rotational symmetric figure; that is, a figure 'overlaps itself' at least two times during
one cycle of rotation. A toolkit was specially designed for this lesson. The toolkit,
which we will call the 'transparency toolkit', consisted of blu-tack, push pins,
overhead transparency, pen and a Styrofoam board (Figure 1). It was designed for the
purpose to verify whether a given/constructed figure has rotational symmetry.
figure (or constructed a figure using given plastic shapes) and
the figure was copied on the overhead transparency. The copied figure acted as an
identical copy of the original figure and was placed on top of the original figure.
pin to locate and fasten the position of the centre of rotation.
While rotating the transparency, the original figure and the rotated copied figure could
be seen at the same time. This synchronic simultaneity enables the concept of
rotational symmetry to become visible and manipulative, thus forming a semiotic
potential for the transparency toolkit.
Transparency Toolkit Figure 2: Sheet of square grids
The lesson structure was basically the same among all the five lessons
there were slightly differences in details. The teacher started the lesson by
giving a brief review on line symmetry through a whole-class discussion and in
transition the idea of rotational symmetry was introduced. Then a figure was given to
they were asked to verify/explore whether the figure has rotational
symmetry using the transparency toolkit. The last part was the main part of the lesson.
plastic square pieces were given to the students (see Figure 1). Students
pair) were asked to design rotational symmetric figures using these four
plastic squares and used the transparency toolkit to verify their works. Afterwards, a
class discussion was led by the teachers along with student presentations.
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designing rotational symmetric figures) in two of the lessons (Lesson A and
B) where the transparency toolkit was used. Sheets of square grids were used in
esson A and plastics squares were used in Lesson B.
Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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stead of the plastic squares and the transparency toolkit.
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esson A and Lesson
s used. Sheets of square grids were used in Theme A – Y.-C. Chan & A. Leung
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Analysis
Part 1: Producing rotational symmetric figures using squares
In Lesson A, square grid sheets were provided for students (in groups) to
produce rotational symmetric figures by colouring selected squares. Generally
speaking, students found it a difficult task to handle. After guidance from the teacher,
some students rotated the square grid sheet and could produce some simple figures
with rotational symmetry. In contrast, in Lesson B plastic squares were given to
students. The plastics shapes were concrete manipulative which seemed to provide
students a more tangible experience to comprehend the meaning of rotational
symmetry. It allowed students to rotate individual square piece and consequently
could produce more complicated figures. Some students could detect and correct
mistakes by rotating the individual square pieces. For instance, a pair of students
initially designed an incorrect figure (Erreur ! Source du renvoi introuvable.).
While rotating (some of) the square pieces, they found that the figure has no rotational
symmetry (figure 4). Then they modified the figure into a rotational symmetric one
(figure 5). In this self-correction process enabled by the tool, the students (implicitly)
experienced the fact that the upper half and the lower half of the figure they produced
have the same distance from the centre but in the opposite direction. This may emerge
in students' mind an intuitive understanding of the meaning of a 2-fold rotational
symmetry.
Figure 3: incorrect figure Figure 4: rotate part of the figure Figure 5: corrected figure
When comparing lessons A and B, it was observed by the teachers and the
researcher that the square grid sheet was not as conducive as the plastics square pieces
to bring about the (intuitive) meaning of rotational symmetry. What made the
difference? One possibility was that square grid sheet could not be separated into
different parts. Whenever the whole square grid sheet was rotated, all the individual
squares were changed in the same way. This lack of variation in parts in the creation
process may result in a limited experience to what are typical in rotational symmetry.
In contrast, the plastics square pieces could be manipulated as separate entities and
changed with respect to each other in different ways. This opened up wider variation
and opportunity to produce more complicated figures. As seen in the 'self-correction'
example of Lesson B, students could rotate parts of the created figure (the middle two
squares). This may lead to discernment of critical features of rotational symmetry and
hence an intuitive understanding of the concept may consequently emerge. Thus a
shift of attention between the parts and the whole of the object of exploration (the
rotational symmetric figure) could occur through manipulation of the square pieces
but may not be the case for the square grid sheet. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22
Part 2: Initiating mathematics discussion through verification with the
transparency toolkit
After the above group activity of figure creation, students were asked to
verify whether their created figures have rotational symmet
transparency toolkit. The teacher then selected a few student groups to report their
works in front of the class and then initiated mathematics discussion which aimed to
bring out the semiotic potential of the tool.
report their works by demonstrating the rotational
toolkit. The teacher only focused on whether the students
not. It was not clear that the semiotic potential of the tool could
teacher led discussion. In contrast,
Lesson B. The structure of reporting for each group was basically the same. First, the
teacher asked the students whether the figure has rotational
students to locate the centre of rotation (which was the location of the push pin).
Then, the students were asked to demonstrate the figure's rotational symmetry using
the transparency toolkit. While rotating the transparency, it was
students paid attention to when the original figure (i.e. the figure composed by the
square pieces) and the rotated figure (i.e. the figure on the transparency) overlapped.
When the two figures overlapped, the whole class counted the num
overlapping. After a cycle of rotation, the teacher asked the whole class whether this
was a rotational symmetric figure and highlighted the reason; that is, the figures
overlapped at least two times in one cycle. It was interesting that th
group initiated further mathematics discussion. The created figure was a 3
rotational symmetric figure composed of three square pieces (Figure 6). Although the
explanation given by the students was basically correct, the teacher int
extended the discussion by pointing out that the original figure and the rotated figure
did not overlapped exactly (Figure 7). He proceeded to ask the whole class how to
modify the figure in order to make these two figures overlapped exactly. Af
thinking for a while, a student suggested that the sizes of the angles between each of
the adjacent squares should be the same. In order to further elaborate this idea, the
teacher compared this figure with a 4
produced by another group (Figure 8). After a brief discussion, it was concluded that
the more number of times of overlapping in one cycle, the smaller the size of the
angle between adjacent squares. This property was actually beyond the mathematics
knowledge that the teacher intended to teach.
Figure 6: a 3-fold rotational
symmetric figure
What could be learnt about tool based task implementation from
episodes? Firstly, as seen in Lesson A, simply focusing on the correctness of students
productions may not bring
the task. In contrast, as in Lesson B,
associated to rotational symmetry by a well
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Part 2: Initiating mathematics discussion through verification with the
After the above group activity of figure creation, students were asked to
verify whether their created figures have rotational symmetry or not by using the
transparency toolkit. The teacher then selected a few student groups to report their
works in front of the class and then initiated mathematics discussion which aimed to
bring out the semiotic potential of the tool. In Lesson A, two groups were asked to
their works by demonstrating the rotational symmetry using the
toolkit. The teacher only focused on whether the students' answers were correct
. It was not clear that the semiotic potential of the tool could be brought out
. In contrast, three groups were asked to report their works in
Lesson B. The structure of reporting for each group was basically the same. First, the
teacher asked the students whether the figure has rotational symmetry and asked the
students to locate the centre of rotation (which was the location of the push pin).
Then, the students were asked to demonstrate the figure's rotational symmetry using
the transparency toolkit. While rotating the transparency, it was observed that the
students paid attention to when the original figure (i.e. the figure composed by the
) and the rotated figure (i.e. the figure on the transparency) overlapped.
When the two figures overlapped, the whole class counted the number of times of
overlapping. After a cycle of rotation, the teacher asked the whole class whether this
was a rotational symmetric figure and highlighted the reason; that is, the figures
overlapped at least two times in one cycle. It was interesting that the work of the last
group initiated further mathematics discussion. The created figure was a 3
rotational symmetric figure composed of three square pieces (Figure 6). Although the
explanation given by the students was basically correct, the teacher int
extended the discussion by pointing out that the original figure and the rotated figure
did not overlapped exactly (Figure 7). He proceeded to ask the whole class how to
modify the figure in order to make these two figures overlapped exactly. Af
thinking for a while, a student suggested that the sizes of the angles between each of
the adjacent squares should be the same. In order to further elaborate this idea, the
teacher compared this figure with a 4-fold rotational symmetric figure which wa
produced by another group (Figure 8). After a brief discussion, it was concluded that
the more number of times of overlapping in one cycle, the smaller the size of the
angle between adjacent squares. This property was actually beyond the mathematics
ledge that the teacher intended to teach.
fold rotational Figure 7: verify the figure by
transparency toolkit
Figure 8: teacher
elaboration

What could be learnt about tool based task implementation from
as seen in Lesson A, simply focusing on the correctness of students
not bring about the mathematical meaning embedded in the tool and
task. In contrast, as in Lesson B, the teacher highlighted the key concepts
associated to rotational symmetry by a well-structured group reporting procedure:
Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
Part 2: Initiating mathematics discussion through verification with the
After the above group activity of figure creation, students were asked to
ry or not by using the
transparency toolkit. The teacher then selected a few student groups to report their
works in front of the class and then initiated mathematics discussion which aimed to
groups were asked to
using the transparency
answers were correct or
be brought out by the
three groups were asked to report their works in
Lesson B. The structure of reporting for each group was basically the same. First, the
symmetry and asked the
students to locate the centre of rotation (which was the location of the push pin).
Then, the students were asked to demonstrate the figure's rotational symmetry using
observed that the
students paid attention to when the original figure (i.e. the figure composed by the
) and the rotated figure (i.e. the figure on the transparency) overlapped.
ber of times of
overlapping. After a cycle of rotation, the teacher asked the whole class whether this
was a rotational symmetric figure and highlighted the reason; that is, the figures
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rotational symmetric figure composed of three square pieces (Figure 6). Although the
explanation given by the students was basically correct, the teacher intentionally
extended the discussion by pointing out that the original figure and the rotated figure
did not overlapped exactly (Figure 7). He proceeded to ask the whole class how to
modify the figure in order to make these two figures overlapped exactly. After
thinking for a while, a student suggested that the sizes of the angles between each of
the adjacent squares should be the same. In order to further elaborate this idea, the
fold rotational symmetric figure which was
produced by another group (Figure 8). After a brief discussion, it was concluded that
the more number of times of overlapping in one cycle, the smaller the size of the
angle between adjacent squares. This property was actually beyond the mathematics
Figure 8: teacher's further
elaboration
What could be learnt about tool based task implementation from these lesson
as seen in Lesson A, simply focusing on the correctness of students'
embedded in the tool and
the teacher highlighted the key concepts
structured group reporting procedure: Theme A – Y.-C. Chan & A. Leung
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locate the center, rotate the figure on the transparency, count the number of times of
overlapping, and lastly, recall prior knowledge, and address the discrepancy caused
by the tool utilization. In this orchestration process, the semiotic potential of the
transparency toolkit as a tool of semiotic mediation is emerged through evaluation of
the students' productions by manipulating the tool and speaking out the key concepts
explicitly at the same time. An interesting double semiotic link occurred when the
teacher made use of the tool's 'inaccurate representation' to extend the conceptual
understanding of rotational symmetry from merely recognizing a descriptive
definition to discerning a critical feature about angle size in rotational symmetry. The
occurrence of this discussion could be regarded as an incidental opportunity created
by the semiotic potential of the tool. The original task was to create a rotational
symmetric figure by using four (identical) square pieces. It happened that one pair of
the students did not follow the instruction and used three square pieces instead of
four. It can be calculated easily (by us not the students) that the angle between two
adjacent square pieces needs to be 30 degrees in order to create a rotational symmetry
figure using three (identical) squares. For Grade five students, it is difficult visually to
arrange the square pieces so that all three gaps are of 30 degrees. Rotating the figure
by the transparency toolkit made this discrepancy explicit which gives rise to an
opportunity for the teacher to orchestrate meaningful mathematics discussion with the
students. This suggests that a tool-based task which can capitalize the potential
discrepancy embedded in the tool may bring about a higher level of conceptual
understanding. The insight and flexibility of the teacher is a key factor of whether the
task can be orchestrated to the emergence of mathematical meaning. In the above
teaching episode, if the teacher ignored the discrepancy or he simply told the students
that all the four squares should be used, the mathematics discussion would not be as
rich and fruitful.
Discussion and Conclusion
In this last section, some lesson episodes on tool-based tasks were analyzed.
In the following, we will extend the discussion to propose some suggestions
(hypothesis) on the design of tool-based tasks which may enhance the bringing out of
semiotic potential of the tools. We will focus on four aspects: 1. the tool, 2. the task,
3. students' productions, and 4. the teacher.
The tool
Semiotic potential of the tool may be enhanced if the tool allows students to
freely shift their attentions between the parts and the whole of the object of
exploration
This suggestion is consistent to the perspective of discernment and variation
(Marton, Runesson & Tsui, 2004). Based on this perspective, Leung (2003) pointed
out that "discernment comes about what parts (features) are being focused and
temporarily demarcated from the whole (background)" (p.198). For instance, in
dynamic geometry environment (DGE), part of the geometric configuration can be
varied via dragging while keeping the other parts fixed. The invariant patterns of the
configuration could then be "separated-out" (Leung, 2008). The lesson episodes
above showed an example of a non-ICT tool. The square pieces (without the blu-tack)
allow students to see the object of exploration (the rotational symmetric figure) in
parts and hence 'separate' some parts from the whole object. Whereas, the square
pieces with blu-tack allow the students to see the object as a whole. This flexibility of Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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the tools may give an opportunity to have a better understanding on the part-whole
relationship of the object and discern the essential feature of the object from the
incidental one.
The task
A task which provides an opportunity to make use of the discrepancy
embedded in the tool may initiate meaningful mathematics discussion which could
lead to deeper conceptual understanding.
Making mistakes may not be a bad thing. Meaningful mistakes may initiate
mathematics discussion which leads to construction of mathematical knowledge.
Some tasks may have higher possibility to make (meaningful) mistakes. For instance,
in the lesson episodes above, creating a rotational symmetry figure by using three
(identical) square pieces is easier to make 'mistakes' than by using four (identical)
square pieces. Though the 'mistake opportunity' occurred in our lesson episodes was
rather incidental, we suggest that the 'mistake opportunity' should be planned
together with the task design. In order to make these mistakes meaningful to the
students, the mathematics implied in the 'mistakes' should be considered beforehand.
The tool used in a task may influence the chance of making mistakes. For instance, if
instead of concrete square pieces, an ICT tool (e.g. DGE) was used for the task
described in this lesson episode, it was unlikely that the 'mistakes' discussed above
would occur because drawing in DGE is usually accurate (at least the discrepancy
cannot be discerned visually). However, there may be other discrepancies in DGE that
are conducive to rich mathematical discussion. (Some empirical studies found that
sometime students deliberately drew a 'wrong' picture in DGE in the process of
exploration and argumentation. See for example, Leung & Lopez-Real (2002);
Mariotti & Antonini (2009); Baccaglini-Frank, Antonini, Leung & Mariotti, (2011).)
Student' productions
Requiring the students to evaluate their productions using the tool while
simultaneously speaking out the key mathematics concepts with respect to the tool
may effectively mediate the progression from students' production to mathematics
knowledge.
The progression from students' production to mathematics knowledge is not
automatic. Teacher, as an expert representative of mathematics culture, needs to use
students' production as "a tool of semiotic mediation" (Bartolini Bussi & Mariotti,
2008). For instance, Mariotti and her colleagues conducted a series of long term
teaching experiments in DGE to develop students' mathematical thinking in
axiomatic approach (Mariotti, 2002). Students were mediated to experience the
development process of mathematical theory by Cabri (a DGE) commands. There is a
correspondence between these commands and the elements of mathematical theory
(definitions, axioms and theorems). Indeed, the 'labels' of the Cabri commands make
the correspondence explicit. However, for non-ICT tools, such correspondence may
not be explicit because this kind of 'labels' may not exist in the tools. It is the role of
teachers to make the correspondence explicit. The teachers in the lesson episodes
made use of a well-structured orchestration procedure which involves students'
evaluation of their own productions by manipulating the tool and speaking out the key
mathematics concepts explicitly at the same time to achieve this 'labelling effect'. In
this orchestration, the teacher deliberately highlighted the mathematics terminologies
realized in the semiotic potentials of the tool. For example, the push pin corresponds Theme A – Y.-C. Chan & A. Leung
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to the centre of rotation. The correspondence of the students' productions and the
mathematics knowledge may be seen through this process of "situated abstraction"
(Noss & Hoyles, 1996).
The teacher
Developing teacher's ability to determine whether student production
(correct or incorrect) is conducive to conceptual learning is an important aspect in
training teachers to use tool-based tasks in the mathematics classroom.
Teachers play a significant role in the process of semiotic mediation. As an
expert representative of mathematics culture, the teacher guides the evolution of
mathematical meanings related to the tool and its use within the mathematics
classroom (Bartolini Bussi & Mariotti, 2008). The lesson episodes above suggested
that the quality of the guidance depends on the insight and flexibility of the teacher. In
the second part of Lesson B, the teacher made use of the students' incorrect (or notso-correct) production to extend the mathematics discussion so that deeper conceptual
understanding of rotational symmetry was evolved. This was incidental in the sense
that it was out of the teacher's original planning. Certainly, not all students' mistakes
are equally worthwhile for extending the discussion. Therefore, a teacher's ability to
determine whether a student's (mistaken or unexpected) production (or response) is
worthwhile for further discussion is one of the criterions of success to bring about the
semiotic potential of a tool-based task. We suggest that developing such ability is an
important aspect in training teachers to use tool-based mathematics tasks.
In this paper, four suggestions on tool-based tasks are proposed according to
our analysis of the design and implementation of a tool-based geometry task at
primary school level. More empirical studies will be conducted in order to verify the
validity of our suggestions.
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Designing tasks within a multi-representational technological environment: An emerging rubric.
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The advent of digital technological environments that offer linked multiple mathematical representations present a substantial challenge to secondary mathematics teachers as they begin to design tasks for their learners. This paper concludes a design rubric for mathematical tasks as one of the outcomes of a longitudinal doctoral study in which fifteen teachers began to integrate such a tool (TI-Nspire handheld and software) within their classroom practice (Clark-Wilson, 2010a). The research, which was carried out in two phases, provided evidence for the predominant use of the technology to support explorations of variance and invariance. It also gave an insight into the process through which the teachers integrated the technology into their classroom practices. This paper offers a joint perspective of the project from the mathematics education researcher (Clark-Wilson) alongside one of participating teachers (Timotheus).
Keywords: Hiccup, instrumental genesis, mathematics education, mathematical generalisation, multiple representational technology, task design, teacher development, variance and invariance.
Introduction
The research reported within this paper was situated within a pilot evaluation project that introduced a prototype technology 'TI-Nspire' (Texas Instruments, 2007) to a group of English secondary mathematics teachers between July 2007 and November 2009. The project community comprised educational technology developers (Texas Instruments), a mathematics education researcher (the first author), mathematics education consultants and secondary mathematics teachers (including the second author). The first author was solely responsible for the design, implementation and reporting of the project and the second author was one of the participating teachers. In common with other teacher development projects, with and without ICT, there was a clear rationale underpinning the approach taken to establish the ethos and ways of working within the project (Ahmed, 1987; Watson, De Geest, & Prestage, 2003). There was a clear sense of 'researching-with', which resulted from privileging the teachers' classroom stories and by supporting teachers to substantiate their own claims of changed mathematical experiences and outcomes on the part of their students. Within the research, the word 'activity' was used to describe the set of Theme A - A. Clark-Wilson & J. Timotheus
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situations that the teacher had designed to initiate the students' mathematical work. These activities would include one or more 'tasks' and the outcomes of these tasks were subsequently described as 'episodes' within the teaching sequence. The broad aims for the study were to try to articulate the nature of the teachers' cognitive and pedagogical learning as they began to use the aforementioned multi-representational technology in their classrooms with students.
Theoretical perspective
The research was framed within an activity-theoretic approach that interprets the Vygotskian notion of activity as a 'unit of analysis that included both the individual and his/her culturally defined environment' (Wertsch, 1981). The elaboration of the instrumental approach within technological environments (Verillon & Rabardel, 1995) was developed further by a body of researchers within the context of technology use in mathematics education, leading to the notions of instrumentation, instrumentalisation, instrumental genesis, instrumental orchestration and documentational genesis (Drijvers, 2011; Drijvers & Trouche, 2008; Gueudet & Trouche, 2009; Guin & Trouche, 1999; Trouche, 2004). These ideas concern the complex and interrelated processes of:
• learning to use a new technology for purposeful mathematical activity;
• designing tasks for students to initiate purposeful mathematical activity;
• collating the various artefacts that comprise the 'document system' for the activity;
• supporting students to learn to use technology for purposeful mathematical activity;
• articulating the teacher's role in supporting the students to navigate their respective routes through the various artefacts that comprise the activity to include interaction with the technology.
It is important to comment that within the research contexts from which these notions emanate, the chosen technologies are best described by Pierce and Stacey (2008) as 'mathematical analysis tools', which include technologies such as computer algebra software (CAS), dynamic geometry software (DGS), graphing software and spreadsheet software. The TI-Nspire technology adopted within this research afforded a range of 'applications' that included: calculator; spreadsheet; dynamic geometry; function graphing; statistical calculation and graphing; built in commands i.e. factor(n); and text editing. In all of these environments, the facility to save numeric outputs as variables supported the linking of variables within and between these different applications.
A second important theoretical construct of significance to the study was that of a multiple representational environment, which was postulated initially by Kaput (Kaput, 1986) as he presented a vision for the way in which technology might support higher-level engagement with mathematics. In the intervening years, the aforementioned genres of technologies have afforded opportunities to engage with mathematics dynamically by observing the simultaneous views of different representations. For example, the representations of a function, its graph and a table of its associated coordinate values. The development of 'dragging' a figural image through the interface of a mouse (or pen or finger) has afforded further forms of mathematical interaction.
Finally, the underlying principle that informed the pedagogical approaches within the study were informed by Mason et al's assertion that "a lesson without the Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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opportunity for learners to express a generality is not, in fact, a mathematics lesson' (Mason, Graham, & Johnston-Wilder, 2005, p.297). This promoted the use of the technology to provide the students with opportunities to explore variance and invariance within a range of mathematical contexts.
Research methodology
In concluding their research study into teachers' perceptions of the use of technology, which involved secondary mathematics departments in seven English schools, Ruthven and Hennessy (2002) , highlighted the need for, naturalistic studies which provide analyses of the perspectives and practices of groups of teachers across different settings within the context of using technology to support teaching and learning of mathematics. (p.50)
Although this study cannot claim to be wholly naturalistic, as the technology had been prescribed, it did aim to gain a deep insight into the perspectives and practices of secondary mathematics teachers as they developed their use of the MRT in the classroom.
The research was carried out in two phases, the first between July 2007 - November 2008 and the second between April 2009 and December 2009. In each of these phases, a group of teachers was selected and a series of methodological tools developed to capture rich evidence of their use of the technology in classrooms to enable the aims of the study to be realised. The methodological approaches for each phase were distinct, and each is described briefly below.
Phase 1
Fifteen teachers were selected (in pairs) from seven English schools where there was some history of use of technology use within mathematics, although some of the individual teachers had limited personal experience. During the first phase of the research (July 2007 to November 2008), the teachers reported 66 lesson activities that had used the TI-Nspire technology, which resonate with Gueudet and Trouche's "document systems" (2009) comprised a number of artefacts that included some or all of the following:
• an activity plan in the form of a school lesson planning proforma or a hand-written set of personal notes;
• a lesson structure for use in the classroom (for example a Smart NoteBook or PowerPoint file);
• a TI-Nspire software file developed by the teacher for use by the teacher (to introduce the activity or demonstrate an aspect of the activity);
• a TI-Nspire software file developed by the teacher for use by the students, which would normally need to be transferred to the students' handhelds in advance or at the beginning of the lesson;
• a activity or instruction sheet developed by the teacher for students' use;
• students' written work resulting from the activity;
• students' TI-Nspire software files captured during and/or at the end of the activity;
• a compulsory post-activity questionnaire, which prompted the teacher to reflect deeply on the lesson(s). (Clark-Wilson, 2008)
This data was used to analyse each of the lesson activities with respect to the teachers' planned instrument utilisation schemes and, as a result nine diagrammatic Theme A - A. Clark-Wilson & J. Timotheus
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'instrument utilisation schemes' were developed. For more detail, see Clark-Wilson (2010b).
The analysis of the phase 1 data led to 268 thickly coded examples of 'teacher learning' from 100 data sources and a process of constant comparison led to 38 categories that were subsequently grouped into four domains: activity (task) design, expectations of students, instrument utilisation schemes and meta-level ideas. As this paper is concerned with the first theme, the broader description of the categories related to task design included aspects concerning:
• the teacher's initial choice of examples, its appropriateness and potential for non-trivial mathematical exploration and extension;
• the features of a good 'first activity' when introducing the technology or new instrumental techniques to the students;
• the need to balance the construction and exploration elements within the design of activities;
• teachers' reflections on their personal instrument utilisation scheme when introducing the activity to the students and during whole class discussions;
• the relationship between the students' learning of relevant mathematical concepts with technology and the traditional 'by hand' or 'paper and pencil' approaches - and reflections surrounding the teacher's role in supporting students to connect these experiences;
• an appreciation of the need for a tighter focus on or attention to specific mathematical generalisations within activities.
The first phase of the project revealed rich data concerning teachers' perceptions of explorations of regularity and variation, which seemed to substantiate Stacey's claim that, from the Australian research, this was the most common form of mathematical activity with technology (Stacey, 2008). There was less evidence of activities that made use of opportunities to link representations. However, the analysis of the teachers' developing instrument utilisation schemes suggested that this could be a function of both teachers' time and familiarity with the technology.
Phase 2
The second phase sought to probe the nature of teachers' situational learning through a focussed and systematic case study of two of the teachers who had participated in the first phase (Eleanor and Tim). These two teachers had a level of technical competency that showed that they had grasped the skills needed to create activities using a range of applications within the MRT. The second consideration was that the teachers adopted pedagogical approaches that placed the students' mathematical experiences at the centre of the classroom environment. By allowing the students more mathematical choices within the activities they developed, they adopted a more socio-constructivist philosophy in which their students could form their own mathematical meanings with a collaborative, supportive classroom ethos. My rationale for this choice was related to my desire for the study to generate new knowledge about the way that teachers conceive and learn from their own innovations with complex new technologies.
The data collected as part of the teacher's 'document system' during phase one was supplemented by the following:
• classroom observation data (video, audio and field notes); Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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• additional data relating to the teacher's interactions with students, for example, a software file at the time of the interaction;
• data resulting from interviews with the teachers, which included pre- and post lesson discussions;
• field notes, which incorporated extracts from email exchanges with the teachers.
A further 14 lessons were observed between May 2009 - December 2009 and for each one, the data set was imported into Nvivo 8 software to support the subsequent data coding and analysis processes (QSR International, 2008). The analysis of the phase two data concluded the important notion of the lesson 'hiccup', that is the perturbation experienced by teachers during lessons stimulated by their use of the technology, which illuminated discontinuities within the teachers' knowledge. The cross-case analysis attributed the hiccups to seven considerations: Aspects of the initial activity design; Students' interpretations of the mathematical generality under scrutiny; Unanticipated student responses as a result of using the MRT; Perturbations experienced by students as a result of the representational outputs of the MRT; Instrumentation issues experienced by students when making inputs to the MRT and

whilst actively engaging with the MRT; instrumentation issue experienced by one teacher whilst actively engaging with the MRT; and Unavoidable technical issues. However, as this paper is concerned with the subsidiary research findings concerning the emergent principles for task design within MRT environments and the tool-driven implications for the design, teaching and learning triad the discussion that follows will focus on these aspects.

What did the teachers learn about designing tasks within the MRT?

A central tenet of this thesis was that activities and approaches that privileged an exploration of mathematical variance and invariance constituted a legitimate pedagogical opportunity for the use of technology in secondary mathematics classrooms. The teachers' evolving conceptions of variance and invariance were revealed in several ways during the study. This included how they conceptualised the initial representation of the variant property under exploration and the related representations that added insight to, or progressed, the exploration and the way in which any invariant properties would become explicit as a result of the exploration.

Eleanor and Tim learned that there were several important aspects when making the choice of initial example within their activity design and its subsequent representation within the MRT. Firstly, it needed to be an example of the generalisation being sought. A hiccup that Tim experienced during the activity 'Linear equations' led him to conclude that his choice of functions for the initial example, which was a spontaneous decision, was unhelpful. His selection of y=1x+1 and y=2x did not support the students to begin to generalise about the gradients of the linear functions, partly because there were two digit ones inherent in y=1x+1, but also due to the way that the MRT displayed the measured equations

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(see Figure 1).

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This hiccup was compounded by the output from the MRT which displayed an unhelpful 'measured' equation. (A measured equation is one which for which the MRT numerically analyses a set of points that define a geometrically constructed line, often revealing unexpected numerical errors.) Theme A - A. Clark-Wilson & J. Timotheus

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Figure 1 Tim's handheld screen during the activity 'Linear graphs' [STP10(tns-T)].

Eleanor and Tim both seemed to grapple with the notion of the 'example space' in ways that were resonant with Goldenberg and Mason's elaboration of the difficulties encountered when trying to identify appropriate examples for different mathematical generalisations (Goldenberg & Mason, 2008). Tim's choice of functions in the 'Linear equations' activity could be interpreted as him having a 'general sense' of the representation of any two straight lines on a dotty grid alongside the functionality of a 'gradient measure' tool as a sufficient space in which to be able to generalise about slope. However, a more informed choice of functions, alongside the utilisation of the MRT's functionality to measure the slope of the line, rather than its equation, all formed aspects of Tim's redesign of this activity.

Having decided upon the initial input representation, another important consideration for the teachers related to their decisions about the syntactic labelling or notation of variable objects. They learned that this was a necessary element of activities when classroom discourse would be initiated to support students to notice and verbalise their generalisations. As both teachers privileged whole-class discourse focussed on key mathematical generalisations in their classroom practices, they both experienced a number of hiccups that indicated that there was insufficient notation or labelling to enable this discussion to be suitably focussed. This is an aspect that Pierce and Stacey have noted in their recent research into the way that Australian secondary mathematics teachers are developing the use of the TI-Nspire handhelds to privilege linked mathematical representations when teaching about quadratic functions (Pierce & Stacey, 2009). They have highlighted the conflict between the labelling of variables within the "ideal' or pencil and paper mathematics alongside the conventions built into the MRT, as an important element for teachers to consider in the design of tasks.

A final consideration was the development of the teachers' knowledge concerning the grouping or sequencing of examples, that is, determining the scale and structure of the example space. This aspect of their learning not only related strongly to their personal mathematical knowledge and interpretation of the mathematical progression of ideas within the topic matter, but was also intrinsically linked to the pedagogic skill of considering the most suitable activity structure for their students.

In general, having selected an initial example for each activity, Eleanor and Tim expanded the example spaces to add insight to, or support the analysis of the variant and invariant properties under exploration in one, or a combination of the following approaches:

• Further examples were explored within the same mathematical representation by making new inputs in the same numeric, syntactic or geometric form.

• Further examples were explored within the same mathematical representation by dragging objects dynamically.

• Further examples were explored within a different mathematical representation by making new numeric, syntactic or geometric inputs. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.

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• The same example was explored within a new mathematical representation.

Conclusion

The scrutiny of over eighty classroom activities within this study has provided a substantial data set on which to draw a number of conclusions relating to their experiences of designing tasks within the chosen MRT environment.

Mason et al's premise regarding the role of expressing generality within mathematical activity could be considered to have been an initial constraint for the teachers involved in this study when designing their activities to explore variance and invariance (Mason et al., 2005, p.297). However, an element of the teachers' epistemological development was related to their realisation that expressing generality was a very important aspect of the activities that they went on to design. The teachers' increasing attentions to the way that the MRT environment supported or hindered this, and the design of the associated supporting resources and their role in mediating the associated classroom discourse, was another element of their professional development.

Evidence from the study suggests that the process of designing tasks that utilise the MRT to privilege explorations of variance and invariance is a highly complex process which requires teachers to carefully consider how variance and invariance might manifest itself within any given mathematical topic. The relevance and importance of the initial example space, and how this might be productively expanded to support learners towards the desired generalisation is a crucial aspect of activity design.

The starting point for any classroom activity is its initial design and the following set of questions, generated as a result of this study offer a researchinformed approach:

• What is the generalisable property within the mathematics topic under investigation?

• How might this property manifest itself within the multi-representational technological environment – and which of these manifestations is at an accessible level for the students concerned?

• What forms of interaction with the MRT will reveal the desired manifestation?

• What labelling and referencing notations will support the articulation and communication of the generalisation that is being sought?

• What might the 'flow' of mathematical representations (with and without technology) look like as a means to illuminate and make sense of the generalisation?

• What forms of interaction between the students and teacher will support the generalisation to be more widely communicated?

• How might the original example space be expanded to incorporate broader related generalisations?

Responses to these questions uncover a generic 'top level' of thinking which makes little sense in the absence of a clear mathematical context. The next level of thinking is closely related to the mathematical concept under scrutiny for which a number of existing structures and approaches can support teachers to develop further this aspect of their practice. Theme A - A. Clark-Wilson & J. Timotheus

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Digital design:
RME principles for designing online tasks

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The theory of Realistic Mathematics Education provides principles that can be applied to task design. In this paper, we investigate how these principles apply to the design of online tasks. To do so, we present examples of tasks on algebra, calculus and geometry designed in the Digital Mathematics Environment. As a result, we conclude that the principles of guided reinvention, didactical phenomenology, and emergent modeling can inform and guide digital design, but that some aspects work out differently compared to the design of paper-and-pencil tasks.

Keywords: Didactical Phenomenology, Digital Design, Digital Mathematics Environment, Emergent Modeling, Guided Reinvention, Realistic Mathematics Education

Introduction

Task design is widely recognized as an important, but complex and subtle activity. Based on the experience of skilled designers, design guidelines and heuristics

have been identified (e.g., see Watson & Mason, 2006). Whereas the design and use
of digital content nowadays plays an increasingly important role in mathematics
education, most of these design principles are based on and applied to the design of
paper-and-pencil tasks. The question, therefore, is how such design principles apply
to digital design, and how the rich experience and knowledge in the field of designing
paper-and-pencil tasks can be transferred to the case of digital design.
To address this question, we limit ourselves to three principles that emerge
from the theory of Realistic Mathematics Education and may inform design: guided Theme A – P. Drijvers et al.
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reinvention, didactical phenomenology, and emergent modeling. We revisit some
tasks designed in the Freudenthal Institute's Digital Mathematics Environment in the
field of algebra, calculus and geometry education.
RME Design Principles
Realistic Mathematics Education (RME) is a domain-specific instruction
theory for the teaching and learning of mathematics. According to RME, mathematics
should be seen as an activity (Freudenthal, 1973), and students, rather than being
receivers of ready-made mathematics, should be active participants in the educational
process, in which they develop mathematical tools and insights by themselves. This
point of departure led to the following RME principles that inform task design: guided
reinvention, didactical phenomenology, and emergent modeling (Freudenthal, 1973;
Gravemeijer, 1994; Van den Heuvel-Panhuizen, 1996).
According to the principle of guided reinvention, students should be given
the opportunity to experience a process similar to that by which a given mathematical
topic was invented. Even if this primarily is a teaching principle, it has consequences
for task design: tasks – or sets of tasks – should invite students to develop 'their own'
mathematics. This process, however, needs guidance from the teacher, to help to
further develop sensible directions, to leave 'dead-end streets' and to ascertain
convergence towards shared knowledge according to the standards within the
mathematical community.
Didactical phenomenology concerns the relation between the thought object
–the 'noumenon'– and the phenomenon –the 'phainomenon'– from the perspective
of teaching and learning. In particular, it addresses the question how mathematical
'thought objects' can help in organizing and structuring phenomena in reality. The
challenge for the task designer, then, is to find such meaningful phenomena that beg
to be organized and structured by the targeted mathematical knowledge.
According to the emergent modeling perspective, a model may play different
roles during different phases of activity. Initially, a model is context-specific: it refers
to a meaningful problem situation that is experientially real for the student, and is a
model of that situation. Then, through working with the model, it gradually acquires a
more generic character and develops into a model for mathematical reasoning that is
possible because of the development of new mathematical objects in a more abstract
framework of mathematical relations that the model starts to refer to. This notion is
elaborated into a four-level structure that represents levels of mathematical activity
(Gravemeijer, 1994). For the task designer, the challenge is to find suitable situations
that ask for the development of such models, and allow for a process of progressive
abstraction.
While applying these RME principles to the design of paper-and-pencil tasks
is not straightforward, their use for the design of digital content is even more
challenging. Although early research on the use of graphing calculators identified
opportunities for an RME-based teaching approach (Drijvers & Doorman, 1996), later
studies describe the tension between RME principles and the integration of computer
algebra software (Drijvers, 2000). Apparently, the match between RME and ICT is
not self-evident.
The Digital Mathematics Environment
As technology for teaching mathematics the Freudenthal Institute's Digital
Mathematics Environment (DME) is used. The DME integrates a content Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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management system, a learning management system and an authoring environment.
The content consists of online modules in the form of Java applets. The learning
management system offers means to distribute content among students and to monitor
student progress.
The authoring tool is the DME's design environment. Authors, such as
teachers, text book authors, or educators, can use the tool for adapting existing online
modules or for designing new ones, based on existing materials and basic tools such
as graphing and equation editing facilities. While designing, the author can split up
the screens in different windows, add applications and tasks, and design feedback.
Knowledge of the underlying Java programming language is not required; rather, an
intuitive and mathematical interface makes the digital design accessible to a wide
audience (Figure 1).
Figure 1 The Digital Mathematics Environment Authoring Tool
Examples of the results of digital design using the DME's authoring tool can
be found at www.fi.uu.nl/dwo/demo/en. In this paper, we will briefly discuss some
examples from algebra, calculus and geometry from the perspective of the above
RME principles.
Digital Tasks for Algebra
As an example of the design of digital tasks for algebra, we consider the
work by Bokhove, available at www.fi.uu.nl/dwo/voho/. The digital tasks focus on
solving polynomial equations. However, the online units go beyond procedural
practice and also focus on the development of symbol sense and strategic skills
(Bokhove & Drijvers, 2010, 2012). Crucial factors in the design are the sequencing of the tasks (with sometimes thought
and its timing and fading in particular
Figure 2 Solving an equation with unexpected difficulties
Figure 2 shows an exemplary equation in line 1, and a part of a student's
work in lines 2-4. The feedback just refers to the algebraic equivalence of the
subsequent equations the student enters; no feedback is provided
solving strategy. This can b
heuristic: this combination of task and tool provides the opportunity for students to
reinvent efficient ways of solving equations. Eventually, students can continue
entering equivalent equations without
exploration space is supposed to elicit a wish for efficiency.
From the didactical phenomenology perspective, this task may seem quite
poor: what is the phenomenon at stake that would motivate students to engage in
mathematics? In defense of this, one can argue that the target group of this online
module consists of students in grade 12, who were to do the national examination
soon, and who were familiar with the 'world of polynomial equations'. This
familiarity makes that these tasks can be appropriate for the mathematization of the
field and the development of new problem solving strategies.
Data shows that students work easily with this type
Drijvers, & Boon, 2011).
use tools to solve the tasks (Artigue, 2002), was an issue to a much lesser extent than
in the case of computer algebra
using CAS puts higher demands on instrumental gen
into account as a designer.
Digital Tasks on Functions and Calculus
As an example of the design of digital tasks for
the online module Function and Arrow Chain,
www.fi.uu.nl/dwo/prootool
conceptual understanding of the notion of functions, where a function is seen as an
input-output assignment, as a dynamic process of co
object with different represen
design is sequencing the tasks so these function models emerge in a natural way of
increasing complexity and abstraction.
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the tasks (with sometimes thought-provoking equations) and the design of f
in particular.
Figure 2 Solving an equation with unexpected difficulties (Bokhove & Drijvers, 2010, 2012)
2 shows an exemplary equation in line 1, and a part of a student's
4. The feedback just refers to the algebraic equivalence of the
subsequent equations the student enters; no feedback is provided on the problem
solving strategy. This can be seen as a manifestation of the guided reinvention
heuristic: this combination of task and tool provides the opportunity for students to
reinvent efficient ways of solving equations. Eventually, students can continue
entering equivalent equations without coming closer to the solution, but this
exploration space is supposed to elicit a wish for efficiency.
From the didactical phenomenology perspective, this task may seem quite
poor: what is the phenomenon at stake that would motivate students to engage in
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soon, and who were familiar with the 'world of polynomial equations'. This
that these tasks can be appropriate for the mathematization of the
field and the development of new problem solving strategies.
Data shows that students work easily with this type of online tasks (Tacoma,
Drijvers, & Boon, 2011). Instrumental genesis, the process of developing schemes to
tools to solve the tasks (Artigue, 2002), was an issue to a much lesser extent than
in the case of computer algebra (Drijvers, 2002; Drijvers et al., 2012). Apparently,
using CAS puts higher demands on instrumental genesis, and this is something to
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on Functions and Calculus
As an example of the design of digital tasks for (pre-)calculus
online module Function and Arrow Chain,
ootool/en. The digital tasks focus on the development of
conceptual understanding of the notion of functions, where a function is seen as an
output assignment, as a dynamic process of co-variation, and as a mathematical
object with different representations (Doorman et al., 2012). A crucial factor in the
design is sequencing the tasks so these function models emerge in a natural way of
increasing complexity and abstraction.
provoking equations) and the design of feedback,
(Bokhove & Drijvers, 2010, 2012)
2 shows an exemplary equation in line 1, and a part of a student's
4. The feedback just refers to the algebraic equivalence of the
on the problem
e seen as a manifestation of the guided reinvention
heuristic: this combination of task and tool provides the opportunity for students to
reinvent efficient ways of solving equations. Eventually, students can continue
coming closer to the solution, but this
From the didactical phenomenology perspective, this task may seem quite
poor: what is the phenomenon at stake that would motivate students to engage in
athematics? In defense of this, one can argue that the target group of this online
module consists of students in grade 12, who were to do the national examination
soon, and who were familiar with the 'world of polynomial equations'. This
that these tasks can be appropriate for the mathematization of the
of online tasks (Tacoma,
of developing schemes to
tools to solve the tasks (Artigue, 2002), was an issue to a much lesser extent than
Drijvers et al., 2012). Apparently,
esis, and this is something to take
)calculus we consider
available at
on the development of
conceptual understanding of the notion of functions, where a function is seen as an
variation, and as a mathematical
tations (Doorman et al., 2012). A crucial factor in the
design is sequencing the tasks so these function models emerge in a natural way of Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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A function as an input-output
assignment: braking distance as
a function of velocity
Investigation of the co-variation
of velocity and braking distance
through tracing the graph

Investigation of a family of
functions, representing braking
distances for three different
vehicles
Figure 3 Different views on function in the Function and Arrow Chain module
Figure 3 shows some screens from this module, in which the function
gradually develops from a numerical input-output engine, to a process of co-variation
and, finally, a mathematical object that is part of a family of functions that can be
compared. This sequence of screen shots reflects the emergent modeling heuristic: the
context, in this case one of a vehicle's braking distance as a function of its velocity,
leads to function models of increasing complexity and abstraction. The digital tool
supports this development by means of offering techniques of increasing richness and
an increasing repertoire of connected function representations. Of course, this
approach requires a context that is suitable for this emergent modeling process.
Finding such a context is a question of didactical phenomenology.
In an online calculus course for university freshmen, the co-variation idea is
supported by a Geogebra applet for tracing graphs (www.fi.uu.nl/dwo/sk/en/).
However, in the context of this intensive and short-period remedial course, the
emergent modeling design heuristics was exploited to a lesser extent than was the
case for the Function and Arrow Chain course.
Digital Tasks for Geometry
An example of the design of digital tasks for geometry is the module
available at www.fi.uu.nl/dwo/dpict/en/. The digital tasks focus on exploring,
discovering, and proving properties of bisectors, altitudes and medians in triangles.
The latter aspect, the proving, provides a particular design challenge: how to design
tasks that offer support and guidance for a proof, but leave room for reinvention? Theme A – P. Drijvers et al.
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Figure 4 Proving that three medians of a triangle intersect in one point
While reading the proving task shown in Figure 4, it should be noted that
Dutch grade 8 students, the target group for this module, have little experience in

proving, and, therefore, need strong guidance: the structure of the proof and the corresponding sequence of images is suggested to the students. Still, writing down the final argument in the reasoning, as is requested in task c, is very difficult to them. The design principle of guided reinvention is easier to apply to tasks in which students explore the properties of bisectors, altitudes and medians in triangles: the dragging options of the dynamic geometry system, in this case Geogebra, in collaboration with DME's feedback, provide a strong learning environment in which students can really experience the geometrical situation and discover the targeted properties.
In the case of this online module, classroom observations show that attention needs to be paid to students' and teachers' instrumental genesis: the interplay between Geogebra and the DME in this module is powerful but it may also be demanding and initially complex to novice users.
Conclusion
In this short paper, we set out to investigate the application of the RME principles of guided reinvention, didactical phenomenology, and emergent modeling for the case of the design of digital tasks in the online DME. To do so, we considered three exemplary tasks. The first example, the task on solving a relatively complex equation, shows that guided reinvention in this case concerns the development of new problem solving strategies, invited by tasks that cannot be solved with the strategies available so far. The didactical phenomenology heuristic here does not lead to the use of real life contexts, but rather takes the world of polynomial equations as a point of departure. In the second example, a real life context –in this case the stopping distance situation– does form the starting point. Emergent modeling heuristics are manifest in the gradual abstraction of the students' view on function. The third example on geometrical proof again takes a guided reinvention perspective, in which the designers chose strong guidance. Again, the didactical phenomenology here does not lead to a real life problem situation, but to a problem in the world of geometry, that is expected to be experientially real to the students. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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What can we conclude about the three RME principles and their application to digital design? The principle of guided reinvention seems to apply well to digital design. ICT offers opportunities for exploration and investigation, and in this way for reinvention. The design choice to confront students with unexpected examples can be seen as a way to invite reinvention as well. In general, this guided reinvention approach might suffer from constraints of the technology that may limit the students' exploration space, such as requirements for input formats and styles, and pre-designed tools that may incorporate too much guidance.
The didactical phenomenology heuristics is also valuable for digital design, but it seems that the phenomena that play a central role in the task do not necessarily come from real life: ICT already forms a meaningful 'world' on its own for the student, in addition to the world of mathematics. Having a real life context as an entrance to these two worlds may lead to cognitive overload. This is a question to be considered carefully in the design process.
Emergent modeling can be a fruitful design heuristics for digital design. As in the case of paper-and-pencil design, the models need to lend themselves for further development towards increasing mathematical abstraction and complexity. Specific for the case of digital design is that these emerging models need to be supported by the digital tools available, for example by an increasing repertoire of representations and techniques in the digital environment, or by increasing options to dynamically use these representations, connect them, and switch between them (Duval, 2006).
As an overall conclusion, this brief exploration of the issue suggests that the three RME principles are valuable for digital design, even if some appropriation is needed compared to the design of paper-and-pencil tasks, such as taking into account the constraints of the digital tools and the fact that the technological environment forms an additional 'world' to the student. The transfer of skills, developed in the ICT environment, to paper-and-pencil, for example, may need specific attention, as well as the instrumental genesis involved in the learning process.
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Designing tasks for a more inclusive school mathematics

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In this contribution to Theme A (tools and representations), we detail our approach to designing tasks to be incorporated into inclusive mathematics learning scenarios. These scenarios also involve tools created to represent mathematical knowledge in forms appropriable by students with sensory disabilities and which are developed to privilege multimodal experiences of mathematical objects, relationships and properties. We begin by introducing the theoretical influences which underpin the processes of task design and our attempts to take into consideration the complex relationships between artefacts, their mathematical affordances and the embodied practices they engender in the context of task resolution. We go on to outline the collaborative approach we adopt to simultaneously develop both tasks and tools, and how teachers, students and researchers bring different, complementary, expertise to this process. To further illustrate our approach, we consider two examples from our work with blind learners and deaf learners.
Keywords: Collaborative design, Inclusion, Blind learners, Deaf learners, Tool mediation, Embodied cognition.

Introduction

In recent years, Brazil has experienced large changes in the educational paradigm. One of these relates to the growing influence of political and social movements that defend inclusive education, and the organization of schools to attend the diverse needs of all students, without any kind of discrimination or segregation. Inclusive schools are those which see difference as a factor which enriches the educational process. They aim to support all learners in overcoming barriers to learning as they become participants in a more equitable system. The political policies related to the process of including students with special educational needs have resulted in a significant increase in their presence within mainstream schools, with statistical data from the most recent school census showing an increase of 234% between 2003 and 2010. At the same time, these policies of inclusion have been associated with taking the educational community out of the "comfort" zone and, amongst the many uncertainties, insecurities and conflict the actors in these communities are facing, questions related to curriculum demands and pedagogical Theme A – L. Healy, S.H.A.A. Fernandes, J. B. Frant.
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actions have a central role. In particular, the increasing diversity of students within the same classroom setting raises questions related to task design: what principles and procedures might be adopted in the design of tasks for the inclusive mathematics classroom?
It is within this context that we began work on a research project aiming to (1) investigate forms of accessing and expressing mathematics which respect the diverse needs of all our students, (2) contribute to the development of teaching strategies which recognize this diversity, and (3) explore the relationships between sensory experience and mathematical knowledge
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. The project involves the development and analysis of inclusive scenarios for mathematics learning, though a collaborative process involving researchers, teachers and students. In this article, we intend to exemplify our approach to task design and present some examples from our work in São Paulo schools.
In this approach, the process of task design is accompanied simultaneously by the development of the material and digital tools that are also incorporated in the learning scenarios. Together, tasks and tools aim to enable the interaction of different students with mathematical objects and relationships. To this end, they are designed to facilitate multiple ways of interacting with these objects and relations and to respect the diverse experiences of the students with whom we work. The approaches we use involve representing mathematical ideas through colour, sound, music, movement and texture, and hence appeal to different sensory canals, and particularly to the skin, the ears and the eyes. The multimodal nature of the mathematical representations reflects our attempts to offer stimuli appropriate to the particularities of each and every student: for those with visual impairments, the tools enable tactile and auditory stimuli, for deaf learners, tactile and visual approaches are privileged and students who can both hear and see have access to all three modes, allowing even those with specific difficulties in learning mathematics to have a variety of ways to think mathematically. Before describing in more detail the process behind the design of task and tools, we begin by delineating our understanding of the term "task" and how this term figures in our view of learning mathematics.

Learning scenarios: Tasks and activity

Our view is that tasks represent one of the elements that compose the learning scenarios we enact within. In a similar way to Laborde (2002), we see learning scenarios as consisting of specific tasks, or sequences of interrelated tasks, the mediating tools available for their resolution, along with the activities of the participating actors (which may include different combinations of students, teachers and researchers). More precisely, we distinguish between task and activity in the same ways as Dejours (1997, p.39). He argues that "a task is that which is to be achieved or that which must be done. Activity is, in the context of the task, that which is actually done by the operator to arrive as close as possible to objectives fixed by the task". That is, tasks are proposed to the collective, and might be realized by differing individuals in different ways. Dejours (1997) was concerned with the work context and how different people might employ different techniques to attain a particular objective, depending on the tools available, of course, but also on the individuals themselves. The same is also the case in educational settings. The task proposed to a

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group of mathematics students might be the same, but the interpretation of the task and the activity that results will be shaped by both intrapersonal and interpersonal aspects of the particular students involved. The actions of each of the individuals who engage in a particular given task, individually or collectively, are a function, not only of the task itself, or the means available to interact with it, but also of the meanings that are associated with the activity itself (Leontiev, 1978).
Here, another question is raised, what is the relationship between completing a task and learning mathematics? In our view, tasks are proposed to motivate learners to engage with practices associated with the set of artefacts that have, historically and culturally, come to represent the body of knowledge we call mathematics. In the socio-cultural perspective which informs this view, learning can be defined as participating in, and appropriating, these practices. Using the concept of activity defended by Leontiev (1978), appropriation is a social process in which participants aim to make their own objects already steeped in cultural meaning. The process of appropriation occurs, necessarily in the case of mathematics, on the basis of actions mediated by semiotic systems. Mathematical activity, then, occurs as a dialectical process, in which individuals interact with the environment and with other individuals to attribute sense to aspects of the knowledge and experiences developed in the course of human history. As a result of this activity, objects of the environment, recognised by the senses, acquire the character of objects of reflection (Fernandes, 2008, p.47).
This brings us to another aspect central to the theoretical framework that guides our approach to task design: the role of the senses in mathematical cognition. We see links between the socio-cultural perspective developed by Soviet psychologists in the last century with the views of researchers in the area of embodied cognition today, who argue that our mathematical understandings, like all others, are structured by our encounters and interactions with the worlds we experience via our bodies and our brains (Gallese and Lakoff, 2005). Indeed, Radford (2006) argues that the body itself

can serve as a semiotic system, through acts of perception, gestures and other
movements. Other semiotic systems for Radford include artefacts, language and signs.
Jointly, then these represent the tools through which task demands are to be mediated.
With this view of mathematics learning in mind, we return to the question of
designing tasks, tools and teaching interventions for inclusive learning scenarios.
More specifically, in this article, we concentrate on learning scenarios in which
students who lack access to one or other sensory field act; that is, we focus on learners
who are deaf and learners who are blind. To enable the participation of students with
sensory limitations, it is important to understand how the different channels through
which they experience the world mediate the processes by which they appropriate
mathematical knowledge, and to recognise that the mathematical practices that are
depend on these experiences. Once again, we draw here from the work of the
Vygotsky and his colleagues in the former Soviet Union and particular to work in the
area of what at the time was called Defectology. Vygotsky (1997) proposed an
approach to understanding the learning of students with sensory, motor or cognitive
disabilities which involves considering how and when the substitution of one (nonfunctioning) tool by another may engender different forms of activity (Healy &
Fernandes, 2011; Healy & Powell, in press). His approach stressed the potential for
development of learners with disabilities, rather than positioning them as deficit in
relation to some supposed "norm".
"The positive particularity of a child with a disability is created not by the failure
of one or other function observed in a normal child but by the new structures
which result from this absence […]. The blind or deaf child can achieve the same Theme A - L. Healy, S.H.A.A. Fernandes, J. B. Frant.
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level of development as the normal child, but through a different mode, a distinct
path, by other means. And for the pedagogue, it is particularly important to know
the uniqueness of the path along with the child should be led" (Vygotsky, 1997;
p.17 - emphasis in the original)
In relation to empowering those without access to one or other sensory field
to participate in social (cultural) activities, for Vygotsky, the solution lies in seeking
ways to substitute the traditional means of interacting with information and
knowledge with another. For example, he suggested that the eye and speech are
"instruments" to see and to think respectively, and that other instruments might be
sought to substitute the function of sensory organs (Vygotsky, 1997). As mathematics
educators, we interpret this message from Vygosky as implying we need to pay
attention to - and where necessary create - a multitude of (substitute) semiotic
systems to mediate mathematics learning.
Our approach to design
The research strategy we adopt is based on the establishing of partnerships
between school- and university-based participants - researchers, school teachers and
school students - who collaborate on the design of tasks and tools for use in the
mathematics classrooms of the teachers and students. In these partnerships,
participants work together to conduct a process of co-generative inquiry (Greenwood
& Levin, 2000), a kind of participatory action research in which all participants cogenerate knowledge through a process of collaborative communication.
The process of designing learning activities is not a simple one and passes
through a number of stages. Not all the participants are necessarily involved in all the
stages, although usually at least one member from the school and one from the
university is present in each. The first stage involves designers in identifying
particular challenges associated with the learning of the mathematical topic in
question and in developing and testing hypotheses about how best to engender the
intended learning. The topics selected are those that are emphasised in the
mathematics curriculum that the schools are following and the starting point for the
design process is twofold, aiming to combine both pragmatic and theoretical
concerns. On the one hand, the teachers and sometimes also the students themselves,
brings examples of particular difficulties and problems they have experienced. At the
same time, we also consult the existing literature to attempt to determine what
previous research tells us about students' conceptions in the chosen topic. On the
whole, we have found relatively little research addressing the mathematical learning
processes of either blind students or deaf students with respect to the majority of topic
areas we have addressed. This means that the first versions of the task are often
developed on the basis of what we know about sighted and hearing learners and hence
may not be fine-turned to the particular strengths of those who do not see with their
eyes or who do not speak with their mouths.
This is one of the reasons that we believe it is crucial to involve the students,
as well as teachers and researchers in the design process. Student participation occurs
early in the design process, as students are invited, either individually or in small
groups, to work on the first prototypes of the tasks and tools under development. For
these first tests, we have tended to work exclusively either with blind students or with
deaf students. The meetings are videotaped and represent a means for reviewing our
developing theoretical models and revising our hypotheses so that they can be
operationalised given the particularities of the schools and students involved. Our
tendency has been to develop tools and task sequences simultaneously and to modify Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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both as the sequence is applied in practice during these tests. It is only after the
scenarios have been tested and analysed, that we consider re-enacting the scenarios in
teacher's mathematics classrooms.
Examples of the design process in practice
To illustrate this process in a little more detail, we have selected two
examples from the collection of learning scenarios we have investigated: one in which
tasks were to be mediated using material objects and a second involving the use of a
tactile, digitally-controlled tool designed to permit the exploration of graphs of
polynomial functions.
The first example involves the topic Matrices, a topic that is introduced to
students in the second year of High School (that is the 11
th
year of compulsory
schooling) in the curriculum currently followed in the state of São Paulo. One
motivation for focussing on this topic comes from the comments of deaf students in
one of our partnership schools and blind students in another. The deaf students, for
example, described Matrices as "something that has brackets and numbers", but were
not confident in manipulating paper-based representations these objects. Their
teacher, fluent in Libras (the Brazilian sign language), suggested that the lack of
specific signs for the vocabulary associated with matrices served as a complicating
factor in teaching the topic. The blind students, too, spoke of their difficulties in
solving tasks with matrices, which they described as "drawings with numbers inside".
We found no research studies which addressed interactions of either blind or deaf
students with matrix representations. We decided to attempt to construct a form of
representing matrices which would permit both deaf students and blind students to
construct them and operate upon them (more details of the design process are
available in Silva, G. G., 2012). The tool MATRIZMAT is a very simple one, made
up of plastic boxes (5cm by 5cm by 3cm) which could be joined to each other by
magnets fixed to each of the boxes' four sides (Figures 1 and 2). In the version for
deaf students, the numbers written on foam-rubber rectangles could be placed in the
cells of the matrices (Figure 1), whereas, for the blind students, we made use of the
lids of the boxes, with numbers in Braille stuck onto the top (Figure 2).
Figure 1: MATRIZMAT with written
numerals
Figure 2: MATRIZMAT with numbers
in Braille
The tasks for both student groups had the aim of introducing the language
associated with matrices, their organisation in rows and columns, determination of the
order of a given matrix, identification of equal matrices and matrix addition. It is not
our intention in this article to present in detail the interactions of the students with
these tasks, but perhaps it is worth stressing that the material tools enabled to both
student groups to develop efficient ways of expressing matrix structure. Figure 3, for
example, shows Maria using the signs developed by the students themselves to
indicate position a12 of a 3 by 3 matrix. It seemed that the layout of the matrix in a
physical, palpable form helped the group to develop ways of talking about its Theme A - L. Healy, S.H.A.A. Fernandes, J. B. Frant.
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structure - something that they had had difficulty to do when operating with the paper
and pencil representation.
Figure 3: Maria signing position a12 of a 3 by 3 matrix
In the case of the blind students, the importance of the tactile tool was most
evident when they were comparing or adding two matrices (see, for example, Figure
4). Being able to explore spatially the positions of the elements of the matrix enabled
them to experience matrices in ways that correspond to those of their sighted
contemporaries. This was rather different to their previous experiences, which had
involved Braille representations in which matrices were presented in a form that did
not emphasise the spatial layout of the elements and in which they had found it very
difficult to locate the elements in different matrices which should be added to each
other.
Figure 4: Adding matrices of order 3x2
The second example evolved as we attempted to develop tasks related to
polynomial functions that would be accessible to blind students. We knew from the
students themselves that tasks involving graphical representations of functions were
something that their teachers tended to avoid assigning (Silva, B. J., 2012). From the
research literature about (sighted) students' conceptions of functions, we conjectured
that a tool in which blind students could experience not only static representations of
the locations of particular points on the Cartesian plane or static representations of the
graphs of specific functions, but could also feel the graphs of different functions
appear as the independent variable changed, might afford more dynamic views of
function and help them understand the dependence relationship that exists between
the independent and dependent variables. The tool that we designed to permit blind
student to engage in such tasks was composed of a digitally controlled board made up
of a rectangular matrix of pins, each of which represented a point on the plane. When
particular point is requested or a graph of a given function plotted, the relevant pins
are raised up (sequentially as the value of the independent variable increases in the
case of the graph of a function), allowing the student to feel the image as it is Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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produced. Figure 5 presents the final version of the tool, while Figure 6 shows a blind
user, Alice, as she feels the graph of a function as it reveals itself to her.
Figure 5:The current version of the tactile graphing tool and its digital interface
Figure 6: Alice feeling a graph as it is plotted
To date, we have only tested this tool with students who do not see with their
eyes. Of course, it could also be used by the sighted, but perhaps represents a rather
expensive option to the on screen dynamic graphing tools which already exist. There
is a question though: are the experiences of seeing a graph as it emerges in front of
one's eyes and feeling it emerge with one's hands cognitively identical? Our
conjecture is that they may not be - the act of moving one's hands to find points in
ways not guided by one's eyes seems rather different to having a kind of global access
to the whole plane upon which the graph appears. The difference in the strategies
afforded by these different ways of perceiving and the properties of the graphs
emphasized in these two conditions is something we believe merits further research.
Reflections on the relationships between task and tool design
The theoretical influences that inform our approach to design indicate that
the processes of creating tasks and the tools by which they are to be mediated are best
tackled simultaneously. They also lead us to recognize, as Cole e Wertsch (1996,
p.255) have pointed out, that the insertion of tools into situations with instructional
intent does not simply serve to facilitate the mental processes that occur within them,
it fundamentally forms and transforms these mental processes, conditioning the
practices of the learners who operate the tools to the particular practices associated
with their use. Moreover, it is not only the learners whose practices are transformed:
the introduction of any artefact into a given situation might offer new - and even
more efficient means - to resolve a problem, but it also changes the very nature of the
task (Béguin & Rabardel, 2000, p.2). In this paper, we have concentrated on the Theme A - L. Healy, S.H.A.A. Fernandes, J. B. Frant.
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process of design and not on the resulting interactions of those who participate in the
learning situations into which the resulting tasks and tools are incorporated, but we
believe that the particularities of the students with whom we work help to illustrate
the extent to which it is not only the material and semiotic tools we provide that
impact upon the practices which emerge in the scenarios. Equally important are the
bodily resources through which tool and task are experienced, with different sensorymotor systems potentially affording different modes of acting mathematically and,
hence, different paths by which mathematical meanings might be appropriated.
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Using computers in classroom mathematical tasks: revisiting
theory to develop recommendations for the design of tasks
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The context for this paper is the ordinary mathematics classroom
lesson, in which the tasks set for students involve the use of computer
software. The paper highlights the importance of paying attention to the
intended mathematical learning of students as they work through the task,
adapting their strategies as they negotiate epistemological obstacles. It
draws on theoretical notions of student activity, or 'dialectics', within the
classroom environment to suggest the sorts of activity that are likely to
provoke mathematical learning, particularly highlighting the role of
computers within these activities. This provides the background against
which approaches to the design of mathematical tasks are recommended.
The approaches focus on the intended mathematical learning of the
students and the obstacles built intentionally and unintentionally into the
tasks.
Keywords: dialectics, computer, validation, feedback, obstacles,
schools
Introduction
This theoretical paper concerns classroom mathematical tasks and activity,
where, in common with the conventions adopted in much of the literature, (e.g.
Ainley & Pratt, 2002; Love & Mason, 1992; Sierpinska, 2004), I use the word 'task'
to mean the work the teacher asks the students to do. The teacher has in mind what it
is she wants the students to learn ('didactical intention'), but, as Brousseau (1997)
explains in detail, she cannot just tell the students and hope that they will learn.
Instead the teacher takes the decontextualised and depersonalised knowledge she
wishes them to learn and recontextualises and repersonalises it by embedding it in a
task. Tasks can therefore be seen as 'the fabric of student learning' (Sierpinska, 2004,
p 25) ; through tasks, teachers provide students with opportunities to engage in
thinking and sense-making in the classroom, which in turn leads to learning (Stein,
Grover, & Henningsen, 1996). Tasks influence learners by directing attention to
particular aspects of content and by specifying ways of processing information
(Doyle, 1983).
The literature distinguishes three main task types; exercises (or routine
problems), problems and investigations (Christiansen & Walther, 1986; Henningsen
& Stein, 1997; Orton & Frobisher, 1996). To some extent, these can be distinguished
by their goals, which are discussed below. However, there is also an argument that
they can be distinguished by the prior learning of the students. As Orton and
Frobisher (ibid) explain 'not all children in a class may view what is taught as a Theme A - M. Joubert
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problem… what is a mathematics problem for one learner may be an exercise for
another' (p 25) and this, they suggest, depends on the prior learning and experience of
the individual students. Christiansen and Walther (ibid) make a similar argument,
suggesting that what is for one student a task that provides potential for learning can
be a routine task for another, depending on their current developmental levels.
For most tasks, the goal for the student is an answer or some answers (Orton
& Frobisher, 1996). These goals can be seen as the end point of the task from the
students' point of view; this is what the student must do to complete the task. Further,
as Sierpinska (2004) and Mason (2000) point out, the assumption is that the students'
agenda is to complete the task; and as a result the design of the task should include the
mathematical thinking and reasoning that are 'strictly necessary and sufficient to
complete the task' (Sierpinska op cit p12, italics in the original); as she suggests,
completing the task may result in problem solving but does not necessarily do so.
Hence, while the goals described above are (usually) explicit products, it is the
mathematics which the task address which should perhaps be seen as more important.
In my empirical research (Joubert, 2007), I observed authentic mathematics
lessons, where teachers chose the tasks they wanted to students to complete, and I
analysed student activity in order to provide evidence of their mathematical learning
(in vain). The lessons I observed could be described as 'disasters of well taught
mathematics courses' (Schoenfeld, 1988); mathematics lessons in which students
were engaged in activity which looked like mathematics but their activity led to
limited learning of mathematics. In all these lessons, computers were used by the
students and my research confirms how difficult and complex it is to integrate
computers into the teaching and learning of mathematics (see for example, the 17
th
ICMI study book, (Hoyles & Lagrange, 2009)).
An analysis of the tasks the students were set emphasised the importance of
understanding a) the mathematics the teacher intends them to learn and the ways in
which the design of the tasks provokes activity through which students learn b) the
relationship between student activity and their mathematical learning, and c) the role
of the computer, which can perform some or all of the mathematics involved in the
task. In this paper, I draw on my research, revisiting some theoretical perspectives
underpinning task design, particularly the work of Brousseau (1997), to propose an
approach to task design taking into account a), b) and c) above, which might begin to
address the challenge of embedding the computer in classroom mathematics.
Intended learning and the design of tasks
The need for a clear idea of the mathematical learning intended to take place
as a result of student engagement with mathematical activity is frequently highlighted
in the literature about tasks; Henningsen and Stein (1997), for example, suggest that
teachers should be attentive to emphasising meaning and explicitly drawing out the
mathematics underlying the activities. Similarly, Christiansen and Walther (1986)
emphasise the need for teachers to be aware of the intended learning as both process
and product and as addressing different 'dimensions of learning'; on one hand the
'object for the activity… the mathematical core of the given task' and on the other
hand 'establishing the schemata for learning' (p 262).
It may seem obvious that teachers should be clear about what it is they want
their students to learn, but, as Brousseau argues, the detail of the intended learning is
frequently not sufficiently well understood by teachers, can be obscured (as, for
example, in the early introduction of decimal numbers through a measuring system Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Brousseau (1997)) or is unclear (Love & Mason, 1992). This point is also made in the
discussion document accompanying the call for papers for the study group:
There is a tacit assumption that the completion of mathematical tasks chosen or
designed by the teacher will result in the student learning the intended
mathematics.
Even when the intended learning is thoroughly analysed, however, students
will not necessarily learn what the teacher intended (Love & Mason, 1992,
Sutherland, 2007). This may particularly be the case when tools, including computer
software, are used and the meanings students make from the feedback on the
computer screen are sometimes at odds with the intended learning (e.g. see Sutherland
(2007) and Hoyles and Noss (2003)). Further, their learning may be influenced by the
design of the software which frequently has some mathematical and design
assumptions built into it (Sutherland, 2007). The implication is that the design of the
task should attend to the mathematics embodied in the tools, and to working out how
this might mediate the learning. Further, the fact that software can perform some of
the mathematical processes can be confusing; if the computer does the mathematics,
what learning is there for the students to do? For example, whereas plotting a
quadratic graph is a valid activity in a mathematics classroom as an end in itself,
generating the same curve using software is perhaps less valid as an end in itself.
To sum up: although the importance of the intended learning of a task seems
to be obvious, to understand what it is that the task is meant to 'teach' requires careful
analysis of the mathematics that might be provoked by the task, of the tools used to
mediate the activity of the students and - crucially - the prior learning of the students
(as argued in the introductory section above).
Obstacles
As discussed above, students will do only what is necessary to complete the
task. It follows that the design of the task needs to 'force' the students to identify,
adopt and adapt strategies to complete the task in order to reach target knowledge
(Brousseau, 1997). Adaptation may take place, Brousseau suggests, if the students
encounter and overcome 'obstacles' as they work through the task. He describes an
obstacle as:
'a previous piece of knowledge which was once interesting and successful but
which is now revealed as false or simply unadapted'. (p 82)
Obstacles can take a variety of forms; Brousseau, for example, identifies
obstacles of ontogenic, didactical and epistemological origin. The first of these relates
to the 'student's limitations' (p 86), including 'developmental' limitations (Swan,
2006) such as the lack of prior learning which is an obstacle because the student is in
some way not ready for the mathematics required to complete the task (Love &
Mason, 1992). For example, suppose a task requires students to generate graphs using
computer software and to 'notice' some feature or other. For students unfamiliar with
'noticing' tasks , this presents an ontogenic obstacle.
Obstacles of didactic origin 'are seen as obstacles for the students because of
ill thought out presentation of subject matter, or 'the result of narrow or faulty
instruction' (Harel & Sowder, 2005, p 34). Harel and Sowder give the example of the
didactical practice of teaching students to look for 'key words' in mathematics
problems (such as 'altogether' signals that addition is required). Theme A - M. Joubert
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Both ontogenetic and didactic obstacles should be avoided (Brousseau, 1997,
Harel & Sowder, 2005). The third type of obstacle, however, the epistemological
obstacle, should not be avoided and is key to the design of 'good' tasks.
Epistemological obstacles are perhaps most clearly explained by Balacheff
(1990, p 264) who discusses how mathematical concepts are learnt through their use
as tools in the process of problem solving with some content; this content is supported
by the students' prior knowledge. However, although the old knowledge is a
necessary basis for the content, it may cause problems for the students as they work
through the problem; in other words it becomes an obstacle which causes the students
to stumble. If, in order to overcome the obstacle, the students are required to construct
the meaning of the new piece of knowledge, then this obstacle is an epistemological
obstacle. Harel and Sowder (2005) suggest that 'MMB' (multiplication makes bigger)
can often be seen as an epistemological obstacle and Sierpinska (1987) identifies a
complete mapping of epistemological obstacles with respect to limit; scientific
knowledge, infinity, function, real number.
As discussed at length in Joubert (2007), it is not always easy to distinguish
between didactical and epistemological obstacles and what may be epistemological in
one context may be didactical in another. A key consideration is the development and
prior learning of the students and the design of any task will need to take these into
account when designing the task.
Student activity and mathematical learning
In completing the task, the students interact with the milieu, described by
Balacheff and Sutherland (1994) as 'the specific part of the environment of the learner
which is accessible and relevant to his or her actions' (p 141). However, each student
experiences the milieu differently, depending on, for example, their prior learning.
The interactions of students can be conceptualised as a 'dialogue' between
the student and the feedback from the milieu. The feedback from the milieu can take
many forms; Brousseau includes verbal feedback from other students and the teacher,
an outcome of a game and drawings produced by a pantograph as examples. The
importance of feedback should not be underestimated;
'The pupils' behaviour and the type of control the pupils exert on the solution
they produce strongly depends on the feedback given during the situation. If there
is no feedback, then the pupils' cognitive activity is different from what it could
be in a situation in which the falsity of the solution could have serious
consequences' (Balacheff , 1990, p 260).
When computers are used, feedback from the milieu will also include
feedback from the computer. The general conception of computer feedback seems to
be that it is beneficial in teaching and learning mathematics; it can be seen to provide
opportunities 'to quickly test ideas, to observe invariants … and, generally, to be
bolder about making generalisations' (Hillel, 1992, p 205). However, students
sometimes interpret feedback from the computer in unexpected ways, which may
have a negative or distracting impact on developing the understandings the teacher
intended (Hoyles & Noss, 2003, Sutherland, 2007), and which may lead to the
development of strategies which are at odds with the intended learning, such as trial
and error, trial and improvement and 'intellectual passivity' (Hillel, 1992, p 217).
A range of different types of interactions between students and feedback
from the milieu may take place. For example, the TIMSS study (Hiebert et al., 2003)
classifies interactions in terms of public and private interactions, and then goes on to Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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categorise the mathematical processes for solving problems as 'Giving results only',
'Using procedures', 'Stating concepts' and 'Making connections'. Frobisher (1994)
provides a more detailed categorisation, including processes such as guessing,
pattern-seeking, predicting, hypothesising and proving. However, these do not take
into account the dialectical
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nature of the student interactions with the milieu which,
as explained above, may be important in understanding and explaining the students'
progress through the task.
Modes of production
Brousseau (1997) uses the notion of 'modes of production' to describe the
different types of dialectic interactions between students and the milieu; he suggests
that as they work through a mathematical task, they will engage action dialectics, in
which the students know how to proceed; they implement a procedure or choose it in
preference to another or apply a knowing (the object of a student's construction
activity), all of these with or without communication. In the context of my research,
an example of an action dialectic was to generate a graph using software and to copy
the computer-generated graph onto a worksheet. In action dialectics within, the
student does not need to extend her mathematical knowledge or understanding to do
so; 'simple familiarity, even active familiarity … never suffices to provoke a
mathematization' (Brousseau, 1997 p 211). Some action dialectics are necessary

within any didactical situation; and Brousseau suggests that it is when the action
dialectics are motivated by the mathematics rather than the didactics, these actions
may lead to a sequence of mathematical dialectical interactions which could include
the development of hypotheses, alternative strategies, justifications and proofs.
Dialectics of formulation occur when students meet a difficulty or problem
as they engage in mathematical activity; Brousseau explains that when a solution to a
problem is inappropriate, the situation should feed back to the students in some way,
perhaps by providing a new situation. The means that the student may become
conscious of her strategies and begin to make suggestions. Brousseau includes in this
category 'classifying orders, questions etc....' (p 61). He goes on to say that in these
communications students do not "expect to be contradicted or called upon to verify …
information' (p 61). When the didactical intention is that the student should make
some observations, or to 'notice', it is likely that the students will begin to formulate
choices; noticing requires the student to make conscious choices about what to notice,
which aspects to attend to, which to suppress (Mason, 1989) and how to express and
articulate these. 'Noticing' can therefore be seen as a key formulation activity and in
some cases noticing may lead to making suggestions or to conjecturing. When
computers are used, an important implication of the computer's ability to 'do the
mathematics' is in the opportunities which can be developed for the students to work
inductively rather than deductively as is more usual in mathematics classrooms. For
example, when computers are used to draw graphs, students may be able to notice

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This is the term Brousseau uses to emphasise the notion that the student interactions can be
seen as a 'dialogue' with the milieu. Sierpinska discusses the term in detail, referring to its roots in
philosophy, and concluding by remarking that, in Didactique, it means the method by which the
student manages contradictions between the expectations from the milieu and the feedback. Feedback,
she explains, is just communication of information; the dialectic turns the information into knowledge.
See Sierpinska, A. (2000) Theme A – M. Joubert
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characteristics of families of graphs, develop conjectures based on their observations
and test their conjectures on the computer.
Both action and formulation involve manipulating 'moves in the game' or
mathematical objects; validation however involves manipulating 'statements about
the moves' (Sierpinska, 2000). Validation therefore takes place when an interaction
intentionally includes an element of proof, theorem or explanation and is treated thus
by the interaction partner (or interlocutor) 'this means that the interlocutor must be
able to provide feedback…' (Brousseau, 1997 p 16). Dialectics of validation include
justification (perhaps of a procedure, a word, a language or a model), organising
theoretical notions, 'axiomization' (ibid p 216), and a range of proofs. An example of
a validation dialectic might be an explanation to justify a prediction that a computergenerated graph will have particular intercepts by noticing a relationship between the
intercepts and the equations of graphs generated previously.
Brousseau, while suggesting that all three modes of production are 'expected
from students', (ibid p 62) argues that it is through situations of validation that
genuine mathematical activities take place in the classroom. Romberg and Kaput,
(Romberg, Kaput, Fennema, & Romberg, 1999) amongst others (for example,
Lakatos, Worrall, & Zahar, 1976) echo Brousseau's sentiments.
Brousseau suggests that situations of validation do not occur very often and
are unlikely to occur spontaneously and it is probable that validation will not take
place unless it is explicitly called for. This has important implications for task design.
Working across different 'fields'
It can be that the mathematical activity of the students relates only to the
physical or concrete characteristics of the mathematical objects they work with. On
the other hand, students may relate their activity to mathematical notions
underpinning the objects. For example, on the one hand students using dynamic
geometry software may construct a tangent to a circle by rotating a straight line so
that it touches the circle, and on the other hand they may use the geometrical property
that the tangent is perpendicular to the radius to construct the tangent (Laborde,
2005).
These different ways of working in different contexts are described in a
variety of ways; for Laborde (1998) the distinction is between 'spatio-graphical' and
'theoretical', for Fischbein (1993) and Mariotti (1995) it is between 'figural' and
'conceptual', for Tall (for example, 1996) it is between 'visuo-spatial' and
'verbal/logical'. While the authors' varying theoretical perspectives imply variations
on these ideas, the point for all of them is that it is useful to make the distinction and
this paper uses the terminology adopted by Noss, Healy, & Hoyles, (1997)
pragmatic/empirical' and 'mathematical/systematic' to make the distinction.
In addition to the perceived need to distinguish between the
'pragmatic/empirical' and 'mathematical/systematic' fields, there seems to be a
consensus that a movement between the two fields is required for mathematical
learning to take place (Balacheff, 1991; Brousseau, 1997, Dörfler, 2000; Laborde,
2005; Noss et al., 1997). Brousseau, for example, describes different types of proofs,
including contingent and experimental proofs and proofs by exhaustion. These, he
suggests, relate to implicit models students hold and therefore they are likely to take
place in the pragmatic/empirical field and not relate explicitly to theoretical
mathematical knowledge. However, for 'mathemization' to take place, according to
Brousseau, mathematical proofs which relate to the theoretical mathematics involved Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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in the situation, are required. Similarly, Mariotti (op cit) claims that the solution of
geometry problems requires continuous moves between the two fields and Laborde
(op cit) suggests that there is a need for interactions between images and concepts.
The implication, in terms of the relationship between the students' dialectics
(particularly of formulation and validation), and student learning is that, where these
dialectics remain in the pragmatic/empirical field, mathematical learning will be
limited.
When computers are used, very often the task students are given requires
them to construct an artefact on the screen, such as, for example, a graph or a
geometric transformation and they begin by working in the pragmatic/empirical field.
It seems that frequently the task does not require movement between this and
mathematical/systematic fields and mathematical learning is limited (Joubert, 2007).
In terms of implications for task design, the theoretical discussion related to
student dialectics above suggested that a task should provoke students to engage in
dialectics of action, formulation and, crucially, validation. Further, the students should
move between the pragmatic/empirical field and the mathematical/systematic field in
order to reach the goal of the task.
Concluding comments
This paper has made an argument that task designers should pay attention to
the mathematical learning a task addresses, emphasising the need for a task to
provoke student activity that does not merely look like mathematics, but actually is
learning mathematics. The task should be designed in such a way that it provides
situations for action, formulation and validation dialectics, so that students begin by
adopting a strategy and, in the course of stumbling over the epistemological obstacles
built into the task, they adapt their strategy and eventually reach the goal of the task.
The importance of understanding the role of the computer software in
classroom tasks was also made. In particular, the paper has suggested that the
designers of tasks need to be clear about what the work of the computer is, and what
the work of the students is. As explained, computer software is able to do the
mathematics that is frequently, in traditional classrooms, seen as the end point of a
task (such as creating a graph), and if computers are used in this role, then teachers
and designers of tasks need to be clear about what mathematics they want the students
to do. Further, because of the potential power of feedback from the computer,
designers need to take into account the meanings students read into this feedback.
The point was also made, several times, that the context and prior learning of
the students is a crucial concern. The implication of this is that no designed task can
be "set in stone" as teachers will frequently want to adapt the task to the needs of their
students (as discussed, for example, by Kieran, Tanguay, & Solares, 2012). My
concluding suggestion is that task designers should therefore take teachers' need to
adapt tasks into account in their design of tasks.
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The interaction between task design and technology design in
creating tasks with Cabri Elem
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Both the design of tasks and the design of technology have been identified as important factors in the effective use of technology-based tasks in the classroom. By analyzing both the design of a sequence of tasks (based on didactical principles from Brousseau's (1998) theory of situations) and the affordances of Cabri Elem software it will be shown that technology can be designed in such a way as to enhance the implementation of didactical principles.
Keywords: Task design, technology, instrumental genesis, feedback, didactical variable
Design of the technology and design of the task
The potential for the contribution of Cabri II Plus and Cabri 3D to student learning is well known. However, Laborde and Laborde (2011) note that there is a gap between situations used in research and everyday teaching practice and suggest the provision of ready-made resources to help teachers take more advantage of dynamic mathematics environments. Sinclair (2003) studied pre-constructed files in a dynamic geometry environment and concluded that the design of the files was critical to their potential to support or impede student learning. There is an increasing concern about the design of technology itself. "Design detail counts" states Jackiw (Butler et al., 2009, p. 431). Research on resources and the problematic of quality indicates the importance of the question (Gueudet et al. 2011; Trgalova et al. 2011). Mackrell (2011) has shown that particular dynamic geometry technologies may have unclear interfaces, a lack of affordances and poor mathematical representations, all of which will hamper the use of such technology as an environment for the design of resources. The above concerns, together with the perceived need to create a version of Cabri that was suitable to the needs of primary students has led to the development of the Cabri Elem technology. A key difference with earlier versions of Cabri is that the focus of the technology is on creating an environment for task design, acknowledging the importance of both the design of the technology itself and of the design of tasks using the technology.
Tasks in Cabri Elem are presented in "activity books" which consist of a succession of pages incorporating a sequence of tasks. Once an activity book has been
created in the Cabri Elem Creator task design environment, it can be used directly by teachers and students in the more restricted environment of Cabri Elem Player. In this paper we will analyze a particular Cabri Elem activity book in order to illustrate the interconnections between the affordances of the technology and the ability to implement particular didactic principles.
Part I. Theories of Task and Software Design
The theory of didactical situations (Brousseau, 1998) offers a theoretical framework and a number of conceptual tools to study and also to design tasks. In this theory, knowledge is a property of a system constituted by a subject and a "milieu" in interaction. Learning occurs through this interaction: the subject acts within and receives feedback from the milieu. But there is an added requirement. The signification of the knowledge that can be constructed in the interaction depends on the existence of a space of uncertainty and freedom for the subject about appropriate action and strategy. If the student has no choice, the learning outcome may have no mathematical meaning. Together with criteria for success or failure, the goal, whether teacher or student determined, is made clear, unlike in the common dynamic geometry task "drag this point and observe" where the student has no choice of action and is uncertain about what is relevant to observe. In the case of mathematical learning with technology, the relation between the technology and the milieu is complex. The milieu cannot be reduced to the technology. Technology may be one component of the milieu, (Laborde and Capponi 1994) but only the part of technology relevant to the mathematics concerned (Brousseau 1998). The same technology will also not constitute the same milieu for every subject. The milieu related to a student changes as student knowledge, both technical and mathematical, develops.
The mathematical problem and the task are key elements of a didactical situation. In the context of this contribution, we characterize a "task" by the following requirements:
• a task involves learning objectives; when a teacher proposes a task to a student, he assumes that achieving the task will cause learning;
• it involves the student encountering a mathematical problem;
• it is performed by concrete and conceptual student actions;
• it corresponds to phases of the didactical situation (in the sense of Brousseau) and is related to different values of a set of didactical variables.
Didactical variables are parameters of the situation, with values that affect solution strategies. The effects can be of three kinds: (i) a change in the validity of a strategy, where a strategy that produces a correct answer with a certain value of a didactical variable will produce an incorrect answer with another value, (ii) a change in the cost of the strategy (for example counting elements one by one is efficient for a small number but much more costly for a larger number) (iii) the impossibility of using the strategy. A combination of the different didactical variable values contributes to the task definition. The learning situation is a choice of different tasks that lead the students to construct the appropriate strategy. Thus task design will consist, for a part, in identifying the didactical variables of the situation and then choosing the succession of appropriate combinations of didactical variable values. This will be one of the foci in describing the task design process.
When creating a learning task in a computer environment, the author has to create all the elements the student will deal with: the objects the student will Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.

manipulate, the possibilities of actions on these objects and the feedback provided by the environment. The elements chosen will determine the possible milieu and the potential for learning. The feedback given by the milieu will be a second focus in describing the task design process.
Instrumental genesis involves the processes of instrumentation whereby a person builds personal utilization schemes for an artefact and instrumentalization whereby a person adapts an artefact to their own purposes, with the result that an artefact becomes an instrument to be used in the pursuit of a goal (Rabardel and Bourmaud, 2003). Instrumentation is clearly relevant to task design using tools, as it has been found to be problematic (Trouche, 2005). However, instrumental genesis is also highly relevant to the design of the technology itself. In the human computer interaction literature, "design is perhaps the most common issue addressed within the approach" (Kaptelinin and Nardi, 2006, p. 110):
"artefacts should be designed to enable efficient transformation into instruments. […] the importance of designing flexible, open artefacts that can be modified by users and adjusted for various tasks, including unanticipated tasks and the need for designers to take into account the actual transformation of practices and the real needs of users over the course of appropriating an artefact".
The most important design principle in the development of Cabri software is that of direct manipulation (Laborde and Laborde, 2011), which involves both action and feedback on action. This is a fundamental concept of human computer interaction, developed in the 1980's when the current almost-ubiquitous graphical user interfaces were replacing interfaces requiring text commands (Kaptelinin and Nardi, 2006, p. 82). The general principles below (Shneiderman and Plaisant, 2010, p. 196) are illustrated by examples from dynamic geometry.
1. Continuous representations of the objects and actions of interest with meaningful visual metaphors. Geometric objects and tools.
2. Physical actions, or presses of buttons, instead of complex syntax. Dragging to move objects, constructing a custom tool or macro.
3. Rapid, incremental, reversible actions whose effects on the objects of interest are visible immediately. The response of any geometric object when one of the points on which it is dependent is dragged.
Some of the benefits claimed for systems designed using these principles are that novices can learn basic functionality quickly, users can immediately see whether their actions are furthering their goals, users experience less anxiety and gain a sense of confidence and mastery (Shneiderman and Plaisant, 2010, p. 196) This has obvious implications for instrumentation.
However, Rabardel (2002, p. 148) suggests that direct manipulation may not always be the most appropriate means of interaction with an artefact in a learning situation. The aim, rather than making the action easier, may be to construct constraints that lead the subject to use and elaborate cognitive constructions of knowledge (p. 148). An implication is that the task designer in a technological environment must have the facility to vary the type of action required and response received as appropriate.
An important additional principle for mathematics educational software is "epistemic fidelity" in which the representations of mathematical objects must avoid any contradictions with the abstract objects they represent, both in appearance and in behaviour when manipulated (Laborde and Laborde, 2011). Rabardel (2002, p. 160) also warns that while instruments may be designed to favour the construction and manipulation of conceptualizations these impose a "world view" on users. This has
consequences for design at many levels, ranging from the choice of perspective (to ensure that a cube does not appear to "pulsate" when moved) in Cabri 3D to the importance of distinguishing geometric labels and variables (Mackrell, 2011).
Laborde and Laborde also feel that physical action is preferable to pressing buttons: that by directly acting on the representation of mathematical objects the user eventually perceives the abstract object itself.
Part II. Analysis of a Cabri Elem activity book
In what follows, we will look at an activity book and discuss the task design decisions and the technology affordances, which enable these decisions.
activity book addresses the French primary school level CE1 (7 year old
deals with the representation of numbers
from comparing counters on a
its position on the board, with the way that the value of a digit depends on its position in a written number. It was designed by a team of ten researchers (including two of the authors of this paper), teacher educators and teachers involved in a French national project

whose purpose is to create resources for the teaching of mathematics in kindergarten and primary
Cabri Elem has the
manipulation and feedback involving geometrical objects and
extended to more effectively model the real world with new objects (such as 3D models) and new tools (such as a realistic compass). It also enables a 3D view, as shown below.
Figure 1. 3D objects and use of the realistic compass tool in Cabri Elem
The affordances for interaction with objects have
enabling restrictions on default behavio
or feedback on student actions may be delayed), and by enabling new possibilities for action, such as feedback given at the click of a button.
A key feature of Cabri Elem for task design is that the
responsible for the student interface, as the Ca
the beginning of the design. Instrumentation and student focus can be better managed by controlling the complexity of the interface, and the author is free to elaborate the milieu by choosing appropriate objects, possi
according to the aims of the task.
scoreboard with three different regions, the
as shown below.

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in collaboration between the IFE Institut Français de l'Educ
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Part II. Analysis of a Cabri Elem activity book
In what follows, we will look at an activity book and discuss the task design
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the French primary school level CE1 (7 year old
deals with the representation of numbers using place value notation. The idea arose
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, with the way that the value of a digit depends on its position
It was designed by a team of ten researchers (including two of
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whose purpose is to create resources for the teaching of
mathematics in kindergarten and primary school.
Cabri Elem has the affordances of earlier Cabri technology for direct
manipulation and feedback involving geometrical objects and numbers.
extended to more effectively model the real world with new objects (such as 3D
models) and new tools (such as a realistic compass). It also enables a 3D view, as
3D objects and use of the realistic compass tool in Cabri Elem
The affordances for interaction with objects have also been extended
enabling restrictions on default behaviours (objects may be locked to prevent changes,
tudent actions may be delayed), and by enabling new possibilities for
action, such as feedback given at the click of a button.
A key feature of Cabri Elem for task design is that the task author is entirely
responsible for the student interface, as the Cabri Elem Player interface is empty at
the beginning of the design. Instrumentation and student focus can be better managed
by controlling the complexity of the interface, and the author is free to elaborate the
milieu by choosing appropriate objects, possible actions and resultant feedback
according to the aims of the task. In our example, the objects are essentially the
with three different regions, the counters, the target number and the score

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Figure 2. Title page and a
The actions on the objects are simple: dragging the
button to get a new target number and reset position of the
button to get an evaluation.
Another key feature o
This enables activities to be planned as a sequence of tasks, in which an evolution in
student strategies may be provoked through changes in the value of the various
didactical variables. The ability t
enables the provision of optional help in tool use, differentiated tasks, notes for
teachers, etc.
In the task design process with Cabri Elem, didactical variables play an
important role. Some are identified
process as the author becomes more aware of what aspects of the situation may be
changed. Once a potential variable is identified, an analysis of the ways in which this
variable may be changed produces a bet
their consequences. It also enables the creation of strategy feedback.
We will also examine three kinds of feedback. Evaluation feedback is related
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the student in the course of task resolution, like scaffolding (Wood and al. 1976). It is
a response to the strategy used by the student.
configurations that are typical of a strategy and hence enable
objects or actions that can be provided to help the student without changing the nature
of the task. Such feedback may consist of help messages, or a graphic enlightening of
contradictory elements. Another possibility is to modify
variables in order to make the student aware of the current strategy limitations. Direct
manipulation feedback is the response of the environment to student action, and may
serve the function of either of the previous types o
Page 2: a page to initiate instrumental genesis
Page 1 is a title page (Figure 2). In page 2, the main objects are presented.
The student may interact with these objects, by dragging counters to different
positions on the scoreboard and noticin
dynamically calculated: one, ten and one hundred for each counter in the green
outside region, the purple intermediate region and the orange central region
respectively. The aim of the page is to give time for instrume
and students. They can explore interactions with the elements that will constitute the
milieu without the constraints of a particular task. It also contains a reset button
which, when clicked, replaces counters in their initial pos
allows students to move on to the next page.
The changing score is direct manipulation feedback that shows students not
only the effect of their action, but also that action on one object (moving a counter to
a different region of the scoreboard) will affect another object (the score). The values
of the different scoreboard regions were chosen by the authors in accordance with
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Figure 2. Title page and a task page from the "Target" activity book
The actions on the objects are simple: dragging the counters, clicking on a
button to get a new target number and reset position of the counters and clicking on a
button to get an evaluation.
Another key feature of Cabri Elem is the ability to include multiple pages.
This enables activities to be planned as a sequence of tasks, in which an evolution in
student strategies may be provoked through changes in the value of the various
didactical variables. The ability to choose the way in which pages are linked also
enables the provision of optional help in tool use, differentiated tasks, notes for
In the task design process with Cabri Elem, didactical variables play an
important role. Some are identified a priori, while others emerge during the design
process as the author becomes more aware of what aspects of the situation may be
changed. Once a potential variable is identified, an analysis of the ways in which this
variable may be changed produces a better understanding of the possible tasks and
their consequences. It also enables the creation of strategy feedback.
We will also examine three kinds of feedback. Evaluation feedback is related
to the achievement of the task or part of the task. Strategy feedback aims to support
the student in the course of task resolution, like scaffolding (Wood and al. 1976). It is
a response to the strategy used by the student. The author needs to identify (i)
configurations that are typical of a strategy and hence enable a diagnosis and (ii) new
objects or actions that can be provided to help the student without changing the nature
of the task. Such feedback may consist of help messages, or a graphic enlightening of
contradictory elements. Another possibility is to modify the values of some didactical
variables in order to make the student aware of the current strategy limitations. Direct
manipulation feedback is the response of the environment to student action, and may
serve the function of either of the previous types of feedback.
Page 2: a page to initiate instrumental genesis
Page 1 is a title page (Figure 2). In page 2, the main objects are presented.
The student may interact with these objects, by dragging counters to different
positions on the scoreboard and noticing how this affects the score. This is
dynamically calculated: one, ten and one hundred for each counter in the green
outside region, the purple intermediate region and the orange central region
respectively. The aim of the page is to give time for instrumentation to both teachers
and students. They can explore interactions with the elements that will constitute the
milieu without the constraints of a particular task. It also contains a reset button
which, when clicked, replaces counters in their initial positions, and a button which
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learning objectives: in a version of the activity book used with younger students the
scoreboard regions only gave values of one or ten. The score is always displayed in
some pages, but displayed only after a specific sequence of actions in other pages.
The score is generated by means of the point counter tool a new tool, which
counts the number of points within geometric shapes (in this case the three regions of
the scoreboard
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). The idea for this tool was initiated by a primary teacher educator
for work with early number: she wanted a means of creating and counting a collection
of objects using technology. Other affordances used in this page are the ability to
evaluate expressions (the total score being calculated from the number of points in
each region), and the ability to reset the position of points.
Page 3: evaluation feedback
On page 3 the student is given a specific task: to reach a score equal to a
target number, randomly generated between 1 and 999 (see Figure 2). Clicking on the
reset button now in addition generates a new target number. Another new action is
that the student may, in addition to comparing whether the score matches the
scoreboard, click on a new button for evaluation feedback: a red frowning face if the
answer is wrong, and a yellow smiling face if the answer is correct. In case of failure,
the student can continue to drag counters and ask for a new evaluation: a new smiley
will appear to the right of the previous one. It is important that new feedback is only
generated at the student's request: otherwise a trial and error strategy not stemming
from mathematical considerations could lead to success.
The key new software affordance used in this page is the Boolean function,
which enables a comparison between the student response and the target number.
When the feedback button is pressed, the value of the resultant Boolean (TRUE or
FALSE) will determine which of the faces is shown.
Pages 4 to 7: suppression of a direct manipulation feedback and evolution of
the task
From page 4 to 7 students are no longer given the direct manipulation
feedback of seeing the score. They hence need to take into account the value of the
counters in the different regions of the scoreboard to determine the score. "Score" was
identified a priori as a possible didactical variable, with two values: visible or hidden.
All objects may be either visible or invisible in Cabri Elem, enabling the author to
control the level of direct manipulation feedback given.
In page 5, the number of counters is reduced so that, if the target number is
over 27, a strategy that consists in placing counters only in the green units region will
fail. A strategy which takes into account that a single counter can have another value
than 1, i.e. using the inside regions of the scoreboard, is necessary. Therefore, another
potential didactical variable is identified: the number of available counters, with two
values, 3x9=27 and =27. In page 6, the target number is a multiple of ten, between 10
and 990. As there are enough counters to either leave the green region empty or to fill
it with multiples of ten counters, a change of strategy is not necessary. In page 7,
however, a single counter is fixed in the green region. Therefore, new strategies are
required, involving the placement of a multiple of ten counters into the units region of

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The outer rings show an example of instrumentalization: as a ring is not a Cabri Elem
geometrical shape, the authors have created non-convex polygons with touching sides. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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the scoreboard. The "fixed counter" didactical variable is identified, with four values:
no fixed counters, or fixed counters in the units, tens, or hundreds region.
Page 8 contains input boxes for the student to enter the values of a counter in
each region of the scoreboard. The aim of this task is to summarize the key idea of the
activity book, i.e. that the value of a counter depends on the scoreboard region.
Other pages of the activity book not devoted to student tasks

The first page is designed with the aim of attracting teachers and students to
the activity book with an iconic representation of some of the main objects.
Pages 9 and 10 contain commentaries for teachers, reporting the main aspect
of the task, the evolution from one page to another, possible student strategies (correct
or not) and also the solution. The structure of the pages of the activity book was used
to organise these notes and the didactical variable analysis helped to determine what
information was useful. Trgalova et al. (2011) also points out that teachers find
teacher notes of value.
Part III. Feedback from the school
The "Target" activity book was trialled in the spring of 2012 in two primary
school classes: CE1 with the version presented here and CP (six year old students)
with a version where the target number size was limited to 99. Teachers used the
activity book as one resource for learning about place value and instrumentalised the
book by printing pages to construct related paper and pencil tasks. They were
enthusiastic about student engagement, mathematical reasoning and the evolution of
strategies, but raised a number of issues. We present here some findings related to
instrumentation by teachers and students, instrumentalization, strategies feedback and
didactical variables.
It was expected that the strong metaphor between the task situation and real
situations involving a scoreboard would, as well as providing a meaningful context,
minimize the need for instrumentation: however, students expected that, as in the real
situation, moving a counter would require tossing it in some way and were initially
uncertain about how to do this using the software. Teachers also proposed that
instrumentation would be enhanced by modifying page 2 to include a target number
chosen either by the teacher according to the constraints of the class, or chosen by
students in order to challenge each other.
Some students used the target number update not only to get a new number
after finding a previous target but also, unexpectedly, to get a number they knew they
were able to deal with. They were able to diagnose their level of expertise and this,
important ability to the learning process must not be ignored by the environment. It is
planned to modify pages to provoke problem resolution, but also to locally enable this
usage. This example of students' instrumentalization of a functionality to adapt it to
their level of expertise is a new, generalizable element in activity book design.
The number of available counters was not a didactical variable for most CE1
students, who used each region of the scoreboard and limited the number of counters
they needed to drag. Many of them did not notice the reduced number of counters on
page 5 and were surprised to apparently have to solve the same task again. However,
for a few CE1 students and many of the younger CP students who used only the units
region of the scoreboard and placed as many counters as the target number the
number of available counters was indeed a didactical variable. The status of page 5
will hence be changed in further developments of the book. Instead of being Theme A – K. Mackrell, M. Maschietto & S. Soury-Lavergne
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automatically displayed to CE1 students, it will only be displayed as necessary, i.e. if
the unit region is repeatedly filled with many more than 10 counters. The strategy
feedback, resulting from our analysis in terms of didactic variables, will consist in
reducing the number of counters and choosing a target number over 50.
Conclusion
In this paper the following elements have been considered: task design,
construction of milieu in the context of a particular technology, didactical variable,
feedback, and instrumental genesis for authors and students. In order to discuss tasks,
we tried to show their mutual relationships. We have shown that software affordances
and theories related to task design together enable the effective creation of resources
for learning and their introduction into the classroom.
However, many elements remain to be controlled and articulated in the
analysis of tasks and task design. This paper, drawn from the initial stages of a study,
attempts to illustrate some of these and to suggest theoretical means to study the
whole process. On the one hand the analysis has shown the potentialities of Cabri
Elem, but on the other hand it has shown the complexity of the process: different
instrumental geneses are involved, subjects' knowledge comes into play, didactical
constraints are present, etc. However, we have also shown that there is not a distinct
divide between software affordances, task design and classroom implementation:
Cabri Elem affordances are based on theories of software design, but also come from
the requests of researchers and educators. In turn, designing a task in the Cabri Elem
environment enables a greater awareness of the potential of different didactical
variables. Feedback from the classroom suggests ways in which the task may be
improved.
The next stage in our study is to explore ways in which activity books could
be usefully modified by any teacher. The Cabri Elem task design environment
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will
enable us to consider this new level of instrumental genesis.
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Cabri Elem also has a range of affordances that were not relevant to this task and is
currently being extended to enable task design at secondary level and also options for teacher
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Designing Tasks to Foster Operative Apprehension for
Visualization and Reasoning in Dynamic Geometry Environment
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Region.
This paper presents how to design tasks in dynamic geometry
environment (DGE) to foster operative apprehension for visualization and
reasoning in Duval's model of geometrical reasoning. The meaning and
significance of operative apprehension in DGE would first be proposed,
and the principle of using soft constructions to design tasks to foster
operative apprehension would then be discussed, illustrated by some tasks
designed in the DGE GeoGebra. Finally a model of task design in DGE to
foster operative apprehension for visualization and reasoning is proposed
for further discussions in future research of task design in DGE.
Keywords: Task design, Dynamic Geometry Environment,
Visualization, Reasoning, Operative Apprehension, GeoGebra
1. Introduction
One of the important aims of the Hong Kong Secondary Mathematics
Curriculum is to develop students' ability to conceptualize, inquire, reason and
communicate (Curriculum Development Council, 1999 p.4). Hence the terms
"explore" and "justify" appear in the learning objectives of many topics. For example,
it is expected that students could "explore the formula for the area of circles" (ibid
p.20), or could "explore and justify the methods of constructing centres of a triangle
such as in-centre, circumcentre, orthocentre, centroids etc" (ibid p.23).
However, it seems that the learning tasks proposed in Hong Kong textbooks
could not always fulfil this aim. For instance, when constructing the circumcircle of a
triangle, students are told directly to first construct the perpendicular bisectors of the
three sides using rulers and compasses, and then use their intersection as a centre to
draw a circle passing through one of the vertices, and finally see that this circle also
passes through the other two vertices (Figure 1). In this task students are neither
provided the opportunity to explore nor asked to justify how the circumcircle could be
constructed, but instead just to verify the correctness of the procedures to construct
the circumcircle given by the textbooks. Figure 1: Construction of the circumcircle in textbooks
As a curriculum officer of the Education Bureau in Hong
design tasks to support teachers to implement the curriculum in their classrooms. In
this paper, I am going to
environment (DGE) to facilitate visualization and reasoning based on Duval
of the role of visualization in the development of geometrical reasoning (Duval,
1998). In particular, I suggest that
effective approach for designing tasks to foster operative apprehension for
visualization and reasoning in DGE. The discussion would be illustrated by some
tasks I designed for Hong K
2. Duval's Model of Geometrical Reasoning
Duval (1998) suggests that geometry involves three kinds of closely
connected cognitive processe
visualization, construction
connections are represented by Figure 2 below.
Figure 2: Duval's model of the underlying cognitive interactions involved in geometrical
In the figure each arrow represents the way a kind of cognitive process can
support another kind in a task. The dotted arrow suggests that visualization does not
always help reasoning. For example, visualization can be misleading if our visualized
image is a special case. Duval states that these three kinds of cognitive processes are
quite different and must be
of geometry is to develop visual representation and reasoning abilities and to favour
the synergy of these processes.
To facilitate visualization as well as reasoning,
of a kind of apprehension of geometric figures called
means operations on the figure or its subfigure, either mentally or physically, that

Figure 1: Construction of the circumcircle in textbooks
As a curriculum officer of the Education Bureau in Hong Kong, I have to
design tasks to support teachers to implement the curriculum in their classrooms. In
to propose a framework on task design in dynamic geometry
environment (DGE) to facilitate visualization and reasoning based on Duval
of the role of visualization in the development of geometrical reasoning (Duval,
suggest that the use of soft constructions (Healy, 2000)
for designing tasks to foster operative apprehension for
ualization and reasoning in DGE. The discussion would be illustrated by some
ng Kong teachers.
s Model of Geometrical Reasoning
Duval (1998) suggests that geometry involves three kinds of closely
connected cognitive processes fulfilling specific epistemological functions, namely,
construction and reasoning. Their epistemological
connections are represented by Figure 2 below.
s model of the underlying cognitive interactions involved in geometrical
activity
In the figure each arrow represents the way a kind of cognitive process can
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image is a special case. Duval states that these three kinds of cognitive processes are
quite different and must be developed separately, and the significance of the teaching
etry is to develop visual representation and reasoning abilities and to favour
the synergy of these processes.
To facilitate visualization as well as reasoning, Duval suggests
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gives insight into the solution of a problem. He emphasizes that operative
apprehension is crucial and teachers have to identify factors triggering or inhibiting it
so as to make visualization possible and gives rise to various transfers.
With regard to the use of dynamic geometry software (DGS), Duval states
that DGS provides enormous possibilities of visualization through the introduction of
the aspect of movement, and allows manipulations of geometric objects and hence
true explorations of geometrical situations. However, the construction-centered design
of DGS does not develop all functions of visualization, in particular the operative
apprehension.
Duval's theory emphasizes the importance of operative apprehension to
facilitate visualization and also reasoning in the teaching and learning of geometry. In
view of his comments on the uses and limitation of DGS in visualization, I shall
discuss how to design tasks in DGE to foster operative apprehension for visualization
and reasoning. I would first define what operative apprehension means in the DGE,
and how the use of soft constructions proposed by Healy (2000) could be an effective
approach of designing task to foster operative apprehension.
3 Operative Apprehension in DGE
A task is a set of pre-designed, environmentally situated materials aims to
engage learners in activities that could transform the ways they see and do
mathematics (Leung, 2011). A task has to be pre-designed in the way that through
these pre-designed means, learners are guided to construct insights and the meaning
of the mathematics knowledge. A task is also environmentally situated in the sense
that the qualities or tools of the environment have been made use to empower learners
with extended or amplified abilities to acquire knowledge which could not be
acquired in the same ways in other environments (Leung, 2011). In what follows, I
shall discuss how to design tasks situated in DGE to foster operative apprehension for
visualization and reasoning in Duval's framework. In particular, I will focus on predesigned DG figures and interpret Duval's operative apprehension as the following:
Operative apprehension of a mathematical concept or problem in DGE is the
insights into the concept or the solution of the problem revealed by operating on a
pre-designed figure in the environment through dragging.
Let me illustrate the significance of operative apprehension in DGE using a
task I designed in GeoGebra (a free DGS). This task originates from the following
problem in a textbook.
A quadrilateral is dissected by a line joining the mid-points of one pair of
opposite sides, and the perpendiculars to this line from the mid-points of the other pair
of opposite sides. (See Figure 3(a) below.) What shape can you get from this
dissection?
A task is designed in GeoGebra to help learners to explore this problem
(http://www.geogebratube.org/student/m3459). In this task, a quadrilateral is
dissected into four pieces as described in the problem. Each piece can be rotated
through dragging the red point at the vertex. In this way we can see that how the four
pieces could form a rectangle (Figure 3). Also, the operation gives us the insights to
reason why this dissection gives a rectangle, by, for instance, thinking about why the
four angles at the vertices give a sum 360˚ (Figure 3(d)).
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(a) (b)
(c) (d)
Figure 3: The GeoGebra task of dissecting a quadrilateral into a rectangle
Besides rotating the four pieces, learners can also operate on the shape of the
quadrilateral. After checking the "Change the shape" box, four green points appear at
the vertices of the quadrilateral and the shape of the quadrilateral could be changed by
dragging them. Through dragging the vertices, I see that the dissection would give a
square for some shapes of the quadrilateral (Figure 4).
Figure 4: Some shapes of the quadrilateral that give a square after the dissection
This problem reminds me the famous Haberdasher Puzzle composed by
English mathematician Dudeney (Dudeney, 1907). This puzzle shows how an
equilateral triangle could be dissected into a square (Figure 5). Although I know this
puzzle for a long time, I never understand how Dudeney could think of this method of
dissection, nor have any idea how to generalize his method to dissect an arbitrary
triangle into a square.
Figure 5: Dudeney's Haberdasher Puzzle of dissecting an equilateral triangle into a square
When I try to compare the quadrilateral problem with Dudeney's puzzle, I
suddenly realize that if I drag a vertex, say the upper-left one, to a position at which it
is collinear with the other two adjacent vertices (Figure 6(a)), the quadrilateral would
be degenerated into a triangle which is dissected into a rectangle. Furthermore, if I
drag this vertex along the side of the triangle (Figure 6(b)), since the shape of this
triangle is unchanged the area of the rectangle is kept constant while its length and
width are decreasing and increasing respectively through dragging. Hence I should
get a square somewhere on this side (Figure 6(c)). Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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(a) (b) (c)
Figure 6: Dissecting a triangle degenerated from the quadrilateral into rectangles or a square
After the above exploration I see how an arbitrary triangle could be dissected
into rectangles of various sizes, and there should be a particular dissection that gives a
square. Through operating on the shape of the quadrilateral, I get important insights of
comprehending how a general triangle could be dissected into a rectangle, and
investigating when the dissection would give a square.
This example illustrates the advantage of fostering operative apprehension in
DGE. If we use a paper quadrilateral, although we could cut it to see how it could be
dissected into a rectangle, it is impossible for us to operate on its shape. In DGE we
can operate on the shape of the quadrilateral so that we can degenerate it to a triangle
to get the insights of how a triangle could be dissected into a rectangle.
4 Operative Apprehension for Visualization and Reasoning: Soft
Construction
At the beginning of research in dynamic geometry, tasks in robust
constructions, i.e. constructions preserve relationships upon dragging, were
recognized as promoting for the learning of geometry. However, Healy (2000)
discovered through observation that, rather than robust constructions, students
preferred to investigate constructions "in which one of the chosen properties is
purposely constructed by eye, allowing the locus of permissible figures to be built up
in an empirical manner under the control of the student". Healy called these
constructions soft constructions.
Healy differentiates the roles of dragging in robust and soft constructions. In
a robust construction, dragging provides a visual verification of the validity of the
construction through dragging. In a soft construction, dragging is not verification but
part of the construction itself. Through dragging, the general can emerge from the
specific by searching empirically for the locus of figures fulfilling the given
conditions. Soft constructions offer a transition from an empirical approach to a
theoretical approach in solving a geometry problem.
In the lens of Duval's model of geometrical reasoning, tasks in robust and
soft constructions can be considered as operative apprehension on figures serving
different functions of visualization: a robust construction provides a verification of the
construction, while a soft construction provides heuristics or insights through an
empirically searched locus which mediates reasoning. I shall illustrate this point with
two GeoGebra tasks of drawing the circumcircle of a triangle, one in robust
construction and one in soft construction.
In the robust construction task, perpendicular bisectors of the three sides are
first constructed using the "Perpendicular Bisector" tool . A circle centred at their
intersection (found by the "Intersection" tool) and passing through either one
vertex (say A) is constructed using the "Circle" tool , and it could be seen that Theme A – A. C. M. OR
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this circle also passes through the other two vertices (B and C). By dragging the
vertices of the triangle, learners can check the validity of the construction by seeing
that the circle always passes through the vertices. They can also see that the
circumcentre lies outside the triangle when the triangle is obtuse (Figure 7).
(a) (b) (c)
Figure 7: A robust construction task of finding the circumcircle of a triangle
In the soft construction task (http://www.geogebratube.org/student/m3958),
learners are first given the triangle and a circle which can be moved by dragging its
centre (in red) and a blue point on its circumference (Figure 8(a)). Learners first drag
the blue point to either one vertex, say A, and a dotted line joining A and the centre
would then be shown (Figure 8(b)). Then they drag the red centre to different
positions at which the circle also passes through another vertex B, and when this
happens a dotted line joining B and the centre would be shown. These positions of the
centre are marked in red, and learners can see that the locus of the centres of circles
passing through A and B is a straight line (Figure 8(c)). Learners can then be asked
what this line of locus should be, and the two dotted lines from the centre to A and B
providing hints for them to reason that this line is the perpendicular bisector of AB
(through looking at two congruent triangles). Once they recognize that the locus of
the centres should be the perpendicular bisector, they can find empirically the loci of
the centres when the circle passes through A, C (Figure 8(d)) and B, C (Figure 8(e)),
and finally visualize that the circumcircle should centred at the intersection of the
three loci, i.e. the intersection of the perpendicular bisectors of the three sides (Figure
8(f)).
(a)

(b) (c)
(d) (e) (f)
Figure 8: A soft construction task of finding the circumcircle of a triangle Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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The above example illustrates how a task in soft construction could foster
operative apprehension by recording the loci of positions at which the eye
construction satisfies the given conditions. These loci of positions provide insights to
solve the problem, and also mediate the reasoning of why the problem could be
solved in this way. I now propose the following principle of using soft constructions
to design task fostering operative apprehension for visualization and reasoning in
DGE.
Principle of using soft constructions to foster operative apprehension
Learners are provided opportunities to perform soft (eye) construction by
dragging. The loci of the dragging satisfying the given conditions, together with the
other elements supporting their visualization and reasoning, would be shown to the
learners so that theoretical elements could emerge from the empirical evidences.
I further elaborate the above principle using a more sophisticated task of
finding the incircle of a triangle (http://www.geogebratube.org/student/m4363). In
this task the triangle and a circle of centre I and passing through P are given, and the
radius IP is also shown. Learners are first asked to drag P to the side BC, then another
dotted line would be shown to indicate that there are two intersections (Figure 9(b)).
By dragging P towards the other intersection learners would visualize that for the
circle to touch BC, the two dotted radii should overlap to form one radius IP
perpendicular to BC (Figure 9(c)). I also anticipate that this process of dragging,
together with the overlapping of the two radii, would help learners to reason why the
tangent of a circle should be perpendicular to the radius.
(a) (b) (c)
Figure 9: A task of finding the incircle of a triangle – touching one side
Once the circle touches BC, P can no longer be dragged and learners are
asked to drag the centre I to different positions so that the circle would also touch AB,
and the locus of I is marked in red (Figure 10(a)). Learners are prompted to identify
this line of locus as the angle bisector at B, and could explain this by looking at the
congruent triangles IBP and IBQ. Similarly learners identify the locus of I at which
the circle touches BC and AC as another angle bisector at C (Figure 10(b)), and see
that the circle would touch the three sides when I is at the intersection of the angle
bisectors (Figure 10(c)).
(a) (b) (c)
Figure 10: A task of finding the incircle of a triangle – finding the incentre
Finally, the three vertices of the triangle are made draggable to the learners
and they are asked to drag the vertex A to change the shape of the triangle, and see that the original circle no longer touches the three sides (Figure 11(a)). They are then
asked to perform robust construct of the incircle by constructing the suitable lines in a
triangle using the given tools (median
perpendicular bisector
can then check the validity of their construction by dragging the vertices (Figure
11(c)).
(a)
Figure 11: A task of finding the incircle of a triangle
5 Discussions and Implications
Based on the above illustrations, I propose a model of task design in DGE to
foster operative apprehension for visualization and
model of geometrical reasoning as follows:
Task Design Model in DGE to foster Operative
Visualization and Reasoning through Dragging
Task design in this model consists of two phases. In Phase 1, the
using Soft Construction to foster Operative Apprehension
students' operative apprehens
strategy to find solutions satisfying the
construction, the trace of the
other support elements that medi
as an example (p.7), the dotted radii and their overlapping through dragging (Figure
9) are the support elements which are shown to students to mediate the insight and
reasoning of perpendicularity of the
Similarly, the traces and the radii shown by the software in Figure 10 support the
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that the original circle no longer touches the three sides (Figure 11(a)). They are then
asked to perform robust construct of the incircle by constructing the suitable lines in a

triangle using the given tools (median , angle bisector , altitude
) and the touching circle tool

(Figure 11(b)). They
can then check the validity of their construction by dragging the vertices (Figure
(b) (c)
Figure 11: A task of finding the incircle of a triangle – from soft to robust construction
Discussions and Implications
Based on the above illustrations, I propose a model of task design in DGE to
foster operative apprehension for visualization and reasoning by modifying Duval
reasoning as follows:
Task Design Model in DGE to foster Operative Apprehension for
Visualization and Reasoning through Dragging
Task design in this model consists of two phases. In Phase 1, the
using Soft Construction to foster Operative Apprehension is applied so as to foster
operative apprehension through soft construction, i.e. to use the
solutions satisfying the given conditions. In the process of soft
of the locus of validity (Leung and Lopez-Real
other support elements that mediate reasoning would be shown. Use my in
as an example (p.7), the dotted radii and their overlapping through dragging (Figure
9) are the support elements which are shown to students to mediate the insight and
reasoning of perpendicularity of the radius and the side when the circle touches it.
Similarly, the traces and the radii shown by the software in Figure 10 support the
that the original circle no longer touches the three sides (Figure 11(a)). They are then
asked to perform robust construct of the incircle by constructing the suitable lines in a
, altitude and
(Figure 11(b)). They
can then check the validity of their construction by dragging the vertices (Figure
from soft to robust construction
Based on the above illustrations, I propose a model of task design in DGE to
by modifying Duval's
Apprehension for
Task design in this model consists of two phases. In Phase 1, the Principle of
is applied so as to foster
i.e. to use the drag to fit
in the process of soft
Real, 2002) and
ate reasoning would be shown. Use my in-centre task
as an example (p.7), the dotted radii and their overlapping through dragging (Figure
9) are the support elements which are shown to students to mediate the insight and
radius and the side when the circle touches it.
Similarly, the traces and the radii shown by the software in Figure 10 support the Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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reasoning that the traces are the angle bisectors of the triangle and in-centre lies on
their intersection.
In this phase dragging and tracing are the cognitive tools (Leung, 2011) to
start a recursive cycle between visualization and reasoning until a solution and its
explanation is reached. In the design of the in-circle task, students are guided to first
visualize through dragging that the radius has to be perpendicular to the side when the
circle touches it. Then with this property students are further guided to visualize
through dragging and tracing that centre of the circle must lie on a certain line when
the circle touches two sides of the triangle. They are then guided to reason, using the
trace and the dotted radii, that this line is in fact the angle bisector. Finally they
further visualize that when the centre lies on the intersection of the two angle
bisectors, the circle would touch all the three sides and at this stage they should be
able to explain why this happens.
When the solution and its explanation are reached in Phase 1, the task is then
transited to Phase two in which students are required to use the construction tools
given by the software to do a robust construction to verify the solution and
explanations they obtained in Phase 1. This is done by releasing the shape of the
figure in the problem so that students observe that the soft construction in Phase 1 no
longer works when the shape of the figure is changed (Figure 11(a)). Students are
then asked to use the tools of the software to construct a robust in-circle that always
touch the three sides (Figure 11(b)(c)). In this phase dragging is a tool for visual
verification of the construction.
This model shows how the different roles of robust and soft constructions
could foster operative apprehension, through which the synergy of visualization,
reasoning and construction can be facilitated. If we agree with Duval that developing
visualization and reasoning abilities to favour the synergy of the three cognitive
processes is of crucial importance for the teaching of geometry, designing tasks to
fostering operative apprehension for visualization and reasoning in DGE effectively
would then be very promising to promote the teaching of geometry. This is also a
great challenge to all teachers, educators and researchers. It is hoped that the principle
and the model of task design in DGE proposed in this paper could provide a useful
initiation for further discussions, challenges and refinement in future task design
research.
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Dynamic representations for algebraic objects available in
AlNuSet: how develop meanings of the notions involved in the
equation solution.
E. Robotti
DIDiMa srl spin-off ITD-CNR Genova (Italy).
This article presents a research work aimed to study the impact of
new representations for algebraic objects handled by AlNuSet, an artefact
of dynamic algebra recently developed. In particular this article focus on
equations and identities. Traditionally, conceptual construction of
algebraic equality is pursued through solving equations using techniques
of symbolic manipulation. Several researches have highlighted that this
approach does not favor the construction of an appropriate sense for the
notion of algebraic equality, identity or for that of solution of equation.
To this aim, teachers need new tools that provide effective representations
for algebraic objects. Thus, this article presents new semiotic
representations available in AlNuSet that support both students in the
conceptualization of algebraic objects and teachers in the design of
innovative educational sequences of tasks allowing to tackle algebra not
only in syntactic aspects but also in the semantic ones.
Keywords : dynamic algebra, representations, dynamic representations,
equation, identity, existential quantifier, universal quantifier
Introduction
With a significant percentage of students, the current teaching of algebra
seems not to be sufficient to effective develop skills and knowledge to master this
domain of knowledge (Sfard & Linchevski, 1994; Kieran, 2006).
To clarify students' difficulties in algebra I refer to Chevallard's (1985)
para-mathematical end proto-mathematical notions.
On the base of these definitions, the algebraic concepts taught at school are
mathematical notions. However, Chevallard highlights that there are some notions,
called para-mathematical notions (i.e. variable, parameter, unknown, universal or
existential quantifier, algebraic expressions, ...) that have a name and that can be
defined in class, but often no educational activity is performed about them. Moreover,
there are proto-mathematical notions (for example, recognize a simple factorization in
an expression of the second degree or the equivalence between expressions), that are
not explicitly defined in class, but they live in educational practice and students are
not aware of their existence. The development of para-mathematical and protomathematical notions, seem to be very important to give meanings to the algebraic
objects and to better control the algebraic manipulation activities. Students'
difficulties in algebra to develop para-mathematical and proto-mathematical notions,
seem to be due to the fact that these notions can be grasp by the control on their Theme A - E. Robotti
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algebraic meaning (Arzarello, Bazzini, Chiappini, 1995). To make explicit protomathematical notions and to ensure that students grasp meaning of para-mathematica
notions, teachers need new artifacts, which make available new algebraic objects'
representations. Our research hypothesis is that AlNuSet (ALgebra on the NUmerical
SETs), can be effectively used to mediate conceptual development necessary to
master the notion of algebraic equality.
AlNuSet was developed in the context of ReMath (IST - 4 - 26751) EC
project and is designed for students of lower and upper secondary school (yrs 12-13 to
16/17). It was developed by the research group of ITD (Istituto per le Tecnologie
Didattiche)-CNR (Centro Nazionale di Ricerca) of Genoa (Italy) that I belong.
Alnuset allows, through visual-spatial approach and dynamic representations, to built
meanings of para-matematical and proto-mathematica notions.
In this article I show how this kind of representations implemented in
AlNuSet can support the teaching and learning algebra making explicit paramatematical and proto-matematical notions involving in equality and equation process
solution. The sequence of tasks was designed in collaborations with teachers involved
in ReMath project and it was experimented with them. This article focus on a
sequence of two tasks, aimed to presents a new educational approach to equation and
identity, which is part of a more completed Teaching Guide for teachers that I have
designed with my group of research (authors of AlNuSet) and experimented in
ReMath project context.
Short description of AlNuSet
Alnuset is constituted of three strictly integrated components: Algebraic Line
component, Symbolic Manipulator component and Functions component. Since this
paper concern tasks where only the first two components are used, I will describe in
the following only the Algebraic Line component and the Symbolic Manipulator
component (for more details see www.alnuset.com).
The main educational characteristics of these components are:
making available, through a visuo-spatial approach, dynamic representations
of expressions and algebraic propositions (in Algebraic Line component);
making available, through an operative and deductive approach, axioms and
rules to transform and manipulate algebraic expressions/propositions (in Manipulator
component).
The main characteristic of Algebraic Line component is the possibility to
represent an algebraic variable as a mobile point on the line, namely, a point that can
be dragged with the mouse along the line. Dragging the mobile point along the line,
the letter associated to that point assumes the values of numerical set instantiated.
This new visuo-spatial approach which exploits dynamic representations, allows to
make explicit the notion of variable as mobile point on the line that assume, in
numerical set instantiated, all its possible values. When dragging the mobile point on
the line, all algebraic expressions containing such a variable move accordingly. This
feature has transformed the number line into an algebraic line where it is possible to
operate with algebraic expressions and propositions through techniques of
quantitative and dynamic nature. This visuo-spatial approach to algebra allows
student to handle dynamic representations as new semiotic representations of
algebraic objects in Algebraic Line. This makes a dynamic algebra possible and
supports students in the conceptualization of algebraic objects. The most important Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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new semiotic representations available in Algebraic Line of AlNuSet that are involved
in the tasks presented in this paper are:
The post-it (see Fig 1b). The belonging of two expressions to the same post-it
can be connected to the notion of equality and equivalence between expressions.
Color of the dot associated to a proposition and/or to the numerical set
constructed by the user (see Fig 1a and 1b). The color accordance between the dot
associated to a proposition and that associated to the numerical set constructed by the
user, can be connected to the notion of truth set of the proposition and can be used to
validate the constructed numerical set as the truth set of the proposition.
The main characteristic of Symbolic Manipulator component is the
possibility to transform algebraic expressions and propositions through a set of
particular commands. These commands correspond to basic properties of operations,
properties of equality and inequality, logic operations among propositions, operations
among sets. Another characteristic is the possibility to create a new transformation
rule once it has been proved. These characteristics support the development of skills
regarding the algebraic transformation and they contribute to assign a meaning of
proof to it. Note that not all the commands are available in the same time: only the
commands that can be applied on the selected part of expression are available. The
user can easily control the whole process of algebraic transformation exploiting
feedback given by the system. Moreover, the user can verify the preservation of the
equivalence in the transformation representing the transformed forms on the
Algebraic Line.
Mediation provides by AlNuSet is profoundly different from that proposed
by the software used for the traditional teaching of algebra: both new dynamic
representations, based on visuo-spatial approach, which allow to reify semiotic
representations and its strictly integrated components, allow students to link semantic
and symbolic nature of algebraic objects. Moreover, AlNuSet's components allow

teachers to make explicit these links and, in particular, to make explicit paramatematical notions (i.e. variable, unknown, universal or existential quantifier, algebraic expressions,...) and proto-matematical notions (for example, choose the appropriate rule to transform an expression to obtain another; reconizing a factorization in an expressions...) involved in equality and equation notions.
Several researches (Chiappini, Pedemonte, B Robotti, 2008; Chiappini, Robotti, Trigalova, 2010;) highlighted the educational potentialities of this software showing how the approaches described above are effective in order to understand the basic mathematical concepts (fractions, expressions, equations,...)
Some particular tasks of the Teaching Guide
Note that, the standard teaching approach to algebra is to find, within the algebraic formalism, the meanings of algebraic notions (for instance, the manipulation of an equation allows to find values that, replaced in the initial equality, make it true; thus, the meaning of solutions of equation is found in the algebraic manipulation itself). The new approach to algebra offered by AlNuSet enables teachers to change this sequence: at first construct the meanings of algebraic notions (for instance, what does mean solving an equation) and, only then, deal with the formalism, finding in it the meanings previously built. This new approach is possible using dynamic representations that are available in AlNuSet, as I will show in the following sessions concerning the sequence of two tasks. Theme A - E. Robotti
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The tasks aim to exploit operative and representative possibilities of the Algebraic Line component to design explorative activities that address the construction of meanings for the notions involved in the solution of equations and identities. Then, tasks aim to consider formal and syntactical aspect of algebraic manipulation involved in the solution of equations and identities. These algebraic manipulations are performed in Manipulator component by the application of rules and axioms available on its interface. The equivalence of the expressions obtained by these transformations can be verified on the Algebraic Line. In this way, the links between semantics aspects and syntactic aspects can be made perceptively evident.
The sequence of tasks presented in this paper is composed of different activities that focus on the solution of a first-degree equation and the comparison between equation and identity. For each activity, I will present the task assigned to the students and I will discuss how we have designed task taking into account the dynamic representations available in Algebraic Line of AlNuSet and the rules and axioms available in Manipulator component of AlNuSet. Finally, I will describe the mediation role of AlNuSet in teaching-learning process of the algebraic notions taken into account.
I underline that Teaching Guide aims to offer indications to better exploit the potentialities of AlNuSet in order to improve teaching and learning algebra (Chiappini G., Pedemonte B., Robotti E., 2010; for more details see Educational Activities in www.ainuset.com).
Linear algebraic equation
In order to master algebraic equality, the conceptual development of notions of equation, identity, truth-value and truth set is necessary. Moreover, to express the way in which a letter can condition the truth-value of an equality, you must be able to consciously use universal and existential quantifiers, even though in implicit way (that is, as para-mathematical notion). For this reason, the sequence of tasks presented in this paper aims to answer the following questions concerning equations, their process resolution and identities:
What does solving an equation mean? What could we intend for truth-value of an equality? What is an algebraic identity and what differentiates it from a conditioned equality?
In order to promote the construction of meanings for the notions of equation, identity, truth-value and truth set involved in answering these questions, the sequence of tasks promotes the para-mathematical notions of unknown, algebraic expressions, variable, universal and existential quantifier, equality. Similarly, it makes explicit the proto-mathematical notion concerning equivalent expressions.
To this aims, I show how the use of Algebraic Line's functionnalities and Manipulator's functionnalities allow students to approach the notions of equation and identity in different ways: perceptive way (by a visuo-spatial approach mediated by dynamic representations) in Algebraic line component, and syntactic way (by operative and deductive approach) in Manipulator component.
To make explicit the connection between AlNuSet, as tool for teaching and learning algebra, the representations and the task design, I highlight that, the design of the following tasks is based on integration of these two different approaches (perceptive and syntactic), which are possible only in the AlNuSet's components, in order to promote the development of semantic competences and operational competences engaged in the solution of equations and identities. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Task 1
Consider the following two polynomials: x+2; 2*x+3.
Explain what does it mean putting the equal sign between them, or, in other words, explain how you interpret the following writing x+2=2*x+3.
Now, represent on the Algebraic Line of AlNuSet the two polynomials: x+2 and 2*x+3 and drag the mobile point x to verify your hypothesis. Insert in Manipulator the equality x+2=2*x+3 and solve it using the appropriate commands. What does it obtains?
The first part of the task is aimed to make explicit students' conceptions on the notion of algebraic equality. As matter of fact, on the semantic plan, equality denotes a truth-value (true/false) related to the statement of a comparison. When the expressions composing the equality are strictly numerical, it is easy verifying their truth-value through some simple calculations (e.g., 2*4+2=10 is true while 2*3+1=10 is false). Experiences with numerical equality contribute to structure a sense of computational result for the "=" sign. This sense can be an obstacle in the conceptualization of algebraic equality as relation between two terms, as highlighted by several researches (Kieran 1989, Filloy et al. 2000). Coherently with this meaning, we expect that the equal sign between two expressions could suggest that the computation related to the two terms of the equality has to produce the same result whatever the values of the variable x is.
The second part of the task aims to discuss this misconception and to construct the idea that the equality between two members (polynomials) is conditioned by the value of x. As matter of fact, when the expressions composing the equality are literal, the equality can present different senses because the value assumed by the letter can condition differently its truth-value. In this cases the "=" sign should suggest to verify numerical conditions of the variable for which two terms are equal. In others words, the equality is conditioned by the values of x.
To make explicit this condition, usually teachers have recourse to substitution of truth-values in the literal equality in order to obtain a true equality. This kind of approach does not seem very effective to grasp the sense of conditioned equality. For this reason, we designed the second part of the task asking students to use Algebraic Line of AlNuSet: by exploiting dynamic representations of the Algebraic Line it is possible to built the meaning of "=" sign between the expressions as conditioned equality in perceptive way. Thus, these representations allow students to grasp the meaning of conditioned equality by means of a visuo-spatial approach rather than a computational approach.
Solution in Algebraic Line component
On the Algebraic Line, students insert the mobile point x, that represent a variable, and the expressions x+2 and 2*x+3. Dragging the point x along the line, students can observe that there is only a value of x for which the points of the two expressions assume the same value. This dynamic representation contributes to build the meaning for the equal sign between the expressions as conditioned equality. As matter of fact, when student try to verify the equality between expressions, the drag of x is made with a specific aim: to ensure that the two expressions take equal values, that is, they are associated to the same point on the line and they belong to same postit (yellow square). If dragging is realized with this aim, then the variable can assume the meaning of unknown expressions have equal value and equality is true.
In Figure 1, two moments corresponding to the drag of x along the line represented (Fig. 1a, Fig. 1b)
2*x+3 are on the same point corresponding to the value 1 and they belong to the same post-it. The expressions are equals so that the equality is true.
Fig 1a
Note that software automatic
color is red whatever value of x different to
color is green for the value
value for the equality). A specific
the truth set associated to the equality
that set. The color red (Fig 1a)
of x is an element/not is an element of the constructed set. So that the concordance of
color between dot associated to the equality and that asso
truth set during the drag of x along the line, is a visual representation
students to construct meaning for
making true the equality.
The functionality
the line: students can perceptively
condition the truth of equality.
The following paragraph discusses t
developed in Manipulator component where
competences necessaries to solve equati
Solution in Manipulator
Manipulator Component allows student to solve equation by different approaches: automatic solution, simplified solution
Theme A - E. Robotti
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unknown: search for value to be assigned to x so that the two
have equal value and equality is true.
In Figure 1, two moments corresponding to the drag of x along the line
represented (Fig. 1a, Fig. 1b). Only when x assume value -1, the expressions x+2 and 2*x+3 are on the same point corresponding to the value 1 and they belong to the same it. The expressions are equals so that the equality is true.
Fig 1a Fig. 1b
Note that software automatically associates a colored dot to the equality:
color is red whatever value of x different to -1 (Fig. 1a: equality is false)
color is green for the value -1 (Fig. 1b: equality is true, so that the value
A specific command allows student to construct and visualize
set associated to the equality (Fig. 1). AlNuSet associates a colored dot to
(Fig 1a) /green (Fig. 1b) of the dot means that the current value
ement/not is an element of the constructed set. So that the concordance of
color between dot associated to the equality and that associated to the corresponding
set during the drag of x along the line, is a visual representation
construct meaning for truth set for equality, as set of the only values
The functionality "Tracking" (Fig 2) allows a more complete exploration of perceptively experiment that x = -1 is the only value that
the truth of equality.
Fig. 2
The following paragraph discusses the syntactic solution of
developed in Manipulator component where the development of operational
competences necessaries to solve equation is promoted.
Solution in Manipulator component
Manipulator Component allows student to solve equation by different
automatic solution, simplified solution and solution by sequence of steps.
so that the two
In Figure 1, two moments corresponding to the drag of x along the line are
1, the expressions x+2 and
2*x+3 are on the same point corresponding to the value 1 and they belong to the same
Fig. 1b
a colored dot to the equality: this
equality is false); instead, the
equality is true, so that the value -1 is a truth
construct and visualize
. AlNuSet associates a colored dot to
of the dot means that the current value
ement/not is an element of the constructed set. So that the concordance of
ciated to the corresponding
set during the drag of x along the line, is a visual representation allowing
set of the only values
lete exploration of
1 is the only value that
he syntactic solution of the equation
the development of operational
Manipulator Component allows student to solve equation by different
and solution by sequence of steps.Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22
The equation can be automatically solved selecting the command
Domain" providing the following result
achieve heuristic or pragmatic aims.
polynomial equation.
A simplified solution can be performed using the "
perform calculus considered standard (Fig. 4).
Moreover, to solve equation, students can use the commands of the interface
that refer to basic properties of operations
inequalities). The solution appears as a sequence of steps
Fig 5
The solution performed in Manipulator component allows students to focus on the meaning of the formal transformation: obtaining the result by means of chain of deductive steps. The expressions obtained in each step
expression maintaining the equivalence. As matter of fact, each step is obtained applying a rule available on the interface.
be verified on the Algebraic line: for instance, the value of x for which the expression x+2 is equal to 2*x+3, is the same value for which the expression x+2
equal to zero (expressions referring to first and second line in Fig. 5).
that Manipulator, making
student's memory: he sha
applicable to the part of the expression
rules and axioms provided by

reach the educational aim.
Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22
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The equation can be automatically solved selecting the command
providing the following result (Fig 3). Teacher can use this command to
achieve heuristic or pragmatic aims. This command can be used to solve any

Figure 3
A simplified solution can be performed using the "Simplify
perform calculus considered standard (Fig. 4).

Figure 4
Moreover, to solve equation, students can use the commands of the interface
that refer to basic properties of operations and propositions (equalities and
inequalities). The solution appears as a sequence of steps (Fig. 5):
Transformation of the equation by the rules
available on the interface of Manipulator
component.
The solution is provided in the same
the Set window presents in the Algebraic Line
component.
The solution performed in Manipulator component allows students to focus
on the meaning of the formal transformation: obtaining the result by means of chain
of deductive steps. The expressions obtained in each step transform the
taining the equivalence. As matter of fact, each step is obtained
applying a rule available on the interface. The equivalence between expressions can
be verified on the Algebraic line: for instance, the value of x for which the expression
"x+3, is the same value for which the expression x+2
equal to zero (expressions referring to first and second line in Fig. 5).
, making available rules and axioms on its interface
he shall not recall to memory the possible rules or
part of the expression selected, but he should consider
provided by Manipulator, are the most convenient and effective
aim. Again, note that the calculus performed in
Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
The equation can be automatically solved selecting the command "Simplify
. Teacher can use this command to
This command can be used to solve any
Simplify" command to
Moreover, to solve equation, students can use the commands of the interface
and propositions (equalities and
quation by the rules
n the interface of Manipulator
The solution is provided in the same form of
the Set window presents in the Algebraic Line
The solution performed in Manipulator component allows students to focus
on the meaning of the formal transformation: obtaining the result by means of chain
transform the form of the
taining the equivalence. As matter of fact, each step is obtained
The equivalence between expressions can
be verified on the Algebraic line: for instance, the value of x for which the expression
"x+3, is the same value for which the expression x+2-(2"x+3) is
Moreover, note
interface, supports
rules or axioms
he should consider which of the
convenient and effective to
performed in application of the rule is completely charged by software. So that student can concentrate
attention on the meaning of the procedure (deductive
calculus. This fact, as well as the
axioms, is particularly interesting for students having difficulties in calculus or
dyscalculia.
The solution obtained in Manipulator component and that obtained in
Algebraic Line component have the same f
Sets in Fig. 1b). Thus, the solution obtained by means the formal approach is charged
of the meaning of truth set (truth value) for the equation. The integration between
visual-spatial approach, developed in
approach, developed in Manipulator
In this way, the teaching and learning of the notions concerning solution of
linear algebraic equation is radically changed: the formal manipulation, performed in
Manipulator component, is arrival point and not starting point in order to develop
meanings related to the
algebraic expressions, variable.
For this reason, it seems
possibility to perform innovative teaching and learning algebra, is realized.
Identity
As presented above, when the expression
the equality can present different senses related to the value assumed by the l
interpret the equality on the semantic plane, it is necessary to distinguish if it has to be
considered as equation or as identity. The "=" sign assigns to the equality the sense of
equation when its two members are equal only for specific values
presented in tasks1. Instead, the "=" sign gives to the equality the sense of identity
when its two members are equal whatever the numerical value of the letter is, as in the
following task 2
Task 2
Consider the following equalities:
Represent on the Algebraic Line of AlNuSet the expressions composing the
two equalities. Drag x along the line and find the values of x for which the
expressions are true.
The following figures (Fig. 6a and Fig. 6b) show
equalities on the line in two moments corresponding to different values of x.
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the rule is completely charged by software. So that student can concentrate
attention on the meaning of the procedure (deductive sequence) rather than in
This fact, as well as the support on students' memory in recall
is particularly interesting for students having difficulties in calculus or
The solution obtained in Manipulator component and that obtained in
Algebraic Line component have the same form (the last line of Fig 5 and the window
Thus, the solution obtained by means the formal approach is charged
of the meaning of truth set (truth value) for the equation. The integration between
developed in Algebraic Line, and syntactical and formal
in Manipulator, is realized.
In this way, the teaching and learning of the notions concerning solution of
linear algebraic equation is radically changed: the formal manipulation, performed in
lator component, is arrival point and not starting point in order to develop
meanings related to the notions of equation, truth-value and truth set,
, variable.
it seems possible to say that our hypothesis, co
possibility to perform innovative teaching and learning algebra, is realized.
As presented above, when the expressions composing the equality
the equality can present different senses related to the value assumed by the l
equality on the semantic plane, it is necessary to distinguish if it has to be
considered as equation or as identity. The "=" sign assigns to the equality the sense of
equation when its two members are equal only for specific values
presented in taks1. Instead, the "=" sign gives to the equality the sense of identity
when its two members are equal whatever the numerical value of the letter is, as in the
Consider the following equalities: 2x + 3 = 5x and 2x + 3x = 5
Represent on the Algebraic Line of AlNuSet the expressions composing the
two equalities. Drag x along the line and find the values of x for which the
following figures (Fig. 6a and Fig. 6b) show the expressions composing
in two moments corresponding to different values of x.
Fig 6a
Fig 6b
the rule is completely charged by software. So that student can concentrate his/her
) rather than in
recalling rules and
is particularly interesting for students having difficulties in calculus or
The solution obtained in Manipulator component and that obtained in
(the last line of Fig 5 and the window
Thus, the solution obtained by means the formal approach is charged
of the meaning of truth set (truth value) for the equation. The integration between
and syntactical and formal
In this way, the teaching and learning of the notions concerning solution of
linear algebraic equation is radically changed: the formal manipulation, performed in
lator component, is arrival point and not starting point in order to develop
value and truth set, unknown,
say that our hypothesis, concerning the
possibility to perform innovative teaching and learning algebra, is realized.
composing the equality are literal
the equality can present different senses related to the value assumed by the letter. To
equality on the semantic plane, it is necessary to distinguish if it has to be
considered as equation or as identity. The "=" sign assigns to the equality the sense of
of the letter as
presented in taks1. Instead, the "=" sign gives to the equality the sense of identity
when its two members are equal whatever the numerical value of the letter is, as in the
5x
Represent on the Algebraic Line of AlNuSet the expressions composing the
two equalities. Drag x along the line and find the values of x for which the
the expressions composing
in two moments corresponding to different values of x. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Dragging x along the line it is possible to verify that the expressions 2x+3x
and 5x refer, for all values of x, to the same point and they belong to the same post-it.
The equality is verified for all values of x.
Note that the colour of the dot associated to the equation 2x+3x=5x is always
green so that the equality is always true and it is an identity. Instead, the expressions
2x+3 and 5x refer to the same point and they belong to the same post-it only for x=1.
This is a conditioned equality that is true for the only value 1 of x. This kind of task
and the support of the Algebraic Line, allow teachers to explicitly speak about
universal and existential quantifiers.
The dynamic representations available on the Algebraic Line allow teacher to
promote meaning of universal and existential quantifiers exploiting the perceptive and
visuo-spatial approach.
Conclusion
The sequence of tasks proposed in this article shows the impact of new
representations performed in Algebraic Line and Manipulator components of AlNuSet
in the solution process of equations and identities. The efficacy of visuo-spatial and
dynamic representations available in Algebraic Line, to built algebraic meanings of
notions involved in solution of equations and identities, is presented. Moreover, in
this paper I described how the formal procedure of equation' solution, performed in
Manipulator component, can be linked effectively to the meanings developed by by
perceptive approach in Algebraic Line component. The algebraic transformations
performed in Manipulator allow students to focus on the educational aim concerning
the equivalence between algebraic propositions. This equivalence is performed on the
base of rules and axioms that have to be chosen among that presented on the interface.
Moreover, the proof of the equivalence is not charged of calculus. The sequence of
tasks in which the use of Algebraic Line of AlNuSet is required, shows how it is
possible to promote the para-mathematical notions of unknown, algebraic
expressions, variable, universal and existential quantifier, equality. Moreover, the
tasks show how the use of Manipulator can support proto-mathematical notion
concerning equivalent expressions. Finally, the sequence of tasks performed by means
of AlNuSet' functionalities shows how teachers were able to innovate the teaching
learning algebra starting with the development of the meanings of algebraic notions
before moving to the formal aspects related to syntactic manipulations. The
discussion of tasks shows that this is possible by means of new dynamic
representations available in AlNuSet.
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Designing Tasks for Use With Digital Technology
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In this paper we use a research-oriented approach to describe
some of the features of technology-integrated tasks with epistemic value.
Some examples of the construction of these tasks are given along with
teacher factors required to employ them. A brief case study of the
collaboration of the authors, a researcher and a teacher, in the
development and implementation of tasks addressing Riemann integration
is presented.
Keywords: Tasks, Technology, Epistemic value, Pedagogy, Teachers
Background
Many educators promote digital technology as having a role to play in
helping students to develop mathematical thinking. However, some of the key
affordances arising from technology use emanate from the tasks we use with it. In
turn the implementation of these tasks depends in large measure on the knowledge
and orientations (Schoenfeld, 2010) of the teacher. Thus while engagement with
digital technologies may change or challenge traditional student learning trajectories
(Anthony & Walshaw, 2007), it can, and will, also challenge teaching trajectories.
Hence, one of the central issues in the use of technology
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is the design and
implementation of tasks that will encourage the learning and understanding of
mathematics, and in particular mathematical thinking (ibid). To design these tasks we
have to consider how this mathematical thinking is mediated by the tasks that are
employed with the technology. In this paper we address the design aspects of such
tasks from a theoretical standpoint, present a brief case study and then discuss some
implications for teachers.
We take a relatively broad view of the idea of a task and take it to mean an
activity that students are asked to attempt (or choose to) in a mathematics lesson (and
possibly complete at another time). It may be short, as in solving a single linear
algebraic equation, or long, such as a structured or semi-structured investigation of
the graphical properties of polynomial functions of degree one and two. Solving the
tasks they are given will require certain techniques from students, which depend on
the task and, in the words of Artigue (2002), may go beyond routine actions to include
a complex combination of reasoning and routine work. To generate a wide range of

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The term technology should be read as meaning digital technology throughout this paper. Theme A - M. O. J. Thomas & C. Lin
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such techniques tasks need to take students beyond the 'routine', placing the task at
centre stage. These classroom tasks will require students to exhibit techniques that
have either epistemic or pragmatic value (Artigue, 2002). In the former the focus is
on the production of knowledge of the mathematical object under study, whereas in
the latter the techniques are perceived and evaluated in terms of their productive
potential. Techniques with pragmatic value alone include what are often termed
procedural tasks. In a technological classroom setting Thomas (2009a, p. 152)
suggests that procedural tasks such as:
1. Solve
3 2 0
2. Draw the graph of

1
2
3. Differentiate
3
9
4. Find
3 2

have little or no epistemic value, since solving these with 'black box'
technology use does not assist students to focus on, or understand, the constructs of
mathematics.
Other tasks that do have well known by-hand techniques can quickly assume
the character of procedural tasks when technology is used. One example of this is
when a well-known by-hand scheme is transferred to a technological tool, with a
subsequent change of technique. Using the scheme for finding the inverse of a
function given as y = f(x) that makes x the subject of the formula and then replaces x
with y can be partly executed on a CAS instrument using the technique of solving the
equation for x. Figure 1 shows two examples of how students given the task of finding
the inverse function by hand have integrated the use of CAS into this scheme,
employing a direct command to perform the first step of making x the subject of the
equation. The use of the CAS is indicated by the ♦ symbol, as requested. We note
that in each case the student writes down precisely the form given by the CAS, and
the first student (top) has also written down the details of each step of the scheme.
Neither student has used the technique of entering the function and simply using the
CAS to provide the inverse function, but instead have integrated it into their by-hand
method.
Figure 1. Integration of CAS in the task:
Given f(x) =
6x+ 1
2x− 3
is invertible (x≠1.5), find f
−1
(x).
What should be considered when constructing tasks employing technology
that have epistemic value? Based on a consideration of the benefits of a TaskTechnique-Theory (TTT) approach (derived from the anthropological theory of Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22
didactics of Chevallard), Kieran and Drijvers (2006) suggest a number of features of a
task with epistemic value.
(2009b), and include addressing a mathematical concept or idea, examining the role
of language and asking students to write about how they interpret their work,
considering dynamic multi
conversions between representations
representations (Thomas, 2008), integrating technological and by
aiming for generalisation, and getti
of mathematical theory. We consider how the following task, based on one in Thomas
(2009a), seeks to follow these research
Which of the following equations have the same solutions?
worked out your answers and write down reasons for your answers. Use a graphic
calculator or computer to help you work out and support your answers with an
explanation. In general, what equations of this type would have the same
solutions, and why?
First, the students are encouraged to think about language and to write about
their work giving reasons. The mathematical theory focussed on in this task includes
the notion of the nature of a solution, the transformations under which solutions
conserved, and the difference between
that are mathematically correct, and
rapidly towards the solution (Hong, Thomas & Kwon, 2000). While this task is
approachable using a graphic calculator, a computer program such as GeoGebra (see
Figure 2) can enhance the dynamic nature of the links to a graphical representation,
allowing one to interact with the representations to ask questions that involve
treatments—Why does the
conversions—What is the effect on the graphs of adding
equation, and why? Thus the aim is that the graphs support the learner in allowing
versatile interactions with the graphs and providing an important algebra
linking of representations (an algebra view is available in GeoGebra). To c
the task requires the integration of technological and by
asked to generalise from the given equations to any equations of the same form, and
the inclusion of the word 'why' may motivate them to think about giving reas
leading to argumentation and possibly the notion of proving a conjecture.
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, Kieran and Drijvers (2006) suggest a number of features of a
emic value. Based on his research, these have been added to by Thomas
addressing a mathematical concept or idea, examining the role
asking students to write about how they interpret their work,
dynamic multi-representations, involving treatments and, especially,
between representations (Duval, 2006) and versatile interactions with
representations (Thomas, 2008), integrating technological and by-hand techniques,
aiming for generalisation, and getting students to think about proof and development
of mathematical theory. We consider how the following task, based on one in Thomas
(2009a), seeks to follow these research-based design principles.
Which of the following equations have the same solutions? Explain how you
worked out your answers and write down reasons for your answers. Use a graphic
calculator or computer to help you work out and support your answers with an
explanation. In general, what equations of this type would have the same
First, the students are encouraged to think about language and to write about
their work giving reasons. The mathematical theory focussed on in this task includes
the notion of the nature of a solution, the transformations under which solutions
conserved, and the difference between legitimate transformations of an equation, ones
that are mathematically correct, and productive transformations, ones that move more
rapidly towards the solution (Hong, Thomas & Kwon, 2000). While this task is
oachable using a graphic calculator, a computer program such as GeoGebra (see
Figure 2) can enhance the dynamic nature of the links to a graphical representation,
allowing one to interact with the representations to ask questions that involve
y does the x solution remain on the same vertical line?
What is the effect on the graphs of adding ax
2
or bx to both sides of an
equation, and why? Thus the aim is that the graphs support the learner in allowing
versatile interactions with the graphs and providing an important algebra
linking of representations (an algebra view is available in GeoGebra). To c
the task requires the integration of technological and by-hand techniques, students are
asked to generalise from the given equations to any equations of the same form, and
the inclusion of the word 'why' may motivate them to think about giving reas
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Explain how you
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solution remain on the same vertical line?—and
both sides of an
equation, and why? Thus the aim is that the graphs support the learner in allowing
versatile interactions with the graphs and providing an important algebra-graph
linking of representations (an algebra view is available in GeoGebra). To complete
hand techniques, students are
asked to generalise from the given equations to any equations of the same form, and
the inclusion of the word 'why' may motivate them to think about giving reasons,
leading to argumentation and possibly the notion of proving a conjecture.Figure 2. Using GeoGebra to investigate adding
Another example of a task that exemplifies using technology to engage
students with a mathematical concept in the manner described above is presented in
Heid, Thomas and Zbiek (2012, in press). This requires a consideration of a linear
transformation, asking for the algebraic form a function
reflected in the line y = k
procedure for writing this down, and may not even have a mental scheme for a
suitable technique, technolo
help them solve the general problem inductively through specific examples. This may
start with reflecting the graph of, say,
can be used to draw the graphs, possibly linking the algebraic and graphical
representations in a dynamic way. In Figure 3 we see an example of how a slider is
used to vary the function (left) and then the value of
up to reflect the function. The first step shows that
functions and y = 2 are invariant under reflection,
and the second that all the points of intersection lie on the graph of the fun
varies. Likewise the x-value of the
The algebra view of GeoGebra also gives access to the algebraic form of the reflected
function. It is hoped that
involving translating the graph vertically by
f(x)), and then translating vertically by +
reflections across representations
Figure 3. Dynamic reflection
Of course a key feature of the task is the encouragement to generalise

algebraically for any function and the line
k every point of the graph (and the plane) is similarly reflected. Thus taking a general
point (x, f(x)), distance n
function, g(x) = k - n = k
graph of the function f(x) in the line
example, reflecting of the graph of
graph of the function g(x)
inductive reasoning from
result, describe it, and seek to prove it, thus engaging deeply with the mathematics.
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Figure 2. Using GeoGebra to investigate adding ax
2
(left) and bx (right) to both sides of an
equation.
Another example of a task that exemplifies using technology to engage
students with a mathematical concept in the manner described above is presented in
Heid, Thomas and Zbiek (2012, in press). This requires a consideration of a linear
ng for the algebraic form a function f would take when its graph is
k, for some real k. Assuming that students do not know a
procedure for writing this down, and may not even have a mental scheme for a
suitable technique, technology, such as a CAS calculator or computer program, can
help them solve the general problem inductively through specific examples. This may
start with reflecting the graph of, say, y = x
2
+ 3x in the line y = 2. The
an be used to draw the graphs, possibly linking the algebraic and graphical
representations in a dynamic way. In Figure 3 we see an example of how a slider is
used to vary the function (left) and then the value of k (right) and the program is set
lect the function. The first step shows that the points of intersection of
= 2 are invariant under reflection, and hence all lie on the line
and the second that all the points of intersection lie on the graph of the fun
value of the vertex is invariant under reflection in both cases.
The algebra view of GeoGebra also gives access to the algebraic form of the reflected
function. It is hoped that students may develop a technique for the ref
ating the graph vertically by -k, then reflecting in the
hen translating vertically by +k. This means linking translation
s across representations, relating graphs to algebraic forms f(x
Figure 3. Dynamic reflection of graphs in the line x = k using GeoGebra.
Of course a key feature of the task is the encouragement to generalise
algebraically for any function and the line y = k. Since g(x) is a reflection of
point of the graph (and the plane) is similarly reflected. Thus taking a general
above the line y = k, n = f(x) - k, and so for our reflected
- (f(x) - k) = 2k - f(x). Hence, the result of reflecting the
) in the line y = k is to obtain a function g(x) = 2
example, reflecting of the graph of f(x) = 3x
4
- 2x + 1 in the line y
) = -4 - (3x
4
- 2x + 1) = 2x - 3x
4
- 5. Involving students in
inductive reasoning from examples might thus enable them to abstract this general
result, describe it, and seek to prove it, thus engaging deeply with the mathematics.
(right) to both sides of an
Another example of a task that exemplifies using technology to engage
students with a mathematical concept in the manner described above is presented in
Heid, Thomas and Zbiek (2012, in press). This requires a consideration of a linear
take when its graph is
. Assuming that students do not know a
procedure for writing this down, and may not even have a mental scheme for a
gy, such as a CAS calculator or computer program, can
help them solve the general problem inductively through specific examples. This may
2. The technology
an be used to draw the graphs, possibly linking the algebraic and graphical
representations in a dynamic way. In Figure 3 we see an example of how a slider is
(right) and the program is set
the points of intersection of the
and hence all lie on the line y = 2
and the second that all the points of intersection lie on the graph of the function as k
is invariant under reflection in both cases.
The algebra view of GeoGebra also gives access to the algebraic form of the reflected
develop a technique for the reflection
then reflecting in the x-axis (g(x)=-
translations and
x) + k and -f(x).
using GeoGebra.
Of course a key feature of the task is the encouragement to generalise
) is a reflection of f(x) in y =
point of the graph (and the plane) is similarly reflected. Thus taking a general
, and so for our reflected
ult of reflecting the
) = 2k - f(x). For
= -2 gives the
. Involving students in
enable them to abstract this general
result, describe it, and seek to prove it, thus engaging deeply with the mathematics. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22
Teacher Knowledge
The ability of a teacher to construct and use tasks with epistemic value, such as
those described above, requires what Thomas
Thomas, 2006) calls pedagogical technology knowledge
emerging framework comprises p
beliefs and attitudes), goals and affect regarding technology use, strong
knowledge for teaching
instrumentation of the technological
the technology, their familiarity with it
mathematics and how to teach it. A teacher with good
principles and techniques required to
technology, with the milieu containing tasks that enable
emerge, mediated by the technology.
A range of factors was identified by Goos (2005) as influencing teacher
technology use. This includes:
and opportunities to learn (pre
beginning teaching, professional development); availability of appropriate teaching
materials; support from colleagues
mathematics teaching; and beliefs about mathematics and how it is learned. Thus to
assist teachers with development of PTK some way of targeting these has to be
devised. One way is through partnerships with r
research involving such a partnership relating to task design are recorded below.
Figure 4. An outline of t
In this research the two authors of this article worked together using the
principles above to develop suitable tasks incorporating technology use that would
assist students to construct a fuller understanding of Riemann integration (see
Thomas, 2011). This included helping students
concept image of definite integral and
Riemann sums and the analytical schema of the limit of the numerical sequence
(Czarnocha, Loch, Prabhu
embodied cognition approach
carefully constructed tasks that would
broaden their concept image of definite in
decision was made to use the GeoGebra software and the first
experienced researcher) passed on to the teacher his ideas on the benefits of the
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Teacher Knowledge
lity of a teacher to construct and use tasks with epistemic value, such as
those described above, requires what Thomas (Thomas & Hong, 2005;
pedagogical technology knowledge (PTK—see Figure 4
comprises positive orientations (Schoenfeld, 2010
goals and affect regarding technology use, strong
(Hill & Ball, 2004) and good instrumentalisation
technological tool. Thus it comprises a teacher's perspective on
their familiarity with it as a teaching tool, and their understanding of
mathematics and how to teach it. A teacher with good PTK can
ples and techniques required to build didactical situations inco
with the milieu containing tasks that enable mathematical learning
emerge, mediated by the technology.
A range of factors was identified by Goos (2005) as influencing teacher
technology use. This includes: skill and previous experience in using technology; time
and opportunities to learn (pre-service education, guidance during practicum and
beginning teaching, professional development); availability of appropriate teaching
materials; support from colleagues; knowledge of how to integrate technology into
mathematics teaching; and beliefs about mathematics and how it is learned. Thus to
assist teachers with development of PTK some way of targeting these has to be
devised. One way is through partnerships with researchers. Some of the results of
research involving such a partnership relating to task design are recorded below.
An outline of the construction of Pedagogical Technology Knowledge (PTK).
In this research the two authors of this article worked together using the
principles above to develop suitable tasks incorporating technology use that would
assist students to construct a fuller understanding of Riemann integration (see
. This included helping students develop a rich and comprehensive
definite integral and the ability to coordinate the visual schema of
Riemann sums and the analytical schema of the limit of the numerical sequence
Czarnocha, Loch, Prabhu, & Vidakovic, 2001). To do this we
embodied cognition approach (Tall, 2008) to the notion of infinity and limit through
carefully constructed tasks that would extend students' previous experience and
broaden their concept image of definite integral (Rasslan & Tall, 2002)
decision was made to use the GeoGebra software and the first-named author (the
experienced researcher) passed on to the teacher his ideas on the benefits of the
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lity of a teacher to construct and use tasks with epistemic value, such as
(Thomas & Hong, 2005; Hong &
see Figure 4). This
Schoenfeld, 2010—includes
goals and affect regarding technology use, strong mathematical
instrumentalisation and
teacher's perspective on
as a teaching tool, and their understanding of
PTK can understand the
didactical situations incorporating
mathematical learning to
A range of factors was identified by Goos (2005) as influencing teacher
skill and previous experience in using technology; time
service education, guidance during practicum and
beginning teaching, professional development); availability of appropriate teaching
; knowledge of how to integrate technology into
mathematics teaching; and beliefs about mathematics and how it is learned. Thus to
assist teachers with development of PTK some way of targeting these has to be
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In this research the two authors of this article worked together using the
principles above to develop suitable tasks incorporating technology use that would
assist students to construct a fuller understanding of Riemann integration (see Lin &
nd comprehensive
the visual schema of
Riemann sums and the analytical schema of the limit of the numerical sequences
To do this we developed an
e notion of infinity and limit through
extend students' previous experience and
tegral (Rasslan & Tall, 2002). A joint
named author (the
experienced researcher) passed on to the teacher his ideas on the benefits of the software and how it might be used in this case to
was enthusiastic about using the software, could see the potential pedagogical benefits
and was committed to trying it out. The decision was made to use algebraic, graphical
and numerical (spreadsheet) features of the
then proceeded through cycles of suggestions by the researcher, experimental
construction by the teacher, and then refinement through discussion with the
researcher, until both were happy with the outcomes. Clear goa
implementation of the technology were agreed.
One of the tasks constructed, relating to the mathematical concept of
Riemann sums is presented in more detail here. The teacher established the following
teaching trajectory, integrating th
Introductory questions and two
covered over a period of time
Obtain a range (minã
Define the minimum (under
maximum (over-estimate) as the "upper sum".
Explore what happens to the estimates when
Explore what happen when we making the time interval real
comparing the results with actual area using

Establish the idea that as the number of rectangles increases, the lower sums are
almost equal to the upper sums.
The task used
14
varying numbers of left
5 to 500) to approximate the area under the curve
hours (see Figure 5). The students recorded the approximate value of the area on each
occasion, as the number of subinterval, or rectangles, was changed dynamical
the slider, in the correct cells in the table
Figure 5. Using the GeoGebra program to investigate Riemann sums.
In line with the task design principles, the students were then encouraged to
write about what they noticed from the program and the results in the table, using the
questions: What did you notice about the difference between the left

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Note the program was used in a form allowing general investigation of polynomial
functions up to degree 4. The spreadsheet view showing numerical sums was also visible
the graph.
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software and how it might be used in this case to construct suitable tasks. The teacher
was enthusiastic about using the software, could see the potential pedagogical benefits
and was committed to trying it out. The decision was made to use algebraic, graphical
and numerical (spreadsheet) features of the GeoGebra program. The task construction
then proceeded through cycles of suggestions by the researcher, experimental
construction by the teacher, and then refinement through discussion with the
researcher, until both were happy with the outcomes. Clear goals for the role and
implementation of the technology were agreed.
One of the tasks constructed, relating to the mathematical concept of
Riemann sums is presented in more detail here. The teacher established the following
teaching trajectory, integrating the technology, for this concept.
Introductory questions and two thought experiments on finding out distance
over a period of time given the rate of change of the distance.
Obtain a range (min&max) of estimates of the area by summing up the rectangle
Define the minimum (under-estimate) of the range as the "lower sum" and
estimate) as the "upper sum".
Explore what happens to the estimates when increasing the number of rectangles
Explore what happen when we making the time interval really small by
comparing the results with actual area using GeoGebra software.
Establish the idea that as the number of rectangles increases, the lower sums are
almost equal to the upper sums.
varying numbers of left-hand and right-hand rectang
5 to 500) to approximate the area under the curve from
hours (see Figure 5). The students recorded the approximate value of the area on each
occasion, as the number of subinterval, or rectangles, was changed dynamical
the slider, in the correct cells in the table provided, shown in Figure 5 (right).
Filling in the table provided
Figure 5. Using the GeoGebra program to investigate Riemann sums.
In line with the task design principles, the students were then encouraged to
write about what they noticed from the program and the results in the table, using the
What did you notice about the difference between the left

Note the program was used in a form allowing general investigation of polynomial
functions up to degree 4. The spreadsheet view showing numerical sums was also visible
construct suitable tasks. The teacher
was enthusiastic about using the software, could see the potential pedagogical benefits
and was committed to trying it out. The decision was made to use algebraic, graphical
GeoGebra program. The task construction
then proceeded through cycles of suggestions by the researcher, experimental
construction by the teacher, and then refinement through discussion with the
ls for the role and
One of the tasks constructed, relating to the mathematical concept of
Riemann sums is presented in more detail here. The teacher established the following
distance
of the area by summing up the rectangles.
estimate) of the range as the "lower sum" and
increasing the number of rectangles.
ly small by
Establish the idea that as the number of rectangles increases, the lower sums are
hand rectangles (from
from t = 0 to t = 10
hours (see Figure 5). The students recorded the approximate value of the area on each
occasion, as the number of subinterval, or rectangles, was changed dynamically with
provided, shown in Figure 5 (right).
Figure 5. Using the GeoGebra program to investigate Riemann sums.
In line with the task design principles, the students were then encouraged to
write about what they noticed from the program and the results in the table, using the
What did you notice about the difference between the left-hand sum and
Note the program was used in a form allowing general investigation of polynomial
functions up to degree 4. The spreadsheet view showing numerical sums was also visible to the right of Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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right-hand sum (Accuracy Difference) as the number of rectangles increases?; As we
increase the number of rectangles drawn between t = 0 to t = 10 hours, what did you
notice about the relationship between the rectangles and the area under the curve?
How does this feature relate to the accuracy of the area approximation?; and Can you
think about other methods to approximate the area to produce a better estimate? A
method using trapeziums followed. This prepared students for presentation of an
algebraic generalisation of the limit of Riemann sums.
Overall the mean improvement of the class based on tests before and after the
teaching was significantly better than that of a second group of students who covered
the same ground but did not do the computer tasks ($t = -2.9095$, $p < 0.01$), and the
students had an improved understanding of Riemann sums and numerical methods
graphically, although they were less able to carry out the method numerically. The
teacher talked to six of the 23 students in the class and when asked how the program
had influenced their understanding. All six commented that they liked the dynamic
graphics view of GeoGebra and that the visual aspect of the software really helped
them learn and understand the integration topic better. Two of their comments were:
Student 1: I think the visual display and the dynamic image of the program
help my understanding of seeing where the formula is derived from. By looking at
the different shapes…I understand better how we found the area using different
shapes and how it is related to the definite integral.
Student 2: I really like it [GeoGebra] when we were learning about the area
approximation using rectangle and also trapezium. I can see it on the graphics
view instantly that the approximation is better when you started changing the
number of rectangles or trapeziums…it is easier for me to understand because it is
not just graphical, it is also instant.
It appears from their comments that the task using GeoGebra particularly
helped these students understand better how the concept of integration can be seen to
begin with area approximations that improve if we increase the numbers of rectangles
or trapeziums, matching more closely the actual area under the graph. This idea can
be difficult to illustrate and explain without the aid of such technology. With regard to
the way their overall understanding from the module of work improved they said:
Student 2: I think my knowledge of the relationship between integration and
area has changed. It [GeoGebra] probably enhances my knowledge of area and
improves it. I mean before I see integration just as a technique or a formula
opposite to differentiation. But now…I understand more why we use integration
for area.
Student 3: Before I think integration just a formula that is opposite to
differentiation. But now I know more why you do integration, like it is for finding
out the area. Basically I see more connection between differentiation and
integration now.
Student 4: At Year 12, I just think integration as a simple algebra algorithm.
Now after you taught us the topic with the computer program, I think I have
learned more about integration such as how it relates to the area. I understand it
better graphically now.
The teacher enjoyed using the approach developed in her lessons, was
confident in doing so, and has continued to use it since. Her growing PTK is partly
evidenced in her comment that: "Teachers need to use GeoGebra…for investigative
activities in order to increase the interactions between students and the software. More
importantly, teachers need to pose interesting and meaningful problems [tasks] so that
students can be actively engaged, and try to answer questions and re-formulate
concepts in their own terms…an ideal way of using GeoGebra is where students
investigate mathematical ideas with the software by making conjectures and testing Theme A - M. O. J. Thomas & C. Lin
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them out." She also comments that although the experience of writing the tasks with
GeoGebra was initially not easy, it improved her understanding of Riemann integral
since she needed to understand the concept thoroughly before she could use all the
program functions available in GeoGebra to achieve what we wanted to show the
students. Overall she felt "It was a very good experience for me collaborating with
[the researcher] trying to write tasks that can improve students' understanding."
In this paper we have taken a theoretical stance to making suggestions on the
kind of features that tasks incorporating technology use might need in order to have
epistemic value. These have been exemplified with several tasks that have their basis
in the research of the first author. We suggest that the development of tasks like these
is not a trivial matter for classroom mathematics teachers and have described how a
teacher-researcher collaborative effort is one possible way to assist teachers to build
the PTK they need to become confident task designers.
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Designing mathematical modelling tasks in a technology rich

secondary school context
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Trevor Redmond
A. B. Paterson College
The potential of digital tools to enhance student learning is well researched, however, the potential of technology to promote students' engagement with mathematical modelling tasks has received limited consideration. This paper draws on a research study that aimed to investigate the possibilities that exist for student learning when teachers from six secondary schools designed tasks that anticipated for the use of digital tools within mathematical modelling tasks. The paper describes and analyses the collaboration which took place in identifying principles of design for such tasks.
Keywords: Mathematical modelling and Applications; Digital tools; Technology
Introduction
While there are strong research traditions in the fields of mathematical modelling and applications and the use of digital tools in mathematics classrooms, few studies have explored the potential of the nexus which exists between these two powerful approaches to thinking (Geiger, Faragher and Goos, 2010). Mathematical modelling is often described as a process involving the formulation of a mathematical representation of a real world situation and then using mathematics to derive results, interpret the results in terms of the given situation and if necessary, revising the model. The purpose of models is to interpret real world situations and/or make predictions about the future or past states of modelled systems (English, Fox, & Watters, 2005).
There is now a large corpus of literature devoted to the way in which digital tools can enhance teaching and learning opportunities in mathematics classrooms. Studies, however, have tended to report on advantages to instruction in mathematical thinking and learning within content specific domains such as number (e.g., Kieran & Guzma

n, 2005), geometry (e.g., Laborde, Kynigos, Hollebrands & Straesser, 2006), algebra and calculus (e.g., Ferrara, Pratt & Robutta, 2006) or social aspects of classroom practice such as collaborative investigative practice (e.g., Beatty & Geiger, 2010). Thus, there is little research on how digital tools can be used in tandem with mathematical knowledge to work on problems that exist in the real world, as

While such innovations [ICTs] have been useful in enhancing understandings of school mathematics, less is known about the transfer of such knowledge, skills and dispositions to the world beyond schools. Given the high tech world that students will enter once they leave schools, there needs to be recognition of the new demands of these changed workplaces. (p. 99)
Given this identified need for students to be provided opportunity to use digital tools when working on real world problems consideration needs to be given to the nature of the learning experiences, and the tasks at the centre of these experiences, students should encounter within school mathematics classes. The aim of this paper is to explore an approach to the design and implementation of tasks which focus on the a mathematical modelling approach to teaching and learning that is supported by digital tools. In doing so, the paper will address Theme A, Tools and Representations, through the following research question.
What are the principles of design for technology rich modelling and applications tasks that result in effective learning experiences for students?
Artefacts as mediators of mathematical learning
In developing principles of design for technology integrated modelling and applications tasks the role of artefacts, in this case the task and the digital tool(s), must be examined. Verillon and Rabardel's (1995) iconic work on the distinction between an artefact and an instrument provides insight into the role of artefacts in mediating learning by distinguishing between an artefact, which includes both physical and sign tools that have no intrinsic meaning of their own, and an instrument in which an artefact is used in a meaningful way to work on a specific task. Different tasks make different demands on the user and their relationship with the artefact. The development of this relationship, and thus how the artefact is used, is known as instrumental genesis. Instrumental genesis is a complex process in which, firstly, the potentialities of the artefact for performing a specific task are recognised which transforms the artefact into an instrument (instrumentalisation), and, secondly, there is a process that takes place within the user in order to use the instrument for a particular task (instrumentation) (Artigue, 2002). Instrumentation generates schemas of instrumented action that are either original creations by individuals or pre-existing entities that are appropriated from others. An instrument, therefore, consists of the artefact and the user's associated schemas of instrumented action. The process of instrumental genesis is also dynamic between the instrument and the user as the constraints and affordances of the artefact shape the user's conceptual development while at the same time the user's perception of the possibilities of the artefact during instrumentation can lead to the use of the artefact in ways that were not originally intended by the designers of a tool (Drijvers & Gravemeijer, 2005).
Instrumental genesis has been used to explain how digital tools are transformed into instruments for learning through interaction with teachers and students (e.g., Artigue, 2002). A teacher's activity in promoting a student's instrumental genesis is known as instrumental orchestration (Trouche, 2005). This process recognises the social aspects of learning as it allows for the sharing of schemas as of instrumented action that individuals have developed within a small group or whole class. A teacher can facilitate the appropriation of these schemas by other students by making the nature of these schemas explicit by orchestration classroom interaction around the schemas through careful and selective questioning
More recently, others have attempted to extend our understanding of an instrumental approach to the role of artefacts in mediating learning by recognising Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.

that the genesis of an artefact into an instrument takes place within highly interactive environments, such as school staff rooms or mathematics classrooms, where a number of artefacts are used simultaneously. Gueudet and Trouche (2009) extend the definition of artefact by introducing the term resources to encompass any artefact with the potential to promote semiotic mediation in the process of learning. Resources include entities such as computer applications, student worksheets or discussions with a colleague. A resource is appropriated and reshaped by a teacher, in a way that reflects their professional experience in relation to the use of resources, to form a schema of utilisation - a process parallel to the creation of a schema of instrumented action within instrumental genesis. The combination of the resource and the schema of utilisation is called a document. The process of documental genesis is an ongoing one as utilisation schemas will be reshaped as a teacher gains more experience through the use of a resource.
A modelling task oriented research project
Six teachers were recruited from six secondary schools; three from each of two different Australian states. Schools were drawn from across different schools systems (government and non-government) and were representative of a range of socio-economic characteristics. Teachers were invited into the project because of their reputations as highly effective teachers with particular skills in the use of digital tools in mathematics learning and their commitment to improving the learning outcomes of their students. The project was managed by two university based researchers - one in each state. The researchers were primarily responsible for the: conceptual development of the project; classroom data collection including lesson observations, teacher and student interviews, and collection of student samples. Teachers were primarily responsible for the development and implementation of technology demanding mathematical modelling tasks. Researchers played a vital role in providing feedback about the effectiveness of tasks trialled in teachers' classrooms. Together teachers and researchers developed principles of design for effective tasks based on their shared experiences while trialling tasks in mathematics classrooms.
This paper reports, specifically, on the work of one teacher and on his students in a Year 11 (15-16 years of age) mathematics class. The curriculum context in which he taught mandated the teaching, learning and assessment of mathematical modelling as a key objective of a state-wide syllabus (educational authorities are state based in Australia). The use of technology in mathematics teaching and learning was also prescribed in the Mathematics B program (incorporating the study of functions, calculus and statistics) in which his students were enrolled. Students had almost unrestricted access to digital technologies including: powerful handheld digital devices with mathematical facilities such as data and function plotters and Computer Algebra Systems; computers with mathematically enabled applications; the internet; and electronic white boards.
The research design consisted of three components: (1) two whole day teacher professional learning meetings which took place at the beginning and middle of the project; (2) three classroom observations for each teacher; and (3) a focus group interview near the end of the project that involved all teachers. The detail and purpose of each of these activities is outlined in Table 1. Further detail on the research methodology can be found in (Geiger, Faragher and Goos, 2010). Theme A – V. Geiger & T. Redmond

Time Activity
Sept-Dec
Year 1
Teacher workshops in each state: research team outline the aims of the project; offer prototype tasks; discussion of principles which underlie prototype tasks.
Jan-April
Year 2
Lesson observations ; teacher and student interviews; collection of student work samples; feedback on effectiveness of trialed tasks in relation to modeling and the use of digital tools.
April-June
Year 2
Lesson observations; teacher and student interviews; collection of student work samples; feedback on effectiveness of trialed tasks in relation to modeling and the use of digital tools.
July
Year 2
Teacher workshops in each state: teachers share exemplars of digital tool and modelling tasks; discussion on principles which underlie teacher developed tasks; research team offer accounts of practice from classroom observations.
Aug-Sept
Year 2
Lesson observations; teacher and student interviews; collection of student work samples; feedback on effectiveness of trialed tasks in relation to modeling and the use of digital tools.
Oct-Dec
Year 2
Final project meeting and focus group interview in each state; teachers share exemplars of modelling and digital tool tasks; further discussion on principles which underlie teacher developed tasks.
Table 1: Research design
Principles of task design in technology demanding modelling tasks
The teacher (the co-author of this paper) who is the focus of this paper, proved to be an effective designer of technology demanding modelling tasks while, at the same time, demonstrated keen insight into his own design processes and how these developed through the duration of the project. This teacher, in particular, contributed to the development of principles for designing modelling tasks. These principles and their descriptions are presented in Table 2. While these are useful insights they confirm rather than extend what is widely accepted as approaches to designing effective modelling tasks or general advice on good teaching practice.
Principles Description
Syllabus compliance The task must meet the requirements of the syllabus for content knowledge and the dimensions related to applications and technology.
Authenticity and relevance
Tasks must be set in an authentic or life-related context. The task must be of interest to the teacher and be of potential interest to the student.
Open-endedness The mathematics necessary to solve the problem set up in the task should not be immediately apparent. The task must be open-ended in nature providing for opportunity for multiple solution pathways.
Connectivity Ideally the task must make links to different content areas within the syllabus.
Accessibility The task must provide opportunity for students to link to their previous learning. There should be provision for multiple entry and exist points. The task should allow for the introduction of scaffolding prompts or hints.
Development The task must provide challenge and so encourage students to go beyond what they presently know and can do through the modelling process.
Students' engagement with the task should provide feedback to the teacher about the development of their understanding.
Table 2: Characteristics of effective modelling tasks
The teacher also provided valuable input into the role technology played in the design of modelling tasks, and indicated that digital tools served as an enabler of each of the identified principles. He provided comment on the role of digital tools in

The use of digital tools is a mandatory element of the state-wide senior secondary mathematics syllabuses. Genuinely authentic problems are mathematically complex. The representational capabilities of digital tools allow students to accommodate this complexity and thus provide access to authentic problems that otherwise might be considered beyond the scope of their capabilities.
If we didn't have the CAS calculators we couldn't do half the stuff that we do. From my perspective it is the integration of the whole lot together. We have a set of data and we try and build a model from that. We do a scatter plot and we make decisions about the model. We build a model and make some sorts of predictions. Digital tools also provide the means for students with gaps in their content

knowledge to access challenging problem scenarios.
Lower achievers may be struggling with differentiation or integration at that particular point in time…but they can still have access to the problem. My lower achieving kids can still engage in the problem and still make some meaningful contributions. If they don't get caught up in all that manipulation they can still be thoughtful about it.
The nature of authentic open-ended problems means there is no clear solution pathway and students need to evaluate options as they progress toward a solution. The teacher argued that digital tools offer facilities that are essential for exploring possible solution pathways. Technology also provides the means for connecting different types of mathematical knowledge, for example, data representations and functional relationships that modelled patterns in the data.
Selecting authentic, open tasks to model generally implies the students will need to make use of technology. Even if the teacher has scaffolded the task to facilitate access to the context, there is a requirement that the task be sufficiently open for there to be multi-representations of the solution and perhaps different solutions.
The authenticity and open-endedness of a problem is enhanced if students are required to collect data relevant to a problem from an original source; a capacity provided by digital tools in his classroom.
There is often a need to collect data and then to determine whether a relationship exists within that data. Students may need to collect primary data, through the use of probes, or from a video that is then analysed using the technology or use secondary data collected from a newspaper, magazine, web site or some other source.
Used effectively, digital tools provide immediate feedback to students about their initial attempts to build models and solve problems thus progressing students' understanding of the underlying mathematics at the core of the task and hence their mathematical development.
Technology has a significant role to play in the provision of feedback to the student in the first instance, about the models they have built and how well they fit the context being investigated. In mathematical modelling it is important to look for consensus between the mathematics and the context, hence, it is necessary to consider the validity of the conclusions in terms of the context.
Exemplar task and commentary
The principles for design of technology demanding modelling tasks are evident in the following description of a task developed and then implemented by the teacher in his Year 11 mathematics classroom – the Algal Bloom Problem outlined in the Figure 1. In developing this task, the teacher had expected his students to build a Theme A – V. Geiger & T. Redmond
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mathematical model for these data by first creating a scatterplot using their CAS active calculator. A plot of this data suggests a piecewise function (one part linear and one part power function) would be appropriate. The teacher anticipated that students would then use the plot to determine the general form of the functions that would best fit the data and, in due course, develop an equation that would best fit the data. Students were then expected to use the model they had created to respond to the question at the end of the task and also to list any assumptions they made in developing their model and also comment on any limitations they believed were inherent in the repose they provided.
In observing the lesson in which this task was used, the researcher noticed that while every student was able to produce a plot of the data using their handhelds, few had drawn the conclusion that a piecewise function was necessary to model the data. Most students attempted to model the data using a single function, generally by trying to generate a model for the data using the digital handhelds regression model facility. When their single functions were plotted on their screens with the original data points it was obvious that their various functions were a poor fit. When students asked the teacher for assistance he simply encouraged them to have a closer look at their data and explore a wider range of possibilities for fitting a model to the data. After a period of time, two students, working together near the researcher, attempted to fit a piecewise function to the data, and after performing fine adjustments to each part of their function were happy with the result. Their success prompted a subdued celebration by the two students which attracted the teacher's attention. After discussing their conjectured model with the teacher students went on to complete the task. A short period of time after his discussion with these students, the teacher called for the attention of the class and asked them about their progress. The two students near the researcher volunteered and were asked to outline their attempt at the task. When they announced they had decided to make use of a piecewise function, sections of the class responded in different ways. A small number of students indicated agreement with the approach the pair of students were proposing even though the details of the functions other students had used differed. Most students, however, expressed exasperation that they had not noticed what was now an obvious feature of the plotted data. These students then returned to the task and were able to develop a piecewise function that fitted the data for themselves. A small minority of students needed more direct help from the teacher but were also able to develop a model based on a piecewise function by the end of the lesson. The lesson concluded when the teacher asked the students to work further on their assumptions and limitations for homework.
The CSIRO has been monitoring the rate at which Carbon Dioxide is produced in a section of the Darling River. Over a 20 day period they recorded the rate of CO2 production in the river. The averages of these measurements appear in the table below.
The CO2 concentration [CO2] of the water is of concern because an excessive difference between the [CO2] at night and the [CO2] used during the day through photosynthesis can result in algal blooms which then results in oxygen deprivation and death of the resulting animal population and sunlight deprivation leading to death of the plant life and the subsequent death of that section of the river.
From experience it is known that a difference of greater than 5% between the [CO2] of a water sample at night and the [CO2] during the day can signal an algal bloom is imminent.
Rate of CO2 Production versus time Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Time in
Hours
0 1 2 3 4 5 6 7 8 9
Rate of CO2
Production
0 -0.042 -0.044 -0.041 -0.039 -0.038 -0.035 -0.03 -0.026 -0.023

Time in
Hours
10 11 12 13 14 15 16 17 18 19
Rate of CO2
Production
-0.02 -0.008 0 0.054 0.045 0.04 0.035 0.03 0.027 0.023

Time in
Hours
20 21 22 23 24
Rate of CO2
Production
0.02 0.015 0.012 0.005 0
Is there cause for concern by the CSIRO researchers?
Identify any assumptions and the limitations of your mathematical model.
Figure 2 : Algal Bloom Problem
Discussion and conclusion
This task satisfies each of the principles developed during the project for mathematical modelling tasks and for the use of digital tools within tasks. The use of modelling tasks and digital tools are consistent with mandatory requirements of the relevant state syllabus. As national scientific bodies monitor the blue-green algae in the various river systems because of the effect on aquatic wildlife this represents a task set in a near authentic life-related context. The task is open-ended in that a variety of mathematical models are plausible and the use of different models will lead to different, but still valid, responses to the problem. The available digital tools provided the facility to trial a range of functions to fit a complex underlying pattern and offered immediate feedback on the appropriateness of a conjectured function allowing students to develop specific solutions from a wide range of possibilities.
Different types of mathematics were necessary to explore the data (data representation, different forms of function) and so, students were expected to make connections to different types of mathematical knowledge. The available technology provided the option of viewing different types of mathematical representations (e.g., scatterplots and function graphs) on a screen at the same time, so enhancing the connecting between these types of mathematical knowledge. Students found the task to be accessible as it linked to mathematical knowledge they had studied in previous classes and the teacher made use of progress made by other students to provide a prompt when many were experiencing difficulty. The opportunity to trial a function against the data and receive immediate feedback provided an entry point to most students and so made the problem accessible. As the task required students to make use of mathematical knowledge they had already studied in previous lessons within an unfamiliar context it provided opportunity for students' development in mathematical knowledge and their capacity to apply this knowledge in real world contexts. Here, digital tools acted as a catalyst for this development by providing feedback which indicated students' first single function conjectures were not consistent with the data. As outlined above, there is an inseparable interplay between the task and digital tools. The teacher has created the task by drawing on principles for developing effective technology active modelling tasks. These principles are based on the Theme A – V. Geiger & T. Redmond
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potentialities of both types of resource – the task and the digital tool. In implementing the task, the teacher anticipated how students would interpret the potentials of the task for learning and of the digital tool to act as a resource. The relationship between student, teacher, task and digital tool represents a documental genesis as each element within this genesis transforms the other in some way. The task is transformed, from the perspective of the students when they realise they need to make use of a piecewise rather than a single function in order to model the data presented in the problem. This transformation occurs as a result of an attempt by the students to use a single function and receiving feedback via the digital device that this was an inappropriate model.
The use of the digital tool changes from that of a device that provided a specific solution for students once they had made a decision on the general form of the function to model the data into a tool used to explore the data and eventually find a model that fitted the data to their level of satisfaction. Students' learning is also transformed during this same process as they realise the purpose of the task and the digital tool is not to algorithmically implement prior learning but to apply their knowledge and understanding in an original way. The teacher had to transform his approach to the lesson when students took a path he had not anticipated – attempting to fit a single function to the data. He changed his approach by orchestrating the resources at his disposal, in this case the two students who had eventually solved the problem, to provide an insight into the problem other students were yet to see.
At the same time, nearly all of the teacher's principles of design, the characteristics of effective modelling tasks, acted as enablers of the process of instrumental genesis of both digital tools and of the task. The principle of authenticity and relevance requires students to recognise the potential of the available digital tools to assist them in exploring and solving the problem described in the task from within both purely mathematical and real world contexts. There was a necessary duality about the schemas of instrumented action required to accommodate the purely mathematical and contextual demands of the task. Students needed to recognise that the real world context demanded the development of a piecewise rather than single function to model the inhalation and exhalation of CO2. This required a specific use of the digital tool that was different from the development of a single function to model the provided data. Having decided that two functions were needed to model the data, a specific instrumentation of the digital tool was needed to find the most appropriate functions for each section of the piecewise function. This second process takes place within a purely mathematical context.
The open-endedness of the task placed students in a position where they were challenged to make choices among multiple potential solution pathways. Thus, students were required to make choices among existing schemas of instrumented action or to generate new schemas. To generate new schemas students must firstly recognising the potential of the digital tool for meeting the challenge defined by the task and then, secondly, develop processes for the use of their digital tool that are specific to the set task.
The principle of connectivity designed into this task required students to generate schemas of instrumented action that were inclusive of different types of mathematical content. The CAS active calculator students used while working with the task included the capacity to link statistical plots with the graphs of specific functions, and these functions could be developed using the regression facility of the calculator. With these facilities available, students needed to find ways of taking advantage of the capabilities of their digital tool in engaging with the demands of the Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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task and pursuing a solution. This is a type of instrumental genesis in which the potential of an artefact is only realised through its instrumented action.
The task was designed to link the demands of the activity to students' previous learning as the separate functions required to build an appropriate piecewise function had been studied and applied to real world contexts in earlier classes. Thus, the task was created to be accessible to students but, at the same time, required students to apply this previous learning in a more complex context – one in which multiple functions were needed to model a phenomena rather than a single function. This meant that students' existing schemas of instrumented action required adaptation in order to accommodate a more complex scenario. The CAS enabled calculator was the tool the teacher believed would mediate this adaptation through the provision of a medium that provided for the representation of multiple functions against complex data.
The development aspect of the design is most apparent in the way the way the teacher invited the pair of students who had found that an appropriate solution required a piecewise function to offer their solution to the whole class and the subsequent realisation by most of the class that this was an insight they had missed.

This revelation changed both the ways in which these students used the available
digital tools and also the way they viewed the task. In this circumstance the teacher
orchestrated changes in students' schemas of instrumented action related to both the
digital tool and also the task
The episode included in this paper demonstrates it is possible to design for
effective technology demanding mathematical modelling tasks, and so the approach
offers direction for curriculum designers, teachers and teacher educators. While the
teacher had designed an engaging task based on principles developed during the
project, students took an approach that was not anticipated by their teacher. The
teacher, however, was able to take advantage of students' original but inappropriate
approaches, generating a dynamic learning environment where students' knowledge
of using mathematics within real world contexts was transformed. This raises a
challenge for teachers in how such triggers can be deliberately embedded in planned
learning experiences in a way that provides space for the type of documental genesis
described in this paper. This also indicates that further research is necessary to
investigate how to take advantage of unanticipated events in a well planned lesson
and in turn for how teacher educators provide advice about task design and
implementation in pre-service and in-service programs.
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Optimizing through geometric reasoning supported by 3-D
models: Visual representations of change
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The representations, tools and task discussed were designed as a
response to a pedagogical challenge: How can rate of change be
investigated meaningfully at stages and ages below calculus? The task
involves 3-D models of volume and surface area, to represent and explore
differences in volume between pairs of open-topped boxes, as well as an
exploration of a 2-D representation with dynamic geometry software.
Reasoning during the task, with the affordances offered by the tools,
learners as young as 14 were able to decide when and why an optimum
volume was reached. Through the lens of conceptual blending, we discuss
what mathematical insight, activity, and understanding is available to
learners via engagement with our task. We further suggest that the
representation of change developed through this task extends as an
accessible spatial reasoning technique applicable in a range of other
problems in 2-D and 3-D.
Key words: task sequence design, optimization, visual spatial
reasoning, 3-D model exploration, dynamic geometry software
Background
In this paper, we report on research about the design of a tool and task
intended to develop concepts of change and rate of change with secondary students,
pre-service and in-service teachers. The mathematical focus of the task is on
geometric reasoning – developing the concept of rate of change through visual spatial
reasoning, without reliance on calculation or computation. We will illustrate how
different types of tools can afford different mathematical activities, representations,
and interactions between representations, as well as how specific tools – in our case 3-
D models supported by dynamic geometry software (Geometer's Sketchpad) – can
impact student learning and understanding of mathematics. Our understanding of
'task' in this case is in line with the definition offered by Theme A: a teacher
designed purposeful 'thing to do' using tools for students in order to activate an
interactive tool-based environment to produce mathematical experiences. This task,
developed by Whiteley and researched by Mamolo and Whiteley, offers a means to
address the following problem: Theme A - W Whiteley & A. Mamolo
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The Popcorn Box Problem
Given a square sheet of material, cut equal squares from the corners and fold up
the sides to make an open-top box (see Appendix A). How large should the square
cut-outs be to make the box contain maximum volume?
A version of this optimization problem was initially developed by the
Ontario Association for Mathematics Education (OAME, 2005) for grade 9 and 12
students (ages 14 and 17), as an introduction to optimization prior to calculus,
primarily with numerical calculations to compare overall volumes of boxes. Our task
affords an avenue of richer investigation of this problem through visual and
kinesthetic reasoning prior to the development of algebraic/symbolic reasoning of
calculus. It includes the use of 3-D models (pictured in Appendix B) made from clear
plastic sheeting and pieces of coloured foam, as well as a dynamic geometry
exploration via Geometer's Sketchpad (sample screens of which are provided in
Appendix C and D). The original tool, task, and an associated novel representation
presented here were designed as a 'proof of concept' that deep, effective reasoning
with change, and rate of change, could be enabled for students prior to symbolic
manipulation and the algebraic techniques of calculus. This was part of the response
by one of the authors (Whiteley) to a challenge during recent curriculum writing in
Ontario: Can we being reasoning about 'change' down to earlier years, with minimal
algebraic load and a focus on big ideas? In addition, do these tasks support unpacking
of concepts of change and optimization by pre-service and in-service teachers?
In what follows, we outline the design principles and development of the
task, highlighting the mathematical epistemological goals and principles to the design,
as well as the pedagogical considerations and modifications which resulted from
implementing the task with diverse sets of learners. These considerations speak to
how different types of tools may afford different learning possibilities for learners –
providing them with different types of experiences and activity, as well as different
ways to represent ideas and concepts. We then go on to address how experience with
this task may impact learners' understanding of optimization and, more generally,
problem solving. We use as a lens of analysis the framework of conceptual blending
developed by Fauconnier and Turner (2002), and offer suggestions of different
possible blends afforded to different participant groups from secondary school pupils
to their teachers.
Design Principles and Task Development
The initial task was to visually/spatially reason about the optimum shape of
an open-topped box (see Whiteley & Mamolo, 2012 for details of the task), enriching
a simple investigation geared for pre-calculus students. This investigation required
developing a new tool of 'paired boxes with physically represented changes' (see
Appendix B). Working with this tool, with a variety of participants from in-service
teachers and curriculum writers, through pre-service teachers, senior high school
students and students just completing elementary school (14 year olds) pushed the
development of the task and associated tools and representations in multiple
directions. In this section we describe specifics of the task, how participants engaged
with the tools, and the emergent principles which informed the task design.
The 3-D models included manipulative tools of paired boxes, with inserts of
foam for volume lost and gained (see Appendix B). The tools were designed, and redesigned after testing, to encourage a visual spatial way of reasoning about rate of
change that could emerge naturally via their use. The task was designed to scaffold Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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from exploring changes in volume with physical models, into reasoning supported by
visual representations through the dynamic geometry software GSP (see Appendix C),
and finally to the important idea of how geometric features can identify non-optimal
objects (leaving the optimal objects as the only remaining choice!).
Testing the design
Given that there were analogous examples in both 2-D and 3-D, and that the
physical models were more difficult to make for the 3-D situation, in the initial pilot,
with a group of classroom teachers, we first presented the 2-D problem in a GSP
sketch. The problem involved optimizing the area of a rectangle along a fixed barrier,
given a fixed perimeter (the fence on the river) and participants worked in pairs on the
computer to solve this. Then a demonstration with the physical model (Appendix B)
was presented, during which attention was directed to the materials filling in
'between' the two boxes as physical representations of the change in volume (and not
the size of the particular individual volumes). The participants took turns examining
the models, taking the 'change in volume' pieces out and comparing them by
overlaying the pieces of foam, and concluding which volume was larger. The
response of these adults was that the 3-D model was more effective in directing their
attention to the change in volume, than the analogous 2-D model had been. As a
result, further models were constructed so that every pair of learners could explore a
pair of physical boxes for themselves.
All subsequent testing and teaching with the task has started with the 3-D
activity, and occurred with high school students (14 and 17 years old), pre-service
teachers, in-service teachers. The novelty of this setting invites more attention to the
physical representation of the change in volume, and less attention to efforts to
convert the problem to symbolic or numerical formulas. The focus on spatial
reasoning provoked discussions of important curricular ideas and the accessibility of
mathematical concepts when lifted from their symbolic representations. In addition,
the unusual appearance of the pairs of models focussed the participants' attention on
key built-in 'errors' to the first-approximation of the change in volume (such as the
visibly missing 'corner volumes', see Appendix B) and away from incidental defects
in the cutting and model construction. This particular feature of the tools (the missing
volume in the corners) triggered a discussion of the roles and implications of
estimation and refinement in a meaningful and concrete way. This discussion led to
the suggestion that the size of the difference in cuts could be reduced by using
different materials – e.g. using Bristol board in addition to foam allows exploration of
thin differences in cut-size (where the Bristol board illustrates instantaneous rates of
change, while the foam illustrates average rates of change).
A bonus of starting with the 3-D tools was that participants made an initial
guess about the optimal shape (the echo of folklore that "the cube has the maximum
volume for a given surface…"). This was quickly found to be incorrect, just through
participants' own viewing of a pair of models. We found during these testing stages
that participants approached the task with the sense that there was a puzzle to be
sorted out that they 'owned', and were now motivated to understand why and what
the maximum might be. In addition, we found that younger students who are less
channelled into 2-D situations, brought stronger memories and naïve intuitions about
3-D settings, were less distracted by one-variable algebra (or any algebra), and posed
novel questions for exploration. As a result they were more willing to engage in the Theme A - W Whiteley & A. Mamolo
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'play' that got them started along a trajectory reasoning with the affordances built into
the models.
Mathematical ideas represented in the task
Via engagement with the task and models, several mathematical ideas and
observations emerge in a manner accessible to learners of various ages and
mathematical sophistication. Here we identify a few key ones:
i.Volume and surface area can be physically represented.
ii.Change in volume is equal to surface area for the gains minus the surface
area for the loss, and these changes in volume (loss and gain) between
pairs can be compared qualitatively by naïve overlay strategies (see
Appendix B).
iii.Reasoning about 'change of volume' shifts to comparing surface areas and
recording the loss and the gain – with a visual focus, not with numerical
calculation.
iv.If the surface area of the loss from one box does not equal the surface area of
the gain of the other box – i.e. the pieces of foam do not overlap
completely – then the volume of the smaller box is clearly not the
optimum (i.e. we have a non-zero average rate of change).
v.Equivalently, the optimum occurs at the shape when the Bristol board

representing 'volume lost' completely covers the Bristol board
representing 'volume gained' (i.e. we have a zero instantaneous rate of
change) - which occurs by creating a box with a "corner cut" of 1/6 of
the way along an edge (or 1/3 of the way to the centre).
vi.Errors are reduced, and then vanish, as the size of the changes in the cut
corners is reduced (a hands-on way of passing to the limit and
representing this limiting value).
vii.The invariance of the optimal shape under scaling (proportional reasoning)
for the optimum is immediate to people working with this tool.
viii.Geometric optimization of other shapes can be explored by the 'geometric
loss-gain' variational paradigm.
ix.For students taking calculus, these steps in the task can then be connected to
algebraic reasoning, including a model-based introduction to the basic
conceptual layers of differential calculus (e.g. average change or secants
(thick foam); small differences as limits (Bristol board); optimal volume
as rate of change equal to zero).
x.For pre- or in-service teachers, this exploration supports unpacking of the
process steps and the big ideas of optimization with differential calculus.
Extending the Task
To the designer, and to the teacher, other questions for investigation arise.
Here are some that we have explored with at least some of the groups of participants:
a.Is there a 2-D analogue? Yes: a 2-D variant starts with a fixed perimeter
seeking maximum area, e.g. the "fence on the river" problem where three
sides of a rectangle must be fenced to produce a maximum area, or
variants;
b. A 3-D qualitative variation: Does the volume get bigger if we tilt the
sides of the box out (as is done in theatres)? Yes: initially the volume Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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increases – explored by a cross-sectional comparison, supported by
dynamic geometry software (see Appendix D);
c.What if we cut corners from a triangle, or a hexagon or…? (There is a
delightful general principle: if the edges of the paper are all tangent to an
in-circle – cut 1/3 of the way to the centre!);
d. If prompted, a deeper puzzle arises within the more common 2-D variant
with all four sides included: Why would the plane shape giving
maximum area when constrained by a fixed perimeter (a square) also be
the shape with a minimum perimeter when constrained to a fixed area?
Interestingly, people anticipate this duality will work ("it sounds right!")
- but the supportive reasoning, linked to a fundamental duality principle
in operations research, requires further shift in representation to points
on an area/perimeter graph, accessible only to the most adept problem
solvers.
We are reminded that problems become 'similar' in the mind of the problem
solver when the representation, or more generally, the reasoning, is transferred. One
key to transferring this geometric 'change in pairs' or 'geometric loss-gain' reasoning
is to provide other problems where it can be applied either quantitatively, or
qualitatively. The above list provides some starting points for exploring this transfer.
In the remainder of the paper, we turn our attention to how engagement with
our task can impact student learning and understanding of mathematics. For our
analysis, we use the framework of conceptual blending (Fauconnier & Turner, 2002),
and offer different possible blends that emerged from participants' engagement with
the task. The blends afforded differ based on learners' mathematical background and
sophistication, along with their own learning goals. We highlight some of the
important blends available for a variety of learners.
Conceptual Blending: A Framework for Analysis
Conceptual blending (Fauconnier & Turner, 1998, 2002) is a theory which
describes how new inferences can arise when two representations and associated ways
of reasoning (or 'input spaces') are brought together in a 'blended concept'. The
"blend" can be thought of as a mapping which combines certain features of the two
input spaces and projects them onto a third (newly formed) mental space. (In the
blend, other features are not mapped, shifting the focus and reducing the cognitive
load for further reasoning.) Blending processes are used to conceptualize actual things
such as computer viruses, fictional things such as talking animals, and impossible
things. Although blends may sometimes be bizarre, "the inferences generated inside
them are often useful and [can] lead to productive changes in the conceptualizer's
knowledge base" (Couson & Oakley, 2005, p.1513). Blending is not a metaphorical
or analogical map, but rather it is a specific way to combine and infer from and about
information from two or more mental spaces (Fauconnier & Turner, 2002). The
partial representations from an individual's perceptions and concepts that are
contained in the prior mental spaces blend by "the establishment and exploitation of
mappings, the activation of background knowledge, and frequently involve the use of
mental imagery and mental simulation" (Couson & Oakley, 2005, p.1513).
The emergent blended space arises in three ways: "through composition of
projections from the inputs, through completion based on independently recruited
frames and scenarios, and through elaboration" (Fauconnier & Turner, 2002, p. 48,
emphasis in original). Specifically, composition creates new relations not previously Theme A - W Whiteley & A. Mamolo
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existent in the separate input spaces, while completion allows the composite structure
in the blended space to be thought of as part of a larger structure in the blend, and
elaboration, or "running the blend" consists of cognitive work performed within the
blend to exploit and elaborate upon the composite structure (Fauconnier, 1997, p.150-
1). The blend continues to offer the individual ways to access each of the original
representations, in a flexible manner. For instance, in our case, the three input spaces,
are (i) the word problem of maximum measured volume, (ii) the classical approach to
optimization through calculus and rate of change (often held as a symbolic procedure
with formulas), and (iii) the spatial reasoning (i.e. the variational exploration of
change in volume represented physically with pieces of foam, and then Bristol board
see Appendix B). Such a blend would allow an individual to:
i.compose - to explore change physically by starting with a pair of
representative examples (e.g. boxes) and focus primarily on the change
in volumes (ΔV for the secant in symbols);
ii.compose - to conceptualize change in volume physically by considering
surface area for loss and gain (ΔV in symbols, pieces of foam in the
model);
iii.complete - to focus on the sign of the change, with a simple physical
comparison via foam inserts, to determine which changes will make the
volume larger;
iv.complete - to consider sources of error – and minimize them (an informal
invitation to a limit process, which is natural in the physical model);
v.elaborate - to consider what boxes cannot be the optimum and why (rate of
change is not zero in both the models and the symbols);
vi.elaborate and combine - to eliminate the boxes with non-zero rates of
changes and determine the single box shape which remains as the
optimum (in both the model and the symbols).
We note that a blend is both an internal cognitive process and a cultural
artefact. Once achieved by someone and shared – this new blend becomes a possible
cognitive approach requiring less cognitive load for others who have the appropriate
parts to develop their own internal blend. Our task provides support for such
transmission because the tools are external and the task focuses attention on key
features listed above. The use of an external representation further supports shared
conversations about the blended concept, the process and the reasoning.
The original background to such model-based visual reasoning was scattered
reflections by Whiteley which evolved over several decades of classroom teaching
practice, observing patterns of deep, but often hidden, connections between: perimeter
and area; and between surface area and change in volume, trying to anticipate why at
answer would be what the symbolic computations produced. The goal was to have
both a symbolic answer and a visually sensible answer that matched! Implicitly – the
key blend was being developed and repeatedly explored in pedagogical contexts.
However, that form of reasoning was not part of shared experiences or curricular
practices, and therefore was not accessible, or even communicable, without the
external support provided by our task. The physical models as tools and
representations provided an essential support for communicating, for refining the
representation, for reasoning, and then for embedding this one clear example into a
widening mix of problems and solutions that fit with the new representation, and the
larger conceptual blend. Whiteley (2011) provides a more extensive discussion of
how the framework of blending gives insight into this modelling and model building
process, illustrated with the task presented here. We turn our attention now toward Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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different blends created by prospective and practicing teachers, university students,
and secondary school students.
Emergent blends from engagement with the popcorn box task
The representation of change and optimization in the model(s) is 'sensible'
to a wide variety of problem solvers. The key steps with the model are similar for a
very wide variety of users. The conceptual blends and the context, however, are very
different:
a.For practicing teachers, we interpret both a completion and composition of
a blend. Specifically, the task allows an opportunity to un-pack the basic
processes of calculus and optimization in an unusual context, supporting
careful reflection on the steps (completion), and an alternative visual
way to reason through the problems (composition). The teachers create
a new blend between a well-grasped symbolic sense of the processes of
calculus, and a novel spatial sense of optimizing in geometric problems.
The 'aha' moment when the 'loss-gain' representation is also found in
the symbolic derivative set to zero is exciting to witness.
b. For prospective teachers, a similar situation arises. The task is also an
opportunity to un-pack the concepts of calculus, and invites reflection to
develop a more flexible and "thicker" conceptual basis for the study of
change and optimization (completing a blend). Individuals are creating
(composing) a new blend between a sometimes fragile symbolic sense of
the processes of calculus, and a novel spatial sense of optimizing and
checking optima in geometric problems, which will strengthen both
(elaboration).
c.Alternatively, for practicing and prospective teachers, the geometric
paradigm available through our task can be blended with the prior
formula-based procedural calculus knowledge, to re-infuse it with sense
and visual estimation that grounds the algebraic solutions to geometric
optimization (elaboration and completion).
d. For senior secondary students or university students starting calculus, or
doing an initial review, the task affords a blend where key processes are
experienced twice, in two representations, supporting an initial
development of both procedural steps and conceptual reasoning.
Students are creating an "immediate" balanced blend which supports
flexible approaches to solving problems (composition and completion).
e.For younger secondary students, even down to age 14, this task provides
an accessible tool to solve problems they had heard about but had not
been able to approach (e.g. why does a square have the maximum
volume for a fixed perimeter?). The task affords a reasoning power, and
a representation, which solves optimization problems and lays an initial
foundation for a later mature blend with the second symbolic / algebraic
approach, with sufficient parallels in the reasoning to make the calculus
'make sense' (composition).
Concluding Remarks
The task and task design are intimately linked to a novel visual
representation of loss-gain in geometric optimization, and to reasoning processes that
provide an alternative to symbolic manipulations – an alternative accessible to young Theme A - W Whiteley & A. Mamolo
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students. The task and activity emergent from the task can be analysed via the lens of
conceptual blending, which can inform the teacher/researcher about both the
conceptual development of his/her students as well as the possibilities or deficiencies
of the tools and models. The task also offers a context in which the actions and
expressed reflections of students/research participants can give insight into spatial
reasoning, and the possibility of new rich blends, given initial procedural symbolic
ways of solving problems. The task was carried out in pairs (and then debriefed in
larger groups), which encouraged development of shared blends, and allowed the
researchers to observe more of the student activity and reasoning. There are many
questions that arise when observing how participants act (and fail to act) with the
materials, which can lead to an enriched understanding of what mathematical
experiences are necessary for the learner.
The tool representing loss and gain, the associated visual representation of
volume changes and the associated task provided a way "to change what we see"
(Hoffman 1998) and "how" we see. With the directed "seeing" flowing from the
affordances in the task, such as the "missing corners", the colour coded pieces of
foam, and the clear plastic sheeting. This change in "seeing" / change in "thinking"
would otherwise not have been accessible to, or at least noticed by, most participants.
With practice, the problem solver internalizes the representations, and can reason
"spatially" in the mind – imaging and imagining situations, and changes, which carry
on the reasoning internally. In the language of Fauconnier and Turner (1998, 2002),
the task affords an opportunity for composing a blend from a physical input space –
the models – and a mental input space – the procedural knowledge of calculus – to
create a newly formed understanding of optimization. This may allow completion or
elaboration of a blend for individuals well-versed in calculus, such as prospective and
practicing teachers. In the blended space, each "visual step" in the task has an
algebraic representation, and each key step in algebraic reasoning has visual support
to confirm that the "solution makes sense". In our view, variants of this intended
blend support a flexible problem solving cycle.
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Appendix A: The popcorn box problem:
(A) The problem being explored is a classical optimization / calculus
problem:
Given a square sheet of material, equal squares are cut out of the corners, and
the sides are folded up to make an open topped box. Which size of corners should be
cut out in order to make the maximum volume? Why?
Appendix B: The 3-D Models
Pairs of clear plastic 'popcorn boxes' and physical representations of loss and
gain in volume (blue/purple and red pieces of foam, respectively):
(i) Two pairs of popcorn boxes with foam inserts representing changes in
volume
(ii) Foam inserts removed and compared via overlaying (iii) 'Missing corners' source of error
in (right)
Appendix C: GSP Screen of
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(iii) 'Missing corners' source of error – volume missing (left) and then filled
Appendix C: GSP Screen of solution method - supplemented by graph Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22
Appendix D: Extending the exploration to make predictions about
boxes with 'tilted' sides
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Appendix D: Extending the exploration to make predictions about
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Appendix D: Extending the exploration to make predictions about Theme A - W Whiteley & A. Mamolo
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Instrumental value and semiotic value of ostensives and task
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This text presents, by means of various examples, how the
notions of instrumental value and semiotic value of an ostensive,
developed in Anthropological Theory of
design or analyse mathematical activity
design.
Keywords:
Anthropological Theory of
Introduction
This text presents a theoretical point of view about conception and
of task design. Our framework
(hereafter ATD).We thus consider
but also the techniques to
paradigm of this theoretical framework, the didactic facts and the mathematics facts
are the two sides of the same coin. Thus, w
serves mathematical knowledge, and in the same way mathematical knowledge
impacts on situations (see Margolinas and Wozniak,
does happen and what could happen in the classroom from these different angles.
this text, we focus on a particular point
are used to do mathematics are considered as elements of constraints and conditions
of the mathematical activity.
In a "classe de cinquième de college" in France
and thirteen years old -, a teacher asks pupils to build a perpendicu
given line segment [AB]. He obtains two kinds of constructions (see figure 1).
Figure 1- Constructions of the perpendicular bisector of the given line segment [AB]
Coming with the required line, several features appear: small parallel
two parts of the line segment [AB],
the perpendicular bisector and the line segment for the first drawing or arcs for the
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Instrumental value and semiotic value of ostensives and task
design
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This text presents, by means of various examples, how the
notions of instrumental value and semiotic value of an ostensive,
developed in Anthropological Theory of the Didactic, make possible to
mathematical activity , in particular tool-based task
Keywords: Instrumental value, semiotic value, ostensive, non
Anthropological Theory of the Didactic.
This text presents a theoretical point of view about conception and
Our framework is the Anthropological Theory of
thus consider the mathematical activity in a large sense: the task,
but also the techniques to perform it and all the discourses about it.
paradigm of this theoretical framework, the didactic facts and the mathematics facts
are the two sides of the same coin. Thus, we consider that didactical knowledge
serves mathematical knowledge, and in the same way mathematical knowledge
impacts on situations (see Margolinas and Wozniak, 2012). Therefore we look what
does happen and what could happen in the classroom from these different angles.
we focus on a particular point about the observed activities: the objects that
e used to do mathematics are considered as elements of constraints and conditions
of the mathematical activity.
In a "classe de cinquième de college" in France – children between twelve
, a teacher asks pupils to build a perpendicular bisector of a
segment [AB]. He obtains two kinds of constructions (see figure 1).
Constructions of the perpendicular bisector of the given line segment [AB]
the required line, several features appear: small parallel
two parts of the line segment [AB], a small square and a cross with the intersection of
the perpendicular bisector and the line segment for the first drawing or arcs for the
Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
Instrumental value and semiotic value of ostensives and task
This text presents, by means of various examples, how the
notions of instrumental value and semiotic value of an ostensive,
Didactic, make possible to
based task
Instrumental value, semiotic value, ostensive, non-ostensive
This text presents a theoretical point of view about conception and analysis
Anthropological Theory of the Didactic
the mathematical activity in a large sense: the task,
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segment [AB]. He obtains two kinds of constructions (see figure 1).
Constructions of the perpendicular bisector of the given line segment [AB]
the required line, several features appear: small parallel lines on
the intersection of
the perpendicular bisector and the line segment for the first drawing or arcs for the Theme A – F. Wozniak
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second one. These graphic ostensives evoke or even "say", the geometrical properties
used to build the perpendicular bisector of the segment. In the left-hand drawing, the
pupil has traced the perpendicular with the segment passing by his midpoint, whereas
in the right-hand drawing, the second pupil has built the whole points equidistant
from the ends of the segment.
These graphic ostensives do not evoke only the geometrical properties that
they incarnate but also others ostensives, materials now: graduated ruler and set
square in one case, ruler (not graduated) and compass in the other case. Resorting to
the ostensives is thus controlled by objects that have no materiality, no perceptibility,
but that are only evoked by the ostensives. In return, depending on the
characterization of the perpendicular bisector used by the pupil, he uses different
geometrical tools and draws lines and signs characteristic of the properties used. The
ostensives embody and reveal the property of the perpendicular bisector underlying
the geometrical construction. Thus, the concept of ostensive is linked in a
consubstantial way to the concept of non-ostensive - notion, concept, idea - which
cannot be handled but only evoked through the associated ostensives. As Chevallard
(1994) emphasized, every implementation of a technique assumes the resort to two
kinds of objects: ostensives and non-ostensives.
We can see through this small example how the tools used to perform a task,
but also the way they are used, lead to a different mathematical activity. Therefore,
the didactical knowledge on the relationship between ostensive and non-ostensive
objects is essential for the teacher when he designs a mathematical activity. In this
example, if a teacher wants the pupils to use a geometrical property instead of
another, he privileges some tools rather than others. Thus, the first stage for
understanding and analysing the task-design, teaching, learning triad is to reveal
ostensives and non-ostensives in each technique used to perform a task. The notion of
ostensive was introduced in Anthropological Theory of the Didactic, (Chevallard
1994, Bosch 1995, Bosch & Chevallard 1999) to describe and analyse the
mathematical activity. A very broad definition is: "One calls ostensives, the objects
which have for us a material form, sensitive, notwithstanding unspecified"
(Chevallard, 1994, p.4, our translation). Therefore, any object which can be handled
concretely by the body, the voice, the vision is an ostensive: material object, gesture,
language, diagram, drawing, graphics, formalism, etc. Then non-ostensives are the
objects which "cannot, strictly speaking, be handled: they can only be evoked,
through the handling of ostensives associated." (Chevallard, 1994, p. 5, our
translation). Thus for example, the concept of quantity related to a thumb, an index
and a major raised is a non-ostensive that is not accessible by perception apart from
its representation by an ostensive like the fingers which were raised, the figure "3",
the word "three", or a drawing like "". Consequently any work of conceptualization
seems the fruit of a work on ostensives which gives access to non-ostensives. The
analysis and the design of a task to teach and learn mathematics go through the
identification of everything that makes signals and "speaks" in the situations, i.e. the
whole ostensives and associated non-ostensives.
The aim of this text is to present how in ATD the artefacts are analysed in
their two dimensions of tool to act and tool to think within task design for the
teaching and learning of mathematics. In the first section we come back on the
concepts of ostensive and non-ostensive as element for describing the mathematical
activity from ATD point of view. In the second section, we clarify the concepts of
instrumental value and semiotic value of ostensive starting from an example; these are
essential concepts for the design and the analysis of mathematical activity. In the third Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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section we show how an ostensive can stop being a tool for taking an action because
the non-ostensive that it evokes is not operational and does not make sense any more.
Then, in the fourth section we show how the *choice* of an ostensive for designing a
task is based on a dialectical play between the instrumental and semiotic values.
Lastly, we will show how the notion of praxeology, developed in ATD, can be a
useful tool to understand the relationship between ostensives and non-ostensives
evoked in a mathematical activity.
1. Ostensives, institution and praxeologies
Let us consider now the following problem
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: if 8 lollipops cost 10 €, how
much are 3 lollipops? Until the middle of the 20th century in France, the expected
answer was based on the theory of the ratios and proportions: "8 is to 10 as 3 is to x".
It results symbolically in writing the proportion "8: 10:: 3: x" and is calculated by
using the fact that the product of the extremes terms is equal to the product of the
middle terms: 8 × x = 10 × 3 thus x = 10 × 3/8 = 3.75. In the 1970's, the reform "of
modern mathematics" has modified the curriculum. Then, the modelling of the
situations of proportionality by a linear function is expected: if f(8) = 10, then f(3) =
f(3/8 × 8) = 3/8 × f(8) = 3/8 ×10 = 3.75. In the 1990's, such problems are solved with
"the cross-product" starting with writing the values in a "table of proportionality" (see
figure 2): x = (10 × 3)/8 = 30/8 = 3.75.
Number of lollipops 8 3
Price of lollipops (€) 10 x
Figure 2 – Table of proportionality
Nowadays, at the primary school the pupil must use the "rule of three" which
uses the return to the unity: if 8 lollipops cost 10 €, then 1 lollipop costs 10 €/8 =
1.25 €. The price of 3 lollipops thus equals 3 × 1.25 € = 3.75 €.
Thus, the techniques used to solve a task and the discourses that justify them
depend on the institutions within which they are used. However the institutions
generate practices and discourses on the practices that are specific because, as
Radford (2002) underlines, the artefacts/ostensives contain within themselves the
culture of the institution that produces them but also the culture of the institution that
resorts to them.These are the two founder points of the Anthropological Theory of the
Didactic. The model of the mathematical activity can be built in terms of praxeologies
within institutions, like any other human activity. The personal relationship with an
ostensive is regulated by the institutional relationship with this ostensive, inside the

institution where it has been activated. In ATD, a praxeology is structured into two components. The praxis component contains techniques to achieve a kind of tasks. The logos component includes the theoretical discourses, called technology, that describe, explain, justify or develop the techniques used, and the justification of technology, called theory. Thus, the mathematical activity is carried out through the handling of ostensives whose instrumental value is perceived through the praxis component whereas the semiotic value nourishes the logos component of praxeologies.

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We borrow the idea of this example from Chevallard (1994). Theme A – F. Wozniak
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2. Instrumental value and semiotic value of an ostensive
In a "classe de CP" of a primary school – children between six and seven
years old – pupils try to calculate the result of the addition "12 + 6". A pupil counts
12, 13, 14,.. 17, 18 and raises an additional finger each time. The fingers that raise
during the counting constitute a gestural ostensive. Raising a finger at the same time a
number name is enunciated allows controlling how much numbers names were
enunciated. Thus the gesture makes the achievement of the task possible: it is an
element of the implementation of the technique. But the fingers that raise gradually
say what the pupil does: he continues the counting from 12. The same ostensive, the
gesture of the fingers raising gradually, has two values: an instrumental value that
expresses what has been carried out by the ostensive and a semiotic value that allows
seeing what is made by the ostensive and reveals the evoked non-ostensive. In the
same way, the number name enunciated when the fingers raise is a linguistic
ostensive which lets us see (or rather understand) the technique used by the pupil: the
counting. But the declamation of number names has also an instrumental value: the
last number name enunciated is the result of the addition. The work made by the pupil
on the words is an element of the technique of addition. Thus the language does not
have only a communication function for describing, for instance, a technique. It is
also an ostensive like another: it is simultaneously a tool which makes possible to
achieve a task and makes apparent the meaning of what is made.
Instrumental and semiotic values of an ostensive "appear, within a given
technique, linked like the recto and back of a sheet of paper" (Chevallard 1994, p.6,
our translation). The mathematical activity is carried out through the handled
ostensives and of the non-ostensives thus evoked. Then, a didactic analysis, in ATD,
takes into account what the ostensives make possible to do, their instrumental value,
and what they let us see of the work done, being done or to be made, their semiotic
value. In this example, we have seen how the efficiency of the process gives sense to
what has been done: the instrumental value of an ostensive nourishes its semiotic
value. With the next example, we observe the case where the loss of sense makes an
ostensive loses its instrumentality.
3. The loss of instrumentality of an ostensive
In a "classe de CM2" of a primary school – pupils between ten and eleven
years old – pupils solve a proportionality problem (Wozniak, 2012). It is asked to
determine the height of a giant in an amusement park from a photograph
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of 16.1 cm
horizontally by 12 cm vertically (see figure 3).

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Copyright Richard Phillips (2001-2009): www.problempictures.co.uk Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22
England. One sees there part of the leg of a giant. Which is
To help pupils, the teacher distributes a docu
1. Observe the photograph. According to you, what is
man in cm?
2. Observe the photograph. What is
3. According to you, what is
4. Observe this table. Give a
information you have found
Calculate the size of the giant in cm, then converted this measure
suitable unit. Sentence
After discussions with the pupils, the
as being 180 cm, the foot of the giant 200 cm and the foot of a man 30 cm.
declares: "Look at this table, it could help you.
accurate calculation". After a
translation):
entire
Size for man 180 cm
Size for giant
Then pupils discover the real challenge of
teacher declares (our translation): "It was the first
continue this work on Monday
calculate the size of the giant
pupils do not allude to a proportional table,
Teacher: Perhaps you remember it.
several times with small
Pupil: Yes
Pupil: Yeah! On the sheet of mathematics. It is like the sheet we have made
there with… where there had written
Pupil: No. About strawberries.
Teacher: There you mix up
we have done a… we have made several small problems where it was necessary
to calculate the price of one kilo
6 kg. It is what is named
Thus it is the evocation of an ostensive and its use
proportionality – which reveals to pupils the kind of problems they solve. At the
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This photo was taken in an amusement park in
England. One sees there part of the leg of a giant. Which is
about the size of this giant?
Figure 3 – Proportionality problem
To help pupils, the teacher distributes a document (our translation):
photograph. According to you, what is roughly the size of an adult
photograph. What is roughly the size of the giant foot in cm?
3. According to you, what is roughly the measure of an adult man foot in cm?
4. Observe this table. Give a headline to each line and fill in them with the
information you have found.

Calculate the size of the giant in cm, then converted this measurement
suitable unit. Sentence-answer:
After discussions with the pupils, the teacher adopts the size of an adult man
180 cm, the foot of the giant 200 cm and the foot of a man 30 cm.
lares: "Look at this table, it could help you. Only with this table we could find a
fter a collective working time two answers are produced
entire foot man giant
180 cm 30 cm Size for feet 30 cm 200 cm
200 cm Whole size 180 cm 1200
Figure 4 – Tables made by pupils
Then pupils discover the real challenge of this collective work when the
teacher declares (our translation): "It was the first step, and from this table we will
on Monday and we will try to see how we can use
of the giant. What does this kind of table make you think
a proportional table, thus the teacher insists:
erhaps you remember it. Some time ago, a few weeks ago, we worked
small problems with tables which looked roughly like this
eah! On the sheet of mathematics. It is like the sheet we have made
with… where there had written Diams and all… tricks like that.
l: No. About strawberries.
mix up. It was a work about durations. You are right, Ouiam
we have done a… we have made several small problems where it was necessary
to calculate the price of one kilo of strawberries and then to find the price of 5 kg,
named a proportionality table.
hus it is the evocation of an ostensive and its use – resorting
which reveals to pupils the kind of problems they solve. At the
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ment (our translation):
the size of an adult
the size of the giant foot in cm?
the measure of an adult man foot in cm?
them with the
into the
the size of an adult man
180 cm, the foot of the giant 200 cm and the foot of a man 30 cm. He
this table we could find an
are produced (our
giant
200 cm
1200 cm
collective work when the
this table we will
can use this table to
make you think?" The
we worked
this one.
eah! On the sheet of mathematics. It is like the sheet we have made
durations. You are right, Ouiam
we have done a… we have made several small problems where it was necessary
ice of 5 kg,
resorting to a table of
which reveals to pupils the kind of problems they solve. At the next Theme A – F. Wozniak
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session, the teacher becomes even more precise to give some meaning to the numbers
arranged in the (proportionality) tables that pupils are obviously not able to make use
of: "How many times we need the size of an human foot to obtain the size of an
adult man? " (our translation). And to make sure of the understanding of the question,
the teacher makes a drawing. Finally, a solution is collectively found and the teacher
concludes: "To calculate the size of the giant, one takes the size of his foot and
multiplies it by six" (our translation). Then pupils must redo individually the work
which was made collectively by using only the photograph given at the beginning.
The analysis of these last writings allows noting that the pupils did not understand
what have occurred: one third answers nothing on the sheet and only another third
gives the good answer (with a badly filled out table for two of them).
If we report this episode, it is in order to illustrate how much an ostensive –
here, the table of proportionality – has lost of its instrumentality when it has lost its
semioticity. A single table of proportionality is not enough to solve the problem
given. However it seems that, for this teacher, the presence of a table of
proportionality alone must be enough to make understandable for the pupils that they
have to solve a problem of proportionality and how to proceed. Thus the teacher has
got rid of the genuine challenge of the learning: recognizing that a situation has to get
modelling by the proportionality. The table of proportionality can become an
instrument only if the pupil has recognized the situation of proportionality: the
instrumentality of ostensive also depends on its semioticity.
With these two last examples – add 12 + 6 or determine the height of a giant
– we observe that instrumental value and semiotic value are inseparable and how they
nourish one another. An ostensive is efficient only if the non-ostensive evoked makes
sense for the mathematical activity. But in the same way, the efficiency of the
technique that uses an ostensive makes sense about what has been done. Therefore,
tool and representation are embedded and the ATD modelling of mathematical
activity as a praxeology allows clarifying the relationships by means of the praxis
component and the logos one.
4. Choose an ostensive for a design task: study the values according to
the learning objective
Now, we illustrate with an example how the concepts of instrumental and
semiotic values make possible to answer two essential questions to design a
mathematical activity: – What mathematics epistemological considerations are taken
into account when designing tasks using tools? How do different types of tools afford
different mathematical activities/tasks, different representations and/or discourses,
and different interactions between representations? –
Let us consider a teacher of a "classe de CP" of a primary school who wishes
to introduce an ostensive representing the integers as a reference tool. This teacher
hesitates between a linear or tabular presentation (see figures 5 and 6).
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
Figure 4- Linear presentation of integers from 0 to 20 Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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0 1 2 3 4 5 6 7 8 9
10 11 12 13 14 15 16 17 18 19
20 21 22 23 24 25 26 27 28 29
30 31 32 33 34 35 36 37 38 39
40 41 42 43 44 45 46 47 48 49
Figure 5- Tabular presentation of integers from 0 to 49
Which ostensive must be chosen? The linear presentation, or number line, is
present in pre-primary school – children between three and six years old – and is
found in various board games where it is necessary to push or to move back a pawn
according to a dice throw. It is a familiar representation whereas the table of numbers
is a tool which belongs to the school culture. The number line enforces the continuity
of the series of numbers and makes possible the association between cardinal number
and ordinal number. It also allows determining for each number what is the previous
and the next number, bounding it between two whole tens, adding or subtracting two
numbers while moving forth or back by analogy with games. They are there the
ingredients of its instrumental value. The same techniques can be used with a table of
numbers. However the tabular structure arranges the numbers according to their

figures of tens and units, and breaks somehow the continuity of the number line. Then it is less easy to perceive than 19 is between 18 and 20 or 10 and 20 with a table of numbers than with a number line.
If the aim of the teacher is to obtain the result by counting, he can consider that the instrumental value is more important for the number line. Of course, the same technique of counting can be used with both ostensives, but the table of numbers makes easier the use of the decomposition of tens and units because of its structure. For example, to add 5 + 12, the pupil can advance of 12 boxes on the number file from 5 (see figure 5). With the table of numbers, it is possible to find the result by moving down from a line from 5 and then by moving forth by two boxes (see figure 6). This last technique is based on the decomposition of 12 into 10 + 2. Obviously, a moving on the number file by a jump of ten boxes and then by two jumps of one box is possible. Nevertheless, this technique is easier to use with the table of numbers than with the number file. If the goal of the teacher is to develop a work on the numeration and the decomposition of the numbers rather than counting, he can consider that the instrumental value is more important for the table of numbers. Therefore, the choice of the representation will be determined by the mathematical praxeologies that the teacher wishes to make alive in the classroom: the greatest instrumentality and the greatest semioticity are then estimated according to the aim of the study. It is finally noteworthy that the semiotic value of ostensive is not "already there". It is a work around the structure of the table which reveals 26 as being at the crossing of the line of 20 and column of 6 and led to the decomposition 26 = 20 + 6. It is because the table of numbers is transformed into a tool making easier the use of the decomposition of the numbers that its semiotic value, connected with the numeration, develops. The instrumental genesis (Rabardel, 1995) does not facilitate only the use, and thus the instrumentality, of an ostensive. It develops also its semiotic value: an increase of the instrumental value allows also an increase of the semiotic value.
We have discussed above that the instrumental value of the table of numbers, from a didactic point of view, is to make easier the transition from techniques relying on counting to techniques based on numeration and additive decomposition. The teacher will determine his choice between the number file and the table of numbers according to the non-ostensives that it evokes and thus techniques that it makes Theme A – F. Wozniak
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possible to implement. The instrumental value of an ostensive from the point of view of the teacher's activity to design a teaching situation is thus based on the semiotic value of this ostensive from the point of view of the pupil's mathematical activity. But not only, because the semiotic value of an ostensive is also built starting from its instrumentalization. Thus, the choice of a particular artefact for a design task is determined according to the analysis of its semiotic and instrumental value to develop particular mathematical praxeologies. The notion of praxeology is thus central in ATD to design a mathematical activity. The teacher must consider the techniques he wants to make alive in the classroom to perform a specific task; he must consider the technological discourses he wants the pupils to develop for describing, explaining, justifying and developing the techniques used. In the above example, the choice of tools is determined by the questions on the kinds of praxeologies that the teacher intends to make alive in the classroom.
Conclusion
Any ostensive is an instrument to do something and the representative of a non-ostensive that allows thinking what is made or has to be made. It is this bivalence that is recognized through the concepts of instrumental value and semiotic value of an ostensive which are inseparable and nourish each other within praxeologies that activate the ostensive. In fact, an ostensive can acquire a higher instrumentality thanks to a technological or theoretical work giving a higher intelligibility to new technical uses. Reciprocally an ostensive can acquire a greater semioticity thanks to a work on instrumentation.The identification of the instrumental and semiotic values seems an essential didactic gesture for the teacher during the design of a teaching situation. It is based on an epistemological analysis of mathematical praxeologies that he wishes to make alive in his classroom depending on the learning challenge. The example in the fourth section illustrates this point of view.
What happens when the instrumental and semiotic values of ostensives are insufficiently taken into account during the design of a mathematical activity? In this case, the milieu of the situation (according to Brousseau, 1997) is insufficient to create a dynamics that allows the construction of the knowledge. In Bulf, Mathé, Mithalal, Wozniak (in press) we show how a teacher in this case uses a specific ostensive, the language. In fact, he resorts to maieutic as a driver of the mathematical activity which cannot start from the others ostensives and non-ostensives of the situation. The teacher intervenes directly with pupils in order to guide them in their activity. Thus, through the questions he asks, he finally ends by suggesting the expected answer. This kind of situation is named by Brousseau (1986) the Topaze's effect.
In fact, the instrumentality of the notions of instrumental and semiotic values of an ostensive to design a mathematical activity is born from the anthropological point of view which we adopted: identification, interpretation and description of the instrumental and semiotic values of an ostensive depend on the studied practices and the institutions in which they are activated. ATD is an epistemological approach and its main theoretical tool is the notion of mathematical praxeology as model of mathematical knowledge. The notion of praxeology allows analysing the tool-driven relationships within the design, teaching and learning triad. What type of techniques does the teacher want the pupils use? What kind of technological discourses does the teacher want the pupils develop? These are fundamental questions to design a mathematical activity and analyse its teaching and learning. Any mathematical Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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activity activates ostensives and evokes associated non-ostensives. Then, the analysis of the instrumental value and the semiotic value of the ostensives allows to choice the tools for performing the task in function of techniques or technological discourses the teacher wants to be used in the classroom. In conclusion, it seems that the notions of praxeologies, instrumental value, and semiotic value of an ostensive, are useful theoretical tools to study task design.
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Theme B
Accounting for student perspectives in task design
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It is obvious that tasks or sequences of tasks are designed to embody mathematical knowledge in ways that are accessible to students, and to improve students' mathematics thinking. However, if we look beyond the intentions of those who design and select tasks, the actual impact on students' mathematical learning raises important questions. One of the aims of this thematic group is to gain insights into students' perspectives about the meanings and purposes of mathematical tasks, and to better understand how appropriate task design might help to minimise the gap between teacher intentions and student mathematical activity.
There is a tacit assumption that the completion of mathematical tasks chosen or designed by the teacher will result in the student learning the intended mathematics. This view is persistent despite research that suggests that this is not a direct relationship (Margolinas, 2004, 2005). This can result in completion of the task (rather than mathematical learning) becoming the priority for students and even sometimes for teachers. This can be particularly true for younger and lower achieving students, who are "helped" by the teacher to complete the task in order to 'keep up' with their peers. Teachers are encouraged to differentiate tasks for different students in order to facilitate learning. However, changes that make it easier for the student to complete the task may have the effect of undermining the designers' intentions, and reinforcing students' attention of completion as the priority.
Research about learners' perceptions of the use of contexts in mathematical tasks has suggested that these can differ considerably from intentions of designers (Cooper & Dunne, 2000). Whilst designers may choose contexts to offer real world models to think with or to illustrate the usefulness of mathematical concepts in real life, pedagogic practice may lead students to adopt 'tricks' to bypass the contextual elements (e.g. Gerofsky, 1996), Verschaffel, Greer, & Torbeyns, 2006)), or fail to appreciate the extent to which everyday knowledge should be utilised in the mathematical task (Cooper & Dunne, 2000). Tasks or sequences which draw on real world contexts, but which do not reflect the purposes for which mathematics is used in the real world, may be perceived by students as evidence of the gap between school mathematics and relevance to their everyday lives (Ainley, Pratt, & Hansen, 2006).
Another issue is a methodological one. One possibility for measuring the impact of tasks or sequences on students' learning is the use of pre- and post-tests. However, since it is highly likely that any teaching may result in some outcome on posttests, it is not so obvious what should be considered as a significant posttest outcome. For instance, if we consider only the mean value of an entire cohort of students, we may not understand whether the low achieving students (as determined Theme B – Introduction
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by the pretest) have really benefited from the task or sequence. Moreover, the goal of the task or sequence may not be easily (or even possibly) assessed in a written test. Often, it is only by observing the evolution of students' strategies that we can understand the effect of a task or sequence (Brousseau, 2008). Task design is generally initially implemented in favourable contexts: the teachers are members of the research team or closely linked to the designers. In this context, the impact on students is not only linked to the tasks but also to the impact on teacher or students of a collaborative way of dealing with teaching (Arsac, Balacheff, & Mante, 1992).
These methodological issues are only examples of those that can be addressed in our group. An aim of this thematic group is therefore to reflect on methodological issues related to studying task impact on students.
Possible questions might be:
• How is it possible to assess the impact of task or sequence on students' mathematical learning?
• What is the intended and actual impact of a task or sequence on low achieving students?
• What do students actually do and attend to when confronted with tasks?
• How do students understand the purposes of tasks they are given in the classroom?
• How do students' reactions influence teachers' adaptation of the task?
• Might what appears to be 'only' a change in presentation convey a different meaning to the student, and result in different mathematical activity?
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Emergent tasks--spontaneous design supporting in-depth learning
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In this paper the concept of emergent tasks is presented. It has
been developed through empirical investigations of interest-dense
situations comparing them with situations where learning opportunities
were not taken up and enables teachers to shape learning opportunities in
everyday lessons in an effective way. Three forms of emergent tasks, their
conditions of success and possible uses are discussed.
Keywords: Interest-dense situations, learning opportunity, social
interaction, epistemic process
Introduction
Interest-dense situations provide epistemic and interest-supporting learning
opportunities to the students by the adaptive behavior of the teacher (Bikner-Ahsbahs,
2003; 2005). So far, we have understood how these situations can be built through
social interaction but we did not exactly know in what way the teacher aligns his
behavior to the students' needs. Investigating this problem brought to the fore a new
task format which is not designed beforehand and hence accompanied by the
problems of given tasks described in the call of ICMI Study 22 (ICMI, 2012) but is
constructed instantaneously deepening mathematical thinking. In the long run, this
investigation has the potential to answer some notable problems of task design, for
example "to better understand how appropriate task design might help to minimize
the gap between teacher intentions and student mathematical activity" (ICMI, 2012,
15) or how tasks developed in favorable contexts can be made more accessible in
ordinary classrooms.
Regularly in mathematics classrooms, tasks designed beforehand are
implemented to initiate learning activities. In this process, time is needed to
understand the task affordances, especially when they are complex and the students
are unfamiliar with their forms. The time on task and with it the possible outcomes
are reduced when a considerable amount of time is needed for understanding its
requirements (Caroll, 1989). However, time on task alone does not induce sucessful
learning, but must be supported by sophisticated teacher instruction (Caroll, 1989, 30;
Christiansen & Walther, 1986, 261). For instance, Prediger and Scherres (2012) show
in a case study that with so called self-differentiating tasks optimal learning takes
place only if it is supported by a teacher intervention that transforms the mathematical
learning potential into a fruitful epistemic situation. The two problematic aspects Theme B – A. Bikner-Ahsbahs & T. Janßen
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discussed so far—the reduction of learning time needed to understand the affordances
of tasks and the necessity to take learning opportunities in the classroom—are
addressed effectively by the concept of emergent tasks presented in this paper. More
aligned to the specific situation than in ad hoc tasks (Christiansen & Walther, 1986,
296), in emergent tasks the teacher conceives the mathematical potential of a learning
opportunity and translates it into a task, so that
1. the students' interest present in the situation is taken up and
2. acute mathematical problems and questions are addressed adaptively.
Investigating emergent tasks and their supporting conditions may elucidate
how the gap between the students' experiences and the given purposes of tasks can be
bridged through the adaptive character of these tasks. In this paper the concept of
emergent tasks will be worked out based on qualitative data. Forms of emergent tasks
and necessary conditions are presented. Furthermore, based on an empirical example,
influences of emergent tasks on further learning processes are illustrated.
Theoretical background and research questions
Given the immediate perceivability of emergent tasks they can contribute to a
more effective use of time in class. With their adaptive character, they are embedded
in an understanding of mathematics learning that places it in the social togetherness of
learners and teacher (Jungwirth, 2003). Both align to the mathematical and situational
requirements not only of the teachers' goals, but especially of the students' learning
processes. This very condition characterizes the social interaction in interest-dense
situations (Bikner-Ahsbahs, 2005). Hence, we assume emergent tasks to be a means
to initiate and sustain interest-dense situations and with it deepened mathematical
learning.
Interest-dense situations can be described by three characteristics: the
learners are deeply involved in working on mathematical questions, they construct
successively deep mathematical meanings, and they explicitly or implicitly see the
mathematical object as important in the situation (Bikner-Ahsbahs, 2003; 2005).
Common to all interest-dense situations is the scheme of students' epistemic actions:
Starting from the gathering of smallest bits of knowledge via the connecting of
mathematical knowledge students get to structure-seeing (GCSt model). Structureseeing can refer to known mathematical structures that are seen in new contexts, but
also includes the development of new structures. Hence, interest-dense situations can
be considered mathematically rich learning processes. The teacher puts aside fixed
answer expectations in favor of the developing situation. The students do not try to reproduce the teacher's expected answers but concentrate on their own learning
processes. They try to grasp the mathematical content in the social interaction with
their peers.
The content-related driving force of such situations is the students' general
epistemic need (GEN) (Kidron et al., 2010; 2011). It is often only implicitly visible in
epistemic actions. Together with a mathematical gap experienced by the learners it
can lead to more specific needs, for example to formalize. In this case, the need can
be satisfied by specific actions. These then lead to epistemic steps, hence to the
experience of competence and a stabilization of the situational interest (Krapp, 2005),
which in turn reproduces the GEN. Thus, a situation-apt emergent task reacting to the
needs of a student should be suitable to bring forward epistemic processes.
However, it remains unclear what characteristics emergent tasks account for
and how exactly they come about. This study tries to clarify the following questions: Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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1.Under which conditions do emergent task come about and are there
specific forms to be identified? More precisely, we look for conditions of
classroom situations and the requirements the teacher must meet to
design useful emergent tasks.
2.What is the use of emergent tasks?
Methodological considerations
The research questions are investigated focusing on processes by the use of
qualitative methods. As emergent tasks seem to be an attendant phenomenon to
interest-dense situations, data from a project dedicated to this subject is used. We
identify scenes where teachers translate learning opportunities into tasks and examine
how these tasks guide further epistemic processes. We consider these processes
successful when students get to mathematical insight regarding the underlying
problem. The GCSt model helps in this process: Success in this sense can be stated
whenever students see structures (which is the case in all interest-dense situations)
and this can be ascribed to the emergent task.
We assume with Brousseau (2008, as quoted by ICMI, 2012, 16) that "it is
only by observing the evolution of students' strategies that we can understand the
effect of a task or sequence", and aim at answering the question what students
"actually do and attend to when confronted with task" (ICMI, 2012, 16) - even more
so as emergent tasks by definition can only occur in everyday classroom interaction
and not in artificial laboratory setups. Thus we ground our ongoing study on the video
transcripts from 16 interest-dense situations of 8 lessons. 5 of the lessons are still to
be analyzed. They are taken from a total of 89 lessons dedicated to fractions that were
conducted for half a year in a 6th grade class. The transcripts are analyzed focusing
the question how the teacher deals with learning opportunities marked by the
students' GEN. By comparison with scenes where a similar learning potential is not
used in an emergent task, fostering and hindering conditions of emergent tasks are
gradually worked out. These may vary depending on the form of emergent tasks and
thus express a teacher-dependent typing. We reconstruct this typing by first analyzing
one rich situation containing an emergent task. In each scene subsequently analyzed a
contrasting permanent comparison to the previous scenes is conducted (Jungwirth,
2003). This leads to a typing of emergent tasks regarding the situational conditions.
To describe the tasks designed by the teacher in the situation we use the
definition of tasks by Bruder (2000). It describes tasks as calls to action with an initial
state I consisting of given information, conditions, etc., a final state F meaning the
solution, conclusion, etc., and a transformation T between the initial and the final
state. Based on this definition we can describe in each situation what is given in the
students' utterances, what the students struggle with, and what is translated by the
teacher.
The method of analysis follows the rules of the interpretative approach
(Jungwirth, 2003). It assumes that subjects act towards things (here: mathematical
content, learning environments, other persons, etc.) according to the meanings these
things have for them, and furthermore, that these meanings are built in social
interactions through interpretation (Blumer cited by Wagner, 1999, 32). Analyzing
then means re-interpreting the interpretations of the subjects. As we work on everyday
classes, we can adopt the reflexivity assumption from ethnomethodology: the reasons
for acting are visible in acting itself (Lamnek, 1995, 51 ff). Regarding the tasks this
means that the persons involved make clear through their actions (i) how the learning Theme B – A. Bikner-Ahsbahs & T. Janßen
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opportunity comes about, (ii) if, why, and how the teacher formulates an emergent
task, and (iii) how this task influences the epistemic process.
Exemplary data analysis
In the following exemplary display of results we will first describe a situation
where the teacher does not see the potential of the learning opportunity and present a
first hypothesis regarding possible reasons. Then one successful emergent task is
presented and its conditions are worked out. Finally these considerations are
completed by a look at the following task sequence which leads from the initial
problem to a learning result for many students.
A missed learning opportunity
The students are asked to find one third of a slip of paper. Rosa proposes to
tentatively fold the slip in three layers. Some students cannot follow this idea as they
have problems implementing it. The teacher, Mr. Kramer, is not content and says:
167 T: ah is there (.) is there yet another
possibility (.) some took (..) the view ,one
could also fold in the middle
168 /S: no
169 /T: and does that work in the middle
170 S: no
171 S: doesn't work.
172 Mira: Yeah ,it works (.) yes
173 Mira: yes but that would only ,work
well. if one would fold it here
174 T: yes'
175 S: uh
176 Mira: and then one would have to
fold once more then there would be four
pieces ,and ,the one piece one would have
had ,to throw that away then that would
work
177 S: ooh
178 T: ah like that yes.
179 S: ooh
180 T: yes so it works yes (...) there one
would have given away one piece right (..)
well okay
181 Eric: but one can also divide the last
piece by three (.) if one then divides by four
and one divides one piece by three then
again
182 /T: yes but then one has the same
problem again one piece divided by three
183 /S: yes exactly
184 /T: right
Transcription key:
w-e-l-l speaking slowly
exact. dropping the voice
exact' raising the voice
,exact with a new onset
EXACT with a loud voice
(.)/(..)/... 1, 2, ... sec pause
/S: interrupts the previous
speaker
T: teacher
The open question of the teacher stimulates the students. They discuss the
question whether it "works" to divide the strip in the middle (168-172). Mira proposes
to bisect twice and to "throw away" one of the four pieces (173-176). According to
the teacher one would have "given away" one piece in this case. He makes clear that
this is a good idea but that not using the whole paper slip is unfortunate—he thus
clarifies the initial state and the final state. In fragile speech, Eric proposes to divide
the surplus piece by three (181). With this proposal Eric starts an approximation Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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process. But the teacher does not take up this idea. Once more he marks the situation
as unfortunate, as one would have the same problem again. "Yes exactly" is the
supporting reaction in the class which in turn gets appreciation by the teacher (182-
184). Eric sees a different final state than the rest of the class. He does not try to get
exactly one third in just one step, but looks for a successive approximation. And his
initial state is different, too: While Mira and even more explicitly the teacher mark the
surplus piece as unfortunate, this very piece constitutes the starting point of Eric's
idea.
Eric's view constitutes a learning opportunity: If one repeats the process of
dividing one of the four pieces by 4 and puts three of the resulting smaller pieces next
to the three remaining larger ones, one is again left with one small piece that can
again be divided. The result is an approximation algorithm for the division by 3.
Fig. 1: The first two steps of the approximation algorithm for the division of the paper slip
by 3
However, the potential of the situation remains unused. Why does the teacher
not adapt to Eric's view and hence does not use this learning opportunity? He is
familiar with approximation algorithms, for example with the bisection method,
because he also teaches calculus. In fact, in another lesson he initiates a similar
method in the very same class to show the density of rational numbers, albeit by
dividing fractions by 2 and not as an approximation. In this case there are some

possible reasons to not follow Eric's idea: Either the teacher really does not see the mathematical potential of the situation, or he does not expect such a complex algorithm at the beginning of grade 6. Anyway, he does not show interest in Eric's idea, possibly also because he has different plans for the class.
An example of an emergent task
Previous to the scene presented here the students had worked on the question how to divide a round licorice stick evenly among three persons. Among other techniques the students discussed the possibility to cut the stick in length, so that the cross sectional area would be divided in three sectors of 120° each. Because angles had not been introduced before the meaning of 360° was presented using two set squares. The division of a circle by 3 was illustrated with a round trash can where the middle was marked by the cast. This is the point where Anji asks her question.
139 Anji: Mister Kramer I have a question here. they have divided it from the top but how do they know what 120 DEGREES are.
140 T: ah okay' you want to go back to the set square once more.
141 /Anji: no (.) well but when they ,that is such a stick and how do they know that because ,they can't do that with the set square.
142 /S: yes that's round now right
143 /Ernst: that is round right
144 /S: it's round right
145 T: Tom yes. that is kind of a practical problem right' how do I do that when it is such a very small one and not such a BIG circle eh'
146 Tom: one has to put the zero in the central point and then it works nevertheless I think (.) then one only has to imagine the lines from the 120 until one can draw.
147 T: ah yeah I ,there must additionally ,we will exercise that
148 /S: yes but how Theme B – A. Bikner-Ahsbahs & T. Janßen
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149 /Rahel: yes Mister Kramer once more a stupid question ,how does one GET the central point how did they GET that. because that is so small.
150 S: that is just
151 /T: that's another problem right. that's a practical problem (..) oh no ,how does one even find the central point in such a small circle right' (.) exactly. those are questions'
152 Anji: a very small compass right'
153 T: yes one can find out with the compass ,only when one is just drawing the circle' one has the central point. but when one has the circle already right'
154 Rahel: yes
155 T: that's exactly what geometry works with.
156 /Rahel: I know that
157 T: there are possibilities to find out a-n-d you can puzzle at home maybe someone finds a possibility'
158 /S: ah I know-
159 /S: one just puts the thing in there
160 /S: one stings in there
161 T: one puts it in
162 S: what'
163 S: oho
164 T: yes but one can act as if one wouldn't have it. how can one find it then. right maybe there's a possibility right maybe you find something. right and then you just try try to divide in thirds. (..) so first draw a circle' then act as if one wouldn't have the central point' one can't find it anymore and then how can one find it again. and when one has it how can one divide in thirds.
The initial question by Anji shows her GEN. The teacher tries to understand and thus shows interest in her learning process (140). The reaction of the teacher seems imprecise as Anji's does not agree and specifies her question (141). Her question is how the division of the circle in three equal parts can be accomplished in practice as the diameter is very small (ca. 1cm). Some students do not see the problem, i.e. the initial state, which Anji and the teacher have agreed upon. They only see the form of the profile as relevant, which is "round" (142-144). So the teacher once more points out Anji's problem by comparing the profile to the larger circle they had worked with before (145). Tom offers a solution. He describes how to measure angles even on very small objects using the set square (146). At this point the teacher probably notices already that the missing central point poses an additional problem and wants to postpone it to an exercise (147). But the students insist on an immediate clarification by asking "but how" (can we exercise that) (148). Rahel reacts by naming the difficulty in dividing the circle without knowing the central point (149). She grasps the epistemic gap and thus sees the mathematical structure. Now the teacher summarizes the two problems: How does one even find the central point in such a small circle? Commenting "those are questions" he documents wonder about the deep involvement of the students that he tries to take up (151). Anji proposes to use a "very small compass" (152). Mr. Kramer tries to reformulate the problem: If one draws the circle the central point is known. But how can the central point be found if the circle is already given (153)? The teacher probably does not know yet how he can work with this situation. He situates the students' questions as problems of geometry. Again, the students react driven by their epistemic need: "I know that" (this is what geometry works with) is a comment that makes clear that the students do not care about the general context, but about their specific problem (154-156). Now the teacher starts drafting a question that picks up the problem and the epistemic need of the class (157). But for some students the initial state is still unclear: they assume the central point to be known, as one can just put the compass there (158-163). The teacher feels prompted to specify his thoughts and poses the following task as a Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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homework assignment: Draw a circle, pretend the central point were unknown, find out how it can be rediscovered (164).
In this situation we see more than just an adaptation of an existing task as a consequence of the students' reactions (ICMI, 2012, 16). Rather, a task emerges as a reaction of the teacher to a central problem of the students. The teacher is the designer taking up the students' epistemic need (Kidron et al., 2010; 2011) that he transforms into an action program. The social interaction is fed by the different views the subjects have of the situation. Initially, there are questions, but a clear initial state for the task is missing. This changes when Rahel asks for the central point. A brief clarification of I and F leads to an emergent task that may seem "to be 'only' a change in presentation" (ICMI, 2012, 17) but proves to be particularly fruitful: At the beginning of the next lesson the teacher asks who has worked alone on the task and gotten to a solution. According to the log of the lesson, 20 of 26 students had tried to solve the task on their own, and 16 students got a solution. Some of the students' solutions are presented. Five different methods can be identified which represent structure-seeing at different levels of abstraction:
1. Two maximal, perpendicular chords are found tentatively, they intersect at the central point.
2. To find the maximal chord, a sampling system with nested measurements is presented.
3. The maximal chord is found using the set square. It is placed in the circle so that the circle line passes through the same numbers, which are then maximized. Finally, the central point is marked by the zero.
4. By twofold, perpendicular use of a ruled transparent the central point is found.
5. Two parallel tangents are applied to the circle. Half of the distance is taken as the radius of two circles with their central points on the circle. Their intersection marks the central point of the initial circle.
The last solution, which is presented by Andy, is closest to the one that is part of the curriculum in grade 7. However, it seems too complex for many of his peers. Hence, the teacher encourages the class to ask Andy about his procedure. The crucial question is why he gets the diameter and the central point this way. The teacher then disseminates prepared paper circles without marked central points and asks the students to use and check Andy's method. The students now focus on the question how one can know where on the circle the compass must be placed. They find out that it is irrelevant as all points have the same distance to the central point (structure-seeing). Only two points on exact opposite sides would be inconvenient. In the end, the teacher gives a homework assignment as an emergent task, which demands an expanded use and application of Andy's method and induces an institutionalization: Andy's and two other methods are to be put down in writing "so we don't forget that". Andy's method is not only used to find the central points of the circles, but also incomplete circles are to be reconstructed. The institutionalization contains both, repetition and consolidation of the method. From our analysis we derive five possible design principles that allow for a sequencing based on an emergent task about learning a method: Theme B – A. Bikner-Ahsbahs & T. Janßen
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Principle Description of the step
Emergent task reacting to a student problem
Presenting and questioning students' solutions to the task are presented, questions are asked
Using and checking an interesting method is used and checked by the students
Expanded use and application the potential in use is evaluated by an expanded task
Institutionalization (individual) textualization
Tab.1: Five principles for a task sequence about lerning a method
What was the difference between this situation and the one presented before? The teacher tried to grasp the problem from the perspective of the students. Posing questions he makes clear what he has yet understood and thus provokes the students to specify and to make the problem (i.e. the GEN) more visible. This does not necessarily bring clarification. But as soon as the teacher has understood the mathematical problem of the students he can translate it into a task: "how does one find the the central point of a circle"? The task is adapted to the students' situation: they have already shown a GEN, and the question is already clear in the situation. The teacher takes the students' situation and their need seriously and makes it the basis of a longer learning process through the emergent task.
Summary of results and concluding remarks
Emergent tasks can be designed whenever the students' epistemic interest can be translated into a task. In most cases both the initial state and the final state are not clear for all participants. It is the teacher's role to translate and thus clarify. Based on the analyses of four interest-dense situations we were able to identify three different forms of emergent tasks. When the students' epistemic need is explicated the teacher can react to it immediately. We presented a case where an emergent task initiated the learning of a method through a task sequence. When the students' need is implicit their problem may be visible to the teacher or not. In the first case emergent tasks may consist of smaller prompts reacting to a student's problem. In the latter case the teacher may help making the problem visible by asking questions like: "show us what you mean", "take the chalk and write (or draw) what you mean". In our data, the students' managed to explicate their problems in this way and especially communicate to the teacher what they were struggling with. On the other hand, we showed how such an opportunity was missed and Eric's idea remained unused. Besides the translations of explicit and implicit problems we also found emergent tasks that unveiled an epistemic gap that initially remained unnoticed by the students.
On the part of the teacher our studies point at three necessary conditions that enable her or him to see situations, where an emergent task is suitable, and to then perform the appropriate translations. The teacher must
• have mathematical knowledge that extends the content of the lesson,
• show interest in the students' learning processes,
• and be open for unusual ways on the part of the students. She or he must be willing to abstain from the planned course.
So far, we have only investigated emergent tasks in one class and the study is still ongoing. Further research investigating interest-dense situations and emergent tasks in other classes and other areas is needed. In addition, it remains an open question whether this task format can also be found outside interest-dense situations, and if yes, how this happens and how it influences learning processes. Emergent tasks can be a powerful tool for the teacher to adapt his behavior to the students' actions Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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and interactions in a given mathematical problem situation. However, the implementation of this tool is yet to be investigated.
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Mathematical Investigations: The Impact of Students' Enacted Activity on Design, Development, Evaluation and Implementation

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This paper is based on a year-long action research study that I conducted in a Form Four mathematics class in a girls' state secondary school in Malta. The focus is on presenting the framework that was used to plan, develop, design, classify, implement and evaluate investigative tasks. This framework hence provides principles and guidelines on examining the differentiation between the tasks as designed and foreseen by the teacher and the actual instructional activities as undertaken by the students. For the teacher-researcher, students' perspectives become crucial within the developmental cyclic process of designing, modifying, implementing and evaluating tasks. The study reported in this paper shows that when students' contributions (students' classroom experiences, responses and understandings) are valued, the teacher-researcher gains more informed knowledge and improved understanding about the design and implementation of investigative tasks.

Investigations; active learning; reflective practice; students' perspectives; task resources

Introduction

This paper draws upon an action research study which I conducted with a Form Four (i.e., 14-year-olds) mathematics class during the scholastic year 2008-2009 that explored the integration of a set of 18 investigations within the mathematics syllabus (see Calleja, 2011). Since then, I have moved to a different state school and have also assumed the responsibilities of Head of Mathematics Department. The focus of this paper is to present a view of didactical goals, guidelines and frameworks that may be useful for collaborative design research projects that are specifically aimed at integrating tasks that promote active learning practices (see Anthony, 1996). The embedded understanding is enlightening my current collaborative project, which commenced in September 2012, on promoting the design and implementation of inquiry-based mathematical tasks with all the Form One students (i.e., 11-year-olds) in my new secondary school. As head of mathematics department, I am coordinating this year-long pilot project and acting as a teacher-mentor within a team consisting of the four other mathematics teachers. At present, we are busy selecting, planning, designing, implementing and evaluating a variety of inquiry tasks, including a range of resource materials (e.g., worksheets, measuring instruments, etc.). Working within our community of practice, we follow a teacher-researcher perspective of 'learning-Theme B – J. Calleja 164
to-develop-learning' (Jaworski, 2006) while implementing our task-driven pedagogy (Walls, 2005).

The Research Area and Methodology

At the time of the action research study in my former school, I undertook the role of a reflective practitioner (see Schön, 1983) in the belief that "good research is not about good methods as much as it is about good thinking" (Stake, 1995, p. 19). This involved taking a reflective stance towards planning, designing, implementing, modifying and evaluating a set of mathematical investigations that I integrated within my scheme of work. Data collection included multiple sources. Apart from writing field notes of my classroom observations, I occasionally discussed classroom situations with a critical friend and kept a reflective journal to give my own interpretations as the events unfolded. However, I also valued students' reflections, thoughts and suggestions. Thus, I encouraged students to keep a learning log book through which they could write about their learning experiences, feelings and learning environment. Moreover, midway through the study, I conducted semi-structured qualitative interviews with the students. Data analysis was then an on-going process of searching for themes as these emerged and resurfaced. This sense-making journey was a search for 'some' truth through qualitative triangulation, namely, an account that presented the classroom situations from three different perspectives: those of the teacher-researcher, the students and a critical friend (McKernan, 1996). My qualitative account emerged as an informed story highlighting the salient features of the classroom community while engaged with a range of investigative tasks as an approach to learning mathematics. At the same time, this study offered me the possibility to better understand and eventually improve my practice.

Conceptual Framework

Reform-oriented approaches claim that doing mathematics should involve sense-making activities by using tasks that provide students with a variety of challenging experiences through which they can actively construct their mathematical meanings (Bishop, 1991). Within this 'active learning' approach – which is associated with experiential, collaborative and inquiry-based learning (see Anthony, 1996) – students gain autonomy and take control over the direction of their learning. Referring to the pedagogical implications, Ernest (1991, p. 288) proposes that the role of the teacher
is understood in ways that support this pedagogy, as manager of the learning environment and learning resources, and facilitator of learning.
Thus, it is within the teacher's remit to include cognitively demanding tasks for their students, encouraging them to work together and to justify their solutions. Learners are thought of in turn as subjects who are responsible for learning, for making decisions and also for their behaviour (Teong, 2002). Put differently, whereas students are trusted and respected as responsible learners of mathematics and as constructors of their own meaningful knowledge, the teacher assumes the role of a facilitator who assists students' learning by observing, listening, questioning and challenging their inquiries.
Learning through active participation also carries an important social dimension. For as Fosnot (2005) claims, learners can negotiate meanings when they engage in cooperative social learning activities. This social constructivist dimension is Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford. 165
based on the understanding that students should be active participants in their own learning by communicating and exchanging ideas with the teacher and other students. Learning is believed to occur by being part of and interacting within a social environment. Discussion with peers can assist learning as students develop conceptual understanding when they articulate their thoughts and points of view, when they learn to listen to others and when they ask questions (Orton & Frobisher, 1996). It follows that learning opportunities arise when student contributions are encouraged and when discussions invite students to challenge, argue and offer explanations.

Defining Tasks as Investigations

Mathematical tasks are an important vehicle through which classroom instruction can enhance students' learning. The tasks teachers present students with and the way in which students negotiate mathematical meaning by working on the set tasks largely determine students' classroom experiences and their learning of mathematics (Hiebert, et al., 1997; Shimizu et al., 2010). As Doyle (1983, p. 161) argues, tasks "influence learners by directing their attention to particular aspects of content and by specifying ways of processing information" and are "defined by the answers students are required to produce and the routes that can be used to obtain these answers". Hence, a key decision for teachers lies in their choice of tasks (Sullivan, 2011). In my action research study, I had chosen investigative tasks. The selected investigations required students to participate actively in their learning as a way of constructing mathematical knowledge within a social setting.
In this paper, an investigative task or investigation is defined as an inquiry into a mathematical situation presented by the teacher, but which can also initiate from a statement or a question posed by a student (Greenes, 1996). The topic of the investigation could arise from real-life or from a mathematically designed problem (e.g., Investigate the sum of angles in polygons). Again, an investigation can be a very open exploration or it can take the form of a more structured task that guides the learner into discovering mathematics (Yeo & Yeap, 2010).
Investigations offer opportunities for students to be more active in their learning. When students engage with such work they are involved in processes of exploration and explanation (Skovsmose, 2001). Students are expected to engage with finding ways to unravel the task assigned and be able to justify their work by presenting their method/s to the whole class. In the process, students engage in thinking critically, learning to ask, sharing ideas and communicating mathematically.

A Framework for Reflection on Design and Implementation

Integrating investigations takes into consideration how different mathematical processes and strategies can be embedded within the core topics which make up the content of a mathematics curriculum (Frobisher, 1994). Consequently, in planning and designing my 18 investigative tasks, I set out to achieve two aims. The tasks had to: (i) be related to the mathematics content prescribed in the Form Four syllabus; and
(ii) engage students in mathematical inquiry, that is, thinking about, developing, using and making sense of mathematics (Breen & O'Shea, 2010). While a number of the selected investigative tasks integrated mathematical topics in order to allow students to experience different areas of mathematics, other tasks were intended mainly to help consolidate mathematical concepts and skills. Furthermore, in the belief that tasks should provoke curiosity in and be meaningful to students, the selection and creation Theme B – J. Calleja 166
of tasks and resource materials took into account the students' interests, mathematical ability and needs. However, when I came to setting future goals I took into consideration the responses/feedback that the students had provided during the study.
Within this conceptual framework, investigative tasks were classified according to the mathematics embedded within the activity, the degree of structure/guidance provided to students and the time devoted for students' activity. The main reason behind the decision to use investigations of varying levels was to smooth the transition for students from working on traditional exercises to engaging in more challenging tasks (Orton & Frobisher, 1996). Moreover, the spread of tasks along this classification helped to gradually introduce students to the cognitive processes of making and doing mathematics. In other words, the different levels offered graded entry points for students to familiarise themselves with the social experiences of mathematical inquiry, discussion and communication.
At the basic level, the investigations were structured tasks that lead students to mathematical discoveries. The given instructions guided students, who worked individually or in pairs, to use particular pre-determined mathematical concepts and apply them to arrive at a solution. At the next level, the investigations were semi-structured. This meant that they were either less structured or students were initially given some guidance in their work but were then free to explore and engage with the task using their own conceptual mathematical understanding and reasoning. Believing that learners benefit from discussing ideas and solutions when working on these more challenging tasks, the students were instructed to work in small groups of two or three. At the third and higher level, the students encountered unstructured investigations that were more process-oriented activities. These required students to investigate the problem posed or the situation presented in as many different ways as they wished and through different methods. These investigations placed greater demands on students to think through a solution, to make inferences and to test their own conjectures. As this type of investigation required students to challenge, argue about and justify their reasoning, the unstructured investigations were set as a group activity involving between three to four students.
Other than the level of structure, the investigations were also classified along the three 'reality levels' identified by Skovsmose (2001). Skovsmose sees mathematical investigations as a landscape that ranges across three levels of real-life contexts. These are: (i) pure mathematics which simply involves working with numbers or geometric figures; (ii) semi-reality which refers to an everyday-life problem that is rendered artificial as it is tackled in a classroom situation where variables can be controlled; and (iii) real-life situations where students are directly involved in carrying out the exercise in the actual setting.
Combining these two classifications, I came up with a rubric consisting of nine different types of investigations. Initially, the main purpose was to produce a template along which I could select and position the investigations (see Table 1). The matrix was eventually also useful in exploring how tasks were enacted in class and whether the students' actual engagement shifted the nature of tasks along the rubric.

Investigation	structured	semi-structured	unstructured
pure mathematics	Type 1	Type 4	Type 7
semi-reality	Type 2	Type 5	Type 8
real-life	Type 3	Type 6	Type 9

Table 1: The nine types of investigationsMargolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford. 167

Investigations of types 1, 2 and 3 were structured tasks that varied according to the level of reality involved. While 'type 1' tasks resembled typical traditional exercises that are similar to those found in mathematics textbooks, 'type 2' and 'type 3' tasks were situated in a context of more practical mathematical experiences. Investigations of types 4, 5 and 6 were semi-structured tasks that again varied from a purely mathematical context to a real-life situation. The unstructured nature of investigations of types 7, 8 and 9, which again differed by context, placed the greatest cognitive demands on students as they were presented as more 'open-investigations'.

Selecting and Planning Investigations within Hypothetical Learning Trajectories

My quest to select tasks with goals in mind (Hiebert et al., 1997) involved thinking about how investigations would provoke inquiry and stimulate learning. As Simon and Tzur (2004, p. 93) argue:
The tasks are selected based on hypotheses about the learning process; the hypothesis of the learning process is based on the tasks involved.
Along these lines, the day-to-day classroom experiences were vital in informing future planning that sought to integrate the learning goals with the trajectory of students' mathematical thinking and learning. This reflective process

included the notion of 'hypothetical learning trajectory' (HLT) through which the teacher considers the learning goals, the instructional activities, and the thinking and learning in which the students might engage (Simon, 1995). A key aspect of this learning trajectory concerns a prediction of how students' thinking will evolve as they participate in the instructional activities. Actually, in classifying the tasks, I always started off with a HLT based on my expectations about students' explorations in learning – indicating the hypothetical learning trajectories from the syllabus (see Table 2). However, the actual learning trajectory cannot be known in advance as it depends on how the teacher and the students enact the tasks throughout the implementation process (see Stein et al., 2000). This eventuality of having students' task-inquiry leading to different learning trajectories than those anticipated by the teacher is indicated in Table 2, which shows a segment of my task-delineated scheme. More precisely, in my study, this occurrence rested upon how students actually went about working on the task and, in particular, their decisions regarding the use of the resources provided. This realization is crucial for the success of my current collaborative project. For I have come to appreciate that designing and implementing tasks requires teacher-decisions not only the number of resources, but also on the type and range of resources to be made available to students. The possibility of differentiating the resources that accompany tasks presents an opportunity for teachers to assign the same tasks to students of different abilities (discussed further in the following section).

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Week
Investigation
Title
Task Resources
Hypothetical Learning
Trajectory
Students' Enacted
Learning Trajectory
6
The Netball
Court
Measuring tape
Squared paper
1 cm grid
Measuring lengths
Ratio notation
Scale drawing
Areas of rectangles,
circles & semi-circles
Measuring lengths
Mean, mode, median
Rounding numbers
Ratio notation
Scale drawing
7 Web Patterns
Handout
Graph paper
Calculator
Plotting coordinates
Finding areas of rightangled triangles
Sequences
Plotting coordinates
Areas of triangles
Pythagoras' theorem
Sequences & nth term

Table 2: A section showing how the scheme of work evolved

The scheme of work thus becomes a crucial reflective document for the teacher. By being responsive to students' inquiries, the teacher can occasionally finetune it as he or she re-plans and re-designs tasks and instructional practices. It is worth mentioning here that tasks might shift horizontally along the rubric presented in Table 1. In my case, a shift from structured to semi-structured to unstructured (or vice-versa) may results from the way students engage with the task.

Incorporating Students' Perspective in Design and Implementation

Reported and used extensively in research literature is the Mathematical Tasks Framework developed from the QUASAR (Quantitative Understandings: Amplifying Student Achievement and Reasoning) project team (Stein et al., 2000). This framework defines mathematical tasks as they unfold from design into implementation. In the framework outlined, mathematical tasks pass through three distinct yet related phases: as written by curriculum developers, as presented by the teacher in class, and as negotiated by students during classroom instruction. The framework is also useful in studying changes in task features and cognitive demands as instruction passes between any two successive phases.

As one might understand, depending on their beliefs, attitudes and experience, mathematics teachers are likely to implement the same task in different ways. For example, during the task presentation phase, different teachers may provide different kinds of instruction (information, guidelines and hints) to students about the task. I would argue that providing instruction towards the process rather than the product of students' inquiry is less likely to influence or modify the cognitive demands within the task. During this phase, teachers also tend to attribute titles to the tasks they assign. Reflecting on an incident from my research, I have come to understand that the task title might convey meaning to students about the kind of mathematical content involved in their investigation. Perhaps this focus indirectly provides unwarranted closure to the activity, thus directing students' attention to specific lines of inquiry and possibly shifting the intended 'open' nature of the task. I believe that presenting 'open' titles, such as, 'Investigate Right-Angled Triangles' rather than 'Investigate Pythagoras' Theorem' may offer more 'open' learning opportunities for students. In this case, a few students who attended private lessons already knew the theorem whereas others seemed puzzled by the title. These students, hence, offered resistance and lacked engagement. Apparently, the kind of title presented altered students' learning dispositions since they had 'good' reasons not to investigate. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.

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During the presentation phase, teachers usually also provide students with resource materials to support task inquiry. Teachers might adapt tasks for students of lower ability by opting to provide more appropriate resource materials possibly without unduly reducing the cognitive demands within the task. As in my current collaborative project referred to earlier, teachers are investigating the possibility of designing and implementing inquiry-based learning tasks, on a weekly basis, along a whole scholastic year. These tasks are accompanied by a range of resource materials (including worksheets and instruments) with the intention to cater for students in our ability classes. For example, for the task 'Classifying Triangles' students in a high ability class are provided with a worksheet, scissors and glue, while students in a lower ability class are provided with cut-out triangles, glue, protractor and a ruler. Although students in the two classes are expected to classify different triangles, the ones in the lower ability class are provided with additional material to support their mathematical inquiry. Within this design principle, the 'process help' provided by a range of resources makes it possible for teachers to adapt tasks for all students. This prospect also offers students multiple entry points in engaging with the task. I therefore contend that, within a research task design project, incorporating a range of resources may be crucial in minimizing the gap between the intended and the enacted activity and in engaging all students in cognitively demanding mathematical inquiry. Students may nevertheless interpret and negotiate tasks in ways that may be different from those intended by the teacher – the process of inquiry may either be undermined or sustained/improved during instruction. Occurrence of the latter trait is more likely to manifest itself in environments where students become truly responsible autonomous learners working within social norms of collaborative learning. When students become independent self-regulated learners, their activity might generate different mathematical trajectories to those intended by the teacher. The 'young mathematician' might be confident enough to choose what resources to use and to consider different lines of inquiry. Alternatively, as a number of studies show, teachers have a propensity in reducing the demand level of the task related to classroom norms, task conditions, and teachers' and students' dispositions (Henningsen & Stein, 1997). For example, Desforges and Cockburn (1987) report this tendency occurring when students are struggling or when they give up. Likewise, during my action research, I faced situations where a particular student continuously asked for help when she fell behind compared to the others. During our interview, this student reported that I usually made it easier for her to finish tasks. My task enactment evidently avoided the student's frustration and speeded up task completion. Bound by time constraints, I occasionally reverted to forms of telling to move on the class to the next phase, namely, the whole-class presentation and discussion. Such teacher interventions direct students' attention towards priority in task completion – at times at the expense of more meaningful and deeper understanding. With hindsight I argue that help directed towards the outcome of the activity hinders the process of learning by investigation and inhibits students' cognitive development.

Concluding Remarks

Implementing investigative tasks essentially involves four-phases: tasks as planned and designed by the teacher; tasks as presented to the students; tasks as negotiated by students; and tasks as concluded by the students and the teacher (Ponte, Segurado & Oliveira, 2003). The study of these inter-related phases has informed practitioners, researchers and academics about design issues originating from Theme B – J. Calleja

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students' engagement with a range of tasks (see Stein et al., 2000). Of foremost importance here is the formulation of principles, guidelines and reflective frameworks for effectively exploring how students negotiate and engage with tasks. As I see it, this awareness is based on a reflective framework that includes: (i) a clear understanding of the purpose of tasks – one that resonates well with active learning; (ii) a classification within which different types of content-related tasks could be fitted – this alignment would be useful in examining how different tasks are designed and presented by the teacher and eventually enacted by students in class; (iii) a taskdelineated scheme that defines the teacher's hypothetical learning trajectories and accounts for students' learning trajectories – this scheme hence also defines the didactical approach undertaken by teachers in class; and (iv) an account of the range and type of resources provided to students in order to render tasks more accessible to all students – this also provides a basis for investigating how students make use of the resource materials supplied and consequently on the ensuing activity.

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Writing the Student into the Task: Agency and Voice
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In this paper, we characterize tasks with respect to intention,
action and interpretation to generate insight into student agency and voice.
It is our contention that a written mathematical problem cannot be
seriously discussed as a mathematical task without specification of the
intended purpose, participants, and product. In addition, differences
between social, cultural and curricular settings, together with differences
between participating classroom communities, shape the performative
realization of a mathematical task. This challenges reductionist attempts
to characterize instructional tasks independent of these considerations.
Given this contextual dependence, any commonalities across context carry
significant weight. Central to an understanding of "mathematical task" as
enacted in classroom settings is the social distribution of responsibility,
agency and voice. Our research shows that competent teachers in several
countries utilize mathematical tasks in a way that maximizes student
agency and voice. We propose that the prioritisation of student agency
and voice should be one of the principles of task design.
Keywords: Task Design, Agency; Voice; Didactical Triangle
Introduction
The classroom performance of a task is ultimately a unique synthesis of task,
teacher, students and situation. The activity that arises as a consequence of a student's
undertaking a task is itself a constituent element of the learning process and the
artefacts (both conceptual and physical) employed in the completion of the task serve
simultaneous purposes as scaffolds for cognition, repositories of distributed cognition
and cognitive products. Task selection by teachers initiates an instructional process
that includes task enactment (collaboratively by teacher and student) and the
interpretation of the consequences of this enactment (again, by teacher and student).
In the context of Theme B, this paper addresses the question: How do students'
reactions influence teacher adaptation of the task? We argue that this question does
not adequately portray the fundamental reflexivity by which the actions of teacher and
student are mutually informing during the performance of a task. There is no doubt
that student response influences the teacher's incremental and iterative adjustment of
the task as performed in the classroom, but, equally, these progressive adjustments by
the teacher iteratively influence the nature of student responses to the task, while
continually re-constructing the task itself in the act of its undertaking. Something of Theme B – D. Clarke & C. Mesiti
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this dynamic reciprocality may be captured by the inclusion of other Theme B
questions as contributing considerations: "What do students actually do and attend to
when confronted with tasks?" and "How do students understand the purposes of tasks
they are given in the classroom." In combination, these questions provide our focus.
Marx and Walsh (1988) identified three essential elements to any
consideration of the role of 'academic tasks': the conditions under which the tasks are
set; the cognitive plans students use to accomplish tasks; and the products that
students create as a result of their task-related efforts. This conception either ignores
the role of teacher intentionality and mediation, or it relegates this to just another
element in the social context in which the task is undertaken. There is a further danger
that student agency and authorship are identified solely with the products of the task
and their role in the performative realisation of the task itself goes unrecognised.
Our conception of the teacher /student /task triad is highly interconnected and
accords significant agency to each in the determination of the actions and outcomes
that find their nexus in the social situation for which the task is the pretext. It is our
contention that a written mathematical problem such as "Find the average of 13, 15,
17, 19, and 21" cannot be seriously discussed as a mathematical task without
specification of the intended purpose, participants, and product. Even with this
specification of intention, the performative realisation of the task is a collective
mathematical performance by the teacher and students in a particular context for a
particular purpose, generating particular products. "Find the average of 13, 15, 17, 19,
and 21" could be referred to as the task statement or the task stimuli, but it is only one
component of the task as enacted by particular people at a particular time and place
for a particular purpose. Even the imprecision of "average" cannot be either praised as
a well-intentioned attempt to elicit information about student understandings of mean,
median and mode, or criticised as mathematical sloppiness, until we know the
teacher's intentions for the task.
We have examined the function of mathematical tasks in classrooms in five
countries. A three-camera method of video data generation (see Clarke, 2006), was
supplemented by post-lesson video-stimulated reconstructive interviews with teacher
and students, and by teacher questionnaires and copies of student work. Our analysis
characterized the tasks employed in each classroom with respect to intention, action
and interpretation and related the instructional purpose that guided the teacher's task
selection and use to student interpretation and action, and, ultimately, to the learning
that post-lesson interviews encouraged us to associate with each task. In this paper,
we draw on some of the findings from that analysis.
Our analysis employed 'function' as the combination of intention, action and
interpretation to examine the functionality of mathematical tasks in classroom
settings. Of particular interest were differences in the function of mathematically
similar tasks when employed by different teachers, in different classrooms, for
different instructional purposes, with different students. The significance of
differences between social, cultural and curricular settings, together with differences
between participating classroom communities, challenges any reductionist attempts to
characterize instructional tasks independent of these considerations. Of equal interest
were differences in learning outcomes arising from the use of fundamentally different
mathematical tasks, such as highly decontextualised or abstract tasks (see the Chinese
examples below) in comparison with contextualized or so-called 'real world' tasks.
In relation to the related advocacies of relevant and authentic mathematics,
Kirschner, Sweller and Clark (2006) make the insightful observation that Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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It may be an error to assume that the pedagogic content of the learning experience
is [should be] identical to the methods and processes (i.e., the epistemology) of
the discipline being studied (p. 84).
In particular, their assertion that "The practice of a profession is not the same
as learning to practice the profession" (p. 83) highlights a critical issue in the design
of instruction in mathematics. How is classroom mathematical activity related to the
activity of the mathematician? While we may classify the tasks of the mathematics
classroom in a variety of ways, we should not confuse those tasks with the tasks of the
mathematician: they are fundamentally different in purpose.
Mathematical tasks employed in educational settings have been variously
categorised under designations such as 'authentic,' 'rich' and 'complex.' The
classification 'authentic' has particularly emotive overtones – suggesting that some
mathematical tasks might be classified as 'inauthentic.' The most common usage of
the term 'authentic' in this regard seems to refer to an assumed correspondence
between the nature of the task and other mathematical activities that might be
undertaken outside the classroom for purposes other than the learning of mathematics.
The value attached to 'authentic mathematical tasks' seems to appeal to a theory of
learning that measures mathematical understanding by the capacity to employ
mathematical knowledge obtained in the classroom in non-classroom ('real-world')
settings and which constructs the process of mathematical learning as 'legitimate
peripheral participation' (Lave & Wenger, 1991) in the mathematical activities of a
community larger than a mathematics class. Such apprenticeship models deny or
ignore the significance of the students' role as agents in their own learning, actively
shaping classroom activity through their participation and progressively becoming
more skilled, not as mathematicians, but as mathematics students.
The eighth-grade mathematics classrooms that provided the sites for our
analysis were drawn from the data set generated by the Learner's Perspective Study
(LPS) (Clarke, Keitel, & Shimizu, 2006). Our initial goal in the analysis of
mathematical tasks undertaken in these classrooms was the selection of tasks that
could legitimately be described as distinctive because of the character of the
mathematical activity or because of the teachers' didactical moves in utilising the
tasks to facilitate student learning. In this paper, we will use a small selection of very
different examples to make two points that we suggest should be central to task
design in mathematics education:
(i) The competent teachers that we studied prioritised student agency in the
classroom performance of mathematical tasks;
(ii) In promoting high quality mathematical activity, the teacher's capacity
to choreograph sophisticated classroom discourse was as important as
the mathematical sophistication of the task statement or question.
Sample Task One: Japan School 1 – Lesson 1 (the Stairs Task)
[For reasons of brevity, 15 minutes of data record has been omitted from the
middle of the table]
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Educational Context of the Task
This was the first lesson in a sequence of lessons concerned with functions, relations and patterns,
where particular emphasis was placed on the special terms used in mathematics. The teacher identified
her global aims for the entire lesson sequence of about sixteen lessons, as: i) identifying functions and
their relationships to everyday life; and ii) understanding how to solve equations using a table, a graph
or formal algebraic techniques. This particular lesson was designed by the teacher to focus on: i)
different variables and their relationships with one another; ii) understanding the form of a linear
equation; and iii) understanding that the investigation of the nature and function of equations is of
utmost importance.
Social Performance of the Task
Teacher Mathematical Task
as presented
Student
The teacher states her goal to the
class, "I'd like to think about
change using these figures."

The teacher asks students to work
on drawing the next two figures in
the sequence.
The Stairs Task
1) The first three figures
have been drawn for you.
Draw the next two figures
by stacking one cm sided
squares on top of each other
(Students attentive)

Nou is invited to the board to
draw the next two figures.
The teacher invites students to
present ideas on identifying what
aspect of the figures is changing.
The teacher invites students to
work in small teams to identify as
many aspects as they can.
2)What changes when the
number of steps changes?
Jitsu immediately suggests, "the
number of steps."
Nobo and Nou add "size" and
"area" respectively.
Taka adds "height."
(The teacher roams the classroom
and speaks with individual
students)
The teacher addresses the class and
invites further suggestions on
"what changes?"
Students working in small
groups)
Nii adds "number of sides" and
"number of squares."
Mika adds "circumference."
Taka says, "shape."
Nou responds with "the length of
the base."
Jitsu adds "the time it takes to
draw the figures."
Nobo adds "sum of the interior
angle" and "the number of
vertices."
The teacher invites students to
suggest how they might go about
examining the relationship between
the number of steps and the
circumference.
The teacher reiterates Mawa's
mention of the use of a table and
reminds students that mathematical
expressions are also useful in
examining relationships.
She proceeds to draw up a table:
3) Examine the relationship

between the number of steps
and the circumference.
Some students suggest,
"graphs."
Mawa mentions "table." Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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She asks students to complete the
table and to identify a
mathematical relationship
(The teacher roams around the
room assisting students)
(Students go to the board
and complete the table or
write an equation)
(~15 minutes)
(Students work on the assigned
task at their desks, while some of
their classmates go to the board)
The teacher asks for other
interpretations - other methods for
getting the answer.
The teacher highlights this method
graphically with Nobo's help:
Nii responds that you can add
four to the previous answer, so
that to get the circumference for
six steps, one adds four to 20.
The teacher assigns the homework
task:
4) Show that 'multiplying by
four' works.
5) Examine the relationship
between the number of steps
and another feature of the
diagrams.
(students attentive)
The Stairs Task is well-known and used by mathematics teachers in many countries. This teacher offered students several opportunities to exercise agency while undertaking the task. For example, the students were able to identify a large number of differing variables. The discussion then converged on the particular connection between the number of steps and the circumference and began to model a mathematical approach to defining this relationship. The teacher's focus on the students and the partnership she had formed with them in their learning was further evidenced by students' contributions at the board: Nou drew steps four and five; Yama traced one of the figures with his finger to illustrate the concept of circumference; Ume completed the table, while Taka wrote the relationship in algebraic terms; and Nobo shared his various conjectures of relevant variables. Once the relationship (y = 4x) was proposed, the teacher's attention turned to how to demonstrate the correctness of this equation, "Think about how to show that multiplying by four works." Her recurrent emphasis was on the collaborative exploration of the problem in the interest of promoting student reflection and learning.
Task Two: China School 2 - Lesson 8 (the Numbers task)
The class was learning about the system of linear equations. The students have spent previous lessons working with inequalities, the relationship between the concept of linear equations in two unknowns and their solution, the transformation of equations and solution methods involving substitution and elimination. This was the eighth lesson in this topic and one of the teacher's goals for this lesson was for students to learn to solve some special linear equations in three unknowns. The task statement consisted of: Theme B - D. Clarke & C. Mesiti
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There is a three digit number. The sum of the three digits is 12. The sum of the hundreds digit and the tens digit is greater than the ones digit by 2. Three times the hundreds digit equals the sum of the tens and the ones digits. Let the hundreds digit be x, and the tens digit be y and the ones digit be z. Set up the equations according to the question.
The teacher's purpose in using this task was to introduce students to the underlying structure of algebraic representations and to assist them to develop appropriate mathematical language. The task was presented to the class and three students were able to correctly provide the individual equations to form the entire system. At this point, the teacher spent some time identifying the exact nature and definitive characteristics of a system of linear equations in three unknowns. The students were not encouraged to actually solve the equations, algebraically or by other means. The task consisted entirely of setting up the system of equations. Students were then presented with the following system to solve:
x + y = 11
y + z = 17
x + z = 10
□
□
□
□
□
Our interest in this task lies in the significant devolution of responsibility to the students to generate solutions and the prioritisation of the development of student facility with the technical language of mathematics.
Task 3. Japan School 3 - Lesson 1 (the Long Task)
In this task, the seemingly simple pair of simultaneous equations 5x + 2y = 9 and -5x +3y = 1 engaged the class for a fifty-minute lesson (and indeed was the discussion point for the first fifteen minutes of the following lesson). A feature of the performance of this task was the extent to which student suggestions, responses and the articulation of their thinking were regarded as instruments for developing understanding.
The teacher, when asked about his aim for this lesson responded, "It's not that I just wanted them to just solve the problems but also, um, I wanted to teach them that there is a need to think about it a little - what solving equations is all about." The emphasis on prompting student reflection, enacted in this class through social interaction, rather than merely on the solution of a fairly trite mathematical problem distinguished this task performance and this teacher's practice.
This third task provides a stark illustration of the crucial nature of the teacher's role. It also problematises the characterisation of a task as a "good task" independent of the social setting and actions through which the task is undertaken. Our analysis prioritises consideration of the social interactive aspects of task performance and, in particular, interactions that constitute the progressive restructuring or reconception of the task itself.
Theoretical Alternatives in Analysing Classroom Task Performance
There are many different theories currently being employed in mathematics education. Activity Theory, for example, is an obvious contender in considering how the classroom use of mathematical tasks might be situated theoretically. Recent developments in the conceptualisation of Activity Theory (eg Engestrom, 2001) have increased the breadth of phenomena and contexts able to be addressed using Activity Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Theory. In particular, mathematical tasks can be situated naturally within the tools available for use in pedagogic activity systems.
Gellert (2008) usefully contrasts 'interactionist' and 'structuralist' perspectives on mathematics classroom practice. In the consideration of the classroom use of mathematical tasks, the interactionist perspective offers insight into the negotiative processes that interact with individuals' use in classroom settings for the socially-mediated constitution of learning. The structuralist perspective potentially offers very different insights into the deployment and function of mathematical tasks in classrooms. Focusing attention on differentiated participation, a structuralist analysis aspires to explain such differentiation in terms of hierarchies and power relationships. In the case of mathematical tasks, these hierarchies reflect the enactment of an entrenched social order and the privileging of particular forms of knowledge. Within the structuralist perspective, particular pedagogies can be seen as embodying systems of social and academic privilege (Bernstein, 1996) and in the mathematics classroom it is primarily through the performance of mathematical tasks that these pedagogies are enacted.
The choice of the theoretical lens focuses analytical attention on some aspects of the role of mathematical tasks and ignores others. This is inevitable. Another entry point employs the three related issues of Abstraction, Context and Transfer. In some discussions, abstract mathematics seems to be treated as simply decontextualised mathematics. Clarke and Helme have argued that there is no such thing as decontextualised mathematics (Clarke & Helme, 1998), since all mathematical activity is undertaken in a context of some sort. If abstraction in mathematics is to have any legitimacy or relevance, then it must reside in some form of generalisability of the mathematical matter under consideration, in the sense that the principle, concept or procedure can be thought of as transcending any particular context or instance. But, to argue that an exercise in Euclidean geometry or in pure number is an abstract task is to deny the social situatedness that has become accepted even from the most cognitivist of perspectives (Lave & Wenger, 1991).
In relation to mathematical tasks, Clarke and Helme distinguished the social context in which the task is undertaken from any 'figurative context' that might be an element of the way the task is posed. In this sense, the task:
Siu Ming's family intends to travel to Beijing by train during the national holiday, so they have booked three adult tickets and one student ticket, totalling $560. After hearing this, Siu Ming's classmate Siu Wong would like to go to Beijing with them. As a result they buy three adult tickets and two student tickets for a total of $640. Can you calculate the cost of each adult and student ticket?
(Shanghai School 3, Lesson 7, Train task)
has a figurative context that integrates elements such as the family's need to travel by train and the familiar difference in cost between an adult and a student ticket. The social context, however, could take a wide variety of forms, including: an exploratory instructional activity undertaken in small collaborative groups; the focus of a whole class discussion, orchestrated by the teacher to draw out existing student understandings; or, an assessment task to be undertaken individually. In each case, the manner in which the task will be performed is likely to be quite different, even though we can conceive of the same student as participant in each setting. Theme B - D. Clarke & C. Mesiti
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The Three Tasks
The three tasks were selected for their disparity across the key attributes: mathematics invoked (both content category and level of sophistication); figurative context (real-world or decontextualised); resources utilised in task completion (diagrams and other representations); and the nature of the role of the task participants. Students were given a significant "voice" in the completion of each task, but the nature of their participation reflected differences in the extent and character of the distribution of responsibility for knowledge constructed in the course of task completion. This distribution of responsibility (or enhanced agency) is a consequence of each teacher's strategic decision, moment by moment, of how best to orchestrate student work on the task. In seeking to understand task performance as the iterative culmination in the joint construction, not only of the task solution, but of the mathematical principles of which the task is model and purveyor.
If we take 'transfer' not as a description of a particular cognitive process, but as a metaphor for a skill developed in one context being used in a different context, then it is reasonable to ask, "Under what conditions (and through the instructional use of what tasks) will the likelihood of transfer be maximised?" A cognitivist might direct attention to the selective variation of task attributes with the intention of successively focusing student attention on salient aspects of the mathematical concept or procedure to be learned. Variation Theory (Marton and Tsui, 2004) identifies learning with an increasing capacity to discern relevant attributes in the object of learning. From such a perspective, particular tasks and particular sequences of tasks can be critiqued as more or less conducive to directing student attention appropriately and thereby to the optimal promotion of the discernment that is identified with learning.
Distributed Cognition (Hutchins, 1995) and other theories with a material semiotic character accord significance to artefacts as participating in cognition. Rezart and Straesser (2012) have expanded our conception of socio-didactical situations in mathematics classes, and include artefacts in addition to the teacher, the students, and the mathematics. Once representational forms are included in the broad class of artefacts, then mathematical tasks cease to be either the objects to which we apply our cognitive tools nor merely the social catalysts for their deployment. Rather, mathematical tasks become the embodiment of performed cognition, integrating, as they do, representational forms, socio-cultural imperatives and mathematical entities. We find it useful to portray mathematical tasks performatively in order to examine the role each task plays in affording or constraining agency and voice in the social settings in which the tasks are communally performed. Our conception of the teacher/student/task triad accords significant agency to each in the determination of the actions and outcomes that find their nexus in the social situation for which the task is the pretext.
In each of the three tasks shown, the teacher prioritised student agency in task completion, but in very different ways: The Japanese teacher opened Task One by stimulating student discussion of what had noticeably changed during the recent holiday break. This discussion, drawing on student personal experience, situated the mathematical activity in relation to out-of-class contexts with which the students would be familiar. In Task Two, the teacher's purpose was to introduce students to the underlying structure of algebraic representations, and the task as posed was stripped of any elements that might invoke a figurative context. However, even lacking any sense of familiar context, other than the classroom setting itself, student Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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active participation in the task performance was still prioritised. Task Three was chosen because of the fascinating use by the teacher of a relatively pedestrian pair of simultaneous equations to scaffold students' developing understanding of the general attributes and properties of systems of linear equations. In the performance of this

task, student suggestions, responses and the elicited articulation of their thinking became instruments for developing understanding.
Conclusions
Tasks have long been recognized as crucial mediators between mathematical content and the mathematics learner. Of particular interest in our analysis were differences in the function of mathematically similar tasks, dealing with similar mathematical content (those relating to systems of linear equations), when employed by different teachers, in different classrooms, for different instructional purposes, with different students. The "entry point" for our analysis was a tabulation of the details related to the social performance of the task (as shown for Sample Task One). Using these tables, our analysis drew on the video-stimulated, post-lesson interview data to identify intention and interpretation and relate both to social performance of the task.
The conception that the community-at-large holds of the mathematics classroom is intrinsically bound up with the type of tasks that characterise such settings. And this conception is not in error. Mathematical tasks are the embodiment of the curricular pretext that brings each particular set of individuals together in every mathematics classroom. In other contexts, individuals come together to engage in musical performances or dramatic performances. The performances of the mathematics classroom are largely the performance of mathematical tasks and if we are to understand and facilitate the learning that is the ostensible purpose of such settings then we must understand the nature of the performances that we find there.
The thread that we pursued through the examples discussed has been that of the social distribution of responsibility, agency and voice. We commenced our analysis disposed from other studies to believe that these issues were important. Our exploration of responsibility, agency and voice in the context of the classroom performance of mathematical tasks suggests to us that competent teachers of mathematics (within the constraints of culture and curriculum) share a belief in the importance of these elements. The valuing of agency and voice is evident in the task performances in the classrooms of these teachers, rather than in any explicit articulation by them in classroom video data or in interview.
If we are to find pattern and structure in the profound diversity of "welltaught" mathematics classrooms around the world, then the attention given by competent teachers to student voice and student agency, and the mathematical tasks that they employ to catalyse that voice and agency, support our belief that the maximization of student agency and voice in the performative enactment of a mathematical task should be recognized as a key principle of task design and delivery.
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Making distinctions in task design and student activity

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In this article, we articulate design principles that have developed, during the time of the authors' collaboration, over a period of fifteen years. The principles are drawn both from the enactivist theory of cognition and learning (Varela, Thompson and Rosch, 1991) and the pedagogic ideas of Gattegno (1987). We exemplify how task design centres around activities that provoke differences in student response, allowing the opportunity for students to make distinctions and for teachers to introduce new skills. We illustrate how the principles operate to inform teacher planning, teaching actions in the classroom and students' mathematical activity. The fact the principles operate on these three levels means that student activity closely matches teacher intentions.

Keywords: task design principles, mathematics teaching, student activity, enactivism, Gattegno, distinctions

Introduction

In this article we draw out implications of task design principles for our understanding of how teacher intentions and student mathematical activity may begin to converge. We address the questions: what do students actually do and attend to when confronted with tasks? how do students' reactions influence teachers' adaptation of the task?

Our design principles have developed over the fifteen-year period of our collaboration; they arise from our epistemological commitment to enactivism (Varela, Thompson and Rosch, 1991) and the pedagogical influence of the work of Gattegno (1987). Following this introduction, we offer a brief sketch of these perspectives, we then set out and exemplify our design principles and present evidence from the practice of two teachers to analyse the students' mathematical activity.

The particular community that is the focus of this article centred around one school (School S) in the Bristol area of the UK, that was in 'partnership' with the University of Bristol, meaning that the mathematics department took student-teachers from the PGCE course on an annual basis. Alf Coles joined this department in 1996, having already begun a research collaboration with Laurinda Brown. Alf became a school-based mentor for the PGCE course and, in 2001, head of the mathematics department. Between 1996 and 2003, Laurinda made this school her main research site and visited, where possible, weekly. During these visits (often for a day) she would work in Alf's classroom, observing, sometimes co-teaching and always Theme B – A. Cole & L. Brown
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reflecting on events afterwards. This collaboration led to publications (e.g., Brown and Coles, 1997, 2010) and research projects (e.g., Brown et al, 2001).

We take, as our definition of task, the one suggested in the discussion document of this study group, 'a task is anything that a teacher uses to demonstrate mathematics, to pursue interactively with students, or to ask students to do something. Task can also be anything that students decide to do for themselves in a particular situation' (Watson and Ohtani, 2012, p.4). In any task, as well as learning some mathematics, students are learning about what learning mathematics is like in this classroom; for us, the choice to use of any task cannot be dissociated from a choice about ways of working.

There is a significant problem, identified in the literature, around the student experience of tasks, compared to the intentions of the designer or teacher. Mason, Graham and Johnston-Wilder (2005, p.131) raise the issue of how an expert's awarenesses get translated into instructions for the learner that do not lead to those same awarenesses. Mason et al (2005) connect this issue to Chevallard's didactic transposition (1988), the problem of moving from the knowledge used in a sphere, such as mathematics, to the knowledge to be taught. The contexts in which mathematical knowledge is used can never be faithfully replicated in a classroom. We see a similar issue being highlighted by Tahta (1980) when he distinguishes 'outer' and 'inner' aspects of tasks. The outer task is what is made explicit by the teacher, the inner task is the relationship or awareness the teacher hopes students will gain. The problem for teachers is that the more the desired behaviours in students are specified, the less these behaviours are likely to emanate from students' own awareness. We see essentially the issue again being described by Watson and Mason (2007):

Engaging learners in activity is important, but in order to learn from that activity they need to experience some kind of shift or transformation in what they are sensitised to notice and attend to mathematically. (p.209)

We also agree with Watson and Mason's (2007) emphasis on:

the importance of developing ways of working, a classroom rubric in which the learners are drawn into patterns of thinking, in which some transforming action takes place (p.210)

In other words, one solution to the problem of how to connect teacher intentions and student activity, is to focus on developing particular ways of working, or patterns of thinking with students. In this article, through analysing examples of what students do when faced with tasks and how teachers respond, we suggest that the issue can also be addressed through a focus on the making of distinctions in task design. However, before getting to this argument, in the next sections we set out our epistemological and mathematical perspectives and the design principles that operated in School S.

Enactivism

Enactivism entails a commitment to the inseparability of thought and action (Varela, Thomson and Rosch, 1991) and a rejection of the view that we operate in the world via making use of inner representations of objects, with thoughts about these representations leading to actions. Rather, we are closed to the world, from an informational point of view. All the world can do is 'trigger'; how we respond is determined by our structure, that is, by the particular relations and physical components that make up ourselves. All living beings are, in this sense, "structure determined" (Maturana 1988, p.12). Through a recursive process of acting in the Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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world and the world acting back we learn to co-ordinate our actions with others, as we 'co-evolve' (Varela, Thomson and Rosch, 1991, p.201) in a process labeled 'structural coupling' (Maturana & Varela, 1987, p.75). In each interaction my structure changes, as does the world. It is our history of structural coupling (both as a species and in each individual lifetime) that accounts for the smooth functioning of so much of our life. Varela writes, on the enactive view, 'perception is perceptually guided action' (1999, p.12), i.e., perception is not a passive receipt of information (which we represent) but part of an active (on-going and recursive) process of structural coupling with the world. Over time, how we act can change how we perceive (an example being the wine taster, who is able to make distinctions literally imperceptible to the untrained palate).

Within enactivist epistemology, practical knowledge linked to action is fore grounded, with more propositional knowledge arising from awareness of action. Maturana and Varela (1987), key enactivist writers, state, 'cognition is effective action, an action that will enable a living being to continue its existence in a definite environment' (1987, p. 29). So an action is effective if it allows me to continue operating in a specific context. Knowledge, which requires cognition, is therefore also equated with effective action. Knowing cannot be separated from the knower nor the context in which the knower acts, hence, '[a]ll doing is knowing and all knowing is doing' (Maturana and Varela, 1987, p.27).

The palate of the wine taster offers a metaphor for the enactive view of learning, in the making of ever-finer distinctions in a sphere of action. If learning is seen as the development of knowing, then learning is equated with being able to act differently (in new ways), which is also the same as saying that learning is equated with being able to perceive differently.

Enactivism also carries implications for how learning can occur, which arise from the notion of structural determinism. In a classroom, the most significant features of the 'environment' for any individual are the other individuals in the room. Any interaction between teacher and student or between student and student must alter the structure of both, in the process. Enactivism is a profoundly social theory, we are quite literally changed through interaction with others, or more precisely, we change ourselves through interaction with others who likewise change themselves. We cannot not change, however minimally, in every encounter. Equally, we cannot specify any change we want to provoke in others. It is in this sense that we understand Stewart (2010), in a book on enactivism, when he writes, 'instruction, in the strict sense of the word, is radically impossible' (p.9). All we can do, as teachers, is provoke, stimulate, trigger; how students respond will be a function of their own structures and histories and cannot be determined by us.

Mathematical thinking

Our view of mathematics has been influenced by Gattegno (1987). Gattegno saw mathematics as the awareness of relationships (1965). Awareness was a technical term for Gattegno, whose purpose is, 'to illuminate our fields of action' (1987, p.25). He turned "awareness" into a countable noun – we can enumerate awarenesses. We are being algebraic whenever we step back from a procedure, or a dynamic, to become aware of it. Writing something as simple as x+y, entails a stepping back from the process of addition, to represent that dynamic. In fact, for Gattegno, all mathematics was algebraic, since awareness of relationships and awareness of dynamics are intimately linked, 'all is algebra in mathematics, because to say Theme B – A. Cole & L. Brown
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"algebra" is to say the awareness of the mind at work on whatever content' (Gattegno, 1965, p.22). Gattegno privileged the visible and the tangible, in the learning of mathematics. He worked with imagery and another quotation of his that we have both lived with, is that mathematics 'is shot through with infinity' (1984, p.20). The connection we make here is that, once we become aware of a relationship (i.e., once we are mathematizing, or thinking mathematically) then it becomes possible to imagine that relationship iterated and hence to infinity. Gattegno also distinguished

'powers of the mind' (1971) that he believed all humans possess, which mean all
humans are able to think mathematically. These powers (also referred to in Brown
and Coles, 2011, p.866) are:
1) extraction, finding 'what is common among so large a range of variations'; 2)
making transformations, based on the early use of language 'This is my pen' to
'That is your pen'; 3) handling abstractions, evidenced by learning the meanings
attached to words; and 4) stressing and ignoring, without which 'we can not see
anything' (paraphrased from Gattegno, 1971, pp. 9-11).
Making distinctions play a role in (1) and (4), linking to our enactivist stance.
Principles of task design
Arising directly from the epistemological and mathematical perspectives
above, the design principles that operated in School S included the following list,
which we will unpack and exemplify. This list is not presented as a finished product
and was not made explicit within the department. However, over the course of the last
fifteen years, within the community around School S, various of these principles were
spoken about and worked on as a group. Two mathematics teachers from School S
have moved to be Heads of Mathematics in different schools, and have established
their own curricula based on similar principles for activities. The principles are:
1.starting with a closed activity (which may involve teaching a new skill).
2.considering at least two contrasting examples (where possible, images)
and collecting responses on a 'common board'.
3.asking students to comment on what is the same or different about
contrasting examples and/or to pose questions.
4.having a challenge prepared in case no questions are forthcoming.
5.introducing language and notation arising from student distinctions.
6.opportunities for students to spot patterns, make conjectures and work on
proving them (hence involving generalising and algebra).
7.opportunities for the teacher to teach further new skills and for students to
practice skills in different contexts.
Principles 2, 3, 5, 7 come from the enactive view of knowing and learning,
which is linked to the making of distinctions. By working with at least two examples,
we support students in sharing distinctions and, through this sharing, making new
distinctions, which is tantamount to learning from the enactive perspective. Language
and notation is introduced to label distinctions students make, to support new ways of
seeing. By 'examples' (principle 2), we mean to capture a wide range of possibilities,
including images, animations and procedures. Principle 6 comes from our view of
mathematics as being essentially about the activities of conjecturing, no matter what
content is being covered. Principles 1, 4 and 7 derive from Gattegno. The closed
activity will, where possible, involved something visible or tangible and which all
students can do. The challenge and opportunity to teach skills in different contexts are Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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linked to the power humans have of extraction. When we do something in different
contexts, it is more likely we can extract the skill are retain it for use another time.
In the next section we exemplify these principles, first with a task in the
scheme of work of School S and then two tasks that arose in lessons.
Principles exemplified
Equable shapes
One exemplification of these design principles, is a task (we call it 'Equable
Shapes') that begins with the shapes in Figure 2 drawn on the board. The origins of
this task are obscure. The version of the task presented below was developed at
School S, from an idea Laurinda had used in her teaching. Often, in School S, tasks
would be borrowed from elsewhere but worked on, by Laurinda and Alf in the earlier
years of our collaboration, and as a department when Alf became head of
mathematics. There was a scheme of work in the school which specified certain
'common tasks' (such as 'Equable Shapes') had to be done in a certain time slot with
a certain year group. Discussion amongst teachers often led to changes in task
beginnings, which were written up for the following year. What follows is a
condensed version of the kind of write up that would be in the scheme of work, for
teachers. We have analysed the write up against the seven design principles above.
10 cm
2.5 cm
3 cm
8 cm
Figure 2: Two contrasting examples
These shapes are 'two contrasting examples' (principle 2). With this image
on the board, the teacher asks students, 'what is the same and what is different'
(principle 3). Generally, in the UK, students will at some point begin to comment
about the different sizes of the rectangles. Students may or may not need support in
remembering that mathematicians use the ideas of 'area' and 'perimeter' to judge the
size of rectangles. This awareness leads the teacher to invite the students to find the
area and perimeter of both shapes (a closed task, principle 1). The results are:
10 by 2.5 cm rectangle: Area = 25cm
2
, Perimeter = 25cm
7 by 3 cm rectangle: Area = 21cm
2
, Perimeter = 20cm
Again, the offer from the teacher is for students to comment on anything they
notice that is the same or different. A student will usually notice that for the first
rectangle, the value of the area and perimeter is the same. At this point, the teacher
introduces the label 'equable' (principle 5) as a name for the 10 by 2.5 rectangle. The
teacher can then ask students what questions they could pose (principle 3) and gather
ideas on the board. The teacher-prepared challenge is: what other equable rectangles
can we find? are there equable shapes that are not rectangles? (principle 4). Just the
work on rectangles, offers opportunities for pattern spotting, generalising, algebra and
conjecture (principle 6). In working on generalising what they notice, students are Theme B – A. Cole & L. Brown
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using skills of distinguishing area and perimeter and, depending on the direction in
which the activity goes, skills may need to be taught to support students in solving
linear equations, using Pythagoras' theorem, using trigonometry, while focused on the
idea of finding equable shapes (principle 7).
In keeping with our own design principles, we now offer contrasting
examples of tasks that arose more or less spontaneously in the practice of two
teachers. These examples allow us to address the questions, when using the design
principles above: what do students actually do and attend to when confronted with
tasks? and, how do students' reactions influence teachers' adaptation of the task?
Transcript 1: Fractions of quantities
The transcript below is from a video recording of a lesson, taken as part of
one of the research projects (Brown et al, 2001) that involved teachers at School S
(among other schools). Laurinda directed the project and Alf was a teacher and
researcher. The transcript shows the emergence of a task spontaneously within a
lesson of Teacher A, who worked in School S. This kind of sequence in a lesson often
occurred in School S, evidenced in video recordings of lessons, discussions amongst
teachers and visitor observations. The original problem for students had been to find
rectangles with area 12. Three examples were on the board, when Student 2 was given
the pen. The students were in year 7 (aged 11-12) and the examples on the board were
a 3 by 4 rectangle, a 2 by 6 and a 12 by 1 (Teacher A refers to these examples in line
42).
[Student 2 takes the board pen and board ruler and draws a half square by 24
square rectangle, giving it the label
1
/2 by 24.]
39 Students: Half a square?
40 Student 2: Half a square.
41 Student 8: Half of 44, half of 48, sorry.
42 Teacher A: Excellent. Oh, lovely. Well done. [Students applaud] So, 3 times 4
is 12, 2 times 6 is twelve, 1 times 12 is twelve and a half times 24 is also 12.
43 Student: And do we do that as well.
44 Teacher A: Pardon.
45 Student 8: And a quarter times 48 is twelve.
46 Teacher A: And a quarter times 48 ...
47 Student 8: And an eighth times ...
48 Student: Three quarters.
49 Teacher A: And an eighth times ...
50 Student: I'm not saying.
51 Student: You can actually go on.
Following this exchange, the teacher then wrote ¼ x 48 and got two students
to confirm the answer. Teacher A then wrote on the board (see boxes) as the
conversation developed.
57 Teacher A: What about a third?
58 Student: What? 1 x
59 Student 9: 36. 3
60 Student: No, you can't really draw a third.
61 Teacher A: Why, how do you work it out
for those that are struggling a bit. [Directed at 1 x 36
Student 9.] How do you know? He's right, it is 3
36. How do you know it's 36?
Two students responded to this question, and then Student 10 said, '3 twelves
are 36'. Teacher A picked up this idea and asked what 4 twelves are, pointing to the ¼
x 48 written on the board. Teacher A asked (line 69), what
1
/6
th
would be, a student Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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responded '72'. Another student asked (line 72), 'What about 100? How could you
draw it though?' and three turns later, a student asked (line 76), 'Sir, what would just
a straight line be?'.
In this short episode, that was not planned, we see the design principles in
action. The questions the class were initially working on are closed (principle 1), for
example, line 49, 'And an eighth times'. The teacher wrote up ¼ x 48 and
1
/3 x 36, so
there were two examples for students to see (principle 2). Students were patently the
ones posing the questions and sometimes answering them, for example in lines 45, 47,
72 and 76 (principle 3). The teacher contributed to posing challenges in lines 57, 61,
69 (principle 4). It is the students who first introduced the language of fractions,
Teacher A provided the notation (e.g., ¼ x 48) to describe the rectangles they were
considering (principle 5). There is clearly scope for spotting patterns in this task, as
many students did; generalizing came with the awareness that you could take any unit
fraction as the height and still make a rectangle of area 12, taken to its limit by a
student in line 76, 'what would just a straight line be?' (principle 6). Part of what
students were doing, in this short episode, was practicing a relatively mundane skill of
finding fractions of quantities, but they were doing it in a novel context and with their
attention on the area of the resulting rectangle (principle 7).
There are sections of the transcript (lines 45-51) where, if names were
removed, it would be impossible to tell if students or teacher were speaking. This
exchange was prompted by a student drawing a shape with a fractional side length.
One connection that is made by a student (Student 10) is that to answer,
1
/3 ? = 12,
you can work out 3 12. We interpret this statement from Student 10 as an example
of the kind of shift in attention that Watson and Mason (2007, p.209) describe as
being central to learning mathematics. Student 10 articulated an awareness of a
relationship that then supported other students in extending the implied pattern (
1
/6
? = 12, means 6 12=?, etc), this awareness of relationship is mathematical thinking.
What is significant for us, in this lesson, is the way the design principles of
the department appear to have influenced how Teacher A adapts in an episode that
arose spontaneously from an unplanned student response to an activity. Teacher A
was creating the task as the lesson unfolded. Perhaps even more striking is the way
the students played a role in creating the task, with Teacher A allowing discussion of
student ideas to run and focusing the whole class on certain questions (e.g., lines 49
and 57). We know, from interviews carried out as part of the ESRC project, that
'going with' student ideas in the way we see in this transcript was something Teacher
A was, at that time, just beginning to experiment with.
Transcript 2: Both Ways
The transcript below is from a video taken in 2008 (as part of an ESRC
Studentship) of a teacher (W) who had been in the department for several years and
who had worked with this group of (aged 12-13) students for a year and a half. The
task we called "Both Ways", W had drawn the image in Figure 3 and invited students
to suggest a number to go in the top left circle.
Figure 3. Both ways
×
×
×
=Theme B – A. Cole & L. Brown
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A student had suggested 74 as the starting number, which the class then
worked through, getting answers of 740 in both circles in the bottom right. Teacher
W looked at these answers on the board as she said the first line of the transcript.
Transcript notation: (.) indicates a small pause, (2) a 2 second pause, ( ) undecipherable
speech
1 Teacher W: oh right (.) okay
2 Student: yeah but why (.) why does it come to that
3 Student: is that meant to happen miss
4 Student: yeah it is
5 Student: no
6 Student: oh yeah miss
7 Student: because if you do seventy four times ( )
8 Teacher W: um (4) any comments (.) any comments (.) yeah

9 Student: cos it can't be seven hundred and forty which is the bottom one (.)
because seventy four times five is three hundred and seventy and then (1) no (1)
no ignore me
10 Student: it (.) the two answers in the little circles no matter what you start up
there will always be the same (.) because if you start there no matter what (.) if
it's times five times two it's like times ten
A little later in the lesson, Teacher W drew a second "Both Ways" image, it
was the same as Figure 3 except the "x5" arrows were replaced by "÷2".
33 Teacher W: okay I've changed the number machine (.) you may not have
noticed (.) okay can we have any thoughts at the moment (1) about what's going
to happen in these two circles (1) any thoughts (.) Student 1 yeah go on
34 Student 1: might be (.) still going to be the same
35 Teacher W: same (1) anything else (1) any other comments
36 Student: I think they're going to be different because they're different order um
(4) what's it called when
37 Teacher W: order of operations
38 Student: ( )
39 Teacher W: so you think it's different (.) timesing by two and adding two is
different to adding two and timesing by two (1)
40 Student: yeah
41 Teacher W: okay so because they're different operations
42 Student: I think it's going to be the same as well (.) because both numbers are
times five and added by two (.) so it's kind of the same thing you're doing
43 Student: I reckon they're going to be different (.) because if you start with one
again (.) if you times by five and plus two it's going to be seven (.) and if you add
two first it'll be three and then you times five which is fifteen (.) so plus-ing on
two first will make it a bigger number
44 Teacher W: okay everybody can you draw that back of your book (.) don't
worry if it's a mess (.) just draw circles squares and two more circles (.) put those
in and I want you to choose your own starting number
This activity meets the design principles presented earlier, however, we offer
this transcript to focus on what students do. In line 2 and 3 we see students asking
questions having made the, perhaps surprising, distinction that there is no distinction
to be made between the final circles of Figure 3. In lines 7, 9 and 10 all the student
comments include the word "because" and in line 10 a statement of proof is offered as
to why the answers at the end are the same. When W sets up the second problem she
asks students what they think will happen. All the responses that follow use the
language of similarity and difference, commenting about reasons the answers at the
end might be the same or different. As in transcript 1, we see evidence here that
students in this department responded to tasks by making distinctions that lead to Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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them asking questions, noticing pattern and generalising. In the classroom of Teacher
W, some students' generalisation has taken the form of mathematical proof.
Discussion
We have demonstrated how our design principles are linked to both
enactivist and Gattegno's ideas about learning. Both perspectives see the making of
distinctions as one of the basic mental functions and a key to learning. The design
principles (2) starting with contrasting examples, (3) students comparing/contrasting
and (5) naming the distinctions that students make, are all linked directly to the
making of distinctions and hence, to learning. We see the principles operating in (at
least) three ways. Firstly, the principles inform teacher planning, for example in the
activity 'Equable Shapes'. In this task, the examples of the two rectangles inevitably
focus students on the distinction between perimeter and area, from which questions
and challenges can be generated that provoke further work with that distinction.
Secondly, transcript 1 provides evidence that, over time, these principles can also
inform teacher actions in the classroom, in adapting tasks in the light of student
responses. We are not suggesting there was necessarily any conscious decision
making on the part of Teacher A, linked to the principles; we see evidence that the
principles have become part of his practice of teaching, as analysed above. Thirdly,
there is evidence in transcript 1 and 2 that the principles can inform (again, implicitly)
student actions in the mathematics classroom. In answering what, given the use of our
design principles, do students do and attend to in response to tasks, the evidence from
these transcripts is that through making distinctions, students notice and extend
patterns, they ask questions and generalise.
In Teacher A's lesson (transcript 1) we see a snapshot of a way of working in
which the design principles of School S have become part of what students see
themselves doing in mathematics lessons. Student 2 offers ½ x 24, and another
student immediately offers ¼ x 48. In this context, the activities that Teacher A
chooses to offer students become less important, as students can make distinctions
and generate questions without prompting. We interpret the students in this short
excerpt as exhibiting 'inquiry as a form of engagement' (Watson and Mason, 2007,
p.213) rather than inquiry being structured into the task. We suggest one reason the
design principles are able to operate on three levels (to influence teacher planning,
teacher actions in the classrooms and student activity) is that they are based on a
theory of learning. The design principles embody how, as a department, we viewed
mathematical thinking; our stated aim, as a department, was to develop students'
mathematical thinking. These design principles helped close the gap between our
intentions for students to be thinking mathematically and what students did in
classrooms. By making distinctions about mathematical objects, which was an
inevitable part of tasks in this department, students were thinking mathematically.
In transcript 2, there is evidence of students posing questions spontaneously
however we see a difference compared to the spontaneity of transcript 1. It was clear
from subsequent discussion with Teacher A that the task students ended up engaging
with in transcript 1 was not planned. In contrast, Teacher W had crafted her two
starting examples on the 'Both Ways' task and there was no surprise in how the
students responded and that the distinctions they made lead to a motivation to explore
different starting numbers and then later different operations on the 'arms'.
We want to suggest that the making of distinctions within mathematics can
become a habit and a normal way of engaging in tasks for students. Creating Theme B – A. Cole & L. Brown
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opportunities for students to make distinctions within mathematics can also become a
habit for teachers and a normal way of both planning activity and informing decisions
in the classroom. When this happens, there is a convergence of planned and actual
activity. With a focus on distinctions, there is a potential route out of the problems
highlighted by Mason et al (2005) around the divergence of teacher intention and
student activity. With a focus on distinctions, the expert (teacher) can plan, initially
via the choice of examples, to support students in making the same distinctions as a
mathematician, leading to the same awarenesses. It is of course no easy skill to be
able to 'run' a discussion in the manner we see Teacher A or W doing in the
transcripts and we are not suggesting that task design is the end of the story; but we
do see evidence that task design based on making distinctions supports teachers in
working to support and develop the responses (distinctions) of their students.
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Applying the Phenomenographic Approach to Students'
Conceptions of Tasks
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Tasks serve a communicative purpose between teacher and
student. The student's perception of the task's purpose is communicated
in the approaches used and work produced. By applying the descriptions
of an outcomes space from a phenomenographic inquiry to student work
samples, an assessment of what students tended to focus on and the
meaning they assigned to the task were analyzed to determine the depth of
student learning. The findings show that the meaning and purpose a
student assigns to a task are aligned with the student's meaning of
learning, approaches to learning, and capabilities sought as a result of
learning.
Keywords: Focal awareness, learning, phenomenography, student
conceptions, tasks
Introduction
Tasks are an important component of any unit plan for learning. They serve
a communicative purpose between teacher and student, by conveying the teacher's
intent for learning and the student's conception of that intent (Crabbe, 2007). Often,
responses or work produced from a task reveal a disconnect between the teacher's
learning expectation and the true depth of knowledge attained by the student.
Attributing to the gap between expected versus actual learning outcomes is that
people experience learning differently, and therefore will conceptualize the object of
learning differently.
Phenomenography is a research methodology with its own theoretical
framework that accounts for the qualitatively different ways people experience
learning. From this theoretical stance, the impact a task has on learning may be
analyzed using the outcome space of student conceptions about the task. The purpose
of this application of phenomenographic inquiry is to present an approach to gauge a
task's impact on learning, solely from the student's perspective. By analyzing a
student's conception of, and approach to learning, the relationship between focal
awareness and task performance is further documented. The analysis is guided by the
following questions: a) what do students attend to when assigned a task, b) how do
students understand the purpose of an assigned task, and c) how is it possible to assess
the impact of the task on students' learning. Theme B – K. Gardner
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Theoretical Framework
Learning
Phenomenography is an empirical approach to studying the various ways
people experience a particular object of learning. The learner's perspective is the unit
of analysis. Learning achievement is based on the quality of the learner's description
of the object of learning, which has often been categorized as ranging from surface
learning to deep learning (Marton & Booth, 1997). Learning is defined as perceiving,
conceptualizing, or understanding something in a new way by discerning it from and
relating it to a context (Marton & Booth, 1997; Pramling, 1996). Furthermore,
learning involves two aspects: i) what is to be learned, and ii) how one goes about
learning (Marton & Booth, 1997). The learner's perspective of what is to be learned
is derived from the student's definition of the direct object of learning. How the
learner assigns meaning to the learning object is determined by the learning strategies
the student recommends for meeting personal learning goals. Since these aspects of
learning will vary, a hierarchical non-linear structure of the qualitative differences is
formed, and is called the outcome space (Marton, 1986).
The outcome space in Table #1 used in the study characterizes the learners'
perspectives of statistics, and it originates from research by Gardner (2007; 2010).
The outcome space has been iteratively refined based on further accumulation of
student responses to the phenomenographic inquiry in various levels of statistics
courses. Phenomenographic studies on the conceptions of statistics conducted by
Gordon (2004) and Petocz & Reid (2003) have yielded outcome spaces with similar
conceptions.
Conception 1: Statistics as facts or algorithms
Definition:
Statistics is a class in which one states terms, evaluates expressions and formulas,
solves equations, and makes and describes graphs.
Approach:
Write and study examples or facts the teacher presents, memorize formulas and
procedures, manipulate a calculator, solve problems the way they are done in class
Capabilities:
Do well on a statistics test, remember formulas and facts after a long period of
time.
Conception 2: Concepts about and procedures for handling data.
Definition:
Statistics is the study of contextualized techniques for collecting, representing, and

analyzing data.
Approach:
Write or state a contextual interpretation of graphs and numerical summaries, execute procedures with and without technology, relate personal experience and knowledge to statistical concepts, determine the appropriate statistical method for a given scenario.
Capabilities:
Explain or teach statistics to another person, read and understand statistics in media, use technology, know when it is appropriate to use a particular procedure or method.
Conception 3: Summarize, estimate, infer and predict
Definition:
Statistics is the study of processes used to estimate population attributes and to generalize or predict trends.
Approach:
Use multiple approaches, utilize technology to differentiate or discover trends, recognize when data need to be collected, explain assumptions, procedures and results to others, assess the reliability of results, provide support for conclusions drawn or estimates made.
Capabilities:
Write or present a detailed analysis of an inference, estimate, or prediction that includes an assessment of assumptions, interpret statistical output from software, appreciate the practicality of statistics. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Conception 4: Adapting, restructuring, changing viewpoint
Definition:
Statistics is a way to acquire knowledge about a population and illuminate trends to improve the quality of life, inform decisions, and change one's outlook of the world. It also comes with the responsibility to use and monitor ethical practices.
Approach:
Adapt to the variable nature of statistics, question the ethical treatment of subjects in studies involving humans or animals, employ the highest ethical standards and design principles, disseminates results of studies to illuminate attributes and inform decisions
Capabilities:
Devise a plan of action to change policies or perceptions based on reliable study results, redefine one's understanding of statistics as new processes are learned, formulate theories, re-structure one's view of the world.
Table 1. Outcome space for conceptions of statistics
Task
Traditionally, a task is defined simply as the work given to the student by the teacher to direct the student towards a specified learning goal (Doyle, 1988). To remain consistent with the phenomenographic definition of learning, a task is further characterized by its relationship to the structural and referential aspects of the learning experience, as defined by Marton and Booth (1997). A task is a situation requiring the learner to experience the object of learning in such a way that the learner must discern components of the situation and how they are related (structural aspect), then assign a meaning to the situation (referential aspect).
Prior knowledge, understanding, skills, and connections reside in the structural domain, and they surface to the foreground as a result of a perceived situation to address, or equivalently complete an assigned task. An analysis of the work produced gives some indication of how knowledge and skills were choreographed to complete the task. The application of multiple approaches or applying and honing skills to tasks with increasing degrees of complexity require students to hold more aspects of the learning object in their focal awareness, therefore they discern more properties of the object of learning and how they are connected (Kirshner, F., Paas, & Kirschner, P., 2011; Runesson, 2006). Therefore, students' work is a depiction of their approach to learning, which reveals structural aspects of their focal awareness.
The referential domain is the learning outcome as perceived by the student, which may be communicated through various forms of assessment. A student's conception of learning informs what is attended to when undertaking an assignment. Therefore, the meaning or purpose of the task assigned by the student is directly related to the student's definition of learning. A student's personal learning goals are made evident by the capabilities they seek to acquire as a result of learning. These capabilities are demonstrated in the product of work. Thus the student's meaning or purpose assigned to the task provides evidence of the depth of learning. The collective structural and referential aspects of students' focal awareness are descriptive evaluations of group performance and learning.
Student Conceptions
Since the student's conception is the unit of analysis, an explanation of what a student is attentive to when engaged in a task is warranted. The basic components of awareness are apprehension, discernment, and simultaneity (Marton & Booth, 1997; Uljens, 1996). Apprehension refers to being conscious of a perceptual or sensual experience in the presence of concrete or abstract entities; discernment involves Theme B – K. Gardner
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recognizing a foreground-background structure of a situation; simultaneity means knowing how the discerned parts are related to the whole structure. The structure of a student's focal awareness directly informs the way the student understands content, which leads the student to perceive that something has been learned. The capacity of focal awareness for an individual is limited, which accounts for the different ways people experience the object of learning (Marton & Booth, 1997).
Methodology
Task Description
Data were collected from one section of a graduate course in data analysis and probability for pre-service and in-service teachers. The task in Figure 1 is an item from the course mid-semester examination. The item assessed the student's performance level on analyzing and reporting summarized data.
Figure 1. Assessment task on descriptive statistics
Four weeks prior to the assessment, students engaged in a five day lesson introducing the data investigation process and data analysis techniques. One of the unit objectives was to recognize the need to generate and analyze data to gain insight to a specific problem. To motivate the lesson, students identified data specific problems, formulated questions, and then generated data from various statistical experiments. Recording the students' scores from the Bop-It game was one such activity, and the dataset was used only for the assessment task. Other experiences students had during the unit lesson were technology labs on producing graphs and numerical summaries of univariate data, and small work groups in which datasets with various features were analyzed and then reported to the class. The lesson activities were informative in determining and expanding the precision and depth of the students' analyses. The Bop-It activity provided a contextual understanding of the assessment task. All student item responses were sorted based on conceptions in the outcome space. One item from each conception was randomly selected for the study, to demonstrate applications of phenomenography. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Approach
To begin the analysis, characteristics of a student's written responses were noted and matched to items in the approach list. The response was categorized by the conception that had the most advanced approaches noted. A grading rubric was applied to rate the performance level (not met, beginning, advancing, accomplished, exceeds) of the response. The conception and performance level were then used to describe the student's focal awareness while engaged in the task, and to explain what the response revealed about the student's understanding of the purpose of the task.
Results and Discussion
Anna
Anna's response to the descriptive statistics task is provided in Figure 2. The performance level was classified as "Beginning" and the approaches used by Anna are aligned with Conception 1.
Figure 2. Anna's task response
Anna's response indicates a focus on surface level capabilities. Her approach to the task is a partial recitation of the written distribution summaries presented in the introduction of the learning unit by the teacher and in the textbook. Based on her use of intervals, "…between 35 – 40.", and approximations, "close to 14", it is evident that Anna fixated on providing visual interpretations of the histogram. She did not state features of the histogram used to conclude the distribution is skewed right, although this response is correct. She also was aware of the effect of a high outlier on the mean, but she did not apply the formula for determining specific outliers. Finally, she did not refer to the table that gave the exact mean of the distributions, and only focused on where she perceived the scores clustered, presumably based on frequencies. She did not provide an overall assessment of variability, nor a summative statement about the contextual meaning of the distribution. The capabilities illustrated by Anna suggest she discerned the purpose of the task to be an assessment of her ability to describe features of a histogram. Anna was able to recall facts about histograms that helped her to properly identify the shape of the distribution, sense the presence of outliers, and approximate the center. As demonstrated by her response, to Anna the purpose of the task is to determine whether she can recall facts about statistical summaries.
Byron
Figure 3 is Byron's response to the task. The response was rated as "Advancing" and the most advanced approach resides in Conception 2. Although Byron approached the task similarly to those presented in the class, he used more than one summary in his response, providing some indication that he could determine Theme B – K. Gardner
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appropriate methods. By referring to the data as scores, he was also able to relate the features to the task context.
Figure 3. Byron's task response
His attention is focused towards two graphs and the table of moments provided in the task. Coming to the forefront is an undisclosed feature of the box plot that lead him to conclude the distribution had "uneven" spread, which to him is not a characteristic of a normal distribution. But the meaning of this is not further clarified. Similarly, an unidentified feature of the histogram informed Byron's description of the distribution shape. His comparison of the mean and median indicates he knew facts about the effect of high outliers on resistant and non-resistant measures of center. He demonstrated correct application of the outlier formula and correctly interpreted its results. Byron's learning is still considered surface level, but advancing towards a higher conception. His conception of the purpose of the task was a means for him to communicate his understanding of concepts about data.
Charles
Charles' task was rated as "Accomplished" because there were a few points deducted for minor misconceptions revealed in his explanations. Based on his approach in Figure 4, Charles attended to all of the summaries provided in the task, thus incorporating multiple approaches to summarize the distribution. He also annotated each graph, thereby utilizing technological output to differentiate or discover trends and to provide support for conclusions drawn. These approaches are in Conception 3. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Figure 4. Charles' task response
Charles' response is a strong indication of his deep understanding of descriptive statistics as demonstrated in his capabilities to write a detailed summary of a distribution and interpret statistical output. Charles' conception of the task is best described as a means for him to demonstrate his ability to summarize data and support the conclusions drawn.
Conclusion and Implications
This research demonstrates a method for determining what students attend to when assigned a task, how they understand the purpose of a task, and what their responses reveal about learning, through the applications of phenomenography. The individual student responses contribute to the collective of varying levels in students' understanding. An analysis of their work through the lens of the outcome space communicates the depth and quality of collective learning, thereby informing the degree to which the task or task sequence impacted learning.
The task used in this research assessed student understanding of descriptive statistics and data analysis. Collectively, the various levels of performance in the class are summarized by the first three conceptions of statistics in the outcome space. The presence of student responses in the three conceptions allow for reflection upon and evaluation of the selection and sequencing of tasks throughout the unit lessons. For example, the lesson task sequence was identifying data problems, formulating questions, generating data, using technology to produce summaries, writing an analysis, and reporting the analysis. A lesson requiring students to first design a statistical activity, and then teach it to the class could be contrasted with the results of the lesson for this study to determine whether more students attained higher Theme B – K. Gardner
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conceptions. Furthermore, since none of the students responded in ways described by Conception 4, supplementing the lesson sequence with complex tasks requiring the compilation of numerous approaches to learning and demanding deeper cognitive focus may enthusiastically move students towards this level. A response indicative of Conception 4 would have included an illuminating effect or call to action based on the analysis. For example, a reflective statement on the Bop It activity explaining the high frequency of 0 scores, or a statement extending recommendations to the manufacturer based on the findings would have been evidence to categorize a response at this level. It is noted that in the researcher's other statistics courses, some students reach perspectives of Conception 4 in reports where they had to design and conduct comparative studies.
This research also shows that the meaning and purpose a student assigns to a task are aligned with the student's meaning of learning, approaches to learning, and capabilities sought as a result of learning. Lau, Liem, & Nie (2008) discuss similar findings in which the value students assigned to task was shown to be associated with the goal of learning. Students may be asked at integral points in a unit to define the object of learning, so that when or where changes occur from task to task can be noted. To encourage students to aim for conceptions that indicate deep learning, the teaching model, Learning Study (Runesson, 2006) provides methods immersed in the phenomenographic perspective.
The Learning Study model begins with planning a lesson aimed at depicting

the critical points of departure in the various ways students understand the object of learning; the lesson is then designed to exploit the patterns of variation to give students multiple ways of experiencing the object of learning (Runesson, 2006). Variation and repeated practice are viewed as effective teaching methods to encourage students to practice varying their perspective. Repeated practice in this context does not mean mundane, rote repetition. Instead, it means to create, invent, adapt, and progress in the light of previous practice where students get numerous opportunities to challenge their perspectives, vary their approaches, and extend or hone the skills they employ (Fazey & Marton, 2002; Runesson, 2006). Designing lessons from the Learning Study perspective inherently leads to developing tasks that provide authentic experiences and require simultaneous, structural focus on multiple details of the learning object.

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On what epistemological thinking brings (or does not bring) to the analysis of tasks in terms of potentialities for mathematical learning

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As recalled by the proposed framework text, the relationship between the completion of tasks by students and the effectively achieved mathematical learning must be questioned. The present contribution will develop and illustrate that this relationship may be adequately analysed by adopting a praxeological perspective (Chevallard, 1999), supported by an epistemological study. Indeed, such a view allows to highlight the fundamental character of the task with respect to the targeted knowledge, and to identify the praxeological level adapted to the level of teaching (Schneider, 2011). Therefore, it allows a certain understanding of the behaviour of students, allowing to assess the impact of the tasks on the learning processes, beyond a possible methodology based on pre/post tests.

Keywords: epistemological obstacle, fundamental task, praxeology, praxeological level, pre/post test

Introduction

Teachers, researchers and the mathematical community in general have an interest in designing tasks to help students, pupils … acquire mathematical knowledge. This goal in mind, a first methodological problem arises regarding their efficiency. A common practice to evaluate the impact of a task is the use of pre/post tests, mimicking procedures found in (hard) sciences like biology and chemistry. The work of Brousseau (Brousseau, 1998) issues a strong warning towards such practices. Although they may be sound in other contexts, in the setting of mathematical education they are subject to great concerns because of sociological considerations pertaining to human behaviour in learning institutions. Brousseau have shown how a teacher and his pupils may engage in some sort of role playing where pupils decode, from the behavior of the teacher, or other hints not drawn from the knowledge they are supposed to acquire, how to answer the given task and how the teacher, consciously or not, gives credit to his pupils doing so, giving an overall illusion of knowledge acquisition. Those initials analysis have been shown to carry over different levels of education (Calmant, 2004, Job, 2011, Rouy, 2007).

Though it puts into perspective the pre/post tests methodology, the sociological viewpoint used by Brousseau might led some people to believe that Theme B – P. Job & M. Schneider 204 mathematical understanding ultimately only relies on considerations where mathematical knowledge doesn't need to be analysed e.g. an epistemological study of the targeted knowledge isn't required. Far away from this conclusion, Brousseau, again, shows, thanks to the concept of epistemological obstacle, how much the inner peculiarities of mathematical knowledge are at the very heart of mathematical difficulties encountered by students and pupils, and, moreover, that the "toughness" of these difficulties leads, to some extent, to the aforementioned role playing between a teacher and his pupils, as a way of trying to do some mathematics despite the encountered difficulties. As a conclusion the work of Brousseau shows beyond any reasonable doubt that an epistemological analysis of the knowledge of concern cannot be bypassed in the design of mathematical task. But what kind of tool do we have to convey such an inquiry? How can we give credit to a task if the pre/post tests methodology is unsound? Those questions can be addressed, at least to some extent, using the praxeology concept.

Praxeologies, a model of (mathematical) knowledge

The work of Chevallard (Chevallard, 1999) extending some parts of Brousseau's work endow us with the concept of praxeology, a model of what (mathematical) knowledge is, whose great strength is to address in a single coherent framework both anthropological and epistemological concerns mentioned above. According to that theory, any activity, including mathematical ones, can be conceptualized as a task, something to do, a technique used to solve it, and a justification of the technique used to solve the task that can be split into a technology and a theory, a theory being a more abstract level of justification than the technology. We will show in the sequel how praxeologies can be put to good use and among other things shed some more light onto the distinction between technology and theory. But before that we should emphasize that one breakthrough permitted by this approach is to allow mathematical knowledge to gain a form of relativism. This doesn't mean that Pythagoras' theorem will sometimes be true and other times false, but that different institutions may have different views on the same knowledge e.g. a given knowledge may be a simple technique in one institution, like the limit concept in many Belgian secondary schools, or a theory developed to give analysis a Euclidean architecture as has been done by Cauchy who in many ways can be considered the father of analysis and the creator of the modern concept of limit (Job, 2011).

Two different praxeological levels

This institutional relativity of mathematical knowledge has led Schneider (Schneider, 2009) to distinguish between two different kinds of praxeology, modelling ones and deductive ones, allowing us, as we shall explain, to understand the dynamic behind some hard to teach concepts like the limit one (Job, 2011), but also to design tasks that transcend the pre/post tests pitfalls.

In the first kind of praxeology, modelling ones, the fundamental task is to compute features of objects like areas of surfaces, volumes of solids whose existence doesn't rely yet on a formal definition. Those objects exist as mental constructions shared, or believed to be, by some institutions. Justifications given in those praxeologies to techniques developed to address the fundamental task often rely on pragmatic arguments. A technique is validated if the results obtained are in accordance with results derived using other valid techniques that may even belong to Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford. 205 other fields of sciences. For instance, early infinitesimals techniques where used and recognized based on the accordance with results obtained using physical arguments of cinematic nature.

In the second kind of praxeology, deductive ones, the fundamental task consist in defining those mental constructions, to make explicit what was left in the shadow in the first kind of praxeology and built a deductive theory. Often the techniques used in modelling praxeology are used in deductive ones as definitions. The definition of integral given by Cauchy is a good example of such a procedure: an approximation procedure is turned into a definition that, in turn, is used to prove theorems about integrals (existence, uniqueness …).

The two kinds of praxeology are distinct but closely related, the second kind often taking place after the first one. With these two concepts we don't aim, nor claim to encompass the whole mathematics, but important parts of its growth like the birth of analysis from calculus. Indeed, it can be showed that calculus can be roughly speaking represented as a modelling praxeology and analysis a deductive one (Job, 2011). These two praxeologies take into account the institutional relativity of various concepts like the limit one and the derivative that may otherwise be seen as concepts that somehow where born at some place in time almost as they appear nowadays whereas they evolved under the guidance of very different viewpoints. The calculus period was mostly guided by the will to be able to compute areas… and analysis was created out of the will to purge calculus from geometry and physics, forge a new area of mathematics whose rigor would equal that of the ancient Greeks.

Epistemological thickness and fundamental character of a task

The praxeological levels introduced above allow us to get back to our initial asking. How do we assess a task? A partial answer given by Brousseau (Brousseau, 1998) is to consider a task fundamental with respect to a given knowledge if the knowledge takes places in a praxeology as a technique where the task cannot be solved without that knowledge. The knowledge is thus seen as a kind of optimal answer to the proposed task. This requirement is legitimated by what we have said earlier in this article about the role playing pupils, students and teachers are prone to engage. It shouldn't be possible to solve the task used to teach a certain knowledge only using hints external to the knowledge like the teachers eyebrows indicating if the students are running along the required lines.

This understanding of the fundamental nature of a task has been shown to be effective to introduce concepts likes the rational numbers (Brousseau, 1998) but doesn't seem to translate well to concepts like the limit one. Indeed, it is one of the twentieth century achievements to have shown with the work of Abraham Robinson that a sound basis could be given to infinitesimal concepts so far rejected as a sound basis for calculus. The limit concept is thus by no means necessary to cast the calculus into a deductive mould.

Anyway, the very heart of Brousseau's idea can be adapted is the following manner, taking into account the institutional relativity of knowledge introduced above. A task is said to be fundamental (in a broad sense) with respect to a given knowledge and a given institution if that institution takes for granted the knowledge is optimal to solve the task. In this new definition there is no more necessity in a "mathematical" way but an anthropological necessity that an institution gives to itself.

At this point, we are now able to understand the leading role played by praxeological levels. The structure of a fundamental task and even the fundamental Theme B – P. Job & M. Schneider 206 character of that task with respect to, for instance, the limit concept depend on what kind of praxeology we place ourselves in. A fundamental task for the limit concept in a deductive praxeology won't be the same as a fundamental task in a modelling one. Before we dive into some characteristics of these tasks, let us first give an example of the consequences of not being able to clearly state whether a task belong to one praxeological level or the other.

The consequences of blurred praxeological levels in secondary school

In (Job, 2011) we study the teaching of the limit concept in secondary school and are able to support the following views. Secondary school tries to teach the limit concept but fails to do so, unable to identify the praxeological level where it should belong.

Secondary school tries to teach this concept giving students elements that belongs to the deductive praxeology of analysis mathematicians use nowadays in order to place itself under the supervision of that institution from which it draws its legitimacy. This deductive praxeology being out of reach to students of that age, the school praxeology mainly consists of elements acting as blazons, that is, parts of the original praxeology that are able to support the illusion of a real teaching of the limit concept from an outside perspective.

Among these blazons, the definition of a limit plays a key role. Secondary school tries to teach this definition using various tricks to make believe students this definition is a somewhat complicated (mathematical) way of saying something very natural. For instance, it gives students tables with values of x and f(x) for a given

function, waiting for the students to recognized some sort of behaviour that should be put into sentences like "as x tends to … f(x) approaches …". Starting from such sentences, teachers gradually turn these into the required forms "f(x) can be made as close as one wishes to …" using arguments that belong more to rhetoric than mathematics.
Such an approach is misleading in nature for the definition of the limit concept was designed by Cauchy to conduct proofs and define other key concepts of analysis like the derivative. But except for a few trivial ones, proofs in secondary school are left aside. So the very use of the limit concept in the deductive praxeology where it belongs is left aside. The school praxeology thus bears no fundamental character whatsoever.
Such a fool's game isn't the consequence of any malicious thoughts on the side of secondary school but the resultant of antagonist constraints. On the one hand, it has to teach the limit concept in a way mathematicians would recognize as valid, which is a daunting task. On the other hand it must succeed in that task. The only way secondary school has to its disposal to conciliate the two is to take the deductive praxeology, strip it from most of its content and wrap it in a discourse that can be accepted by students even if the cost is to propose tasks that have no fundamental character. This wrapping is partly a consequence of its unawareness of the existence of another praxeology (a modelling one) where the limit concept is legitimate.
So secondary school's praxeology with respect to the limit concept lies in a no man's land, not being in a deductive or in a modelling praxeology. Similar conclusions are drawn in (Rouy, 2007) regarding the derivative also based on praxeological considerations. This analysis sheds a new light on the pre/post-tests methodology. How could we give credit to a task succeeding a sequence of pre and post tests if that task isn't epistemologically consistent? Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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What praxeological level for secondary school?
The section above asks a crucial question. Is there a place left for the limit concept in school that would be mathematically legitimate? The answer might be positive if we place ourselves in a modelling praxeology. Although Schneider is critical towards some of the tasks they designed (Schneider, 2001), AHA (AHA, 1999) has proposed a fundamental task for the limit concept in a modelling praxeology, which is declined at the various levels of application of the concept in sub-tasks (areas, speeds, tangents).
On the other hand, Job has studied the teaching of the limit concept in a deductive praxeology (Job, 2011). Its results show how much a deductive approach to the limit concept is a very demanding task. In a few words, the students were asked to propose definitions of a certain behaviour of sequences of real numbers and then to proof properties related to this behaviour. The students were mostly unable to make their definitions evolve. They stayed stuck with definitions that are "descriptions" of what they see of the studied behaviour. They couldn't possibly envision their definitions as something to be chosen to allow proofs despite the many contradictions pointed out by the teacher. This inability is related to epistemological obstacles. Students see definition as a description of some mental concept they believe everyone of them share. They therefore don't understand the rules of the game they are asked to play, feeling they are asked something unnecessary complicated because "everyone agree with the found properties", "nothing has to be proved". This situation seems like a dead end because the teaching school has given them tends to reinforce their vision of mathematics, depriving them from the need to cast theories into a deductive mould.
Different understandings of the task concept
Let us give a second example of the use of praxeologies that will put the task concept itself into question, showing it should sometimes be understood at a different level than is usually done, thus clarifying the concept of a fundamental task understood in a broad sense.
We shall illustrate our views through a task used by our team (Job & Schneider, to be published) to teach negative numbers and specifically the multiplication rule to 12 years old pupils. Being as concise as possible, pupils are asked to devise a single formula that allows them to encompass the motion of two vehicles, being flashed by a radar, driving different roads, but at the same constant speed of 2km/min. A first formula p=2t emerges for positive times where p denotes a location and t a time
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. They are then asked to elaborate a formula that would also be valid for negative times e.g. times before the two cars are flashed. This requirement of a single formula brings pupils face to face with expressions like -6=2 x (-3) and therefore to an extension of the multiplication rule for positive numbers to negative ones. Pupils are then asked to deal with cars driving in the direction opposite to the one the first two where driving. This introduces "negative" speeds, the minus sign telling which direction the car is driving. The same requirement of a single formula leads in turn to expressions like 6=(-2) x (-3) a completes the multiplication rule for negative numbers.

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Aside the multiplication rule, the task allow us to make pupils distinguish between distance and position among many other things we have no space to elaborate on. Theme B – P. Job & M. Schneider
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Such an introduction of negative numbers and their multiplication meets pupils' global assent but what we are trying to emphasize lies somewhere else. The peculiarity of our task doesn't rely so much on the pupils' assent, but on a characteristic where they are not involved in the first place. This task tries to expose pupils to a choice made by mathematicians/physicists to allow them to model with a single formula the various incarnations of the same motion, in terms pupils should be able to understand. Pushing the structure of our task to the extreme, it doesn't matter so much if the pupils agree with the decision made by mathematicians/physicists as long as they understand there is a choice to be made and its consequences, because it is not their assent we are seeking. We simply try to make as explicit as possible choices made by some institution they have no impact on. Learning mathematics and physics also means learning the conventions of those institutions whether we agree or not with them. It is not to say that pupils have nothing to understand. On the opposite, there is something to understand which is located at a level that is subtle to explain, not to pupils who are living the task, but to the mathematical learning community: if you want to learn mathematics you have to accept its conventions whether or not you agree with them as long as you understand why those conventions have been adopted.
Conclusion
We have argued that a pre/post-tests methodology is unsound to assess the efficiency of a task and that the distinction between two kinds of praxeologies (modelling and deductive ones) plays a key role in designing tasks and understanding the dynamic of ordinary lessons. A task should clearly identify whether it belongs to one praxeology or the other in order to be meaningful. A task that doesn't belong to any of those two levels should be handled with great care, its fundamental character being dubious. Being able to state to what kind of praxeology we belong allow us to interpret students' work in the light of a solid epistemological background, therefore giving us tools to avoid misinterpretations that pre/post-tests a prone to commit due to their very structure: a post test result better than a pre-test one isn't obviously a sign of better understanding but may only be the result of an accommodation from the students that have understood how to answer the tasks without using the targeted mathematical knowledge. Taking advantage of the distinction made between modelling and deductive praxeologies and the relativity of knowledge, we have put into question the very concept of a task showing how much its understanding can be and should broadened as soon as we are dealing with the teaching of concepts like the limit one or the negative numbers. Those tasks should be understood in a broader sense than usual, as a way to highlight choices made by an institution and the reasons underlying these choices.
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Designing Covariation Tasks to Support Students' Reasoning about Quantities involved in Rate of Change
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This paper articulates theoretical and practical considerations for designing a sequence of covariation tasks to support students' reasoning about quantities involved in rate of change. Adapting the well-known bottle problem during two iterations of implementation and analysis, this researcher-developed task sequence incorporates dynamically linked geometric and graphical representations of covarying quantities and prompts fostering students' coordination of quantities that are changing together. Taking into account students' perspective of quantities involved, this task sequence is designed to support students' progression in using nonnumerical quantitative reasoning to make predictions and create representations indicating how one quantity might change in relationship to another changing quantity.
Keywords: Task design, Reasoning, Quantity, Covariation, Rate of Change
Researchers using mathematical tasks involving dynamic representations of covarying quantities have supported middle and high school students' consideration of relationships between quantities involved in rate of change (e.g., Johnson, 2012b; Monk & Nemirovsky, 1994; Saldanha & Thompson, 1998; Stroup, 2002). In this paper I articulate theoretical and practical considerations for designing a sequence of covariation tasks to support middle school students' reasoning about quantities involved in rate of change. The design of this task sequence accounted for students' perspective of quantities involved to support their progression in using nonnumerical quantitative reasoning to make predictions and create representations indicating how one quantity might change in relationship to another changing quantity.
Background
Drawing on Sierpinska's (2004) characterization, by mathematical task I mean a purposefully designed problem intended for a particular audience. By problem, I mean a situated problem (Gravemeijer, 1994) involving a particular context. I take an interpretive stance on context, drawing on Van Oers' articulation of context: "What counts as context depends on how a situation is interpreted in terms of activity to be carried out" (1998, p. 481). Acknowledging that individuals for whom a task is intended can interpret the task in myriad ways, I assume that an individual's perspective on the nature of the problem to be solved can influence an individual's reasoning about mathematics he or she perceives to be involved in the task. Theme B – H. Johnson
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I consider mathematical reasoning to be an individual's purposeful mental activity situated within a particular context. The purposeful activity includes making sense of how a mathematical situation holds together (Simon, 1996), making relationships between objects involved in a situation (Thompson, 1996), and engaging in operation that involves carrying out actions both mentally and physically (Piaget, 1970). When characterizing reasoning as quantitative, I consider a quantity to be an individual's conception of the measurability of an attribute of an object (Thompson, 1994). Because individuals do not need to determine actual measurements to reason quantitatively, quantitative mental operations are nonnumerical (Thompson, 1994). By articulating that the object of the reasoning is quantities involved in rate of change, I do not assume that individuals will reason about rate of change as single entity. Focusing on covariation (Carlson et al., 2002), I attend to how individuals make sense of and make relationships between quantities that are changing together.
Adapting the well-known bottle problem to design covariation tasks
I designed covaration tasks by adapting the well-known bottle problem developed by the Nottingham University's Shell Centre (Swan & the Shell Centre Team, 1999). Given the context of a bottle filling with liquid being dispensed into the bottle at a constant rate and a picture of a bottle, the bottle problem requires students to sketch a graph of the changing height of the liquid as a function of the changing volume. Researchers have used the bottle problem to investigate the reasoning of undergraduate and graduate mathematics students (Carlson et al., 2002) and prospective elementary (Carlson, Larsen, & Lesh, 2003) and secondary (Heid, Lunt, Portnoy, & Zembat, 2006) mathematics teachers. My adaptations to the bottle problem have had two iterations of implementation and analysis. The task sequence reported in this paper is from the second iteration.
In the first iteration I developed a covariation task by adapting the bottle problem in two ways: (1) Providing students with a graph and asking students to sketch a bottle that the graph could represent, and (2) Using a graph that represented the changing volume of the liquid as a function of the changing height. Prompts included in this covaration task were: (1) How is the volume of the liquid in the bottle changing as the height of the liquid in the bottle increases? (2) Sketch a bottle that the graph could represent. Intending to implement the task with high school students who had not yet taken a calculus course, I provided students with a graph because previous

research (Carlson et al., 2002; Heid et al., 2006) found that even students with extensive mathematics background have difficulty creating graphs. I chose to represent volume as a function of height in part because preservice elementary teachers working on the bottle problem operated with the independent variable, volume, as if it were time (Carlson, Larsen, & Lesh, 2003). I hypothesized that representing volume as a function of height might reduce the likelihood of students treating the independent variable as if it were time.

In the second iteration I adapted Thompson, Byerly, and Hatfield's (in press) version of the bottle problem that dynamically links a pictorial representation of a filling bottle with a graph relating the volume of liquid in the bottle to the height of liquid in the bottle. Key to Thompson et al.'s adaptation was the use of a dynamic environment linking pictorial and graphical representations. I hypothesized that such an environment could foster students' reasoning about covarying quantities by explicitly linking a change in one representation with a change in another representation. Hence, I adapted Thompson et al.'s bottle problem (implemented with Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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beginning calculus students) for use with 7
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grade pre-algebra students. Anticipating that 7
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grade students might have limited conceptions of volume, I altered the context of the task from a filling bottle to a two-dimensional shape being filled with area. In making this change, the two dimensional pictorial representation would represent a two dimensional rather than a three dimensional quantity.

The adaptation for 7
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grade pre-algebra students resulted in a sequence of four tasks. Accompanying each task was a dynamic sketch I developed using Geometer's Sketchpad software (Jackiw, 2009). The filling rectangle sketch (see Fig. 1) linked a rectangle with a graph that related the amount of shaded area to the height of the shaded area. Students could vary the height of the rectangle by animating or dragging point H. Students could drag point F to vary the width of the rectangle, then predict and create corresponding graphs representing the amount of shaded area as a function of its height. The filling triangle sketch (see Fig. 2) linked a right triangle with a graph that related the amount of shaded area to the height of the shaded area. Students could vary the height of the triangle by animating or dragging point D, then predict and create a corresponding graph representing the amount of shaded area as a function of its height. By affording students' manipulation of dynamically linked representations, the dynamic sketches seem to foster students' consideration of relationships between covarying quantities.

Figure 1. Filling Rectangle Sketch Figure 2. Filling Triangle Sketch

Task sequence

The task sequence is designed to support students' progression in using nonnumerical quantitative reasoning to coordinate covarying quantities. The description of the task sequence includes: (1) Statement describing the context for the task sequence (2) Identification of dynamic sketch used with each task (filling rectangle or triangle), (3) Quantitative reasoning (QR) objective for each task (italics), and (4) Prompts fostering students' coordination of quantities that are changing together. The context for the task sequence describes the situation on which the tasks are based. A QR objective is distinct from a learning objective because it indicates purposeful activity intended to support a way of reasoning rather than an intended mathematical understanding. These QR objectives indicate purposeful ways of making sense of and making relationships between covarying quantities. Prompts refer to questions and directives designed to foster students' making of relationships between quantities and predictions about characteristics of linked representations. Theme B – H. Johnson
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Context for task sequence

Imagine that a shape (rectangle/triangle) is being filled with area.

Task #1: Filling Rectangle Sketch

Create and use non well-ordered tables of values indicating measurements of related quantities to predict characteristics of graphs relating those quantities.

Press Animate Point to run the animation of the filling rectangle. What changes and what stays the same?

Given a non-well ordered table of heights for a rectangle with a given base, determine the different amounts of area. For example:

Imagine the side length of EF was 4 cm. Complete the table:

Length of Side EH 1 cm 3 cm 5 cm 7 cm 10 cm

Area of Rectangle EFGH

Imagine you created a graph relating the side length of EH and the filling area of EFGH. Would the graph be linear? Explain why or why not.

Repeat bibc for bases of different lengths. How would the graphs be similar/different?

Task #2: Filling Rectangle Sketch

Use an amount of change in one quantity to predict an amount of change in a related quantity.

Given a non-well ordered table of heights for a rectangle with a given base (e.g., the table in 1b), determine the different amounts of area.

Determine at least two different ways to complete this statement: When the height increases by _____, the area increases by _____. How many ways can this statement be completed?

When given the two different heights for the same base, determine the amount of increase in area. For example,

Imagine the side length of EF was 3 cm. Complete the table:

Length of side EH 14.5 cm 16.5 cm

Amounts of increase in area of rectangle EFGH

Task #3: Filling Rectangle Sketch

(1) Use a graph relating two quantities to predict a measure for a third, related quantity (not represented explicitly by the graph). (2) Use a relationship between two quantities to predict characteristics of a graph relating one of those quantities to a third quantity.

Given graphs representing the amount of filling area as a function of the length of side EH, predict the length of the base of the rectangle.

Given a relationship between changing area and changing height (e.g., As the length of EH increases by 2 cm, the area of rectangle EFGH increases by 3 cm
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predict what a graph relating area and the side length of EH would be like.

What would a graph be like for a rectangle with (1) a very short base, (2) a very long base? Why? Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Task #4: Filling Triangle Sketch

Use a dynamic geometric representation of covarying quantities to predict characteristics of a graph relating the quantities.

Press Animate Point to run the animation of the filling triangle. What changes and what stays the same?

How does area change as the height increases?

Imagine you created a graph relating the side length of AD and the filling area ABCD. What would the graph be like?

Press Animate Point to sketch the graph. Was it what you expected? Why do you think it looks that way?

Design principles

Three principles guided my design of the task sequence. Central to each of these principles was my consideration of how students might perceive the nature of the quantities involved and how students' perspectives might influence their consideration of relationship between those quantities.

(1) Anticipate students' perspectives on relationships between changing quantities

Drawing on results of analysis of students' work on the first adaptation of the bottle problem, I anticipated two distinct perspectives students might have on relationships between change in the covarying quantities involved in the task sequence: (a) Quantities can change simultaneously with change in one quantity being independent of change in a related quantity (Johnson, 2012a) and (b) Change in one quantity depends on change in a related quantity (Johnson, 2012b).

(a) By relating covarying quantities as if each quantity were changing independently of the other quantity with respect to time, students can make comparisons between amounts of change in each quantity (Johnson, 2012a). For a filling rectangle, if 7
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grade students could determine amounts of area for given amounts of height (The height being the length of EH – see Fig. 1), they could make comparisons between amounts of change in area and amounts of change in height. By sequencing filling rectangle tasks that afforded calculation of amounts of area prior to a filling triangle task that problematized calculation of amounts of area, I intended to support students' gradual move away from making numerical calculations.

(b) Reasoning about change in one quantity as being dependent on change in a related quantity can support students' attention to variation in the intensity of a change (Johnson, 2012b). If 7
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grade students were able to envision the area of the filling triangle as varying in relationship to the height (The height being the length of AD – see Fig. 2), they could make claims about the area of the triangle increasing more slowly as the height increased. By requiring students to predict how area would change as height increased when manipulating dynamically linked geometric and graphical representations, I intended to support students' reasoning about variation in area as being dependent on variation in height. Theme B – H. Johnson
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(2) Incorporate key aspects to support students' attention to covarying quantities

The task sequence incorporated three key aspects: (a) Tables that were not well ordered, (b) Questions supporting students' attention to covarying quantities, and (c) Questions supporting students' making of predictions about change.

(a) In the filling rectangle tasks I incorporated tables that were not well ordered. By indicating that a table is not well ordered, I mean that the independent variable contained in the table (in the case of the filling rectangle tasks, the length of EH) does not increase by a uniform amount. When working only with well ordered tables, secondary students did not necessarily attend to change in both independent and dependent variables (Lobato, Ellis, & Munoz, 2003) and tended to pay attention to numerical patterns (Ellis, 2007). I anticipated that including tables that were not well ordered could promote students' attention to and making of relationships between covarying quantities.

(b) Both the filling rectangle and the filling triangle tasks incorporated prompts supporting students' attention to and making of relationships between the changing quantities of area and height. The prompt "What changes and what stays the same?" provided an entry point into the task and fostered students' attention to different quantities involved in the task. Subsequent prompts supported students' making relationships between quantities that were changing together. For example, in task 2b, completing the statement "When the height increases by _____, the area increases by _____" in multiple ways supports students' consideration of multiple relationships between amounts of increase in height and area for a rectangle with a given base. Once students could begin to focus on relationships between quantities, it seems reasonable that students could then draw on those relationships to engage in related activity.

(c) Each task supported students' activity of making predictions about (1) characteristics of a linked representation (tasks 1, 3, 4) or (2) a related amount of change (task 2). Making predictions can foster students' use of nonnumerical reasoning by supporting students' envisioning of running through calculations without actually making the calculations to make relationships between the changing area and the changing height. For example, task 2c required students to predict amounts of change in area given two different heights without actually determining amounts of area. If students determined amounts of area, subtracted those amounts, then arrived at amounts of change, it could indicate that students were not yet relating changes in height with changes in area. By requiring students to predict characteristics of graphs representing rectangles with differently sized bases, task 3c supported students' use of relationships between area and height. Such predictions were intended to support students' attention to relationships rather than to results of calculations.

(3) Sequence tasks to support students' progression from numerically based reasoning to nonnumerically based reasoning

I sequenced the filling rectangle tasks prior to the filling triangle task based on two research hypotheses related to students' perspectives on quantities involved: (a) Students can use numerical calculations (with or without actually engaging in calculations) to make comparisons between amounts of change in covarying quantities, and (b) By focusing on covariation, students can attend to situations Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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involving constant rate of change in ways that can support their attention to variation in the intensity of change in situations involving varying rates of change.

(a) While numerical calculations can be an entry point into the filling rectangle tasks, the design of each filling rectangle task is not intended to support students generalizing from numerical calculations. In contrast, intent is to support students' engagement in the nonnumerical operation of linking a changing area with a changing height. Students working from numerical calculations could begin to imagine running through calculations (without actually completing the calculations) to relate the changing area to the changing height.

(b) The filling rectangle tasks incorporate constant rates of change, and the filling triangle task incorporates a rate of change that increases at a decreasing rate. Although it may seem obvious to position tasks involving constant rate of change task prior to a task involving varying rate of change, research has questioned whether situations involving constant rate of change are sufficiently complex to engender students' reasoning related to varying rate of change (Stroup, 2002). Filling rectangle tasks incorporating constant rate of change supported students' focus on the covarying quantities of volume and height. I anticipated that such tasks would contain sufficient

complexity to support students' consideration of situations involving varying rate of change because they could foster students' coordination of covarying quantities.
Task implementation and analysis
During May 2012, I implemented the filling rectangle tasks with 4 sections of 7
th
grade students at an urban middle school in a large Midwestern U.S. city. The district has identified the school as high performing based on students' academic performance, with approximately 45% of students identified as English Language Learners and over 90% of students receiving free or reduced lunch. I implemented tasks 1-3 during three consecutive days of whole class instruction. Following the lessons, I conducted 40-minute task-based interviews with 7 pairs of students, selecting at least 1 pair of students from each of the 4 sections. I purposefully chose student pairs based on the students' participation in classroom instruction and on evidence of reasoning about quantities involved in rate of change. During the taskbased interviews I followed up with task 3 and presented task 4. For this paper I report results of analysis of students' work on task 4.
Analysis of students' work during task-based interviews revealed two main findings: (a) Students may depend on numerical calculations to make claims about how quantities are changing together, and (b) Students may create graphs relating covarying quantities (not including time) as if one quantity were elapsing time.
(a) Students who depended on numerical calculations had difficulty making predictions about how the area of the triangle would change as the height increased. The responses of two students, Navarro and Myra (who participated in different interview pairs), provide insight into the kind of difficulty students might have. When Navarro and Myra were presented with the filling triangle task, both of them attempted to determine amounts of area. Even after prompting to not worry about making calculations, Navarro's persistence in trying to calculate amounts of area made it seem as if he depended on calculating amounts of areas to make such predictions. Unlike Navarro, after my prompt to not worry about how to calculate the area, Myra smiled and exclaimed "Oh, I get you now!" When I asked her to explain, she said "the area is getting bigger, but how much it increases is getting smaller." By

variation in how the area was increasing. Future iterations of implementation and analysis could provide further explanation as to how students' nonnumerical reasoning develops when making relationships between quantities.
(b) Although the filling triangle sketch related area with the length of AD, when predicting features of a graph some students seemed to operate as if AD represented time. I designed the animation (See Fig. 2) so that the graph would begin to be sketched from a point when the triangle was partially filled. When predicting the shape of a graph relating area and side length for the filling triangle, Tomas sketched a graph in the air, indicating the graph would curve and then begin to increase again (which is not consistent with the graph shown in Fig. 2). When asked to explain why, Tomas said that area would fill more slowly and then start filling more quickly again. Even though Tomas attended to both area and side length when using the animation, he seemed to sketch the graph as if it were relating increasing area with elapsing time. Future tasks dynamically linking multiple graphs relating changing quantities to a single geometric representation might support students' consideration of the independent variable as something other than time.
Implications for task design in research investigating students' reasoning
Designing a sequence of tasks to support students' reasoning involved theoretical considerations including how students might make sense of and coordinate covarying quantities and practical considerations including how students might manipulate quantities represented with dynamic geometry sketches. The task sequence supports students' progression in using nonnumerical quantitative reasoning to make predictions and create representations indicating how one quantity might change in relationship to another changing quantity. Researchers could draw on and expand design principles underlying this task sequence to develop other task sequences focusing on quantity and covariation. Future research involving implementation of this and other task sequences could support development and expansion of frameworks articulating progressions in quantitative and covariational reasoning.
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Designing tasks that enhance mathematics learning through creative reasoning
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This paper presents a research framework for a design research approach to constructing creative learning opportunities and some related empirical studies.
Keywords: Learning difficulties, rote learning, creative reasoning, task design.
Introduction
A central problem in mathematics education is that we want students to understand mathematics and to become efficient problem solvers, but even after 30 years of research and reform many students still do inefficient rote thinking (Hiebert, 2003; Lithner 2008). This can be seen as one of the main reasons behind learning difficulties in mathematics. The reason that the rote learning problem is (largely) unsolved in many countries is a combination of several factors related to the immense complexity of mathematics learning (Niss, 1999) and to the lack of research insights concerning the effectiveness of different teaching designs (Niss, 2007).
This design research addresses some of the most central mathematics learning goals, which we (teachers, researchers and other actors in the educational systems) largely fail to help students to reach: problem solving ability, reasoning ability and conceptual understanding. Problem solving is defined as "engaging in a task for which the solution method is not known in advance" (NCTM, 2000, p. 51) and includes identifying, posing, and specifying different kinds of problems and solving them, if appropriate, in different ways (Niss 2003). Reasoning is a fundamental aspect of mathematics (NCTM, 2000). It goes beyond constructing reasoning, and includes abilities like following and assessing chains of arguments, Theme B – J. Lithner et al

knowing what a proof is and how it differs from other kinds of reasoning, uncovering the basic ideas in a given line of argument, and devising formal and informal arguments (Niss 2003). The notion of understanding is very complex (Sierpinska 1996), and will not be pursued here beyond noting that several of the theoretical constructs denoted understanding concern relations between rote learning and deeper understanding, e.g. Skemp (1978) and Hiebert & Lefevre (1986). The notion will here be used in a relatively intuitive way, referring to insights in the origin, motivation, meaning and use (Brousseau 1997) of a mathematical fact, method, concept or other idea.
Rote learning
Rote learning is "the process of learning something by repeating it until you remember it rather than by understanding the meaning of it" (Oxford Advanced Learner's Dictionary). The characteristics, causes, and consequences of rote learning in mathematics can to a large extent be connected to an unwarranted and far-reaching reduction of complexity in terms of an algorithmic focus (Skemp, 1978; Hiebert & Carpenter, 1992; Tall, 1996; Vinner, 1997; Hiebert, 2003; Lithner, 2008). Referring to "massive amounts of converging data" in studies from USA, Hiebert (2003) suggests that the baseline conclusion is that students are learning best the kinds of mathematics that they are having the most opportunities to learn, which is simple calculation procedures, terms and definitions through memorization. Similar opportunities to learn were found in a Swedish large-scale study including observations of 200 mathematics classrooms (Boesen et al., 2012).
Memorising facts and procedures, sometimes without understanding, is a central aspect of mathematics learning. The problem is when rote learning becomes dominating since it is not possible to develop other central competencies like problem solving ability and conceptual understanding by rote learning alone. For example, it is well known (e.g. Schoenfeld, 1985) that there is no transfer from rote learning of basic facts and procedures to the ability to solve non-routine mathematical problems. From literature reviews (e.g. Hiebert, 2003) and from the empirical studies exemplified below, it is reasonable to draw the conclusion that rote learning is one of the main causes behind the difficulties to learn mathematics that large groups of students of all age levels encounter.
Creative reasoning
This and the next section consist of a summary of selected parts of a research framework (Lithner, 2008) that is based on the outcomes of a series of empirical studies on the relationship between reasoning and learning difficulties in mathematics. Reasoning is defined in this paper as the line of thought that is adopted to produce assertions and reach conclusions when solving tasks. Reasoning is not necessarily based on formal logic and is therefore not restricted to proof; It may even be incorrect as long as there are some sensible (to the reasoner) arguments supporting it. A basic assumption is that the design of the students' tasks (and of the classroom context) affect the reasoning the students will activate which in turn affects their opportunities to learn. A "task" includes most of the work requested from students in classrooms, such as exercises, tests, group work, etc. Arguments can be anchored in either surface or intrinsic properties, and the relevance of a mathematical property can depend on context. In deciding if 9/15 or 2/3 is largest, the size of the numbers (9, 15, 2, 3) is a surface property that is insufficient to resolve the problem (a conclusion Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.

based on this property alone is that 9/15>2/3 since 9 and 15 are larger than 2 and 3), while the quotient captures the intrinsic property. The intrinsic/surface distinction was introduced because one of the reasons behind students' difficulties was found to be the anchoring of arguments in surface properties (Lithner, 2003). The aspect of creativity that is emphasized in this framework is not 'genius' or 'exceptional novelty,' but the

creation of mathematical task solutions that can be modest but that are original to the individual who creates them (Silver, 1997).
The discussion above leads to a definition of Creative Mathematically Founded Reasoning (CMR) that fulfils all of the following criteria. i) Creativity; a new (to the reasoner) reasoning sequence is created, or a forgotten one is re-created, in a way that is sufficiently fluent and flexible to avoid restraining fixations. ii) Plausibility; there are arguments supporting the strategy choice and/or strategy implementation explaining why the conclusions are true or plausible. iii) Anchoring; the arguments are anchored in the intrinsic mathematical properties of the components that are involved in the reasoning.
Imitative reasoning
The empirical studies that form the basis of this framework have identified three main types of mathematically superficial imitative Algorithmic Reasoning (AR), which may lead to rote learning. The term 'algorithm' includes all pre-specified procedures (not only calculations), such as finding the zeros of a function by zooming in on its intersections with the x-axis with a graphing calculator. "An algorithm is a finite sequence of executable instructions which allows one to find a definite result for a given class of problems" (Brousseau, 1997, p. 129). The importance of an algorithm is that it can be determined in advance. The n:th transition does not depend on any circumstance that was unforeseen in the (n-1)st transition - not on finding new information, any new decision, any interpretation, or thus on any meaning that one could attribute to the transitions. Therefore, the execution of an algorithm has high reliability and speed (Brousseau, 1997), which is the strength of using an algorithm when the purpose is only to produce a task solution.
However, if the purpose is to learn something from solving the task, the fact that an algorithm is independent of new decisions, interpretations or meaning implies that all of the conceptually difficult parts are taken care of by the algorithm, and thus only the easy parts are left to the student. This may lead to rote learning. In particular, the resultant argumentation is normally superficial and very limited, as seen in the main AR types that are found in studies: Familiar AR includes a strategy choice that can be characterized by attempts to identify a task as being of a familiar type with a corresponding known solution algorithm or a complete answer. In Delimiting AR, the algorithm is chosen from a set of algorithms that are available to the reasoner, and the set is delimited by the reasoner through the included algorithms' surface property relationships with the task. For example, if the task contains a second-degree polynomial p(x), the reasoner can choose to solve the corresponding equation as p(x)=0 even if the task asks for the maximum of the polynomial. In Guided AR, the reasoning is mainly guided by two types of sources that are external to the task. In person-guided AR, a teacher or a peer pilots the student's solution. In text-guided AR, the strategy choice is founded on identifying, in the task to be solved, similar surface properties to those in a text source (e.g., a textbook). Argumentation may be present, but it is not necessary because the authority of the guide ensures that the strategy choice and the implementation are correct. Theme B - J. Lithner et al
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In students' attempts to resolve problematic task solving situations, the CMR criteria i-iii were found to capture the main differences seen in reasoning characteristics between AR (where i-iii are absent) and constructive CMR (Lithner, 2008). Students often use superficial imitative reasoning of the types presented above in laboratory tests and when working with tasks (e.g. textbooks or assessment) in regular classroom contexts, which is a major obstacle both when it comes to learn and to use mathematics (e.g. Lithner, 2000; 2003: 2008; 2011, Bergqvist, Lithner & Sumpter, 2008; Boesen, Lithner & Palm, 2010). In addition, teaching, textbooks and assessment mainly promote rote learning in the sense that Guided AR is provided by teachers and textbooks, and that most practice and test tasks can be solved by AR (e.g. Bergqvist 2007; Palm, Boesen & Lithner, 2011; Bergqvist & Lithner, 2012; Boesen et al., 2012). Judging from the quote by Hiebert in the introduction this may be the case also outside Sweden, for example as found in common American calculus textbooks (Lithner, 2004).
Design research and the theory of didactical situations
The ongoing research described below can be characterised as design research which in this paper refers to the use of scientific methods to develop theories, frameworks and principles of innovative educational designs. Although the meaning of design experiments have not been settled in the literature (Schoenfeld, 2007) Plomp (2009) argue that authors may vary in the details of how they picture design research, but they all agree that design research comprises of a number of stages or phases: preliminary phase (development of framework), prototyping phase and assessment phase. A key characteristic of design research is thus that it is strongly aligned with effective models linking research and practice (Cobb et al. 2003), which, according to Burkhardt and Schoenfeld (2003), "the traditions of educational research are not". According to Gravemeijer & Cobb (2006) "the purpose of the design experiment is both to test and improve the conjectured local instruction theory that was developed in the preliminary phase, and to develop an understanding of how it works."
The underlying characterisations of students' reasoning in this paper emanate from a cognitive psychology perspective, but extend into sociocultural considerations when addressing potential causes and consequences. The theoretical foundation for the attempts to design better learning opportunities is Brousseau's Theory of Didactical Situations (1997), which is a theory of how mathematics can be learnt through non-routine problem solving. It emphasises "the social and cultural activities which condition the creation, the practice and the communication of knowledge" (p. 23). One central construct is the devolution of problem. The student has to take responsibility for a part of the problem solving process, but she cannot in general learn in isolation. The teacher's task is to arrange a suitable didactic situation in the form of a problem. Between when the student accepts the problem as her own and the moment when she produces her answer, the teacher refrains from interfering and suggesting the knowledge that she wants to see appear. This part of the didactic situation is called an adidactical situation. The student must construct the piece of new knowledge and the teacher must therefore arrange not the communication of knowledge, but the devolution of a good problem.
The teacher may (e.g. to reduce complexity) try to overcome learning obstacles and force learning by devolving less of the problem to the student. Brousseau exemplifies this by the Topaze effect (p. 25) when the teacher lets the Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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teaching act collapse by taking responsibility for the student's work and letting the target knowledge disappear (as in Guided AR). Telling the student that an automatic method exists relieves her of the responsibility for her intellectual work, thus blocking the devolution of a problem. If this is the normal didactic situation the student meets then the didactical contract is formed accordingly, which may not be the teacher's intention. The teacher expects the student to learn problem solving reasoning, while the student expects that an algorithm should be provided that relieves her of the responsibility of engaging in the adidactical situation.
The key issue with respect to this paper is to find a suitable devolution of problem, with the aim of providing learning opportunities through CMR instead of AR. It is in general easy to design AR tasks, since the structure of the task is based on repeating the algorithmic procedure and follows therefore directly from the procedure. For example, after the procedure to solve linear equations (ax+b=cx+d) is described then a large number of AR tasks are obtained trivially by just formulating different equations. If the purpose is just to design any mathematical non-routine problem suitable for a particular student group, then the situation is a bit trickier but the literature and the Internet is full of good mathematics problems. However, if the purpose is to design a problem that can help the student to construct a particular target knowledge then the design becomes much more complicated. In addition, the central target knowledge within mathematics curricula is often such that a set of tasks (and adidactical situations) rather than a singular task is required. For example, if the goal is that the student shall by herself construct a general method for solving linear equations it is unrealistic that this can be done in a single adidactical situation.
A design experiment
This design experiment is a part of a larger project that studies teaching designs that give students different opportunities to learn with respect to imitation or construction of knowledge. In this experiment two ways of teaching are compared: I) An algorithmic method for solving a type of tasks is presented, and students apply this method on a set of practice tasks. The structure is founded in the framework for AR and the tasks have the same structure as common textbook tasks. II) Guiding the individual into by herself constructing a solution method. This structure is founded in the devolution of problem and in the framework for CMR. See below for examples.
In order to be able to compare these two ways of teaching, it is prioritised a) that similar target knowledge can be reached by both ways and b) that the target knowledge may be learnt both by rote and by CMR. A suitable form of target knowledge is task solving methods that can be economised as mathematical procedures. This is a central aspect of mathematical knowledge (Kilpatrick, Swafford & Findell, 2001). The teaching of such procedures seems to constitute some 50-100% of mathematics teaching (Lithner, 2008; Boesen et al., 2012), at least in Sweden but maybe also in other countries (Hiebert, 2003). Thus the overall background question posed is: 'how to best learn mathematical task solving procedures'? Is it to practice standard algorithms by large amounts of drill exercises, or by the students' own construction of the procedures? Concerning this issue the discrepancies between research and practice, and between different research perspectives seem large (Arbaugh et al., 2010). In addition, there seems to be little empirical evidence backing the rather few theoretical claims made.
The teaching mode I is hypothesised to lead the subject into rote learning of algorithms by AR without understanding the foundations of the algorithm. In the Theme B - J. Lithner et al
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mode II the subject is not given a method that can be directly applied to solve the tasks. Instead, a sequence of exploratory tasks is given. This devolution of problem is intended to make pure rote learning impossible and the subject has to understand the method in order to solve the task. One argument behind the hypothesis that tasks that require CMR will lead to a constructive adidactical situation with a real devolution of problem is related to the three defining criteria of CMR: i) Novelty, that the task cannot be solved by familiar imitative reasoning, ensures the devolution of some kind of reasoning that the student has to be responsible for. ii) The presence of arguments, supporting the plausibility of the conclusions, is necessary to guide and verify the construction of new insights. iii) The necessity to anchor the reasoning ensures that the mathematical obstacles are addressed and that the resolutions are based on properties of relevant mathematical facts and concepts.
The research question of this experiment is: What are the characteristics of an adidactical situation that leads to a devolution of problem where learning through CMR is more efficient than learning through AR in the format common in school?
The present pre-clinical (Schoenfeld, 2007) experiment is in order to reduce complexity carried out without peer-peer or peer-teacher interaction, and serves to clarify basic phenomena as a preparation to pose the same question in a real classroom context. Several iterations and revisions of task designs have been carried out. In one of the designs two groups of students (n=99) are matched by grades and basic cognitive tests. An example of an AR practice task is given in Figure 1. The corresponding CMR practice task consists of the same introduction and same question at the end, but the sentence with the formula ("If x is…) and the solved example is removed. One week after the practice session all students from both groups take the same post-test.
When squares are put in a row it looks like the figure to the right. 13 matches are need for four squares:
If x is the number of squares then the number of matches y can be calculated by the function y=3x+1
Example: If 4 squares are put in a row then y=3x+1=3.4+1=13 matches are needed.
How many matches are needed to get 6 squares in a row?
Figure 1, example of an AR practice task.
One may note that compared the AR group could have an advantage since they are provided with more information. However, the empirical studies mentioned above show that if students are given an algorithmic solution method to a task, they will mainly apply AR to solve the task without considering the underlying meaning of the concepts, representations or connections. Thus they will probably not even try to understand meaning of the algebraic formula, which in this example is the relation between the figure of matches and the formula y=3x+1. Then the Theory of Didactical Situations implies that the CMR group may learn better. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Figure 2
The results show that this is the case, in the sense that the CMR group on average has significantly higher test results and shorter response times (Figure 1). The test tasks "Formula" ask for recalling the formula, "Short numerical" for recalling and applying a solution method and "Long numerical" for (re)constructing a solution method.
In addition, the common belief that only the best pupils can benefit from learning through their own construction of solution methods is not supported by this experiment. On the contrary, in the 30% with lowest cognitive index (a composite grade and pre-test score) the difference to the advantage of the CMR group is even larger than for the whole sample (Figure 2).
Figure 3 Theme B - J. Lithner et al
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Parallel to the experiment above, other complementary studies are carried out within the research project. One example is an ongoing study using functional Magnetic Resonance Imaging (fMRI) to compare brain activity for students (n=40) from AR and CMR training groups. This study is exploratory with the aim to analyse non-behavioural information about students' thinking processes. One question asked is if students from the two groups activate different neural networks, and how this relates to earlier research findings about the brain and the learning of mathematics. Another question is if students from one group show higher brain activity in specific regions, and what the causes may be. For example, brain activity in the CMR group could be higher if they have created some kind of richer neural networks or lower if they have developed more rational solution methods and/or more efficient understanding. Another example of ongoing research uses eye-tracking methods to compare the strategies used by students learning through AR and CMR designs.
The work reported in this paper has been presented in research journals (see the references), in a large number of Nordic teacher conferences, in international

research conferences (e.g. Kristiansand 2011, London 2012, Seoul 2012) and has
attracted some interest from national and local media (TV, radio, newspapers and
popular scientific journals). During the spring 2012 discussions with leading local
school administrators about collaboration in design research and educational
development has been initiated, and financial resources are allocated for this purpose.
The research is externally and internally financed and is carried out in
collaboration with researchers, teacher educators and teachers from different areas
(mathematics education, psychology, neuroscience) in Umeå, Falun and Karlstad. The
roles of the authors can be summarised as follows. Johan Lithner (Professor in
Mathematics education): Project leader, educational theoretical and conceptual
background, design principles, construction of initial task versions. Bert Jonsson
(Senior university lecturer in Psychology), psychological theoretical and conceptual
background, cognitive measures. Carina Granberg (Senior university lecturer in
Educational work) and Jan Olsson (School teacher, Licentiate research student),
designing ICT interactive tasks for the classroom. Yvonne Liljekvist and Mathias
Norqvist (School teachers, teacher educators, PhD students in Mathematics
education), construction of testable tasks. All are engaged in the cyclic designrevision-testing-evaluation-redesign of the tasks.
The ongoing research presented above reside in the pre-clinical stage
(Schoenfeld, 2007) and concerns the design of mathematical tasks that are suitable for
devolution of problems where students may solve the tasks by CMR. One aim is to
form a basis for clinical (classroom) studies. However, it is not just to take tasks
designed and evaluated in the pre-clinical phase into the classroom. Stein, Engle,
Smith & Huges (2008) argue that teachers who attempt to use inquiry-based, studentcentred instructional tasks face challenges that go beyond identifying well-designed
tasks and setting them up appropriately in the classroom. Thus one major challenge
for the further research is how the task and teaching designs can incorporate class
interactions. A second challenge is to design tasks that are more open to the students'
own initiatives, and a third to design adidactical situations that encompass wider and
deeper target knowledge than the algebraic formulas in the design experiment above.
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Shortcomings in the milieu for algebraic generalisation arising
from task design and vagueness in mathematical discourse

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This paper presents how the milieu for students' engagement
with an algebraic generalisation task is constrained by two factors: first,
by the task design; second, by the students' unawareness of the nature of a
mathematical statement, combined with the teacher's use of a generic
example without the students' awareness of it.

Keywords: Milieu, adidactical situation, algebraic generalisation,
mathematical statement, generic example.

Introduction

The formulation of a task, as well as its mathematical, social, psychological,
and didactic contexts, are important factors for students' responses on the task
(Sierpinska, 2004). This paper presents an analysis of three student teachers'
collaborative engagement with a task on algebraic generalisation of a shape pattern.
The task is designed by their mathematics teacher educator.

The author has had
influence neither on the design nor on the implementation of the given task. 'Task' is
here understood as an assignment given to students to which they are expected to
produce a solution. The paper deals with the question of how students understand the
purpose of the task they are given in a regular teaching situation (i.e., it is not the
result of didactical engineering, Artigue & Perrin-Glorian, 1991). I show how the
formulation of the task and the interaction between the teacher and the students about
the task constitute a gap between the teacher's intention with the task and the
students' mathematical activity.

Inspired by the writing of Whitehead (1947), Devlin (1994), and others, I
view mathematics as the science of patterns. A shape pattern in school mathematics is
usually instantiated by some consecutive geometric configurations in an alignment
imagined as continuing until infinity. Radford (2006) provides a useful
characterisation of algebraic generalisation of patterns when he proposes that
generalizing a pattern algebraically rests on the capability of grasping a
commonality noticed on some elements of a sequence S, being aware that this
commonality applies to all the terms of S and being able to use it to provide a
direct expression of whatsoever term of S. (Radford, 2006, p. 5)

In the rest of the paper, "students" is used to refer to student teachers, and "teacher" is
used to refer to a teacher educator. Theme B – H. S. Måsøval
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Theoretical framework

In Brousseau's (1997) theory of didactical situations in mathematics, an
adidactical situation is a situation in which the student takes a mathematical problem
as his own and solves it on the basis of its internal logic without the teacher's
guidance and without trying to interpret the teacher's intention with the problem. The
devolution of an adidactical learning situation is the act by which the teacher
encourages the student to accept the responsibility for an adidactical learning situation
or for a problem, and the teacher accepts the consequences of the transfer of this
responsibility (Brousseau, 1997). The student cannot engage in any adidactical
situation; the teacher attempts to arrange an adidactical situation that the student can
handle.

In the devolution process, which is part of the broader (didactical) situation,
the teacher is faced with a system, itself built up from a pair of systems; the student
and a milieu that lacks any didactical intentions with regard to the student (Brousseau,
1997). The milieu is a subset of the students' environment with only those features
that are relevant with respect to the knowledge aimed at by the teacher in the
didactical situation. The concept of milieu models the elements of the material or
intellectual reality on which the students act and which may be an obstacle to their
actions and reasoning (Laborde & Perrin-Glorian, 2005). That is, the milieu of a
didactical situation is the part of the environment that can bring feedback to students'
actions to accomplish a task.

An adidactical situation is part of the didactical situation that is the broader
situation with the system of interaction of the students with the milieu arranged with
the purpose of the students' appropriation of the target knowledge without the
teacher's intervention (Brousseau, 1997). The teacher can act on the milieu by
providing new information or new equipment, for example by asking a question or
directing students' attention to certain factors in the classroom situation. When the
teacher acts on the milieu, she changes the knowledge needed to solve the problem
(Perrin-Glorian, Deblois & Robert, 2008). Whether the student can handle an
adidactical situation depends upon two conditions: first, that the student has prior
knowledge that enables him to engage with the situation; second, that the milieu
created by the teacher provides the student with knowings that enable him to develop
the knowledge aimed at (by the teacher).

Succeeding the devolution phase, the didactical situation consists of four
situations (or phases) in which the role of the teacher and the status of knowledge
change (Brousseau, 1997): Situations of action, formulation, and validation are
intentionally adidactical situations, whereas the situation of institutionalisation is not
adidactical. In the following paragraph, these situations are briefly described (for a
more elaborated explanation, see Brousseau, 1997, or Måsøval, 2011, Chapter 2).

The situation of action is where the students engage with the presented
problem on the basis of its internal logic without the teacher's intervention. The
students construct a representation of the situation which serves as a "model" that
guides them in their decisions. In the situation offormulation the students exchange
and compare observations between themselves, where the main purpose is to develop
language to formulate their observations and agree on some common meanings. Here,
the teacher re-enters the scene to chair the exchanges and make sure that all
formulations are made "visible" in the classroom. In the situation of validation the
students try to explain some phenomenon or verify a conjecture. Here, the teacher acts
as a chair in a scientific debate and intervenes only to structure the debate and Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI
encourage the students to use more precise mathematical concepts.

Institutionalisation is where the teacher
terminology and highlights definitions and theorems considered important for the
contextualised knowledge
situations) to gain the status of cultural knowledge so that it can be used in settings
other than in the original one

The adopted epistemological perspective in the paper is rooted in the theory
of didactical situations: First, teaching involves
adidactical, appropriate situation; learning is the student's adaptation to this situation
Second, teaching involves the transformation of the student's
knowledge which can be used beyond the situatio

Methodology

The reported research is derived from the author's PhD project (a case study)
reported in (Måsøval, 2011), where the addressed research question was:
constrain students' appropriation of algebraic general
with which the paper deals (Task 4, presented in Figure 1) was the last of four tasks

(during eight lessons of 45 minutes)
Task 4 is designed for students' collaborative self
subtasks which are part of the milieu for
validation, respectively:
finding a "model" that guides the students towards the in
action); identifying what figurate numbers that are part of the invariant structure of
the pattern represents developing language to formulate the students' observations
(Task 4b: formulation); and, expressing what the pattern tell
verifying the intended theorem (Task 4c: validation).
Figure 1. Task 4 given to the class for work in small groups
It is relevant to notice that this categorisation
author's conceptualisation. A
designed the task was not acquainted with Brousseau's theory. According to the same
teacher, the mathematical knowledge aimed at
mathematical statement (a theorem) represented in algebraic notation (e.g.,
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to use more precise mathematical concepts. The situation of
is where the teacher informs the students about conventional
terminology and highlights definitions and theorems considered important for the
contextualised knowledge (developed by the students through the preceding
e status of cultural knowledge so that it can be used in settings
other than in the original one set up by the teacher.
The adopted epistemological perspective in the paper is rooted in the theory
of didactical situations: First, teaching involves the devolution to the student of an
adidactical, appropriate situation; learning is the student's adaptation to this situation
Second, teaching involves the transformation of the student's responses
knowledge which can be used beyond the situation in which it is produced
The reported research is derived from the author's PhD project (a case study)
reported in (Måsøval, 2011), where the addressed research question was:
constrain students' appropriation of algebraic generality in shape patterns?
with which the paper deals (Task 4, presented in Figure 1) was the last of four tasks
45 minutes) on algebraic generalisation of shape patterns.
Task 4 is designed for students' collaborative self-engagement. It is divided into three
which are part of the milieu for situations of action, formulatio
Inventing a continuation of the shape pattern represents
finding a "model" that guides the students towards the intended theorem (Task 4a:
action); identifying what figurate numbers that are part of the invariant structure of
the pattern represents developing language to formulate the students' observations
(Task 4b: formulation); and, expressing what the pattern tells them represents
verifying the intended theorem (Task 4c: validation).
Thorvaldsen's museum in Copenhagen contains
several floor mosaics with mathematical content.
We looked at one mosaic last Friday, and here we
shall look at another one. This
thought of as built up by equal
containing bright and dark mosaic tiles.
On the shape below the pattern is reproduced
schematically with for each of the dark squared
tiles and for each of the bright ones.
a) If this shape were part of a sequence
what would the next one look like?
b) What kinds of figurate numbers do
the bright and the dark areas, and in the shape as
a whole?
c) Express what the shape tells you abo
numbers in terms of a mathematical statement.
Figure 1. Task 4 given to the class for work in small groups
It is relevant to notice that this categorisation of the task is
conceptualisation. At the time the data were collected, the teacher who
igned the task was not acquainted with Brousseau's theory. According to the same
teacher, the mathematical knowledge aimed at in Task 4 was the formulation of a
mathematical statement (a theorem) represented in algebraic notation (e.g.,
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The situation of
about conventional
terminology and highlights definitions and theorems considered important for the
(developed by the students through the preceding
e status of cultural knowledge so that it can be used in settings
The adopted epistemological perspective in the paper is rooted in the theory
olution to the student of an
adidactical, appropriate situation; learning is the student's adaptation to this situation.
s into a piece of
produced.
The reported research is derived from the author's PhD project (a case study)
reported in (Måsøval, 2011), where the addressed research question was: What factors
ity in shape patterns? The task
with which the paper deals (Task 4, presented in Figure 1) was the last of four tasks
on algebraic generalisation of shape patterns.
is divided into three
situations of action, formulation, and
Inventing a continuation of the shape pattern represents
tended theorem (Task 4a:
action); identifying what figurate numbers that are part of the invariant structure of
the pattern represents developing language to formulate the students' observations
s them represents
Thorvaldsen's museum in Copenhagen contains
mosaics with mathematical content.
Friday, and here we
shall look at another one. This pattern can be
thought of as built up by equal, squared areas
containing bright and dark mosaic tiles.
On the shape below the pattern is reproduced
of the dark squared
for each of the bright ones.
If this shape were part of a sequence of shapes,
one look like?
What kinds of figurate numbers do you find in
dark areas, and in the shape as
Express what the shape tells you about these
mathematical statement.
Figure 1. Task 4 given to the class for work in small groups
is based on the
t the time the data were collected, the teacher who
igned the task was not acquainted with Brousseau's theory. According to the same
was the formulation of a
mathematical statement (a theorem) represented in algebraic notation (e.g., Theme B – H. S. Måsøval
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2
T T n
n n ––– 1
, where 1 2 3 T n
n L denotes the n-th triangular number). This
intention was not communicated to the students.
The data are Task 4 and a video-recorded observation of three students'
collaborative engagement with Task 4 (with teacher intervention). The students are
Anne, Helen, and Paul (pseudonyms), who were in their first academic year on a
teacher education programme for primary and lower secondary school in Norway.
The observed teacher (the one who had designed the task) is my colleague, an
experienced, male teacher of mathematics. My role during data collection was to be a
silent observer while video-recording the classroom interaction analysed in the paper.
The teacher interacted with the students during the observed episode on the basis of
the students' difficulty in understanding two of the concepts used in the task. The
video-recorded episode has been transcribed and analysed through a process of open
coding (using an adapted grounded theory approach, Strauss & Corbin, 1998) where
concepts from the theory of didactical situations (Brousseau, 1997) have been used to
make sense of what factors constrain the students' algebraic generalisation of the
actual shape pattern (in this way addressing the research question).
Analysis of students' engagement with a task on algebraic generalisation
Anne, Helen, and Paul have drawn the first three elements of a shape pattern
(Figure 2) which is a continuation of the element given in Task 4. They have found
that for the first element of this shape pattern (the 5x5 square given in the task), the
number of black components (represented by black x-es) is equal to the sum of the
first four natural numbers, and the number of white components (represented by
turquoise x-es) is equal to the sum of the first five natural numbers. The students have
observed that this is a regularity that applies also for the next two elements of the
shape pattern. That is, they have verified by inspection that for the second element (a
6x6 square), the number of black components is equal to the sum of the first five
natural numbers, and the number of turquoise components is equal to the sum of the
first six natural numbers, and likewise for the third element. They have, however, not
identified the sums of consecutive natural numbers to be triangular numbers.
Figure 2. Continuation of the shape pattern invented by Anne, Helen, and Paul (Task 4a)
When they come to Tasks 4b and 4c, they wonder what is meant by "figurate
number" and "mathematical statement", and get the teacher to help them. The teacher
explains that the question about figurate numbers is about being down on the
"bedrock" looking for standard numbers that it is common to have in one's "toolbox".
The teacher thereafter asks what the students recognise if they look at the first
element (given in the task) as a whole. The following exchange takes place:
591 Paul: Well, that it is five squared.
592 Teacher: Right.
593 Anne: Yes, it is indeed squares, and then [Pause 1-3 s]
594 Teacher: Yes, it is indeed squares, square numbers.
595 Paul: And then you have nine and sixteen as the numbers of
[Pause 1-3 s] no, ten perhaps? Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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596 Teacher: And then there are the black and white ones? Do you recognise
them?
597 Anne: Fifteen, twenty one [Pause 1-3 s] ehm
In turn 591 Paul focuses on the first element of the shape pattern (the 5x5
square). I interpret his words here to suggest that he continues to look at this element
in turn 595 and refers to its number of black and white components. The numerical
values he suggests at first are wrong, but he then makes a new suggestion which is
right for the number of black components of the first element. The teacher does not
directly respond to Paul's answer, and when the teacher asks if they recognise the
black and white components (turn 596), Anne responds by giving the number of black
and white components of the second element. This seems to make Paul insecure about
what the teacher asks for; he wonders whether it is only the first element or it is the
sequence of elements they are supposed to consider:
598 Paul: If we are supposed to see the connection, it is only this very shape we shall
look at now? [Draws a curve with his pencil around the element
given in the task] It is not the next shapes we have made [points
at the succeeding elements drawn in his notebook when he says
"next"]?
599 Teacher: You may well look at it as it stands there [Pause 1-3 s] uh [Pause
1-3 s] [indecipherable]
600 Paul: Not further, ok.
The teacher's response in turn 599 I interpret as confirming that it is
satisfactory that the students look at the element given in the task (a 5x5 square) as a
basis for finding answers to Tasks 4b and 4c. It is plausible that the teacher takes this
stance as a consequence of seeing the 5x5 square as a generic example. This element
of the shape pattern is an example which illustrates that the sum of the fourth and the
fifth triangular numbers is equal to the fifth square number. It is generic (Rowland,
2000) in the sense that it is a representative of a class of elements which have the
property that they are squares which (by the two colours) illustrate that the n-th square
number is the sum of the (n ––– 1)-th and the n-th triangular numbers.
These general properties are however not addressed in the classroom
situation. The teacher does not express to the students that he uses the 5x5 square in
the sense of a generic example, nor does he use the term "generic". What I interpret as
the teacher's implicit utilisation of a generic example contributes to vagueness in the
discourse: The stance taken by the teacher about the sufficiency of looking at one
element of the shape pattern (genericity of the 5x5 square) is consistent with the
formulation in Tasks 4b and 4c (reproduced below), a correspondence which may be
expected since the task is designed by the same teacher. Application of singular
number in the noun "the shape" indicates that the shape presented in the task is seen
as generic:
What kinds of figurate numbers do you find in the bright and the dark areas, and
in the shape as a whole? [Task 4b, emphasis added]
Express what the shape tells you about these numbers in terms of a mathematical
statement. [Task 4c, emphasis added]
After having observed that the 5x5 square contains ten black components and
fifteen white components, the students describe the structure of the next elements of
the shape pattern. They observe that the elements develop by adding to the white
components an extra row (at the top) with one more component, and that the number
of black components of a successive element is the same as the number of white
components of the present element. The teacher reminds the students that they have Theme B – H. S. Måsøval
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earlier written ten as a sum of the first four natural numbers, and further, tells them
that numbers with this structure are referred to as triangular numbers. He refers to
what I interpret as (for him) a generic example when he continues:
640 Teacher E: So this is actually the clue here. That this element, I think I'll just
tell you, that this shape represents a kind of connection between
triangular numbers and square numbers.

This is succeeded by a comment by Anne that she had been insecure what
was meant by the concept of "figurate numbers". After some exchanges between the
teacher and her, she (re)turns attention to the concept of "mathematical statement"
which so far has not been addressed explicitly by the teacher:
651 Anne: Express what the shape tells about these numbers in terms of a
mathematical statement [recitation from the task]. Are we
supposed to write it as a formula or shall we formulate it?
Based on the students' conclusion on Task 4c (an explanation in natural
language of the structure of the first element of the shape pattern), it is plausible that
Anne in turn 651 is trying to figure out whether the teacher wants them to present the
solution to Task 4c as a formula (potentially in mathematical notation) or as a
formulation (potentially in natural language). The teacher responds by reinforcing
attention towards the first element of the shape pattern, which I suggest he continues
to use as a generic example:
652 Teacher: Well, then you can think of that one [points at the 5x5 square
presented in the task]. If you look at it as a whole, what square
number is it that it [Pause 1-3 s] shows us? What position?
653 Helen: Five or?
654 Anne: What number in the series or?
655 Teacher: What number in the series of square numbers, right.
656 Anne: Well, I can imagine it is [Pause 1-3 s] the fifth then.
657 Teacher: The fifth, right.
658 Anne: Because that would have been good for us [smiles]
659 Teacher: Yes. [Students laugh] Well, but here we don't have much choice,
really. It is the fifth, it is twenty five, it is square number five.
(Anne: uh huh). And if we think of it as composed by triangular
numbers (Anne: yes) then you can think of [Pause 1-3 s] what
position in the series of triangular numbers is that which these
black and white [components] represent?
I interpret the teacher's utterances in turns 652, 655, 657, and 659 as an
incidence of the Topaze effect (Brousseau, 1997): The answer that the students must
give is determined in advance (the theorem asserting that the n-th square number is
equal to the sum of the (n ---1)-th and the n-th triangular numbers); the teacher
chooses questions to which the answer can be given (turn 652). The knowledge
necessary to produce these answers changes, so does its meaning. Faced with the
student's continued difficulty in giving the answer, the teacher poses easier and easier
questions: It is possible to answer the teacher's question in turn 659 without having to
formulate the intended theorem. Hence the target knowledge has disappeared, a
phenomenon referred to as the Topaze effect.
There are features in the milieu which I interpret as giving rise to the Topaze
effect: First, the nature of the concept of a mathematical statement is not known to the
students and is neither explained to them in plain text. Second, the teacher's use of a
particular example in a generic sense, apparently without the students' consciousness
about it, contributes to the students' comprehension of the particular example as
representing a mathematical statement in its own right. The teacher leaves the Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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students after turn 659, and the students collaborate to find the positions of the
triangular numbers from which the fifth square number is constructed (the new task).
The outcome of their engagement with Task 4c is the expression in natural language
of the property of one particular shape: the fifth square number is constructed from
the fourth and the fifth triangular numbers. They make no attempt to generalise this
characteristic to apply to an arbitrary element of the shape pattern, neither in natural
language, nor in algebraic notation.
682 Paul: Well, a person who could figure out a formula for this, he would be good
[laughs].
683 Anne: No, but it is not written (Paul: no) that we shall have a formula
(Paul: right). We are supposed to express it as a mathematical
statement. We have done that now. It is not very good, but we
have emphasised what is relevant, I think.
Recall that Anne asked the teacher if they were supposed to write the
mathematical statement as a formula or just formulate it (turn 651). Paul's and Anne's
utterances (turns 682 and 683) indicate that they have interpreted the teacher's
response (turn 652) to Anne's question (turn 651) to mean that a mathematical
statement is a formulation (in natural language) about the numbers in the shape given
in the task. Further, Anne seems to conclude that a formula is different from a
mathematical statement in the way she claims that they are not asked to find a formula
(turn 683). It is likely that she by "formula" understands an expression in
mathematical notation.
Discussion
The students remain unaware that the aim of Task 4 is to establish a theorem
about a general relationship between numbers (or sequences of numbers). The
generalisation process is obscured by two interrelated factors, which are interpreted as
weaknesses in the milieu.
The first factor is about the design of the task: It is problematic that the task
presents only one element of an imagined pattern, combined with the use of singular
number ("the shape") when referring to the pattern. Further, there is a problem with
the design of the task because the students produce appropriate solutions to the first
two subtasks (with input from the teacher on the concept of figurate numbers); this,
however, does not afford them with knowings that enable them to formulate the
intended theorem. In the context of algebraic generalisation of shape patterns, the
knowledge at stake is algebraic generalisation of arithmetic relations mapped from the
elements of the pattern. For epistemological reasons, the focus in tasks on algebraic
generalisation of shape patterns therefore should be on those arithmetic relations; e.g.
in Task 4 (here efficiently represented in mathematical notation): that
2
4 5 T T 5 ,
2
5 6 T T 6 and so on, to subsequently encourage generalisation by algebraic thinking.
The second factor that constitutes a weakness in the milieu is about the
students' unfamiliarity with the concept of mathematical statement. The students had
identified the structure of the fifth, sixth and seventh elements, even if they had not
been explicit about their rank (that is, they had not made the point that the fifth
element is the sum of the first four natural numbers and the first five natural numbers,
and likewise for the next two). They had got the teacher to come to them because they
did not know what was meant by the concepts of figurate number and mathematical
statement. Anne's recitation of Task 4c and her subsequent question (turn 651)
indicates that the adidactical situation devolved to the students is not appropriate Theme B – H. S. Måsøval
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because it depends on knowledge they do not have (the concept of mathematical
statement). The teacher, however, instead of explaining the nature of a mathematical
statement, directs attention to the 5x5 square. It is relevant here that the teacher
believes that the students know the concept of mathematical statement (articulated in
conversation with the teacher after the lesson). It is therefore plausible that the teacher
interprets Anne's question in turn 651 to signify a problem with seeing the invariant
structure of the elements of the shape pattern, and not a problem with the concept of
mathematical statement per se. For that reason, when he acts on the milieu, he tries to
help them discover the structure of the elements (by utilising a generic example) so
they can develop the knowledge aimed at: an equivalence relation between square
numbers and the sum of two triangular numbers. But, as described above, the
students' interpretation of the teacher's (generic) example as complete in itself,
without attention to general properties, terminates the generalisation process. The
milieu is changed, so is the knowledge needed to solve the (new) task. A gap has been
created between the teacher's intention with the (original) task and the students'
actions on the milieu.
Måsøval (2011) has identified that tasks on algebraic generalisation of shape
patterns are of two different types, based on the mathematical object they aim at: The
first type (arbitrary shape patterns) aims at a formula for the numerical value of the nth member of the sequence mapped from the shape pattern; the second (conjectural
shape patterns) aims at a theorem which asserts equality between two different
algebraic expressions for the n-th member of the sequence mapped from the shape
pattern. It is therefore important that those who design (or choose) tasks analyse the
target the tasks aim at, whether it is a formula for the numerical value of the n-th
element of the actual pattern (a functional relationship), or it is a general numerical
statement (a theorem) decontextualised from the actual pattern. This is in order for the
milieu to be designed such that the desired relationship can be explored and
explicated by the students (e.g. through decomposition of elements according to the
invariant structure of the pattern).
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Order of tasks in sequences of early algebra
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This paper is about the selection, construction and application of a
sequence of tasks involving patterns, in order to promote algebraic thinking,
in two classes, in the period starting in the last trimester of the 3
rd
year and
the first trimester of the 4
th
year of schooling. Each sequence was composed
by twelve tasks (six structural and six sequential). One class got the
sequential group of tasks first, and then the structural, while the other group
got the reverse order. One of the goals of the study was to ascertain the
cognitive suitability of designed sequences of tasks as a situated process.
Keywords: Algebraic thinking; sequence of tasks; patterns
1. About task design and algebra study
We consider task design as a crucial element of the learning environment, and
we propose to explore the role that it plays for learners without forgetting the potential
role of the teacher in encouraging whole class discussion around tasks. Our perspective
relates to one aspect of Realistic Mathematics Education, in which the designer
conducts anticipatory thought experiments by envisioning both how proposed
instructional activities might be realized in the classroom, and what students might learn
as they engage in them. Central instructional design strategies were task analysis and
the construction of learning hierarchies in order to identify reification development. Our
starting conjecture in terms of designing didactical sequences of tasks, is that we need
epistemic and cognitive analysis not only to criticize each task itself, but to adapt its
connections as best as possible to the students abilities, found by analyzing teaching
experiments as cumulative and with anticipatory purposes. It's important for our task
analysis to identify difficulty factors providing frameworks for hypothesizing
instructional designs inspired by developmental cognition.
We decided to use an early algebra task as an explanation for a situated study
supporting the perspective in which algebraic reasoning could be highly promoted
(Kaput, 1995; Malara & Navarra, 2002; NCTM, 2000) as a tool intertwined with
arithmetic (Lins & Gimenez 1997) building on their interconnection (Mason, 2008) to
promote success by developing together both arithmetic and algebra, one implicated on
the other. The study supporting this paper has been done with 8-9 years old students. Theme B – P. Palhares, L. Vieira, J. Gimenez
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About the teaching experiment
The basis to build our sequence of tasks and test analysis, is to promote
algebraic thinking by overcoming relational apprehension (Smith, 2011) and the use of
patterns (Orton, 1999; Carraher, Martinez and Schliemann, 2008), used here in
connection with a search for order or structure and therefore regularity, repetition and
symmetry are frequently present (Frobisher et al., 1999), because of the relevance to the
development of abstraction, generalization and the establishment of relations (Lins &
Gimenez 1997; Mulligan et al., 2006).

The first part of the study consisted of the construction and validation of a questionnaire-task that later was used as pre- and post-test control strictly validated as it's usual in empirical studies (Vieira, Palhares & Gimenez 2012). Next step concerns the experimental process based upon a refined sequence of tasks. The principles for our task design are: (1) assuming the possibility of using arithmetic number sense related to algebraic reasoning; (2) assuming suitability criteria for analyzing mathematical activities; (3) mathematically inspired by using relations and diversity of representations but not letters for the unknowns; (4) hearing the voice of the students for analyzing and promoting mathematisation and retention.
A set of nineteen short tasks was designed, from which twelve tasks were chosen, six of the structural type and six of the sequential type. These separation is closely related to strands 1 and 2 within Core aspect A of algebraic reasoning (Kaput, 2008). In each type, half was figural and the other was numeric. In our sequence of tasks, the main aim is to reflect about the use of some knowledge to build and/or transfer to other mathematical knowledge. Our presentation leaves the problems from everyday life for a subsequent time. Therefore, we don't explain other algebraic activities more related to promote modeling processes. We are facing the problem of complexity and connection of tasks in designing sequences.
The tasks were meant to be diverse, some leading to an exploratory and investigative open activity to improve meaningful construction (Thompson, Carlson, & Silverman, 2007), and also as a problem solving approach (Arcavi & Friedlander 2011), some others involving structured generalization rules (Stacey, 1989). In the experiment, the teacher introduces always the problem by focusing heavily on exploring the situation, leaving the student a maximum degree of freedom for discussion.
About mathematic/epistemic suitability
The activities integrate various mathematical aspects. The tasks of the sequential type were characterized by the existence of the first terms of a sequence, asking for the next term or a more advanced as a kind of generalization. The tasks of the structural type include different processes and properties: equality and order properties, finding rules from examples, identify equivalences, assume relational identities, acting by assumption, particularization,… among others.
Cognitive suitability
The implementation sequence was such that numeric and figural were alternated, with an increasing mathematisation in the case of the sequential type. Figural aspects are interpreted as supported by images or pictures in the sense of Pallascio (1992). The patterns and questions are not usual in textbooks, but involve addition and multiplication and addition rules. Only in one case one pattern can be considered as a division. We now give one example for each subtype:Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Task type 1 (structural and figural)
In such a task, the aim is to observe how the arguments reveal an algorithmic process (based on Femiano, 2003).
Observe carefully the four dish scales that are in equilibrium.
Substitute the question mark by the number that allows the scale to remain in equilibrium. All equal figures have the same value.
Explain how you have found the value to substitute the question mark. You can use words or calculations to describe what you have done.
It's expected that students find the value of a star, and after finding values, identify that the square and triangle sum is 8. Then they can use trial and error, or change the order and assume that two other variables have the same value. Four equal properties are used in this case. We must notice that in such figural tasks, the role of the teacher is to pay attention about the consideration of the same value for the same figure. The teacher asks suddenly the students to explain their ideas in order to focus on mathematical argumentation used.
Task type 2 (structural and numeric)
In such a task, it was planned that the most difficult aspect is the interpretation itself because the sequence of starting numbers is not in order and for the fact that it's not a usual class task.
Observe carefully the four 'number machine'.
Substitute the question mark by a number that follows the rule of the other three machines.
Explain how you have found the number to substitute the question mark. You may use words or calculations.
We already knew that the use of additive comparison tended to be somewhat challenging to students of this age.
Task type 3 (sequential and figural)
This task was proposed to initiate table representations to establish combined rules made by two operations. The idea is to identify the ways in reaching the rule, and the way to explain. This is similar to those explained by Radford for these ages (Radford 2010) to provoke the use of arguments based on numbers. In this case, the figure can help to find the answer for the first questions.
Observe carefully the sequence of figures below.
Fig. 1 Fig. 2 Fig. 3 Fig. 4 Fig. 5 Theme B – P. Palhares, L. Vieira, J. Gimenez
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Construct the two following figures 4 and 5. Without drawing or constructing it, say how many squares of each colour will have the 8
th
figure. Complete the table.
Figure number 1 2 3 4 5 … 10 20
Number of white squares 2 6 …
Number of gray squares 1 1 …
Explain how you have got to the number of squares of each colour in the 8
th
figure, using the table. Is it possible to find in this sequence a figure with exactly 50 white squares and one gray square? Explain what your thoughts to answer the question were. Did you find any regularity in the sequence of white squares? And in the gray squares? Write a small text on the conclusions you have reached. Can you establish some relation between the number of the figure and the number of white squares? Justify your answer. What is the rule for this pattern?
Task type 4 (sequential numeric)
It's a typical task in which the students are required to identify the pattern
Observe carefully the sequence of numbers below:
6, 10, 14, 18, 22, ____, ____, ____, …
Place numbers on the lines. What will be the 10
th
term of the sequence? And what will be the 20
th
term of the sequence? Explain how you have found the 20
th
term of the sequence. Will the number 63 be a part of this sequence of numbers? Justify your answer.
The tasks have been answered in a group work setting. After the groups have solved, they had to present their solution with the justification to the whole class and a discussion followed, contributing to provide a sense of purpose and ownership (Ainley and Pratt, 2011). In these classes, one group was chosen from and followed from the start, with their dialogues being recorded. The second author participated as a second teacher, after obtaining permission from the school board and from parents.
These tasks have been introduced in two classes after the pre-test. In one of the classes, the students were confronted with the sequence of sequential tasks, and in the other class students were confronted with the sequence of structural tasks. This has happened near the end of the schooling year and it was followed by the application of the post-test, which happened in the end of May. After vacation time, in September, being the students already on the 4
th
year of schooling, the test was applied again. The intention was to check for retention. Then students were confronted with the sequence of tasks they had not solved yet. We had then one class solving six sequential tasks and then six structural tasks and another class solving six structural tasks and then six sequential tasks. All students in the two classes solved the same set of twelve problems. Finally, the test was applied one last time, in November.
2. Results and implication for redesigning
The main intention is to introduce a cycle of redesigning interpreting the students' answers both as qualitative and quantitative data. A systematic analysis of student work is not thoroughly described because of space limitation. Contrary to our expectations what these results show is that it is not indifferent to start with sequential or with structural tasks. Sequential tasks are better for starters and apparently provide a solid foundation for the work with structural tasks. As to the two groups that have been followed in more detail, the results have been these: Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Structural/sequential group
April 2011 June 2011 September 2011 November 2011
Daniel 146 156 156 156
Bert 114 144 120 152
Arthur 54 104 53 134
Cindy 42 72 57 69
Sequential/structural group
April 2011 June 2011 September 2011 November 2011
Gisela 72 102 144 120
Helen 74 93 117 132
Earnest 42 128 117 156
Frank 114 141 156 156
We have taped and later transcribed the dialogues in one group for each class.
We will focus on the first two tasks of each type and compare their resolution in the two classes. In terms of the designed tasks, both pairs reached, at the initial stage of predictions, generalizations expected by the designers. In particular, we found some inductive generalization based on the collection and analysis of data (as followed by the sequence of tasks in this activity), but nobody did any deductive reasoning. Let's explain some observations about the tasks above exemplified.
Concerning task 1 described above, which for the structural>sequential class (StSe) was the first task and for the sequential>structural class (SeSt) was the seventh both groups initially fell for attempting trial and error. Both groups easily found the value for the star. Group StSe had some difficulties in concluding that the triangle and the rectangle had to have the same value. The main reason apparently was that being the shapes different, they assumed the value had to be different. They made several attempts with different values. The researcher asked them if they had observed all the scales and from this prompt they returned to the observation of the scales and concluded immediately that the triangle and the rectangle had to have the same value and solved the rest easily. Group SeSt tried at first to guess the final scale value. But abandoned this strategy, found the value for the star and through the analysis of the third scale they found the values of the triangle and rectangle and solved the problem.
Concerning task 2 described above, it was the second task for the StSe group and the eighth for the SeSt group. Group StSe has approached this problem with a strategy based on the difference between the entry number and the exit number. As one of the machines presented the number 10, they started conjecturing with it, as calculations were easier. Eventually they perceived the strategy was not working. Next they tried to establish the difference between the exit number from one machine and the entry number of the following. They gave it up and tried two things: one was comparing the sum of entry and exit numbers of one machine and the entry number of the following, the other was to compare the sum of the three entry numbers of three machine and the exit number of the third. They continued attempting adding and subtracting numbers until they moved to a multiplicative strategy, but this move resulted from an ill constructed comment from the teacher, giving too much away. After this they revealed no difficulty in finding the relation or the rule. Group SeSt rapidly found the value that should be placed in the question mark. Therefore we could not conclude if that was due to the establishment of relations between the numbers or to luck on the first attempt. In the explanation that followed they just describe the calculations performed:
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Helen – it's 5 times 3.
Frank – 5 times 3?
Helen – plus 1.
Earnest – it's 5 times 3.
Helen – 7 and 7 is 14 plus 7, 21; 21 plus 1, 22… it's 28 (pointing with the finger to the question mark). 9 and 9 is 18, plus 9, 27.
Earnest – It's 28 (pointing to the question mark).
Helen – 28 (while pressuring Frank's arm, who is responsible for the writing).
Frank – calm down!
Helen – we have to write it down.
Earnest – now we have to explain why.
Concerning task 3 described above, it was the second task for the group SeSt and the eighth for the group StSe. In the group SeSt, one of the elements refers that it is always 'plus two' (speaking to the group partners) while the researcher was still presenting the task, adding that it was always one above and one below:
Earnest – Ah! Figure six has to have on the top 7. Figure…
Frank – eight…
Helen – it has to be 6 down here and the figure eight? Figure seven how much does it have?
Earnest – figure seven, figure seven, figure seven… ah, figure six has 7 here and 6 here and figure seven…
Frank – 8, it has 7 here and here 8.
Helen – 9 and 8
Earnest – what? Figure eight? Then write it.
They work line by line, interiorizing that the top line has one more square than the line below. They do not reveal any difficulty filling the table, establishing the relation 'double' between the figure number and the number of white squares, identifying the invariant (gray square). In the question about wherever there was a

figure with exactly 50 white squares and one gray square, one of the group elements refers immediately that it is possible, since they're even (white squares) plus one gray square. In the question about the relation between the figure number and the number of squares they find the relation and manage to establish a distant generalization in the oral conversation, referring that it is the double plus one, however they fail to say so in the written answer.
The group StSe identifies immediately the number of white and gray squares in figure 8, and they reveal no difficulty in filling the table. As to the question about the existence of a figure with 50 whites and one gray, they rapidly answer that it is figure 25. In the question about the relation between the figure number and the number of squares, they find a near generalization mentioning that "the number of whiter squares doubles is two more than the previous one and the gray stays". This group has no difficulty finding the relation between the figure number and the number of squares, but does not offer a justification.
Concerning task 4 described above, it was the fourth for the group SeSt and the tenth for the group StSe. Group SeSt have initiated solving this task by searching for a regularity between consecutive terms (plus 4 than the previous term) and later they found that to find a distant term they could multiply the term number by 4 and add 2. They justified that 63 was not a part of the sequence because it was an odd number. In the last question about the regularity found they wrote: "the order number times 4 plus 2 gives the number for the term. They are all even numbers". Group StSe does not reveal difficulties in finding that the difference between consecutive terms is 4. Meanwhile they discover they can multiply the term number by 4 and then add 2, and have written 20x4+2=82 to find the 20
th
term. However in the written explanation they show some Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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confusion and get into contradictions. As to 63 being part of the sequence, they claim that it is not multiple of 4 (omitting the ulterior adding of 2 units). They say: "all multiples of 4 are even". Last, in relation to the regularity, they write: "the quadruple of the number for a term plus 2 is equal to the number of the sequence of the term".
Conclusions
First of all, we know from many previous experiences that structured investigative activities indicate that they provide opportunities for meaningful learning of mathematical concepts. We also know that differences between a designer's planned actions and student work should be expected. One can argue that, these students were unfamiliar with structural algebraic tasks, and they answer them as open questions, making their own interpretation. Cognitive analysis seems not to be enough to decide about ordering. The study is considered as a first step for reconsidering the tasks for redesign in which a new cycle of testing could lead to small or big changes. Apparently, it seems that only because of epistemic values, we consider structural tasks after. It is however clear students that started with the sequential tasks seem to be capable of establishing distant generalization when the other group couldn't. And certainly the group that started with sequential tasks appears to retain their performance more robustly stable across time. This type of task design proved to be a rich starting point for significant classroom discussions on early algebraic situations.
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A Task Design for Conjecturing in Primary Classroom Contexts

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The purpose of the study was to design tasks for conjecturing to support students engaging in the activities of proving in primary classroom contexts. A task involving perimeter and area of figures as an example was designed on the basis of nine principles of designing for conjecturing and proving suggested in F. L. Lin's team work (Lin, et al., 2012). Twenty seven third grade students' written work was collected for the main data of the study. It was found that the task was characterized as four features in accordance with students' justification, which were analyzed by three components of proofs suggested by Stylianides and Ball (2008) .

Key Words: mathematical conjecturing, proving, task design, primary,

Introduction

Mathematical tasks are recognized as the tool for teachers to shape a teaching design and for students to develop, utilize and understand a certain concept (Stein, Grover, Henningsen, 1996). Thus, mathematical instruction is generally organized and delivered through students' activities on mathematical tasks. Nevertheless, students respond to mathematical tasks very differently, depending on the structure and demands shaped by tasks enacted by teachers. This indicates that to reach high quality of mathematics instruction, mathematical tasks play a crucial role. Thus, selecting and designing appropriate tasks is essential to the success of teaching mathematics (Doyle, 1988; Stein & Lane, 1996).

Doyle (1983) defined academic tasks as (a) the products that students are to formulate, such as the answers to a set of questions; (b) the operations that are to be used to generate the product, such as classifying examples of a concept; and (c) the "givens" or resources available to students while they are generating a product. Doyle (1988) further argues that tasks with different cognitive demands are likely to induce different kinds of learning. According to this definition, tasks can vary not only with respect to mathematics content but also with respect to the cognitive processes involved in working on them. Only worthwhile tasks offer students the opportunity to extend what they know and stimulate their learning. Tasks that require students to solve complex problems can be considered to be cognitively demanding tasks. In contrast, cognitively undemanding tasks are those that give less opportunity for the students to engage in high-level cognitive processes. The conjecturing tasks require Theme B – Pi-Jen Lin & Wen-Huan Tsai
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high cognitive demands because they are involved in three components: a set of true statements, valid modes of argumentation, and appropriate representation of modes of argumentation (Stylianides, 2007). Therefore, it is worthwhile to develop teachers' knowledge of conjecturing tasks design for enhancing students' proofs. Moreover, proof is a vehicle to enhance students' understanding of mathematics concepts and promote mathematical proficiency and reasoning (Hanna, 2000). Proving is an important means of exploring in mathematics. Research shows that engagement in proving can support students to explore why things work in mathematics and explain their disagreements in meaningful ways, thus providing them with a solid basis for conceptual understanding (Stylianides, 2007). The previous studies suggest that students should have early and appropriate opportunities to incorporate proof into their mathematical learning (Ko, 2010; Kilpatrick, Swafford & Findell, 2001;Stylianides, 2007).

Conceptual Framework for Designing Conjecturing Tasks

Conjecturing and proving can promote mathematical thinking and launch mathematical inquiry; therefore, tasks of conjecturing and proving should be designed to be embedded into any grade level of classrooms (Kilpatrick, et al., 2001). The tasks of both conjecturing and proving are involved in students' conjecturing, but F. L. Lin and his colleagues distinguished the distinction of the principles for conjecturing tasks design from those for proving task design (Lin, et al., 2012). The four principles for designing conjecturing tasks and the five principles for designing proving tasks are considered as the conceptual framework of the study, since they are achieved to two fundamental functions: relating to the learners' roles or hypothetical learning trajectories and the practical function of easily evaluating.

In the tasks for conjecturing, students are asked not only to generate conjectures according to the given information which could be ill-defined, but also search for proofs to verify or justify whether the conjectures they made are true or not. They further suggest that the designing efficient tasks for conjecturing should consider the provision of opportunities to: (1) observation, (2) construct, (3) transform, and (4) reflect. The observation-based conjecturing refers to activities that involve purposeful or systematic focus on specific cases in order to make a generalization about the cases. The construction is a principle that encourages students to construct new knowledge based on prior knowledge which may lead to conjectures. The transformation design principle means that the task gives students the opportunities to generate conjectures by transforming given algorithms or formula. The conjectures by transformation may lead students to incorrect or meaningless statements. Thus, the reflection principle is essential to design the tasks for conjecturing.

The tasks for proving can promote proofs and proving. The type of tasks for proving is characterized as to ask students to justify whether the given statement in the existing conjectures is true. The false or true statement is given by the instructor's. Lin and his colleagues suggest that the tasks for proving have different difficulties, depending on the sources of conjectures are either from instructor's or students' themselves (Lin, et al., 2012). In the case of conjecture given by the instructor, students may feel it is not necessary to prove it, because they accept the truth based on either epistemic values (Duval, 1998) or instructor's academic authority (Lin &Tsai, 2012a; Reid, 1995). In the case of conjecture given by students themselves, they may Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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have unclear distinction between conjecturing and proving. For instance, they may consider empirical arguments as deductive proof (Stylianides and Stylianides, 2009). The five principles for designing proving tasks suggested by Lin's team have to do with modes of argumentation and its representations. They include: classifying mathematics statements, expressing arguments in several modes, changing roles in a task, defining efficient and necessary proof, and creating and sharing proof. The first principle is to provide the opportunity for students understanding six possible types of statement between universal and existential statements. It could be always true, sometimes true, and never true for true or false statement, respectively. The second principle is to provide the opportunity for students to understand various modes of argumentation to be used appropriately for different types of statement, such as counter examples, supportive examples, deductive and inductive proof via various

representations. The third designing principle promotes the opportunities of switching the role of students as an instructor or an evaluator of the proposed justification from traditionally being seen as the learner. The fourth principle provides students the opportunities to be aware of the sufficiency and necessity for a legitimate proof. The final principle is to provide students the opportunity of creating a proof and then present it to the whole class in public for verifying its validity and truth.

A statement, existing in the tasks for proving, proposed by the instructor can be logically true or false. The existing empirical studies suggest that students at primary or secondary level do not accept counterexamples as refutation, rather, they offer more than one counterexamples for refuting a false statement (Reid & Knipping, 2010). Student even at high school level or undergraduates still have difficulty with mathematical proof in school mathematics (Lin &Tsai, 2012a; Ko, 2010). Some of them accept the truth of empirical induction from finite number of discrete cases for verifying a true statement. Besides, students or future teachers also perform better on false statement than true statement (Lin &Tsai, 2012b). The different difficulties may result from the nature of proving, because the verification of a false statement is easier than a true statement in that one counterexample is enough to refute a false statement. A true statement needs to be verified by inductive or deductive reasoning. The processes of proving a true statement demand rigor and complex argumentation.

On the basis of literature review, it could be a good start for the beginners at primary level for the study to learn proofs with a false statement instead of a true statement for designing proving tasks. However, it is new experience and knowledge that the task design for conjecturing are conducted by the primary teachers involving in the study, because their students have little experience with what conjecturing and proving looks like. Thus, the purpose of the study is intended to support teachers to design various conjecturing tasks in line with the mathematics contents to be taught in primary classroom contexts for student exploring the activities of proving. The paper describes not only the development of the task design but also the effect of conjecturing task on enhancing students' engaging in the activity of proving in the context of the relationship between perimeter and area in two figures. The task referred to in the study was the activities or artifacts such as teaching aids exploring in classroom contexts. Theme B - Pi-Jen Lin & Wen-Huan Tsai

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Method

Participants and Context

The task involved in the study was designed by one of the teachers who participated in the first year of a three-year project that was designed to help teachers to create conjecturing tasks for engaging students in the activity of proving. Hence, the tasks design for students engaging in valid proofs is a new experience and novice learning for the teachers, but they were mutually supported in the professional team consisting of six teachers and two researchers, the authors of the paper.

Twenty-seven third grade students in the class have separately learned the concepts of perimeter and area of a figure before engaging the task. They engaged in several activities of conjecturing in the first semester. The task was conducted in the second semester, so that some of the students had slight experience of finding out a counter example to refute a false statement.

They were grouped heterogeneously in groups of 4 or 5. After given the task, the students first worked independently and jotted down their judgement and verification on B4 paper; then they came together in groups to compare their solutions, and finally they shared their arguments to the whole class. The lesson was videotaped throughout the entire class. Each student's written work was collected throughout the whole year.

Designing Conjecturing Tasks for Students Engaging in the Activities of Proving

The task designed by the teacher was to ask students to make a conjecture and verify whether it is true. The statement is that "In any two figures, if the area of one figure is bigger than the other, then its perimeter of the figure is greater than the other, too. Do you agree? Why? Show your work on the grid paper. " The task for conjecturing is initiated from a false statement. The task design was on the basis of the four principles of the task for conjecturing and five principles of the task for proving, suggested by F. L. Lin's team work (Lin, et al., 2012), since it is potential to launch the following activities for students engaging in conjecturing and proving: (1) The task provided students an opportunity to engage in observation through finding out a pair of two figures and making a generalisation about the cases; (2) The task provided students an opportunity to engage in construction. For instance, to solve the task, students needed to create two figures with different areas but same perimeter; (3) The task encouraged students an opportunity to transform prior knowledge of the perimeter and area of an irregular figure by counting the number of small squares on the grid paper; and (4) The task provided students an opportunity for reflection. For instance, "Show your work" as part of the task is to ask students to explain why they believe their conjecture is true for the given condition.

In addition, the task was also characterised as the following features: (1) The task promoted students classifying various statements, such as, the statement of same area in two figures result in same perimeter; (2) The task had the potential to require students to express same argument in several representations of modes of argument. The counter examples for refuting the false statement can be expressed by either "bigger area but smaller/same perimeter" or "same area but greater perimeter"; (3) The task engaging in the classroom provided the students as the role of an evaluator for the justification proposed by them; (4) After the false statement was proved, the students were asked to modify it and rephrase it as a true statement; and (5) The task Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.

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engaged in the classroom context provided students an opportunity with creating and sharing their own proofs.

Data Collection and Analysis

The result session was aligned to the Stylianides and Ball's (2008) three components of proof by using students' written work. Taking the consideration of students as mathematical learners, proof suggested by them can be defined as a set of accepted statements, known modes of argumentation, and accessible modes of argument representation to a classroom community. The three components of proofs as the framework of analyzing the data collected for the study, students' written solutions were first split into two piles and then a pile was assigned to each group of the two groups consisting of six school teachers studying in a master program. Afterwards, they took turns to review the other pile for increasing the validity and reliability of analysis.

Results

The Set of Statements Accepted by the Third Graders

After the task was explored, 8 (30%) students made incorrect judgment by accepting the teacher's statement, while 19 (70%) students judged correctly and verified successfully by finding out a pair of figures to reject the conjecture given by the teacher.

The set of statements referred to the statements accepted by the classroom community. Once they figured out a pair of figures, various accepted set of statements as part of their arguments were generated by 13 students, such as "bigger area but smaller perimeter", "bigger area but same perimeter", "smaller area but same perimeter", or "same area but greater/smaller perimeter". For instance, Ming, Jenny, Ron, and Huei, as the examples, successfully showed their accepted statements by their classmates, as shown in Table 1. Thus, possessing various types of statements for students' classification is the first feature of the task.

In addition, the statement "The bigger area in one figure, it is not necessary to be greater in perimeter." that 6 students used was another type of statement. Starting from the condition "bigger area in one figure than the other" given in the instructor's statement was the most common statements accepted by those who were in favor of the conjecture (in total, 68%, 13 out of 19). Theme B - Pi-Jen Lin & Wen-Huan Tsai

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Table 1: The Set of Statements Accepted by the Third Graders.

Modes of Argument Known by the Third Graders

Modes of argument are the ways of verifying or justifying a statement.

Overall, the mode of argument for the task used mostly by the third graders was the use of counterexample. They seemly knew that one counterexample is sufficient to refute a false statement. Their modes of argument for this task were not like other tasks that students are used to offer more than one counterexamples for refuting a false statement (Lin & Tsai, 2012a). Thus, the task provided the best opportunity for promoting students' understanding on proving that a single counterexample is sufficient to refute a false statement. This is the second feature of the task.

For verifying or refuting the statement given by the teacher, the students needed to find out a pair of figures such that one area in one figure is bigger/smaller than the other. It is followed by observing the relation of their perimeters. To fulfil the work, students needed clear distinction between the two concepts: area and perimeter. Sometimes, they needed to attempt several times, as shown in Figure 1. The Figure 1 displayed that the task was not only providing students an opportunity for exploring the activity of conjecturing and proving but also for clarifying students' confusion of perimeter with area. This is the third feature of the task.

Figure 1: Student' Tries for Finding Out a Pair of Two Figures

Representations of Modes of Argument Accessible by the Third Graders

The representations of modes of argument are the forms of expression for communicating with the classroom community. This conjecture task involving the relationship of perimeter and area, word expressions with figures was the most popular form of argument accessible by the third graders. As shown in Table 2. Eighteen (67%) students verified the statement by drawing at least an irregular figure in the pair of figures on the grid paper. It seems that third graders readily draw an irregular shape on the grid paper. As a consequence, it made students successful in conjecturing and justifying the statement. This is the fourth feature of the task. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.

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Figure 2: Student' Representation of Figures

Conclusions and Discussions

The quality of task design was evaluated by students' justification on the basis of the three components of mathematical arguments, suggested by Stylianides and Ball (2008). It is said that the task maintained high cognitive demands while it was implemented into a third grade classroom, since the task was characterized as the following four features.

(1) The task provided students new experience that there were various types of statement instead of a single type for a counterexample to refute a false statement. It has to do with the first component of proofs: a set of statements. The students have developed knowledge of proofs for refuting a false statement by unique type of statement for a counterexample from prior tasks. This feature meets the first principle of task designing for proving suggested by F. L. Lin's team (Lin, et al., 2012).

(2) The second feature was that the task provided the best opportunity for promoting students' understanding of a single case as a counterexample to be sufficient to refute a false statement. This feature is documented from the second component of proofs suggested by Stylianides and Ball (2008). The task was beneficial for developing students' knowledge of the way of refuting a false statement. This was different from previous studies in which most of the students at primary or secondary level were used to utilize more than one counterexamples to refute a false statement (Lin &Tsai, 2012a; Reid & Knipping, 2010).

(3) The task, which provided students an opportunity not only engaging the activity of conjecturing and proving but also clarifying students' confusion of perimeter with area, was characterized as the third feature. The feature of clarifying students' misconception or confusion between concepts is not on the list of task design principles suggested in F. L. Lin et al.'s work. The result indicated that the task could be a powerful instructional approach via conjecturing for clear understanding on perimeter and area. However, the effect of conjecturing on learning the relationship of perimeter and area needs further study in the future.

(4) The final feature was that the provision of grid paper as part of the task has potential to make students' successful in conjecturing and proving. The pictorial representation in several modes of argument was matched with the second principle of task design for conjecturing suggested in F. L. Lin et al.'s work (Lin, et al., 2012).

The study suggested that knowledge of students' mathematics concepts and teachers' knowledge of proofs embedded in the conjecturing tasks were two essential factors affecting the quality of proving exploring in the conjecturing activity. Without solid mathematical concepts underpinning the arguments, it is impossible for students Theme B - Pi-Jen Lin & Wen-Huan Tsai

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to produce logical proofs. The study also suggested that the provision of false statements instead of true statements made it more possible for students to learn successfully on acquiring the knowledge of conjecturing and proving.

Acknowledgement

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Using Student Solutions to Design Follow-up Tasks to ModelEliciting Activities

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Model-Eliciting Activities (MEA) are open ended mathematical
tasks in which students develop a mathematical model to solve a real
world problem. Their open-ended nature often results in students
developing and articulating a great diversity of mathematical ideas. What
tasks can help students extend these ideas after the MEA has been
completed? We describe an innovative approach for designing Follow-up
tasks to MEAs, which requires students to analyse other students'
solutions to the MEA they have just completed. As students critique and
analyse the mathematical strengths and shortcomings in other students'
solutions, they simultaneously challenge and extend their own
mathematical thinking.

Keywords: Mathematical modelling; task design; model eliciting
activities

Introduction

Peter's story: As a teacher, I've often come across intriguing contexts and
wondered if I could turn them into mathematical tasks for the classroom. An
advertisement in the situations vacant column, asking for a chicken sexer: "An
extremely high level (98%) of accuracy is imperative. Speed is vital, sexing
approximately 1000 chicks per hour." Or the following description of a YouTube
video: "Alexander Overwijk draws a perfect freehand circle 1m in diameter in less
than a second." Experience has taught me that turning these contexts into effective
mathematical tasks is not an easy thing. Often, the context outshines the mathematics.
Other times, the mathematical content is too fleeting, trivial, broad or complex. I have
folders filled with this kind of raw material but for some time, I lacked the design
tools to craft them into mathematical tasks. I was intrigued when I heard Caroline
describe some task design principles she was using in her research, while I was on a
teaching sabbatical at a local university.
Caroline's story: In that talk, I described a sequence of four calculus tasks
that I had developed together with Tommy Dreyfus and Mike Thomas. Two of the
calculus tasks (described in Yoon, Dreyfus & Thomas, 2010) were created using wellestablished principles for designing special modelling tasks called Model-Eliciting
Activities (MEAs) (Lesh, Hoover, Hole, Kelly & Post, 2000). But the other two
calculus tasks were created without any specific guiding design principles. I had just Theme B - P. Radonich & C. Yoon
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written a grant proposal to develop design principles for the latter kinds of tasks—
tasks that followed MEAs and extended students' thinking further. As fate would
have it, I was looking for teacher-designers to participate in the project at the same
time that Peter was looking for task design principles.
A team of designers: The proposal was funded, and we spent the next two
years (with two other teacher-designers, Anne Patel and Nikki Sullivan) designing,
testing and revising tasks and their underlying design principles. We applied Lesh et
al.'s (2000) design principles to the context of the perfect freehand circle YouTube
video to create an MEA we called the Giotto MEA (described below). When we
tested the Giotto MEA in classrooms, students typically developed and articulated a
diversity of mathematical ideas, but their written solutions often had numerous
shortcomings. Peter proposed that these imperfect student-generated solutions were
rich fodder for designing Follow-up tasks to MEAs. He proposed that students could
extend their own mathematical thinking by analysing the gaps and strengths in other
students' solutions to the MEA. We will describe how we used this approach to
design Follow-up tasks to MEAs, as illustrated in one Follow-up task to the Giotto
MEA. We identify with Theme B working group, which aims to "understand how
appropriate task design might help minimise the gap between teacher intentions and
student mathematical activity". Our Follow-up tasks address this aim as they are
designed around students' perspectives (indeed their very solutions) to the initial
MEA.

The Giotto MEA

Model-Eliciting Activities (MEAs) are a class of mathematical modelling
tasks where students develop a mathematical model in response to a real world
problem. MEAs do not stipulate what that model should look like, but only what the
model should be able to do. This open-endedness typically leads to MEAs eliciting a
diversity of mathematical approaches from students. Consequently, MEAs have been
used to identify creatively gifted mathematics students (Chamberlin & Moon, 2005)
and to investigate equity issues in undergraduate engineering courses (Diefes-Dux,
Hjalmarson, Zawojewski & Bowman, 2006).
We created the "The Giotto MEA" (see website http://icmi22radonich-yoonpaper.wikispaces.com/home) according to Lesh et al.'s (2000) principles for
designing MEAs, which are summarised in Table 1. The problem begins with a comic
that tells how the Renaissance artist, Giotto, gained the pope's attention by drawing a
perfect freehand circle (see Figure 1). After reading the comic, students are asked to
draw their own freehand circles and choose the best among them. Next, they watch a
short YouTube video of a mathematics teacher who professes to be the world's
freehand circle drawing champion and appears to draw a perfect freehand circle. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI
Figure 1 Students read this comic about the
artist Giotto drawing a perfect freehand circle
Students then meet
Bonnie, who is holding a circle drawing competition at the local Pancake House. The
students are asked to work in teams of three to
attempts from most circular to the least circular, which Bonnie can use to judge the
circle drawing attempts on the night of the competition. Students are
their method on some examples of circle attempts (
must also work for any circle attempt that could be drawn on the night of the
competition. The student teams
Bonnie.
Table 1: A summary
Reality principle
The activity should present a problem situation
that students can interpret usi
personal knowledge and experiences.
Model-construction principle
The activity should create a convincing need for a
mathematical model.
Self-assessment principle
The activity should articulate the criteria that are
used to judge the students' final models.
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Figure 1 Students read this comic about the
artist Giotto drawing a perfect freehand circle
Figure 2 Students are asked to
demonstrate their method on some circle
attempts.
meet the problem statement, which introduces them to a client,
Bonnie, who is holding a circle drawing competition at the local Pancake House. The
work in teams of three to develop a method for ranking circle
from most circular to the least circular, which Bonnie can use to judge the
circle drawing attempts on the night of the competition. Students are
examples of circle attempts (e.g. Figure 2), but their method
k for any circle attempt that could be drawn on the night of the
The student teams write their final method in the form of a letter to
A summary of Lesh et al.'s (2000) six principles for designing
The activity should present a problem situation
that students can interpret using their own
personal knowledge and experiences.
Model-documentation principle
The activity should require students to describe
their mathematical models to others.
The activity should create a convincing need for a
Model-generalisability principle
The activity should require a reusable and sharable
mathematical model.
The activity should articulate the criteria that are
used to judge the students' final models.
Effective prototype principle
The activity should be as simple as possible, while
still creating the need for a mathematical model.
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Figure 2 Students are asked to
demonstrate their method on some circle
the problem statement, which introduces them to a client,
Bonnie, who is holding a circle drawing competition at the local Pancake House. The
develop a method for ranking circle
from most circular to the least circular, which Bonnie can use to judge the
circle drawing attempts on the night of the competition. Students are asked to test
), but their method
k for any circle attempt that could be drawn on the night of the
orm of a letter to
for designing MEAs
documentation principle
The activity should require students to describe
their mathematical models to others.
generalisability principle
The activity should require a reusable and sharable
The activity should be as simple as possible, while
still creating the need for a mathematical model.Mathematical and epistemological perspective
solutions
When analysing the mathematical approaches in students' solutions to
MEAs, we draw on theoretical perspectives that focus on the
that students attend to, such as modelling perspectives (Lesh & Doerr
Blum and Galbraith, 2007) and Mason's theory of the structure of attention (2004).
These perspectives characterise mathematical structure as consisting of
objects or elements (such as counts, measures, sets),
few, large, open), operations (e.g. combine, enlarge, invert), and relationships (e.g.
greater than, equivalent to). Accordingly, when analysing students' solutions, we
concentrate on identifying
and relationships that students perceive and work with.
Additionally, we
structures can only be identified through the observable signs the person produces and
works with. According to Arzarello, Paola, Robutti and Sabena (2009), these signs or
semiotic resources may involve spoken language, gestures, written text, symbols,
diagrams, graphs and physical artefacts. Consequently,
resources that students produce
students appear to be perceiving.
Student solutions to the Giotto MEA
In order to demonstrate the diversity of mathematical approaches that are
often elicited by the Giotto MEA, we describe some student solutions from one
classroom implementation. This class consisted of 29 year
from a New Zealand suburban
Prior to the implementation, t
the class. Students worked in groups of three
50-minute class period. A researcher collected the ten group
notes students made while wor
The ten group letters described ten different methods, involving more than
different mathematical elements, three

different constructions of a new shape to be used to measure the cir
circularity, and six different
mathematical thinking is too large to present fully in this short paper, so we highlight
some of the different mathematical elements, relationships and operation
from three groups in Table 2 below.
Penny and Carol: Construct a line segment connecting any two points A
and B on the perimeter of the circle attempt. Construct two
perpendicular line segments from points A and B to the where they
intersect the perimeter (C and D). Construct a line segment between
points C and D. Construct a perpendicular line from point C. Measure
the angle θ between line CD
process 2 or 3 times, and find the average angle. The circle attempt with
the smallest average angle wins.
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Mathematical and epistemological perspectives for analysing student
When analysing the mathematical approaches in students' solutions to
MEAs, we draw on theoretical perspectives that focus on the mathematical structures
that students attend to, such as modelling perspectives (Lesh & Doerr
Blum and Galbraith, 2007) and Mason's theory of the structure of attention (2004).
These perspectives characterise mathematical structure as consisting of
(such as counts, measures, sets), attributes of those
few, large, open), operations (e.g. combine, enlarge, invert), and relationships (e.g.
han, equivalent to). Accordingly, when analysing students' solutions, we
ing features the mathematical objects, attributes, operations
that students perceive and work with.
Additionally, we acknowledge that a person's perceived mathematical
structures can only be identified through the observable signs the person produces and
to Arzarello, Paola, Robutti and Sabena (2009), these signs or
may involve spoken language, gestures, written text, symbols,
diagrams, graphs and physical artefacts. Consequently, we analyse
that students produce in order to identify the mathematical structures the
students appear to be perceiving, constructing, recalling and manipulating.
Student solutions to the Giotto *MEA*
In order to demonstrate the diversity of mathematical approaches that are
the Giotto *MEA*, we describe some student solutions from one
classroom implementation. This class consisted of 29 year-11 (15 year old) students
uburban school with a cross section of socioeconomic groups
Prior to the implementation, the teacher had taught properties of circle geometry
Students worked in groups of three (or two) on the Giotto *MEA*
A researcher collected the ten group letters as well as
made while working on the activity.
The ten group letters described ten different methods, involving more than
hematical elements, three different ways of constructing a "centre", four
different constructions of a new shape to be used to measure the cir
different measures of the circle attempt. This diversity of
mathematical thinking is too large to present fully in this short paper, so we highlight
some of the different mathematical elements, relationships and operation
from three groups in Table 2 below.
Construct a line segment connecting any two points A
and B on the perimeter of the circle attempt. Construct two
perpendicular line segments from points A and B to the where they
intersect the perimeter (C and D). Construct a line segment between
and D. Construct a perpendicular line from point C. Measure
CD and the perpendicular line. Repeat this
process 2 or 3 times, and find the average angle. The circle attempt with
the smallest average angle wins.
Author-generated diagram
of Penny and Carol
solution
for analysing student
When analysing the mathematical approaches in students' solutions to
mathematical structures
that students attend to, such as modelling perspectives (Lesh & Doerr, 2003; Niss,
Blum and Galbraith, 2007) and Mason's theory of the structure of attention (2004).
These perspectives characterise mathematical structure as consisting of mathematical
attributes of those elements (e.g.
few, large, open), operations (e.g. combine, enlarge, invert), and relationships (e.g.
han, equivalent to). Accordingly, when analysing students' solutions, we
attributes, operations
acknowledge that a person's perceived mathematical
structures can only be identified through the observable signs the person produces and
to Arzarello, Paola, Robutti and Sabena (2009), these signs or
may involve spoken language, gestures, written text, symbols,
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constructing, recalling and manipulating.
In order to demonstrate the diversity of mathematical approaches that are
the Giotto *MEA*, we describe some student solutions from one
11 (15 year old) students
school with a cross section of socioeconomic groups.
had taught properties of circle geometry to
Giotto *MEA* during one
letters as well as written
The ten group letters described ten different methods, involving more than 60
ys of constructing a "centre", four
different constructions of a new shape to be used to measure the circle attempt's
measures of the circle attempt. This diversity of
mathematical thinking is too large to present fully in this short paper, so we highlight
some of the different mathematical elements, relationships and operations in solutions
generated diagram
Penny and Carol' Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI
Mike, Mark and Justin: Mark 6 points on the perimeter at 60
adjacent points with line segments. Measure the length of the last line
segment, and compare with the lengths of the remaining li
The circle attempt whose last line segment is closest to the other line
segments wins.
Nate, Ward and Jim: Construct an exscribed square around the circle
attempt and use the intersection of the square's diagonals as the centre.
From this centre, find the longest "radius" of the circle attempt and
this to construct a perfect circle from the centre previousl
Measure the area of the perfect circle that is not covered by the circle
attempt – this is called the margin of error. The circle attempt with the
smallest margin of error wins.
Table 2: Our summaries of
Penny and Carol's method focuses on relationships between interior angles
of an inscribed quadrilateral. They compare the size of the final constructed interior
angle of the inscribed quadrilateral to the ideal value of 90
also set about trying to construct an inscribed polygon (this time a hexagon), which
should have certain symmetrical properties in a perfect circle. Whereas Penny and
Carol focus on the symmetry of the interior angles, however, *Mike*, Mark and Justin
compare the length of the final constructed hexagon side to the lengths of the other
sides. Thus, the two groups' measures of circularity focus on different mathematical
elements (interior angles and lengths of sides).
Nate, Ward and Jim's measure of circularity
between the circle attempt and a perfect circle, which is constructed using the longest
"radius" in the circle attempt, and centred on the circle attempt's "centre". Their
measure does not, however, take into account diff
consequently favours small circles (as does *Mike*, Mark and Justin's). Other solutions
in the class attempt to mitigate this by using ratios rather than differences to compare
actual and ideal measures. Our Follow
mathematical diversity expressed in students' solutions.
Designing and testing Follow
We used a Design Experiment
& Schauble, 2003) to develop the
principles. This methodology aligned well to our dual focus on theory and practice
Design experiments have both a pragmatic bent
of learning – and a theoretical orientation
by systematically studying those forms of learning a
them (Cobb et al., 2003
We sought to "engineer" principles for designing Follow
be implemented immediately after a given *ME*
that students encountered in the *MEA*.
A core team of four designers with diverse expertise (teachers, researchers,
and teacher educators) met every 3 weeks over two years to design, test and revise
these tasks and their underlying principles. We tested tasks with intermediate,
secondary, and tertiary-level students, as well as with pre
primary and secondary school teachers. We
video and audio recordings, an
tasks and the underlying design principles. The results of these implementations led to
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Mark 6 points on the perimeter at 60°, and join
adjacent points with line segments. Measure the length of the last line
segment, and compare with the lengths of the remaining line segments.
The circle attempt whose last line segment is closest to the other line
Construct an exscribed square around the circle
attempt and use the intersection of the square's diagonals as the centre.
From this centre, find the longest "radius" of the circle attempt and use
construct a perfect circle from the centre previously found.
Measure the area of the perfect circle that is not covered by the circle
this is called the margin of error. The circle attempt with the
Table 2: Our summaries of three groups' solutions to the Giotto p
Penny and Carol's method focuses on relationships between interior angles
of an inscribed quadrilateral. They compare the size of the final constructed interior
angle of the inscribed quadrilateral to the ideal value of 90°. *Mike*, Mark and Justin
lso set about trying to construct an inscribed polygon (this time a hexagon), which
should have certain symmetrical properties in a perfect circle. Whereas Penny and
Carol focus on the symmetry of the interior angles, however, *Mike*, Mark and Justin
the length of the final constructed hexagon side to the lengths of the other
sides. Thus, the two groups' measures of circularity focus on different mathematical
elements (interior angles and lengths of sides).
Nate, Ward and Jim's measure of circularity focuses on the difference in area
between the circle attempt and a perfect circle, which is constructed using the longest
"radius" in the circle attempt, and centred on the circle attempt's "centre". Their
measure does not, however, take into account differing sizes of circle attempts, and
consequently favours small circles (as does *Mike*, Mark and Justin's). Other solutions
in the class attempt to mitigate this by using ratios rather than differences to compare
actual and ideal measures. Our Follow-up activities capitalise on this rich
mathematical diversity expressed in students' solutions.
Designing and testing Follow-up activities
Design Experiment methodology (Cobb, Confrey, diSessa, Lehrer,
& Schauble, 2003) to develop the Follow-up activities and their underlying design
principles. This methodology aligned well to our dual focus on theory and practice
Design experiments have both a pragmatic bent – "engineering" particular forms
and a theoretical orientation – developing domain-specific theories
by systematically studying those forms of learning and the means of supporting
(Cobb et al., 2003, p. 9).
We sought to "engineer" principles for designing Follow-up tasks that could
be implemented immediately after a given *MEA*, and that built on the mathematics
that students encountered in the *MEA*.
A core team of four designers with diverse expertise (teachers, researchers,
and teacher educators) met every 3 weeks over two years to design, test and revise
r underlying principles. We tested tasks with intermediate,
level students, as well as with pre-service and in
primary and secondary school teachers. We collected participants' written work and
ings, and analysed these data to assess the effectiveness of the
tasks and the underlying design principles. The results of these implementations led to
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groups' solutions to the Giotto problem
Penny and Carol's method focuses on relationships between interior angles
of an inscribed quadrilateral. They compare the size of the final constructed interior
Mike, Mark and Justin
lso set about trying to construct an inscribed polygon (this time a hexagon), which
should have certain symmetrical properties in a perfect circle. Whereas Penny and
Carol focus on the symmetry of the interior angles, however, *Mike*, Mark and Justin
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"engineering" particular forms
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A core team of four designers with diverse expertise (teachers, researchers,
and teacher educators) met every 3 weeks over two years to design, test and revise
r underlying principles. We tested tasks with intermediate,
service and in-service
participants' written work and
assess the effectiveness of the
tasks and the underlying design principles. The results of these implementations led to over 40 Follow-up tasks and at least seven significant revisions of the underlying
design principles (Yoon & Ra
3. Of these 40 tasks, about 25 were based on student
and we describe one such task in the next section. We regard the practice of using
student solutions a special desig
than a design principle for all Follow
The Focus principle
The Follow-up should focus on a particular aspect
of mathematical communication,
conceptual understanding that is
the initial MEA
The Challenge principle
The Follow-up should challenge students to
improve on areas of weakness or misconception
that are commonly found in students' responses to
the initial MEA
Table 3: Our principles for designing Follow
A follow-up task
solution
We created a Follow
by Sonya, Jun and Hayley (SJH), a group of students from the classroom
implementation that was described previously (for the full task, see website
http://icmiZZradonich-yoon
3) begins with paper-folding to find a "centre" of a given freehand (imperfect) circle
attempt. This "centre" is then used to obtain four "diameter" measurements of the
circle attempt. Next, SJH instruct r
circle whose diameter is the average of these four "diameter" measurements. A score
of circularity is obtained:
The (imperfect) circle attempt that yields the score closest to zero is declared
the winner of the circle-drawing competition.
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up tasks and at least seven significant revisions of the underlying
design principles (Yoon & Radonich, 2011), the latest of which are described in Table
3. Of these 40 tasks, about 25 were based on student-generated solutions to MEAs,
and we describe one such task in the next section. We regard the practice of using
student solutions a special design feature or strategy of some Followfor all Follow-ups.
should focus on a particular aspect
of mathematical communication, modelling, or
conceptual understanding that is encountered in
The Consolidation principle
The Follow-up should ask students to
consolidate what they learn in the form of a
sharable rule or guideline
should challenge students to
weakness or misconception
that are commonly found in students' responses to
The Flow principle
The Follow-up should flow on from the context
in the initial MEA so that students are likely to
relate their new knowledge to that which they
developed in the MEA
Table 3: Our principles for designing Follow-up activities to MEAs, from Yoon & Radonich
(2011)
task to the Giotto MEA based on Sonya, Jun and Hayley's
We created a Follow-up task that focuses on the solution to the
by Sonya, Jun and Hayley (SJH), a group of students from the classroom
implementation that was described previously (for the full task, see website
yoon-paper.wikispaces.com/home). SJH's method (see Figure
folding to find a "centre" of a given freehand (imperfect) circle
attempt. This "centre" is then used to obtain four "diameter" measurements of the
circle attempt. Next, SJH instruct readers to calculate the circumference of a (perfect)
circle whose diameter is the average of these four "diameter" measurements. A score
The (imperfect) circle attempt that yields the score closest to zero is declared
drawing competition.
up tasks and at least seven significant revisions of the underlying
donich, 2011), the latest of which are described in Table
generated solutions to MEAs,
and we describe one such task in the next section. We regard the practice of using
-up tasks, rather
should ask students to
consolidate what they learn in the form of a
should flow on from the context
in the initial MEA so that students are likely to
relate their new knowledge to that which they
up activities to MEAs, from Yoon & Radonich
MEA based on Sonya, Jun and Hayley's
up task that focuses on the solution to the Giotto MEA
by Sonya, Jun and Hayley (SJH), a group of students from the classroom
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folding to find a "centre" of a given freehand (imperfect) circle
attempt. This "centre" is then used to obtain four "diameter" measurements of the
eaders to calculate the circumference of a (perfect)
circle whose diameter is the average of these four "diameter" measurements. A score
The (imperfect) circle attempt that yields the score closest to zero is declared Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI
Figure 3: Excerpts from Sonya, Jun and Hayley's method.
Although most aspects of this method are mathematically correct, one
shortcoming is the final ranking, which assigns the attempt that scores closest to "0"
as the winner, when in fact it should be the one closest to "100". This error formed the
basis of the Follow-up task that we now describe.
After reading SJH's written letter, students are instructed to determine
(merely by looking) which of two hand drawn circle attempts appear more circular
(see Figure 4). These circle attempts were intentionally chose
is more circular than the other (attempt A). Then, students are asked to determine
which circle would be ranked as being more circular under SJH's method (see Figure
4).
Which circle attempt looks more circular?
A B
At this point the students are led to notice that SJH's scoring system ranks
circle attempt A as more circular, which contradicts their earlier, common
assessment that attempt B is more circular. Students
components of SJH's scoring formula to identify its mathematical flaws. As part of
this process, students are instructed to apply the scoring system to a perfect circle,
which should score 100. The final instruction in the task i
SJH's scoring system so that it works effectively, and to rewrite SJH's method.
Discussion and conclusion
The open-endedness of tasks like MEAs is both a blessing and a burden: such
tasks elicit a multiplicity of ideas from studen
these ideas further (Stein, Engle, Smith & Hughes, 2008). We have presented a design
approach that utilises this diversity by incorporating real student solutions to MEAs
into Follow-up tasks for the same MEAs. One
requires designers to analyse the thinking behind the students' solutions in great
detail. This detailed analysis can yield fresh insight into what the students understand
(or misunderstand), which can inform the desig
misconceptions and mistakes can be particularly rich material for designers as they
expose new mathematical avenues to pursue that the designers (who would not
typically make those same mistakes) might not have consi
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Figure 3: Excerpts from Sonya, Jun and Hayley's method.
Although most aspects of this method are mathematically correct, one
shortcoming is the final ranking, which assigns the attempt that scores closest to "0"
as the winner, when in fact it should be the one closest to "100". This error formed the
up task that we now describe.
After reading SJH's written letter, students are instructed to determine
(merely by looking) which of two hand drawn circle attempts appear more circular
. These circle attempts were intentionally chosen so that one attempt (B)
is more circular than the other (attempt A). Then, students are asked to determine
which circle would be ranked as being more circular under SJH's method (see Figure
Which circle attempt looks more circular? When Sonya, Jun and Hayley wrote:
They should have written:
"Find the circumference of a circle whose diameter is
the average length of the four diameters in the circle
attempt."
Apply this to the circle attempts A and
the formula: C = πd
Figure 4: Two questions in the Follow-up activity
At this point the students are led to notice that SJH's scoring system ranks
circle attempt A as more circular, which contradicts their earlier, common
assessment that attempt B is more circular. Students are invited to examine the
components of SJH's scoring formula to identify its mathematical flaws. As part of
this process, students are instructed to apply the scoring system to a perfect circle,
which should score 100. The final instruction in the task is for students to correct
SJH's scoring system so that it works effectively, and to rewrite SJH's method.
Discussion and conclusion
endedness of tasks like MEAs is both a blessing and a burden: such
tasks elicit a multiplicity of ideas from students, but teachers may struggle to develop
these ideas further (Stein, Engle, Smith & Hughes, 2008). We have presented a design
approach that utilises this diversity by incorporating real student solutions to MEAs
up tasks for the same MEAs. One advantage of this approach is that it
requires designers to analyse the thinking behind the students' solutions in great
This detailed analysis can yield fresh insight into what the students understand
(or misunderstand), which can inform the design of subsequent learning tasks. Student
misconceptions and mistakes can be particularly rich material for designers as they
expose new mathematical avenues to pursue that the designers (who would not
typically make those same mistakes) might not have considered themselves. For
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"Find the circumference of a circle whose diameter is
the average length of the four diameters in the circle
and B. You can use
At this point the students are led to notice that SJH's scoring system ranks
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misconceptions and mistakes can be particularly rich material for designers as they
expose new mathematical avenues to pursue that the designers (who would not
dered themselves. For Theme B – P. Radonich & C. Yoon
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example, we found that the mistake in SJH's scoring system provided an opportunity
to explore ways of comparing actual measures of circularity to ideal measures.
Another advantage of using students' written work in Follow-up tasks is that
it shines the spotlight on student knowledge. Some of the Follow-up tasks we have
designed are more typical of teacher-led approaches, which reinforce circle properties
such as cyclic quadrilaterals and angles at the centre and circumference. Such tasks
run the danger of emphasising the already prevalent perception that teachers are the
source of mathematically correct solutions (English & Doerr, 2004), and students
could interpret such tasks as subtle hints that these properties of circles should have
featured in their solutions to the Giotto MEA. The use of student solutions to design
Follow-up tasks can help avert the danger of students thinking the teacher-presented
ideas are necessarily more correct than their own.
When other students read Sonya, Jun and Hayley's letter with its original
handwriting and diagrams, they may make what is essentially a cultural connection.
The Follow-up task is situated in the context of Sonya, Jun and Hayley: three
mathematics students who have struggled with the Giotto MEA. Students working on

the Follow-up based on SJH's solution may relate to them, as they too are
mathematics students who have just worked in groups on the same problem. This
connection may lead students to be more engaged and curious about exploring the
mathematical concepts that other students described, which in turn, could help
students reflect more deeply on their own method (Boaler 1993).
As part of our research, we obtained permission from students to use their
solutions in developing future tasks, and we used pseudonyms to protect their
identities. However, a teacher may wish to use their own students' work as a resource
for creating Follow-up tasks, possibly even with the intention of using those tasks in
classes where the students' identities may be revealed to their peers. In such cases,
teachers need to be sensitive to students' fears about making their work public. Effort
will be required to create a culture where discussing student work is a natural and safe
part of the teaching and learning process. Our experience is that students are often less
concerned about sharing group solutions with the class, and are more inclined to share
individual solutions if names aren't revealed. Even if students give permission to
share their solutions, however, teachers may not have enough time to prepare Followup tasks for the next lesson. In this case, we advocate using Follow-up tasks based on
student solutions from previous years or from other schools or even countries.
This paper has emphasised the diversity of mathematical ideas that are
evident in student solutions to MEAs. However, in our research we have found that
student solutions also reveal a range of competencies in other related areas, such as
mathematical modelling and communication. We have successfully used the same
approach of using student solutions to create Follow-up tasks whereby students
evaluate the strengths and shortcomings of other students' mathematical
communication and modelling as a way of improving their own. A valuable feature of
our design approach is that it enables us to create tasks that extend and develop a wide
range of mathematical competencies in students.
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Tasks to Promote Holistic Flexible Reasoning about Simple
Additive Structures
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In this paper, we focus on an approach that promotes holistic
flexible reasoning with simple additive structures in arithmetic problem
solving in elementary school. We propose to distinguish two paradigms in
which additive problem solving tasks can be seen. We also propose an
ethno-mathematical model to analyze the tasks and their implementations.
In our experimentation, a new form of problem solving task was designed
and tested. The results of our experimentation show that the proposed
approach promotes the holistic additive reasoning in students.
Key words: elementary mathematics; problem solving; additive
structures; holistic reasoning
Introduction
The current math curriculum for elementary school in Quebec pays a special
attention to the development of students' problem solving skills. Some problems that
involve using one addition or one subtraction operation can be difficult for some
students until the age of 12-14 (Vergnaud, 2009). Many researchers (Carpenter,
Fennema, Franke, Linda, & Empson, 1999; Carpenter, Moser, & Bebout, 1988; De
Corte & Verschaffel, 1980; Gerofsky, 2004; Julo, 2002; Nesher, Greeno, & Riley,
1982; Riley, Greeno, & Heller, 1984; Vergnaud, 1982a) (to name few) studied the
subject from different perspectives and stated that students have difficulty in
acquiring a full and flexible understanding of addition and subtraction. Thus, it seems
relevant to look at how to improve students' learning by helping teachers to develop
new teaching practices.
Our team is conducting a 3-year research project funded by the Quebec
Ministry of Education on additive problem solving in early grades of elementary
school. The goals of the project are: 1) to develop a pedagogical approach that would
promote holistic and flexible reasoning about simple additive structures; 2) to design
and test a set of tasks and didactical scenarios that implements the new approach; 3)
to propose a related teacher professional development program. Our research team Theme B - A. Savard et al.
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consists of two researchers (Savard and Freiman), a designer (Polotskaia), and a
school board consultant responsible for the teachers' professional development
(Gervais).
Answering to the request of Theme B of the study group, we would like to
clarify the following question. How can the task of additive problem solving
transformed the way to minimise the gap between teacher intentions and students
activity? First, we will discuss the nature of additive problems as mathematical tasks
to better understand their possible educative purposes. Then, we will provide a
theoretical framework to analyze students' and teachers' behaviours regarding this
kind of tasks in school settings. It will help us to see why students' actual problemsolving activity can differ from teacher's intentions. A short overview of alternatives
found in the literature will provide us with specific pedagogical recommendations for
the new ways of designing tasks related to word problems having additive structure.
As support to our principles for such design we briefly discuss some preliminary
findings from the first year of experimentations with elementary school teachers.
Two paradigms in additive problem solving
Nowadays, there are two paradigms in which additive problem solving can
be seen. The first one, which we will call the Operational Paradigm, puts the focus
on addition and subtraction as arithmetic operations. From this position, additive word
problems can be seen as exercises where the knowledge about arithmetic operations
can be applied or further developed. It corresponds to an order of teaching
mathematics where students first learn about operations (how to add and subtract) and
then try to solve different word problems to practice this knowledge and to get the
conceptual understanding of these operations. This paradigm is clearly formulated by
Brissiaud (2010) who argues that, "to have a conceptual knowledge of subtraction"
means "to have different senses of the subtraction operation" like: finding the
difference, finding the complement and take over. However, knowing what it means
to add or to subtract two quantities, or how to add or subtract two numbers, is not
enough to solve additive word problems (Vergnaud, 1982b). As follows from the
research (Carpenter, Fennema, Franke, Linda, & Empson, 1999; Gerofsky, 2004;
Nesher, Greeno, & Riley, 1982; Riley, Greeno, & Heller, 1984; Vergnaud, 1982) , the
senses of subtraction-as-finding-the-difference or subtraction-as-finding-thecomplement, are not easy to construct directly from the first and quite intuitive sense -
subtraction-as-a-take over.
Several studies conducted in 1980-90s aimed to understand students'
difficulties with word problems by proposing classifications of word problems
according to their semantic and mathematical structures (Carpenter et al., 1999;
Nesher et al., 1982; Riley et al., 1984; Vergnaud, 1982a). Contemporary research
(Barrouillet & Camos, 2002; Nunes, Bryant, Evans, Bell, & Barros, 2011; Pape, 2003;
Thevenot, 2010) shows that some problems are particularly difficult because they
require a flexible and holistic analysis of their mathematical structure while easy
problems do not require such analysis. Yet in the Operational Paradigm, additive
structures are not seen as a primary mathematical knowledge, but only as different
senses of two arithmetic operations.
A different paradigm, which we will name the Relational Paradigm, appears
in the works by Davydov (1982) and more recent studies (Iannece, Mellone, &
Tortora, 2009; Xin, Wiles, & Lin, 2008). According to Davydov (1982), the concept
of additive relationship is, "the law of composition by which the relation between Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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two elements determines a unique third element as a function" (p. 229). Davydov
(1982) advanced that an adequate understanding of the additive relationship is the
basis for the learning of addition and subtraction and should be taught prior to
counting. According to this view, arithmetic operations are not the mean to
understand a situation but serve as tools to modify the situation, one understood, in a
desired way.
In the Relational Paradigm, a problem solving task should be an occasion to
analyse the additive relationships present in the situation. This yields the following
task design principles.
1.The task should be based on a situation involving a simple additive
relationship between three quantities.
2.The task should involve students in the mathematical analysis of the
described relationship as a whole. It should help students to discover
different properties of the relationship, and to see how different
arithmetic operations can be used in the described situation for different
purposes.
Existing formats of presenting tasks with additive word problem usually
limits student's activity to the goal of finding a solution as final step of problemsolving process, and thus does not directly imply any in-depth analysis of the
problem's structure which would help to develop more holistic view. Furthermore,
problem solving as a school task has some psychological and social characteristics
that, themselves and together with the classroom norms (Bosch, Chevallard, &
Gascón, 2005), can potentially contribute to either widening or reducing the gap
between pedagogical intentions and students' activity. The socio-cultural analysis is
therefore needed to explicit teacher-student behaviour patterns during the activity and
related didactic tensions.
Modeling teaching and learning situations for developing mathematical
and citizenship competencies
A well-designed curriculum should help students develop profound
mathematical knowledge. Furthermore, students should develop competencies
applicable in an outside classroom context. To respond to these fundamental
requirements, a good mathematical task should be anchored in a socio-cultural
context, that reflects the culture and the society including the student's experiences
inside and outside the classroom (Savard, 2008).
In order to analyze learning situations, and related tasks as a socio-cultural
phenomena, we used an ethno-mathematic model - the Math and Citizenship
competencies learning model - created by Mukhopadhyay et Greer (2001) and further
developed by Savard (2008). According to this model, implementing problemsolving tasks should start with an analysis of the given "real world" situation
involving its socio-cultural context. Then, the focus should move towards the
mathematical context. The model compels students to take into consideration the
relationships between the quantities involved. Mathematical operations are used to
find a result. Then, the result should be evaluated and interpreted within the initial
"real world" context (corresponds to the cycle described by (Novotná, 1998)) and
eventually in a larger socio-cultural and political context to develop students' critical
thinking (Lipman, 2003). Completing the learning cycle real world-mathematics-real
world can potentially promote meaningful learning in students. Does the traditionally Theme B - A. Savard et al.
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designed problem-solving task help the teacher to organize students' activity in this
cyclic way?
The explicit main goal of a problem solving tasks is finding the answer. As
described by Gerofsky (2004), very often the answer has no real value in the sociocultural context of the child. Therefore, students don't see problem solving as an
occasion for a meaningful discussion about real-world situations. They often see it as

an exercise of addition or subtraction operations. This students' perception of the task
prevents the learning cycle to be completed. The task is often finished at the moment
when the numerical answer is found and validated by the teacher. A special effort is
required from the teacher to organise the class work in a way where the learning cycle
can be completed.
Moving too quickly to finding an answer can provoke a specific didactic
contract (Brousseau, 1988) in students. For example, students might think that the
teacher's expectation is to find the numerical answer the easiest way possible. Under
the pressure of this contract, students can develop different strategies that help them
rapidly translate the text of the problem to a mathematical expression. According to
Hegarty et al. (1995), some students can recognize numerical data and key-words,
such as more, less, increase, take over, and construct a mathematical expression in a
straightforward way, translating these words to arithmetic operations. However, by
using this "direct translation" strategy (Hegarty et al., 1995), students move too
quickly to the arithmetic operation and do not pay attention to the underlying
mathematical concepts and relationships. This problem solving strategy is very
efficient for many problems but not for all of them. Yet, the success of using this
strategy can prevent students from acquiring a more profound understanding of the
mathematical structure of the problem.
The proposed above socio-cultural model (Savard, 2008) helps recognize the
difficulties teachers may have while implementing problem solving tasks and
obtaining the desired educational goal – a profound and flexible understanding of the
additive structures. Furthermore, based on this model, some more task design
principals can be deduced.
3.The task should use a socio-cultural context in which students can identify
themselves as active agents.
4.The task should not contain any explicit and immediate questions that
could be answered by finding one particular number. This criterion is to
prevent students from immediately calculating the answer. However, the
task should include an intriguing element, which would support students'
natural interest and commitment.
5.The goal of the task, which is learning to analyze the situation, should be
explicitly communicated to students.
The formulated task design principles do not correspond hundred percent to
the traditional word problem task organisation. What other organizations of such tasks
can we find in the literature?
Task organizations overview
As we mentioned earlier, in order to solve difficult or complex problems,
students should be able to see the mathematical structure of the problem in a flexible
holistic way. In the literature, we can find many approaches and methods potentially
supporting this ability development in students. Some of these studies (Bartolini
Bussi, Canalini, & Ferri, 2011; Ducharme & Polotskaia, 2008; Gamo, Sander, & Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Richard, 2009; Julo, 2002; Nguala, 2005) concern the use of different representations
for the problems. Gamo et al. (2009) demonstrate that the comparison of problems
and the use of different representations can help students develop efficient problemsolving strategies. Any graphical or schematic representation potentially gives
students a rapid visual access to the entire system of quantitative relationships
described in the problem. Therefore, using diagrams should promote the holistic
vision of the problem in students. Comparison of problems (Bartolini Bussi et al.,
2011) and multiple wording of the same problem (Julo, 2002; Nguala, 2005) are also
shown to be beneficial for elder students. However, these tasks can be linguistically
difficult for very young students (Ducharme & Polotskaia, 2008).
Some other studies (DeBlois, 2006; Neef, Nelles, Iwata, & Page, 2003)
propose particular didactic management and class work organisation. Neef and her
colleagues (2003) have shown that learning about the roles of each data element in the
problem greatly improves the success in problem solving among students with
developmental disabilities. DeBlois (2006) suggests that a request for feedback on the
solved problem may provoke coordination between representations and procedures
and may lead students to reorganize their thoughts. Both approaches clearly reflect the
effort to reorganise students' reasoning about the situation in a holistic flexible way.
All mentioned above teaching approaches, explicitly or implicitly, promote
in students the ability to see the problem as a whole and to better coordinate
relationships between quantities involved. To reinforce this important aspect of the
problem solving activity in young students the following principles should be
followed.
6.The text of the task should be very short and should contain simple words
and expressions that the students are familiar with.
7.The mathematical discussion of the situation should integrate appropriate
graphical representations as a method of analysis.
Keeping in mind all formulated above principles, what concrete task
organization do we propose?
Task description
We provide here one example of the task that we named 360° situation to
highlight the main goal – holistic analysis of the mathematical structure of the
situation. This is an example of a text proposed to students.
Peter, Gabriel and Daniel are playing marbles. Peter says, "I have 5 marbles."
Gabriel says, "I have 8 marbles." Daniel says, "Peter has 4 marbles less than
Gabriel".
We introduce this text as a strange situation or as a situation where one of the
persons made a mistake. Students are invited to explain why the text is unrealistic and
how it can be corrected considering different quantities involved.
The objective of the first is to make explicit the fact that all three quantities
are related to each other and that the choice of two values implies one (and only one)
third value. At the next step, we invite students to construct a graphical
representation, which can support discovering of the appropriate arithmetic
operations. Each quantity should be evaluated to figure out a correct numeric value in
the condition where the other two quantities are fixed. At this step, the formal use of
arithmetic operations can be discussed. Finally, the numbers in the text can be
replaced with different ones to further generalise the initially discussed quantitative
relations. This will complete the 360° tour around the situation. Theme B – A. Savard et al.
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Preliminary observation in the classroom
The first year of our project was implemented in rural public schools in
Quebec. We worked with two experimental groups of grade 2 elementary students (32
students, age 7-8 years). One other group of students of the same grade (14 students)
was observed as a control group. In the experimental group, teachers worked with
new and traditional tasks of additive problem solving. The teachers from the
experimental group received 6 follow-up sessions during the school year. In the
control group, only traditional problem solving tasks design was used and the teacher
did not participate in follow-up sessions.
Experiments with the 360° situation tasks started with manipulative activities
where we discussed with students some methods and sense of comparison of lengths
of two physical objects. We used coloured ropes and paper strips. The 360° tasks
were used then to organize the analyse-and-representation activities for students to
discuss different comparison situations. One of the central part of these activities was
the construction of an "Arrange-all" diagram (Ng & Lee, 2009; Polotskaia, 2010) – a
representation of the mathematical structure of the problem similar to the ropes we
have just manipulated with. These diagrams were used as an analytical tool to find
appropriate arithmetic operations for the given situation. In total, three 360° situation
tasks were worked with students prior to solving traditional problems. During our
sessions with teachers we recommended the use of similar discussions about all
additive problems till the end of the school year.
Although a detailed presentation and analysis of collected data lies beyond
the scope of this paper, we would like to share some observations made in the
experimental classrooms. Below is the partial script of the first lesson when the
described above task was proposed to students.
Teacher writes the story of Peter, Gabriel, and Daniel on the blackboard and asks
students to read it aloud. She explains that one of the three friends said something
wrong or made a mistake.
Teacher, showing the three affirmations on the blackboard: Who said something
wrong?
A: Gabriel
Teacher: You say Gabriel. Why?
A: There are 8. It's a lot. There are 5 and 4.
Teacher: Ok, Who agree with A that Gabriel is wrong? [Some students raised
their hands]
Teacher: Does anybody think differently? Is it another friend who made a
mistake? [Many students raised their hands]
Teacher: You B, what do you think?
B: I think it is Daniel
Teacher: You think it is Daniel who is wrong. Why?
B: Because… Because, let us take the 5 marbles of Peter. Daniel says that Peter
has 4 marbles less than Gabriel. Because, 4 plus 4 is 8, but 5 plus 4 will be 9.
…
Teacher: Ok. Now we will check each of your propositions.
The intention of the teacher for this part of the lesson was to engage students
in the discussion about additive relationship between three quantities. The semantic
meaning of the situation described in the task was in contradiction with the numerical
values indicated in the text. The numbers were small, so it was easy for students to
see this contradiction. We can see from the script that students, while giving their
arguments, tried to analyse the three numbers altogether. Therefore, the form of the
task helped minimize the gap between the teacher's intentions and the students'
activity. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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We should mention here that at the end of the year, students from the
experimental group have generally succeeded to demonstrate a progress in solving
problems in which the holistic and flexible mathematical analysis was really
necessary. At the same time, students from the control group have progressed more in
solving problems that could be solved without such analysis.
Regarding the implementation of the 360° situation tasks, we observed that
teachers have a tendency to return to the traditional teaching behaviours as soon as
they start to work with traditional problems. For example, once the numerical answer
was found for the problem, the discussion of the problem often ended abruptly. Thus,
the focus of the activity was often shifted towards the use of the correct (recently
discussed) representation or the calculation of the numerical answer.
Conclusion
As response to the call for contribution to the Theme B of the study, we
presented our theoretical perspectives on the new task design for problem solving. By
collaborating with teachers on the use of different types of tasks (we called them 360°
situation), we expected to minimize the gap between teachers' intentions and
children's mathematical activity. We are convinced that the Relational Paradigm can
be a powerful theoretical tool in didactic engineering and task designs. It helps to
construct tasks in which relational analysis is explicitly targeted. The tasks, having
this important property and implemented throughout the Math and Citizenship
competencies learning model, can play a major role in the reorientation of the
classroom work with word problems from operations-oriented toward relationsoriented. It can also help teachers and students to reconsider their immediate
teaching/learning targets in mathematical activities. While our preliminary data on
students' results and teachers' practices show some progresses, we agree with De
Corte (2012), that teachers need a closer follow-up in the new approaches for a
considerably long time in order to profoundly change their practice.
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Theme C: Design and use of text-based resources

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This theme focuses on the design of textbooks, downloadable materials, and
other forms of text-based communication designed to generate mathematical learning.
We recognise that most teachers use textbooks and/or online packages of materials as
their total or main source of tasks. Hence the design and use of tasks presented in
textbooks is central to many school students' experience of mathematics. The
scholarly study of task design should include consideration of theoretically-based
textbook development, and can take place at different grain-sizes from individual
tasks, through sequences of tasks, to a whole textbook series (Usiskin, 2003).
Some analyses of textbooks draw attention to differences in the use of
language, illustrations, cultural and social allusions and some focus more on the
mathematical and epistemological content (Askew, Hodgen, Hossain, & Bretscher,
2010; Haggarty & Pepin, 2001; Sutherland, 2002; Thompson, Senk, & Johnson, in
press). Significant differences have been found in the conceptual coherence,
mathematical challenge, consistency of images, and ordering of tasks between, for
example, UK and Singapore textbooks. For example, in some textbooks a new
concept is introduced through some everyday questions which are gradually refined to
focus on a formal presentation; in others, practice of a technique precedes application
through word problems (Ainley, 2010). The design of the order, development,
representation and presentation of content is therefore a suitable topic for this ICMI
study.
Another way to look at textual presentation is to analyse the content of
individual questions or sequences of questions, and variation theory has been used as
a tool both for design and analysis at this fine-grained level (Watson & Mason, 2006).
For example, control of variation among examples can be used to direct learners
towards inductive generalisations about concepts; example sequencing with
controlled variation can lead learners towards some cognitive conflict. Textual
presentation could be informed by research about how features of page and screen
layout affect learners' attention (Ainsworth, 2009; Poole & Ball, 2006).
A third way to look at textbook tasks is to view them as the shapers of the
curriculum rather than merely presenting a given curriculum (Senk & Thompson,
2003). The underlying commitments about the nature of mathematics, mathematical
activity, and how mathematics is learnt, vary between textbook series and between
countries. How these are promoted in the design and content of the tasks in the
textbook is an important area of study because a textbook series might have more
influence on learners and learning than a national curriculum. Different designers may
interpret national standards or recommendations in different ways so that
understanding the principles on which they instantiate these recommendations is an Theme C – Introduction
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important area of study (Hirsch, 2007). Various components of mathematics will be
prioritised or marginalised differently through different kinds of tasks and there will
be legitimate debate about how students come into contact with mathematical
absolutes (if there are any) (e.g. Harel & Wilson, 2011).
Authors' intentions can be different from how tasks or sequences of tasks are
used in classrooms, and in this theme we could also look at pedagogic suggestions,
particularly for innovative or unusual tasks, and information about conceptual
intentions (Thompson & Senk, 2010). Many textbooks now refer users to online
resources and tasks, and there is a professional development element to their use.
There may be a difference between the adventurousness of students and the
conservatism of teachers in their use and vice versa. (See chapters in Reys, Reys, &
Rubenstein (2010) for issues related to curriculum and tasks in terms of intentions and
enactments.)
Throughout the following set of questions, we consider a textbook and/or
online resource to be a collection of tasks, generally sequenced in a given way, and
often surrounded by related narrative and/or questions:
• How do curriculum expectations influence authors' design principles?
• How does an intention to promote change influence design?
• How do designers' expectations of teacher knowledge inform the design of
dual purpose tasks: to teach students and to facilitate teacher learning?
• How can authors and teachers learn from alignments and misalignments of
teachers' adaptations and authors' intent, and the implications for
students' learning?
• How can or should new digital formats influence textbook design: e.g. use
of podcasts, twitter, and other social media; implications for design and
coherence of materials (either original digital design or transfer from
print) if teachers are able to select tasks in varied orders?
• How do cultural considerations about instruction and pedagogy influence
design: for example, whether teachers are seen as "facilitators" or
"givers" of knowledge?
• How can designers take account of the language of instruction not being
students' home language?
• What research about design of textbooks and other materials should be
undertaken to inform the next generation of designers? In particular, how
might design experiments (e.g., Clements (2007) or teaching
experiments (such as Japanese lesson study)) influence task design in
curriculum materials?
• How can design principles from software design, advertising, graphical art
and eye-gaze research be used to improve text-based materials?

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Designing Tasks for Engaging Students in Active Knowledge Organization

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Mathematical tasks aim at supporting students to engage in a
range of mathematical activities with specific didactical goals. Task
design has to take into account the specificity of these different didactical
goals (e.g., exploration, concept formation, practising skills). In this study,
we focus on tasks intended for the didactical aim of mathematical
knowledge organization ("organizing tasks"). In our learning pathways,
phases of knowledge organization usually follow a phase of open
exploration, of constructing individual concepts; they aim at regularizing
and systematizing the students' singular ideas and results. Because such
organizing processes conducted in whole-classroom discussion often fall
short of engaging every single student, our design research study sets out
to develop task formats that promote adequate cognitive activities and
formats for this organizing phase. The article describes the efforts in
constructing and evaluating organizing tasks and presents – as a result of

our study – a conceptual framework for the delicate balance between individual engagement and convergence.
Keywords: organizing tasks, organizing knowledge, regularizing and systematizing
The challenge of organizing knowledge
The construction of mathematical knowledge is a multistep process of "organizing fields of experience" as Freudenthal pointed out (1973, p. 123). In subsequent work of the realistic math education approach (de Lange, 1996), examples of organizing processes are abundant, comprising "horizontal" and "vertical" mathematization (Treffers, 1987, p. 247). Although these ideas lie at the heart of most contemporary approaches for mathematics education, it is undeniable that in mathematics classrooms such organizing processes can rarely be found explicitly (Hiebert et al., 2003). However, there is a need for development of activities which

promote systematizing, regularizing, and preserving the results of exploration with several goals:
• structuring the singular and divergent results and connecting them to other facets of knowledge (systematizing),
• transforming results into regular and consolidated mathematics (Brousseau, 1997, calls this phase the "institutionalization", we call it regularizing);
• writing down in a form that is accessible later (preserving).
When and how do such procedures of organizing knowledge occur? When observing German mathematics classrooms, we often find phases of discovery and individual problem solving. For the subsequent phases of regularizing, we actually encounter two different types of classroom procedures.
In the first type of procedure, the teacher conducts a whole-classroom discussion in a Socratic dialogue, collects and evaluates students' contributions, and leads the class to an organized and structured knowledge. This procedure requires whole classroom conversation techniques and proves precarious with respect to the cognitive activation of every single student. In the second type of procedure, the teacher may avoid whole-classroom discussion and instead refer students to the information boxes in the textbook where the correct mathematical concept or result is stated. This ensures a common basis, but there is a danger that the individually constructed knowledge cannot be integrated in this "ready-made-mathematics".
What are alternative options for supporting students in organizing their knowledge, in mastering the step from the singular and individual knowledge to the regular and commonly accepted mathematical knowledge while preserving students' engagement? In our design research study, we developed approaches and tasks on the basis of three premises: (1) Teachers predominantly work with textbooks, so the organizing tasks should be embedded into a comprehensive textbook curriculum; (2) Students must be actively involved in the learning processes; (3) Teachers are not supposed to give up their role in moderating the process of organizing with the whole class. We consider the communication processes within the class as extremely important for attaining a high level of mathematical insight.
Hence, our goals are to construct tasks that support students in actively organizing their knowledge and simultaneously support teachers in guiding this process in an effective manner. We call this task type "organizing task" or, when we need to avoid the misunderstanding of organizing as a purely external, administrative activity, as "knowledge organizing tasks". The guiding questions for our study are:
Q1. Specification of learning goals:
What elements of knowledge have to be organized and preserved?
Q2. Types of Task:s
Which types of tasks can support students' active knowledge organization?
Q3. Principles for the Task Design:
Which principles guide the construction of organizing tasks?
The framework: design research for a middle school curriculum
The study on organizing tasks is embedded in the long term design research project KOSIMA (2006-2016, cf. Hußmann et al., 2011). It is briefly presented here with its methodological framework and the conceptual framework for the design products. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.

Methodological framework of the long-term design research project
The project KOSIMA (Hußmann et al., 2011) follows the methodology of Didactical Design Research (Gravemeijer & Cobb, 2006; McKenney & Reeves, 2012) with its dual aim of designing teaching-learning-arrangements for a complete middle school curriculum (grades 5 to 10 of German Realschule, Gesamtschule, Sekundarschule) and empirically researching the teaching-learning-processes and their conditions. The developed curriculum is published as the textbook Mathewerkstatt from 2012 to 2017 (Barzel et al., 2012 ff.) and a comprehensive teachers' manual.
All teaching-learning-arrangements of the textbook-to-be are developed in iterative cycles of design, evaluation (by expert discussions and classroom experiments), and redesign. Whereas the design and evaluation steps of the project refer to the entire implementation of the textbook, the deeper research is organized in several smaller design research studies that necessarily have to address more narrow research questions. These studies use different concrete research methods and designs (e.g., intervention studies in quasi-experimental designs, design experiments in laboratory settings with up to four cycles, e.g., Leuders & Philipp, 2012; Prediger & Schnell, 2013). An overall evaluation of summative effectiveness commenced in August, 2012. Results of the quasi-experimental intervention with pre-post-test over two years can be expected in 2014.
The community involved in these processes is a large group of people from different backgrounds who collaborate fruitfully:
• researchers (the four authors of this paper, being the editors of the textbook and leaders of the design research project, supported by many PhD students and student researchers)
• authors of the teaching-learning-arrangements (about 20 experienced reflective practitioners, together with the editors)
• the publisher (with 2-4 copy editors who finalize the design products)
• project teachers (about 10 teachers with their classes, who teach with the curriculum and the textbook material continually in their regular classes).
Conceptual framework for the design product
The design of the middle school curriculum is guided by certain design principles. We only state those which are relevant for the focus of this paper, that is for designing organizing tasks (cf. Hußmann et al., 2011; Prediger et al., 2011).
Following socio-constructivist theories of learning, we emphasize the importance of students' active engagement and sense-making by starting from meaningful context problems, and developing conceptual understanding (Leuders et al., 2012, following Wagenschein, 1977 and Freudenthal, 1973). For realizing these principles, every teaching-learning-arrangement (each for 2-6 sessions) is structured into four main phases: activation, exploration, organization of knowledge, and practice.
Activation of students' experiences. For including students' pre-instructional experiences into the learning pathways, every arrangement is situated in an everyday context that allows problems for ready-made mathematics to be reinvented and students to construct meanings for the intended mathematical topics.
The following is an example referred to throughout this article. "Constructing Packages" provides a context for students to think about solid figures and their

characteristics, such as parallel and perpendicular lines. To solve problems related to this topic, students put themselves in the role of a package designer who has to create new packaging for a toy. In considering a good design for a package, a designer has to think about what criteria are relevant to create a good package-box. Apart from price, other possible criteria are look, stackability, and ease of construction. Stackability specifically leads to the necessity of having packages with parallel and perpendicular lines.
Exploration. In this extensive phase, the problems are operationalized into open and rich exploration tasks (Fiewelling & William, 2001; Freudenthal, 1973). They allow students to actively and collaboratively re-invent ideas, concepts, procedures and relations in the sense of horizontal mathematization. Due to the openness and student-centricity of this phase, it often results in a large diversity of individual ideas, strategies, solutions, findings, pre-concepts, etc. The concrete design principles for exploration tasks that were developed or refined during the design research project are not reported here. Looking at our example in this phase: Students assume the role of the package-designer and they actually construct a package-box for a specific item, e.g. a toy. During this process of construction, many students realize that the "box must be straight and precise, not awry or askew". These experiences prepare the systemization of the concept of perpendicular and parallel in the next phase.
Organization of knowledge. The goal of the subsequent phase of organization (in German "ordnen" which also means "ordering", "arranging") is to establish a shared understanding of the core concepts, theorems and procedures and to preserve and document this understanding in the self-written "knowledge-storage" (in German "Wissensspeicher").
The packaging example highlights the kinds of activities required in this phase. Exploration often leads to a great diversity – in our example a lot of different boxes, ideas, images, and students' thoughts. All of these products and ideas have to be shared and compared in order to systematize the new knowledge. To stimulate students´ mental processes, we created focused cognitive activities (called acquisition activities) according to the new knowledge.
The type of acquisition activity depends on the types of knowledge. For the concept of perpendicular and parallel, one can differentiate between the following aspects, which have to be learned:
• Students must learn the technical terms "perpendicular" and "parallel". These words are names and just conventions, which cannot be discovered or explored. Students have to be informed about these new words. This has to be done in a language, which can be easily understood by students and with words which link to the experience of the exploration phase (see the first lines of the task in Fig 2).
• Another convention in our example is the marking of a right angle with a dot (a convention that differs throughout several countries). This is information that students must learn (see picture on the left side in Fig. 2).
• Students have to learn to recognize perpendicular and parallel lines. Task 2b (in Fig. 2) involves concretisation and distinction of the new concept. Pupils have to recognize which pairs of lines show parallel or perpendicular ones.
• The extent to which individual students can verbalise the new concept varies and can be described by the extrema: On the one side students Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.

formulate a definition by themselves, a very ambitious task; on the other side students have to copy a given definition (see Fig 3). Choosing a correct definition from several given ones is an activity in between these extrema, and this is demonstrated in our example. (See (2b) in Fig 2.)
When inspecting regular lessons and textbooks, we have only rarely encountered tasks that were constructed for supporting such organizing processes (e.g., Swan, 2005). That is why we had to conduct a design research study for specifying principles for a systematic construction and composition of organizing tasks. The results are reported in the section on Findings.
Practice. In the fourth phase of practice (in German "vertiefen" = deepen, intensify), the students are supposed to render their knowledge and skills more stable and flexible by repeated practice and transfer. Our design principles for these tasks refer to didactical approaches of productive exercises, structured tasks and reflective practising (Büchter & Leuders, 2005; Watson & Mason, 1998; Winter, 1984; Wittmann & Müller, 1990).
It is important to note that the construction of "organizing tasks" as a "carrier of the organizing" phase has only become indispensable within our didactical approach that distinguishes these phases explicitly to give students space for more individual mathematical activities. Within a more integrated approach, concept exploring and organizing activities could be combined or integrated more flexibly.
Methods of the design research study on organizing tasks
The construction of "organizing tasks" and the development of an overarching didactical approach for the principled construction of the tasks are embedded in the larger design research project depicted previously. For the concrete study, the cycles of constructing and evaluating organizing tasks were passed through topic by topic, each in four cycles (mostly including further microcycles).
For each topic, the first and second cycle of design, evaluation, and redesign is conducted by expert discussions. After specifying the goals of the learning arrangements, the writing team (authors and one editor) suggests a first draft for the formulation of target knowledge that students are supposed to save in their "knowledge storage". After discussing the selection and priorities by the editor team with respect to didactical-conceptual considerations, the (exploration and) organizing tasks are formulated, discussed and further developed. The details of formulation are edited by experienced copy editors of the publisher who optimize readability and coherence. The third cycle of evaluation is conducted in classroom experiments with 3-10 teachers in their regular classrooms. The data base for the investigation consists of teachers' written and oral feedback, scans of students' written texts for tasks, knowledge storages and classroom assessments as well as some videos of classroom interaction and design experiments in laboratory settings on selected tasks. For the current study, the qualitative data analysis is conducted with respect to connections between forms of tasks and a) students' engagement within the processes, b) the convergence of the processes and c) the results of organized knowledge as stored and especially as performed in the assessments. In the fourth cycle of redesign, with theoretical feedback authors, editors and copy editors again are involved in finalizing versions for the textbook and the teachers' manual. The final version is ready for widespread implementation and effective everyday use.
During these four-step processes for many different mathematical topics, we iteratively accumulated the reflections and experiences, generalized from the specific

topics, and developed a conceptual framework for organizing task design. Whereas the answers for research question Q1 on the relevant kinds of knowledge were mostly generated in the first and second steps of theoretical and conceptual evaluation,

answers for Q2 and Q3 on effects of forms of tasks mostly rely on the empirical
investigation, involving deeper insights into the mechanisms of the teaching-learning
processes as well as practical experiences on robustness for different classroom
conditions. Out of the whole conceptual framework for designing organizing task, we
present two major aspects in the next section.
Findings
Specification of learning goals: modes and facets of knowledge
The didactical base of a systematic task design is the exact specification of
the intended learning goal, here concretely the mode of knowledge that is supposed to
be systematized, regularized, and preserved. For concretizing this specification
process for each mathematical topic, we developed a conceptual framework of
different knowledge elements as printed in Figure 1 (Prediger et al., 2011).
The horizontal dimension follows the classical distinction (Hiebert &
Carpenter, 1992) among knowledge about facts, concepts (e.g., numbers, operations,
relations), and connections as codified in theorems (conceptual knowledge) on the one
hand, and the knowledge about mathematical and technical procedures (procedural
knowledge) on the other hand. We added metacognitive knowledge which includes
problem solving strategies or steps in modelling processes.
Explicit
Verbalisation
Concretisation
& Distinction
Meaning &
Connections
Conventions
Concepts definitions examples/
counterexamples
mental models /
representations
technical terms
Connections theorems examples/
counterexamples
(visualized)
explanation/
proof
names of
theorems,
conventional
rules
Mathematical
procedures,
algorithms
instructions conditions for
applicability,
special cases,
knowledge about
errors
mental models /
reasoning as link
to conceptual
meaning
non-justifiable
specifications
Technical
procedures
instructions realization,
conditions
non-justifiable
determinations
Conceptual Knowledge
Procedural Knowledge
What? (Modes of Knowledge)
Metacognitive Knowledge
Which part? (Facet of Knowledge)
Fig. 1. Conceptual framework for specifying learning goals (Prediger et al., 2011)
During the design research process, we realized the importance of specifying
a second (here vertical) dimension that we call facets of knowledge: Whereas a piece
of knowledge is often only represented by its explicit verbalisations (in definitions,
theorems or instructions for procedures) or underlying conventions (like the technical
terms), didactical research has often shown that knowledge acquisition must also
comprise concretization and distinction (like examples and counterexamples for
concepts, cf. Winter, 1984 and Fig. 2 Task 2a) or knowledge of possible errors in
procedures (cf. Vollrath, 2010)) and meanings and connections to other elements of Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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knowledge, given by visual representations and mental models, explanations and preformal proofs.
As mathematical understanding is conceptualized as individual construction
of relations, dependencies, or connections between mathematical ideas, procedures
and concepts (Hiebert & Carpenter, 1992), we promote that the organizing tasks and
the knowledge storage must always include various facets of knowledge. It is the first
step of the construction process of an organizing task to specify which cells of Fig. 1
shall be addressed in the task. These modes and facets of knowledge are then subject
for initiating the processes of systematizing (= connecting different facets of
knowledge systematically), regularizing (= transforming individual constructions
from the exploration phase into regular mathematical concepts, connections and
procedures), and preserving (= documenting facets in the knowledge storage so that
they can be recalled some months later).
Types of tasks: initiating acquisition activities in a balance between students'
engagement and convergence
As made explicit previously, students must be actively involved in the
processes of systematizing, regularizing, and preserving knowledge. The necessity of
active engagement is explainable within the socio-constructivist framework and
reconstructable in the empirical investigations, because simple inputs (e.g., teachers
dictating the information) were not suitable for activating students' mental processes.
That is why for each piece of knowledge (cells in Fig. 1 selected for a
specific topic) that is to be systematized and preserved, a focused cognitive activity
(acquisition activity) must be initiated that supports students' active acquisition of this
piece of knowledge.
Fig. 2 shows some formats for how such activities could be initiated. First the
technical terms (here parallel and perpendicular) are given - in the frame and context
of the experiences and discoveries which have been done before (here the context of
exactly folding straight boxes). In 2a), examples and counterexamples are to be
identified. In 2b), students shall choose between possible definitions. Independent
personal definitions would produce again very divergent solutions, but finding a
correct and fitting one among some examples allows an active engagement with
higher convergence. Theme C – B. Barzel et al.
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Fig. 2. Examples and counterexamples in 2a), Finding a correct definition in 2b) (Barzel et
al., 2012)
The classroom experiments showed clearly that the balance between
students' engagement and convergence of the process is delicate. If the openness of
the activity is high, students can intensively engage with the content; like in the
exploration tasks, they develop very divergent ideas and entries for the knowledge
storage. These divergences produce either the need for the teacher to moderate the
processes in funnel-like patterns (Bauersfeld, 1988), or to give individual feedback to
each individual attempt to write a knowledge storage, which is too much work for
each task. If tasks are optimized only with respect to convergence, it might risk
students not being engaged enough. The adequate balance between convergence and
engagement depends on the concrete topic and the concrete piece of knowledge.
The result of the generalization process was the specification of a range of
acquisition activities for each piece of knowledge, as illustrated for two exemplary
pieces of knowledge in Fig. 3. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Fig 3. Range of acquisition activities in the balance between convergence and engagement
Discussion
Due to the fact that our design research in the KOSIMA project results in a
new textbook, the design part of the work itself has important impact on several
communities. First of all, it affects the students, who get the chance to develop and
reinvent mathematical concepts by relating relevant contexts with individual,
sustainable conceptions. The developed learning arrangement offers structured tasks
so that the teacher can moderate the learning processes and can support students in
organizing their knowledge. It has to be mentioned that the Mathewerkstatt is one of
only a few textbooks in Germany that are put to the test and fully revised and
reviewed by sample classes for guaranteeing usability by teachers.
The research that was conducted in the iterative interplay with design,
evaluation, analysis and revision of the learning arrangements showed the potential of
the structure of activation – exploration – organization – practice. The insights gained
into the deeper structures of the initiated learning processes allow us to contribute also
to didactical theory. A further evaluation on the generated learning progress was
started in August 2012 for a two year study.
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Developing Young Students' Geometric Insight Based on
Multiple Informal Classifications as a Central Principle in the Task
Design
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The geometrical (and more generally, mathematical) insight of
students, as defined in our previous research, is characterized by four main
parameters. Such insight should be, in our opinion, the focus of educators at
any level of mathematics. This implies the need for teaching/learning
materials designed to serve this purpose. In this paper we illustrate how task
design principles aimed at students' mathematical insight are implemented
in a series of geometrical textbooks for grades 2-6 of primary school.
Keywords: mathematical insight, teaching solids at primary school,
teaching geometry at primary school, informal classifications
Introduction: our model of mathematical insight as a conceptual basis for
the task design

In our past and present research, we follow previous studies (see e. g., Barabash & Guberman, 2008; Griffiths, 1971; Sternberg & Davidson, 1999) in the search for characteristics and ways of development of mathematical insight as a feature reflecting the depth of a person's mathematical thinking and understanding. In doing so, we infer that these characteristics should not be dependent upon the person's formal mathematical knowledge (FMK); we found in our research support for this inference.
The concept of mathematical insight is closely related to mathematics understanding and comprehension as a result of learning. Our concept of insight is closely associated with learning theories dealing with procedural vs. conceptual knowledge and includes features of both types in accordance with recent studies indicating that they actually should be intertwined in a fruitful process of learning mathematics (see e.g., Schneider & Stern, 2005).
This may be a good point to indicate our personal professional profiles, since they have a direct implication on our tasks and on their further implementations with various populations. One of us has a PhD in mathematics, and the other one has a PhD in mathematics education. Both of us are teacher educators with more than 20-years of experience. One of us has been a primary school teacher; we both are active in inservice educational programs for mathematics teachers, and have written a number of textbooks and learning materials for primary and secondary school, for mathematicsteachers-to-be, and for teachers educators. Theme C – M. Barabash & R. Guberman
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According to our research (Barabash & Guberman, 2008), the characteristics of mathematical insight (MI) are:
Implementation: the ability to implement, to employ the material being currrently learned in its close "neighborhood", and the measure of closeness of this implementation to the form in which it has been taught in the class ("the farther the better"). This indicates the student's ability to grasp the idea beyond its immediate presentation in a lesson or a handbook.
Skills: the variety of mathematical skills needed or demonstrated by a student in relation to the current issue, and the level of mastering these skills.
Extension / generalization: the ability to extend the acquired knowledge and/or to generalize it, or to incorporate the issue beyond the obvious mathematical context in which it has been originally studied.
Mathematical language: usage of newly acquired terminology; reasoning using relevant mathematical argumentation (appropriate to the formal mathematical (FM) development of the student); usage of formal-mathematical language and ability to appropriately relate it to verbal-non-mathematical expressions and vice versa, etc.
Principal considerations in the basis of our task design
Mathematics learning is known to develop in a spiral-like process, in which each coil is based upon previous ones and unfold out of them. The teaching and learning tasks should support this process in their didactical design and mathematical vision. Tasks can vary not only with respect to mathematics content but also with respect to the cognitive processes involved in working on them (Shimizu, Kaur, Huang, & Clarke, 2010).
In view of these and other similar reference points and in consistency with our own professional and academic experience, we define a task as a series of questions and assignments united by a big mathematical idea. We regard textbooks for primary school as a self-consistent series of such tasks on topics outlined by the curriculum, each task comprising a structured well-planned succession of problems, situations, explanations and exercises aimed at the on-going spiral-like learning of mathematics. By "learning of mathematics", we mean two intertwined processes: the development of students' FM knowledge (i.e., the corpus of notions, facts, techniques, ways of reasoning); and understanding of this corpus, which we characterize as mathematical insight.
Accordingly, we claim that the teaching procedures and materials should be aimed at the current state of the students' insight and designed so that they permanently advance it, regarding each of its characteristics as an ongoing objective. It is in view of this intention that we refer to the 5-year perspective of teaching solids in primary school as it is worked out in our series of textbooks for 2nd through 6th grade (excluding 3rd grade).
In our design of learning tasks, we keep in mind mathematical, epistemological and curriculum-related (MEC), cognitive-developmental (CD), and pedagogical and didactical (PD) objectives of the curriculum issue for which the tasks are being designed, all of them having direct implications on the insight development (Fig.1). We refer here briefly to each of these principles in relation to our textbooks. The abridged notation will serve for further reference. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Fig.1. Schematic representation of task design principles aimed at the development of students' mathematical insight at any level of FMK
MEC: When we plan a teaching task, for whichever level of FM learning it is aimed, we first consider it from the perspective of its mathematical content and meaning. This is done in view of curriculum indications, on the one hand, and of our mathematical knowledge and of our pedagogical content knowledge (PCK), on the other hand. Obviously, the mathematical and epistemological considerations concerning the primary school geometry textbooks are curriculum-related. They are:
MEC1: Mathematical correctness. We decline any possibility to present a mathematical notion in a way, however seemingly clear, which contradicts the mathematics of the issue, and hence, will render the notion to be sooner or later learned anew in a different way not consistent with the present one. Mathematical correctness is not a synonym to mathematical precision, rigor, and formality. Any mathematical notion, idea or fact intended for the primary school can be learned in a mathematically correct way at an informal, pre-formal or more-or-less formal mathematical level.
MEC2: The mathematical perspective of past and future development of the issue at school and (possibly) in future higher-education studies (longitudinal perspective), in compliance with the "big mathematical idea" being studied.
MEC3: The links of the new topic to other mathematical or outermathematical topics being currently learned (cross-curricular perspective).
MEC4: Intuitive-inductive acquaintance with figures and solids and their properties, with implicit or explicit suggestions for generalization. Where possible and appropriate, we do our best to reveal to the teachers (in the teachers' guide) and/or to the students (in the textbook itself) the motivations for definitions, ways of reasoning, ways of computing, etc. These motivations are conceived so as to pave the future way to more formal mathematical approaches (in compliance with *MEC2*).
CD: An ability to discern various characteristics of mathematical objects, or to find similarities and differences between them is the basis of the classification ability, which is one of the central thinking abilities. Piaget regarded cognitive developmental ability to be one of the most important cognitive abilities. Following Piaget (1962), class inclusion is an understanding of the fact that a set of objects is simultaneously a Theme C – M. Barabash & R. Guberman
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subset of another, bigger set. The development of this ability starts at the preoperational stage (ages 2-7). At this stage a child may classify objects by one property. At the concrete operational stage (ages 7-11), the child is able to perform classification by a number of properties, and is also able to rank the objects by one of the properties, for example, by size. At this stage, one of the aims may be to develop the Decentration ability (Piaget, 1962), i.e. an ability to shift between the classifications. The classifications may be formal-mathematical or not.
PD: The results and analysis of TIMSS (see e.g., Valverde et al., 2002) indicate that school mathematical textbooks are the main basis for the lesson planning by teachers and have a powerful influence on what is learned and how it is learned (Ball & Cohen, 1996; Yang, Reys, & Wu, 2010). Textbooks more and more tend to form the actual classroom curriculum and guide teaching practices.
According to the National Curriculum, the Israeli students start getting familiar with three-dimensional geometric solids during second grade. The Curriculum specifies the topics to be learned, but suggests no leading principle as to how this is to be done. Hence, textbook authors are more or less free to implement their own didactic principles.
We discern two major possible approaches to teaching solids:
• Allocating a lesson to each solid, e.g. a lesson to the parallelepiped, another lesson to the cube. The comparison between them may be the topic of the concluding lesson.
• Handling together all the solids included in the curriculum, studying their common properties, and the properties special for some of them. This approach leads to classifications by various geometrical properties.
Our decision was to choose the second approach, i.e. non-formal classifications as the basis for our teaching solids. To make things clearer, we outline here the issue of classifications from the mathematical point of view prior to presenting the examples.
Formal classification of mathematical objects is based on rigorous definitions which reflect the hierarchy of mathematical objects by pure inclusion in a settheoretical sense. The primary school handbooks are not based on rigorous definitions; at this stage of learning formal definitions are, generally speaking, implausible.
Unlike formal classifications based on rigorous definitions, informal classifications are based on properties not necessarily defining purely inclusive sets, because they do not necessarily either imply or mutually exclude each other. Though informal classifications may not lead immediately to rigorous mathematical hierarchy, we find them important from the didactical point of view, at least at early stages of mathematics learning (PD), because they enable the concept images to be gradually (*MEC2*, *MEC4*) and mathematically consistently (*MEC1*) constructed starting from the initial stages of geometry learning.
The following examples from our textbooks clarify how the task design principles serve to enhance the geometrical insight of students. We intentionally chose an example from the second grade and an example from the sixth grade in order to demonstrate the self-consistency of our approach throughout primary school geometry teaching (*MEC2*).
Example 1: 2
nd
grade
The kit: This is the students' first encounter in their school studies with six solids included in the curriculum for the 2
nd
Grade: the cone, the cylinder, the ball, the square pyramid, the parallelepiped, the cube. A kit of these solids is attached to every Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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student's copy of the textbook. For the sake of the first encounter, all the revolution surfaces are of the same radius; the square bases of the polyhedra are congruent. The heights of all the solids are equal.
The story: A group of intruders (who are actually the six solids) has broken into the math classroom and left it in a great mess. Fortunately, the intruders have left traces: stains on walls and floor, signs of different forms, etc. The famous Mathy the Detective uses these signs to discover who the intruders were. To succeed, he also watches them at night because they keep breaking into the classroom night after night, so he sees their shadows in the night. This "story" is a rich basis for the variety of classifications by a number of features, among which are the following:
The sign left by a solid on a plane. By a sign we mean the result of a momentary contact of a solid with the plane, as if the solid were a stamp. Having experimented with the solid, the students discover that all of them can leave a sign in the form of a point, whereas the segment-like sign may be left by some of them (e.g., a cone) but not by others (e.g., a ball). Actually, the point is the only sign a ball may leave.
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Fig.2a. "Signs" as a result of a contact Fig.2b. "Stains" as a result of " rolling"
Stains left by a shape on the plain, when one "rolls" the shape on the plane (without sliding). See Fig 2a, b for some examples of signs and stains.
The outline of the solid: which solids have "shadows" of similar form, and which do not? By shadows we actually mean the outline of orthogonal projections. For example, the pyramid and the cone may have the same isosceles triangle-shaped shadow, while the parallelepiped and the cylinder may have congruent rectangular shadows.
These and other similar classifications are used to identify the intruders. At the first step, the students must fill in under each shadow the names of all possible "candidates" among the solids (Figs. 3a, b). Later, they must use all their recently and previously acquired knowledge to recognize the solids. One of the main ways for the students to help the detective is to solve logical problems like: "Who is it who has a rectangular shadow and who has left a circular sign?", or "Can we recognize for sure the intruders whose shadows appear in the following two sets?" (Figs. 3a, b). In order to recognize the intruders by their shadows as they appear in Fig. 3a, the students have to reason more or less as follows: there are three circular shadows, hence in this setting the cylinder, the cone and the circle are certainly involved, though we cannot pinpoint any of them. Hence, the rectangular shadow must belong to the parallelepiped. As to the square shadows, they are of the cube and of the pyramid, but we can't know which is which. Thus, in the first setting one recognizes for certain the shadow of the parallelepiped. To compare, in Fig. 3b there are two triangular shadows; hence, these are of the cone and of the pyramid, but we cannot know which is which. The square shadow then belongs to the cube, and the circular shadows are of the ball and of the cylinder, though again we cannot know which is which. Hence, we have pinpointed the Theme C – M. Barabash & R. Guberman
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cube. To be sure, the wording of second graders is different, but grosso modo this is the reasoning they must use, including the necessarily correct naming of plane figures and of solids, which by itself may be regarded as a success.
Fig.3a Fig.3b
Fig 3. Using shadows to identify potential solids
Linguistic and thinking skills needed for the solution of these problems are clearly aimed at the students' insight. We will refer later to the generalization possibilities of this task.
Example 2: 6
th
grade
According to the sixth grade curriculum, the solids are to be studied from several points of view: volume and surface computations; closer acquaintance with solids of revolution and with various types of polyhedra; unfolding solids into plane nets. In addition, we also include a number of assignments whose purpose is to form initial understandings related to the three-dimensional version of isoperimetric problems (*MEC2*). For some of these mathematical purposes (volume and surface computations and related issues, including the isoperimetric issues; nets folding and unfolding), it is worth regarding prisms as generalized cylinders, i.e. as surfaces formed by parallel sliding of a straight line along a plane directrix. A circular directrix leads to a circular cylinder, a polygonal directrix leads to a prism; other directrices may lead to

other generalized cylindrical surfaces. Similarly, pyramids are generalized cones: both surfaces are formed by a straight line whose one point is fixed and another is moving along a directrix (circular, polygonal or other) (MEC1). This clarifies the identical methods of the volume and surface computations for cones and pyramids, as well as for prisms and cylinders, and is actually an example of a "big mathematical idea" on which the whole series of textbooks is focused concerning 3-d measurements. Naturally, such grouping differs from the classification of solids of revolution vs. polyhedra which serves different purposes. Nets unfolding, in addition to the previous grouping, suggests yet another approach to classifying the solid surfaces: those who may be unfolded into a net and those who may not.

One of the widely used primary-school approaches to the volume computations of cones (and the one recommended by the Israeli school curriculum) is to fill a conic vessel with water or sand to show that three such vessels fill the cylindrical vessel of the same base and height. The same is to be done with a pyramid and a prism, when the volume of a pyramid is studied. This demonstration may be persuasive enough to show that the volume of the cone or of the pyramid is three times less than that of a cylinder or of a prism, but does not give even a hint as to why it is so. This is not compatible with our concept of insight as a result of meaningful learning. Ignoring the similarity of volume computation formulas for cones and pyramids and for cylinders and prisms seems to be an aggrieving omission of the possibility to teach a mathematically meaningful approach to the volume computations in a way that accounts for the future perspective of this topic. The approach which we undertook in our textbooks is actually based on an intuitive mathematical principle known as Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Cavalieri's principle, which is an example of a big mathematical idea actually going back to Archimedes. It states: Given two solids of equal heights whose bases have equal areas, if the plane sections of these solids at any height have equal areas, then these solids are of equal volumes. We do not propose to explicitly formulate this principle in the class, but this understanding is our mathematical and epistemological source for the overall approach to volume computations in primary school. The grouping of cones with pyramids and cylinders with prisms serves the initial perception of the intuitive 3-d version of the isoperimetric problem, as well as the consistent understanding of how surfaces of similar structure may unfold into plane nets. One has obviously to discern the revolution solids from the polyhedra in order to study intrinsic properties of each type of solid, which renders a completely different grouping.
The following excerpt from the 6
th
grade volume-and-surface-computation
assignment includes also the intuitive minimal-surface-maximal-volume issue. It originates from designing a tent (MEC3). How do we design a tent (Fig. 4a)? We want a tent to be as light as possible (one has to carry it), to provide as much sleeping place as possible on its floor, and to hold as much air as possible (one expects to breathe while sleeping). What do we need to compute in order to fulfill each of these demands? Which tent will we prefer amongst the following three which are of the same height and of equal bases areas (Fig. 4b)?
Fig.4a How do we design a tent? Fig.4b Which tent will we prefer?
The problem thus posed is a situation leading to motivated computation-based decisions: when one needs to compute the basis area, when the surface area is relevant, and when the volume computation is the answer to the question (MEC4). Moreover, it is obvious from the context what the net unfolding is for and why it is essentially the same for all the tents. Here, all the tents are of a conic structure, whatever their base forms are. Alternatively, further in the book the pottery is brought as an example in the context of solids, thus distinguishing solids of revolution from polyhedra.
Since the textbooks have been published and approved by the Ministry of Education, about 2000 school teachers have been using them in their primary school geometry lessons. Interviews with the teachers imply that their students understand that the properties of solids may be presented in different ways through two- and threedimensional notions. The students are able to combine verbal and visual representations, the words they use being similar or identical to mathematical terms.
Here is an excerpt from the feedback of one of the teachers:
"… a student is expected to be familiar with solids already at the primary school. That is, he (or she) must understand them from a number of viewpoints, to be flexible with them.… Flexibility, I would say, means – to feel them, to be able to see them from various directions, to see what they enable us, to see their meanings, when it is convenient to use them… your approach, from all the books I've used, and I've used some textbooks, enables this flexibility for the students, gives them confidence and freedom to "play" with the solids…"
As a result of encouraging reactions of practicing teachers, we included our textbooks as the learning material in our geometry course for students who are being educated as future primary school mathematics teachers or kindergarten teachers.
Analysis of the examples of the tasks from the textbooks Theme C – M. Barabash & R. Guberman
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We will use now the examples to illustrate how the previously formulated objectives serve to advance the students' insight by each of its parameters.
Implementation. The situation in which a second-grader must follow and apply the classification immediately after its presentation in the class by the teacher or by one of the classmates is an important opportunity to enhance his or her aptitude for implementation. Different behaviors may indicate different levels of insight. For example, examining the stains left by the "rolling" solids, a child may follow closely the "rolling" manner of the teacher, or may decide to roll a solid (e.g., a parallelepiped over its longest side), thus providing a stain in the form of a long and narrow rectangle instead of a shorter and wider one. Performing these activities, the student has to master his knowledge of plane figures acquired previously (during the 1
st
grade) and discern
various signs and stains left by the same solid (MEC2; CD).
Skills. The approach which we propose develops the aptitude of comparing and classifying solids by their properties and adequately using these properties for various purposes, which requires much more than mere recognition. We regard these skills as essentially more meaningful for geometry learning, and for mathematics learning in general (MEC). Examples of such skills are: an ability to discern a special vertex of a pyramid analogous to the vertex of a cone (which is indispensable for the volume computations); discerning specific parts of solids, including those not directly accessible, like the diameter of a sphere; folding and unfolding nets and recognizing the solids by their nets. These skills comprise the integral learning (CD).
Generalization. The approach based on analysis of properties common to a group of objects (such as signs and shadows in 2
nd
grade, common structure for cones
and pyramids vs. structure common to cones and balls in the example of the 6
th
grade)
has direct implication for generalization on this basis. We attempted to demonstrate that this approach is both mathematically (MEC) and didactically (PD) justified.
Language. We regard the mathematical language to be one of the central factors in the development of one's mathematical insight. The linguistic abilities of the student develop when he or she has to explore, formulate hypotheses, explain, motivate, and relate mathematical ideas, as in the activities described previously. These abilities comprise another component in a student's cognitive development (CD). The example of reasoning needed to recognize solids by their shadows speaks for itself, to our opinion, as far as all these aspects are concerned.
Discussion
We hope to have demonstrated that the mathematical and didactical features of our texts, based on the principal considerations formulated above (MEC, CD, PD), serve the on-going purpose of fostering the mathematical insight of young students with that insight as characterized by its four parameters. Moreover, they help to achieve specific objectives which comply with our general line as reflected in these considerations:
Advancing the young students 'half-way' to future formal definitions, ideas, notions and formulas in geometry; in particular, to the general understanding of volume and surface area computations and to intuitive perception of the interrelation between the form of a solid and its minimal-surface-maximal-volume properties (MEC1, MEC2).
Well-motivated outer-mathematical foundations for correct perception of central notions in geometry of solids which sometimes cause a great deal of ambiguity for students, such as surface area vs. volume (MEC1, MEC3, CD). Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Flexibility in the students' perception of solids in ways corresponding to specific mathematical and not-necessarily mathematical contexts (CD), and more.
The Pedagogical-Didactical (PD) objective which is less referenced in the course of the presentation and discussion is obviously central to the series.
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How Do Students Learn Mathematical Proof? A Comparison of Geometry Designs in German and Taiwanese Textbooks
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The learning of mathematical proof varies significantly among topics, tasks, curricular materials, and teaching. In this research, we focus on comparing the content of mathematical proof provided by curricular materials, especially school textbooks, between Germany and Taiwan. We first discuss how mathematics textbooks are designed and then report on the differences of geometric content structure between them (focused on grades 7-9) for unveiling a further comparison of the design relative to two common statements, the Pythagorean theorem and the sum of interior angles of a triangle. We compared six different textbook series, three from each country, using three different principles, continuity, accessibility, and contextualization, to inspect the content of these two statements. The results reveal that German textbooks employ a more generic approach to lead pupils to a theoretical position in mathematical proof and offer tasks in divergent contexts whereas Taiwanese textbooks prefer a visual-algorithmic approach to guide pupils to the transition from the pragmatic to the theoretical position by providing physical experiments or algorithmic tasks in the same context.
Keywords: mathematical proof, geometry, design, textbook
Introduction
The design process and usage of curricular materials, e.g., textbooks, are similar in both Germany and Taiwan. However, students' performances on international assessments are discrepant. To study the differences, it might be worthwhile to look deeper into the intended design of mathematics instruction in addition to comparing teaching and learning in classrooms. The importance of textbooks in school cannot be ignored. Although there is an increasing number of comparison studies in mathematics textbooks, most focus on comparing the differences of semantic features or textual presentations; only some discuss the details of the design for specific topics (e.g., Charalambous, Delaney, Hsu, & Mesa, 2010; Stylianides, 2005, 2009; Thompson, Senk, & Johnson, 2012).
Balacheff (2010) indicated the importance of acquiring mathematical knowledge as a process and not only receiving it as a fact. Theme C – Yu-Ping Chang, Fou-Lai Lin & K. Reiss
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…mathematical ideas do not exist as plain facts but as statements which are accepted only once they have been proved explicitly; before that, they cannot be instrumental either within mathematics or for any application (p. 9).
As Balacheff (2010) said, getting involved in mathematics means for pupils (learners) to change their intellectual position and to become a theoretician. He provided two different types of shift, say from practical geometry (the geometry of drawings and shapes) to theoretical geometry (the deductive or axiomatic geometry) and from symbolic arithmetic (computation of quantities using letters) to algebra.
Mathematical proof is viewed commonly as a core activity in mathematics (Heinze & Reiss, 2007). The basic features of the curriculum, being content, organization, and sequencing, have an impact on pupils' conception of proof (Chazan, 1993; Healy & Hoyles, 2000; Hoyles, 1997; Stylianides, 2007), and the activities of problem solving might provide an easy way for pupils to experience the process of proof (Balacheff, 1988; Pólya, 1981; Schoenfeld, 1992). Tall, Yevdokimov, Koichu, Whiteley,

Kondratieva, and Cheng (2012) collected eight distinct methods of English, German and Taiwanese pupils' proof of the statement that the sum of the interior angles of a triangle is 180°. However, the question remains how the curricular sequences in the three countries differ from each other. Though there are various methods to prove this statement, e.g., empirical argumentation (Lin, 2000), or heuristic worked example (Reiss & Renkl, 2002), it does not mean that all pupils receive the same opportunity to do proof. As we said above, mathematical knowledge (ideas) influences students' learning; once they ascertain the validity of mathematical knowledge (prerequisites) they can apply it to a new statement. In addition, the alignment between designed tasks and how teachers select and use them influences the learners' engagement (Watson & Chick, 2011). Therefore, inspecting the designed tasks is important.
We will argue in the remainder of the paper that the different sequences of acquiring knowledge influence the design of tasks and have an implicit impact on students' opportunities to learn mathematics. We initiate an analysis on comparing the designs of geometry content in different textbooks between Germany and Taiwan and discuss the details of textual presentations as a precursor.
A snapshot of mathematics curricula in Germany and Taiwan
The school system (before collegiate education) in the Federal Republic of Germany differs according to each federal state. However, in general there are three different tracks in secondary school with differently designed curricula: Gymnasium (grades 5-12), Realschule (grades 5-10), and Hauptschule/Mittelschule (grades 5-9). For example, geometry proofs are primarily taught in the Gymnasium track. In contrast to Germany, mathematical proof is supposed to be introduced to all pupils in the unitary school track, junior high school (grades 7-9), in Taiwan.
Role of textbook and design process
Regarding the educational ideas and actual materials used in classrooms, it is necessary to allocate and define the position of related elements of them. In Figure 1, we present three layers with their respective trajectory of involved elements from ideal situation to reality. The dashed lines link the implicit relationships between three layers. The middle layer of three different roles of curriculum is the bridge between ideas and actual materials. Textbooks in German or Taiwanese schools can be viewed as intended curriculum or implemented curriculum depending on how teachers use them in classrooms. They are designed based on national standards and teachers use Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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them as a tool to write their own lesson plans or directly include them in their teaching.
The processes of designing mathematics textbooks are similar in Germany and Taiwan. The contents of textbooks are designed by a group which might encompass researchers, mathematicians, mathematics educators, and school teachers, and finally edited by the responsible editor(s). All written textbooks should be designed based on the national standards (and state syllabus in Germany) and approved by the ministry of education, and then can be published. Each individual textbook is designed based on the textbook editors' stated intentions.
Figure1. The role of German and Taiwanese school textbooks
Distribution and structures of geometric contents
Geometry content is not in the same sequence in German and Taiwanese textbooks. In Germany, most geometry content is arranged in 7
th
grade, and part in 8
th
and 9
th
grades. In Taiwan, geometry is not taught in 7
th
grade or the second semester of 9
th
grade, but takes a large amount of classroom work in the second semester of 8
th
grade and the first semester of 9
th
grade. A comparison of both countries in different topics is shown in Table 1.
Table 1. Distributions of geometric contents between Germany and Taiwan
Comparing the content structures in all the textbook series adopted in this study, we developed a representative structure for each as listed in Table 2.
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Germany	Taiwan
Construction 1	Symmetry
Perpendicularity	Pythagorean theorem
(Angle) Bisector	
Construction 1	
Perpendicularity	
Angles	Parallel postulate (Angle) Bisection
Construction 2	Congruence postulate (Line) Symmetry
Special triangles	
Isosceles triangle	
Solid	
Geometry	
Surface area	
Right-angled triangle	Volumes [prism, cylinder]
Thales theorem	Length of perimeter
Construction 3	
Special lines of triangles	Interior and exterior angles
(Centers of triangles)	Construction 2 Congruent postulates
Similarity	
Projection	Side-angle relation of triangles
Intercept theorem	Parallelism: Parallelogram
Pythagorean theorem	
Hypotenuse-Leg theorem (Kathetensatz)	
Similarity	
Proportional segment	
Leg-Leg theorem (Höhensatz)	(Intercept theorem)
Trigonometry	sine, cosine, tangent Circles
Solid	
Geometry	
Regular	
Oblique	
Prism	
Cylinder	
Pyramid	
Cone	
Angles	Geometry Proof
Surface area	
Volumes [Cavalieri's principle]	
Length of perimeter	

Table 2. Content structures of German and Taiwanese textbooks
Method
Textbooks are the main materials we used in this study. They are developed and designed with different editors' intents. Therefore, the presentations vary between different textbook series. Nevertheless, they are based on conjoint curricular goals across different settings for instruction. To examine different textbooks from different countries, it is necessary to set principles for comparing them in the same phase, specifically whether they are grounded on the same conceptual knowledge, learning opportunities, and context. Below, we list the materials used in this study and give our exposition of the principles we implemented for the comparison.
Analytical materials
We selected six representative textbook series used in school years 2009-2010, three from the state of Bavaria in Germany and three from Taiwan. In this study, we focus on grades 7, 8, and 9 of the Gymnasium track in Germany and of junior high school in Taiwan. We chose two common statements: the sum of interior angles of a triangle, which is introduced in grade 7 in Germany and the second semester of grade 8 in Taiwan; and the Pythagorean theorem, which is introduced in grade 9 in Germany and in the first semester of grade 8 in Taiwan. The texts from the textbooks can be generally separated into three parts: the corpus texts, which provide various activities and tasks to introduce mathematical knowledge; the summary part, which summarizes the point of one section or one chapter; and the exercises pool, which provides numerous exercises for pupils to practice. We focus only on the corpus texts introducing both statements, and exclude the summary and final exercises pool in order to avoid repetition (summary) and subjective selection without considering the continuity of former mathematical ideas (exercises).
Three principles of content comparison
We set three principles in inspecting the task design to be able to discuss the similarity and dissimilarity of German and Taiwanese tasks on the same scale. Principle one examines the continuity of the mathematical knowledge (ideas) involved, the flow of concepts. This principle contemplates the Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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related concepts of a peculiar task from dissimilar societies. Principle two scrutinizes the accessibility, which means the learning opportunities provided by the designed tasks. It addresses whether the presentation allows the students access in a reachable way; in other words, it addresses clarity of the final resultant (goal) and the transparency of figural or textual representations. Principle three reviews the contextualization of tasks. The contextualization refers to the sociocultural perspective, which is regarded as stable physical and discursive elements of a setting in which a learning activity takes place. It refers also to the constructivist perspective, meaning that the personal, cognitive context shaped by the learner's personal interpretations of an activity is taken into account (Nilsson & Ryve, 2010).
Results
Two different approaches to the common statements
The geometric content of German and Taiwanese textbooks varies in topics and content structures as mentioned above. In this part, we show the common statements which relate to different mathematical concepts and are designed in different trajectories in both countries. In general, German textbooks design a new statement validly following the origin/hierarchy of related concepts. We call this a generic approach. In our view, the generic approach can be linked to the idea of Balacheff's (1988) generic example, "involving making explicit the reasons for the truth of an assertion by means of operation or transformation on an object that is not there in its own right, but as a characteristic representative of its class" (p. 219). In Taiwan, the visual-algorithmic approach relies highly on the figures and algorithm and is commonly used for geometry content.
Though some German textbooks simultaneously introduce the same content with a visual-algorithmic approach, it is not the mainstream of knowledge arrangement but an alternative method for teachers/pupils to teach/learn mathematics (and reflects probably the influence of the present international discussion). In this study, we compare only the main approach, the intersection of practice, among textbook series in one country. Below, we present the designs regarding both statements between German and Taiwanese textbooks.
Approaches to the Pythagorean theorem and the design
In both countries, instruction of the Pythagorean theorem should lead to the algebraic formula,

, where , and are three sides of any right-angled triangle, and then the application of this formula in different types of problems. The relationship between the areas of three similar figures, usually squares as Euclid's Elements presents, generated from three sides of a right-angled triangle, is an important method to prove the theorem.
However, the methods of proving the Pythagorean theorem differ in German and Taiwanese textbooks (see Tables 3 and 4). The German curriculum arranges the Pythagorean theorem as part of a series of theorems including hypotenuse-leg theorem (Kathetensatz) and leg-leg theorem (Höhensatz). These are highly connected to the conditions of similarity of triangles, therefore learning the Pythagorean theorem presupposes the topic of similarity. The Taiwanese curriculum arranges this statement after learning the expansion of perfect squares:

and

, with various permutations of geometrical figures. In the Taiwanese design, the visual figures and the algorithm coexist to influence the reasoning. Concerning the conceptual continuity, the routes to this statement are obviously different in both countries.
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Some German textbooks provide additional tasks to construct
19
the squares generated from each side of the right-angled triangle and then encourage students to discuss the relationship between the areas.

German Main Approach	Taiwanese Main Approach
Generic	
Statements	Former ideas
Similarity: - ! ! - ! -	
Similarity: - !	Visual-algorithmic
Visual-algorithmic	Visual-algorithmic
1. Deduced from above formulae	

2. Relations between
area

4
5
¾±
.

4 5
¾±
.

2
.....(1)

2...(2)

Table 3. Stereotypical approaches to the introduction of the Pythagorean theorem
Germany Taiwan
Continuity
Main idea:
• Conditions of similarity
• Additional ideas:
• Figural construction
• Area formulae
Main idea:
• Area formulae of triangle and
square
• (Skills: Perfect squares)
Accessibility
• High accessibility to the stereotypical introduction • High accessibility to the
stereotypical introduction
Contextualization
• Various tasks (mainly, deductive reasoning)
• Divergent contexts (e.g., figural construction;
algorithm on figural areas)
• Various tasks of algorithm on
figural areas, but stable context
Table 4. Three principles in analyzing the Pythagorean theorem
Approaches to the sum of interior angles of a triangle and the design
The topic connecting angles and geometric shapes is first introduced in 5
th
grade in both German and Taiwanese schools. Textbooks in both countries provide a
physical experiment for students to experience the quantity of angles by measuring
them with a protractor. The difference is that only the Taiwanese curriculum gives the
sum of interior angles of a triangle, 180˚, in the textbook in this school year. Then the
formal reasoning on this topic is set in 7
th
grade in Germany after the statement of
parallel postulates and in second semester of 8
th
grade in Taiwan.
In Tables 5 and 6, we show the related concepts introducing and applying
this statement, and the arrangements of the design. In German textbooks, it is
connected to the formerly learned concept that alternate interior angles of a pair of
parallel lines with a transversal being equal (parallel postulates) and the concept of
straight line (angle). In Taiwanese textbooks, it is not the main work to prove the truth
of this statement (the sum of interior angles of a triangle equals 180˚) but to apply this
"authorized" knowledge in different situations. The introduction of this statement is

There are ample opportunities of experiencing geometric construction (based on ruler and
compasses) in German textbooks. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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connected again to the students' experiences (physical experiment in 5
th
grade) to
"prove" visually/intuitionally with the concept of straight angle.
Table 5. Stereotypical approaches to the introduction of the sum of interior angles of a
triangle
Germany Taiwan
Continuity
Main ideas:
• Parallel postulate
• Straight angle
• Further:
• The sum of interior angles of a polygon
Main idea:
• Straight angle
• Further:
• The sum of interior angles of a polygon
• The sum of exterior angles of a polygon
Accessibility
• High accessibility to the stereotypical
introduction
• High accessibility to the stereotypical
introduction
Contextualization
• Various tasks (mainly, deductive
reasoning)
• Divergent contexts (ex., to experience
the invariance of the sum with
transformation; physical operation)
• Various tasks with physical operation, but
stable context
Table 6. Three principles in analyzing the sum of interior angles of a triangle
One point that needs to be mentioned is that the German design ends with the
generalization

of this statement to the sum of interior angles of other polygons by
the construction of auxiliary line(s). The follow-up of the Taiwanese design continues
to apply the three straight angles and this statement:
180˚ 3 180˚ 360˚, to
introduce the sum of exterior angles of a triangle by calculation, and then to
generalize to all polygons by doing algebraic proof, that is,
180˚ + ,180˚

+ 2- 360˚.
Discussion
There are many factors which influence the design of curricular materials in
different countries, such as design intentions for curriculum, content structures,
educational goals, policy issues, etc. In this study, we restricted ourselves to a single
aspect, the presentation of geometry content in textbooks. It turned out that the
approaches to introducing mathematical proof through geometry content significantly
differed in Germany and Taiwan. This was especially true for the details of the
exemplifications of textbooks above. By examining the designed tasks for the same

The Taiwanese design also presents the similar generalization as the German design does. Theme C – Yu-Ping Chang, Fou-Lai Lin & K. Reiss
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statements, the Pythagorean theorem and the sum of interior angles of a triangle, we
could identify important differences.
We found that the opportunities to learn mathematical proof differ from
Germany to Taiwan. In Germany, deductive reasoning with hierarchical concepts in
geometry seemed to be pivotal in receiving new mathematical knowledge. In addition,
the German textbooks validated two statements by deductively reasoning with generic
mathematical ideas (theoretical position) and provided various operational tasks, like
construction or physical experiments (pragmatic position). These operational tasks in
German textbooks expected students to experience the concrete knowledge. Yet, this
design of tasks was not always consistent with the central mathematical ideas
(divergent contexts). In Taiwan, authorized knowledge accompanying various
applications dominated the learning... The Taiwanese textbooks used a physical
experiment or reasoning with algorithms accompanied by figures to access both
statements, and then arranged various tasks to apply them by using the algorithm in
different situations. The truth of the statements was only shown with some empirical
examples to "prove" (argue) the conjecture and it seemed the pupils would accept
them. Though the differentiation between a pragmatic position and a theoretical
position in mathematical proof was not clear in either statement, the consistency
between concepts and activities was strong (stable context). Moreover, the application
of algebraic proof, which followed these two statements (authorized knowledge), was
emphasized more in Taiwanese textbooks. In summary, these differences might
correspond to the research which indicates that German students lack strategies in
proving while Taiwanese students lack principles to explain why a property is true
(e.g. Heinze, 2004; Lin et al., 2003; Reiss et al., 2002).
We do not intend to over-generalize our judgement to the quality of tasks.
The usage of tasks might differ with teachers' intentions in using them in class, and
hence differ from teacher to teacher. Moreover, this work should not be seen as a
general cross-cultural comparison, but it provides some evidence for different
teaching styles. We provided an overview of how the same statement can be
introduced in different arrangements and in different cultures; we see this as a first
step for a more profound and more general comparison of textbooks.
This paper is based on part of the first author's doctoral dissertation work
supervised by the other two authors.
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Designing the Practical Worksheet for Problem Solving Tasks
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Problem solving task design is not only the design of a nonroutine problem to be solved by students. We conjecture that, in addition, task design also requires a supporting document that would act as a cognitive scaffold for students in the initial stages of the problem solving process before they can internalize the metacognitive strategies and automate the use of these strategies when faced with a new problem. In this paper we describe the design and use of such a document, which we have called practical worksheet, within our ongoing problem solving research project.
Keywords: problem solving, cognitive scaffold, practical worksheet
A fundamental aspect of doing mathematics is to solve mathematics problems. Mathematical problems are inherent in the structure of the subject itself and are the raw materials for problem solving, which is a highly valued process in mathematics education. There is now a mounting body of literature pointing to the fact that problem solving is still not implemented in mathematics classrooms, or if implemented, then certain routine approaches to heuristics are being adopted (see English, Lesh, & Fennewald, 2008; Lesh & Zawojeski, 2007; Schoenfeld, 2007; Silver, Ghousseini, Gosen, Charalambous, & Strawhun, 2005). Stacey (2005) has suggested that:
To get closer to the goal requires research directed to understanding the problem solving process for mathematics (in all its aspects), developing effective classroom processes, and designing excellent tasks. Moreover, the research needs to be closely intertwined with curriculum development and teacher development projects so that it can make an impact on practice. (p. 341) Theme C – J. Dindyal et al.
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Our research team has identified several issues in the literature in the international context as well as those in the Singapore local context that need to be addressed to facilitate the implementation of problem solving in schools. Our approach still values the problem solving model of Pólya (1945) and the insights from Schoenfeld (1985). To aid in the implementation of problem solving in schools, we have come to the realization that the design of specific problems or problem solving tasks cannot be the only focus of problem solving; rather, cognitive scaffolds (see Holton & Clarke, 2007) that allow students to solve a wider range of problems should also be an important focus. Accordingly, in this paper, we turn our attention to the design of the practical worksheet that can be used as a cognitive scaffold in problem solving tasks (see Figure 1). We document the development of the practical worksheet based on our design principles and feedback from teachers.
The Problem Solving Task Design Principles
Based on the literature, our research team drafted the following design principles for the problem solving tasks:
1. Each task is challenging, focused around important mathematical concepts, and interesting to the students.
2. Each task would offer the students an opportunity to extend and/or generalize.
3. The task, although fundamental to the design, is only the means to the loftier goal of transfer of problem solving skills to other situations.
4. A task is most meaningful to a student when presented as an assessment task for which the student earns some credit.
5. The process of solving the problem is as important, if not more important, than the final solution of the problem. As such, assessment of problem solving should consider carefully the problem solving process. The students should see that they earn enough credit for specific steps and/or intermediate strategies that they use.
6. Pólya's problem solving model and Schoenfeld's ideas about problem solving guide the development of the problem solving tasks.
Scrutinising the above principles, we came to the conclusion that, no matter how well problems (as tasks for students) themselves are designed, they can at best address the first two design principles. Good selection of problems alone would have little effect in fulfilling Design Principles 3-6, which are about the need for students to attend to the process of problem solving and the motivation for problem solving. As such, in considering task design for problem solving, we need to include elements of design that attend directly to Design Principles 3-6. To address these, we conjectured that the task for students should include a supporting document that would act as a cognitive scaffold for the students in the initial stages of the problem solving process before they could internalize the metacognitive strategies and automatically use these strategies when faced with a new problem. This is in line with the view that any scaffold should be gradually withdrawn as the learner becomes more competent (Rittle-Johnson & Koedinger, 2005; Yelland & Masters, 2007). We have followed Holton and Clarke's (2007) ideas: Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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...cognitive scaffolding allows learners to reach places that they would otherwise be unable to reach. With the right word or question or other device a teacher may put in place the scaffolding that will allow new knowledge to be constructed, incomplete or wrong concepts to be challenged or corrected, or forgotten knowledge to be recalled. (p. 129)
Thus, our team's effort was directed towards the development of this important document that we have since called the practical worksheet. The practical worksheet is developed along the same lines as a worksheet that science students would generally use in a science class to help them carry out experiments, record observations, and make inferences. We have also incorporated ideas from Yelland and Masters (2007), who have also used the term cognitive scaffolding in the context of technology use.
We have used the term cognitive scaffolding to denote those activities which pertain to the development of conceptual and procedural understandings which involve either techniques or devices to assist the learner. These include the use of questions, modelling, assisting with making plans, drawing diagrams and encouraging the children to collaborate with their partner. (p. 367)
The way a task is imagined and intended by the teacher may be quite different from the way it is construed and carried out by the students (see Mason & Johnston-Wilder, 2006). The intended learning by students may not happen if the tasks are misconstrued by them. Also, if teachers give too many directions, then the solution process may become too trivial for the students and the solution process may be reduced to a sequence of steps. If the teachers give too few directions, then the students may focus on different things and the implied learning may not happen. As such, the implementation of problem solving in the classroom ultimately hinges on the classroom teacher; in our designing process, we paid careful attention to teacher preparation for problem solving, including the use of the practical worksheet.
Teachers were taught three levels of scaffolding 0 Pólya stages, specific heuristics, and problem specific hints 2 to be used when advising students who are doing problem solving (Toh, Quek, Leong, Dindyal, & Tay, 2011). The levels are hierarchical and the next level of scaffolding should be given only after an earlier level has failed. Consider the Lockers Problem given below.
The Lockers Problem
A new school has exactly 343 lockers numbered 1 to 343, and exactly 343 students. On the first day of school, the students meet outside the building and agree on the following plan. The first student will enter the school and open all the lockers. The second student will then enter the school and close every locker with an even number. The third student will then 'reverse' every third locker; i.e. if the locker is closed, he will open it, and if the locker is open, he will close it. The fourth student will reverse every fourth locker, and so on until all 343 students in turn have entered the building and reversed the relevant lockers. Which lockers will finally remain open?
In the crucial level, which we call Level 0, we emphasise the student learning and reinforcing of the Pólya model (see Table 1). We may ask the student if he or she knows what Pólya stage he is in, and what would one normally do in such a stage. We help by asking these Control questions. In Level 1, we suggest specific heuristics to get the work moving. Level 2 is to be avoided as much as possible and is included only for the important aspect of ensuring that the self-esteem of the student is not Theme C – J. Dindyal et al.
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seriously damaged by his or her perceived failure and helplessness on the problem. Here, we give problem specific hints, which essentially is a throwback to the 'usual' help afforded by mathematics teachers.

Level	Feature	Examples based on the Lockers Problem
0	Emphasis on Pólya stages and control	What Pólya stage are you in now? Do you understand the problem? What exactly are you doing? Why are you doing that?
1	Specific heuristics	Why don't you try with fewer lockers (use smaller numbers)? Try looking for a pattern.
2	Problem specific hints	Think in terms of the locker rather than the student – which student numbers get to touch the locker?

Table 1. Level of scaffolding
The objective of the practical worksheet is for students to internalize Level 0, and ask for Level 1, and to a much lesser extent Level 2, hints only when pressed for time. Assessment of problem solving is certainly another issue that guides students in problem solving tasks. To this end, an accompanying assessment rubric was developed to focus students on what is valued in the problem solving process. At the same time, the rubric gives them feedback on their strengths and weaknesses.
The Practical Worksheet
Holton and Clarke (2007) have claimed that scaffolding is an act of teaching that supports the immediate construction of knowledge by the learner and as well provides the basis for the future independent learning of the individual. These authors have added that scaffolding does not necessarily require the teacher and the student to be actually physically present together. Furthermore, it is essential for an individual student to be able to scaffold for himself or herself when solving a new problem, termed self-scaffolding by Holton and Clarke (2007). Accordingly, our aim was to find a way of developing the learner's autonomy in taking charge of his or her own learning when faced with an unfamiliar mathematical problem, whether the teacher was present or not. The artifact that we have developed is called the practical worksheet. We relate the evolution of this artifact from 2005 to the present, to scaffold mathematical problem solving behavior within the story of our efforts to teach mathematical problem solving in the schools.
The Development of the Practical Worksheet
In 2005, we developed a problem solving module within the mathematics curriculum of a high ability school in Singapore (see Appendix B). This initial design consisted of a 1.5 hour lecture on problem solving covering Pólya's problem solving model (Pólya, 1945) and Schoenfeld's framework (Schoenfeld, 1985). We had a second attempt in the same school in 2006. In 2009, the research team was invited by another high ability school to develop a module on problem solving. We adopted a design experiment approach (Brown, 1992; Collins, 1999; Gorard, 2004; Wood & Berry, 2003) to produce a workable "design" (an initiative, artefact or intervention, for instance) that could be adapted to other schools. In Gorard's (2004) words, "The emphasis [in design experiments], therefore, is on a general solution that can be 'transported' to any working environment where others might determine the final product within their particular context" (p. 101). We have argued the following theoretical justification for a design experiment for our research on problem solving (Quek, Dindyal, Toh, Leong, &Tay, 2011): Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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1. Obstacle - Current instruction on problem solving consists mostly of the teaching of heuristics only.
2. The theoretical basis of Pólya and Schoenfeld remain sound.
3. Mathematical problem solving must include the Looking Back stage of Pólya's model.
4. Mathematics problem solving is valuable enough to be adequately assessed and must be adequately assessed to be valued.
5. Mathematics problem solving is for every student of mathematics.
We then worked on the curriculum package by referencing the classical view of design where the parameters of the product are to be specified a priori (Ullman, 1992). We thus stated the following parameters, underpinned by the earlier theoretical justification, for the design of a package for teaching problem solving:
1. Place in the curriculum: The problem-solving module must be part of the mainstream mathematics curriculum.
2. Model of mathematical problem solving:
i. Pólya's model – all four stages
ii. Schoenfeld's framework – teach Heuristics and emphasise Control
3. Teacher autonomy: Teachers in school will ultimately teach the module themselves. Build teachers' capacity in problem solving and to teach it.
4. Infusion into regular mathematics content: Problem solving skills and habits learnt in the module must be infused into other mathematics modules to prevent

atrophy.
5. Assessment: A valued component in school assessment
The following features were then built into the first prototype of the package
to satisfy the demands of the stated parameters. From our experience with the first
school, we realised that we must first look for a way out of the perennial quandary of
the undervaluation of assessment of problem solving. As before, our package
promoted mathematical problem solving as learning the processes of mathematics akin
to the established science practice as a way to learn science processes. We designed a
mathematics practical worksheet (Figure 1 shows a typical response by a student on a
test which used the practical worksheet) and added a rubric for assessment (further
details are available in the next section). Parameters 1 and 5 were addressed. A series
of lessons to teach students how to use Pólya's model as a scaffold, to teach various
heuristics and choice of heuristics, and to make students aware of the need for control
in problem solving was developed. Parameter 2 was addressed.
To us, it appeared necessary that the teachers make the proposed
instructional approach a routine sufficiently familiar to them so that the approach
becomes classroom practice. To reach this stage, it seemed essential for the teachers
to adapt the researchers' ideas to make them their own, in the sense that their beliefs
of mathematics and of problem solving in mathematics are transformed. Such a
process would pass through a stage where the teachers negotiate and change the
problem solving lesson. Finally, a community of practice would develop among the
teachers to support the change process by providing opportunities to learn to engage
the proposed ways - thinking, talking and reflecting on the new teaching experiences
and ways of doing mathematics (Shulman & Shulman, 2004). The entire process of
transforming an externally proposed instructional approach and curricular change into
classroom and school practice appeared to be cyclical, incremental, and emergent in
nature. This was our approach to Parameter 3.
To address Parameter 4, problems that were from the regular mathematics
curriculum and were rich enough for extended work would be crafted and infused into Theme C – J. Dindyal et al.
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the regular schedule. The justification was the reference to science practicals going in
tandem with science theory lessons, and the motivation was that these difficult
problems could be used as assessment for learning.
Assessment Using the Scoring Rubric
Traditionally, the assessment of problem solving in the classroom has
focused on assessing the products rather than the processes of problem solving. Our
efforts to meet the challenge of teaching mathematical problem solving to
students call for a curriculum that emphasizes the processes (while not
neglecting the products) of problem solving. The assessment strategy must match
it so as to drive the mode of teaching and learning of mathematics.
It is common knowledge that most students study mainly those curricular
components which are to be assessed. Accordingly, there needs to be a corresponding
assessment strategy that drives the teaching and learning of problem solving as
described in the preceding paragraphs. Effective assessment practice begins with and
enacts a vision of the kinds of learning we most value for students and strives to help
them achieve. To assess the students' problem-solving processes (which we value),
we developed a scoring rubric based on Pólya's model and Schoenfeld's framework.
The scoring rubric focuses on the problem-solving processes highlighted in
the practical worksheet. There are four main components to the rubric, each of which
would draw the students' (and teachers') attention to the crucial aspects of an attempt
to solve a mathematical problem. In establishing the criteria for each of these
components of problem solving, we ask the question: What must students do or show
to suggest that (a) they have used Pólya's approach to solve the given mathematics
problems, (b) they have made use of heuristics, (c) they have exhibited "control" over
the problem-solving process, and (d) they have checked the solution and extended the
problem solved (learnt from it)?
The rubric is outlined below. The complete rubric is in Appendix A.
• Pólya's Stages [0-10 marks]. This criterion looks for evidence of the use of
cycles of Pólya's stages (Understand the Problem, Devise a Plan, Carry out
the Plan), and correct solutions.
• Heuristics [0-4 marks]. This criterion looks for evidence of the application of
heuristics to understand the problem, and to devise/carry out plans.
• Checking and Extending [0-6 marks]. This criterion is further divided into
three sub-criteria:
Evidence of checking of correctness of solution [1 mark];
Providing for alternative solutions [2 marks];
Extending and generalizing the problem [3 marks]. Full marks for this part
is awarded for one who is able to provide (a) two or more problems with
solutions or suggestions to solution, or (b) one significant related problem
with comments on its solvability.
The rubric was designed to encourage students to go through Pólya stages
when they are faced with a problem, and to use heuristics to explore the problem and
devise a plan. They would return to one of the first three stages (see practical
worksheet) upon failure to realize a plan of solution. Students who show control
(Schoenfeld's framework) over the problem-solving process gain marks. For example,
a student who did not manage to obtain a completely correct solution would be able to
score up to eight and three marks each for Pólya's Stages and for Heuristics, making Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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a total of eleven, if they show evidence of cycling through the stages, use of
heuristics, and exercise of control.
The rubric allows the students to score as much as 70% of the total 20 marks
for a correct solution. However, this falls short of obtaining the top marks for the
problem. The rest would come from the marks in Checking and Extending. Our
intention is to push students to check and extend the problem (Stage 4 of Pólya's
stages), an area of instruction in problem solving that has not been largely successful
so far (see for example, Silver, Ghousseini, Gosen, Charalambous, & Strawhun,
2005). Theme C – J. Dindyal et al.
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Figure 1. Student's solution using the practical worksheet Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Conclusion
The practical worksheet holds promise for teachers who want to elevate
problem solving to a prominent position in the mathematics classroom. They can
now not only encourage problem solving in their classes, but they can also make
transparent to students the criteria for assessment and the processes that are valued.
Taken altogether, our task design, which includes the practical worksheet, assessment
rubric, a set of problem solving lessons, and teacher scaffolding, has shown great
potential in developing student self-scaffolding in problem solving.
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Appendix A
RUBRIC FOR ASSESSING PROBLEM SOLVING
Name: _____
Pólya's Stages
Descriptors/Criteria (evidence suggested/indicated on practical
sheet or observed by teacher)
Marks Awarded
Correct Solution
Level 3 Evidence of complete use of Pólya's stages – UP + DP + CP;
and when necessary, appropriate loops. [10 marks]
Level 2 Evidence of trying to understand the problem and having a clear
plan – UP + DP + CP. [9 marks]
Level 1 No evidence of attempt to use Pólya's stages. [8 marks]
Partially Correct Solution (solve significant part of the problem or lacking rigour)
Level 3 Evidence of complete use of Pólya's stages – UP + DP + CP;
and when necessary, appropriate loops. [8 marks]
Level 2 Evidence of trying to understand the problem and having a clear
plan – UP + DP + CP. [7 marks]
Level 1 No evidence of attempt to use Pólya's stages. [6 marks]
Incorrect Solution
Level 3 Evidence of complete use of Pólya's stages – UP + DP + CP;
and when necessary, appropriate loops. [6 marks]
Level 2 Evidence of trying to understand the problem and having a clear
plan – UP + DP + CP. [5 marks]
Level 1 No evidence of attempt to use Pólya's stages. [0 marks]
Heuristics
Descriptors/Criteria (evidence suggested/indicated on practical
sheet or observed by teacher)
Marks Awarded
Correct Solution
Level 2 Evidence of appropriate use of heuristics. [4 marks]
Level 1 No evidence of heuristics used. [3 marks]
Partially Correct Solution (solve significant part of the problem or lacking rigour)
Level 2 Evidence of appropriate use of heuristics. [3 marks]
Level 1 No evidence of heuristics used. [2 marks]
Incorrect Solution
Level 2 Evidence of appropriate use of heuristics. [2 marks]
Level 1 No evidence of heuristics used. [0 marks]
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Checking and Expanding
Descriptors/Criteria (evidence suggested/indicated on practical
sheet or observed by teacher)
Marks Awarded
Checking
Level 2 Checking done – mistakes identified and correction attempted
by cycling back to UP, DP, or CP, until solution is reached. [1
mark]
Level 1 No checking, or solution contains errors.
[0 marks]
Alternative Solutions
Level 3 Two or more correct alternative solutions. [2 marks]
Level 2 One correct alternative solution. [1 mark]
Level 1 No alternative solution. [0 marks]
Extending, Adapting & Generalizing
Level 4 More than one related problem with suggestions of correct
solution methods/strategies; or
one significant related problem, with suggestion of correct
solution method/strategy; or
one significant related problem, with explanation why method
of solution for original problem cannot be used.
[3 marks]
Level 3 One related problem with suggestion of correct solution
method/strategy. [2 marks]
Level 2 One related problem given but without suggestion of correct
solution method/strategy. [1 mark]
Level 1 None provided [0 marks]
Hints given:
Marks deducted: _____

Total marks: _____ Theme C – J. Dindyal et al.
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Appendix B
Table 1: Development of the Practical Worksheet
Year Research Emphasis Objective Notes/Issues
2005 Development of a problem solving module in school XX (a 1.5-hour lecture followed by eight 1-hour lessons) To help students to: understand a problem before rushing in to solve it; devise a plan to attack the problem; use heuristics to understand the problem and as a plan of attack; check a solution attempt; pose new problems with reference to a successful solution of the original problem Moderate success. Students were reluctant to apply the stages of the Pólya's model. Even higher achieving students did not check and extend the problem
2006 Second attempt in same school. An approach to treat the mathematics lesson as a 'mathematics practical' lesson. Development of a worksheet (first version of the practical worksheet) Accommodate the different problem solving strategies of students. Not to force all students to go through each of the four Pólya's stages Some issues still persisted. Some students were unfamiliar with the problem solving heuristics. Some students could not quite differentiate among some of the Pólya's stages.
2007 Teachers in the same school were left on their own to implement the problem solving module. Teachers to develop autonomy and work independently of the researchers. Develop such a problem solving module sustainable in a mainstream school. Some students were still not filling the worksheet as desired. The lead teacher decided to put tick boxes for the students to tick rather than write out certain parts.
2009 In another school for more able students Adopt a design experiment approach in the research and produce a workable design not only of the problem solving module but as well of the practical worksheet. Have a greater focus on teacher development for use of the worksheet in their lessons. There was better success in this school. A scoring rubric focusing on the processes of problem solving had to be developed to be used in tandem with the worksheet.
2010–2012 Dissemination of the problem solving module to more schools Adapt the problem solving module using the practical worksheet for implementation in a wider range of schools. Positive feedback so far on the use of the practical worksheet Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Textbooks' Design and Digital Resources
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In this paper we report on the comparison of the design and conceptualization of two very different French lower secondary mathematics textbooks: one which was developed, as it is 'traditionally' done, by 'experts' (teacher educators and researchers); and one which was developed, innovatively, by teachers using a digital platform. These different designs and conceptualizations had repercussions on the content, structure, potential and intended use of the books, which we investigated using specially designed questionnaires given to the two groups of textbook authors. Our results point to re-conceptualisations of the notions of 'quality' and 'coherence' of resources, such as textbooks, taking into consideration teachers' documentation work, in particular their often collective work with digital resources.
Keywords: digital teaching resources, mathematics teaching resources, quality process, textbooks authors' intentions; teacher communities; textbooks' design; teachers' documentation work; textbook use
Introduction: digital means and evolutions of the textbooks' design
As stated in the ICMI study 22 discussion document, "most teachers use textbooks and/or online packages of materials as their total or main source of tasks." Moreover, the textbook content and structure influence teachers' choices (Haggarty & Pepin, 2002; Pepin, 2009), for example, in terms of their choice of tasks or sequencing of the topic area; both content and structure depend on the textbook's design. We investigate textbook design and its developments brought about by digital means. Textbooks are now often complemented by digital materials: files to be projected during the lesson by the teacher; animated figures and exercises using different software and providing feedback to students, available on a CD or on a website. We focus on the following question:
"How can or should new digital formats influence textbook design: e.g. use of podcasts, twitter, and other social media; implications for design and coherence of materials (either original digital design or transfer from print) if teachers are able to select tasks in varied orders?" (ICMI study 22, discussion document, p.19).
We consider that this question is a complex one. It encompasses indeed several aspects of the evolutions resulting from digital means, in particular the Theme C – G. Gueudet, B. Pepin & L. Trouche
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following: digital means provide new opportunities for the structuring of textbooks for their use by teachers. They also open up new possibilities for design and further evolution. In previous works, we provided evidence that digital means foster collective teacher work (Gueudet & Trouche, 2012): teachers discuss by e-mail; or online teacher associations create resources to be shared by all, not only members. Like many other kinds of teaching resources, now the textbook can also be designed by groups of teachers: "bottom-up" designs (in contrast to a traditional "top-down" design) are now developing, in particular in countries where national policies allow it. This constitutes a major evolution, which we retain as a central focus of our study.
Considering these developments, what does the concept of "coherence of a textbook" mean? The discussion document mentions (p. 17), "conceptual coherence." This can be understood as the correctness of the mathematical content; an alignment or 'consistency' with the official curriculum; a sequencing of the notions and properties introduced to avoid gaps in the mathematical progression; a correct articulation between the text of the course and the associated exercises and problems, or an emphasis of 'abstraction' as an 'umbrella concept' for the coherence of the textbook. In the case of a collective design, coherence can also mean that the individual mathematical intentions of the individual authors, their epistemological stances, are well coordinated; and that the whole textbook corresponds to the same "mode of address" (defined as the positioning of the user induced by the material (Remillard, 2012)). Moreover, in terms of implications for learning, the most important issue linked to coherence is certainly the 'coherence' of what the teacher produces when drawing on the textbook. Similar to Shield and Dole (to appear), we consider that analysing textbooks can only inform about their potential to assist in teaching and learning, because teachers interact with textbooks in various ways. Investigating the potential of a resource leads to an exploration of the concept of quality. We consider that the intrinsic quality of a resource has to be distinguished from its adequacy with respect to institutional and users' expectations (Trouche, Drijvers, Gueudet, & Sacristan, 2013). The intrinsic quality encompasses mathematical, didactical, and ergonomic (ease of use) aspects. It also depends on the mathematical topic considered. For example, Trgalová, Soury-Lavergne, & Jahn (2011) assessed the quality of dynamic geometry resources, and they differentiated between nine dimensions of quality: mathematical content; technical aspects; instrumental aspects; added-value of dynamic geometry; didactical implementations; pedagogical implementations; the potential of the resource integration into a teaching process/sequence; ergonomic aspects (e.g., presentation and adaptability); and metadata (e.g., accuracy facilitating searchability).
Traditional textbook analysis (e.g., in TIMSS (Valverde et al., 2002)) proposes three aspects: content (e.g., number, measurement, geometry); performance expectation (knowing, using routine procedures, problem solving, mathematical reasoning, and communicating); and perspective (attitudes, careers, participation, interest, and habits of mind). Most textbook analyses focus on tasks (e.g., exercises and working tasks), and many studies have analysed problem-types (Zhu & Fan, 2006), problem solving procedures (Fan & Zhu, 2007), procedural complexity (Vincent & Stacey, 2008), cognitive demand (Jones & Tarr, 2007) or concept treatment (Cai, Lo, & Watanabe, 2002).
However, how textbooks deal with depth of understanding, for example in terms of mathematical abstraction, is largely left untouched. Without going into deep philosophical discussions, textbooks are without doubt didactical materials, and as such can be seen as providing 'tools and products of abstraction' (e.g., in their tasks, Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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representations, contexts). Textbooks use different registers of representation (Duval, 2006), e.g. usual language symbols; figures; representations of technological tools. This variety, and the need for conversion between different representations, can be associated with 'depth of understanding'. Another way of analysing depth of understanding might be by conceptualising 'understanding mathematics' in terms of 'making connections' (Pepin, 2008): e.g., connections to what the pupils already know; to authentic situations; across mathematical topics; across other subjects. This is in line with the literature on 'learning mathematics with understanding' (e.g., Hiebert et al., 1997). They contend that students build mathematical understanding by 'reflecting and communicating', and tasks should allow and encourage these processes. This means that such tasks should have the following features:
"First, the tasks must allow the students to treat the situations as problematic, as something they need to think about rather than as a prescription they need to follow. Second, what is problematic about the task should be the mathematics rather than other aspects of the situation. Finally, in order for students to work seriously on the task, it must offer students the chance to use skills and knowledge they already possess. Tasks that fit these criteria are tasks that can leave behind something of mathematical value for students." (p.18)
In summary, and considering the above, the notion of 'quality' is complex: it involves the notion of coherence and depth of understanding in textbook analysis, and this in the light of the evolutions brought on by digital means. We discuss this further in what follows, drawing on the comparison of two differently designed textbooks.
The main research questions are: What are the differences between a textbook

designed by a team of experts (researchers, teacher trainers, etc.); and a textbook designed by a group of teachers using a digital platform? What are the 'consequences' of these different designs, in particular in terms of coherence and quality of the textbooks produced?
In the next section, we present the study and subsequently the findings and results. The findings are presented under three headings: (1) the two contrasting teams of authors and their different conceptualisations of their respective textbooks; (2) authors' choices concerning content and structure; and (3) authors' intentions concerning the use of their textbooks.
The study
Our study took place in France, where no institutional control of textbooks exists. France has a National Curriculum, which is presented as a text detailing mathematical objectives and accompanied by detailed comments for teachers, in addition to a booklet giving a structured list of pupil competences. For our study we selected two very different textbooks on the basis of contrasting cases. Both textbooks were grade 6 books (first year of lower secondary school, collège, in France). Because no official statistical figures were available on the most commonly bought mathematics textbooks for this grade, we cannot claim that these textbooks were the most used by teachers. However, they were the most commonly used by the teachers we worked with, and most mathematics teachers in the region knew of them.
The differences between the two books not only concerned their content or the material they offered; the differences were also linked to the authors' teams and the design processes. Therefore, we designed a questionnaire for each of the authors' teams. The questionnaires were designed by drawing on our knowledge of previous studies on textbook analysis (e.g., Pepin & Haggarty, 2001) and on our knowledge of the books, including what teachers told us about their use. The questionnaires Theme C – G. Gueudet, B. Pepin & L. Trouche
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included questions on the textbook's design mode; the authors' perspectives on mathematics and their teaching; and on the design choices, on a general level and on specific aspects of the textbook studied. The results we present here draw on a cursory analysis of the textbooks, but more importantly on the analysis of the responses to the questionnaires. The analysis involved the identification of similarities and differences, category generation and saturation based on constant comparison using a procedure similar to that described by Woods (1996). In particular, we have chosen to focus on the differences in this paper.
Different authors, different conceptualizations
In terms of textbook design, Hélice 6
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(Chesné, Le Yaouanq, Coulange, & Grapin, 2009) has been developed by a team of four "experts": three teacher educators, two of them with a masters' degree in mathematics education; and a researcher in mathematics education. A grade 6 teacher (not considered an author) 'tested' some of the tasks in his class. Asked about the way they evaluated the relevance of the content, the authors declared that they trusted "research results, or [their] training experience" – we call it 'expert evaluation'. These experts were clearly aware of their expert position.
Sesamath 6
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(Sesamath, 2009) has been developed by a large group of authors: approximately 57 lower secondary school mathematics teachers (being involved in producing both the paper version and the digital complements), their work being coordinated by members of the Sesamath association
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: teachers who are involved in the design of online resources. The Sesamath association (Gueudet & Trouche, 2012, Sabar & Trouche, 2013) designed many other resources: online exercises; adigital geometry software; and a complete virtual environment, LaboMep
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. LaboMep allows the co-ordination of various kinds of resources, from Sesamath or from other sources on the web, and their preparation for student use. The "Sesamath 6
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" textbook was published under a free license; it can be downloaded from a website
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, or used online (Figure 1). A paper textbook exists, corresponding to the text files (which are available in .pdf and .odt). Other complementary files (e.g., dynamic geometry files, spreadsheets, slides, online exercises) can only be accessed using the website.
In terms of evaluation of the content relevance, the teacher authors and other members of the association used the developed textbook in their classrooms and observed and evaluated it, in particular in terms of their students' involvement with particular features of the book. Referring to a distinction introduced in the field of computer supported collaborative learning (Dillenbourg, Baker, Blaye, &O'Malley, 1996), we consider that the collective work of the authors in the case of Hélice was collaborative (the authors sharing each part and step of the work). In the case of Sesamath, we consider the authors' collective work as co-operative: different tasks had been assigned to different authors.
Another important difference which we develop in the discussion was that the content of the digital textbook continuously evolved, according to the experiments

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The name of the association itself, Sesamath, is interesting and linked to « Open sesame », the famous phrase from the Arabian Nights. The motto of the association is "mathematics for everybody".
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LaboMep- Laboratory for Mathematics in the Pocket
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http://manuel.sesamath.net/Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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and contributions of the teacher users. In fact, Sesamath proposed a website, 'Sesaprof', which was open for teachers and comprised of a 'discussion forum' section (discussing the textbook). These online discussions led to modifications of the textbook's content: for example, when a large number of users asked for the solutions of the textbook's exercises, these were added in the digital textbook.
Beyond these differences, both teams claimed that they were constrained by commercial publishing and user expectations. For example, the Hélice team explained that they had wanted to write more guiding comments for the teacher, but had been asked by the publisher to limit these comments. For the Sesamath team, the grade 6 book was the last in a series; they had started with and accomplished textbooks for grades 7, 8 and 9 (in this order) due to curriculum changes. Thus, the structure of these earlier textbooks gradually became a model familiar to users, and the authors were obliged to keep a similar structure for the grade 6 textbook.
Figure 1. Sesamath online textbook for grade 6. The text on the screen background corresponds to the content of the .pdf, or paper version. The "complements" window opens when the mouse is placed on a selected activity.
Authors' choices concerning content and structure
The analysis of the textbooks and the authors' answers to the questionnaire showed many differences concerning the textbook content. We illustrate these statements with examples from the mathematical topic of 'area'.
Organisation of the content
Concerning the textbook's global structure and the organisation of the mathematical content, Sesamath provided a "classical" organisation: chapters were organised according to the headings of the official National Curriculum. As an example, the topic of 'area' was in the section entitled "Area and Perimeter", which was the second section of the chapter entitled "Quantities and Measures".
Hélice had a very special, spiral organisation, hence the title "Hélice" (meaning "Helix"). The book was not structured in chapters, but in "units", working with different 'thematical' lessons, and a 'unit' finished with problems and exercises linking the different notions learnt. The authors of Hélice presented this (spiral Theme C – G. Gueudet, B. Pepin & L. Trouche
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organisation) as their central and original structure. Indeed, in France Hélice was the only textbook retaining this composition. The authors specified that the intention had been that the learner revisited and deepened the same notion at different stages – a spiral curriculum. At every stage the notions were associated with different representations; links between different chapters were frequently made; and differentiation in terms of pupil learning (e.g., pace) was taken into account. The topic 'area' appeared twice in the table of contents: in the unit entitled "Distance and areas" (Unit 4; lesson called "Area: comparison and sharing") and in the unit entitled "Division and computation of area" (Unit 5; lesson called "Areas: measure and computation"). Moreover, the area of a circle was in fact presented in Unit 7, as an activity demonstrating that the area of a circle and the square of its radius are proportional.
We claim that this difference in terms of structure was influenced by the different design modes and author teams. Following a complex and coherent structure, such as the one retained by Hélice, was only possible for a 'steady' author team, and likely to be very difficult for a large and 'variable' author group, such as the Sesamath group. A large 'collective' process, like Sesamath, required splitting the content into different parts, which then were designed by different authors who did not necessarily communicate. This splitting of tasks, we argue, has also influenced the coherence of the textbook. Bringing together the work of more than fifty authors and achieving a coherent didactical structure would require an enormous coordination effort, more than the one organised by the Sesamath team.
Different methods, or a single expert method
Hélice authors stated that it was important for them to propose different representations and a rich vocabulary in their book. They also favoured exercises with different possible solutions.
In Sesamath, our analysis showed that some additional research activities suggested and fostered the search for several solutions. However, we contend that most of the exercises led to a single and final solution. In particular, the worked examples were all one-solution exercises, the authors called them 'expert solution' in the questionnaire (Figure 2).
Figure 2. Extract of Sesamath 6
e
. Method 2: evaluating an area.
Example 1. Determine the area of this figure by choosing the yellow triangle as the unit, then calculating the area of this rhombus. To find the area of the figure, you have to count the number of area units of the figure. The purple figure is made of 9 triangles. Its area is thus the area of 9 yellow triangles. The rhombus is made of two yellow triangles. The area of the purple figure, in number of rhombi, is thus twice as smaller. Hence the area of the figure equals 4.5 rhombi. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Primary-secondary link
Another important aspect, given that the textbooks were for grade 6, was whether the primary-secondary link was explicitly addressed. Some of the Hélice authors were teacher educators, both for secondary and primary school level. Thus, they had considerable knowledge about the content taught at primary and secondary school. They identified several crucial cognitive and didactical changes in terms of the transition from primary to secondary school mathematics. Concerning 'area', one important choice, according to them, was to start with activities on the comparison of areas of two figures, without using any kind of measures. The activity in Figure 3 illustrates this choice.
Figure 3. Extract of Hélice 6
e
. Lesson 12. Areas: comparing and sharing.
Activity 2: Cutting and arranging. The objective of the activity is to demonstrate that all these figures have the same area as the square, by cutting and adjusting to obtain a superposition with the square.
The official curriculum mentioned "geometrical comparison of areas" as a teaching objective. This aspect was particularly developed in Hélice; and this was a deliberate choice of the authors, in order to provide a better link to primary school notions where comparing areas without using measure was a common task. Sesamath authors declared that they had only limited knowledge of primary school mathematics curricular aspects when they started writing the grade 6 book. Concerning 'area', the textbook focused on area computations, whereas the official curriculum also stipulated "geometric comparison of areas". This geometric comparison, without measures, was not evident in the Sesamath textbook. With a better knowledge of primary school, where such tasks were frequent, the authors may have inserted it. However, the Sesamath authors paid attention to one 'classical' difficulty: the potential problem to distinguish between area and perimeter of a figure. This difficulty was known to be important for primary school, and also for grade 6 students, and as teachers of grade 6 students the textbook's authors knew about it. Writing the book raised the authors' awareness of transition questions (e.g., the importance of mental arithmetic for grade 6.). The paper version of Sesamath did not reflect this, but in the associated online resources, in particular online exercises, the authors attended to this aspect of primary-secondary transition. Whilst the paper book has remained the same since 2009, the authors claimed that they had attended to particular 'shortcomings' and that the online complements had considerably developed since that time (e.g., in LaboMep many online exercises of mental arithmetic for grade 6 had been added). Theme C – G. Gueudet, B. Pepin & L. Trouche
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Authors' intentions concerning the use of their textbooks
In the questionnaire, the Hélice designers adopted a general stance about teachers' adaptations of the book to their specific contexts. They said that "it [was] impossible to anticipate all possible adaptations" (in terms of contexts) and that they did not regard it as their "responsibility" to attend to these. This view was reflected in several statements throughout the questionnaire, e.g. the following: "the gap between what is planned and what happens in class is large - the gap between the authors' intentions and the teachers' use is even larger". They also declared that they would anticipate that teachers would combine Hélice with the use of other textbooks. However, the spiral progression made such a practice difficult and the complex spiral structure was clearly an obstacle for the adaptation, or combination of several textbooks.

For Sesamath, the possibility of adapting the content was an important issue.
The authors conceptualised the book, from its inception, as a digital textbook, each
chapter file in .odt format. This offered opportunities for teachers to modify the texts
of the exercises, and of the lessons. The book also offered a large amount of exercises,
with the intention of leaving their choice to teachers. According to the Sesamath
authors, the digital textbook was a "collection of bricks". Moreover, they stated that it
should be thought of as belonging to a more general set of different kinds of resources
available in the virtual environment LaboMep. Sesamath authors considered that it
was the teachers' responsibility to 'build coherent lessons' and a 'coherent
progression'. In their view this was made possible by providing a large range of
resources to choose from, the textbook being only one of these. In addition, they
wanted to provide 'efficient tools', such as LaboMep, for teachers to build their own
teaching from these 'initial bricks'.
Discussion
Going back to our initial questions ("What are the differences between a textbook
designed by a team of experts (researchers, teacher trainers, etc.) and a textbook
designed by a group of teachers using a digital platform? What are the 'consequences'
of these different designs, in particular in terms of coherence and quality of the
textbooks produced?), we answer these by comparing the two contrasting textbook
cases (Helice and Sesamath).
The investigation of the two textbooks showed that there are differences
concerning essentially six levels (at least):

• the modes of design: more collaborative in the first case, more co-operative in
the second;
• the nature of the structure: the first book is a single whole, with an organised
structure (organised by the team of experts); the second is an atomistic system
that can be arranged differently by different users;
• the organisation of the content: more didactically original, linked to the
didactical choices of a 'homogeneous' team in the first case; more aligned
with the institutional instructions in the second;
• the content: more open to a variety of ways for solving a given problem in the
first case; more driven by an expert solution in the second;
• the integration into the whole grades 1-9 mathematics curriculum; links with
primary school more taken into account in the first case than in the second;
and Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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• the links to the users; the textbook provided as a final product given to the
teachers in the first case; and as a proposal to be enriched by teachers'
contributions in the second.
For Helice, the coherence is insured by the authors' didactical expertise, i.e.
the mastering of the concepts at stake and of the potential difficulties and
misconceptions for learning. It could be said that it is a transcending coherence. For
Sesamath, the coherence is insured by the link with the curriculum and the
institutional prescriptions; by teacher evaluation in class; and by the discussions
among authors faced with the different contributions. Sabra and Trouche (2011)
describe, for example, the discussion in another Sesamath author team (for a grade 10
book) in terms of reaching a coherence and consistency between the introduction of
equations (in one chapter), and the introduction of function (in another chapter). We
argue that this is a collective and institutional coherence.
Helice was, we claim, of high didactical quality. It offered many rich tasks,
organized according to a carefully considered and complex structure. It took into
account central aspects of the primary-secondary transition. The Sesamath textbook
appeared to be, in its initial version, of a lower intrinsic quality: it offered fewer
problems and fewer rich tasks. In terms of structure it simply followed the structure of
the official National Curriculum.
However, the "digital additions" and possibilities of Sesamath prompted us to
re-consider the notions of 'quality' and 'coherence'. Helice remained the same, in
both its paper and pdf versions, we call this static quality. In contrast, the online
version of Sesamath had already been modified several times, to take account of 'user
comments', i.e. users' experiences and needs. The digital means offered possibilities
for modifications, and these were integrated by Sesamath in the process of re-design.
The association perceived this as a necessity for meeting users'/teachers' needs in
order to insure the quality of the textbook, we call this dynamic quality. Both Helice
and Sesamath authors recognized that teachers would select and adapt elements of the
textbook for their teaching. However, only Sesamath supported these adaptations and
drew on user contributions.
We argue that this was a major development linked to digital means: the
involvement of a large group of teachers in the design of resources, which in turn
continuously evolved. This also deepens our knowledge of already established
phenomena: even teachers who were not involved in the design of textbooks could be
considered as 'designers' of their own teaching materials, as teachers selected
resources, combined them, and set them up in class – a process that Gueudet &
Trouche (2009) called teacher documentation work. This leads us to consider that the
question raised in the discussion document, "[Which] implications for design and
coherence of materials […] if teachers are able to select tasks in varied orders?",
does not sufficiently recognize the complex link between 'design' and 'coherence'
and the evolving role of teachers. Exploiting the potential of digital means for the
design of teaching resources, including textbooks, requires the acknowledgement of
this new role.
Drawing on the results presented here, we argue that textbooks can be
considered as lived resource (Gueudet, Pepin, & Trouche, 2012), or even living
resources, as they get continuously enriched and renewed by teachers' experience.
This new conceptualization of the textbook is likely to be associated with new forms
of design, for example, in terms of reflection on meta-design (Fischer & Ostwald,
2005), resources which support the design by teachers. It can also lead to favour Theme C – G. Gueudet, B. Pepin & L. Trouche
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teams of textbooks designers who combine and involve different 'experts' and
expertise (e.g., teachers and other 'experts' such as researchers). From the literature
(e.g., Kieran, Tanguay, & Solares, 2012), it is clear that teacher documentation work
and professional knowledge are intrinsically intertwined, one leading to the evolution
of the other (e.g., Gueudet & Trouche, 2012; Pepin, 2012). This has implications, also
in terms of policy, for teacher professional development (of 'users' and of
'designers'); mathematical task design and digital means/possibilities; and the reconceptualization of the quality of resources in mathematics education.
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Pedagogical Content Analysis of Mathematics as a
Framework for Task Design
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This paper describes a pedagogical content analysis framework
for task design in mathematics. This framework has been developed and
applied through the author's ongoing work to develop textbooks,
classroom materials, professional development materials and programs,
and standards, particularly in the area of discrete mathematics.
Keywords: mathematics, discrete mathematics, tasks, teaching,
learning, textbooks, curriculum
Introduction
Mathematics instructional tasks are at the heart of teaching and learning mathematics.
Effective tasks are the core of good textbooks, classroom lessons, and professional
development programs. Designing worthwhile tasks is challenging, but doing so has
immense educational payoff. Student learning is developed, deepened, and extended
through engagement in rich mathematical tasks; teacher skill and knowledge are
strengthened through teaching with and designing tasks; textbooks become teachingand-learning books rather than just reference books and sources for homework
problems when they are developed around worthwhile instructional tasks.
It is difficult to design mathematics tasks that are effective for teaching and
learning. This paper describes a useful design framework developed and applied by
the author to design sequences of tasks in a variety of settings: for textbooks, through
the Core-Plus Mathematics Project and the Transition to College Mathematics
Project; through individual tasks for classroom lessons and teacher professional
development; through projects for the U.S. state of Iowa (Every Student Counts and
Iowa Core Mathematics); through projects for the National Council of Teachers of
Mathematics (Illuminations website project); and through a state-funded lesson study
project (Important Mathematics and Powerful Pedagogy). Most of the tasks are in the
area of discrete mathematics, designed to be used at the high school level.
There are many types and sizes of mathematical tasks. In this paper, the
discussion ranges from sequences of tasks to individual tasks, including tasks that
generate or clarify a definition and those that develop major concepts or procedural
skills. Task in this paper means textbook material or classroom lesson material that a
teacher uses to help students gain a deeper understanding of mathematics. Several of
the examples are what can be called "problem-based instructional tasks," as defined in
Iowa Core Mathematics (2010, p. 3):

Problem-Based Instructional Tasks:
• Help students develop a deep understanding of important mathematicsTheme C – E. W. Hart
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• Emphasize connections, including to the real world
• Are accessible yet challenging to all
• Can be solved in several ways
• Encourage student engagement and communication
• Encourage the use of connected multiple representations
• Encourage appropriate use of intellectual, physical, and technological tools.
Effective mathematical instructional tasks must take into account the nature of mathematics and knowledge, as well as issues of teaching and learning. A key idea in all these domains is connections. Mathematics is a connected and coherent subject, it should be taught that way so students can learn it that way. As Schoenfeld (2009) states, "I truly believe that (a) mathematics is special because of the way it coheres-it really does make sense!-and (b) mathematics can be taught so that students come to see it so" (p. 168). Similarly, "mathematics is an integrated field of study.... When students can connect mathematical ideas, their understanding is deeper and more lasting" (National Council of Teachers of Mathematics, 2000, p. 64).
Understanding is of course our goal. A key perspective on understanding is the construction of knowledge. "Understanding involves the construction of knowledge by individuals through their own activity, so that they develop a personal investment in building knowledge" (Carpenter & Lehrer, 1999, p.23). As stated by Gardner, "When you've encountered an idea in your own way and brought your own thinking to bear, the idea becomes more a part of you ... it's a part of your own experience" (1993, p. 6). Teaching for understanding is not just a platitude, there is a large body of research that helps identify what it means, why it is effective, and how one does it (for example, see Hart, 2010). Teaching for understanding can be operationally defined as in the Every Student Counts Project (ESC, 2008, p. 3):
Teaching for Understanding means
• Developing deep conceptual and procedural knowledge of mathematics
• Posing problem-based instructional tasks
• Engaging students in the tasks and providing guidance and support as they develop their own representations and solution strategies
• Promoting discourse among students to share their solution strategies and justify their reasoning
• Summarizing the mathematics and highlighting effective representations and strategies
• Extending students' thinking by challenging them to apply their knowledge in new situations, including in real world situations
• Listening to students and basing instructional decisions on their understanding.
The design framework described in this paper takes the above perspectives into account. The resulting tasks are used in textbooks (Fey et al., 2010; Hirsch et al., in press), in teacher resource books and journals (e.g., Hart, Kenney, DeBellis, & Rosenstein, 2008), in classroom lessons (e.g., Resources for the Iowa Additions of Iowa Core Mathematics, 2012), and as objects of reflection for professional development (e.g., in the Every Student Counts and Important Mathematics and Powerful Pedagogy projects). They are used by teachers to teach, students to learn, educators to provide professional development and teacher training, and scholars to study (e.g., Ziebarth et al., 2009). The framework will now be described, with examples. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Pedagogical content analysis framework for task design
This framework is based on analyzing the target mathematics (i.e., the specific mathematics to be learned in the task or sequence of tasks) from the perspective of mathematics, learning, teaching, curriculum, tools, models, and applications.
Outline of the framework
The framework can be outlined as follows. First, for the targeted mathematical content, analyze the mathematics by answering these questions:
• What is it? [deeply, simply, in essence]
• How do you do it? [compute it, compute with it, find it, operate on it, operate with it]
• What is it connected to? [interconnected web of mathematical ideas, concepts, methods, representations]
• What is it good for? [applications, contexts, models]
Next, analyze the mathematics pedagogically. That is, understand, identify, and develop strategies for addressing:
• Common misconceptions
• Common student content difficulties
• Sequencing
• Questioning
• Scaffolding
• Tools
• Pedagogical mathematical language.
These analyses are carried out with more or less formality. If time permits, a lesson study approach is very effective. It can be helpful to engage in an explicit process of unpacking and then repacking. The analysis may also take place informally in real time during teaching. In all cases, ultimately the perspective and process become a habit of mind and habit of practice.
Example: multiplicative inverses in modular arithmetic (Zn)
In this example, we see instances of analyzing the mathematics, and addressing the issues of misconceptions, common difficulties, sequencing, scaffolding, questioning, and tools. A sequence of tasks is designed through a process of systematic analysis, development, and revision based on results from classroom tryouts. The tasks are designed for use with high school students in a fourth-year (final year) math course that is an alternative to a pre-calculus course, but with similar prerequisites.
As background to this sequence of tasks, previous analysis and design includes establishing an engaging context (cryptography, in particular the RSA public-key cryptosystem), and developing a prior sequence of tasks for the ideas of equivalence mod n, reducing mod n, addition and multiplication mod n, and the integers mod n with the notation Zn. Also, although the target is multiplicative inverses, because this is what is needed for RSA, additive inverses are also developed to provide an accessible lead-in to multiplicative inverses and to more fully develop the key mathematics.
To begin, a mathematical analysis includes considering the four questions outlined above: What is a multiplicative inverse in Zn? How do you find a multiplicative inverse? What are connections to other and previous ideas in mathematics? What are multiplicative inverses in Zn good for? Of course, a huge Theme C – E. W. Hart
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amount of mathematics emerges from the answers to these questions. This can be thought of as "unpacking." Then we must "repack" to identify the target mathematics for this particular sequence of tasks as it is situated in a particular curriculum and classroom. In this case, the target mathematical ideas are analyzed to be (a) multiplicative inverses in Zn are similar to and different from multiplicative inverses for real numbers with "regular" multiplication, (b) not every number in Zn, for all n, has a multiplicative inverse, and (c) there are important properties of numbers that determine if a multiplicative inverse exists.
The most fundamental and universal design decision is that all tasks are structured sequences of questions. Key decisions pertain to the issues of which questions to ask, in what sequence, and with how much structure.
The first design decision is easy – additive inverses are easier and will help lead into the more difficult idea of multiplicative inverses. Even here, careful decisions are made. For example, how much mathematical formality should be developed, in particular, where and how should the idea of additive identity fit in? How much scaffolding should be provided? Scaffolding refers to guidance in terms of open-endedness of questions and size of steps in the step-by-step sequence of questions.
The task is initially designed with no explicit mention of additive identity, just use of 0, and little scaffolding. The initial attempt is based on trying out the hypothesis that students will find additive inverses in Zn to be straightforward analogs of additive inverses with real numbers. Thus, after a question about informally finding additive inverses with real numbers, students are asked to find additive inverses in Zn, for a few different values of n:
a. What is the additive inverse of 6 in Z10?
b. What is the additive inverse of 3 in Z8?
Classroom testing showed that students struggled mightily, they could not make the jump from their previous work with real numbers to inverses in Zn. So the questions were redesigned to provide more scaffolding initially, and then gradually relax the guidance:
2a. Find a number in Z10 that you can add to 6 to get 0 mod 10. Such a number is the additive inverse of 6 in Z10.
2b. Find a number in Z10 that you can add to 2 to get 0 mod 10. Such a number is the additive inverse of 2 in Z10.
2c. What is the additive inverse of 3 in Z10? Explain.
2d. What is the additive inverse of 3 in Z8? Explain why this answer is different than the answer you got for the additive inverse of 3 in Part c.
Students seemed to have no difficulty with the idea of an additive identity (probably because it is 0 for real numbers and Zn). For the target high school students, formal understanding of mathematical structures, such as rings and fields, is not deemed appropriate. Nevertheless, for the integrity of the mathematics, additive identities are also more formally defined in the first question about real numbers:
1. Think about additive inverses. The additive inverse of a number is the number that you add to it to get 0. For example, with real numbers, the additive inverse of 3 is -3, since 3 + -3 = 0. The number 0 is called the additive identity. Find the additive inverse of each of these real numbers:
5.

.
, and -1.5. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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After the previous two tasks, students are getting comfortable with inverses in modular systems. They see some differences compared to additive inverses with real numbers, but it is still true that every number has an additive inverse.
Now to the harder idea of multiplicative inverses. Through an iterative process of pedagogical content analysis and classroom testing, the following design issues and decisions emerged.
Start with a task similar to the first additive inverse task (#1 above), to anchor the investigation in students' past experience with real numbers. For Zn, begin by investigating Z7, a prime modulus, in which all non-zero numbers have a multiplicative inverse, so that students get introduced to the idea in the simplest setting. They only have to deal with one new idea – a modular system, without yet needing to face the issue that some non-zero numbers in some systems do not have a multiplicative inverse. This task ends with a question to find the multiplicative inverse of 6 in Z7. Because 6 is its own inverse, students are confronted with their next carefully-planned disequilibration, because no real number (other than 1 and -1) is its own multiplicative inverse.
Which modular system would be best to investigate next? The first try is Z6. This choice was quite confusing for students. Why might this be so? Part of the reason might be because there are too few numbers with inverses, only 1 and 5, and both are their own inverse. This is a bad pedagogical step because the example is too special; it only illustrates numbers that are their own inverses, and thus raises the danger of students' over-generalizing this pattern.
The next try is Z8. This choice is better because more numbers have inverses. However, the numbers without inverses are all the even numbers, which could again lead to over-generalizing a pattern. Once again, every number with an inverse is its own inverse. Also, the modulus is a power of 2, that is, the only prime factor is 2, which will not be helpful when looking for properties of numbers that do and do not have inverses. Thus Z8 also is pedagogically problematic because it has too many special characteristics.
The analysis thus far points to the need for a modular system in which the modulus has more than one prime factor, not all numbers with inverses are their own inverse, not all numbers without inverses are even, and the modulus is not too large so that computing is not a barrier. This leads to the decision to use Z10 next.
Now the next big step is to find a pattern for which numbers have an inverse and which do not. Consider Z10. The following design decisions are made:
• For the first time thus far, give students an empty multiplication table, ask them to complete it, and see which numbers have a multiplicative inverse and which do not. It is an important decision when to give support, such as this multiplication table. In previous problems, the intent is for students to think it through on their own in their own way. They may decide to use a multiplication table or not. Such decisions are important for students to consider before the decision is made for them.
• Ask first about patterns for numbers that do not have multiplicative inverses. [2, 4, 5, 6, 8]
• Then, after students work for a while, if they need more guidance, ask progressively more focused probing questions:
How do such numbers relate to 10?
Do you see any factors of 10? Are they all factors of 10?
Do you see any connections between the factors of 10 and the factors of these numbers? Theme C – E. W. Hart
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• Ask for patterns for numbers that do have a multiplicative inverse.
At this point, referring back to the three target mathematical ideas identified at the beginning of this discussion, students are expected to have made substantial progress learning that (a) multiplicative inverses in Zn are similar to and different from multiplicative inverses for real numbers (in ways that students can describe), (b) not every number in Zn, for all n, has a multiplicative inverse, and (c) there are important properties of numbers that determine if a multiplicative inverse exists. Teachers may decide that complete attainment of (c) is not necessary, or that it is not attainable in a timely manner, for their students. To help provide differing support for this goal, for some or all students, the design decision is to first ask for a pattern in Zn in an open-ended question. Then provide guidance to help get the general result, as follows:

You discovered in previous problems that not all numbers in a given
modular system have a multiplicative inverse. Think about when multiplicative
inverses exist in Zn.
a. Make some conjectures about which numbers have multiplicative inverses
in Zn, either for a general n or for particular values of n. For each
conjecture, try to prove it or disprove it. (You can disprove it by finding a
counterexample.)
After trying some of your own conjectures, complete and prove the
following three statements.
b. When n is _____, then every nonzero integer in Zn has a
multiplicative inverse.
c. If n and m have a particular relationship to each other, then m does not have
a multiplicative inverse in Zn. What is that relationship?
d. m has a multiplicative inverse in Zn if and only if _____.
This concludes this example of designing a sequence of tasks using the
framework of pedagogical content analysis. We have seen an application of the
mathematical analysis outlined in the description of the framework, along with
consideration of many of the issues in the pedagogical analysis. These issues and
examples from this sequence of tasks include common misconceptions (overgeneralizing, for example, that all numbers have a multiplicative inverse), common
student content difficulties (making the jump from inverses for real numbers to
inverses in Zn), sequencing (doing additive inverses before multiplicative inverses, or
asking about patterns for numbers that do not have inverses before patterns for those
that do), scaffolding (numerous examples of strategically providing more or less
guidance), tools (when to introduce a multiplication table), and questioning (examples
of trying to find the right question, at the right time, with the right open-endedness).
Brief examples illustrating aspects of the pedagogical content analysis
framework
The one issue not yet addressed in this task design framework is so-called
mathematical pedagogical language. This refers to the use of language that is
mathematically accurate, though not conventional, and it is pedagogically powerful.
This is language that helps students understand a mathematical idea through the name
of the idea or the notation used for the idea.
For example, consider graph theory. The word graph is used in many ways,
and for students it usually means a graph in the coordinate plane, or perhaps a data
plot. But these are not at all what is meant by graph in graph theory, where a graph is Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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a collection of vertices and edges. So why not modify the language, maintaining
accuracy, to make the name more sensibly reflect the object. Thus, the term vertexedge graph is used when designing tasks for high school students in graph theory.
Another example is recursion. This idea, before it is mathematically
formalized, is intuitive for most students. They find it quite natural, for example, to
think of the next counting number in terms of one more than the number you have
now. So why not make use of the language of “next” and “now.” Consider the
following incomplete function table:
x y
0 3
1 7
2 11
3 15
4 19
Based on many classroom trials, when students are asked to find patterns in
this table, they are likely to say things like “it goes up by 4” or “add 4 each time.”
That is, they are seeing the recursive pattern in the y-values. When asked to write an
equation that describes that pattern using the words NOW and NEXT, students have
little trouble writing NEXT = NOW + 4. In contrast, students have great difficulty
with the conventional subscript notation for recursion. Of course, students should also
be able to find the x-y pattern: y = 4x + 3. To see one strength of the NEXT–NOW
language, notice how concretely and meaningfully the slope is shown in the recursive
equation, as opposed to the more abstract representation of slope in the y = 4x + 3
equation.
This example illustrates some of the characteristics of pedagogical
mathematical language, and why it is an important feature of task design. The
language must be mathematically accurate and also pedagogically powerful. In the
case of NOW/NEXT, this language captures the essence of recursion, as used to
describe sequential change. The language helps make an otherwise notoriously
difficult idea accessible to all students. It also has substantial payoff in terms of
deepening understanding of linear and exponential functions, constant rate per unit
interval versus constant percent rate per unit interval, and making concrete the
connection to arithmetic and geometric sequences. It promotes “semantic learning” as
opposed to only “syntactic learning” (the latter is a danger when moving too fast to
subscript notation). It is a “bridging language” in that the formal notation will be used
at an appropriate later time. Students will definitely need understanding and facility
with subscript notation to utilize the full modelling power of recursion. Finally, note
that such pedagogical mathematical language may have limitations that must be
carefully monitored. In the case of NOW/NEXT, this notation is not so useful for
modelling change related to quadratic functions, and one must be careful to consider
the incremental change in the x-values as well when looking for a NOW/NEXT
relationship in the y-values.
For one last brief example, consider a lesson on the slopes of perpendicular
lines developed as part of a lesson study project. The first iteration of the lesson used
transparencies to rotate several lines 90˚, gather and record the data on slopes of pairs
of perpendicular lines, look for a pattern, and conclude that the slopes are negative
reciprocals. After classroom tryouts, several features in the pedagogical content
analysis design framework became particularly evident.
First, the use of the transparencies (an instance of the tool issue) proved to be
confusing to students and needed to be refined. For example, if no axes are drawn on Theme C – E. W. Hart
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the transparency, which is on top of the paper with a line drawn, then when the paper
is rotated students get confused about how to interpret the new position of the axes. If
the transparency is rotated, then students are confused because it seems to them like it
is the “same” line so it must have the same slope. The best solution proved to be
putting axes on the transparency, and then rotate the paper underneath. This way, two
lines appeared on the transparency with the axes fixed. Such small adjustments can
often make a big difference in a lesson. Second, it turns out there had been a lack of
mathematical analysis, specifically with regards to the first question outlined in the
description of the framework, namely, with respect to the target mathematics, What is
it? In this case, the answer to the question, “What is the slope of a perpendicular
line?,” is not that it is a pattern in the data. Rather, the slope is a consequence of the
definition of slope and the nature of a 90˚ rotation, which is exhibited as a pattern in
the data. This was an important learning episode for both teachers and students – that
finding patterns in data is important, but you must try to understand those patterns in
terms of the underlying mathematical concepts and relationships that caused the
pattern.
Conclusion
This paper presents a framework for task design in mathematics that has been
gradually developed and applied over the last two decades by the author in work on
several research and development projects for textbooks, classroom lesson
development, and teacher professional development programs. The framework is
called a pedagogical content analysis to highlight that it involves both analyzing the
mathematics and analyzing the mathematics pedagogically. Through iterations of task
design and classroom tryouts, specific features that operationally define this
framework have been identified, refined, and applied.
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Task Modification and Knowledge Utilization by Korean
Prospective Mathematics Teachers
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It has been asserted that mathematical tasks play a critical role in
the teaching and learning of mathematics. Modification of tasks included
in intended curriculum materials, such as textbooks, can be an effective
activity for prospective teachers to understand the role of mathematical
tasks in the teaching and learning of mathematics; designing of new tasks
requires more knowledge and experience. This study aims to identify the
patterns that Korean prospective mathematics teachers seem to follow
when they modify the mathematical tasks in textbooks. Knowledge
utilized by prospective teachers while they modify textbook tasks is
identified and characterized to understand the possible factors that have
impact on Korean prospective mathematics teachers' modification of
tasks.
Keywords: task modification, prospective teachers, teacher knowledge
Introduction
It has been asserted that mathematical tasks play a critical role in the teaching and
learning of mathematics (Crespo, 2003; Henningsen & Stein, 1997; Stein, Grover, &
Henningsen, 1996; Watson & Mason, 2007). Henningsen & Stein (1997), for
example, pointed out that the nature of tasks may provide different opportunities to
develop students' mathematical thinking and reasoning skills. Furthermore, they
argued that teachers' goals, mathematical knowledge, and knowledge of students can
influence how the teachers set up the tasks included in intended curriculum materials,
such as textbooks. This indicates that students' mathematics learning can be positively
influenced by tasks that teachers modify or develop through their content and
pedagogical knowledge. Therefore, the issue of teachers' competency to modify and
pose meaningful tasks needs to be addressed more actively in teacher education.
However, according to Watson & Mason (2007), who reviewed the submitted
proposals for the special issue of the Journal of Mathematics Teacher Education, only
a few papers dealt with issues in teachers' critical analyses such as task analysis, task
evaluation, or posing tasks. This tendency is in line with the comment by Prestage and
Perks (2007), which is that the issue of how teachers pose and organize tasks to
educate students in classrooms has been ignored. Theme C – K.-H. Lee, E.-J. Le & M.-S. Park
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Teachers' ability to analyze and modify tasks to be used in the classroom or
pose worthwhile tasks may be a critical factor affecting students' mathematics
learning. However, such skills cannot be developed in just a short time, so teachers
need continuing and systematic support to develop the skills (Crespo, 2003; Prestage
& Perks, 2007).
In the case of Korea, especially, most teachers, in fact about 90% of secondary
mathematics teachers, tend to use textbooks as a major resource in classrooms
(Mullis, Martin, Gonzalez, & Chrostowski, 2004). Therefore, it seems very important
for prospective teachers to have opportunities for performing critical analyses of tasks
and the modification of them. The purpose of this study is to identify the patterns that

Korean prospective mathematics teachers seem to follow when they modify the mathematical tasks in textbooks. Knowledge utilized by prospective teachers while they modify textbook tasks is identified and characterized to understand the possible factors that have impact on Korean prospective mathematics teachers' modification of tasks.

Theoretical Background

Tasks can be designed in different forms according to the objective and orientation, and every single task form provides a different learning opportunity for students. Prestage & Perks (2007) suggested that transforming only the conditions of closed problems in school textbooks by adding or deleting conditions can foster students' mathematical thinking. When conditions are deleted, students can decide for themselves. When conditions are added, students can extend the knowledge from the national curriculum. Thompson (2012) attempted the transformation of textbook exercises according to the following strategies to reflect the new curriculum's point of view, which emphasizes both reasoning and communication: a) reframe a basic problem by including one or more conditions; b) use relationships to find patterns or predict other results; c) generate conjectures for students to investigate; d) encourage students to solve a problem in multiple ways; e) evaluate student solutions; f) write a question appropriate for a given answer; and g) connect procedural and conceptual knowledge. The different learning opportunities from each task can be identified when tasks in textbooks are modified to require high cognitive demand (Kaur & Yeap, 2009) or to develop critical thinking and creativity (Krulik & Rudnick, 1999). Competency in task design has been taken into consideration in mathematics teacher education. Zaslavsky (2008) showed changes in the teaching and learning of mathematics after task modification. Analyzing different versions of tasks promotes teachers' development of adaptability; fosters their awareness of similarities and differences; helps them learn how to cope with conflicts, dilemmas and problem situations; encourages them to learn from the study of practice; teaches them to select and use (appropriate) tools and resources for teaching; helps them to identify and overcome barriers to students' learning; and allows them to share and reveal self, peer, and student dispositions. Teachers can foster their professionalism after analyzing the tasks. Voica & Pelezer (2009) investigated the difference between prospective teachers' and inservice teachers' task design of problem posing. Inservice teachers considered the students, reflected the curriculum, and applied their pedagogical knowledge and mathematical knowledge, whereas prospective teachers mostly focused on the context without considering the students or the level of difficulty. Differences between inservice and prospective teachers are typically caused by differences in teacher knowledge. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.

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Ball and her colleagues suggested Mathematical Knowledge for Teaching (MKT) based on Shulman's categories of teacher knowledge (Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2008). According to them, MKT is categorized into two main domains: Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). SMK is subdivided into three categories: Common Content Knowledge (CCK), Specialized Content Knowledge (SCK), and Horizon Content Knowledge (HCK). PCK is also subdivided into three categories: Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT), and Knowledge of Content and Curriculum (KCC). Each type of teacher knowledge constituting MKT is explained briefly as follows (see Ball et al., (2008) and Hill et al., (2008) for more details). CCK, which is included in SMK, is defined as mathematical knowledge regardless of teaching, whereas SCK is mathematical knowledge and skill for teaching. To distinguish mathematical knowledge for teaching as a mathematics teacher from mathematical knowledge, SMK was conceptualized by Ball and her colleagues, who emphasized the importance of SMK. KCS, which is included in PCK, is the combination of knowledge about students and knowledge about mathematics; for example, teachers with KCS can identify and predict the common errors that students are more likely to commit as well as their misconceptions. Finally, KCT is defined as the combination of knowledge about teaching and knowledge about mathematics. Teachers who possess KCT can identify appropriate examples they should first use to help students understand mathematical concepts or appropriate teaching sequences and methods for the design of instruction.

Method

Participants were 38 prospective secondary mathematics teachers who enrolled in the course, "A Study of Teaching Materials and Teaching Methods," taught by the first author of this paper. In the course, participants learnt about the Korean mathematics curriculum and teaching materials. At the end of this course, they were asked to analyze the tasks presented in the introduction of each unit of the secondary textbooks and modify them. The selection of tasks and the direction of modification were left up to the prospective teachers. Hence, they chose tasks from different units, such as function, geometry, algebra, or numbers; in particular, the function and geometry units were most frequently selected.

Most of the prospective teachers analyzed and modified more than 2 tasks, so all of the tasks that were done by each participant were analyzed in this study. In addition, the prospective teachers tended to use different types of teacher knowledge simultaneously in the processes of task analysis and modification, so all of those types of teacher knowledge were counted in the data analysis.

In this study, the task modification of prospective teachers was classified into three types: context modification, condition modification, and question modification. Context modification refers to modification by changing the context of tasks, making them student-friendly or diverse. Condition modification refers to modification by adding, deleting, or transforming the conditions in tasks (Prestage & Perks, 2007). It can be related to 'what-if-not' strategies of Brown & Walter (1990), who mentioned the manipulation of the conditions of a problem when posing a new problem. Question modification refers to modification by changing what students are required to answer. We can find an example of question modification in Crespo (2003), who suggested changing the task into a more open-form or one that requires a process of investigation. Theme C – K.-H. Lee, E.-J. Le &, M.-S. Park

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The authors of this study classified the modified tasks presented by the prospective teachers into types, and the classifications not agreed on by all of the authors were discussed until we finally reached an agreement. In addition, this study analyzed the types of teacher knowledge that the prospective teachers used for the task analysis and modification, according to the classification of teacher knowledge suggested by Ball et al. (2008). The types of teacher knowledge used by the prospective teachers were also analyzed in the same way that the classification of the modified tasks was analyzed.

Findings

Types of task modification

As aforementioned, the modified tasks presented by the prospective teachers were analyzed by the following criteria: context modification, condition modification, and question modification. Figure 1 shows the number of each modification type. As can be seen in the figure, the conditions of the tasks were the most often modified by the prospective teachers.

Figure 1 The number of each task modification type made by the prospective teachers

In addition, some prospective teachers changed only one among the three: context, condition, and question; others simultaneously changed more than two for their modification. Therefore, to analyze the results from the prospective teachers' modification types in detail, both cases are described in the following section.

The case of focusing on only one among context, condition, and question

The number of tasks in which only the conditions were modified was 26 out of 64; most of the prospective teachers modified the conditions to correct misconceptions that students might have or to adjust the difficulty level of the tasks. Most prospective teachers who modified the conditions of the tasks to adjust the difficulty level tended to add graphs or pictures to help students' understanding of the tasks. The other prospective teachers noted that students might have misconceptions about mathematical ideas from the tasks in textbooks, so they modified the conditions of the tasks to prevent the misconceptions. For example, Hee (hereafter, all names presented in this study are pseudonyms) mentioned that the task in the introduction of the function unit (Figure 2) might cause some misconceptions. Hee was concerned that students may misunderstand that function as discrete, because the given domain of the task was not continuous as it asked for each moving distance after 1 second, 2 seconds, 3 seconds, and 4 seconds, respectively. Thus, Hee modified the task by changing the condition to 'Show the graph of the moving distance for 4 seconds after Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.

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P-waves started moving'. This modification focused only on preventing a misconception that students might have, that might have made it difficult for students to perform a task given in the introduction of the unit.

Figure 2 A textbook task to introduce concept of function

The number of tasks in which only the context was modified was 10 out of 64, and in most of the cases, the context was changed to attract students' interests. The prospective teachers who performed context modification claimed that the contexts used in the tasks in the textbooks could not draw students' interest. Thus, they suggested that materials in which students were interested or contexts related to students' real life experiences should be used in tasks. For instance, Nam analyzed an existing task (Figure 3); the context of the task was related to real life but could not engage students' interest. Thus, Nam changed the context by using electronic devices such as MP3 players that students often used. This finding from the context modification indicates that the Korean prospective teachers who participated in the study believed that contexts in mathematical tasks should be related to real life and students' direct experiences, and they tended to use context in a very limited view.

Figure 3 A textbook task to introduce a real life application of function

The case of focusing on more than two among context, condition, and question

There were some prospective teachers who modified the task by focusing on more than two among context, condition, and question. Because condition was the most frequent aspect of change as mentioned above, in this section, we show the responses of prospective teachers for the cases of modifying condition and context, condition and question, or context, condition, and question.

The number of cases in which the prospective teachers modified condition and context at the same time was 2 out of 64. Sol modified the task from the Introduction Theme C – K.-H. Lee, E.-J. Le &, M.-S. Park

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of the function unit (Figure 2). She considered that the task did not show arbitrariness and univalence, so it would lead to the misconception of the function. She thus modified the context by selecting material which shows the function concept properly and provides appropriate difficulty and interest to students, and added a condition by presenting one more new relation. By considering the condition and context at the same time, this prospective teacher presented a much richer and more meaningful task than people who considered only the condition or context. Her modified task is given in Figure 4.

Figure 4 Sol's modified task of the original task in Figure 2

The number of cases in which the prospective teachers modified both the condition and question at the same time was 15 out of 64. Jong modified the task from the introduction of an isosceles triangle in geometry (Figure 5, left). He raised the question of the problem in the textbook being focused only on the shape of the triangle rather than on the properties of the figure. Thus, he changed the condition to focus on the same properties without mentioning the side or the angle by modifying the question so it was an open-ended question asking in general about figures with the same properties (Figure 5, right). In that sense, Jong provides an opportunity for discovering propositions and allowing students to explore the figure, requiring more deep mathematical thinking through changing the question altogether.

Figure 5 A textbook task to introduce properties of a triangle (left) and Jong's modified task (right)

The number of cases in which the prospective teachers modified context, condition, and question at the same time was 6 out of 64. Young attempted to modify the condition to complement the weak point of the problem (Figure 2), and in that procedure, she also tried to modify to arrive at a proper context and question (Figure 6). Because she pointed out that the problem in the textbook only treated the direct Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.

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proportion restrictively, she added a condition to extend to the reciprocal proportion. She presented a distribution situation which shows the reciprocal proportion and is familiar to students. In addition, by asking them to share their thinking about the relation of x and y, she tried to make students more explorative.

Figure 6 Young's modified task of the original task in Figure 2

Teacher knowledge

As aforementioned, teacher knowledge can influence how teachers actually set up tasks in textbooks used in classrooms (Henningsen & Stein, 1997). Similarly, when the prospective teachers modified tasks, teacher knowledge of content and pedagogy might affect the direction and process of modification.

According to what prospective teachers considered when they modified tasks, we classified teacher knowledge. When prospective teachers considered the difficulty of the task or misconceptions and interest of students, they demonstrated KCS. When they focused on what example should be given or the way of teaching, they demonstrated KCT. KCC includes consideration of curriculum or the order of the mathematical content, and SCK includes the good usage of mathematical terms, mathematical ideas, mathematical history, and appropriate examples. CCK is related to general mathematical knowledge, and HCK is related to the relevance of the whole curriculum.

Figure 7 shows types of teacher knowledge used by the prospective teachers in the modification process. As can be seen in Figure 7, the prospective teachers used PCK more frequently than SMK; among PCK, KCS and KCT were more often used than the others. SCK among SMK was more frequently used than CCK and HCK. The SCK used by the prospective teachers was knowledge about mathematical ideas, its historical backgrounds, and various representations.

The fact that the prospective teachers used SCK more frequently than CCK indicates that they considered content knowledge as more important for teaching than using only mathematical content knowledge for their modifications. SCK related to mathematical ideas seemed to have an impact on the identification of misconceptions that students might have, and the prospective teachers tried to modify tasks by suggesting teaching methods or procedures to prevent those misconceptions; in other words, SCK might affect KCS and KCT. The prospective teachers, furthermore, tried to modify tasks by applying the historic-genetic principle or asking students to present different representations on the basis of SCK related to historical backgrounds and

various representations. From these findings, it can be clearly seen that content
knowledge is closely related to pedagogical content knowledge. Theme C – K.-H. Lee, E.-J. Le &, M.-S. Park
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As mentioned earlier, the prospective teachers participating in this study
modified the conditions of tasks more frequently than the context or question. The
types of teacher knowledge that the prospective teachers used to change the
conditions were KCS related to the identification of misconceptions and consideration
of difficulty level, KCT related to teaching methods, and SCK related to mathematical
knowledge for teaching. As can be seen in Figure 1 and Figure 7, therefore, most of
the prospective teachers changed the conditions of existing tasks, and teacher
knowledge types such as KCS, KCT, and SCK played an important role in the task
modifications.
Considering teacher knowledge utilized in the task modifications in this study,
it is noteworthy that the prospective teachers tended to use several types of teacher
knowledge simultaneously to modify one task, rather than using only one type of
teacher knowledge. It is presumed that Korean teachers' high level of mathematics
content knowledge (Park, 2004) might help them activate teacher knowledge, such as
KCS relevant to the difficulty of task and misconception, and SCK relevant to various
representations, good usage of mathematical terms and mathematical ideas.
Consequently, their awareness of this kind of knowledge led them to consider
appropriate teaching methods or strategies. Therefore, it can be said that different
types of teacher knowledge are closely interconnected and influence each other to
activate knowledge. This suggests that an activity, such as task modification, may be
a good way to develop teacher knowledge by encouraging teachers to consider
different types of knowledge at the same time.
Figure 7 The number of teacher knowledge by types that the prospective teachers used
Conclusion
In this study, we examined the types of task modification conducted by the
prospective teachers and the types of teacher knowledge that were activated during
the modification process. By classifying the types of modification into context
modification, condition modification, and question modification, we found that
condition modification was performed at the highest rate. In addition, we found that
when prospective teachers focused on more than two among context, condition, and
question rather than focusing on only one, they modified the tasks in a more
appropriate, richer, and meaningful manner. KCT and KCS were the types of
knowledge that prospective teachers used most frequently during modification, and
they also used SCK at a high rate. However, other types of teacher knowledge were
hardly displayed.
Prospective teachers modified the conditions mostly when they considered
students' misconceptions and difficulty level, and intended to provide students with an Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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opportunity to make a discovery. This result is consistent with the highest rate of the
usage of the KCT and KCS teacher knowledge types when prospective teachers
modified the tasks. Also, because cognitive ability is necessary when modifying the
conditions, it can be related to the high rate of usage of SCK.
Prospective teachers who used KCS as their type of teacher knowledge,
especially those who considered students' interest, modified the context. So, when
they focused only on the context, they usually modified the context on a superficial
level by presenting students with a familiar situation. When prospective teachers
focused also on the condition or question along with context, they attempted to
modify by using their KCS, especially the knowledge of misconceptions, and KCT. In
this sense, prospective teachers who focused on more than two aspects at the same
time were able to generate a more meaningful task because they activated various
factors of teacher knowledge.
The number of prospective teachers who modified the context, conditions, and
question all at the same time was not that high, surprisingly. To modify tasks in a
meaningful way by considering all three factors, prospective teachers not only need
KCT, KCS, and SCK but also need knowledge of the curriculum or attempts to relate
to other mathematical concepts. Therefore, the reason that many prospective teachers
did not consider all the aspects at the same time is that they possess narrow
knowledge rather than balanced teacher knowledge.
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The Lemon Squash Task
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The Lemon Squash Task affords many learning opportunities in
connection with algebra and proportionality. In this paper, we show that
although opportunities are embedded in the task, they do not necessarily
surface when the task is treated in a classroom setting. Depending on the
context in which a task is placed and the knowledge and intentions of the
teacher, the task may contribute to very different kinds of learning.
Modeling real world situations often has its limitations and can, as in this
case, make the problem unsolvable unless it is accepted as a 'textbook
task' disguised as real but adjusted to the norms of school mathematics.
Keywords: textbooks, tasks, mathematics, proportionality, algebra
Introduction
When a mathematics textbook task is designed for school use, it is generally intended
to offer an opportunity to support the learning of some particular mathematical idea.
One branch of research about textbook tasks concerns the tasks themselves in relation
to, for example, creativity (Lithner, 2008) or cognitive demand (Brändström, 2005),
or how tasks appear in the context of assessment such as the TIMSS framework
(Mullis, 2005). Quite often, the way tasks are treated in classrooms is absent in this
research. Additionally, classroom studies often take textbooks for granted and do not
analyse textbook tasks in detail. According to the didactic tradition in the Nordic
countries, most studies assume curricula and teaching materials as the basis for their
analysis and there is a need for content related classroom studies (Klette, 2007). We
argue that few Nordic studies analyse how specific textbook tasks are interpreted in
the classroom and we relate this to the affordances and constraints of the tasks per se.
What happens with the intended learning opportunities of a textbook task when
transposed by the teacher in a classroom setting?
In this paper, we use the term task to mean tasks found in mathematics
textbooks in line with Chevallards' (2006) definition. Tasks are also seen as cultural
artefacts, influenced by the mathematics culture at large as well as by the time and
place in which they are created and used. Tasks are a valuable instrument in
mathematics education, particularly in a textbook intense teaching culture as the one
in Sweden (Johansson, 2006). We seek in the following sections to analyse one
particularly interesting task related to algebra and proportionality found in a Swedish
grade 6 mathematics textbook and try to answer the following questions: Theme C – A. L. V. Lundberg, C. Kilhamn
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• What are the affordances and constraints of the chosen task?
• How do these come into play when the task appears in the classroom?
We will also discuss how the results of our analysis could be of use for textbook
authors.
Being a teacher and a teacher educator, we take special interest in what types
of tasks appear in the classroom and how teachers use the tasks. We are particularly
interested in proportionality tasks because proportionality has been a new concept in
the Swedish national curriculum for grade 6 since fall 2011 (Lgr11, 2011).
Proportionality and proportional reasoning
Proportionality can be defined in several ways and we chose to define it as a linear
relation such that
- where y and x are magnitudes and a is the proportional
constant. Miyakawa & Winsløw (2009) call this dynamic proportionality, as opposed
to static proportionality expressed as a multitude of equal ratios
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&

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Proportionality can be expressed as internal or external depending on whether it is a
relation within or between measure spaces, according to Freudenthal (1983).
Proportional reasoning is used to denote reasoning in a system of variables between
which there exists a linear functional relationship (Karplus, Pulos, & Stage, 1983).
The Lemon Squash Task studied in this paper involves a mixture of juice,
sugar, and water. Tasks involving mixing substances in different proportions have
been studied by several researchers (e.g., Mellar, 1991; Noelting, 1980; Nunes, Desli,
& Bell, 2003). These are often called mixture problems by Tourniaire and Pulos
(1985) and differ from rate problems in that the elements in a mixture create a new
object and different quantities are often expressed in the same unit. Tourniaire and
Pulos (1985) also describe three context variables that affect the difficulty of a
mixture task; firstly the mixture as such, being an object where the different parts are
not distinct and therefore more difficult to handle; secondly, it is more easy to
visualize discrete quantities than continuous quantities; and finally, if the context is
familiar the task is often found to be easier.
The Anthropological Theory of Didactics
The Anthropological Theory of Didactics (ATD) postulates an institutional
conception of mathematical activity (Chevallard, 2006). It starts from the assumption
that mathematics, like any other human activity, is produced, taught, learned and
diffused in social institutions, and designed in terms of mathematical organisations
(MO). Knowledge content and form in these activities is a consequence of the
didactic transposition process, i.e. a change or adaptation of a selected existing
knowledge to 'teachable' knowledge. Mathematics treated in school can be analysed
as several types of knowledge (Bosch & Gascón, 2006):
1. Scholarly knowledge;
2. Knowledge to be taught, described in curricular documents;
3. Knowledge actually taught which can be gleaned from the teachers' classroom
discourse and the tasks he or she prepares for the students;
4. Knowledge actually learned.
The MO is constituted on two levels, the 'know-how' and the 'know-why'
related to a given task, and has a praxeology comprised of four components: type of
tasks, techniques, technologies, and theories. The praxis 'know-how' contains types
of tasks to be carried out and techniques to do so, technique being considered here in Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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a general sense of 'ways of doing'. The logos or 'know-why' includes technology and
theory (Bosch & Gascón, 2006). In this paper, we focus on the 'know-how' of the
Lemon Squash Task and the transposition between steps 2 and 3.
Contextualised tasks
In a contextualised task, the realistic touch of the question is supposed to increase the
meaning of the task, but it also increases the cultural burden on the student in the
solving process, in light of the task being a cultural artefact (Wyndhamn & Säljö,
1997). Everyday contexts will trigger the use of out-of-school knowledge, everyday
concepts and common sense, but these may not always be applicable. The Lemon
Squash Task is interesting because it entails a switch between everyday concepts and

scientific concepts (Vygotskij & Kozulin, 1986). A contextualised textbook problem
intended to be a 'realistic' problem is, argues Jablonka & Gellert (2007), neither
mathematics nor a real world problem. With reference to the work of Dowling, they
discuss the "myth of reference":
"It is conveyed through problem settings that are constructed mathematically and
only retain a trace of non-mathematical significations. It does not remain possible
for the learner to evaluate the solution of the problems from a practical point of
view."(ibid, p. 2)
The interpretation of a task is also constrained by certain norms of school
mathematics, regulated by the so called didactical contract (Brousseau, 1997).
Method and setting
Data for our analysis in this paper is taken from a larger set of video data collected
within the project VIDEOMAT (Kilhamn & Röj-Lindberg, 2012), a comparative
video study including students from Sweden, Finland, Norway and California about
the introduction of algebra in grades 6 and 7. In each country, 4-5 classes were video
recorded for 4 consecutive lessons. There was no intervention because the aim was to
record authentic instruction and classroom activity concerning introduction of
variables. When looking through the Swedish data, the Lemon Squash Task caught
our interest because teacher-student interaction around the task appeared seven times
in the recordings of one of the grade 6 classrooms. These episodes were selected to
form a case study. In addition to the video recorded lessons, a post lesson interview
was conducted with each teacher; six months later, a focus group discussion was held
with all the teachers where this particular task was discussed.
The choice of analysing the Lemon Squash Task was naturally influenced by
our background as teacher and teacher educator (Goodwin, 1994), and our method of
analysis can be described as "a whole-to-part inductive approach" (Erickson, 2006).
After viewing the seven episodes, they were carefully transcribed, reviewed many
times, and also viewed in groups to establish agreement about the phenomenon.
The Lemon Squash Task in our data was found in the algebra unit in a
commonly used grade 6 textbook (Carlsson, Liljegren, & Picetti, 2004). With the aim
of investigating the transposition of "knowledge to be taught" into "knowledge
actually taught", content analysis of the task as it appears in the textbook was first
made, focusing on embedded aspects of proportionality and algebra, since it appears
in the algebra unit and concerns proportional relationships. The algebra unit in the
textbook spans over 26 pages, including 127 tasks of various kinds. It has a strong
emphasis on the learning of algebraic symbolic language and the meaning of the
terms equation, expression, and variable. There are 6 pages in the teacher's guide on
the algebra unit and an additional 8 worksheets. Tutorial notes are given in general
terms and no guidance is given in relation to any particular task. The text in the guide
concerning the page where the Lemon Squash Task appears is as follows:
"An expression consists of one or more variables written with letters and
sometimes one or more numbers. An expression does not have a fixed value until
the value of each variable is known. That is a big difference compared to an
equation. The letters in an equation have a fixed value and it is that value you find
out by solving the equation. Therefore the letters in an equation are not variables."
(Carlsson et al., 2004, p. 64).
The local curriculum, where the core content is explained in more detail,
states as learning goals for the unit that students should be able to: interpret and write
expressions with variables, work with equalities, and solve simple equations. The
teacher in this particular class is a generalist teacher for grades 4-6. She has been
teaching the class since grade 4 and said in the post lesson interview that she feels
insecure teaching algebra. When she last taught grade 6 three years ago, the national
curriculum did not emphasise algebra until grade 7; so she had actually never taught
algebra as explicitly as she now does, in spite of her 10 years of teaching experience.
The notion of proportionality is part of the national curriculum but not treated as an
explicit topic in the textbook. In this classroom, mathematics lessons consist of a short
whole class introduction followed by individual deskwork where the teacher walks
around helping students when they get stuck. Students work through the textbook unit
at their own pace, which means they are seldom working on the same task at the same
time, a common practice in Sweden (Carlgren, Klette, Myrdal, Schnack & Simola,
2006).
The Lemon Squash Task
The Lemon Squash Task
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appears in the textbook under the heading "Expressions".
There are two pages where students are asked to write expressions translating from
verbal to algebraic representation, followed by one page with tasks where a number is
to be inserted in the place of a letter in an expression. The last task is the Lemon
Squash Task. A contextualised task like this is a cultural artefact. We believe that the
authors of the textbook chose this context because mixing and drinking this type of
drink, usually called saft, is a common point of reference for Swedish children
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Figure 1: The Lemon Squash Task in the textbook (Carlsson et al., 2004, p. 103), our
translation.

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We have chosen to use the name "Lemon Squash" although the Swedish name for the task is
"Törstsläckare" which literally means "Thirst reliever".
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Saft is a Swedish sweet drink, which is close to the British cordial. Traditionally it is made
from juice of berries mixed with sugar and water.
47 You want to mix "Lemon Squash". You have 5 dl
lemon juice, x = 5 dl. How much do you need of
a) water? b) sugar?
48* How much lemon juice is there in 7dl
mixed "Lemon Squash"?
"Lemon Squash"
x lemon juice
2x water
x/2 sugar Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Results
The results will be presented in two parts. First a content analysis of the Lemon
Squash Task is made, highlighting its affordances and constraints. The next section
presents how this task was treated in a grade 6 classroom in a Swedish school.
Content analysis
The Lemon Squash Task presented here includes tree subtasks. The first two subtasks
(47a, b) are both closed questions with a unique solution. Proportions of the three
ingredients are given in symbolic representation in the 'recipe', and the student is
asked to interpret the algebraic expressions x lemon juice, 2x water and x/2 sugar.
There are two ways of solving these tasks. The interpretation of 2x water and x/2
sugar leads to proportional reasoning through a translation into "twice as much water
and half as much sugar as the amount of lemon juice." Translating from a symbolic to
verbal representation gives x a meaning and makes use of proportional reasoning
within the recipe, i.e. how the parts relate to each other. Another solution technique is
to stay within the algebraic representation, exchange x for 5 in each expression and
interpret water as 2 · 5 = 10 and sugar as 5/2 = 2.5. In the second case, it is not certain
that x has any meaning outside the fact that is it to be replaced by a number, and no
proportional reasoning is necessary. The two tasks could be used as an opportunity to
develop proportional reasoning and meaningful interpretations of algebraic
expressions, or to practice the procedure of replacing x in an expression to calculate
the value. Task 48, however, is much more complex involving the total volume of a
mixture of fluids and solid matter. Here we find the first and most striking constraint
of the task, namely the fact that mixing these ingredients in reality and the intended
mathematics content are not aligned. We will deal more with this constraint in the
classroom analysis and the discussion, and for now analyse what we interpret as the
intended mathematical content.
Lemon Squash is made of three ingredients: lemon juice, water, and sugar.
Task 48 gives the proportions of these ingredients as algebraic expressions and the
total amount as a set quantity asking for the amount of water. It can be seen as a
proportionality task by considering how the total number of parts (one batch) is
related to the total quantity. To find out the number of parts in a batch, the algebraic
expressions in the recipe need to be translated into a verbal representation giving x the
meaning of one part. One part of lemon juice, two parts of water, and half a part of
sugar adds to 3½ parts. We then want to find out how much one part is when 3½ parts
are 7 dl. This can be described as an external proportionality where 7 dl:3½ parts = x
dl:1 part, or as an internal proportionality where 1 part· 3½ parts = x dl:7 dl. It can be
solved using different techniques, for example by using number facts (7 is 2 · 3½) or a
cross multiplication algorithm (7·1 = 3½· x). Proportional reasoning in combination
with a trial and error approach is also possible, i.e. if each part is 1 dl the total would
be 3½ dl, but 7 is twice as much as 3½ so each part must therefore be twice as much
as 1. In this case the proportionality is static.
When Task 48 is interpreted as a linear relation, the total quantity
can be
expressed as
· . If a is a scalar operator (Vergnaud, 1988) then
4
· 4
1. We know that 4
1 3½ and we want to find 4
7; we see
that 7 · 3½. In this case the meaning of x is the number of times we need to take
the whole mixture (all the parts) to get 7. However, if a is a function operator, then
4
3½ · and the meaning of x is what quantity I need to take 3½ times to get 7.
In this case, the proportionality is dynamic. Whichever way the linear relation is
interpreted, a graphic representation could be a useful solution technique.
A more primitive example of proportional reasoning is to consider only the
internal proportions among the ingredients and use a trial and error solution
technique. If we have 1 dl of lemon juice, we would need 2 dl of water and ½ dl of
sugar, which adds to 3½ which is less than 7; so we try with 2 dl of lemon juice.
Despite the way proportional reasoning is used, the translation from algebraic
to verbal representation of the expressions presented in the recipe becomes an
essential part of the solution technique. The task can be said to afford different types
of proportional reasoning and different solution techniques, making use of translation
between symbolic and verbal representations.
A different approach to the task is to stay within the algebraic representation
and model an equation by adding the expressions: 2 2 7 ⁄ . Again the
solution technique can either be one of trial and error, or of formally solving the
equation for x to get 2. In this case the task affords the opportunity for modeling
and solving of an equation with one variable. Because the expressions are already
given, the modeling part is limited. No switch of representation is needed, although it
is possible to translate the algebraic equation into a visual or concrete representation
instead of using a formal equation solving procedure.
The Lemon Squash Task involves all the three context variables that often
appear in mixture problems according to Tournlare and Pulos (1985). It is a mixture
where parts are not distinctly visible, it deals with continuous quantities, and students
are familiar with making such mixtures.
Classroom Analysis
The classroom analysis is grounded on the first two episodes where the teacher
explains the Lemon Squash Task to individual students (in total she does it 7 times).
In the first episode, the teacher first misinterprets the task. She points to the recipe in
reasoning proportionally about the amount of water, lemon juice and sugar, and starts
discussing doubling and halving 7. The textbook section is about expressions, not
equations; suddenly the Lemon Squash Task appears, challenging students to make a
mathematical model using an equation. Task 48 seems difficult because there are no
clues in the preceding pages on how to model an equation. Our interpretation is that
the teacher therefore uses the same technique as in the preceding task. She exclaims
"Ah!" when she realises that 7 dl is the total amount, not a part (see excerpt 1).
Excerpt 1:
T: So, then you can first calculate how much, (.)
Ah! it also says seven decilitre of mixed, yes.
[…]
T: and water gives twice as much (.) and it will be (.) mmm in itself. I wonder
if sugar also gives some amount there? (.) I have to think, does it really it's
no amount of sugar it just ends up in the liquid. Eh, this will be seven
together.
S1: mhm
T: something plus something plus something will be seven.
When the teacher has realised her mistake, she considers the modeling of the
task. Knowing that sugar dissolves in water, she contemplates whether it contributes
to the total volume. She wants to use the given algebraic expressions in the recipe but
her experience tells her that sugar dissolves in water and does not increase the
volume. Initially she discards the sugar and considers only the lemon juice and the
water, which means, she needs to divide 7 dl into 3 parts. When she realises that 7/3
will be a long decimal number
be treated as if it did contribute to the total volume in the same way as the fluids
excerpt 2). In fact, the sugar does contribute, but not as m
avoiding the continuous variable, expecting the task to include only discrete numbers.
Excerpt 2:
T: That was a difficult calculation!
S1: mm
T: Wasn't it? It would have been easier if it were six (.
Maybe they count yes eh I think they eh mean that they consider the sugar
to give some amount. Because then I can work it out!
After this the teacher solves the task by translating from an algebraic into a
visual representation, drawing four lines represen
The lengths of the lines are in proportion to the recipe. One line for lemon juice, two
for water, and a line half as long for sugar, adding ½ under the last line for

clarification (see figure 2)
what number is the right one, thus choosing a solution technique using a trial and
error approach. She suggests 2 and adds 2 + 2 + 2 + 1 = 7. When leaving to attend
another student, the teacher mutters "
technique to represent numbers as lines, she informs us in the post
was something she learned when she was a scholar herself, and she still uses this
technique when solving algebra tasks.
Figure 2: Copy of
The next episode shows the teacher explaining
Referring to her earlier experience, she tells the student that the task is tricky and
explains the mistake made about the sugar contributing to the total
Excerpt 3
T: This is a bit tricky. When I solved this I had to use trial and error.
Do you know why I thought it was difficult to solve? Because I don't
think or I guessed that sugar did
squash. But of course it does. This is how I did it.
[T4 takes the pen and paper from the student and starts to draw lines.]
T: If it is seven decilitres mixed lemon squash. Uh, then it is, they write here
that, something plus something two times is the secret number
half of the secret number. I write like this, so I will remember, it will be
seven decilitres. A number plus a number plus a number plus half of the
secret number will be seven. Can you try f
When the teachers in the VIDEOMAT project discussed opportunities for
learning in connection to this task, they indicated the task appeared very abruptly in
the textbook without sufficient preparation. The solution techniques they sugges
for the task were trial-andin the lesson reported on here commented that the task is about proportionality,
meaning the doubling and halving, and that it was an everyday example of the use
simple equations. She said:
"A trial-and-error technique is not the only possible one but I use it automatically
in my head. I try putting 1 in the place of
how one solves it in an everyday situation. […] You
but that is not the first thing a student would do." (Focus group discussion)
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will be a long decimal number, she states that "they" must have intended the sugar to
be treated as if it did contribute to the total volume in the same way as the fluids
. In fact, the sugar does contribute, but not as much as the fluids.
avoiding the continuous variable, expecting the task to include only discrete numbers.
That was a difficult calculation!
Wasn't it? It would have been easier if it were six (...) decilitres there.
Maybe they count yes eh I think they eh mean that they consider the sugar
to give some amount. Because then I can work it out!
After this the teacher solves the task by translating from an algebraic into a
visual representation, drawing four lines representing the amount of each ingredient.
The lengths of the lines are in proportion to the recipe. One line for lemon juice, two
and a line half as long for sugar, adding ½ under the last line for
). After drawing the lines, the teacher says she will try out
what number is the right one, thus choosing a solution technique using a trial and
error approach. She suggests 2 and adds 2 + 2 + 2 + 1 = 7. When leaving to attend
the teacher mutters " ... what a complicated task that was!".
technique to represent numbers as lines, she informs us in the post-lesson interview,
was something she learned when she was a scholar herself, and she still uses this
technique when solving algebra tasks.
Copy of Task 48 from a student's notebook including the lines drawn by the
teacher.
The next episode shows the teacher explaining Task 48 for the second time.
Referring to her earlier experience, she tells the student that the task is tricky and
ke made about the sugar contributing to the total (see excerpt 3
This is a bit tricky. When I solved this I had to use trial and error.
Do you know why I thought it was difficult to solve? Because I don't
think or I guessed that sugar didn't contribute to the volume in lemon
squash. But of course it does. This is how I did it.
[T4 takes the pen and paper from the student and starts to draw lines.]
If it is seven decilitres mixed lemon squash. Uh, then it is, they write here
ething plus something two times is the secret number x. (.) plus
half of the secret number. I write like this, so I will remember, it will be
seven decilitres. A number plus a number plus a number plus half of the
secret number will be seven. Can you try finding the number?
When the teachers in the VIDEOMAT project discussed opportunities for
learning in connection to this task, they indicated the task appeared very abruptly in
the textbook without sufficient preparation. The solution techniques they sugges
-error, making a table, or setting up an equation. The teacher
in the lesson reported on here commented that the task is about proportionality,
meaning the doubling and halving, and that it was an everyday example of the use
simple equations. She said:
error technique is not the only possible one but I use it automatically
in my head. I try putting 1 in the place of x first, checking if that will fit. That's
how one solves it in an everyday situation. […] You could do an equation as well
but that is not the first thing a student would do." (Focus group discussion)
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she states that "they" must have intended the sugar to
be treated as if it did contribute to the total volume in the same way as the fluids (see
uch as the fluids. She is
avoiding the continuous variable, expecting the task to include only discrete numbers.
Maybe they count yes eh I think they eh mean that they consider the sugar
After this the teacher solves the task by translating from an algebraic into a
ting the amount of each ingredient.
The lengths of the lines are in proportion to the recipe. One line for lemon juice, two
and a line half as long for sugar, adding ½ under the last line for
the teacher says she will try out
what number is the right one, thus choosing a solution technique using a trial and
error approach. She suggests 2 and adds 2 + 2 + 2 + 1 = 7. When leaving to attend
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lesson interview,
was something she learned when she was a scholar herself, and she still uses this
including the lines drawn by the
ask 48 for the second time.
Referring to her earlier experience, she tells the student that the task is tricky and
see excerpt 3).
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If it is seven decilitres mixed lemon squash. Uh, then it is, they write here
. (.) plus
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seven decilitres. A number plus a number plus a number plus half of the
When the teachers in the VIDEOMAT project discussed opportunities for
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error technique is not the only possible one but I use it automatically
first, checking if that will fit. That's
could do an equation as well
but that is not the first thing a student would do." (Focus group discussion)Theme C – A. L. V. Lundberg, C. Kilhamn
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In the discussion, one teacher assumed a simple proportionality of 1 part
sugar, 2 parts lemon juice, and 4 parts water making 7 parts, but without realising that
interpreting the x/2 as one part is not a trivial matter for students. One of the other
teachers said she skipped the task during the algebra unit, using it in a class discussion
about recipes later on.
Discussion
When the teacher starts to explain the task, she uses the recipe in the margin because
she expects the solution technique to be found in the textbook (Lithner, 2008). The
didactical contract says that the task is solvable. The teacher considers the amount of
sugar and how it will disappear in the total volume, but she continues to use the recipe
even if she realises that it is unsolvable. She constructs a model of the recipe using
lines with different lengths and successfully solves the task, subsequently repeating
this solution technique each time. However, the constraints of her model are never
made explicit. The Lemon Squash Task affords many opportunities of learning
proportionality, internal and external, dynamic as well as static, but they stay hidden.
The Lemon Squash Task differs from other mixture problems reported in
previous research literature (Tournlaire & Pulos, 1985), which are often comparison
problems. When making lemon squash, there are three quantities involved (sugar,
water and lemon juice), and these ingredients are in a proportional relation to each
other. If you add more sugar, the sweetness of the lemon squash will increase
proportionally; if you increase lemon juice, the taste will again be less sweet by
proportion. The amount of lemon juice is inversely proportional to the taste of
sweetness because the juice of a lemon is sour. When the mix is blended with water,
again the taste will be weaker. So in order to mix lemon squash, there are both direct
and indirect relations between quantities. In this case the within quantity is taste and
the between quantity is the amount of sugar, water, and lemon juice. As the task is
presented, the sweetness is irrelevant and only the between quantities are focused.
Different mixtures are not compared, but the whole idea of mixing water, juice and
sugar is an implicit aspect of familiarity relevant as a context variable (Tournlaire &
Pulos, 1985) that clearly affects the solution procedure, at least for the teacher.
In this paper we have studied a teacher in the process of transposing the
knowledge embedded in a textbook task into knowledge taught. We argue that the
teachers' interpretation of the Lemon Squash Task is constrained by the context in
which the task is placed and by an existing didactic contract. The task appears in the
algebra unit focusing on the use of symbolic notation. Proportional relations, such as
static or dynamic proportionality, embedded in the task become a background feature
and do not stand out as affordances to the teacher. A relevant solution technique is
expected to be found earlier in the unit and difficult calculations are not expected to
appear in the algebra unit. These features of the didactical contract lead her to discard
her everyday experiences (the sugar does not contribute to the volume) as well as the
scientific concept of volume when she says, "I think they mean that they consider the
sugar to give some amount". She interprets the task as a mathematics task, not a
realistic problem, a common feature in so called 'realistic mathematics' where "The
focus is on the mathematics, not the 'realistic' situations from which the
mathematisation is hoped to be derived" (Jablonka & Gellert, 2007, p. 4). Although
the task appeared in the algebra unit, the teachers found that setting up an equation
was too difficult for the students, so they resigned to working it out using trial-and-Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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error. Getting the correct answer became the main goal. Solving it as if it were an
everyday situation became the goal instead of some specific mathematics learning.
We have shown that a textbook task can be seen as a cultural artefact greatly
dependent on the context in which it is placed. Consequently, a task cannot be judged
as good or bad in itself, and the learning opportunities a task designer sees in the task
may not stand out as affordances to the teacher. We conclude that textbook authors
need to elaborate more on tasks in the teachers' guide, making affordances of a task
explicit and include tutorial notes about learning, especially concerning modeling
tasks. The Lemon Squash Task becomes exceptionally difficult because the intended
mathematical relations are in conflict with both the scientific and the everyday
concept of volume when mixing fluids and solid matter. We see it as an unfortunate
choice of realistic context and suggest task designers pay a good deal of attention to
both everyday experiences and scientific validity of the chosen context.
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Designing Interdisciplinary Tasks in an International Design Community

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A major difference between the world of school and the real world is that, in the former, knowledge application to solve problems is clearly connected to a single 'academic' discipline whilst, in the latter, solving problems requires interdisciplinary knowledge application. The gap between these "worlds" provides a challenge, particularly from a materials design perspective. In this paper, a theoretically based design process of a package of ten interdisciplinary tasks and associated interactive resources is discussed. First, the theoretical foundations behind the design process are introduced. Second, our concept of iterative cycles of improvement, with a structured approach to formative assessment, is presented. Finally, results and outcomes from the analysis and implementation of the package are discussed.

Keywords: interdisciplinary teaching and learning, inquiry-based learning, iterative cycles of improvement.

Background: Interdisciplinary teaching in day-to-day teaching

Mathematics and the sciences in school are often taught as isolated subjects. This, however, does not meet the needs of the world of work, which demands integrated, interdisciplinary solutions to complex problems often requiring the use of technology. In consequence, students should not only be prepared for application of knowledge and skills in single 'academic' domains but learn to solve complex real world problems.

Although in many countries school curricula contain some aspects of interdisciplinary learning, particularly across the sciences
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, in general lessons remain

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See e.g., the so-called Fächerverbund (Connection of subjects) in Baden-Württemberg, Germany; the new integrated science subject (Advanced Science, Mathematics and Technology) in the Netherlands since 2007; methodological guidelines in the regional curriculum of Andalusia; and new mathematics and science curricula in Cyprus. Theme C - K. Maas et. al
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mono-disciplinary. In the reality of day-to-day curriculum implementation in schools, there are many obstacles to interdisciplinary approaches to teaching and learning. These include, amongst others, the high degree of organisational work required of teachers (in organising interdisciplinary teaching with their colleagues), training in single disciplinary areas for most teachers, interdisciplinary problems viewed as too complex and time consuming by teachers for their students, and few materials that support mathematics and science from an interdisciplinary perspective. In this paper we present the design work carried out within the EU-project COMPASS
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, which addressed the issues raised above by aiming to provide teachers with a set of challenging interdisciplinary tasks relevant and usable within day-to-day teaching. Fundamental to the project was the development of new approaches to teaching that aim to motivate learning both within and across the disciplines. Pragmatically, given that there is little official demand for such curriculum in most countries, this required careful understanding of how such an innovation might be motivated in ways that ensure current curriculum demands are met.

Our design community and its aims

The COMPASS design community aimed to develop teaching materials connecting the sciences and mathematics with each other and most crucially with the lives of individual students and their communities. We wanted to foster in young people a desire to learn mathematics and science throughout their lives by coming to realize that such learning is essential if they are to be empowered as active citizens with critical scientific inquiry and problem solving skills. In achieving the above aims, the project focused its activities mainly on designing and developing materials to support teachers with their day-to-day teaching.

The design community was comprised of 15 mathematics and science educators from six countries, geographically spread across Europe (the Netherlands, Spain, England, Cyprus, Slovakia, Germany), ensuring a range of complementary expertise. The consortium included design experts in the fields of interdisciplinary tasks, real-life-based tasks, modeling tasks, and ICT mediated tasks, together with partners having considerable experience in the design and implementation of continuing professional development courses. Further, the expertise of the participants was enriched with stakeholders from school authorities and teachers from schools who contributed to the further refinement of the materials during iterative cycles of improvement.

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A theory-based design approach

In this section, the theoretical foundation adopted in the design of the COMPASS tasks is summarized.

Interdisciplinary work

In general terms, "interdisciplinary work" – in relation to science and mathematics – refers to some connection between these 'academic' disciplines. However, there is much debate in the literature about what might be considered interdisciplinary teaching. Many authors base their classification systems on the tightness or looseness of the connection between disciplines in an integrated curriculum. These connections range from no connection at all between disciplines, to some coordination, or to total integration into a new subject (Nikitina, 2006). Geraedts, Boersma and Eijkelhof (2006) mainly consider the context in which interdisciplinary work is established and distinguish between interdisciplinary work implemented at the macro, meso, or micro levels ranging from interdisciplinary work being demanded by curriculum specification to ad-hoc implementation by teachers in their day-to-day teaching). A totally different way of looking at interdisciplinary approaches is proposed by Nikitina (2006) who does not look at the bonding between disciplines but rather at the concrete focus of the interdisciplinary tasks/teaching units. Here, three different ways of doing interdisciplinary work are identified:

- Contextualising, when the material is embedded in a cultural, personal, philosophical or historical context in order to gain a better understanding of the social and cultural development of knowledge.
- Conceptualizing, when core content that is central to two or more disciplines (e.g., change, linearity) are considered, with the intention of going, beyond facts, to the underlying concepts. The goal is to understand essential laws of the world and establish a connection among them.
- Problem-centering, when an ill-structured real-world problem is used as an axis of connection among disciplines. The aim is to generate possible solutions for this problem, bringing together different disciplines.

From this perspective the COMPASS tasks were developed taking a problem centred approach (first major design principle), starting with a problem that is initially ill-structured but situated in a real-world context and of immediate relevance and importance to European citizens. Contexts such as saving energy, environmental pollution, and biodiversity have been used in developing the COMPASS tasks. Our choice of problem-centering as a design principle had a major impact on task design; we envisage that the tasks would have looked quite different if we had chosen to work within conceptualizing or contextualising perspectives. Conceptualising approaches are often taken when developing interdisciplinarity in mathematics and science. For example, proportionality and linearity may be explored from mathematical and scientific viewpoints emphasising scientific principles and applications such as Ohm's, Newton's and Hooke's Laws. Alternatively a contextualising approach would have emphasised connections between mathematics and science and the social and historical situation and not necessarily to strong links between mathematics and science. An example Theme C – K. Maas et. al
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might involve exploration using mathematics and science of the impact of the recent catastrophe in Fukushima on the policy for atomic power plants.

Inquiry-based learning

Our second major design principle focused on inquiry-based learning pedagogies that we sought to underpin classroom implementation. Inquiry-Based Education (IBE) refers to a student-centered paradigm of teaching mathematics and science, in which students are invited to work in ways similar to how mathematicians and scientists work. Students are guided to observe phenomena, ask questions, seek mathematical and scientific ways of answering related questions (for example, carrying out experiments, systematically controlling variables, drawing diagrams, looking for patterns and relationships, making conjectures and generalizations), interpret and evaluate their solutions, and communicate and discuss these effectively (Dorier & Maaß, 2012).

Added value through the optional inclusion of digital-technological tools

Our work builds on the previous wave of research into use of information and communication technologies (ICT) in science and mathematics teaching, which has focused on the design of pedagogical principles for applications such as simulations, dynamic geometry environments, and spreadsheets (Andersen, 2006; Hoyles & Lagrange, 2009; Linn, Davis, & Bell, 2004). These principles are summarized in the following possible activity triads: Predict, test, explain; Tell, explore, check; Construct, validate, prove; Observe, find a pattern, generalize; Explore, validate, document.

Underlying all these principles (which have also been advocated by software developers and implemented in their systems) is the notion that direct manipulation of abstract representations of concrete objects and phenomena can assist students in exploring and testing out their ideas about the natural world in comparison with the theoretical world of science and mathematics (Hoyles & Lagrange, 2009).

Based on these reflections the third major design principle was that all COMPASS tasks are designed to include ICT that supports inquiry-based learning. A library consisting of nine applets has, therefore, been provided as complementary to the COMPASS materials. The applets have been designed as interactive microworlds, offering idealized, dynamic and visual representations of physical phenomena and experiments.

An example of a task (at the end of several cycles of improvement)

During the project, ten interdisciplinary tasks were developed for students age 14 – 16, each comprising six to eight lessons. Each task is introduced by one overarching question placed in a meaningful, problem-centred context. Tasks could be used following a project-based-learning approach, or a more guided series of lessons in both mathematics and science. Each task is divided into sub-tasks, which are all introduced by guiding questions themselves and allocated to the different subjects (mathematics and the sciences) involved. As an example, the following table gives an overview for one of the tasks Desertec
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All tasks are available at http://www.compass-project.eu/ Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Desertec task

Overarching question: To what extent do solar power plants potentially contribute to European energy needs?

Task 1: Science

How efficiently is solar energy converted into
electricity?
Concepts: Solar vs. gas power station, energy
production, energy conversion
Task 2: Mathematics
Solar power plants consist of mirrors with
curved surfaces. These mirrors are constructed
with the help of computer technology.
Imagine you are a constructor: Find a function
that describes the form of the mirrors to be
used for computer-based construction.
Concepts: Parabolas and their construction,
circles, focus point, quadratic functions
Task 3: Science
What are the advantages and disadvantages of
cylindrical/spherical mirrors as compared to
parabolic mirrors?
Concepts: Spherical mirrors, parabolic
mirrors, reflection of light rays
Task 4 – Mathematics
Can a power plant like Desertec really provide
15% of the energy needed in Europe by 2050?
What surface area of mirrors would be
needed?
Concepts: Area, proportional reasoning
Task 5 – Science
How can the energy be transported?
Concepts: Alternating current, continuous
current
Returning to the overarching question: Can solar power stations in the desert – such as the
Desertec project – make a meaningful contribution to the energy needs of Europe?
All materials include pedagogical guidelines and explanations for teachers, in
addition to worksheets for students. Moreover, for most of the tasks, interactive
software in the form of applets have been designed so that students can inquire about
important aspects of the task with a focus on developing important conceptual
understanding. For instance, the applet in the Desertec task offers a simulation where
students can inquiry about how mirrors of different shapes reflect parallel sunlight or
beams from focused light sources (see Figure 1).
Fig. 1. Desertec task: interactive applet Theme C – K. Maas et. al
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Our concept of iterative cycles of improvement with a structured
approach to formative assessment
The design of sustainable and robust materials for use in day-to-day practice was core
to the project objectives. A design that works in the laboratory, or can be used by
experts, may not work in the average classroom (Walker, 2006). For this reason we
studied the resource requirements thoroughly and involved stakeholders in judging the
quality of the design (Walker, 2006) during iterative cycles of design and
improvement. Our process followed Nieveen's (2009) six steps of formative
assessment and improvement to result in the final version of the materials:
a. Analysis of the context: To meet the requirements of the users we carried out an
analysis of the context following a systematic approach based on the
Anthropological Theory of Didactics (Chevallard, 2006). This analysis ensured
careful understanding of the conditions and constraints that might favour or hinder
the use of interdisciplinary approaches to the learning of mathematics and science
across the different nations of the consortium.
b. Screening: Each national design team identified problem contexts relevant, both
at a local community level, and also at a more general European level, that might
allow the development of the core concepts of mathematics and science. Each
team developed a first draft of its chosen tasks working within constraints agreed
by the consortium as a whole, relating to the important characteristics of the tasks
and their structure. Following these characteristics, one national team checked the
tasks of another team and provided feedback.
c. Expert appraisal: During this first cycle of improvement, the first version was
presented to stakeholders from school authorities and schools. They were invited
to comment on the tasks and problems posed during a focus group.
d. Micro-evaluation: In each country a small number of teachers (between 3-10)
read the materials, piloted them and commented on them raising a number of
questions, which lead to a third version of the tasks.
e. Try-out: In each country, about 20 teachers piloted this version in their
classrooms and gave feedback comprising both qualitative and quantitative data to
inform the development of the final version of the tasks.
f. Evaluation of the dissemination: Following Kelly (2009) that design research
should continue to explore models for diffusion of innovation, we implemented
several dissemination activities, including teacher conferences, professional
development courses, newsletters, and other information materials, such as posters
and flyers. The impact of our dissemination activities was also evaluated, in an
attempt to contribute to the design of the materials.
Impact of the steps on the design principles
The design process itself and the feedback that was generated informed a further
iteration in the development of design principles which we outline below.
The analysis of the context showed that competency-oriented curricula have
been introduced in most countries of the COMPASS consortium. Most of these also
include a clear orientation towards interdisciplinarity, suggesting integrated and/or
project oriented approaches. Thus curricula specifications provide favourable
conditions for interdisciplinary teaching and learning, although this is still
underdeveloped in many countries. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Despite this, there are also several constraints that pertain across the
consortium. A major constraining factor is the lack of interdisciplinary tasks in
national tests. This testing is particularly important in England and Germany, where
these tests have a strong systemic influence, but also in the other countries of the
consortium. Testing has a particularly negative effect on the introduction of
interdisciplinary tasks within the teaching of mathematics, particularly because
assessment is often seen as particularly high-stakes in the curriculum.
In addition, day-to-day school systems and practices, such as the distribution
of subjects, the existence of different teachers for mathematics and science, their
initial training or the limited collaboration between them, act as barriers for the
implementation of interdisciplinary oriented methodologies. Further, data from
different studies and reports show that highly structured teaching practices are
dominant, to the detriment of student-oriented practices (OECD, 2009). These act as
difficult obstacles to the introduction of interdisciplinary oriented tasks that follow an
inquiry-based learning approach.
In our analysis, a tension (sometimes even a contradiction) between
curricular intentions and societal demands, on the one hand, and current school
organization and teaching practices, on the other hand, were identified. Teachers can
play a crucial role in the reduction of such tensions.
These insights led to the following design principles:
• The materials are designed so as to signal to teachers a clear vision of
interdisciplinarity throughout the tasks.
• The materials are well-engineered, carefully planned, realistic in terms of
their requirement of time and students' and teachers' efforts, and
optimally adapted to each national context, including supporting an
adequate use of ICT resources.
The screening and the expert appraisal lead to the following more refined
principles:
• A clear reference to the official curriculum specification at the beginning
of the task should be provided, so as to support teachers in deciding
whether or not to use the materials.
• The tasks should offer the opportunity for true interdisciplinary work, but
teachers should also be supported to understand how the whole task,
which would need several lessons, can be subdivided into interlocking
sub-tasks that can be used in separate subjects.
• An overview about the sequence of the required lessons should be
provided.
• The tasks should be written in a way that teachers can use them directly in
their lessons.
• Interesting and motivating guiding questions for students should be
provided for each sub-task.
• The individual tasks should contribute to the solution of the whole
overarching question and the links should be made clear.
• A national adaptation to the specific cultural context of the tasks might be
needed and this should be signalled to teachers.
The micro-evaluation (via formative questionnaires, interviews in workshops
and written reviews) resulted in further revision and again more detailed design
principles, namely:
• A short, yet concrete, theoretical part needs to be included in each
activity. Theme C – K. Maas et. al
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• The materials need to have clear descriptions of the tasks for students,
including possible solutions, along with pedagogical guidelines for
teachers.
• The materials need to provide a list of mathematical and scientific
competencies addressed in the activities (as to align them with the
curricula which are competence based).
• The materials need to provide suggestions and guidelines for
differentiation.
• Students' worksheets need to be provided separately. The layout of the
materials needs to facilitate their use; it must be ensured that teachers can
differentiate at first glance between student tasks, background
information, solutions, etc.
• The materials need to provide pedagogical alternatives, e.g. for both
teachers who want to follow a very open approach (by giving only the
overarching questions) and those who want to follow a more structured
approach.
Summing up, the initial overarching 'visionary' major design principles
guided early drafts of materials; feedback, following trials of these, led to more and
more detailed design principles to inform the eventual development of final versions
of the materials. A final try-out (step e) suggested that no further improvement
seemed necessary, with the design being considered as complete and the final phase
of dissemination and exploitation process of the materials (step f) being started.
Results of the try-out and evaluation of the dissemination
For the evaluation of the trial of our materials, we used a mixed method-design, using
both qualitative formative questionnaires and quantitative questionnaires in a pre-post
design.
The teachers and the students who participated in COMPASS highly valued
the developed materials. Further, the majority of teachers reported positive attitudes in
using the materials after the end of the project. In the teachers' view, the materials
contained motivating and interesting practical activities and dealt with relevant
problems for society and realistic questions of sustainability. Furthermore, after
testing the COMPASS units, COMPASS teachers showed interest in further materials
for interdisciplinary teaching. However, the degree of satisfaction in relation to the
practicality of use in day-to-day teaching as well as teachers' opinions about how well
the tasks were described was only mediocre. These reactions show what a big step it
is for teachers to include interdisciplinary problems in their daily teaching, bringing to
the foreground, once again, the systemic conditions that hinder the use of these kinds
of activities (mainly through the general organization of schools that supports well
established school subjects). Both student and teacher questionnaires revealed that
teaching had changed towards a more student-centred and application-orientated way
of teaching.
It is difficult to determine the effectiveness of the dissemination and
exploitation of the materials. However, we can claim that the dissemination activities
carried out have been quite successful, although counting participants at an event,
subscribers to a newsletter, the number of distributed flyers, and so on, is not a clear
indicator of the success. The question remains whether that really gives a reliable
insight into the success of the dissemination. For example, more than 250 teachers
from across the consortium nations, but mainly from the host nation Germany. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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attended the final project conference. The project may have an impact on these
individual teachers; further 'reach' can be secured by engaging the interest of just one
individual, such as the representative of the Baden-Württemberg school authority,
responsible for the education of all teacher trainers in the region, who attended the
conference and has adopted use of the materials with this group. Altogether,
COMPASS ideas and designed tasks and software have reached more than 9500
teachers through dissemination activities, such as conferences, professional
development courses, video conferences, etc. Website visits, tasks' and applets'
downloads, and positive teacher reactions in workshops and conferences provide us
with confidence that COMPASS interdisciplinary tasks have been well adopted and
are used in various classrooms in a number of European countries.
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Enabling Education for Values with Mathematics Teaching

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In the fall of 2007, we conducted a workshop for high school
teachers that served as a spring board for a 3 year multi-faceted study
focusing on the integration of values education in the ordinary teaching of
mathematics in high schools in Israel. This paper is focused on the
guidelines and principles underlying the development of text-based tasks
for the study. Each of them has the potential for contributing slightly to
values education, in addition to the learning of some mathematical topic
of the mandatory curriculum. The paper also includes a few relevant
results of the main part of the study, which included action-research
carried out by 12 mathematics teachers at 8
th
grade who worked with us
for a year interweaving in their regular math teaching various tasks of that
sort.

Keywords: values and math education, personal values, social values,
text-based tasks, action research, thought experiment

Introduction

As in many other countries, the educational system in the state of Israel is
based upon the law of mandatory education (1949) and the law of national education
(1953, am. 2000, 2003). These laws state that education for values is in the forefront
of the national goals of upbringing youth towards becoming literate citizens: some of
them are social values like equity, tolerance, social justice; others are personal values,
such as being rational, achieving, and exhausting one's intellectual potential. In
practice, the study of subject-matter usually takes over, and values education subsides
to a negligible corner, in high school even more so than in primary education. The
majority of the professional teachers, and particularly high school mathematics
teachers, devote their classtime to instruction of their disciplines, and see themselves
exempt from values education. Some math teachers may even argue that this
combination is in contradiction to the "objective" nature of mathematics.
The mandatory curriculum for high school in Israel is centralized, and
includes details about the various subjects to be taught in each grade-school. In the
introduction, there is a statement about the general goals for teaching mathematics.
These goals set the stage for teaching a large variety of mathematics content,
emphasizing that teaching this content should (i) improve students' mathematical
thinking, (ii) provide skills and competency in problem solving, (iii) take into
consideration differences among students' learning styles and capabilities, avoiding as
much as possible students' frustration and failure, and (iv) make students like
mathematics and appreciate its beauty (Israel Ministry of Education, 2012). This
introduction to the syllabus leaves room for interpretation about the place of values
education in mathematics education.
Analysis of formal documents of the ministry of education, the mandatory
mathematics curriculum, and introductions to mathematics textbooks shows that there
is a gap between the legislators' intentions and the availability of means for their
realization. Indeed, it is well known that knowledge impartation, particularly
preparation for the matriculation exams, take precedence, in most cases over
education for values.
We challenged the possibility for combining values education with
mathematics teaching in a multi-faceted research study (Edri, 2010)
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. This is a
further step along the line of research initiated by A. Bishop (1988, 2008), who
offered a theoretical model of values in mathematics education, and by Harmin,
Kirschenbaum & Simon (1973) and Shechtman (1980) who offered a model of three
levels of teaching for Clarifying Values Through Subject Matter. The basic drive for
this study stemmed from the a-priori assumption that if a possibility for combining
values education with mathematics teaching exists, it is necessary to exhaust it.
The Basic Idea
Basically there are three settings in which each secondary school teacher
treats values education:
1. As a role model (not necessarily in class, but also in the courtyard, during a
fieldtrip, in a preplanned personal meeting, or a meeting by chance). Teacher's
personal norms are the basis for the education s/he gives at such settings.
2. As a classroom manager, regardless of the discipline the teacher is supposed to
teach. A mixture of different methods of classroom management and personal
norms of the teacher, which usually come from personal background, gives rise to
values education at this level, for example, teacher's tolerance to differences in
level of achievements among students, teacher's culture of listening to others,
teacher's respect to time, etc.
3. As a subject matter teacher, utilizing the curriculum as a lever to education for
values. Surely, teacher's personal norms are reflected here, but along with that the
discipline itself, in our case mathematics, has a crucial effect on the selection and
exploitation of the opportunities that the discipline allows for values education.
The first two settings are common to teachers of all school subjects. The
third is dependent on the specific discipline and the opportunities embedded in it for
values education. In this paper, we focus on the third setting, addressing opportunities
for values education from within mathematics. Needless to say, in practice it is
inconceivable to separate the three settings, as they naturally complement each other.
Hence, it is a matter of emphasis and priority that we put on the third one.
The study

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A preliminary enquiry through interviews with mathematicians and
mathematics educators, and through written questionnaires administered to
mathematics teachers, and the top percentile graduates of the educational system,
revealed various ways these populations perceive the relevance of values education to
mathematics and its learning/teaching. The common thread among these groups,
representing an accessible sample of the literate society in Israel, was skepticism
about the feasibility of connecting the two. It is noteworthy that many of the high
school mathematics teachers considered it important to combine values education
with the teaching of any subject. However, they found it difficult to connect it to their
own subject-matter and cumbersome to intentionally do it in mathematics teaching,
beyond the role-model each of them obviously plays as an adult human being who
manages a mathematics classroom. This enquiry also revealed that personal values,
such as critical thinking, accuracy, and persistence, are mostly perceived as inherent
to mathematics instruction, whereas the social ones, such as tolerance, social justice,
and empathy, are perceived as not inherent to it.
In the professional literature, we were able to find some less sceptical views:
Because mathematics is a discipline that nurtures common sense, it has the potential
for education for values (Bishop, 2008; Ernest, 2004; Vinner, 2009) including to
social ones (D'Ambrosio, 1990; Gutstein & Peterson, 2006; Osler, 2007; Taplin,
2009; Voss, 2012). Despite the risk, our study originated from the hypothesis that
embedded in the mathematics curriculum is a potential not only for personal values,
but also for social values education. If this is indeed the case, ways must be found to
exhaust this potential. We found some support to this hypothesis in a number of
examples given by Mistrik &Thul (1997), Gutstein & Peterson (2006), and later on by
Taplin (2009) and by Voss (2012).
The first stage
To start, we conducted a thought experiment, in which an analysis of the syllabus was
carried out to identify places in the curriculum suitable for linking to various values.
This search yielded "The values-annotated math curriculum". This work was based
upon the following guidelines established in a workshop that took place prior to the
thought-experiment:
• Replacing the context of 'word problems' (keeping the underlying mathematical
model) so that they can yield, beyond the mathematical solution a discussion that
has bearing on values, such as environmental conservation and protection,
preventing drugs, smoking and drinking, and more.
• Linking mathematical concepts and terms to values (e.g., equality, inequality,
true vs. false statement, identity, negative and positive numbers, rational and
irrational numbers).
• Assigning higher level tasks and training students to persist in solving harder
problems, to join their effort, to collaborate in solving difficult problems, to not
give up easily, to seek and experience the fulfilling intellectual satisfaction
attained by a solution reached following an investment of effort.
• Providing education for life in a democratic society through mathematics:
- Axiomatic systems can be compared with The Constitution. Acceptance and
follow-up, rather than argue and quarrel.
- Laws /rules/mathematical theorems can be employed as a springboard for a
discussion of values in a more general way - distinguish between lawful and
unlawful actions, observing mathematical laws prevents mistakes, and Theme C – N. Movshovitz-Hadar & Y. Edri
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observing civil laws prevents other trouble. A law in a democratic society is
accepted by the majority and can be changed by the majority; in
mathematics, there is no democratic vote, there is a proof.
- Connecting mathematical actions to parallel ones outside of mathematics For
example, in court there is a demand for a proof for any claim; in court, a
proof is only beyond a reasonable doubt. In mathematics, a proof stands on
logical basis; once proven, a theorem ("verdict") cannot be reopened for
consideration (although there is always room for purification of the proof
itself).
- Conflicts and conflict resolution among friends, siblings, rivals or political
parties can be brought up following tasks focused around mathematical
paradoxes. Emotions, such as frustration, can be reflected. Accepting
differences in opinion as a basis of friendship without forcing a consensus, as
opposed to the necessity to resolve cognitive conflict and mathematical
paradoxes resulted by some hidden errors.
- Encouraging debate in mathematics through tasks involving paradoxes is a
challenge in itself. What is right and wrong in mathematics is not determined
by a majority vote.
- Freedom of thought with tools and limitations linked to such tasks. There can
be distinction between random actions and arbitrary choices, action based on
intuition or on rational explanation.
- The ability to reason logically and the power of being rational, rather than
emotional or impulsive, in explaining your stance and in drawing
conclusions.
- Accountability is another important issue that can be integrated.
• Using a clear and accurate language orally and in writing can be linked to
learning mathematics. More precisely, the ability to define clearly a concept, and
to sequence logically a chain of arguments which follow one another, can be
enhanced through mathematics. This includes:
- Alternative (equivalent) definitions
- Finding "holes" in arguments presented by others (in a civilized manner),
- Responsibility for accurate wording, and liability
- Elaborating on idioms in the language borrowed from mathematics. For
example, "Squaring the circle" (for an impossible mission), "They have a
common denominator" (for comparing two issues), Two things are
"orthogonal" to each other (meaning cannot go hand in hand) or parallel (go
hand in hand but never meet), "This is an axiom" (meaning - one should
accept it as true)
• Bringing up human values, such as equity, freedom, determination, and modesty,
through interweaving stories about the lives of mathematicians and the history of
mathematics that have to do with values. For example:
- The life story of some female mathematicians, for example, Sophie Germain
(1776-1831) who used the pseudonym of a man, Anthony La- Blank, to find
her way among the mathematicians of the time and win their respect. Such
stories can lead to a short discussion about gender these days, and about
"masculine" vs. "feminine" professions.
- Cartesian coordinate system leads naturally to Rene Descartes. He was a
sickly child and his parents let him get up late in the morning and stay home
a lot. He used it to think and create mathematics. He is quoted as saying: "I
prefer truth over beauty." He dealt with the difficulties and even managed to Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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create despite the difficulties. He was known as a humble man who lived a
Spartan life but never criticized others for having a different life style.
• Addressing matters inherent to teaching, particularly to the teaching of
mathematics, that teachers need to be aware of, to emphasize during the teaching
process, and to leverage explicitly or implicitly to values education, such as:

- Self-confidence and lack of it, the freedom to make mistakes and learning
from a mistake, acknowledgement of potential ability, patience and
tolerance, consistency, persistency, self-control, delayed gratification,
intellectual courage.
- Express feelings of anxiety and fear of failure in mathematics and compare
them to similar feelings in students' lives. Hope and despair, frustration and
joy of achievement, a sense of power and confidence versus inferiority - how
do you deal with them?
- Exceptions, such as the prohibition of division by zero, can be linked to
treating exceptional cases in society.
- Quantification of the size of an error (in learning statistics) can be linked to
the courage to speculate, to guess, to take the risk of mistake. (The freedom
to err is the freedom to learn.)
- Unexpected solutions to a mathematics problem or some other mathematical
surprises can be used to talk about situations in life where a problem was
solved unexpectedly, the joy of discovery.
The next stages
The second stage was a development of 23 exemplary text-based tasks that teachers
would be able to assign to their students while teaching mathematics in the 8
th
grade.
In the main part of the study, 72 experimental lessons were taught in 7 high schools
by twelve 8
th
grade teachers who applied some of the exemplary tasks in their classes
and created more tasks inspired by the exemplary tasks and following the guidelines.
The rest of this paper is devoted to 3 of the 23 exemplary tasks. An intentional effort
was made to keep the part of values education limited to a small part of the
mathematics lesson, so that progress in teaching and learning the mandatory
mathematics curriculum would not be harmed as a result of assigning these tasks to
the students.
Sample Task 1
This task is basically a word problem. At the mathematical level, it is a
practice task in basic calculations of percentage rate adapted from a common
textbook, by changing the context and adding two parts -- no. 1 and 4, which add the
values education aspect (see below). At the values education level, the goal is to
educate for social equity by raising students' awareness to an affirmative action taken
by the government of Israel to promote appropriate representation of minorities in the
civil service. Theme C – N. Movshovitz-Hadar & Y. Edri
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The Task
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In 2007, minorities (Arabs, Druze, and Circassians) were one fifth of the
population in Israel. Despite this, only 6.2% of all civil service employees in this year
were minorities. Over the years, the Israel government has made decisions (in 2004,
2006, and 2007) to promote suitable representation of minorities in the civil service,
setting 10% as a target for the percentage of employment of minorities in the civil
service.
1. What do you think about the goal that was set by the government?
2. The Ministry of Housing and Construction had 741 employees in 2007. Had the
target set by the government been achieved, how many members of minorities
would have worked in the Ministry of Housing and Construction?
3. Twelve employees in the Ministry of Housing and Construction were minorities in
2007. What percentage of all employees in the ministry were minorities?
4. What do you have to say about the two results you obtained?
Integrating Education for Social Justice in the Lesson
The answers to Part 1 varied. Some students explained that because minorities were
one fifth of all citizens, their proportion in the civil service should be 20% which
represented a fifth of the citizens. Other students claimed that people are selected for a
job according to their capabilities, and it may be difficult to find suitable people
among the minorities for some jobs. They maintained that there is no reason to hire a
person who is not qualified only because he belongs to a minority group.
The mathematical solutions of parts 2 and 3 of the task
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were handled in the
common way, starting by independent student-work, followed by a dialogue between
the teacher and the students to elucidate the particulars.
Responses to Part 4, which was intended to make students aware of the huge
gap between the situation in 2007 and the goal set by the government, indeed gave
start to the social-values-oriented discussion that followed.
In the course of the class discussion, the teacher provoked questions about
the issue of affirmative action for minorities in Israel. For example, the teacher asked:
"Why at all was the government promoting employment of minorities in the civil
service?" There were various answers:
• One student said that the principle of equality among people should lead to
equality in representation of minorities in all areas.
• Another said that the principle of equity among people should be interpreted
as equality in the criterions for hiring for a job.
• Another student said that Arab parliament-members were exerting pressure on
the government to take action in this area to improve the economic and social
status of the Arabs in Israel.

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The figures in this problem were taken from a report by a parliamentary investigative committee on
the subject of accepting Arab employees in the public service, headed by MP Ahmed Tibi (2008).
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The mathematical solution to Part 2: 10% of all the employees in the Ministry of Housing should
have been members of minorities, i.e. 74 employees.
The mathematical solution to Part 3: 12 employees constitute 1.6% of all the employees
in the Ministry of Housing. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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• Some others claimed that improving the economic and social status of Arabs
in Israel is in our interest, if we wish to live in peace.
• Two girls had a dispute between them: one claimed that promoting equality of
Arabs was liable to harm the interests of Jews in Israel which was established
as a homeland for the Jewish people. The other girl said that when Jews were
a minority in the Diaspora they suffered from injustice; now in Israel, we
ought to give equal rights to the minorities in understanding their suffering.
A few clarification issues were also addressed: What is civil service work?
What reasons could cause difficulties in implementing the government decision to
promote employment of minorities in the civil service?
The task drew students' attention to the question of representation of
minorities in The Ministry of Housing and Construction, and in the government in
general. Nevertheless, in the course of the discussion about these questions, several
related issues came up. One was the kind of work minorities do in Israel. The
discussion diverted to other workplaces: schools, hospitals, post offices, the Israel
Broadcasting Authority, and more. For example, one student mentioned that in this
school an Arab woman worked as an Arabic teacher, and 12 more Arab women
worked as cleaners. Another student said that a few Arab doctors can be seen in the
hospitals, but many Arabs working there are cleaning staff.
The teacher summarized, acknowledging the importance of the discussion
and leaving some of the issues open for further discussion in the future and outside
the mathematics classroom. Then the teacher moved to another activity related to
percentage rate.
Sample Task 2
At the beginning of the chapter on solving linear equations, students were assigned
for homework a Google search or any dictionary/encyclopaedia search of words
related to "equation," summarizing the definition of the words they found.
Students came back to class reporting about the definitions of equator,
equality, equilibrium, equalizer, and more.
The teacher indicated that upon completion of the study of solving
mathematical equations, students would be asked to compare their understanding of
the processes involved in equation solving with their understanding of the related
non-mathematical terms they found.
This final task, which apparently consumes a minimal part of in-class time,
was a trigger for summarizing the mathematical aspects of solving linear equations
from a new perspective and quite naturally opened the door to the discussion of
inequality - gender inequality, racial inequality and even inequality in family
relationships. This discussion also served as a nice transition to the next mathematical
topic – solving linear inequalities.
Sample Task 3
This task is adapted from One Equals Zero (Movshovitz-Hadar and Webb
1998). It is appropriate for presentation either before or after the study of area of a
parallelogram, or earlier when the notion of area is introduced.
1. In Figure 1, four strips of width d are marked 1, 2, 3, 4. They connect
the two parallel lines PP' and QQ'. Which strip has the largest area?
Which one has the smallest area? Theme C – N. Movshovitz-Hadar & Y. Edri
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2. Copy the parallelogram (2) so that its left side coincides with the left
side of the rectangle (1). Express the area of the parallelogram as a
difference between two areas, and reconsider your response. Would
you now change your mind?
3. Read the following explanation:
Were you convinced? Did reading this explanation change your mind?
4. Share with your peers the feelings this task gave you. Did you feel
that it was unfair to present the task the way it was presented?
5. Would you agree that the freedom to err in mathematics is actually
the freedom to learn?
The teacher can stop the individual activity after part 1 and have a vote or
even rank order the strips by students' votes.
Many students tend to choose Strip 1 as the smallest in area and Strip 3 as the
largest. It would take "Poker-Face Pedagogy" on the teacher's part to let students
continue working on the next two parts. (See more about "Poker-Face Pedagogy" as
related to tasks, in Movshovitz-Hadar, 2011.) This work might put at stake the
students' knowledge about area and hopefully yield a deeper understanding of it. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Nevertheless, some students may feel cheated. Some of them may suspect the teacher
is intentionally misleading them, while actually the teacher is intentionally
challenging the fragility of their knowledge. (See more about "Knowledge Fragility"
as related to tasks, in Movshovitz-Hadar, 1993.) Of course, pedagogically speaking,
there is nothing wrong or unethical in presenting this task or others like it. However, it
is up to the teacher to handle such tasks with care, so as to prevent possible frustration
from creeping in, and exploit the potential opportunity for values education by
moving to parts 4 and 5 of the task.
In part 4, students are asked to express their feelings about the task. Have
their senses ever tricked them? - Hearing some noise that seemed frightening but
proved to be nothing more than a cat; entering a swimming pool and feeling that the
water is much colder than it feels after a few minutes; eating a sweet candy after salty
French fries. Through tasks of this kind, students can become aware of the power of
being rational rather than emotional or impulsive in drawing conclusions. They would
gather that they should be careful about their intuition and about taking what they see
as a basis for decision making too quickly. The value of trusting one's sense of reason
is enhanced as more fundamental than trusting one's intuitive eye-sight.
Emotions such as surprise and frustration can be reflected. This has an
embedded potential for connecting to values education as follows. Students can be
encts ncaged to talk about other incidents of frustration following conflicts which they
encountered or heard about, such as conflicts among friends, siblings, rivals or
political parties. The role of conflict resolution can be discussed and the relief it
brings when it is achieved. Some students may even point at a difference between the
necessity to resolve a cognitive conflict resulting from some hidden mathematical
error or misconception and the unnecessary enforcement of consensus among friends
who respect and tolerate each other despite different political views they hold.
Encouraging debate in mathematics paradoxes is a challenge in itself. This
task brings up the point that right and wrong in mathematics is not determined by a
majority vote. Mathematics has its own tools. Unlike spoken language in which
misunderstandings occur because interpretation often depends on the context, and
may sometimes lead to conflicts and unfortunate incidents, in mathematics there is
very little (if any) room for dispute.
The moral of this task is that the freedom to err yields an opportunity to
learn.
Effect of the Experimental Work on the Teachers
The findings of the study (Edri, ibid) indicated that the teachers' involvement in the
experimentation contributed to their personal and professional development. Here are
a few statements from the reflections of participating mathematics teachers
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:
…At first, I was very dubious about integrating values education in my
mathematics teaching. Values such as accuracy or intellectual courage, are values
that I could see fit in with mathematics, but social values?!? When I heard about
the study, I thought that if social values were to be integrated, it would be
artificial… Up until now, I never considered integrating values education in the
framework of mathematics lessons at all. I didn't think this could really be done.
It surprised me that it was possible to integrate values such as "social justice" in
mathematics lessons… I thought it could be done only at the margins and only

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The study was conducted in Hebrew and the quotes were translated to English by the
authors. Theme C – N. Movshovitz-Hadar & Y. Edri
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indirectly… When I gave the lessons in statistics concerning violence among
students, it opened the way to a lively discussion on the subject… Without my
expecting it, students spoke about personal experiences related to violence…
During our in-service preparation meetings the examples presented to us started to

inspire me… however, when I started employing them in my class, integration of values in teaching mathematics subjects became gradually a second nature to me … the students dared saying to me that my teaching changed, became much more interesting, and relevant to life, and I also enjoyed it a lot… They say that you don't really absorb material until you teach it, and that's exactly what happened to me. Until I experienced activities that combined values education with mathematics in class, I didn't understand how much I could affect the students, and what values discussion could develop around mathematical activity… Today, I make sure to include some task that has to do with values education in almost every lesson I prepare…
Note that all 12 teachers who took part in the study succeeded in integrating personal as well as social values education in their teaching without harming the results of teaching mathematics according to the mandatory curriculum. They attested that their involvement in the experimental teaching contributed to their professional growth and to their personal development as educators.
Summary and relevance to ICMI22 Theme C
In this paper, we focused on a description of sample tasks aimed at integrating values education in teaching subjects from the mandatory mathematics curriculum. The integration is achieved by following the guidelines mentioned at the beginning of the paper into text-based tasks for students, which teachers can adapt to their own classes and teaching style. This was quite a challenge.
Examining the tasks and employing "reverse engineering" analysis
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, the
following are suggested as the design principles underlying our work on the development of text-based tasks:
1. The tasks are mathematics-curriculum based assignments, which can be completed within the framework of one class period or one homework session.
2. They include a clearly phrased introduction followed by two kinds of short questions: (i) Mathematical problem solving, preferably of exploratory nature or thought provoking nature; (ii) Dialogue promoting questions intended as a vehicle for values clarification through bridging between the contents and student's feelings, views, and behavior (Harmin, Kirchenvaum & Simon, 1973; Raths, Harmin & Simon, 1978).
These questions are sometimes provocative, however "neutral" in nature, so as to allow the surfacing of various personal attitudes about human values and avoid any implicit hints as to "the desired" response. These questions may serve as triggers for a short whole-class debate connected to the mathematical work, but not necessarily in a strict way.
3. The text is carefully expressed so as to avoid in as much as possible obstacles in the form of editorial faults (see more in Movshovitz-Hadar, Inbar, & Zaslavsky, 1987).
4. Tasks can be assigned as individual work, or as group work, for class time or for homework, depending upon the teacher's preference. The teacher's manual

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Reverse engineering as a tool for validating a goal-oriented task development was introduced in Amit and Movshovitz-Hadar (2011)Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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includes recommendations for facilitating and handling values-related discussions, which usually are not an ordinary part of mathematics teachers' background.
The amazing fact that values education can be integrated even in mathematics, a leading rational field, was confirmed by the team of teachers who participated in the experimentation of the idea. Beyond the feasibility of the integration itself, it was also proven that it is possible to do so without lagging behind the mandatory curriculum on the one hand, and on the other hand without harming students' achievements, even improving them in many cases.
As we discussed in an earlier occasion (Edri & Movshovitz, 2009), turning the integration that has been proved possible into a tool regularly used by mathematics teachers is essential, but it will obviously require (i) preparation of suitable tasks for the various age levels and many subjects appearing in the curriculum, and their introduction into textbooks, and (ii) preparing inservice teachers as well as future teachers to use them and possibly invent more of the kind.
The constant search for opportunities to integrate values education in mathematics teaching adds a missionary aspect to the mathematics teacher's work, and justifies the great investment, from which the resulting satisfaction enriches not only the students, but also the teacher.
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Designing Interdisciplinary Curriculum for College Algebra
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Interdisciplinary curriculum, long viewed as a "best practice" for early undergraduates, is usually delivered through team teaching or learning communities. However, the complexity and expense of these approaches preclude wide implementation. This manuscript proposes an approach to designing supplemental, interdisciplinary curriculum for college algebra textbooks that a single mathematics instructor can deliver. Key elements of the model include an essay and disciplinary introduction co-developed by a mathematics instructor, a disciplinary specialist, and a creative writer; explicit learning goals to limit the responsibilities that a teacher feels for expertise in another discipline; and interdisciplinary homework questions. We discuss a sample module that was designed to engage students pursuing social sciences and human services majors.
Keywords: interdisciplinarity, algebra, modeling, undergraduate education
Introduction
This paper proposes an approach to designing interdisciplinary curriculum to supplement college algebra textbooks. Higher educational leaders in the United States have called for increased interdisciplinary undergraduate education for decades, but its implementation has been relatively uncommon (Ellis, 2009; Leskes & Miller, 2006). We outline an approach to designing curriculum for first-year algebra courses that is authentically interdisciplinary, that is, the curriculum supports learning goals that are significant within the partner discipline as well as in algebra.
Our work takes place in college algebra classes that serve students pursuing studies in education and other human services disciplines, such as social work, family social science, sport management, and human resource development. For most of these students, college algebra is their final mathematics class. Teaching students at the end of their formal educational journey in mathematics poses significant dilemmas of responsibility. Within the framework imposed by the institution, how can a teacher help students strengthen their skills and their sense of purpose sufficiently so that mathematics may remain relevant in their academic work and in their lives?
In addressing this dilemma, we focus our curriculum experiment on connections between algebra and the social sciences and the humanities rather than the sciences (Usiskin, 2003). The curriculum module described below takes the unusual step of linking the algebra of finance with a classic theory in psychology, one Theme C – S. Staats & J. Johnson
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that students often encounter in an introductory psychology course. Erik Erikson's theory of life stage development holds that people experience identity crises throughout their lives; productive engagement of these crises results in the person's positive psychosocial development. As an uncommon curriculum partner for algebra, psychology provides a useful case for probing the problems and limits of creating interdisciplinary curriculum, more so perhaps, than partnering algebra with a scientific discipline in which the applications are well-understood.
This module is intended to be used as a homework supplement to a college algebra textbook, and so it occupies the curriculum level that Usiskin calls a "lesson" (2003). In this paper, however, we focus primarily on principles of organizing modules so that they support interdisciplinary teaching by a single mathematics instructor. We consider this curriculum as a type of task that is designed according to specific principles and that creates a non-traditional learning environment (Becker & Shimada, 1997).
Staats is an associate mathematics professor in the College of Education and Human Development (CEHD), University of Minnesota, USA, where she teaches college algebra and interdisciplinary seminars, primarily to undergraduates enrolled in education and human services fields of study. One of her research strands involves designing interdisciplinary algebra curriculum and analyzing student reactions to it (Staats, 2005, 2007; Staats & Robertson, 2009). Staats' doctoral degree is in cultural anthropology; her experiences teaching with anthropology curriculum influenced her thinking about interdisciplinary algebra curriculum. Jason Johnson is a graduate student studying educational public policy. Johnson taught mathematics at Henry Sibley High School in Minneapolis, Minnesota for nine years, where he became interested in interdisciplinary curriculum development through founding and coaching the school's debate team.
The theoretical background for this experiment in task design is the pedagogy of mathematical modeling and studies of mathematical discourse. Mathematical modeling involves cycles of simplification, solving, interpretation, validation and generalization, or critical reflection leading to improved solutions and methods (Lesh, et al, 2003; Maaß, 2006). The module discussed below is assigned within a college algebra class that is dedicated to modeling. Interdisciplinary writing assignments are presented as a particular style of interpreting and validating mathematical work.
Studies of mathematical discourse also inform this work. Gerofsky (1996, 2004) and Morgan (1998) have detailed ways in which mathematical writing and speaking rely on stylistic and grammatical patterns or genres that may influence the responses that students consider appropriate to offer. The genres of mathematical

writing thus problematize the goals of interdisciplinary learning. In the curriculum
design model discussed here, a creative writer constructs the core essay for the
module, helping to establish expectations that are less centered in one discipline.
Interdisciplinary Math in Higher Education
The Curriculum Foundations study by the Mathematical Association of
America (1999-2001) identified interdisciplinary, team-written curriculum for early
undergraduate mathematics curriculum as a high priority (Ganter & Barker, 2004).
However, typical means of organizing interdisciplinary education are complex,
expensive, and require an enormous commitment from institutions and instructors
(Boix Mansilla, Miller & Gardner, 2000; Burrill & Hernández-Gantes, 2003; Klein, Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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1990; Lattuca, Voigt. & Fath, 2004; Usiskin, 2003; Wentworth & Davis, 2002). In a
team-teaching model, two or more faculty members provide instruction on an
interdisciplinary theme to a single class of students. In the learning community model,
all the students who enrol in one class also enrol in a class in a different discipline.
Although both models ensure that students receive instruction from disciplinary
experts, the complexity of scheduling classes, registering students, and achieving an
adequate balance between faculty compensation and generated tuition means that the
best practices for interdisciplinary education are beyond the means of many higher
education institutions.
Strong disciplinary boundaries also make interdisciplinary work challenging.
Instructors often feel uncomfortable leading explorations outside their area of
expertise, particularly in mathematics. Mathematics curriculum may be defined more
rigidly than in other fields (Grossman & Stodolsky, 1995). Mathematics faculties
often have limited professional interactions with faculty in other disciplines (Ewing,
1999). At times, even the philosophical basis of mathematics is held in contrast to the
integrative goals of interdisciplinary education (McGivney-Burelle, McGivney &
Wilburne, 2008; Siskin, 2000). This sense of disciplinary authority and curricular
boundedness can impede the adoption of deeply interdisciplinary curricula in
mathematics classes (Staats, 2007). When the ideal delivery models for
interdisciplinary education are not feasible, carefully-designed, supplemental texts
may empower a single mathematics teacher in a standard college algebra class to
deliver short, restricted units of interdisciplinary content.
Design Principles
Drake (2010) found that teachers often introduce supplemental materials
when their textbooks lack varied activities and when they wish to engage students
with far-ranging needs and interests, needs that interdisciplinary curriculum can help
fulfill. First, an interdisciplinary curriculum clearly must enable students to
demonstrate competency on topics drawn from each discipline. Correspondences
between the goals of interdisciplinary education and several contextual approaches to
mathematics education can further guide development (Staats, 2007). The modeling
approach emphasizes the creative process, critical reflection, and refinement. Realistic
mathematics education seeks scenarios that are authentic and relevant to students.
Social justice approaches seek to create greater awareness of social issues. Each of
these themes -- relevance, awareness of social complexity and taking a critical
perspective on one's own solution -- are key goals of interdisciplinary learning as
well. A supplemental interdisciplinary curriculum for college algebra should also be a
useable module that a single mathematics teacher can deliver. The following design
features help to fulfill these goals:
1. A module is co-authored by a mathematics teacher, disciplinary specialist,
and creative writer.
2. The module core is an "essay," written in any genre, which poses questions or
presents scenarios that require consideration of algebra and another discipline.
The role of the creative writer is to present the interdisciplinary scenario in a
lively and engaging way, and to ensure that the module avoids framing
questions within the ideology of a single discipline.
3. An introduction can convey disciplinary content in more direct fashion than
the core essay. This allows for more expressive latitude for the essay. Theme C – S. Staats & J. Johnson
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4. The essay is followed by a list of explicit learning goals for both algebra and
the partner discipline. The learning goals help limit the content from the
partner discipline that the mathematics teacher will discuss in class.
5. Homework includes scaffolding questions on algebra and on the partner
discipline, and interdisciplinary activities.
6. The final component is a short bibliography to support further reading by
either instructor or student.
This structure for curriculum writing is modeled in part after a style of
undergraduate curriculum writing that is prevalent in the social sciences. In a typical
introductory anthropology class, for example, students use a textbook which conveys
the course content directly and a reader with case studies in the subfields of cultural,
biological and linguistic anthropology and archaeology. A reading has an introduction
and a case study, either one written as curriculum, or an excerpt from a published
work. This style of curriculum writing allows instructors to present material that
diverges widely from their own specialty. Scientifically-oriented biological
anthropologists, for example, regularly teach about linguistics or symbolic analysis,
perhaps having taken only a single class in this subfield. The case of anthropology
curriculum development and delivery suggests that it is possible for a single instructor
to teach across wide epistemological differences.
In the proposed algebra curriculum project, enlisting a creative writer to
develop the focal essay helps to ensure interdisciplinary integrity. The creative writer
chooses a genre -- fiction, life history, poetry, academic prose, nonfictional
exposition -- that conveys the perspectives, philosophies, and curricular needs of the
two disciplines in a lively way. The essays must invite students to demonstrate
analytical skills in two disciplines, but they should not privilege one perspective over
another. Creating this dynamic tension between different viewpoints and different
ways of organizing ideas is at the heart of the writer's craft and, perhaps, of
interdisciplinary thinking as well.
Mathematics teachers are not expected to be experts in the linked discipline. It
is, however, reasonable to expect mathematics teachers to be able to learn a limited
amount of first-year university-level material and to lead a critical, not authoritative,
classroom conversation based on this reading. Some mathematics teachers do this
when they use newspaper clippings as a basis for mathematics lessons. Some
universities assign instructors from across the disciplines to lead discussions on a
book that all entering students read. These teaching activities are relatively
commonplace situations in which mathematics teachers lead students in discussion of
material outside their area of scholarly specialization. They position mathematics
instructors as lifelong learners who have developed basic intellectual and civic
competencies through the liberal education process that their first-year students are
about to commence.
The sample module that connects financial mathematics to Erikson's theory of
psychosocial development can be viewed at z.umn.edu/icmi22. The core of the
module is a short story titled Indebted, in which a young man wrestles with the
question of how to pay for his college education. The young man visits his
grandfather, who suffers from Alzheimer's disease and lives in a nursing home. The
grandfather hoped to contribute to his grandson's education, but instead, he was
obligated to use his life's savings for his own health care. The young man considers
mathematical scenarios associated with indebtedness, such as rapidly rising college
tuition and the per capita value of the national debt. Finally, he uses an elegant writing
pen, a gift from his grandfather, to sign his college loan papers. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Relevance in K – 12 Settings
In primary grades, interdisciplinary learning is often experiential, sometimes
drawing from the foundation of community knowledge. For example, an elementary
school in Arizona explored a wide range of mathematical, environmental and literacy
topics by creating a class garden (Kahn & Civil, 2001). These active lessons are
sometimes called predisciplinary curriculum (Applebee, Adler, & Flihan, 2007, p.
1005). The interdisciplinary activities in this design model could certainly suggest
experiential or ethnographic activities. Students could educate peers on financial
literacy through social media, record a financial life history of an elder in the family,
or interview adults about their methods of reconciling monthly wages and payments.
Still, the project described here focuses on principles of organizing written
texts for use by individual students. For this reason, it seems to be most applicable in
an upper secondary setting as a bridge experience into university life. The modules
could introduce students to a wider range of disciplines than they would encounter in
secondary school. They would help students reflect on the theoretical foundation of
disciplines, a goal of a recent curriculum reform in Danish schools which calls for the
development of multi-disciplinary learning activities (in contrast to integrated,
interdisciplinary ones) (Andresen & Lindenskov, 2009). Interdisciplinary activities
can also support opportunities to write from a variety of perspectives, including
research-based, persuasive, fictional, or experiential. Overall, this curriculum
approach seems more appropriate for students making the transition to the writingbased academic experiences that are typical of university learning.
In some settings, secondary schools are working towards interdisciplinary
learning in ways that could be enhanced by this curriculum. A case study of 11
interdisciplinary teaching teams serving over 500 students in middle and high schools
found a variety of levels of curriculum integration, from simply juxtaposing subjects
to fully reconstructing the curriculum (Applebee, Adler, & Flihan, 2007). The team
that included a mathematics teacher represented relatively little integration. A team of
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th
grade teachers developed substantial integration across English, social studies, and
art through the theme of teaching respect and understanding for varied cultures and
peoples. Team members reported a sense of being co-learners as their own
interdisciplinary understanding emerged. They developed interdisciplinary exam
questions together, but many student activities were developed "individually out of
each teacher's sense of the issues shared by the team as a whole" (Applebee, Adler, &
Flihan, 2007, p. 1027).
Portions of the proposed design model would capture some of this intensely
interpersonal work so that it could be shared in new settings or with new team
members. Artifacts of the team's work that would prove useful to others include
learning goals for each discipline, student activities, and disciplinary materials
(corresponding to items 4 – 6 in the previous section). The teaching team's reflections
on the disciplinary ideas that informed student activity could form the introduction of
the module. The focal essay, however, would not develop from the work of the
teaching team in a natural way. A creative writer would have to construct a scenario
that affords consideration of all the subjects.
The structure of the design model might make interdisciplinary learning more
acceptable within the institutionalized curriculum frameworks which are prevalent in
K-12 settings. Teaching across disciplines is at times controversial and politicized. In
Boyle & Bragg's 2008 study of combining curriculum in the U.K., mathematics and
history were taught as single subjects more often than other subjects, even during time Theme C – S. Staats & J. Johnson
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periods of increasing cross-curricular teaching. The organization of the proposed
curriculum, in particular, listing explicit learning goals for each discipline, may help
empower teachers to justify connections to required curriculum.
The Indebted Module: Research in Practice
Staats has used this module in her college algebra classes over the last three
years, revising components of it at the end of each semester based on the quality of
students' work. This section briefly outlines some of the design choices made during
this research-in-practice (Schoenfeld, 2009). Revisions were governed by four
questions:
1. What format would make this module a usable homework supplement to a
college algebra textbook for an instructor who does not work in an
interdisciplinary program?
2. What types of interdisciplinary questions encourage students to synthesize
ideas from algebra and psychology?
3. What kinds of interdisciplinary questions stimulate students to pose and
answer algebra scenarios that are more complicated than textbook word
problems?
4. What kinds of interdisciplinary questions do students seem to enjoy?
The financial mathematics section of the class textbook (Harshbarger &
Yocco, 2010) was both engaging and frustrating for students. Many students find this
topic relevant and interesting, particularly when problems allow them to understand
the high accumulated interest costs of major purchases, such as houses and cars.
However, many students also find it difficult to decide which equation -- compound
interest, present or future value of annuities, amortization -- to use for different
scenarios. Occasionally a student made a wry comment about homework problems
that presented overly positive scenarios of parental savings for children's higher
education. In the spring of 2009, Staats asked her colleague, Gary Peter, if he would
write a short story that would help students develop a critical perspective on the
financial mathematics section of the textbook. Gary Peter is a literature and sociology
of law instructor who also writes fiction.
The first version of the module, then, was the short story itself. The story
begins:
"For all the important documents in your life," my grandfather said. "Your
marriage license, your first mortgage…you know. A young man about to go out
into the world should have a really nice pen for those things."
As the story progresses, the young man comes to understand his
grandfather's deeply held polarity between hard work and personal debt. He learns of
his own father's financial troubles. Trying to grasp the relative sizes of debts, the
young man calculates his own share of the U.S. national debt. Near the end of the
story, he receives an application for a student loan:
Neither a borrower nor a lender be. Sorry, Grandpa. We all should have listened
to you. But it was too late now.
I picked up the pen, the pen that my grandfather gave me, and signed my name.
During the first semesters of the module's use, several students decided to
model tuition increases at the University of Minnesota. In accordance with revision Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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principle 4, we added a small bit of text to the story in response to this student
interest:
Maybe I should just work for a while before college. Save up my own nest egg,
pay for tuition straight up. Less debt. But then I remembered the newspaper
article last month: University tuition increases 3%. Would it be worse if I waited?

This revision to the story was intended to encourage students to consider
modeling debt issues in matters related to their lives.
A mathematical dilemma that has always been present in implementations of
this module is that the underlying mathematics of the formulas for this section of the
class requires consistent payment and interest values. The algebra of geometric series
limits the variety of scenarios that students can mathematize algebraically.
Consequently, students most commonly pose a series of financial problems for
various life stages. At the end of each life stage, the student reconciles the account
and uses the output as the input for another problem. This can assist students'
mathematical understanding because it deepens their understanding of how to choose
a formula that applies in a particular situation, one of the typical difficulties.
However, students typically do not extend the algebraic formula, for example, by
deriving a new equation. In the following semester, we plan to use a spreadsheet
exercise to encourage students in creating more flexible numerical models to
accompany their short stories (revision principle 3).
In the original version of this module, its interdisciplinary character was not
clear. Was it economics, sociology of aging, or literature? Establishing the
disciplinary partner for this story explicitly was a necessary stage in making the
module usable by other teachers (revision principle 1). In the spring of 2010, Staats
asked a retired psychology professor (her mother, Sara Staats of Ohio State
University!) to read the story and comment on any themes that are addressed in an
introductory psychology course. Erik Erikson's model of lifelong identity struggles
seemed to provide an academic means of analyzing the positions of the young man
and of his grandfather. Staats and Staats developed the introduction to the module so
that it became more formally an experiment in linking the disciplines of algebra and
psychology. In this module, because the story was written first, all disciplinary
content had to be placed in the introduction, but in the future, maintaining these as
two separate components will likely prove useful, as well. The introduction provides a
space for explaining framing ideas, and this allows the essay to present focused case
studies and lively exposition. At this point, the learning goals and the scaffolding
questions in algebra and in psychology were also added to the unit. Both the learning
goals and the introduction are written to communicate to the instructor as much as to
the student, to highlight direct disciplinary content as well as the "intentions and
purposes" of the module to aid instructor implementation (Mason & Johnston-Wilder,
2006, p. 28).
Once the introduction and learning goals were added to the module, students
could be held accountable for content in psychology. The introductory material made
it easier to ask students to think about the emotional stories that lie unspoken behind
the assumptions of an equation. Interdisciplinary homework questions were revised to
create thinking around life stage dilemmas, personal relationships, financial
behaviors, and modeling the results of these behaviors. The interdisciplinary
assignments have changed or been refined with almost every semester as Staats
learned more about the scholarship of interdisciplinary teaching and learning. For
example, one of the original questions was: Theme C – S. Staats & J. Johnson
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When David's grandfather entered the nursing home, the family didn't have time
to develop a different savings plan. What if the grandfather's illness happened
earlier and the family did have time to change their savings plan?
Make up an example in which a family uses two different savings plans to save
for college. Try to estimate how much they will be able to save. You can make
whatever assumptions about income and interest rate that you want, as long as
you state them clearly.
This question prompt was no longer adequate because it did not ask students
to engage a psychological perspective. Each semester, there were always quite a few
students who seemed to enjoy writing fictional stories, and so new interdisciplinary
questions (revision principle 2) asked students to write stories that portray life stage
dilemmas:
Create a character representing a person moving through several of Erikson's life
stages: adolescence, young adulthood, middle adulthood, and senior. Discuss the
identity conflicts that this person experiences in each stage, and describe the
character's financial behaviors, with a sample calculation, for each stage. Can you
create a scenario using a single equation that produces a similar financial outcome
for this character?
The last question in this prompt, "Can you create a scenario using a single
equation that produces a similar financial outcome for this character?" proved to be
problematic. The question asks students to reflect on their story and calculations and
then to create a single equation that would create the same financial outcome as their
series of posed equations. It was inspired by the scholarship of interdisciplinary
teaching and learning that suggests that students should take a reflective stance
towards their solutions, to recognize that alternative solutions are possible. The
question also responded to revision principle 3, as an attempt to create a deeper
modeling orientation in students' mathematical work. In practice, though, most
students ignored this part of the question, a sign that they didn't understand it or that it
was uninteresting to them. They wrote stories that illustrated lifelong identity
questions supported by sample calculations, but only a few students produced a final
"life equation." This question might be more successful if it were separated from the
first one. It could be a follow up question for students to answer after they completed
their stories.
Interdisciplinary question prompts also created a dilemma for students who
wished to write creatively while at the same time demonstrating objective knowledge
of Erikson's theory. Some students resolved this problem by incorporating phrases
from Erikson's model, as when a student wrote in her story, "She was restless, and
had a very difficult time connecting with people, struggling with intimacy versus
isolation." Here, the student draws phrases directly from the essay introduction. Other
students portrayed conflict or life choices less directly, through descriptions of the
characters' relationships.
That same lady is on a bus many years later…at the time she was not aware she
was pregnant with her first child…people were waiting for her, her parents,
brothers and sisters, friends and her best friend…who she knew she loved (but
didn't always know it). Sitting in the sun she realized that she was happy to go
home and meet up with everyone who was missing her.
The woman saved money for her child once a year.
Regular payment =1000, Interest rate = 6%, length of time = 18 years
[Note: typed version of handwritten mathematical work]
1000 8
5 9
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=> =
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>=< =?
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A =
.BCDCEBD .FE.BE
.GGE .GC
= $387,353.19 30,905.65 Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Characterizing an adequate interdisciplinary synthesis of both algebra and
psychology is an open research question. In this selection, the student poses the
character's dilemma subtly, and leaves it to the reader to acknowledge that this
represents a resolution of Erikson's young adulthood conflict of balancing intimacy
with isolation. The student initially wrote a problem representing monthly deposits of
$1000, and then changed the problem to coordinate with the annual payment schedule
that she posed. She avoided a typical difficulty of understanding the relationships
among variables in the formulas. While the student has arguably demonstrated
disciplinary understanding in both algebra and psychology, the question of whether
the response is an interdisciplinary synthesis is no doubt debatable. To some, the
narrative itself, in which mathematical examples represent the outcome of a life stage
crisis, are evidence of synthesis. Other readers may desire a tighter synthesis. For
example, if the student had explained how the character's choice of annual payment
was connected to her resolution of her relationship with her partner, it might have
achieved a greater synthetic quality.
Conclusion
Textbook word problems on annuities and amortization use geometric series
which assume consistent investment patterns — implied behaviours that are
unrealistic for many people. Setting financial mathematics problems within a
psychological theory of lifelong identity challenges allows students to pose and
answer more realistic and humanistic mathematical questions, and it allows them to
incorporate mathematics into a more personal vision of a human life-span.
Currently, we are developing additional interdisciplinary modules that align
with standard college algebra textbooks, seeking reviewers from disciplines of
psychology and mathematics for this module, and seeking grant funding to allow
classroom testing of these modules. The long-term goal of this project is to establish a
peer-reviewed, open-access website for interdisciplinary algebra curriculum that
would help provide professional recognition for university level teacher-designers
(Wiitman, 1985). Such a website would also allow algebra teachers to incorporate
low-cost interdisciplinary teaching into their classes.
The general, liberal education experiences of an undergraduate degree
program set students on the pathway of becoming lifelong learners who can evaluate
complex social and political questions. Students' instructors, having already
completed this training, can be expected to model this perspective. The project is
based on the assumption that a mathematics instructor can engage many of the topics
of the first-year undergraduate curriculum, not as a disciplinary expert but as a welleducated, thoughtful adult. By asking this of instructors, we can better engage nonSTEM majors, the largest university mathematical audience, and make their last
formal mathematical training more deeply relevant.
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Assessments Accompanying Published Curriculum Materials:
Issues for Curriculum Designers, Researchers, and Classroom
Teachers
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Many classroom teachers use the assessment tasks that
accompany their adopted curriculum materials, assuming that these
assessments align with the curriculum. In this paper, we share results
suggesting that these assessment tasks may limit students' opportunities to
demonstrate mathematical understanding and we share a framework that
curriculum designers, researchers, and teachers could use to inform future
development and use of such resources.
Keywords: ancillary resources, mathematical assessment, conceptual
understanding
Introduction
Curriculum materials, specifically textbooks, are an integral part of the mathematics
classroom throughout the world. The nature of textbooks, including the types of
lessons (e.g., inquiry vs. direct instruction) and emphasis of problem sets for student
practice (e.g., focus on procedural skills or conceptual understanding), influences the
opportunities students have to learn mathematics. In addition, the assessment tasks
that accompany those materials potentially provide a variety of insights to teachers
about their students' understanding of mathematics (e.g., ability to complete skills or
engage with mathematical processes). According to the National Council of Teachers
of Mathematics (NCTM) in the United States, assessment that is aligned with the
current vision of school mathematics should provide students "with opportunities to
formulate problems, reason mathematically, make connections among mathematical
ideas, and communicate about mathematics" (1995, p. 11). In particular, assessment
should be "an integral part of instruction that encourages and supports further
learning" (NCTM, 1995, p. 13).
As noted by Valverde et al. (2002), textbooks translate the intended
curriculum of national standards into the potentially implemented curriculum of the
classroom. Thus, textbooks provide evidence for teachers of the recommended focus
of instruction. In the United States, adopted curriculum materials are typically
accompanied by a suite of ancillary materials, including at the minimum a teacher's
edition with suggestions on instruction and assignments, supplementary practice Theme C – D. R. Thompson, P. D. Hunsader & B. Zorin
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activities, and assessment resources (i.e., unit tests). These ancillary materials are
often used extensively by classroom teachers. In fact, some researchers (e.g.,
Delandshere & Jones, 1999; Hunsader, Thompson, & Zorin, 2012b; Senk, Beckmann,
& Thompson, 1997) have documented that many classroom teachers use the unit tests
accompanying their curricula, often with only minimal adjustments and
modifications.
Assessments that occur as summative measures at the end of a unit of
learning "should promote valid inferences about [students'] mathematics learning."
Given that "an inference about learning is a conclusion about a student's cognitive
processes that cannot be observed directly" (NCTM, 1995, p. 19), those assessments
need to provide opportunities for students to demonstrate the depth of their
mathematical understanding and knowledge. So, an issue for focus in this study
conference is whether curricular assessments have been designed in ways that support
the instructional goals of the curriculum, and by extension national/state standards.
That is, if it is important during instruction for students to engage with mathematical
processes (e.g., reasoning and proof, communication, connections, representation),
then assessments need to provide opportunities for students to demonstrate
performance with those processes. Because tests are often developed at the end of the
design cycle, they are not necessarily given the same attention by curriculum
designers or researchers as other aspects of the curriculum. Consequently, the nature
of such tests may undermine the instructional goals of the curriculum as teachers use
them for gauging students' mathematical achievement, and possibly make incomplete
inferences about the nature of their students' understanding of the mathematics being
studied.
Our perspective on tasks
In this paper, we use the term "task" to refer to the entirety of a unit test designed to
assess students' mastery of the concepts, processes, and procedures of the
mathematics under study. Clearly, an individual item provides insight into a particular
concept, yet no one item can or should be expected to assess knowledge across all
aspects of a concept. By focusing on the set of items comprising a unit test, we are
able to consider what might be lacking from that set of items, an analysis that would
not be appropriate at the level of an individual item. Reviewing comparable analyses
across a set of unit tests for a particular curriculum then provides insight into the
extent to which the assessment tasks for that curriculum focus on important aspects of
mathematical understanding, including both content and mathematical processes.
We have not designed the tasks (that is, the unit tests) that are the focus of
this paper. Rather, we have developed a means to analyze the tasks that accompany
published curricula and have thought about how that analysis can potentially be used
with teachers, researchers, and curriculum designers to improve them. Our views and
the subsequent issues we discuss are based on our detailed analysis of tests
accompanying the elementary grades 3, 4, and 5 (ages 8-10) curriculum from three
different publishers in the United States.
Integrating processes into curriculum and instruction
Skemp (1971) provided a means of classifying instructional tasks as focused on
instrumental learning (i.e., procedural learning) or relational learning (i.e., conceptual
understanding). With the release of the Curriculum and Evaluation Standards for
School Mathematics (National Council of Teachers of Mathematics, 1989) in the Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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United States, curriculum materials began to be designed so that both of these types of
learning were explicitly addressed. In addition, the Standards identified not only
content foci (e.g., number and operations, algebraic thinking), but mathematical
processes that needed to be part of both curriculum and instruction, specifically
representation, connections, communication, reasoning and proof, and problem
solving. These processes are essential to students' demonstration of their conceptual
understanding of mathematics. As students explain their thinking, teachers are able to
gain greater insight into their level of mathematical understanding than is possible
solely from the manipulation of procedural skills.
It is relatively straightforward to determine if specific content
recommendations are implemented into the textbooks or ancillary materials, such as
tests. Designers, researchers, and teachers can do a simple checklist to indicate
whether the content is present because evidence of content is explicit. However,
identifying whether the mathematical processes are part of the textbook or the tests is
more difficult because, by their nature, evidence of the mathematical processes tends
to be implicit. Yet, it is through the mathematical processes that students are able to
demonstrate their relational understanding. Without opportunities on assessment tasks
to demonstrate understanding through the mathematical processes, it is often difficult
for teachers to see beyond a surface level what students know about mathematics.
Just because the curriculum and the instruction based on it provide
opportunities for students to engage with the processes, one cannot assume that the
task used to assess mastery of that curriculum (i.e., the unit test) also includes
opportunities to engage with the processes. Indeed, in the United States the
development of ancillary materials is often outsourced to development firms. So,
consumers of the materials, the teachers, may assume that the task provides the same
opportunities for engagement with the processes as is true with the materials
themselves, but this is not necessarily the case. Teachers may also assume that the
task will provide them with adequate evidence about their students' conceptual
understanding of the content to inform future instruction.
In a survey of 43 teachers, mathematics coaches, and mathematics
supervisors in our own state, 81% used the assessments that accompanied their
curriculum and 68% of these reported using them because they ensured alignment
with the curriculum (Hunsader et al., 2012). Furthermore, when asked to indicate the
extent, on a scale of 1 (low) to 5 (high), to which they perceived the tests engaged
students with the mathematical processes, only problem solving (mean = 3.4) and
representation (mean = 3.2) had a rating over 3. But both the ratings and the rationale
for using the assessment tasks accompanying the curriculum are based on perceptions.
What evidence can or should exist that these perceptions are on target? What evidence
would inform curriculum designers or researchers about future design or research
about these tasks? What evidence could be used to persuade teachers to take a critical
and objective look at their assessments rather than assuming that these tasks give
them the needed insight into students' understanding to guide future instruction?
These are the questions we explore in the remainder of this paper.
A framework to guide the analysis and design of mathematical processes
on assessment tasks
As a means to analyze assessment tasks in an objective manner, we developed the
Mathematical Processes Assessment Coding Framework (MPAC Framework) in
Figure 1 to determine the extent to which the mathematical processes were embodied Theme C – D. R. Thompson, P. D. Hunsader & B. Zorin
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in assessment tasks accompanying published curricula (Hunsader, Thompson, &
Zorin, in preparation).
Reasoning and Proof
N The item does not direct students to provide or show a
justification or argument for why they gave that response.
Y The item directs students to provide or show a justification or
argument for why they gave that response ('Check your work'
by itself is not justification).
Opportunity for Mathematical Communication
N The item does not direct students to communicate what they are
thinking through symbols (beyond a numeral answer),
graphics/pictures, or words.
Y The item directs students to communicate what they are thinking
through symbols, graphics/pictures, or words.
V The item only directs students to record a vocabulary term or
interpret/create a representation of vocabulary.
Connections
N The item is not set in a real-world context and does not explicitly
interconnect two or more mathematical concepts (e.g.,
multiplication and repeated addition, perimeter and area).
R The item is set in a real-world context outside of mathematics.
I The item is not set in a real-world context, but explicitly
interconnects two or more mathematical concepts (e.g.,
multiplication and repeated addition, perimeter and area).
Representation: Role of Graphics
N No graphic (graph, picture, or table) is given or needed.
S A graphic is given but no interpretation is needed for the
response, and the graphic does not explicitly illustrate the
mathematics inherent in the problem. (superfluous)
R A graphic is given and no interpretation is needed for the
response, but the graphic explicitly illustrates the mathematics
inherent in the problem.
I A graphic is given and must be interpreted to answer the
question.
M The item directs students to make a graphic or add to an existing
graphic.
Representation: Translation of Representational Forms
N Students are not expected to record a translation between
different representational forms of the problem.
SW Students are expected to record a translation from a verbal
representation to a symbolic representation of the problem or
vice versa.
GS Students are expected to record a translation from a symbolic
representation to a graphical (graphs, tables, or pictures)
representation of the problem or vice versa.
WG Students are expected to record a translation from a verbal
representation to a graphical representation of the problem or
vice versa.
TG Students are expected to record a translation from one graphical

representation of the problem to another graphical
representation.
A Students are expected to record two or more translations among
symbolic, verbal, and graphical representations of the problem. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Figure 1. Mathematical Processes Assessment Coding Framework.
Source: Hunsader, Thompson & Zorin (in preparation).
The framework focuses on four of the five processes from the Principles and
Standards for School Mathematics (NCTM, 2000); for representation, we generated
two criteria to reflect different aspects of this process standard – the role of graphics
and translation between and among representational forms. We were interested in
analyzing the assessed curriculum as reflected in written tests, without the need to
analyze the intended written curriculum of the textbook or the enacted curriculum of
classroom instruction. Hence, the framework does not address the problem-solving
standard because whether an item on an assessment requires problem solving or is a
routine exercise depends on prior experiences students have had, which would require
analyzing the textbook and instruction.
We have applied the framework to analyze over 100 complete unit tests and
over 2000 individual items across those tests. Table 1 summarizes the extent to which
these assessment tasks (i.e., tests) provide opportunities for students to engage with
the mathematical processes.
Process Standard Minimum % Maximum % Median %
Reasoning & Proof 0 23 0
Communication (yes) 0 85 13
Communication (vocabulary) 0 73 0
Connections (R or I) 0 100 44
Representation (role of graphics: M or I) 0 100 22
Representation (translation of forms) 0 67 9
Table 1. Range and Median Percent of Items Per Assessment Task Addressing the
Mathematical Processes.
Note: The unit of analysis is an entire test.
With the advent of the Standards movement in the United States, most
curriculum materials purport to integrate those standards, both content and process.
However, the results in Table 1 suggest that there is a wide range in the extent to
which the process standards are integrated into the assessment tasks.
How might designers and researchers use this coding framework and
related analysis?
The potential value of this framework is that it provides an objective look at
assessment tasks – either to confirm perceptions or suggest where changes are needed.
Consider the item in Figure 2 cloned from an item on one grade 3 assessment task.
Five friends have 20 pieces of candy to
share equally. How many pieces of
candy will each friend get?
Figure 2. Clone of an Item from a Grade 3 Assessment.
We would classify this item as providing no opportunity for reasoning or
communication because students are simply asked to provide a numerical answer but
not to explain what they were thinking or why their answer is correct. For the role of
graphic, we would classify the item as S because no interpretation of the graphic is
needed to answer the item and the graphic does not explicitly illustrate the
mathematics inherent in the problem. Note that the student does not need to consider Theme C – D. R. Thompson, P. D. Hunsader & B. Zorin
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the graphic to solve the problem nor could the student make sense of the problem
simply by interpreting the graphic.
A closer analysis of the item raises some interesting issues. Although the
graphic is not needed to answer the item, it does not appear to be random. The
designers did not use a single candy, or 5 or 10 candies. They used the exact number
needed in the problem, and yet, they never explicitly asked students to use the
graphic. Is the graphic there to support struggling students or those who are English
language learners? Is the graphic there because the curriculum encourages students to
use graphics to help solve problems? Is it there simply to liven up the page? Is the
graphic possibly distracting to students who do not need it?
The issues raised by this one item provide potential research opportunities.
One might ask designers why they chose to include this particular graphic but not to
ask students to use it in any way. Researchers or teachers might observe students as
they answer the item to determine how many use the graphic without leaving any
written record of its use as well as how many actually show their solution on the
graphic. Researchers might ask teachers how they perceive their students will attend
to the graphic. Another question might be whether such related, but not needed,
graphics are a regular part of the curriculum and whether this type of use encourages
students to ignore graphics in their mathematics textbook. What might be the
unintended consequences later in students' mathematical careers if they become
accustomed to ignoring the graphics in their curriculum?
Figure 3 adapts the item from Figure 2 in three different ways. How do these
adaptations engage students with the process standards and what further insights into
students' thinking do they provide teachers?
Adaptation 1. Five friends have 20 pieces of candy
to share equally. How many pieces of candy will
each friend get? Write a number sentence to show
how many pieces of candy each friend will get.
Adaptation 2. Five friends have 20 pieces of candy
to share equally. How many pieces of candy will
each friend get? Write a number sentence to show
how many pieces of candy each friend will get. Use
the picture to explain your thinking about the
problem.
Adaptation 3. Five friends have 20 pieces of candy
to share equally. Draw a picture to show how many
pieces of candy each friend will get. Write a number
sentence to represent your picture.
Figure 3. Adaptations of the Item from Figure 2.
Note: Adaptation 3 from Hunsader, Thompson, & Zorin (2012b).
Adaptation 1 does not resolve the issues about the use of the graphic raised in
the discussion of Figure 2. But having students write a number sentence does provide
insight into what they were thinking. So, when students write a number sentence to
translate the problem from words to symbols as part of the solution, they
communicate what they were thinking and give teachers more insight about their
understanding than simply providing the answer 4.
Adaptation 2 explicitly asks students to use the graphic to explain their
answer, thus requiring them to interpret a graphic in the sense that the student must
illustrate a solution on the graphic. Hence, this adaptation engages students in both Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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communication and reasoning by having them show what they were thinking (e.g., the
number sentence) and why they gave that response (e.g., the solution on the graphic).
Adaptation 3 also engages students in both communication and reasoning,
while also requiring them to make a graphic to illustrate a solution. Thus, Adaptation
3 extends the requirements of Adaptation 2 so students independently demonstrate
their ability to connect verbal, symbolic, and visual representations of a problem.
All three adaptations have the potential to provide teachers insight into
students' conceptual understanding as well as the procedural understanding expected
from the original item. To some extent, there is also a sequence to these items so that
researchers could investigate whether achievement varies for different versions of the
item and different explicit references to use of the graphic.
If the original item (Figure 2) were the only such item in a set of items
comprising an assessment task, one might overlook the apparent haphazard use of
graphics or the lack of explicit engagement with the process standards. After all, no
one expects every item on an assessment task to integrate one or more of the process
standards. However, if such items are the norm on a given assessment or are the norm
across assessments, as suggested by the results in Table 1, then consumers of those
tasks might rightly question whether the task is providing the insight into students'
mathematical thinking and understanding that is expected from an assessment. Thus,
the framework and analysis can help curriculum designers ensure that assessments
maintain the same focus on processes as they design into their curriculum.
Researchers can use the framework and analysis to understand how variations of
items might enhance student achievement.
How might teachers and teacher educators use this coding framework
and related analysis?
Classroom teachers might be encouraged to use such a framework and the
analysis it provides to better understand the nature of the assessment tasks they use
with students. As indicated previously, teachers often use the assessment tasks that
accompany their curriculum because they assume such assessments are well aligned
to the textbook. Thus, having them reflect on the nature of these assessments can be a
powerful influence on their future practice.
For instance, after we have worked with practicing or prospective teachers to
help them recognize the need for more mathematical discourse in the classroom and
for use of the mathematical processes, we have had them analyze several items or
tests associated with their curriculum. The results typically lead to a stark reality –
there is often little or no integration of the process standards. Although the
assessments clearly relate to the content of the related unit, opportunities for
reasoning, communication, representation, or connections may be limited. Teachers
then consider how they might adapt some of the items comprising the overall task so
that opportunities to engage with the processes are more evident. The potential impact
of such work with assessment tasks is reflected in the following comments from
teachers:
"The [MPAC] framework reminded me to consider areas of the problem that I
sometimes overlook."
"…actually doing the test item analyses really made me think outside of the
box in how I would teach students about math and assess them."
"[The experience] will greatly impact my formulation of chapter tests in
mathematics when I am a teacher." Theme C – D. R. Thompson, P. D. Hunsader & B. Zorin
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Conclusion and relation to the study conference
Obviously, the design of assessment tasks to accompany a curriculum can
only occur after the curriculum has been designed. At that point, there may be fewer
financial resources available for development, or assessments may not have been the
intention of the designers but are required for commercial publication. Our analysis of
assessment tasks from three different elementary publishers found some variation
across publishers, content domains, and grade levels, but the overall pattern was
consistent – connections (i.e., real-world problems) were moderately integrated into
assessment tasks, but other process standards were less so. Subsequent analysis of
tests associated with U. S. middle grades curricula has found a similar pattern.
The issues raised in this paper suggest that more attention needs to be paid to
the assessments associated with curriculum, particularly if those assessments are used
to judge the effectiveness and quality of a curriculum and to make inferences about
the student learning that such a curriculum facilitates. If the curriculum and the
assessments are not aligned in terms of the process standards, then accurate
information about learning is not obtained. Consequently, teachers may have an
incomplete picture of their students' mathematical knowledge as they design future
instructional tasks. They may assume students have a depth of knowledge about
mathematical concepts that is not supported by the type of performance elicited on the
assessment tasks, and thus fail to develop appropriate instruction to extend or enhance
mathematical knowledge.
One issue for the study conference to address is how research such as that
described here can or should inform future curriculum designers and the next round of
curriculum revision. A second issue is how to ensure that all aspects of the curriculum
cycle – textbook or material development, instructional implementation, and
assessment tasks – are aligned and receive equal attention in the design process.
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Different features of task design associated with goals and
pedagogies in Chinese and Portuguese textbooks: The case of
addition and subtraction
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This paper presents different features of task design for addition

and subtraction in Chinese and Portuguese textbooks and their different goals and pedagogies. Through comparison, we address the question, "what is the effect of different cultural backgrounds on tasks and task design for addition and subtraction", and propose some conjectures that arise from answers to this question.
Keywords: variation theory, example and exemplification, addition and subtraction, Chinese mathematics education, textbook design, Portuguese mathematics education

Introduction

China has a clear textbook-centred tradition. Textbooks play multiple functions in the Chinese mathematical education system, such as tools for teachers` professional development by studying textbooks (e.g. Ma, 1999), self-learning instruments for out-of-school learners, and the main medium for teaching and learning in the classroom. In these ways, textbooks play a central role in shaping students' learning and teachers' teaching through their central role in the interactions of local groups of teachers and academic organizations. Although Chinese textbooks play a critical role in schooling, it is worth noting that Chinese textbooks are designed by local experts based on their experience and local curriculum standards. Principles of task design are generally not explicitly articulated. However, Marton (2008) has identified a specific implicit underlying principle of task design in Chinese culture as expressed in the following words:

There is a very powerful pedagogical tradition in the Chinese culture. The careful molding of 'the space of learning', the blending of necessary different components of learning and of different ways of supporting it, might occasionally run counter to the ever shifting pedagogical fashion. But what is proven to work well in practice and what is theoretically not only defensible, but even positively implied, should not be replaced, but developed further. ...Chinese students do very well when compared to students from other cultures. Teachers spend much more time on planning and reflecting than teachers in other countries, and they develop their professional capabilities by the teaching, in which patterns of Theme C – S. Xuhua, T. B. Neto & L. E. Ordóñez
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variation and invariance, necessary for learning (discerning) certain things, usually brought about by juxtaposing problems and examples, illustrations that have certain things in common, while resembling each other in other respects. By such careful composition, the learner's attention is drawn to certain critical features and each problem and example make a unique contribution to the things that the learner will hopefully pay attention to in the future, instead of just going through problems that are supposed to be examples of the same method of solution. (p. 1)

These principles were expressed and illustrated by Liu (2004). More recently, Sun (2007, 2011) indicated that two kinds of problem variation (变式题) could describe these features of example design in China: OPMS (One Problem Multiple Solution methods) and OPMC (One Problem Multiple Changes), both of which are local routine teaching terms,. Liu's examples were concerned with absolute value, and Sun's with division of fractions. In the present paper, we compare textbooks from two countries using the question "what varies?", and ask whether this is helpful to identify differences in the space of learning.

According to variation theory (Marton & Booth, 1997), which emphasizes "simultaneity," the "one-thing-at-a-time" design might miss the chance to discern critical aspects among two or more topics. More importantly, the contemporaneous variation approach, to a greater extent, emphasizes "general relationships" more than others. Thus, problem variations aim to discern, compare the invariant feature of the relationship among concepts and solution methods, and provide opportunities for making connections; comparison is considered the pre-condition to perceive the structures, dependencies, and relationships that may lead to mathematical abstraction. The emphasis on "general relationships" reflects the soul of variation theory, emanating from the work of Marton, who argues that "learning will take place through discernment of variation in simultaneous events"; "variation is a necessary condition for effective discernment" (Bowden & Marton, 1998, p. 35).

As mentioned before, Chinese teachers are required to study textbooks intensively (研究教材) and this is regarded as an important method for professional development. Textbooks, together with their teachers' guidebooks and a professional research group (Yang, 2009), are the predominant force in a teacher`s professional life. This professional development environment is rarely known outside of the Chinese community, so to study the detail of design in textbooks is a strong proxy for studying the way content is shaped through professional development. In this study, we explore how example design, and the associated curriculum standards and reference books, might influence teachers' teaching, and hence, the space of learning... For comparison, a Portuguese textbook and teaching guide are used as a "mirror". The research questions of this study are:

What are the different features of task design in the topics of addition and subtraction within Chinese / Portuguese textbooks?

How might features of task design in textbooks influence the space of learning in Chinese / Portuguese classrooms?

We focus on a Chinese textbook (Mathematics Textbook Developer Group for Elementary School, 2005) that has been used for over 30 years by the majority of Chinese students from diverse backgrounds. It represents the Chinese national curriculum and is seen, with its teaching guide, as an authoritative guide on what to Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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teach/learn. However, all national examination problems are required to be "from textbooks, but above textbooks"; hence, the material is supposed to extend knowledge beyond the procedures that might be carried out in the textbook. The Portuguese textbook (Gregório, Valente, & Calafate, 2010), with supporting teacher guide (similar to the Chinese teacher guide in that it informs teachers about appropriate goals and pedagogies), reflects typical practices of teaching and learning these topics at this level in Portugal.

Different features of task design for addition and subtraction within Chinese /Portuguese textbooks

The content structure of Chinese textbooks is fixed. The organization is consistent without repetition:

• knowing numbers 1-5 as a foundation, then knowing addition and subtraction from 1-5;
• knowing numbers 6-10, then addition and subtraction from 6-10;
• knowing numbers 11-20, then addition and subtraction from 11-20;
• knowing numbers within 100 (1000,10000), then addition and subtraction algorithms within 100(1000,10000), step by step.

Invariant concept vs. variant concepts embedded in Chinese / Portuguese textbook examples

Addition and subtraction are almost always elicited together in examples of OPMC in the Chinese textbook
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, rather than separated in different chapters as they are in the Portuguese textbook. The two concepts of addition and subtraction are always connected by problems with variation. Here are two typical "prototype" examples of OPMC in the Chinese textbook.

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In Chinese, OPMC references "changes"; in English, variation theory would likely use the word "transformations" to mean the same thing. Theme C – S. Xuhua, T. B. Neto & L. E. Ordóñez
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Fig.1 from Mathematics Textbook Developer Group for Elementary School, 2005, 68

Fig.2 from Mathematics Textbook Developer Group for Elementary School, 2005, 57

It is interesting to note that the Chinese textbook authors did not separate the subtraction concept from the addition concept, even in the first lesson (Sun, 2011). Figure 1 shows a paradigmatic example of problem variation: 10+3=13, 13-3=10, 3+10=13; 13-10=3.The problem set intends to help learners recapitulate the relationship between addition and subtraction, and the meaning of "equal". The textbook design offers visual models enabling learners to understand the underlying additive relationship.

Every example in the Portuguese textbook introduced the concept of addition and subtraction without any connection, and without a visual model to situate the problem. The pages below are from two separate chapters. Addition and subtraction concepts are introduced as counting in the following examples. While the number line model provides a method of calculating the answers, it is unclear whether it enables learners to use the underlying meaning to connect addition and subtraction.

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Fig. 3 The example introducing addition concept by counting in one of the Portuguese textbooks (Segredos dos Números 1, Matemática 1º ano o Ensino Básico, 2010, p. 75)

Fig. 4 The example introducing subtraction concept identifying inverse operation in one of the Portuguese textbooks (Segredos dos Números 1, Matemática 1º ano o Ensino Básico, 2010, p. 76)

Although the Chinese textbook authors appear to use multiple concepts in every example, the underlying invariant concept is of part-part-whole relations and the tool being used is knowledge of numbers. –In contrast, the addition examples in the Portuguese textbook use multiple underlying physical concepts, such as "counting" (as in Figures 3 and 4), "combining", and "adding". The subtraction examples in also use multiple concepts such as, "taking away", "comparing", and "identifying the inverse operation of addition" but they do not relate these to the addition concepts in any direct way. In the Chinese textbooks, the meaning of the additive relation is invariant, but the way it is represented and enacted varies. In contrast, in the Portuguese textbook the meanings of the addition and subtraction concept appear to vary, and they are not connected except by the notion of 'inverse'. Thus, the space of learning (i.e., what is available to be learnt) is different in the two countries.

Single solution vs. multiple solution methods embedded in every Chinese / Portuguese textbook example

In the Chinese textbook, multiple solution methods are almost always elicited together, illustrating the OPMS principle, rather than a single solution method as in the Portuguese textbook. Theme C

Fig. 5. Three solution methods for variations of 4+1=5 (Mathematics Textbook Developers Group for Elementary School, 2005,1(

Figure 5 is a typical example in the problem variation, of addition. Within the problem set in the example, there are three solution given. The first one is that of addition by counting from the addend that of counting from the addend with 4 and 1. Figure 6 shows solution method is that of addition by regrouping with regrouping with 9 and 1, decimal system. Compared with example in the Portuguese solution method without necessarily connecting that method to other approach the book. For example, as shown in Figure 7 through a specific single concept on each of these two pages does not vary; however, a learner might not connect this method to other models of addition, nor to subtraction, nor be able to choose when to use these methods without juxtaposing the examples, as Marton suggests (2008), with some that are not solvable using 'doubles' or 'doubles plus 1'. The patterns of variation and invariance from which the learner might discern the underlying conceptual relations are unclear.

Fig. 7 The example elicits addition by the solution method of in one of the Portuguese textbooks (Segredos dos Números 1, Matemática 1º ano o Ensino Básico, Theme C – S. Xuhua, T. B. Neto & L. E. Ordóñez
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hree solution methods for variations of 4+1=5 (Mathematics Textbook Developers Group for Elementary School, 2005,1(16))

Fig. 6. Three solution methods for variations of 9+5=14 (Mathematics Textbook Developers Group for Elementary School, 2005, 1(30))

typical "prototype" example of OPMS in the Chinese textbook

In the problem variation, 4+1=5 is designed to introduce naturally a solution system of addition. Within the problem set in the example, there are three solution . The first one is that of addition by counting from 1 to 5; the second solution is counting from the addend 4 to 5; the third is that of addition by regrouping shows the problem variation of OPMS, 9+5 solution method is that of addition by regrouping with 5 and 5; the second one is , both of which highlight the "make-10 method"

ompared with the Chinese multiple-solution method approach uese textbook is intended to be carried out using without necessarily connecting that method to other approach the book. For example, as shown in Figure 7 a typical design to introduc single solution method of "doubles" or "doubles plus 1 concept on each of these two pages does not vary; however, a learner might not connect this method to other models of addition, nor to subtraction, nor be able to choose when to use these methods without juxtaposing the examples, as Marton

008), with some that are not solvable using 'doubles' or 'doubles plus 1'.
The patterns of variation and invariance from which the learner might discern the
underlying conceptual relations are unclear.
elicits addition by the solution method of "doubles" or
textbooks (Segredos dos Números 1, Matemática 1º ano o Ensino Básico,
2010, p. 99)
Three solution methods for variations
of 9+5=14 (Mathematics Textbook
Developers Group for Elementary School,
example of OPMS in the Chinese textbook.
e naturally a solution system
of addition. Within the problem set in the example, there are three solution methods
he second solution is
addition by regrouping 5
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concept on each of these two pages does not vary; however, a learner might not
connect this method to other models of addition, nor to subtraction, nor be able to
choose when to use these methods without juxtaposing the examples, as Marton
008), with some that are not solvable using 'doubles' or 'doubles plus 1'.
The patterns of variation and invariance from which the learner might discern the
or "doubles plus 1"
textbooks (Segredos dos Números 1, Matemática 1º ano o Ensino Básico, Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Invariant solution method vs. variant solution methods embedded in Chinese
/ Portuguese textbook examples
Although Chinese textbook authors use multiple solution methods in every
example, the particular methods in the Portuguese textbook, which depend on
counting and doubling, are rarely introduced. Only one specific solution method,
"make-10"," is addressed explicitly among all the addition /subtraction examples
within the first 6 chapters. In contrast, the addition examples in the Portuguese
textbook suggest multiple solution methods, such as "doubles", "doubles plus 1",
"compensation" (6+8=7+7=14), and "reference number" (6+7=5+1+5+2=10+3=13).
The subtraction examples use multiple solution methods, such as "counting back",
"the use of tables for the addition to subtraction", and "identifying the inverse
operation of subtraction as addition".
Different goals and pedagogies in teaching reference books
Different cultural traditions have different educational goals and traditional
pedagogies. The Chinese curriculum goal is "two basics", i.e. knowledge and skills;
these are the central aspect of the unified teaching outline of the Ministry of
Education (1963, 1991, 2001).The basics include important mathematical facts and
experiences, mathematical thinking, and skills for further development. In this case,
the basic knowledge is the "part-part-whole" or additive relation as a foundation for
algebra, and the "make-10" foundation for place value and the decimal system. The
textbook emphasises these as basic knowledge and skills and the examples illustrate
how this is done.
In general, to realise the "two basics", the Chinese develop specific
pedagogies, starting from analyzing the focal points of the curriculum content (重点),
then identifying the difficulties that students might encounter (难点) and the critical
points for learning (关键点) (as derived from students' difficulties in learning), and
finally building in relevant variations in the teaching that focus on these three points.
Because this theory is at too general a level for teachers to enact in every lesson, the
concrete goals and pedagogies for every lesson are clearly presented in the teachers'
guide. For example, Figure 8 shows the focal points for knowing numbers 6-10
(EMD, 2005, p. 67). Theme C – S. Xuhua, T. B. Neto & L. E. Ordóñez
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Fig. 8 The concept struture of knowing numbers 6-10 (EMD, 2005, p. 34)
The Chinese example design principle
Based on the Chinese examples and analysis above and knowledge of the
teacher guide, we could infer the underlying design principle of 'two basics' by
noticing: the foundation of knowing numbers; the connection between addition and
subtraction; and the invariant concept/ solution: part-part-whole /make-10 solution.
Portuguese curriculum goals and pedagogies
The Portuguese textbook and supportive guide does not reflect such a
coherent knowledge structure (Gregório, Valente, & Calafate, 2010). The initial work
about addition and subtraction =focuses =on addition understanding it as combining,
and performing it using counting, doubling, compensation, or using and adapting
known facts, with subtraction also having several meanings. The underlying design
principle is not made explicit; we could infer that it is more fragmented, more focused
on learning one method at a time, and less dependent on foundational principles for
future work.
Summary
The task design discussed in this paper shows a pedagogical phenomenon in
organizing a curriculum with an emphasis on discerning relationships through the
variation approach. Many readers may argue that the variation approach may be
confusing and that a sequential organization with time gaps ("one-thing-at the-time")
should be preferred. In fact, the variation approach might come from different kinds
of pedagogical traditions and philosophies developed for centuries. The issue of
variations in problem sets directly reflects the old Chinese proverb, "no clarification,
no comparison" (没有比较就没有鉴别), rather than "to consolidate one topic, or
skill, before moving on to another," a notion broadly used in most textbook
development (Rowland, 2008) in Europe and throughout the world. In contrast, the
"one-thing-at-a-time" design would clearly provide fewer opportunities for making
connections compared to those of contemporaneous variation approaches. The "onething-at-a-time" design might possibly reflect the hidden belief that learners would Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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naturally make the relevant connections. In this context, the role of the curriculum in
presenting connections might be taken for granted and remain implicit.
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Theme D
Principles and Frameworks for task design within and across
communities
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Considerations and principles for designing and sequencing tasks depend
highly upon the context of the design activities. Various design communities, such as
those consisting of researchers, teachers, professional developers and teacher trainers,
or textbook writers, have different aims and agendas for task design. Thus, principles
for task design vary across the context in which the communal practice is situated. In
addition, principles for task design can vary as to whether they are applied to the
initial creation of tasks or to the shaping and modification of existing tasks
(Remillard, 2005), as well as to whether they are applied to the design of a single task
versus a sequence of tasks. Moreover, tasks can be designed not only by members of a
singular community but also by groups whose members cut across two or more design
communities (see, e.g., the international examples of such efforts (Kieran, Krainer, &
Shaughnessay, 2013). For example, recent projects where teachers are regarded as
key stakeholders in research (i.e., as (co)producers of professional and/or scientific
knowledge) and where they have a significant role to play in the design of tasks have
been shown to yield not only rich task designs for mathematical learning, but also
make the link between research and practice more fruitful for both sides. In this
working group, we address the diversity and the interactions between design
principles and communities that are involved in task design and attempt to make
explicit those principles of task design within and across design communities that
have up to now been largely tacit.
This Working Group has a twofold aim:
• To solicit papers that delineate principles and frameworks for task
design within singular design communities so as to illuminate
differences and commonalities across the specific contexts of the
various communities.
• To solicit papers that delineate principles and frameworks for task
design by teams that cut across the various diverse communities so as
to illuminate the nature of, and thereby aid in encouraging the further
emergence of, such interactive, cross-community approaches to task
design. Theme D – Introduction
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Papers being submitted to this working group should specify which of the
two above aims is the main focus of the paper. Papers being proposed for this group
should also address and develop some subset of the following questions, in addition to
whatever other issues might be considered relevant to the given theme:
• If you identify yourself as a member of a singular design community,
which one is it? Or if you identify yourself as a member of a design
group that cuts across communities, which ones are they? If the latter,
how did this cross-community come to be formed?
• When you or your group engages in designing tasks, what are you
trying to achieve? What are your primary considerations?
• Do the principles applied to task design depend on the nature of the
mathematical activity inherent in the tasks (i.e., tasks for exploration,
concept development, practicing, generalizing and reflection)? If so,
in which ways?
• In which ways do the principles for task design interact with the issue
of the time factor, that is, whether a task sequence is to occur across
several lessons or within one given lesson?
• Which theoretical, mathematical, pedagogical, technological, cultural,
and/or practical aspects are taken into account when designing a task
or a task sequence? Which aspects are considered primary?
• Is there a particular framework or theory of learning that is drawn
upon in designing a task or task sequence, and how is this framework
reflected in the task design?
• What is the extent to which individual/communal value systems and
beliefs about how mathematics is to be learned enter into the
designing of tasks?
• What is the extent to which the inclusion of digital-technology tools
within a task or task sequence is reflected in the principles employed
in designing the task or task sequence?
• Are the designed tasks subject to revision in later cycles of the work?
If so, what is it that specifically leads to the redesign? On what basis
and according to which principles is the redesign carried out?
• What constitutes the main differences and commonalities between

design principles for different design communities?
• What constitutes the main differences and commonalities between
design principles for different age groups and school levels?
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Task design within the Anthropological Theory of the
Didactics: Study and Research Paths for pre-school.
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Abstract. Task design has become a central issue within the
Anthropological Theory of Didactics, connected to the necessity of a
renewed paradigm in the teaching and learning of Mathematics. It is the
aim of this paper to situate "task design" within the Anthropological
Theory of Didactics, to introduce how the term "task" is conceptualized
(study and research paths and activities), to outline key design principles
and, finally, to exemplify the design process in the case of a task for preschool.
Keywords. Anthropological Theory of Didactics, Study and Research
Path, Study and research Activities, Pre-school.
Introduction
The Anthropological Theory of the Didactic (ATD, from now on) has been developed
over the last 30 years, giving rise to a rich and complex theoretical model. In its
evolution, initial constructs have been developed and enriched. The emergence of new
problems and the enlargement of initial questioning have led to new developments of
the theory, in a continuing process that is still going on.
The origins of the ATD can be found in the reconceptualization of
mathematics as a human activity (Chevallard, 1999). This fundamental hypothesis
gave rise to the notion of praxeology, which captured the essential components of any
(mathematical) activity: tasks, techniques, technologies and theories. "Tasks" and
"techniques" considered as the "know-how", whilst "technologies" and "theories"
interpreted as the "knowledge" that explains and justifies the techniques used to deal
with kind of tasks.
(…) no human action can exist without being, at least partially, "explained", made
"intelligible", "justified", "accounted for", in whatever style of "reasoning" such
an explanation or justification may be cast. Praxis thus entails logos which in turn
backs up praxis (Chevallard, 2006, p. 23)
Therefore, from the very beginning the notion of "task" has been central
within the ATD. In this initial approach, the word "task" has to be considered in a
very general sense and institutionally situated.
Thanks to the notion of praxeology, and the subsequent development of the
theory, researches within the ATD have been able to analyse the teaching and
learning of mathematics, to identify didactic phenomena, to formulate didactic Theme D - J. G. García & L. Ruiz-Higueras
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problems, to find tentative answers and, of course, to create new didactic
infrastructures. Looking backwards, an important part of this research has to do with
the design of tasks (for instance, in the doctoral dissertation of García (2005),
Barquero (2009), or Ruiz-Munzón (2010)).
It is the aim of this paper to summarize the notion of "task" used in our
research within the ATD, linked to a deep understanding of the "didactic system" and
to an explicit epistemological and didactic paradigm of school mathematics, to reveal
the design principles behind, and, finally, to exemplify them in the case of a study
process designed and experimented in preschool.
Towards a new paradigm of mathematics at school
The notion of "task" we will introduce in the next section is strongly connected to a
particular paradigm
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about the teaching and learning of mathematics. This "new
paradigm" is the rationale for this kind of "tasks" and, reversely, the design and
implementation of this kind of "tasks" shows to what extent this "renewed paradigm"
could be possible.
Let consider mathematics as a human activity, institutionally situated, and
modelled in terms of mathematical works or praxeologies, being the mission of
school to make possible the encounter and appropriation of some of them by new
generations. Then, analysing how school and mathematics are organized in order to
make this encounter and appropriation possible, and also developing new ways of
making this process more effective, could be considered as one of the main aims of
the research in the didactics of mathematics.
The traditional paradigm of organizing this encounter and appropriation, still
dominant in many places, is what Chevallard (2012) called the paradigm of visiting
works. According to it, mathematics is split in several little works, which students
visit at school (like monuments), being the teacher the guide. Considered as cultural
and historical works, students are expected to admire and enjoy this visit. Some side
effects of this paradigm are students' difficulties to give sense to the works they are
visiting (lack of their raison d'être), the atomization of mathematics in little works
that can be linearly programmed in a school syllabus, or the disconnection and
deficient articulation between some of this pieces.
In contrast, Chevallard (2012) pled for a (counter)paradigm of questioning
the world. If we model the didactic system as a triplet (X, Y, O), being X the set of
students, Y the set of teachers (normally restricted to a single person) and O the
didactic stake (learning outcome), the main change of this paradigm is that students'
access to O has to be through meaningful and generating questions Q. It is through the
study of Q, and the subsequent questions that might arise from it, that students will
find O, or at least some elements of O, in order to get to possible answers to Q. This
study process might need considering, studying, and even evaluating, already existing
works Oj as well as already existing answers H
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, created and diffused by other
institutions (which form the didactic milieu M). At the end, the aim is to get to an
answer
♥
, which will depend on the path followed by the community of study and

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The word paradigm is considered here in the sense of "a set of rules prescribing, however
implicitly, what is to be studied—what the didactic stakes O can be—and what the forms of studying
them are" (Chevallard, 2012, p. 2)Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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on the works and answers found in the journey. This study process can be synthesized
in what Chevallard calls the Herbartian schema:
In this paradigm, works are not desirable by themselves, but insofar they
contribute in the process of getting to a better answer
♥
. They do not arise as
meaningless objects but, on the contrary, they get their meaning from the questions
they are contributing to get to an answer. The division and isolation of school
mathematics is reduced, as meaningful generating questions give rise to a connected
activation of several praxeologies, even extramathematical ones. Also because the
analysis and evaluation of existing answers, as well as the development and
justification of new ones, situates the activity beyond the routine application of
techniques, giving place to the technological dimension of the mathematical activity
which, in turns, originates that students' work is placed, at least, at the level of local
praxeologies
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.
The paradigm of questioning the world offers some clear connections with
what it is named as inquiry-based learning. Like this one, is still rather marginal in
many educational systems. It is beyond the aim of this paper to analyse this issue.
However, it is clear that the adoption of this paradigm needs a completely renewed
understanding of the "tasks" used at school. And, reversely, only through the design
of a totally different kind of tasks will there be a chance for this paradigm to exist.
Study and Research paths and activities
Very briefly, in this section we will introduce the main features of the kind of tasks in
accordance with the paradigm of questioning the world. Therefore, we make our
definition of "task" explicit.
Within the ATD, the term study and research path and activity has been
coined to refer to tasks in which:
1. The starting point is a crucial and "alive" question Q0. Therefore, the
community of study (X, Y) considers that the study of Q0 is worth it, and
not just an excuse to introduce some pieces of mathematics.
2. Q0 is a generating question. The study of this question will lead to new
questions Qi. That might make the study process open and even
undermined in advance.
3. There is a collaborative and shared study process, looking for good
answers as well as for good questions. In this process, the classical
distribution of responsibilities between X (students) and Y (teachers) will
be continuously renegotiated. Specifically, Y avoids been the questionposer, the unique source of information (media) and the ultimate source of
validity (milieu).
4. The study of the questions Qi will need to consider intermediate answers.
Some if these answers might be already available within the community of
study, others might be "imported" from other institutions, and, even others

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Within the ATD, punctual praxeologies are those characterised by a single technique to
solve an isolated task (like solving linear equations in grade 9 by the technique of transposing terms
from one side to the other). Local praxeologies are characterized by a technology that articulates and
connects several punctual praxeologies (like solving polynomial equations with one variable). Beyond
these, regional praxeologies are characterized by a unifying theory for a set of different local
praxeologies (like the theory of algebraic equations). Theme D - J. G. García & L. Ruiz-Higueras
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might have to be constructed ad hoc. In particular, extra-mathematical
works might have to be considered.
5. The access to intermediate answers will be mediated by the media
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accessible to the community of study. As far as these answers might have
been developed elsewhere, the study and adoption of some elements of
theses answers should include their possible validation and justification
against the didactic milieu available to the community of study (dialectic
process between media and milieu, Chevallard, 2006).
6. The study process will culminate with the development of a possible,
sometimes tentative, answer. This answer might include both
mathematical and non-mathematical contents, which are the didactic stake
(Chevallard, 2012). It will be meaningful as it offers an answer to the
initial question Q0, as well as to the subsequent questions Qi
. Therefore,
students in X will learn in order to think about and interact with the world
they live in, and not just about a world already thought by others.
The whole process can be visualized under the light of a modelling process.
Questions do not live isolated, but embodied in systems (biological, economical,
social…). The study of Q0 involves the study of at least one system and some of its
relationships. Answers (already built, or created throughout the study process) entail
considering or creating models (not just mathematical ones). Intermediate questions
might arise from the original system but also from the progressive models considered.
Particularly, the process would lead to the development of praxeologies of increasing
complexity (García, 2005). The final answer will depend on how the initial system
has been modelled: which relations have been considered and which ones ignored
(structuring and simplification), which kind of model has been used (setting up a
model), the kind of information produced by the model (working inside the model)
and how this information has been interpreted and validated against the starting
system.
Within this context, a distinction between study and research activities
(SRA) and paths (SRP) has to be made. In a SRA, the didactic stake (learning
outcome) has been set in advance. That means that, starting from a generating
question, the study process has been designed to ensure that students meet some
praxeologies already decided. On the contrary, SRP are undetermined by definition.
Starting from a generating question Q0, it will be the responsibility of the community
to decide which path they will follow, which will determine the praxeologies they
meet with. Particularly, an SRP might include one or several intermediate SRA in
order to build some needed praxeologies to make the study process advance.
Designing study and research paths and activities: principles and
process
In accordance with the paradigm of questioning the world, the identification of
generating and crucial questions is a critical issue. It is not just finding nice and
motivating questions. At least, three conditions must be considered: mathematical,
functional and social legitimacy (García, 2005).
The mathematical legitimacy implies that studying and looking for answers
to the initial question should give rise to the mathematical praxeologies the study
process is aiming at, in the case of SRA, or at least to a set of possible mathematical

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In the sense of "mass media", like books, press, TV, or Internet. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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praxeologies identified in advance (the didactic stake). In that way, mathematical
praxeologies will emerge as motivated objects, as far as they are tools to deal with the
initial question, as well as with the questions derived from this.
Identifying questions with generating power for some mathematical

praxeologies need to carry out a deep work of mathematical engineering. The design process needs to consider and explicitly question all the steps in the didactic transposition process
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. That is, instead of accepting the dominant and culturally unquestioned transpositions of scholarly knowledge as the unique ones, researchers need get out of school (considered as an institution) and question the meaning of the mathematical objects they are working with, their origin and evolution, their relation with other parts of mathematics, their integration in one or another praxeology. The result will be an explicit epistemological model used as the reference for the design of the SRP/A. Specifically, this mathematical engineering process is crucial to identify the meaning (one or several) of the mathematical praxeologies that will be integrated within the SRP/A
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. It is this meaning which will inform the kind of generating questions that might be considered.
The social legitimacy implies that questions should overcome the limits of mathematics and school, and be connected with society and its problems (questioning the world). Many of the questions currently studied at school have a pretended social legitimacy that rapidly vanishes. Extra-mathematical context are used mainly for motivational purposes, and disappear quickly to give rise to a decontextualized intramathematical activity. The functional legitimacy means that the questions should be selected so that its study does not lead to a dead end. On the contrary, crucial and generating questions should allow the study community to explore new territories, interesting both from a mathematical and social perspective. From the mathematical side, the epistemological analysis and the construction of the epistemological reference model will be again crucial to determine the functional legitimacy of a question. From the extra-mathematical side, the functional legitimacy is connected with the social one.
The initial design of SRP/A mainly involves researchers in the field of didactics, especially to carry out the mathematical engineering process. But the collaboration of teachers has been quite productive in our research. In particular, to identify and consider restrictions emerging from school that might provoke that the final product does not fit in school. However, the ecology of the SRP/A is a main issue that we cannot discuss here.
In line with the didactic engineering methodology (Brousseau, 1997), the apriori design just explained has to be complemented with the piloting of the SRP/A and its a-posteriori optimization. Researchers and teachers normally carry out this step jointly, and it is structured in advance. During the piloting, it is tested both students' mathematical activity (mathematical praxeologies) and teacher's teaching

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Scholarly Knowledge Knowledge to be taught Taught Knowledge Learned,
available knowledge
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For instance, in the SRA designed in García (2005), the epistemological analysis questioned the isolation of proportionality in the Spain curriculum. Proportionality is culturally modelled arithmetically and considered more as a static relation between magnitudes than a functional one. The epistemological reference model we built integrated proportionality in the world of functional relationships, as a possible (and not unique) relation in a variation system. As a consequence, in the SRA designed, students had to work in a variation system (savings plan) in which different variations hypotheses were possible. So, proportionality emerged as a kind of functional relation, not as a privileged one. (García, Gascón, Ruiz-Higueras and Bosch, 2006). Theme D – J. G. García & L. Ruiz-Higueras
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activity (didactic praxeology). In the students' dimension, it is analysed: how the different elements of the intended praxeologies emerge; the creation of techniques to deal with problems; the identification and formulation of new questions; the exploration of mathematical techniques, including determining its validity and limits; how the mathematical activity is described and justified (technological moment); how the mathematical objects are evaluated by the community of study; the institutionalization of the intermediate and final answers. From the teacher's dimension, it is analysed: the "place" of the teacher and students during the study process (topogenesis); the management of the didactic time and its consequences in the evolution of the mathematical activity (chronogenesis); the management of the learning environment (mesogenesis)
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. Further, piloting also offers evidence about the ecology and the economy of the designed task.
Study and research path for pre-school: taking care of our silkworms
In this section we will introduce, very briefly, a study and research activity for preschool (4-6 years old students). Researchers and pre-school teachers have designed it jointly and it has been implemented several times
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From the mathematical perspective, the didactic stake is learning about elementary arithmetic. In this level, students are supposed to develop quantification skills and the cardinal sense of numbers, languages and forms of expression to communicate about quantities, as well as start building the sense of some arithmetic operations (addition and subtraction mainly). Following Ruiz-Higueras (2005), which relies on the work of Brousseau and his colleagues in the 80s and 90s, an epistemological reference model was developed. Briefly, the cardinal sense of numbers will emerge in measuring situation with discrete magnitudes. Numbers emerge as models to express the measure of a set, to verify its conservation, to manage it, to remember the quantity, to reproduce or produce a set of a known quantity and to compare two or more sets. Numbers (as mathematical objects) and codes to express them (numerals) will emerge naturally in communicative situations where the aim is not just to measure a discrete set, but to communicate about it so that another person can produce a set similar to the initial one without having access to it (neither visually nor manipulatively)
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Having in mind this reference model, many a-didactical situations have been designed elsewhere. However, they are normally isolated and, sometimes, the context is quite fictitious. So, we wanted to embed numbers in the study of a system that would be real and meaningful for pre-school students, and also rich enough to allow a generating activity that might give rise to the emergence of new questions and different mathematical and extra-mathematical praxeologies. This context came up from the collaboration with a pre-school teacher. She suggested using collections of silkworms, considering that it was springtime and students were bringing some of them into the classroom.
The initial question faced by students was: if we've got N silkworms, how many leaves do we need to feed them? Firstly, they could go to the playground and

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For the SRC described in the next section, a detailed description of the analysis of the mathematical and didactic dimension can be found in Ruiz-Higueras and García (2011).
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In fact, some elements of this work have given rise to the publication by the teacher of a textbook for pre-school (Aguilar, 2010).
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Definition of a fundamental situation for cardinal numbers (Ruiz-Higueras, 2005). Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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collect the leaves from a mulberry tree. But after a few days, they had to ask the gardener to collect the leaves for them, using a written message. That provoked the need of using codes to express quantities
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In this stage, the mathematical activity is meaningful and significant for pupils, but restricted to elements of a single praxeology. The turning point is when the biological system starts to evolve: silkworms turn into cocoons, then moths arise and, finally, they die. Pupils have to control a heterogeneous collection made of silkworms, cocoons and moths, ruled by a conservative law
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. This is challenging for pupils:
• Firstly, because they need techniques to record the evolution of the system.
In the experimentation, the teacher prepared a table to record and control the evolution of the system (Fig. 1). She introduced a new artefact into the didactic milieu so that students could take the control of the evolution of the system under their own responsibility.
Fig.1: Table to record control system's evolution
• Secondly, because addition, in the sense of "combining", appears and new techniques are also needed. Due to the conservative law of the system, pupils need to control that the total initial amount remains constant.
Additionally, an expected issue emerged from the study of the system. The magnitude "time" was not expected a-priori (undetermined character of an SRP). However, pupils raised the question of how long will it take to a moth to rise since the cocoon was formed. This task involved the need of developing techniques to record and control the time. In the experimentations carried out, pupils used a calendar and their counting skills as a technique to measure time. Starting from an a-priori hypothesis from the media (answers H
◊
, given by their parents
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), they had the opportunity to evaluate this "already existing answer" against their learning milieu, getting to their "own" answer
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. Finally they found that a silkworm inside a cocoon needed 12 days to become a moth (later, they discovered a little variation in this time).
Tables like the one in Fig 1. had to be adapted as the system was changing. Moths had to be included and, finally, alive moths and dead moths as well.
At the end, when all the moths died, the system disappeared. However, pupils had lots of information about its evolution. And a final task was proposed: reconstructing the system. From a modelling perspective, the process is inverted. Now, interpreting models make it possible to recover information about the life of system that will never be back again.

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In this early age, many pupils identify written numerals but do not give them a meaning in terms of quantities. Indeed, most of them use self-invented codes (drawings) to communicate about quantities. Gradually, through activities like this one, they start giving some sense to the Arabic numerals and leave their primitive codes.
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Each state of the system can be described with the vector (t, n(t), c(t), m(t)) where t is time, n is the number of silkworms, c is the number of cocoons and m is the number of moths. A conservative law rules the system: for every t, n(t)+c(t)+m(t)=N, being N the original number of silkworms.
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Different parents gave different number of days. Theme D – J. G. García & L. Ruiz-Higueras
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Fig. 3. Reconstructing the system from the model
Conclusions
Task design within the ATD is a complex process. On the one hand, tasks are intrinsically linked to an epistemological conception of mathematics as a human activity, modelled in terms of praxeologies. On the other hand, it is connected with an explicit paradigm about mathematics in school, the so-called paradigm of questioning the world, in contrast with the paradigm of visiting works, still dominant in many school systems.
Study and research activities and paths within the ATD constitute a promising field of research and design. Among others, affordances of this approach are that it is aiming at a renewed school epistemology sensible to the world and its problems, opened to a more democratic flow of knowledge between school and society (Chevallard, 2012). Also, it contributes to a better articulation and integration of topics that are normally separated in school mathematics (García et. al., 2006), and even topics from different disciplines.
Also, it contributes to a better articulation and integration of topics that are normally separated in school mathematics (García et. al., 2006), and even topics from different disciplines.
On the contrary, limitations of the approach are connected with the cognitive dimension. As far as cognition is considered institutionally within the ATD, what remains unexplored is the issue of the cognitive demand of a designed SRP. Further research in this direction would probably clarify and strengthen the scope of this design approach.
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Task design for systemic improvement: principles and frameworks
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Tasks play many roles in mathematics education, some of them unintentional. They come in many forms, spanning many dimensions, including those of mathematical content, processes and practices, length, and modes of working. In this paper we point out the crucial role that tasks play in forwarding or preventing the process of improvement of teaching and learning - and thus of an education system. We argue that multi-dimensional schemes of task classification have a powerful role to play in the design of tasks and task sequences.
Keywords : classification, dimensions, balance, curriculum, assessment, sequences
Tasks in this paper will mean "activities that students are asked to do, in which mathematics has an important role". Students meet tasks in the course of classroom learning, in informal tests or examinations, in the course of their everyday lives, and in other school subjects. Task design for all these roles has been a central concern of the work of Shell Centre team over the last 30 years, which the authors have led (Swan and Burkhardt, 2012). Our mathematical and epistemological perspectives are eclectic, reflecting the priority in the team's research given to direct impact on systemic improvement.
In discussing tasks and their design, exemplification is absolutely essential to clarify the meaning of descriptions. As so often with print, the limitations of length in this paper mean that we cannot give examples here; we shall accept the opportunity to extend the range on-line, with links in the text to examples of various kinds.
1. Tasks and their various roles
The major focus of this Study will be on the nature and use of tasks in teaching and learning in classrooms. From the systemic viewpoint of this paper, however, it is appropriate to describe and discuss their roles starting with the system, and moving through assessment in its various forms to the classroom. Theme D - H. Burkhardt & M. Swan
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In specifying a curriculum Currriculum
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specifications are normally in the form of descriptions of principles, amplified by an analytic model of the domain. The latter in particular, vary greatly in length, from a few pages on competencies in the Denmark to long and detailed lists of mathematical content in British or US documents. The one common feature is that they are largely expressed through language. Such descriptions do not, in fact, specify learning or performance goals - for example, lists of mathematical content could be taught and assessed entirely through short tasks on separate elements, or through substantial projects in which the student chooses and uses appropriate elements of content and process for the investigation in hand - or, more sensibly, for a balanced variety of types of performance. Task exemplars can play a crucial role in reducing this ambiguity. We have argued (Burkhardt, 1990) that a curriculum specification needs three different elements: an analytic model of the domain: an exemplar task set, with each task linked to the model; a list of the range of classroom learning activities that should be involved (for a brief example, see Cockcroft Report, 1982, paragraph 243).
In high-stakes examinations In countries that have tests where the results have life consequences, the range and balance of types of task in the tests have a strong influence on the range and balance of classroom learning activities (see e.g. OFSTED 2012). Indeed they often seem to define the de facto "implemented curriculum" in most classrooms, whatever the intended curriculum of the last paragraph may say. So high-stakes assessment, and the tasks the tests contain, plays three roles:
• A: to 'measure' performance - ie "to enable students to show what they know, understand and can do" but also, with high-stakes assessment that impacts students' and teachers' lives, inevitably
• B: to exemplify the performance goals - assessment tasks communicate vividly to teachers, students and their parents what is valued by society, and thus
• C: to drive classroom learning activities (What You Test Is What You Get)
These roles carry responsibilities for test designers and those who commission tests - responsibilities that are widely ignored. Psychometricians, too, focus on measurement and statistical error, ignoring the systematic error that comes from assessing only a part of what you want students to learn. Ignoring roles B and C, and the systemic responsibility for test design that they imply, is a major source of the mismatch between intentions and outcomes in school systems.
In classroom assessment In some countries including ours, classroom assessment has traditionally reflected the formal tests, with similar task sets. This is a natural way for teachers to check progress towards an important goal. Some teachers, aware of the limitations of the tests, have always used a broader range of task types in the classroom. In the last decade there has been growing awareness of the power of formative assessment, when well done, in forwarding student learning (see Black and Wiliam 1998, 2001). This approach integrates assessment and teaching in a form where the design of task sequences plays a crucial role. This is challenging for teachers so there has been work on the design of support, initially through live

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By curriculum, we mean the whole set of learning activities that a student experiences in school. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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professional development and, more recently, through classroom materials (Swan et al., 2011)
In teaching and learning We expect the roles of tasks in teaching and learning to be the focus of most of the papers in this study - so here we shall be brief. That their range and variety should cover all the learning and performance goals of the intended curriculum is clear. What makes this area rich are the issues of, and principles for, designing task sequences that will lead students along the road to the understanding and performance goals. One may view task sequences as the spine around which all teaching is built, whether they be the succession of closely related exercises of "incremental learning" and its behaviourist relatives or, at the other extreme, the mathematical microworlds of "open investigation", where creating a task sequence by posing questions is part of the student's responsibility. We shall say something more in this in Section 3.
2. Task difficulty
The issue of task difficulty is often ignored but is important in all aspects of task classification and design. It is known from research that the difficulty of a task depends on various factors, notably its:
• complexity - the number of variables, the variety and amount of data, and the number of modes in which information is presented, are some of the aspects of task complexity that affect the difficulty it presents.
• unfamiliarity - non-routine tasks (those which arent just like the tasks one has practiced solving) are more difficult than routine exercises.
• technical demand - tasks that require more sophisticated mathematics for their solution are more difficult than those that can be solved with more elementary mathematics.
• student autonomy - guidance from an expert (usually the teacher), or from the task itself (e.g., by structuring or "scaffolding" it into successive parts) makes a task easier than if it is presented without such guidance.
Assessments of student performance need to take these factors into account. For example, these factors imply that, in order to design a task for a given level of difficulty, a relatively complex non-routine task that students are expected to solve without guidance needs to be technically easier than a short exercise that employs a routine skill. Focusing on technical aspects alone can lead to rich tasks as being dismissed as "below grade".
The difficulty of a task is determined by trialling the task with a random sample of students drawn from the target population. All assessment tasks, whether for use in the classroom or in summative tests, should be developed in this way, establishing their level of difficulty without undermining their validity as good mathematics.
3. Task variety and task classification
Given the range of roles that tasks play, outlined above, it is clear that appropriate forms of task classification may be useful for various purposes. In this section we set out some schemes and ways they have proved useful in supporting various task roles. There is a constructive duality between holistic and analytic dimensions of classification. Theme D - H. Burkhardt & M. Swan
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Novice, Apprentice and Expert tasks
This simple holistic dimension of classification (Swan et al 2011) has proved useful in drawing attention to the mismatch between widely accepted goals of mathematics education and current practice in both assessment and curriculum
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Mathematical skills and practices can be taught and/or assessed partly in isolation, partly under scaffolded conditions, and partly when students face substantial problems without scaffolded support. We call tasks that assess these three different types of performance novice, apprentice, and expert tasks respectively. More specifically:
• Expert Tasks. Experts solve problems as they arise. Expert tasks are rich tasks, each presented in a form in which it might naturally arise in mathematics, science or daily life. They require the effective use of problem solving strategies, as well as concepts and skills. Performance on these tasks indicates how well a person will be able to do and to use mathematics beyond the mathematics classroom. Expertise is the end goal of mathematics education.
• Novice Tasks. Novices are learning the tools of the trade. Novice tasks are short items, each focused on a specific concept or skill. Reflecting the highstakes assessment, mathematics teaching and learning in Britain and the US is mainly focused on novice tasks.
• Apprentice Tasks. Apprentices solve problems, but usually carefully structured problems with guidance from an expert. Apprentice tasks are substantial, often involving several aspect of mathematics, and structured so as to ensure that all students have access to the problem. Students are guided through a "ramp" of increasing challenge to enable them to show the levels of performance they have achieved. Because the structure guides the students, the strategic demands and the range of mathematical practices involved are at a comparatively modest level. Apprentice tasks have a role in developing expertise.
A Framework for Balance
Clearly, classification needs to go well beyond this. The NSF-funded project Balanced Assessment for the Mathematics Curriculum aimed to design assessment that reflected the goals set out in the NCTM Standards (NCTM, 1989, 2001). We set out two supervening design principles: Curriculum balance: a test should be such that a teacher who "teaches to the test" is led to deliver a curriculum balanced in accord with the Standards. Curriculum value: doing the assessment tasks should be a worthwhile learning experience. To articulate what this means we developed the Framework for Balance, summarised in the table below.
The headings are self-explanatory except, perhaps, for reasoning length.
This is the time envisaged for the student to work on the longest prompted section of the task - so a 10-minute task that is structured into many equal parts may have a short reasoning length. (Driven by the naïve criterion-referencing behind the National Curriculum, this is common in the UK, where tasks often consist of a sequence of

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The reader can find examples of each at http://map.mathshell.org.uk/materials/tasks.php Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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short items, set in a common context.). The most important features of the Framework for Balance are:
• the classification is multidimensional, addressing the major aspects of performance
• the dimensions are both analytic (content, process, etc.) and holistic (task type, openness, goals, etc.)
• it provides a method of choosing a set of tasks for a tests to meet conditions, particularly balance, for this
• the analytic dimensions are handled semi-quantitative, with the elements of content or process in a task given rough proportions
• an associated "balancing matrix" can be used to ensure that, while every combination of properties cannot be assessed, the main dimensions are samples with appropriate weight.
This approach was first used for balancing collections of classroom materials (Balanced Assessment 1997-99). It also produced a way around a design dilemma: the more constraints you impose on a task designer (such as a cell in a content matrix to assess), the poorer the holistic quality of the tasks that result. The alternative approach is to free designers to design good mathematical tasks, classifying them later and choosing a balanced set for each test.
Framework for Balance
Mathematical Content Dimension
• Mathematical content in each task will include some of:

Number and Operations including: number concepts, representations relationships and number systems; operations; computation and estimation.
Algebra including: patterns and generalization, relations and functions; functional relationships (including ratio and proportion); verbal, graphical tabular representation; symbolic representation; modeling and change.
Measurement including: measurable attributes and units; techniques tolls and formulas.
Data Analysis and Probability including: formulating questions, collecting, organizing, representing and displaying relevant data; statistical methods; inference and prediction; probability concepts and models.
Geometry including: shape, properties of shapes, relationships; spatial representation, location and movement; transformation and symmetry; visualization, spatial reasoning and modeling to solve problems.
Mathematical Process Dimension
• Phases of problem solving include some or all of:
Modeling and Formulating;
Transforming and Manipulating;
Inferring and Drawing Conclusions;
Checking and Evaluating;
Reporting.
• Processes of problem solving, reasoning and proof, representation, connections and communication, together with the above phases will all be sampled.
Task Type Dimensions
• Task Type will be one of: design; plan; evaluation and recommendation; review and critique; non-routine problem; open investigation; re-presentation of information; practical estimation; definition of concept; technical exercise.
• Non-routineness in: context; mathematical aspects or results; mathematical connections. Theme D – H. Burkhardt & M. Swan
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• Openness -tasks may be: closed; open middle; open end with open questions.
• Type of Goal is one of: pure mathematics; illustrative application of the mathematics; applied power over a practical situation.
• Reasoning Length is the expected time for the longest section of the task.
Circumstances of Performance Dimensions
• Task Length: in these tests most tasks are in the range 5 to 15 minutes, supplemented with some short routine exercise items.
• Modes of Presentation, Working and Response: these tests will be written.
This idea has been taken further: Daro and Burkhardt (2012) proposed the development of a "population of tasks" that epitomises the curriculum goals, and from which tests will be drawn as balanced samples.
While we do not present these classification schemes as definitive (though they have worked well for specific purposes), we do see task classification as an important part of task design.
4. Principles for the design of tasks and task sequences
We have recently written in some detail
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on the principles and processes of
task design (Swan and Burkhardt, 2012). Here we have space for a bare list of principles. We argue that curriculum and assessment should be built on tasks that:
1. Reflect the curriculum in a balanced way. Assessment should be based on a balanced set of tasks that, together, provide students with opportunities to show all types of performance that the curriculum goals set out or imply.
2. Have 'face validity'. Assessment tasks should constitute worthwhile learning activities in their own right. The tasks should be recognizable as problems worth solving – because they are intriguing and/or potentially useful.
3. Are fit for purpose. The nature of the tasks and scoring should correspond to the purposes of the assessment. Individual tasks should assess students' ability to integrate as mathematical practices their fluency, knowledge, conceptual understanding, and problem solving strategies. These aspects should not be assessed separately.
4. Are accessible yet challenging. Tasks should be accessible with opportunities to demonstrate both modest and high levels of performance, so the full range of students can show what they can do (as evidenced by high response rates with a wide range of levels of response).
5. Reward reasoning rather than results. Tasks should elicit chains of reasoning, and cover the phases of problem solving (formulation, manipulation, interpretation, evaluation, communication) even though their entry may be scaffolded with short prompts to ensure access.
6. Use authentic or 'pure' contexts. Assessment should contain tasks that are 'outward-looking', making connections within mathematics, with other subjects, and to help one to better understand life and the outside world. As in the real world, they may contain insufficient data (where the student makes assumptions and estimates) or redundant data (where the student makes selections). Students may be asked to respond in a given role: e.g. a designer, planner, commentator, or evaluator. Tasks that use contrived contexts should be avoided.
7. Provide opportunities for students to make decisions. Tasks should be included that encourage students to select and choose their own methods, allowing them to

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See http://www.educationaldesigner.org/ed/volume2/issue5/article19 Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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surprise or delight. Some may be open-ended, permitting a range of possible outcomes.
8. Are transparent in their demands. Students should be clear what kinds of response will be valued in the assessment.
Task sequences in curriculum design
More broadly, we see the concept of task sequence at the heart of curriculum design. Too large a topic for this paper with its systemic focus, it will be the focus of a forthcoming article. Here we will just reinforce the idea with a few examples.
We have already noted that the "incremental learning" approach to curriculum design, characterised by small steps, is strongly reinforced by the nature of assessment. Instead, we have cotinued developing an approach that is very different (see Swan, 2006).
We distinguish whether task sequences are designed primarily to foster conceptual development or problem solving processes. The focus of the first is on discussing different interpretations of mathematical ideas; the second is on the contrasting alternative approaches that may be taken. In both cases we begin by seeking to find out students prior knowledge, by asking them to tackle a carefully chosen task individually, unaided. Their responses are assessed by the teacher, outside the classroom, who must then prepare a series of questions (tasks) designed to prompt students' deeper reflection. We provide a set of questions matched to typical responses and to assist the teacher in this.
In a problem solving lesson, students are then invited respond to these questions and form small groups to produce joint solutions that both combine the best of their individual ideas, and that address the teachers' questions. A sharing of alternative approaches is then undertaken, akin to the Japanese practice of 'neriage'. Often, students do not consider the most powerful problem solving approaches without further prompting. We therefore provide students with some "sample student work", chosen and collected by ourselves. This work is designed to show more sophisticated attempts at the problem. Students' task is now to critique, improve, complete and extend suggested solutions – a challenge to their existing thinking.
The concept-focused lessons are similar in structure to the problem solving lessons, but here we identify the different task genres that promote concept development and select a rich of that kind. Examples are given in the table below:

Task genres Description of tasks
Classifying and
defining
Students devise classifications for mathematical objects, and/or apply classifications devised by others. They discriminate, recognise properties and develop mathematical language and definitions.
Interpreting and
translating between
multiple
representations
Students match cards that show different representations of mathematical objects – words, diagrams, algebraic symbols, tables, graphs. They share interpretations, compare and group the cards in ways that made connections between underlying concepts. The discussion of common 'misconceptions' is encouraged by the inclusion of distracters.
Testing and
evaluating
mathematical
statements and
conjectures
Students are given short mathematical statements or generalisations, are asked to make posters that describe their domain of validity and provide examples, counterexamples and explanations to support their decisions.
Creating and solving
variants of
Students devise new or variants of existing problems, prepare solutions then challenge other students to solve them. They offer support when the solver Theme D – H. Burkhardt & M. Swan
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mathematical
problems
becomes stuck. This promotes awareness of the structures underlying problems, and focuses attention on the doing and undoing processes in mathematics.
Analysing reasoning
and solutions
Students compare different methods for doing a problem, organise solutions and/ or diagnose the causes of errors in solutions. They begin to recognise that there are alternative pathways through a problem, and develop their own chains of reasoning.
These activities are conducted in a collaborative atmosphere, with the teacher acting as a provoker, using the prepared questions to prompt students to argue and refine their interpretations, with a whole class 'neriage' discussion as wrap-up.
Challenges for research and development
Currently, the tasks presented by high stakes examinations and textbooks, (which in the UK are often written by examiners who focus on repetitive practice of examination-type questions) largely determine the types of task that are used within classrooms. We need to challenge this state of affairs at policy level using such classification schemes as we have described above in curriculum documents to describe learning objectives. A vital component, often missing form such documents, is the vivid exemplification that is necessary to show exactly what such tasks might look like.
At a deeper level, further refinement and illustration of the task-types we have described here is needed; in particular, further classroom evidence of their individual impact on teacher and student practices and performances is required. In addition, research is needed to show how student performances on conceptual and problem solving tasks might be reliably measured and reported. Otherwise examiners and teachers will continue to assess fragments rather than complete performances.
Background and context
While all members of the team contribute to the various aspects of the Shell Centre's work, the authors have played central roles. Malcolm Swan has led the design of tasks and the elicitation of design principles (see e.g Swan and Burkhardt 2012) while Hugh Burkhardt has played a leading role in the development of the analytic frameworks for describing and balancing tasks and the strategic design of tests and their curriculum support (see e.g. Burkhardt 2009). The work has had ongoing national and international support over the past 25 years.
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Task design in a school-based professional development programme
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equal contributions.
In this paper we report on two principles of task design arising
from our study of a school-based mathematics teacher professional
development programme in Shanghai, China. The two principles are: a)
developing a 'hypothetical learning structure' for the topic; and b)
developing tasks within a web-like structure of knowledge connections.
This paper provides an example of each and connects the literature
review with the described research design.
Keywords: task design; school-based professional development;
expert teachers; Chinese mathematics pedagogy; collective task design.
Introduction
This paper focuses on the issue of how published tasks (sourced from textbooks) are
appropriated by teachers for instructional purposes and hence how task design
influences mathematics teaching. In particular, we focus on the central aim of the
Working Group Theme D (in the ICMI Study 22 Discussion Document) by reporting
on the design and implementation of tasks by a team drawn from three communities:
academic researchers; the local school district professional developer, who is an
expert teacher with considerable skill in school-based teacher professional
development; and a group of Chinese teachers from Shanghai Soong Ching Ling
School, Shanghai. In doing so, we take the stance highlighted by Kieran, Krainer and
Shaughnessy (2013) that teachers are key stakeholders in research and can play a
significant role in the design of rich tasks for mathematical learning. To add to this,
we argue that the expert teacher/professional developer can also play a significant
role in task design and its 'implementation'.
In China, the 2011 version of National Mathematics Curriculum Standards
(briefly called Standards in this paper) emphasises the shift from the traditional "Two
Basics" (basic knowledge and basic skills) to the current "Four Basics" (adding basic
mathematical thinking, and basic activity experiences). As a result, teachers are Theme D – L. Ding; K. Jones; B. Pepin 442
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expected to carefully consider and prepare the teaching of particular mathematical
knowledge and skills, in addition to the development of students' individual
understandings and thinking, and to building up students' learning activity
experiences in mathematics (Shi, 2012). The problem now facing educators and
teachers in China is which mathematical tasks, and task sequences, should be
selected/considered (for particular topics), and how to work with such tasks, so as to
implement effectively the range of intentions laid out by the Standards.
Theoretical framework and key pedagogical features
Simon (1995) proposed the notion of Hypothetical Learning Trajectory (HLT) to help
mathematics educators (i.e. teachers, researchers, curriculum developers) to think
about the design and use of mathematical tasks to promote mathematical conceptual
learning. The HLT comprises of three components (ibid p. 136): the learning goal;
the learning activities; and the hypothetical learning process – a prediction of how the
students' thinking and understanding might evolve in the context of the learning
activities. Simon (1995) further points out that teachers need to develop skills to
generate hypotheses about students' understandings (which go beyond soliciting and
attending to students' thinking), to generate HLTs, and to engage in conceptual
analysis related to the mathematics that they teach.
Ma (1999, p. 97) reports that one feature of Chinese teachers' knowledge is
their well-developed "knowledge packages" of the range of ideas needed to teach a
topic like arithmetic. Within such a package, there is a certain "key" piece of
knowledge that is fundamental to enable students to learn other knowledge. Another
kind of key piece of knowledge in the package is the "concept knot" (p. 98), which
links to several different concepts of the learning topic. Ma (1999) considers that
teachers' knowledge packages reveal their understandings of the longitudinal process
of opening-up and developing mathematics of a particular field in students' minds.
Gu, Huang and Marton (2004) identify two types of teaching with variation in
the Chinese mathematics classes. One is conceptual variation (CV), concerned with
understanding concepts from multiple perspectives; the other is the procedural
variation (PV), focused on developing insights into the hierarchical features of
mathematical activity. Gu (2012) further interprets that the PV can be used to
connect, in a dynamic way, different mathematical processes as a whole process. That
is, a teacher can divide the process/es of a mathematical activity into a number of
sub-activities, and then use variation as a means of "pu dian" (i.e. scaffolding) of
knowledge between the sub-activities.
Based on the literature (e.g. Ma 1999; Gu et al. 2004), we identify the
following key features of Chinese mathematics pedagogy: 1) understanding concepts
from multiple perspectives; 2) designing for individual student's actual learning
processes and responses; 3) gradually deepening learning through an orderly-layered
teaching procedure (a teaching/learning method called "Xun Xu Jian Jin" in Chinese).
In our project we utilised a school-based teacher professional development program
(TPDP) (as detailed in the next section) as a means to enable researchers, expert
teacher and school teachers to work together. In the TPDP, we are aware of
Gravemeijer's (2004) notion of a teacher's "local instruction theory" (highlighted in
Choppin, 2011), which is concerned with student thinking about complex tasks and
how this is situated within a broader instructional 'space'. For more on the
mathematical and epistemological basis of our task design study, see Lin (2011). Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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The task design study
The main study was arranged in Shanghai Soong Ching Ling School located in
Qingpu district, a western suburb of the city. The school was an international
laboratory school funded by China Welfare Institute with the key mission of
launching innovative and experimental educational classroom studies aimed at
improving the quality of compulsory education for children in the country.
The reasons for this study are twofold. First, the school planned to launch its
own innovative mathematics curriculum to complement the strengths of the Shanghai
(SH) mathematics curriculum. Second, an innovative school-based model of
curriculum was expected to bridge the gap between curriculum developers' intentions
and school teachers' classroom practice.
The methodology of this study is based on a Design-Based Research (DBR)
(e.g. Design-Based Research Collective, 2003) approach to study, document and
advance a model for sustainable teacher professional development. The approach
involved processes such as iteration and feedback loops in such ways that
development and research took place through cycles of design, enactment, analysis,
and redesign (Cobb et al. 2003).
The tasks in the pilot study were sourced from published material from two
countries: the SH official textbooks; and the online professional development
resources from New Zealand (NZ):
http://www.nzmaths.co.nz/professional-development
The participant groups of the pilot study were: (1) three researchers (the three
authors); (2) an expert teacher (called 'Mr Zhang' in this paper); and (3) three
teachers, one of them a teacher-researcher (first author). The three researchers
(authors) designed the study and provided the theoretical background (literature,
research design, etc.). Next, in the school-based TPDP, the first author (a teacherresearcher) designed the first intervention based on ideas from the NZ online
resources. The expert teacher, *Mr* Zhang, commented on it together with other
teachers in the TPDP. Two Grade 4 teachers (one experienced teacher, and one less
experienced teacher) then took Mr Zhang's comments into account in their redesigned lessons of the same topic in two other grade 4 classes. *Mr* Zhang reflected
upon both lessons, together with other teachers. Subsequently, the three researchers
analysed the data, which consisted of data on the context; observations and videos of
the three lessons; discussions/interviews with *Mr* Zhang after each lesson; and
documents, such as curriculum materials and textbooks; also official documents,
guidelines for teaching, syllabi, etc..
In this paper we define a "task" as a learning situation with a specific
teaching goal in a single lesson. The main body of the task design is therefore to
create a sequence of multi-layered learning situations within a broader instructional
plan and to develop a general picture of the students' learning paths. In the pilot study
the teachers were actively engaged in the work of how to select, modify, sequence,
teach, observe and reflect on a single lesson and on a sequence of lessons.
As laid out in the SH Grade 4 mathematics textbook (term II), the chapter on
decimals (30 pages in total) consists of the following sub-topics: decimals in real life;
the meanings of decimals; comparing decimals; the properties of decimals; the
movement of the decimal point; the addition and subtraction of decimals; and the
application of the addition and subtraction of decimals in problem solving. In their
teaching teachers were expected to appreciate the key connections of the
teaching/learning goals embedded in the SH textbook as follows: (for pupils) Theme D – L. Ding; K. Jones; B. Pepin 444
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1. to learn/understand the relationship between decimals and fractions (to
understand the base-ten place value system of decimals and the meaning of
0.1, 0.01, and 0.001);
2. to learn/understand the units of decimals (e.g., 0.1) (to understand the
meanings of decimals with one, two, three digitals; and to know a whole
number with a decimal);
3. to learn/understand the concept of place value of decimals (to understand the
place value table of decimals).
In what follows, we focus on delineating the two major principles that
emerged from the data analysis of the three teachers' lessons on 'place value of
decimals' (briefly called 'decimal value') and of *Mr* Zhang's comments during the
post-lesson TPDP.
Findings: two principles of the task design study
Principle 1: Developing a 'hypothetical learning structure' for a particular
topic
The first teacher intended to use the NZ online teaching resources to complement the
strengths of the SH mathematics textbook and pedagogy. For the lessons of decimal
value, the teacher's "local instruction theory" (Gravemeijer, 2004) was to enable her
students to accumulate learning experiences with the big idea that the decimal
number system is a base-ten place value system. Thus, she used the NZ online tasks
to enable students to examine two sub-ideas associated with the base-ten place value
system.
1. The places (or columns) in the number system are based around groupings of
ten. For example, 10 ones = 1 ten, 10 hundreds = 1 thousand.
2. The decimal point is a convention that indicates the units place.
The teacher started the lesson by presenting the place value of whole number
(briefly called whole number value) on the blackboard (see the left half in Figure 1).
While students observed it, they recalled previously learnt knowledge such as series,
and a 10-to-1 relationship between the values of any two adjacent places (or
columns). For instance, some students discussed that 10000 equals 1000 of ten.
Others argued that 10000=10×1000. Still others added that the "Wans" (tens of
thousands) are 3 places to the left of "Shi" (tens). This means that there is a
10×10×10 relationship between "Shi" and "Wan".
… Qian
(Thousands)
Bai
(Hundreds)
Shi
(Tens)
Ge
(Ones)
Shi Fen
Zhi Yi
(Tenths)
Bai Fen Zhi
Yi
(Hundredths)
Qian Fen Zhi
Yi
(Thousandths)
…
… 1000 100 10 1 1/10
0.1
1/100
0.01
1/1000
0.001
…
Figure 1. The decimal number system
Next, the teacher drew students' attention to the decimal value (see the right
half in Figure 1). For instance, according to the newly learnt knowledge of decimals,
students were able to discuss in the class that 10 of 1/100 is 1/10. Some represented
such a relationship by decimals (10 of 0.01 is 0.1). The teacher then led students to
pay attention not only to a 1-to-10 relationship between adjacent places to the right
where the places were getting smaller by a factor of ten, but also to the relationship
between any two places in the number system. In observing such relationships as
0.1×10=0.01, 0.1×100=0.001, and 0.1×1000=0.0001, students were also led to think Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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about the movement of the decimal point. Moreover, the teacher posed tasks to
enable students to see the connection between the movement of the decimal point and
the units of a number. For instance, 6501.4 (in this case the "ones" is
assumed),650.14 tens (in this case the units are tens),65014 tenths (in this case
the units are tenths),and 6.5014 thousands (in this case the units are thousands).
In the first post-lesson profesional activity (PPA), *Mr* Zhang firstly
commented on the teacher's lesson from three perspectives, namely (1) the HLT of
students; (2) the learning methods; and (3) the degree of learning difficulty.
I found that the (first) teacher dealt with the textbook in her lesson differently
from those in our common lessons. … There is a starter in the HLT. That is, the
anchor of knowledge. For instance, what types of knowledge, experience and
methods of learning the students already have? And to what degree? The starter
of this lesson was at an abstract level, such as the comparison of the place values,
enlarge or reduce a number, the relationship of places, etc. … Some students in
the class may be lost at this abstract level. … Moreover, the learning method is
different. In this lesson, the teacher provided a bit of context of the problems,

students then developed a discussion in the class and then solved some points of problems by themselves. It's considerably random. The logical structure of learning itself is loose. For instance, what is the first step and then the next step of learning? In a whole, what is the general goal of learning? The learning was designed in a macro way.
Next, Mr Zhang drew the teachers' attention to the HLT from a micro perspective of the "concept knots" (the meaning and the places of decimals) (Ma 1999) in the decimal chapter of the SH textbook.
To view the learning from a micro perspective, students are expected to master a number of factors from this topic. For instance, can a number be read on the decimal value (dv) table? Can a decimal be put on the dv table? Can the form of the decimal be explained on the dv table? In the last lesson [according to the SH textbook], it is about the structure of the meaning of decimals. In this lesson, a shift is to be made to the structure of the places of decimals. In the end of this lesson, the two structures should be connected.
From such a micro perspective, Mr Zhang also addressed the importance for teachers to recognize the connection of the "key" piece of knowledge (the whole number value) (Ma 1999) and new knowledge (the decimal value) in the HLT.
When pupils learn the decimal value, their cognitive anchor is on what they have previously learnt of the place value of whole numbers [learned in Grade 4 –Term 1]. Teachers thus should create cognitive conflicts of the new topic for their students, to go beyond their previous knowledge of whole numbers. It is because … to 'fill' a decimal is something new for students. In so doing, their intellect is challenged and they can be engaged in thinking about how to create a decimal value which is the topic of this new lesson.
In our study, we use the term "hypothetical learning structure" (HLS) to distinguish the Chinese expert teacher's concept of HLT from Simon's (1995) HLT, in terms of two considerations: (1) The HLS in our study is not based on the western constructivist theory, but largely on the Chinese expert teacher's expertise in predicting students' learning processes and responses through observing the same topic many years during authentic classroom practice. (2) The use-aim of the HLS is not only to address the teacher's well-developed "knowledge packages" (Ma, 1999) in mathematics, but also to distinguish the learning methods from those of the HLT (Simon, 1995). That is, in Mr Zhang's view, pupil learning could be more 'efficient' (in the sense of 'whole class learning') if students do not randomly use their previous knowledge to respond to the teacher's questions in a set of points, but are engaged in
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the teacher's well-designed mathematical tasks in a set of 'blocks'. Mr Zhang offered the following 'blocks of tasks' for the decimal value lesson:
1. Write decimals and then write decimals with whole number on the number line;
2. Write units of decimals on the number line (e.g., 0.1, 0.01, and 0.001);
3. Write whole number in the whole number table and then write decimals with whole numbers in the whole number table;
4. Create the decimal table, design the place value in the decimal table and recognize the role of the decimal point;
5. Develop an understanding of the units of decimals in the decimal table;
6. Write the form of decimals in mathematical notation (e.g., 0.23 = ( ) × 0.1 + ( ) × 0.01);
7. Write the form of decimals in word language;
8. Define two types of decimals (decimals and decimals with whole numbers).
Principle 2: Developing tasks within a web-like structure of knowledge connections
Two teachers followed up the conversations with Mr Zhang by carefully planning the second and the third lessons for each of their classes on the same topic of decimal values according to the HLS (outlined above). Their lessons were quite different due to the teachers' different instructional intentions and the different students in each class. The second teacher (with two years teaching experiences in primary mathematics) sought to develop students' understanding of the form of decimals on the concrete geometrical model (the number line), and then their abstract thinking in the decimal table. For instance, in the starter of his lesson, the teacher used a considerable amount of time for students to write decimals on the different sections of the number line (e.g. those between 0 and 1, 0 and 1/10, 0 and 1/100, 0, 1 and 2, and 61, 62 and 63).
The third teacher (with ten years teaching experiences in primary mathematics) did not use the number line, but developed her students' abstract thinking of decimals directly from the form of decimals to the decimal table. For instance, in the starter of her lesson, the teacher posed a number of decimals and a whole number on the blackboard (e.g. 0.23, 0.63, 1.08, 61.52, 88.888, 1045). She then asked students to discuss the form of these numbers. Some of the examples are given below:
0.23 = ( ) × 0.1 + ( ) × 0.01
1.08 = ( ) × 1 + ( ) × 0.1 + ( ) × ( )
61.52 = ( ) × 10 + 1 × ( ) + ( ) × ( ) + ( ) × ( )
In the second PPA, Mr Zhang mainly commented on the two approaches applied by the two teachers, in order to help teachers to develop an insight into selected didactical ideas in mathematics teaching which are advocated in the latest national curriculum reforms. The new curriculum advocates developing students' complex thinking in a web-like knowledge structure, so as to enable students to see the most important feature of how mathematical knowledge is constituted. Mr Zhang said the following:
Different from the second teacher, the third teacher's lesson was from number to table. In fact, there are a considerable number of concepts in the (decimal) table. For instance, 10-to-1 relationship between the values of any two adjacent places,
the place of decimal, the number on the unit of decimal, the unit number of decimals, etc. The (third) teacher's instruction was clearly concerned about the first two concepts, but did not talk about the other concepts in the lesson. The main 'missing point' of the lesson is the lack of concrete diagrams to support students to understand the connection of the abstract concepts, such as the unit of the decimal place and its meaning (as the second teacher did). The advantages of the combination of decimal table and the number line are as follows: firstly, the infinity in dividing a small segment on the number line can be connected to the concept of number place in the table. At the same time, it is an opportunity for students to develop the concept of infinity. In such a way the teacher helps students to see the connection between the representation of the geometrical model and the mental representation of the decimal table. Consequently, the teacher makes the scaffolding for students to understand the abstract concepts and ideas such as infinity, set, 'approaching'. A lesson like this is likely to be a deep learning lesson. … In the latest pedagogy reforms, 'gathering thinking' is highlighted in teaching. That is, thinking is developed as a web, rather than the linear line/progression of knowledge connections. The 'gathering thinking' is based on a rich web of knowledge and a web of experiences. New knowledge is the result of the richer/more connections. The point is to understand what kinds of web new knowledge is based on. Then teachers could see how to help students to develop multiple connections of knowledge and experiences within the web. The connection of a new point of knowledge to the multiple points of knowledge on the web would enable students to enrich knowledge representations, make knowledge transformation and application, and develop complex thinking, namely the 'gathering thinking' in mathematics.
Mr Zhang then drew two web-like structures of knowledge connections of the decimal table. Figure 2 is a macro-level of the knowledge structure of the decimal table. Figure 3 is a micro-level of the knowledge structure of the decimal table.
Figure 2. A macro-level of the knowledge structure of the decimal table
Figure 3. A micro-level of the knowledge structure of the value place of decimal table
The two figures indicate two key theoretical ideas of this expert teacher in helping teachers to develop knowledge and ideas to make the multi-layered task design: first, the conceptual variation (Gu et al., 2004) for understanding concepts from multiple perspectives; second, the "Xun Xu Jian Jin" method to develop the complex mathematical thinking within the whole structure of knowledge connections.
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Implications
In this paper, we have delineated two principles which had emerged in our pilot study of the school-based tasks and instruction design project in Shanghai. However, there are challenges for the main study that concern the difficulties of the connection between theoretical frameworks and principles for our task design across communities:
(a) It is challenging to situate knowledge from different design communities into the 'living context' – the classroom in a broader culture (in our study the Chinese culture). In particular, if we wish to introduce innovation in teaching, we need to understand more comprehensively the alternative "local instruction theories" (Gravemeijer, 2004) that different teachers (researchers, professional developer and school teachers in our study) appear to hold.
(b) The HLT (Simon, 1995) features a teacher's design decisions based on her/is best guess of how learning might proceed. In our study the HLS addresses the importance for teachers to develop their "knowledge packages" (Ma, 1999) and to try things out in practice. It would be necessary to understand how the expert teacher's HLS may help teachers to make the connection between the unpredictable nature of individual learning and the pedagogic practice/ mathematical-didactic structures (macro and micro) suggested by the expert. Further, it would be necessary to develop new insights into the different teaching/learning methods underlying Simon's (1995) HLT in the US classroom and the HLS in the Chinese classroom. For instance, the distinction of the western construction (i.e., scaffolding) and the Chinese "Xun Xu Jian Jin" and "pu dian" (i.e., procedural variation, in Gu et al., 2004, p. 340), the theoretical ideas of the 'proper potential distance' (Gu et al., 2004, p. 343), and the micro and macro perspectives of learning viewed by the expert teacher in our study.
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Hybrid Tasks: Promoting Statistical Thinking and Critical Thinking through the Same Mathematical Activities
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Even though statistical thinking and critical thinking appear to have strong links from a theoretical point of view, empirical research into the intersections and potential interrelatedness of these aspects of competence is scarce. Our research suggests that thinking skills in both areas may be interdependent. Given this interconnection, it should be possible to stimulate both forms of thinking through the one task. An exploratory qualitative study has been undertaken into thinking processes when working on tasks encompassing both these areas. This paper explores the implications of this study for the design of tasks that simultaneously stimulate critical thinking and domain-specific thinking.
Keywords: Statistical and Critical Thinking, Task Design, Hybrid Tasks
Background
Mathematics classroom instruction is generally organized around and delivered through students' activities on mathematical tasks (Doyle, 1988). Notably, in all of the seven countries that participated in the TIMSS 1999 Video Study, eighthgrade mathematics was most commonly taught by spending at least 80% of lesson time in mathematics classrooms working on mathematical tasks (Hiebert et al., 2003). For our purposes, complete description of a mathematical task requires specification of the intentions, actions and interpretations of both teacher and student/s, together with details of the context in which the task was undertaken and by whom (Mesiti & Clarke, 2010). In this paper, "task" mostly refers to the written stimulus, although we also provide the details required to understand the performative realisation of the task

as a mathematical activity. Classroom activities are coherent actions shaped by the instructional context, in general, and, in particular, by what is taught through the use of tasks (Stodolsky, 1988). The tasks that teachers assign can determine how students come to understand what is taught. In other words, tasks serve as a context for students' thinking, during and after instruction. Doyle argues the point that tasks influence learners by directing their attention to particular aspects of content and by specifying ways of processing information (Doyle, 1983, p.161). Theme D – E. Aizikovitsh-Udi, D. Clarke & S. Kuntze
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To achieve quality mathematics instruction, the role of mathematical tasks to stimulate students' cognitive processes is crucial (Hiebert & Wearne, 1993). Contemporary curricula prioritise more than just procedural knowledge. In the mathematics curricula of countries as culturally and geographically distant as China, Australia and Finland, student thinking is promoted in more sophisticated terms than simply knowledge of facts and procedures. Korea and Singapore, in particular, seek to retain the high level of mathematical competence documented in TIMSS and PISA results, while improving higher order thinking and problem solving expertise. This poses the question as to whether both can be achieved. The possibility is explored in this paper that the same tasks might be employed to realise both goals. The tasks employed came from the research of Kuntze and his colleagues, the actual interviews were conducted by Aizikovitsh-Udi, the analyses of the task responses with respect to Statistical Thinking and Critical Thinking were coordinated by Kuntze and Aizikovitsh-Udi and the task-specific interpretive analysis was done by Clarke.
In summary, the centrality of tasks in mathematics classroom is evident from theoretical perspectives as well as in empirical results from international comparative studies. The role of mathematical tasks provides a key to any attempt to understand teaching and learning in research on classroom practices in mathematics. But can a given task stimulate and promote both discipline-specific thinking and more generic forms of higher-order thinking? To explore this question, we take Statistical Thinking (ST) and Critical Thinking (CT) as our exemplars of the two modes of thought.
Statistical Thinking and Critical Thinking
In a well-known definition of Statistical Literacy by Gal (2004), a "critical stance" is included among the key attitudes for successful statistical thinking (ST) – hence, Gal includes such attitudes in his definition of statistical literacy (cf. also Wallman, 1993; Watson, 1997; Reading, 2002). However, being critical in statistical contexts is not only an attitude. It is possible to describe specific abilities that have to be used in order to critically evaluate statistical data. Two key concepts or overarching ideas in statistical thinking relevant for a critical evaluation of data are manipulation of data by reduction (Krügel, Peschek & Schneider, 2000) and dealing with statistical variation (e.g. Watson & Callingham, 2003). Successfully manipulating data by reduction requires the awareness of such things as that calculating a mean value affords an overview on the original data, but it reduces the initial information. Hence, the resultant statistical value is (only) an indicator corresponding to a specific mathematical model, and we should not forget that it reflects only a part of the information. In order to critically evaluate the data, we might need additional information about the distribution, such as the variance, or information about extreme values.
Critical thinking (CT) skills rely on self-regulation of the thinking processes, construction of meaning, and detection of patterns in supposedly disorganized structures (Ennis, 1989). Critical thinking tends to be complex and often terminates in multiple solutions, each with advantages and disadvantages, rather than a single clear solution. It requires the use of multiple, sometimes mutually contradictory criteria, and frequently concludes with uncertainty. This description of CT already suggests links with ST, such as dealing with uncertainty, contradictions and a critical evaluation of given claims (cf. McPeck, 1981). Dealing critically with information – a crucial aspect for both domains – demands critical/evaluative thinking based on Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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rational thinking processes and decisions (Aizikovitsh-Udi, 2012; Aizikovitsh-Udi & Amit, 2008). Can a single task be used to elicit and promote both forms of thinking? In-depth analyses are required into how CT and ST may interdepend. In order to design the type of hybrid tasks proposed here, it is essential that we understand the connection between the two forms of thinking that provide the specific goal for the use of such tasks in mathematics classrooms and how these connect to task characteristics.
Investigating Hybrid Tasks
In attempting to stimulate particular thinking skills, it might be thought best to target either discipline-specific thinking (eg. Twelve numbers have a mean of 10 and a standard deviation of 2, what might the numbers be?) or generic critical thinking skills in discipline-free contexts (eg. Five people are isolated by flood, under what circumstances would it not be appropriate to share the available food equally?). Hybrid tasks seek to promote both. This paper explores the actions such tasks promote, the rationale for their use, and their design characteristics.
In order to explore thinking processes related to tasks in the domains of both Statistical Thinking and Critical Thinking, individual semi-structured interviews were conducted with mathematics teachers. By using mathematics teachers as subjects, basic content competence can be assumed and it becomes possible to examine their content-related higher order thinking skills, both in terms of statistical thinking and critical thinking. The interviews focused on thinking-aloud when solving tasks and each lasted about 40–50 minutes. Beyond solving the tasks, the interviewees were also free to give their personal views on the tasks.
In the following section, the results from one interview with a single teacher are used to exemplify the sort of data generated and the type of task likely to stimulate Statistical Thinking and Critical Thinking. The analysis concentrated on identifying Statistical Thinking (ST) and Critical Thinking (CT) as employed by the interviewees. Our interest is not just in the capacity of a single task to stimulate both ST and CT, but whether the interaction between ST and CT made them mutually supportive or disruptive. A first analysis was done focusing on ST only (Watson, 1997), then a second analysis employed a CT point of view (Ennis & Millman, 1985). We then carried out a combined interpretative analysis in order to examine relationships between CT and ST elements. In this methodological approach, the analyses were done by two coders working in parallel. Figure 1 shows the first task analysed. Theme D – E. Aizikovitsh-Udi, D. Clarke & S. Kuntze
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Figure 1: Task "population"
(Kuntze, Lindmeier & Reiss, 2008)
Nena was an experienced U.S. secondary Mathematics teacher with 20 years of mathematics teaching experience. She was dedicated to improving her teaching and had been participating in a year-long professional development program for mathematics teachers at the time this interview occurred. In the interview, Nena was asked to solve the problem in Figure 1, while thinking aloud:
Interviewer: What do you think? From which year on has the population been decreasing?
Nena: Well…. the population has been decreasing since 1963 until about 1973, where the population begins to rise a bit….. but it is still down since 1963.….
Interviewer: Are you confident about it?
Nena: Of course… I am sure! You can look at this problem mathematically, anywhere there is a positive slope you could say the population is increasing, where there is a negative slope the population is decreasing. But compared to 1963, it's always been down.
Dealing with assumptions is one of the key elements of Critical Thinking (Ennis & Millham, 1985). Nena initially assumed that the graph of births completely determined the population development. Even when asked to reflect on this assumption, Nena did not generate possible counter-arguments for testing her initial assumption, nor did she appear to question this assumption. She tended to seek confirming evidence rather than evidence that might challenge her initial assumption.
Seen from the perspective of Statistical Thinking (ST), Nena chose an inappropriate statistical model for interpreting the data given in the diagram (cf. description of this task in Kuntze, Lindmeier & Reiss, 2008). She appeared to focus on the data related to the births only, and she deduced her conclusions from a mathematical consideration of slopes. Even when encouraged by the interviewer, she did not check this model against the full data given in the diagram.
Nena's answers show deficits both in Critical and Statistical Thinking.
Looking at the relationships between CT and ST: At the very beginning, Nena shows only a partial perception of the evidence, focusing on the birth data from 1963 on. This selective focus may have been a result of the headline given in the diagram ('The Germans don't have enough children'). This headline may have triggered Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Nena's misinterpretation of the births as determining the population, from the turning point in 1963. It is interesting that Nena emphasised that it was possible to "look at this problem mathematically", which suggests that she saw a discrepancy between looking at the situation from the perspective of a mathematical model and looking at it from the perspective of the context (population and children born in Germany). Possibly the mathematical or statistical model is taken as an authority that is used to justify the appropriateness of the assumption instead of questioning the model chosen initially. It is possible to conclude from this combined analysis that the elements of reasoning in both domains interfere and interact. Given this interconnectedness, help in either domain, either in CT and ST, may have had a positive impact on the thinking process as a whole. Moreover, the CT and ST perspectives offer not only simultaneous and parallel ways of interpreting the reasoning process, but, through a combined analysis, can explain how CT and ST can be mutually beneficial, reinforcing related reasoning approaches. Tasks that stimulate the use of both CT and ST are consequently of practical importance.
However, CT and ST are not always interdependent in an obvious way, as the following example suggests (cf. Figure 2 and the corresponding interview section):
Figure 2: Task
Interviewer: So, what do you think, with which of the following statements do you agree?
Nena: I would agree with the 3rd statement.
Interviewer: Can you explain, please?
Nena: Yes… 400 computers is not a large sample when talking about computers so I would go with the 3rd statement and just listening to the comments of her friends and the consumer's magazine, it is possible that both the computers could be just so.
Interviewer: Are you sure?
Nena: Sure. I like this question!
Seen from the perspective of CT, Nena questions not only the experiences of the friends, but also the results from the study with the 400 laptops. On this basis, she expressed agreement only with the third statement, which highlights questioning evidence as a sub-aspect of CT. However, considering Nena's answers under the lens of ST, she appears not to acknowledge the statistical power of the sample of 400 laptops. Nena remarks that it is not possible to make a prediction on the base of the Theme D – E. Aizikovitsh-Udi, D. Clarke & S. Kuntze
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data, and she appears to compare the number of the 400 laptops to the number of all laptops. She does not reflect in depth on the statistical power of the magazine study. From a joint perspective, Nena's dominant critical attitude may have blocked her use of elements of ST, e.g. reasoning related to the sample size and representativeness. This example gives insight into how CT and ST may interfere. In this case, CT practice acted to the detriment of ST. Conversely, ST can be dominant over CT, as the following example (associated to the problem in Figure 3) suggests:
Figure 3: Task "crimes"
Interviewer: …and here… Do you agree with the interpretation of the police president? Why or why not?
Nena: I do not see a big difference with the number of crimes for any given year that would warrant extra police force to be hired…
Interviewer: Can you justify it, please?
Nena: It looks like the average is approximately 520 which is close to all the numbers so I do not think anything different is really happening from any given year.
According to Ennis' taxonomy (Ennis, 1989) one crucial element of CT is raising questions, having doubts, and exploring key definitions, like "crime". In this task, the nature of the crimes is a key consideration. For example, if we knew that all the crimes were murders, we might decide differently than if the crimes related to paying taxes or fraud. No question about such a definition was raised by Nena and the focus was purely numerical. In this task, Nena employed a restricted set of CT skills.
From the point of view of ST, Nena used an appropriate model and showed an awareness of statistical variation. By these means, she arrived at the conclusion, that, given the variation of the data of the past years, the rise of the crime number is not significant. Consequently, seen from the ST perspective, Nena showed an appropriate understanding of the statistical situation.
Looking at this part of the interview in the joint CT and ST interpretation mode, the analysis suggests that Nena successfully questioned the statement of the police president by using the data given in the problem and a statistical argument. She appeared to remain in the statistical domain, giving more details related to the model she had chosen (distances to the average value). This focus on ST may have hindered her use of any of the CT skills, such as analysing and questioning the definition of 'crime' as the key notion here, questioning the evidence (i.e. the way the data had been collected), etc.
Questioning data plays a role also in the following example related to the problem in Fig. 4:
Year 2000 2001 2002 2003 2004 2005 2006
Number of crimes 504 528 525 499 529 518 538
The police president shows the following diagram and says: "This diagram shows, Number of crimes that since 2005, the number of crimes in the city center has increased, so that we have to expect that it will further go up in the next years. We need more policemen for patrol in the city center."
In a more extensive press communication, Fred has found the data of the last 7 years:
Do you agree with the interpretation of the police president?
Why or why not? Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Figure 4: Task "tablets" from Kuntze, Lindmeier & Reiss (2008)
Nena: And here… I think tablet one takes too long to get rid of the headache.
Tablet two seems to get rid of the headache a lot quicker for the majority of the
people. You don't really know the age or weight of the people. Many factors play into
the reason why a headache might occur so the statistics are poor. Based on the chart, I
would have to pick tablet number two in hope of a speedy recovery.
From the perspective of CT, Nena not only evaluates the given statements,
but she also shows CT elements when going beyond the data given: She gives
examples of relevant influencing factors, and questions the data provided in the
diagram ("the statistics are poor").
From the point of view of ST, the analysis of Nena's short answer yields that
Nena chose an appropriate model and was aware of the key elements of the problem,
even if she did not explicitly discuss the minority of cases with very slow recovery
for tablet 2. These considerations led to her personal conclusion to pick tablet number
two, as she obviously sees the chance of a "speedy recovery" as more important than
the risk of a very slow recovery.
Looking at both CT and ST, the example appears to highlight how elements
of CT can contribute to ST, e.g. when evaluating data, its presentation and analysis,
planning data collection, etc. In the example, Nena suggests an analysis that takes
into account the age or weight of the persons in the study. Conversely, aspects of ST
like dealing with statistical variation and uncertainty can contribute to CT, especially
when it comes to decisions in non-determinist situations, where full data is
unavailable. These examples are intended to illustrate how both ST and CT skills can
be evoked by the same task. We suggest that this models authentic and useful
thinking practice more effectively than a more closed task that stimulated only
statistical thinking and the application of taught procedures. Our question now, is
"what are the messages for task design?"
Conclusions: The feasibility, utility and design of hybrid tasks
The tasks used in this exploratory study have certain distinctive
characteristics and we would argue that each of these characteristics constitutes a key
principle of task design:
• each uses a "real-world" situation as its "figurative context" (see
Clarke & Helme, 1998); Theme D – E. Aizikovitsh-Udi, D. Clarke & S. Kuntze
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• each provides succinct statistical information relevant to that
context (Kuntze, Lindmeier & Reiss, 2008);
• the problem is stated very simply;
• some form of evaluation is integral to the problem (Ennis &
Millham, 1985);
• the task affords many reasoning approaches.
The exploratory study demonstrates that connections clearly exist between
Statistical Thinking and Critical Thinking at the level of individual reasoning
practices. In seeking to stimulate both forms of thinking we suggest that an individual
employing Statistical Thinking has access to a structured framework of analytical
principles that guide and support their reasoning. That is, the relationship between
measures of central tendency and variance, for example, structure any consideration
of distribution of data that might be invoked in drawing evidence-based conclusions
or making evidence-based judgements. On the other hand, the components of Critical
Thinking are not related in such a structured fashion and an individual's inclination to
employ one strategy (e.g. Questioning Evidence or Questioning Assumptions) can be
given expression without any obligation to also invoke other components of Critical
Thinking. Some Critical Thinking skills resemble the "heuristics" that were the focus
of the enthusiasm for problem solving in the 1980s and 1990s (Clarke, Goos, &
Morony, 2007). Catalogues of such heuristics were similarly fragmented.
Ennis and others have catalogued critical thinking skills (Ennis, 1989) and
even arranged these categories in a form of hierarchy, but the connection between
specific critical thinking skills is under-theorised in comparison with Statistical
Thinking. Nonetheless, the forms of Critical Thinking identified in such classificatory
schemes are clearly of significance, both as aspects of reasoning and as potential
curriculum content. If it were possible to develop a structure for Critical Thinking in
which the component elements were not only identified, but also their relationship
established, then to invoke one aspect of Critical Thinking would serve to catalyse
the use of other related aspects, because the connections between elements would be
well known and understood. The question of how best to conceptualise these skills,
how to integrate or connect them with other curricular goals, and how best to
promote them and nurture their development in the classroom has been a major
challenge. An earlier study (Aizikovitsh-Udi, 2012), using similar tasks, has
documented efforts to produce CT through a program of instructional immersion in
the related topic of probability. In this paper, we argue that particular tasks can
stimulate the use, promotion and development of both Statistical and Critical
Thinking. We would like to suggest that an instructional program of hybrid tasks
could be devised that provides the opportunity to employ Statistical Thinking, while
simultaneously introducing students to the practices and structure of Critical
Thinking. The design characteristics of such hybrid tasks have been identified.
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Effects of variations in task design using different
representations of mathematical objects on learning: A case of a
sorting task
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The goal of our study was to examine how variations in task
design using tools may affect the process and object of learning. The
study focuses on sorting tasks, i.e., learning tasks that require grouping a
given set of mathematical items, in as many ways as possible, according
to different criteria suggested by the learners. We present an example of a
sorting task for which the items to be grouped are related to basic
concepts of analytical geometry that are connected to the notion of loci of
points. Based on an experimental study
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with three groups of in-service
secondary school mathematics teachers, we report on intended and
enacted objects of learning inherited in three versions of the task.
Empirically-based suggestions are drawn about design principles of
sorting tasks that potentially to evoke desirable enacted learning.
Keywords: Analytical geometry; classifications; sorting task; variation
theory
Introduction
The study aims at examining how variations in task design may affect the process and
object of learning. We focus on sorting tasks, i.e., learning tasks that require grouping
a given set of mathematical items (e.g., graphical, symbolic or word representations
of mathematical concepts), in as many ways as possible, according to different criteria
suggested by the learners (Zaslavsky, 2008a; Zaslavsky, Chapman & Leikin, 2003).
In particular, we analyse a sorting task based on a set of items related to basic
concepts of analytical geometry that are connected to the notion of Loci of Points (we
refer to this task as the LP sorting task).
The study can be seen as an application of variation theory to design of a
task aimed at enhancing learners' awareness of mathematics as a connected field of
study. Briefly, in variation theory terms (Marton & Booth, 1997; Runesson, 2005),

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This study was conducted as part of a Ph.D. dissertation by the third-named author under
the supervision of the first- and second- named authors (Dolev, 2012). Theme D – B. Koichu, O. Zaslavsky & L. Dolev
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awareness of mathematics as a connected field of study was an intended object of
learning, a particular sorting task was a medium of learning, the number and content
of mathematical items included in the task were subjects of didactical manipulations,
and the learners' ways of handling the task, which was offered in three different
versions, constituted a variation space or an enacted object of learning. The latter is
the focus of our study.
The study was guided by the following research question: What is the nature
of learning that may occur in small groups of mathematics teachers while working on
the different versions of a sorting task? Based on the empirical answer to this
question, we address at the end of the paper the more general one: What are the
characteristics of a set of mathematical objects for sorting that can enhance
opportunities for the learners to achieve the intended object of learning and what
design principles underlie the construction of such a set?
Theoretical Background
Sorting tasks
Sorting or classification tasks are recognized as a useful teaching/learning tool either
in a mathematics classroom or in a teacher professional development workshop. For
instance, Sawada (1997) found that students who were not motivated to take part in
traditional mathematics lessons were responsive when given a sorting task in the
context of geometry and could classify the given items in the variety of ways.
Zaslavsky, Chapman and Leikin (2003) discuss the learning potential of
sorting tasks in the context of symbolic and graphical representations of functions.
They pointed out the open nature of the task and its potential to evoke different
mathematical solutions. Zaslavsky (2008a) analyzed a sorting task based on a set of
graphical representations of functions and argued that engaging in the task can draw
students' attention to subtle similarities and differences between the items.
Zaslavsky and Leikin (2004) described how a group of mathematics teachers
coped with a sorting task based on a set of algebraic equations and inequalities. They
found that the participants first suggested ways of sorting based on surface features
(i.e., features that can be observed without solving the statements), and only after a
while turned to more structural features (i. e., those that may be identified only by
solving the statements). A compatible result was found by English and Sharry (1996)
for high school students working on a similar sorting task. In their study, the
participants tended to classify equations by arithmetic manipulations involved in
solving them, but, as a rule, not by deep-level mathematical structure of the equations.
In spite of an increasing attention to the potential of sorting tasks, none of the
studies on this type of tasks examined iterative aspects of task design; rather they look
at a final version of a task. As a rule, the processes of designing sets of objects for
sorting remain salient, though the design processes of some other kinds of open
learning tasks have been usefully unpacked (Zaslavsky, 2008b; Liljedahl, Chernoff &
Zazkis, 2007). Our study focuses on the task design iterations as well as the

differences in the learning experiences inherited in the different versions of the task.
Variation theory of learning
A central premise of a variation theory of learning is that "there is no
learning without something being learned" (Runesson, 2005, p. 70), or, in other
words, "learning always has an object" (ibid). The theory encompasses three types of Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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objects of learning (Marton & Booth, 1997; Runesson, 2005). A lived object of
learning denotes what is actually learned from the point of view of a learner. It is
difficult to reveal. An intended object of learning denotes the capabilities the teacher
wants the learners to develop. An enacted object of learning is constituted in the
interaction between learners and the teacher or between the learners and themselves,
and denotes what is possible to learn in a particular situation. Intended and enacted
objects of learning may or may not coincide.
Analyzing the enacted object of learning requires identifying the affordances
and possibilities for developing a certain capability as inherited in the learning
situation. In connection to task design, revealing the enacted object of learning means
that the design team should critically evaluate, based on empirical evidence, which
experiences the task affords and to which extent those are beneficial for coconstructing the intended object of learning.
An additional premise of the variation theory is that objects can be perceived,
understood or experienced by a learner or a group of learners in different ways,
depending on which aspects of the object one's awareness is directed to and how.
Consider an example in the context of sorting tasks. When deciding whether or not
two mathematical items belong to the same group, a group of learners can direct their
attention to the appearance of the common words or symbols in the items'
formulations. Alternatively, they can direct their attention to small differences in the
items' formulations and eventually find out that similarly looking formulations
describe different mathematical objects. Such possibilities constitute a space of
variations inherited in the learning situation or task. In variation theory, variations in
ways by which a certain object may be perceived or experienced by an individual or a
group are considered fundamental for learning. Accordingly, our analysis focuses on
the diversity of the teachers' ways of experiencing the different versions of the task as
well as on their final responses.
The study
The settings
An LP sorting task was developed by the third-named author for the use in 90-minute
workshop for in-service secondary school mathematics teachers as learners and
potential users of the task in their classrooms. The LP task design was driven by the
following learning objective: to facilitate learners' awareness of mathematics as a
connected field of study by directing their attention to structural similarities and
differences among the basic concepts of analytical geometry and loci of points.
Each workshop had the following structure: a brief presentation of different
representations of conic sections – 10 minutes; instructions for working on a sorting
task – 5 minutes; small group work on the LP task – 40 minutes; reflective discussion
in a whole group – 15 minutes; discussion of those cards that were perceived by the
teachers as the most difficult ones – 10 minutes. The instructor refrained from
intervening in the main part of the workshop.
Data sources and analysis
Overall, 53 secondary school mathematics teachers participated, as learners, in the
study. Each of them took part in one of three 90-minute workshop. Each workshop
used a different version of the LP task. Nineteen learners divided into 6 groups took Theme D – B. Koichu, O. Zaslavsky & L. Dolev
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part in the first workshop; 18 learners divided into 5 groups – in the second one and
16 learners divided into 5 groups – in the third one.
The data consisted of transcribed videotape-recorded small-group and whole
group discussions, in addition to their sorting actions. For each small group, the
completed sorting sheets were collected and analyzed.
The between-group variations in the sorting criteria and the order, in which
they appeared in the course of the learners' actions were deduced from the data and
considered to be the variation spaces for each version of the task. Patterns of the
learners' experiences while working on the task were identified by juxtaposition of the
participants' discourse, sorting actions and corresponding sorting sheets. The main
two patterns were: (i) making sorting decisions by recognizing the objects as familiar
from past study of analytic geometry, and (ii) making sorting decisions by 'solving'
the item, i.e., by applying algebraic manipulations in order to obtain a familiar
symbolic or pictorial representation of the loci of points. The variation spaces of each
version of the task were compared to determine to which extent they are compatible
with the intended object of learning: to facilitate awareness of structural similarities
and differences among the basic mathematical concepts of analytical geometry rather
than similarities and differences based on surface features of the items.
Three task iterations
Intended and enacted objects of learning: The first version
Generally speaking, the initial process of the set construction for the LP task was
compatible with that described by Watson and Mason (2006) for designing teaching
sequences of exercises based on the students' (presumed) perceptions. Specifically, it
started from the analysis of how the chosen concepts are represented in textbooks and
included decisions about which controlled variations should be considered so that the
learners "might observe regularities and differences, develop expectations, make
comparisons, have surprises, test, adapt and confirm their conjectures within the
exercise" (Watson & Mason, 2006, p.109).
The first version of the LP task consisted of 24 items (see Table 1 in the
Supplementary Material (SM) file
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). It was created so that three types of controlled
variations would be maximized. The first type of variation was related to the
mathematical objects described in the cards for sorting: a straight line, circles,
parabolas, ellipses, hyperbolas and the empty set. Two items disguising the empty set
and one item disguising a straight line were included in order to increase the
participants' awareness of the internal mathematical features of the mathematical
objects under consideration; we refer to these items as "pathological" ones. The
second type of variation was related to the type of representation: symbolic, graphical
and verbal. Furthermore, the verbal items varied with respect to two factors: the main
operation involved in the loci generation and the generating elements given in the
descriptions (see Table 2 in SM). The main desirable way of classification was by the
names of loci of points. This is because this way of classification required from the
learners to unravel structural similarities and differences among the items.
The third type of variation was related to the types of experience needed to
handle the task, in relation to the (presumed) mathematical knowledge of the

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The Supplementary Material file is available at the following website:
http://edu.technion.ac.il/docs/KoichuZaslavskyDolevThemeA_Supplementary_material.pdfMargolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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participants. It was hoped that the participants could identify most of the loci of points
based on their prior knowledge and on the information given in the introductory
presentation, and thus, would have enough time to look for structural similarities and
differences. The reality turned to be more complicated than our plans.
First, though all the groups considered several ways of sorting as requested,
only two groups wrote more than one way in their sorting sheets and one group did
not write in the sheet at all. This finding suggests that the offered format of the task
was not comfortable for most of the participants, in spite of (or may be due to) its
openness. Second, all the groups started from the most apparent way of sorting, by the
types of representations, but only one group wrote it in the sorting sheet. This
suggests that most of the participants did not perceive this way as worthwhile sharing.
Third, the desirable way of classification, by names of the loci of points, was
considered by all the groups only in advanced stages of the work on the task and was
documented by 5 out of 6 groups. Forth, the need to "solve" the verbal items
manifested itself stronger than we expected. There were 69 attempts to classify 18
verbal items, and in 44 cases (59%) the sorting decisions were made based on
algebraic manipulations. Only 21 sorting decisions (36%) were made based on the
prior knowledge. This means that the task was much more "technical" for the
participants than we expected and that an essential part of the enacted learning was
not related to sorting, but to the ways of "solving" the verbal items by means of
algebraic manipulations. Finally, it became clear that the overall number of the items
was overwhelming. The characteristic assertions on this issue included: "There are too
many cards. It is impossible to consider them all!" and "It is a lot of work… we still
have 5 cards to sort, but I don't want to do it any more…"
In conclusion, the first version of the LP task, which was designed in
accordance with the principle of maximizing the chosen types of variations, resulted
in partial success: the desirable way of classification was considered, but a lot of time
was devoted to the technical work and to classifying the items by surface features.
Intended and enacted objects of learning: The second version
The second version consisted of 18 items (see Table 1 in SM). The items, which have
been approached in all the groups only algebraically, were excluded. (The intended
characteristics of the second set of cards are presented in Table 3 in SM).
We found that in spite of a smaller intended variation space, in comparison
with the first version of the task, the enacted variation space became richer. The main
sorting criteria were: "by the type of representation – symbolic, graphical or word" –
9 appearances; the (desirable) criterion "by the name of locus of points – parabola,
hyperbola etc." – 5 appearances; the criterion "by key words in the items descriptions
– 'sum of distances', 'ratio' etc." – 7 appearances, and the criterion "by the ability of
the group to identify the loci without 'solving the item'" – 3 appearances.
This time all the groups completed the sorting sheets, and 2 out of 5 groups
documented in the sheets more than one way of sorting. The desirable way of sorting,
by names of the loci of points, was considered by all the groups earlier than in the
first workshop and was documented by 4 out of 5 groups. This finding suggests that
the teachers were more engaged in looking for structural similarities and differences
than in the first version of the task.
The analysis of the teachers' actual experiences with the task supports this
suggestion. Fifty-five attempts to classify 11 verbal items were made, and in 29 cases
(53%) the sorting decisions were made based on the prior knowledge. In 7 cases Theme D – B. Koichu, O. Zaslavsky & L. Dolev
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(13%) the decisions were made based on algebraic manipulations (5 decisions were
correct), and in 12 cases (22%) algebraic manipulations were used to verify the
decision based on prior knowledge (for instance: "I've solved no. 12. It is a straight
line. It was for nothing – I could see [from the formulation] that it is about a
midline…").
With respect to the teachers' experiences, it is interesting to point out that in
7 cases (13%) the teachers fixed, by means of algebraic manipulation, their earlier
wrong decisions based on their prior knowledge. For instance, the "pathological"
items were treated in this way and summoned attention and interest of the
participants. Overall, it can be concluded that in the second version of the task the use
of algebraic manipulations supported the teachers' classification actions, but was not
the main experience they gained from the task.
Still, 4 out of 5 groups started from considering the apparent, but not
especially instructive criterion "by the type of representation." Moreover, it was
evident that the presence of the well-familiar pictorial and symbolic representations in
the task postponed and probably hindered the learning experiences related to making
sense of the verbal items. For this reason, we decided to leave in the third version of
the task only 11 verbal items (see Table 1 in SM).
Intended and enacted objects of learning: The third version
As mentioned, in the third version of the LP task our intention was to supress the
appearance of sorting criteria by surface features, in favour of criteria related to
identification of structural similarities and differences. The prospective
characterization of the set of items can be found in the "verbal representations" part of
Table 3 in SM. The findings show that this goal was achieved.
As in the second workshop, all the groups completed the sorting sheets, and 2
out of 5 groups wrote in the sheets more than one way of sorting. As in the second
workshop, all 5 groups considered the desirable way of sorting, and 4 out of 5 groups
wrote it in the sorting sheets. The overall number of the considered sorting criteria
(26) was also compatible with that from the second workshop (30), though the
number of items to be classified was considerably smaller, 11 instead of 18.
One essential difference between the second and third versions appeared to
be related to the types of experience utilized for making sorting decisions. Out of 55
sorting decisions, 41 (75%) were based on the use of prior knowledge, and only 8
decisions (15%) – on algebraic manipulations. Another difference was related to the
use of the "by key words" criterion (15 out of 26 appearances). In the first and the
second versions this criterion reflected the learners' attention to certain words in the
formulations, but usually without particular attention to their mathematical role in
generating the loci. In the third version, the criterion was used more meaningfully
and, as a rule, on the way to the desirable criterion "by names of the loci of points."
In closing, the two main enacted subcategories of the "by key words"
criterion in the third version of the task – "by the main operation for generating the
locus" and "by the main generating elements" – were remarkably close to one of the
intended types of variation of the LP task.
Discussion: Articulation of the principles underlying design of the LP
task
In each successive version of the LP task, the enacted variation space was closer to
the intended variation space than in the previous one. In this section, we attempt to Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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pinpoint this phenomenon by extracting from the findings the characteristics of the
sets of items for sorting and, in a more general mode, by formulating seven design
principles underlying their construction, in addition to the principles formulated in
past research.
From only partial success of the implementation of the first version of the LP
task, two design principles can be deduced:
1. An amount of work needed to handle the task should be feasible for learners
under the given time constraints and their incentives to be engaged in doing
the task meaningfully, and not only formally.
2. In order to preserve the central role of the main intended activity (in our case –
classifying), the task should not be overloaded with mathematically
challenging items that may require "solving" by means of complementary
mathematical techniques (in our case – by algebraic manipulations).

The second and the third versions of the task also taught us that:
3. Inclusion of a small number of mathematically challenging items constructed
so they can create a feeling of surprise for the learners (in our case,
"pathological" items fulfilled this role) is beneficial for learning.
The next design principle, which arises from comparison of all three versions of the
task, is in the spirit of the proverb "sometimes less is more:"
4. Well-informed reduction of the intended variation space of a task does not
necessarily lead to reduction of the enacted variation space, and, in some
cases, may even increase it.
In addition, we observed that:
5. Having items based on different types of representations of the same concepts
does not necessarily enhance an experience of looking for structural
connections between the chosen concepts.
This principle may seem counterintuitive in light of the professional literature, which
calls for engaging the students into experiences that require connecting different
representations of the same mathematical concepts (e.g., NCTM, 2000). One possible
explanation is as follows. Well-familiar symbolic and pictorial representations in the
first and the second versions of the task served as reference points for looking for
ways of how to sort the verbal items by names of the loci of points. By reference
points we mean pieces of knowledge the learners hold as true and use as an anchor for
planning or monitoring (cf. Harel, Koichu & Manaster, 2006, for the compatible use
of a 'reference point' notion in the context of a problem-posing task). Coming back to
the LP task, it seems that from the learners' perspective, the verbal items in the first
and the second versions should have been "solved" in order to reveal their symbolic
or pictorial representations to be then compared with the items presented in symbolic
or pictorial forms from the beginning. Symbolic and pictorial representations were
unavailable as reference points in the third version, but an idea to sort the items by
names of the loci of points was still apparent, given the participants' prior knowledge.
Consequently, the participants were forced to look for the subtler reference points in
the items' formulations. The main operations for generating the loci and the main
generating elements were considered as such.
The next two principles stem from the observed regularities in the order, in
which the teachers considered different sorting criteria.
6. Task design should take into account that learners tend to start from easy-tomake decisions on their way to making more effortful (and meaningful)
decisions. Theme D – B. Koichu, O. Zaslavsky & L. Dolev
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7. The easy-to-make decisions can either hinder or support the intended object of
learning, thus, the effort should be made to anticipate and organize them so
they would serve as reference points on the way to more meaningful decisions.
Simply put, these two principles are in the spirit of the well-known saying "a
journey of a thousand miles begins with a single step." A more sophisticated
argument is presented below. To recall, we found that in the first and the second
versions of the task most of the groups started from sorting the items by the most
apparent criterion, "by the types of representation – symbolic, pictorial or verbal". We
first observed that in many cases this criterion had been surface-level only for the
learners and as such it might hinder the desirable deep-level reasoning around the
verbal items. However, when the possibility to consider the types of representations
as a sorting criterion was suppressed in the third version, the learners began to
consider instead the next most apparent criterion, "by key words". A closer look at the
data reveals that this criterion was not always only a surface-feature one for the
participants. As we have argued above, in many cases it could be seen as an indicator
that the participants looked for the reference points to start with on their way to more
mathematically meaningful criteria. Having this idea, we started to reconsider the role
of the "by the types of representations" criterion in the first and the second versions.
We now are inclined to think that the criterion that appeared at first was not only due
to some weakness of the early versions of the task, but also due to some fundamental
mechanisms embedded in cognitive activity of classifying or, more generally, in
human thinking (cf. Zaslavsky, 2008a, for an elaborated argument).
Back to the LP task, we assume that in virtually any version of the task we
can think of, the participants would probably always start from considering some
easy-to-see criteria. To recall, English and Sharry's (1996) and Zaslavsky and Leikin's
(2004) results indirectly support this assumption. And only after critical evaluation
whether the first criterion is interesting enough to be written, the participants might
start digging dipper. The pedagogical implication of this simple idea is formulated
above as Principle 7. To put it in other words, instead of trying to supress the
immediate (as a rule, not especially deep) ideas of the learners coping with an open
learning task, the task designers should worry about whether the task provides
opportunities for developing these initial ideas into more sophisticated, desirable,
ones. The presented case suggests that this goal is feasible, though it is not easy to
achieve.

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Principles of task design to foster proofs and refutations in
mathematical learning: Proof problem with diagram

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Proving is an essence of mathematical activities, and therefore
educational studies should seek effective tasks that foster students'
proving activities. Among various aspects of proof and proving in
mathematics education, this study aims to develop a set of tasks by which
students can experience an authentic process of proofs and refutations. In
particular, this paper focuses on proof problems with diagrams, and
elaborates task design principles for such problems from a theoretical
perspective based on a notion of deductive guessing which Lakatos
formulated as one of the heuristic rules. In later cycles of this study, the
authors and teachers will jointly scrutinize the design principles from
practical perspectives.

Keywords: proof and proving, refutation, Lakatos, deductive guessing,
proof problem with diagram

Introduction

Mathematical tasks are one of the most essential tools for promoting students'
learning. In particular, what kinds of problems should be posed to students and how
teachers should deal with the problems in their classrooms are vital for quality
mathematical learning (Henningsen & Stein, 1997; Hiebert & Wearne, 1993). It is
also necessary to make clear which mathematical activities are targeted at, because
effects of a task depend on the nature of mathematical activities (e.g. conceptual
understanding, procedural fluency, mathematical modelling or inquiry, and so on).
This study chooses proof and proving as a main target of mathematical
activities, because proving is an essence of mathematics, and therefore it should be a
core of students' experiences at all grades (NCTM, 2000). Several mathematics
educators have already conducted important researches about tasks related to proof
and proving (e.g. Bieda, 2010; Buchbinder & Zaslavsky, 2011; Lin et al., 2012;
Stylianides, 2009). For example, Stylianides and Stylianides (2009) develop an
instructional sequence which consists of a series of tasks and associated instructor
actions so that students can recognize a limitation of empirical arguments and a need
for deductive proofs. They deal with one of the most prevailing difficulties that
students do not feel a need to learn deductive proofs.
Among various aspects of proof and proving, this study focuses on processes
of proofs and refutations that task design researches do not seem to have examined so Theme D – Kotaro Komatsu & Yosuke Tsujiyama
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far. In particular, we intend to develop task design principles for an authentic learning
which mirrors mathematical processes that Lakatos (1976, 1978) described. He
insisted that "informal, quasi-empirical, mathematics ... grow(s) through ... the
incessant improvement of guesses by speculation and criticism, by the logic of proofs
and refutations" (Lakatos, 1976, p. 5). He described processes of mathematical
development through rational reconstruction of actual histories about Descartes-Euler
conjecture on polyhedra and about uniform convergence. Thus, relying on Lakatos's
research could lead to design tasks by which students can experience developmental
mathematical activities, which gradually progress through conjectures, proofs and
refutations.
As a task which can allow us to realize such authentic learning in regular
classrooms in junior or senior high school, this study focuses on "proof problems with
diagrams" (the definition and illustration will be provided in the following sections).
We first refer to the context of this study briefly, that is, mathematics education in
Japan. Japanese students start to learn geometric proofs from the eighth grade (13-14
years old), and eighth and ninth graders learn to prove geometric statements about
various properties of triangles, quadrilaterals and circles, using conditions for
congruent or similar triangles. In Japan, proof problems with diagrams are standard
for students and teachers in the sense that most of the proof problems in school
geometry include diagrams that indicate meanings of problem sentences (such
problems seem to be common in other countries as well; for example, see Herbst &
Brach, 2006). This study attempts to reflect on whether and how an authentic
mathematical learning based on proofs and refutations can be realized through such
standard problems.
This paper is the first report of a larger study that aims to develop, through
collaboration between researchers and teachers, a set of proof problems with diagrams
and associated teachers' guidance which prompts students to engage in processes of
proofs and refutations. Toward this overall goal, this paper focuses on the first aim of
the Working Group on Theme D, that is, to delineate task design principles within a
singular community of researchers. Current roles of the authors are to elaborate task
design principles for proof problems with diagrams from theoretical perspectives and
to develop several tentative tasks, and we address these aspects in this paper. Later on,
the authors and teachers will jointly re-examine the principles and tasks from practical
perspectives, enact the tasks in classrooms, and refine these principles and tasks from
results of the enactment.
Before taking up the main subject, it is better to clarify the definition of
"task" in this study. We regard task as a problem such that at once students try to
solve it and teachers plant an educational aim of getting students to experience certain
mathematical processes during solving it (e.g. Fig. 1 and 3); problems are usually set
by teachers or textbooks, but in some cases students may formulate problems by
themselves. In task design, it is essential to select or develop problems that have high
possibilities to achieve the intended educational aims. In addition, teachers'
interaction with students plays a vital role for quality mathematical learning because it
is unrealistic to expect that only posing the problems can facilitate students' activities.
Therefore, task design in this study involves not only selection or development of
problems but also teachers' instructional guidance related to the problems. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Deductive guessing in Proofs and Refutations

In elaborating task design principles, this study takes into account a mathematical
perspective primarily, in particular a mathematical process which Lakatos (1976)
described in Proofs and Refutations. Among various methods shown in the literature,
this study focuses on "increasing content by deductive guessing" (hereafter, deductive
guessing) that has not been sufficiently deliberated in mathematics education
literature so far. Deductive guessing indicates that after one proves conjectures and
then faces their counterexamples or non-examples, one invents deductively more
general conjectures which hold true even for these examples (Lakatos, 1976, p. 76).
Lakatos formulated it as one of five heuristic rules which summarized several
methods for coping with counterexamples. The word "heuristic" seemed to have a
normative meaning in his philosophy of mathematics, because he was largely
influenced by Polya's mathematical heuristic (Davis & Hersh, 1981; Lakatos, 1976),
and Polya (1957) stated that "(m)odern heuristic endeavors to understand the process
of solving problems, especially the mental operations typically useful in this process"

(pp. 129-130, emphasis is original). Thus, Lakatos appreciated deductive guessing as a useful and productive action for advances of mathematical activities.
Since deductive guessing is a mathematical notion, some examination from pedagogical perspectives is also necessary to regard deductive guessing as a central component for task design principles. For a pedagogical rationale of the principles, this study adopts "the intellectual-honesty principle" that Stylianides (2007) indicates. On the principle, he argues that when conceptualizing the notion of proof in school mathematics, it is essential to take into consideration "both the normative aspects of proof in mathematics … and what is known or conceptually accessible to the learners under teacher guidance or collaboration with peers" (Stylianides, 2007, p. 3). Further, he implies that the intellectual-honesty principle should be applied to not only the notion of proof in school mathematics but also other notions or activities.
It is already mentioned in the above that Lakatos regarded deductive guessing as a normative action in mathematical research. In addition, there is some evidence that shows the accessibility of deductive guessing to primary or junior high school students (Komatsu, 2010, 2011). For instance, Komatsu (2011) carried out an experiment in which, after a pair of ninth graders proved their conjecture and then confronted with counterexamples of the conjecture, they could invent a more general statement that held true for the counterexamples. Thus, selecting deductive guessing as a central component for task design principles would enable mathematics teachers and educators to attain an authentic mathematical learning which at once mirrors processes of proofs and refutations and is accessible to students.
Principles of task design for proofs and refutations: Proof problem with diagram
Proof problem with diagram
In order to develop certain kinds of tasks which lead to mathematical processes advocated by Lakatos, this study focuses on "proof problems with diagrams" in geometry. A proof problem with diagrams is a problem in which a statement is described with reference to particular diagrams with symbols (one diagram in most cases) and solvers are required to prove the statement (Fig. 1). In the following, we briefly summarize the nature of such problems and then illustrate that specific kinds
of such problems could give students opportunities to find and deal with counterexamples and non-examples.
Not only in Japan but also in many other countries, students encounter proofs within geometry at junior or senior high school, and proofs in school geometry have their origin in Euclidean geometry. In Euclid's Elements, propositions are stated in general and abstract words without any symbol or diagram. To prove such proposition, one needs to draw particular diagrams, attach symbols to points, construct deductive arguments according to these diagrams and symbols, and finally examine whether the arguments can demonstrate truth of the given general proposition. In the final phase, it is necessary to see each diagram as "a representative special case" (Polya, 1954) of the general proposition (on this dual nature of diagrams, see Tsujiyama, 2010).
In contrast, in many cases in school geometry, students are given proof problems to which diagrams and symbols are already attached. Mathematics educators in Japan have discussed the educational values of such problems. They argue that the attached diagrams bring some diversity in interpretations of such problems and that the diversity can give students an opportunity to engage in productive inquiry. In particular, Shimizu (1981) discusses that, after students solve such problems, it is important for the students to further inquire "of what (mathematical) relations the given diagram is a representative special case" (Shimizu, 1981, p.36) by utilizing already-obtained proof. He illustrates such inquiry by using a specific kind of proof problems with diagrams as below.
Fig. 1 shows an example of proof problems with diagrams. We consider a case in which one completes the proof of the statement by showing ΔBPA ≡ ΔAQC, BP = AQ and AP = CQ. One might have a doubt on meaning of "in the diagram" or "a straight line l through point A" in the statement, and this doubt will bring an occasion to inquire what may happen if one changes the position of the straight line l. This inquiry leads to encounter non-examples and counterexamples such as the diagrams shown in Fig. 2; line BP does not exist in Fig. 2-a (but, BP + CQ = PQ in a sense), and BP + CQ K PQ in Fig. 2-b and c. These examples disclose an implicit assumption in the statement in that "in the diagram" means a case where the straight line l and triangle ABC share only point A. On the other hand, if one observes Fig. 2-b and c with the already-obtained proof, one will find that |BP CQ| = PQ holds in the diagrams because ΔBPA ≡ ΔAQC, BP = AQ and AP = CQ are still true. Furthermore, it is possible to organize the relations among segments PQ, BP and CQ in the overall cases in terms of "sum" and "difference" (Shimizu, 1981, pp. 33-36).
In the diagram, angle A is a right angle and AB = AC. To a straight line l through point A, we draw perpendicular lines BP and CQ from points B and C respectively. Then prove that BP + CQ = PQ
Fig. 1: Proof problem with diagram (Shimizu, 1981, p. 30) Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.

Fig. 2: A non-example and counterexamples of the problem (ibid., pp. 34-35)
This illustration shows the nature of proof problems with diagrams that the attached diagrams sometimes make obscure the domains of the statements because the diagrams may include some implicit assumptions. This feature enables us to practice processes of proofs and refutations which Lakatos advocated, for example, to find counterexamples and non-examples through changing the attached diagrams, to restrict the domains of the statements by articulating the hidden assumptions, and to investigate new conjectures which hold for the counterexamples and non-examples. But, students will need some support from teachers in order to practice these activities successfully, and in next section we elaborate task design principles which would facilitate students' activities.
Principles of task design for proofs and refutations: Proof problem with diagram
According to both deductive guessing in Lakatos's research and the nature of proof problems with diagrams, this study derives three principles of task design which aim to achieve an authentic mathematical learning based on proofs and refutations:
1) Educators and teachers should select or develop a certain kind of proof problems with diagrams where students can find counterexamples or non-examples and engage in deductive guessing through changing the attached diagrams.
2) Teachers should promote their students to change the attached diagrams with keeping the conditions of the statements so that the students can find counterexamples or non-examples of the statements.
3) After students face the counterexamples or non-examples, teachers should plan their instructional guidance with which students can utilize their proofs of initial problems to invent more general statements that hold true for these examples.
The first principle is about selection or development of problems, and it seems to be quite an obvious principle from the intention of this study. Nevertheless, all of proof problems with diagrams do not allow us to find counterexamples or nonexamples and engage in deductive guessing by changing the diagrams. Thus, it is necessary for mathematics educators and teachers to select or develop a certain kind of proof problems with diagrams that enables students to practice such activities.
The second is about teachers' instructional guidance for discovery of nonexamples and counterexamples. Before deductive guessing, students need to confront with counterexamples or non-examples of the statements, and as shown in Fig. 2, changing diagrams attached to proof problems with keeping conditions of statements leads to finding such examples. However, it would be difficult to expect that students change the attached diagrams spontaneously, because it seems that the diagrams are typically given by teachers or textbooks in ordinary classrooms, and that there are few opportunities where students vary the shapes or places of the diagrams (Herbst &
Brach, 2006). Hence, teachers need to
diagrams so that the students
The third is about teachers' instructional guidance
which indicates to invent
counterexamples or non-examples
students, more active intervention by teachers would be necessary. In fact, Komatsu (2010) analyzed an experiment in which a pair of fifth graders was able to perform deductive guessing successfully, and
specific guidance by the experimenter.
was intended to prompt the students
This experiment illustrates that the proofs of initial problems may become an aid for deductive guessing, and that teachers
utilize the proofs. Thus, teachers should plan their instructional guidance with which students can utilize their proofs of initial problems to engage in deductive guessing.
This study has so far developed several proof problems with diagrams according to these principles, and in next section we illustrate these principles with a familiar problem about parallelogram
An illustration of task design principles
Fig. 3 is another example of proof problems with diagrams, and it has many textbooks and has been discuss
Larson et al., 2007; Okamoto et al.
an illustration lies in the first design principle of this study. That is, as shown later, this problem allows students
deductive guessing by changing the
examines this familiar problem from the task design principles based on processes of proofs and refutations.
In parallelogram ABCD, we draw perpendicular
lines AE and CF to diagonal BD from points A
and C respectively. Then prove that quadrilateral
AECF is a parallelogram
Fig. 3: Proof problem about parallelogram (Okamoto et al, 2012, p. 133)
The task will range across two lessons; an initial problem and its proof in the first lesson, non-examples and
described in the following are hypothetical, but the hypothetical process is based on an informal pilot study which the authors conducted with undergraduate students.
In order to prove the problem, studen
AEB and CFD are right angles (the hypothesis of the problem), and refer to AB = CD and ∠ ABE = ∠ CDF (properties of parallelogram). Next, they could deduce CDF because the hypotenuses and acute angles of the two right triangles are congruent with each other. They then could infer that segments AE and CF are equal in length (a property of congruent triangles) and parallel (equality of alternate angles, ∠ AEF = ∠ CFE), and therefore quadrilateral AECF becomes a parallelogram from one of the conditions for parallelograms.
According to the second design principle of this study, teachers should prompt their students to change
students would consider particular parallelograms such as rectangles, rhombuses and
Brach, 2006). Hence, teachers need to encourage their students to va
students can find counterexamples or non-examples.
The third is about teachers' instructional guidance for deductive guessing which indicates to invent deductively more general conjectures which hold examples. Since deductive guessing is usually unfamiliar to
students, more active intervention by teachers would be necessary. In fact, Komatsu (2010) analyzed an experiment in which a pair of fifth graders was able to perform ive guessing successfully, and their success was largely dependent on experimenter. Moreover, this guidance by the experimenter
the students to reflect their proof of their primitive conjecture.
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Thus, teachers should plan their instructional guidance with which utilize their proofs of initial problems to engage in deductive guessing.
This study has so far developed several proof problems with diagrams according to these principles, and in next section we illustrate these principles with a out parallelograms.
of task design principles
Fig. 3 is another example of proof problems with diagrams, and it has has been discussed by some researchers (e.g. Chino, 2005; 2007; Okamoto et al., 2012). The reason why we select this problem as an illustration lies in the first design principle of this study. That is, as shown later, students to find non-examples of the statement and
deductive guessing by changing the shape of parallelogram ABCD. This paper re
examines this familiar problem from the task design principles based on processes of
In parallelogram ABCD, we draw perpendicular
lines AE and CF to diagonal BD from points A
respectively. Then prove that quadrilateral
AECF is a parallelogram
Fig. 3: Proof problem about parallelogram (Okamoto et al, 2012, p. 133)
The task will range across two lessons; an initial problem and its proof in the examples and deductive guessing in the second. Students' actions
described in the following are hypothetical, but the hypothetical process is based on an informal pilot study which the authors conducted with undergraduate students.
In order to prove the problem, students would firstly use the fact that angles AEB and CFD are right angles (the hypothesis of the problem), and refer to AB = CD CDF (properties of parallelogram). Next, they could deduce
CDF because the hypotenuses and acute angles of the two right triangles are congruent with each other. They then could infer that segments AE and CF are equal in length (a property of congruent triangles) and parallel (equality of alternate angles, CFE), and therefore quadrilateral AECF becomes a parallelogram from one of the conditions for parallelograms.
According to the second design principle of this study, teachers should prompt their students to change the shape of parallelogram ABCD. First of all, students would consider particular parallelograms such as rectangles, rhombuses and their students to vary the attached
examples.
deductive guessing,
which hold for
. Since deductive guessing is usually unfamiliar to
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students would consider particular parallelograms such as rectangles, rhombuses and Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22

squares. Among them, quadrilateral AECF does not exist when quadrilateral ABCD is
a rhombus or square, because points E and
(Fig. 4-a and b). Moreover, students may slant parallelogram ABCD to the left side,
and in this case perpendicular lines from point A or C do not intersect with diagonal
BD (Fig. 4-c).
The diagrams in Fig. 4 are non
because it is assumed in the problem in Fig. 3 that quadrilateral AECF exists or
perpendicular lines from point A or C intersect with diagonal BD. However, students
may recognize these diagrams as counterexamples. They may also feel that the
statement has to be modified by articulating the above assumptions, because these
assumptions are embedded implicitly in the problem sentences. Therefore, providing
students with opportunities where they
them to find out various non
of hidden assumptions of the problems.
The third design principle of this study requires teachers to help their
students examine whether they are able to produce more general statements that hold
true even for the case of Fig. 4
perpendicular lines from point A or C intersect with line BD (Fig. 5
investigate whether quadrilateral AECF becomes a particular type or not (Fig. 5
particular, if students draw diagrams by hand, it seems difficult for them to judge the
type of quadrilateral AECF from the diagrams because the diagrams would not be
clear. Then, it is crucial for teachers to help their students think about whether they
can apply their previous proof to this case
their proof. In fact, by adjusting slightly the parts in which
// CF were deduced, it is possible to conclude deductively that quadrilateral AECF
Fig. 5-b also becomes a parallelogram. Through this problem and teachers' guidance,
students would become able to engage in deductive guessing
general statements that hold true even for the previous non
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squares. Among them, quadrilateral AECF does not exist when quadrilateral ABCD is
a rhombus or square, because points E and F coincide with each other in these cases
a and b). Moreover, students may slant parallelogram ABCD to the left side,
and in this case perpendicular lines from point A or C do not intersect with diagonal
Fig. 4: Non-examples of the problem
The diagrams in Fig. 4 are non-examples rather than counterexamples,
because it is assumed in the problem in Fig. 3 that quadrilateral AECF exists or
perpendicular lines from point A or C intersect with diagonal BD. However, students
ize these diagrams as counterexamples. They may also feel that the
statement has to be modified by articulating the above assumptions, because these
assumptions are embedded implicitly in the problem sentences. Therefore, providing
ies where they can alter the attached diagrams could promote
them to find out various non-examples or counterexamples and to become conscious
of hidden assumptions of the problems.
he third design principle of this study requires teachers to help their
udents examine whether they are able to produce more general statements that hold
true even for the case of Fig. 4-c. Students may extend diagonal BD so that the
perpendicular lines from point A or C intersect with line BD (Fig. 5
er quadrilateral AECF becomes a particular type or not (Fig. 5
particular, if students draw diagrams by hand, it seems difficult for them to judge the
type of quadrilateral AECF from the diagrams because the diagrams would not be
crucial for teachers to help their students think about whether they
previous proof to this case as well and, if not, how they should vary
their proof. In fact, by adjusting slightly the parts in which ∠ ABE = ∠
re deduced, it is possible to conclude deductively that quadrilateral AECF
also becomes a parallelogram. Through this problem and teachers' guidance,
students would become able to engage in deductive guessing where they invent more
general statements that hold true even for the previous non-examples.
Fig. 5: Deductive guessing about parallelogram
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squares. Among them, quadrilateral AECF does not exist when quadrilateral ABCD is
F coincide with each other in these cases
a and b). Moreover, students may slant parallelogram ABCD to the left side,
and in this case perpendicular lines from point A or C do not intersect with diagonal
examples rather than counterexamples,
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as well and, if not, how they should vary
∠ CDF and AE
re deduced, it is possible to conclude deductively that quadrilateral AECF in
also becomes a parallelogram. Through this problem and teachers' guidance,
where they invent more Theme D - Kotaro Komatsu & Yosuke Tsujiyama
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Concluding remarks
This paper elaborated three task design principles for proof problems with diagrams
by which students could experience an authentic mathematical process of proofs and
refutations, and illustrated these principles with a familiar problem on parallelogram.
Here, we refer to implications of this paper on teachers' community, keeping in mind
the context of our study. In Japan, as stated earlier, proof problems with diagrams are
standard for students and teachers, and learning activities where students inquire into
domains of statements by changing the attached diagrams has been already discussed
(e.g. Shimizu, 1981) and also may have been practiced in some classrooms. This
paper can provide teachers with theoretical supports that show the mathematical
authenticity of their practices according to a Lakatosian perspective. In addition,
teachers can utilize the principles of this study as a guideline for their task design
including necessary instructional guidance when they use proof problems with
diagrams to attain mathematical learning based on processes of proofs and refutations.
This paper still remains only a theoretical consideration by a single
community of researchers. In order to develop a set of appropriate tasks for regular
classrooms, the team of the authors and teachers will later scrutinize the theoretical
consideration from practical perspectives. For example, this study plans to re-examine
the design principles and tasks shown in this paper from teachers' abundant
experiences about teaching and learning of proofs. It is also necessary to enact the
tasks in classrooms, and to refine these principles and tasks from results of the
enactment.
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The "Language and argumentation" project: researchers and
teachers collaborating in task design
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This contribution illustrates the "Language and argumentation"
project, carried out since 2008 by the Mathematics Department of the
University of Genoa. The project is aimed at designing, experimenting
and refining task sequences for a smooth and meaningful approach to
proof in lower secondary school. Two examples illustrate the way of
working of the team (the cycles of experimentation and refinement) and
some special tasks explicitly aimed at promoting students' reflection on
processes and products.
Keywords: language, argumentation, proof, low-secondary school,
sequence of tasks, students' processes, cycles of experimentation
Introduction
This contribution presents the "Language and argumentation" Project (lower
secondary school strand), carried out by the Mathematics Department of the

University of Genoa since 2008. The aim of the paper is to describe the structure of the Project team, the theoretical assumptions underlying the task design activity carried out by the team, the special tasks that were created and the way they were progressively refined throughout cycles of experimentation.
The contribution aims at addressing the following questions, as presented in the ICMI Study 22 Discussion Document (Theme D): If you identify yourself as a member of a design group that cuts across communities, which ones are they? How did this cross-community come to be formed? When you or your group engages in designing tasks, what are you trying to achieve? What are your primary considerations? Which theoretical, mathematical, pedagogical, technological, cultural, and/or practical aspects are taken into account when designing a task or a task sequence? Are the designed tasks subject to revision in later cycles of the work? If so, what is it that specifically leads to the redesign? On what basis and according to which principles is the redesign carried out?
In reference to the Discussion Document, we take tasks as the mediating tool between teaching and learning, and we illustrate the way in which, in our project, tasks are used in order to achieve specific educational goals (fostering the approach to argumentation and proof).
Background: proof and task design
Scholars agree on the fact that teachers should set up proper actions so as to arouse students' need for proof and proving (Zaslavsky et al., 2012), and also point out that Theme D – F. Morselli
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teachers need preparation and support in doing that. Indeed, teachers face many challenges when dealing with proof in the classroom (Lin et al., 2012b): they must establish suitable socio-mathematical norms, choose and manage the good tasks, or even create their own tasks, guide the students towards deductive thinking without turning proving into a "ritual" activity. Also teachers' beliefs about proof and its role in the teaching and learning of mathematics play a key role in influencing the effectiveness of the teaching of proof (Furinghetti & Morselli, 2011). For instance, many teachers seem to believe that geometry is the most suitable domain to teach proof and that proof is an advanced mathematics issue, to be taught only in secondary school.
In the literature we can found many examples of task sequences that were created with the explicit aim of improving the teaching and learning of mathematical proof (Stylianides, 2007). Lin et al. (2012a) list a series of principles for task design for conjecturing, proving and the transition between conjecture and proof. Concerning conjecturing, we mention the importance of providing an opportunity to engage in observation, construction, and reflection. Concerning proving, the authors point out the importance of promoting the expression of arguments in different modes of argument representation (verbal arguments, symbolic notations etc.), asking the students to create and share their own proofs and to evaluate proofs produced by the teacher (thus "changing the roles"). Finally, concerning the transition from conjecture to proof, the authors suggest that the teacher should establish "social norms that guide the acceptance or rejection of participants' mathematical arguments" (p. 317).
This contribution will illustrate the way the team of the "Language and argumentation" project designed and experimented task sequences aimed at arousing students' "need for proof" in lower secondary school.
The "Language and argumentation" project
In 2004 the Italian Ministry for Instruction, University and Research (MIUR) founded the national Project "Lauree Scientifiche" ("Scientific degrees") (PLS in the following), whose aim was fostering the enrolment in university courses with scientific orientation, stimulating young people's interest in studying sciences and providing a better education in the base sciences. The project had several strands, going from special interventions for "high-achieving students" to pre-university orientation programs. Among them, the so-called PLS Laboratories, that is to say special lessons, performed in the school environment through a collaborative work between university researchers and school teachers.
Within this framework, in 2008 the Mathematics Department of the University of Genoa started the "Language and argumentation " project, a special case of "PLS Laboratory" aimed at designing and experimenting task sequences with a special focus on argumentation and proof.
Three main features characterize the "Language and argumentation" project:
1) task design is a central part of the collaboration between university and school; 2) argumentation and mathematical proof are the core of the task sequences; 3) teachers of different school levels (and not only higher secondary school) are involved, since the project members share the belief that argumentative competence should be developed in a long-term perspective, starting from the very first years of school and throughout all the school levels.
For each school level a team (university + school) was created. The different
teams met regularly in order to share theoretical references on argumentation and take Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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advantage of the exchanges and discussions. This contribution specifically refers to the work of the lower secondary school team. The next section illustrates the organization of the teamwork and the theoretical tools that were shared within the team and that helped to perform the didactical and methodological choices.
Task design: the team
Structure of the lower-secondary school team
The lower-secondary school team is currently made up of 7 members: the author FM, a researcher in mathematics education, three teachers with at least 10 years of teaching experience (two of them, MT and EZ, have a university degree in mathematics, one, EQ, has a university degree in chemistry), two teachers with less than 10 years of teaching experience (one, EP, has a degree in mathematics, the other one, GA, has a degree in biology), one retired teacher, AS, with a long experience in collaborative research in mathematics education. The teachers entered the project voluntarily and their participation was strongly supported by the school head. All the teachers (except for AS, retired, and GA, who changed the school after the first year but kept the work in the team) work in the same school.
The team was born in 2008-09: this means that the team just finished its fourth year of work. The first two years were dedicated to the development of a common frame (both in terms of theoretical references and of didactical methodologies). Starting from the third year, task design became a crucial activity for the team. This contribution will especially deal with this part of the project. We also point out that the teachers could design and experiment more than one task sequence for the same group of students
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, throughout the school years from 2008-09 to 2011-
12. This means that a sort of mini-curriculum with a focus on argumentation and proof was created, and that students could experience more than one task sequence.
The way of working of the team
The author organized all the team meetings (one scheduled meeting per month, starting from November and until June) and acted as an observer during the class sessions. She also made video recordings of the sessions and collected all the students' written productions. Besides the team meetings, she had individual meetings with each teacher, before and after the class experimentation.
The way of working may be synthesized as it follows: during a preliminary meeting, the researcher proposes the theme of the task sequence and sketches a first draft of the core task. The core task is discussed and the teachers, together with the researcher, set up the sequence of tasks, with a special care in the sequencing of tasks. Afterwards, a first experimentation is carried out. Teacher and researcher perform the analysis of the experimentation immediately after the experimentation; the whole team performs an additional analysis during regular meetings. The analysis may lead to the refinement of the task sequence and, thus, to a new experimentation. Two modalities of experimentation were tested: parallel experimentations of the same sequence, and sequential experimentations. In the first modality, two teachers realized

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In Italy, lower secondary school is made up of three years. Usually the
teacher teaches the same group of students throughout all the three years. Theme D – F. Morselli
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the task sequence in their classes, almost in the same period. Regular meetings during and after the experimentation allowed a continuous exchange between the two experiences. In particular, students' processes were compared and the actual development of the task sequence in the two classes was analysed and discussed by the whole team. In the second modality, a first experimentation was carried out in one class, afterwards the whole team discussed the way the experimentation was carried out. Possible modifications to the task sequence were discussed, thus leading to a modified sequence to be experimented. In this way, a cycle of planningexperimenting-analyzing-modifying-testing the modified sequence was realized.
In both types of experimentation, the degree of variability left to each teacher is quite high, provided that his/her choices are discussed a priori or analysed a posteriori by the whole team.
In Italy teachers are often involved into research in mathematics education, under the Italian paradigm of the "Research for innovation" (Arzarello & Bartolini Bussi, 1998). Within this paradigm, teachers (who are called "teachers-researchers") collaborate with the researchers in the planning and analysis of the teaching experiments, and theoretical reflections and teaching experiments are performed dialectically, so that the analysis of the teaching experiments may lead to the evolution of the theoretical framework itself. In our case, the teachers were at their first experience of collaboration with researchers. We may say that the "Language and argumentation" project had also the final aim of fostering the professional growth of a new generation of teachers-researchers. Indeed, during the project the teachers did not only receive and implement in their classes the innovative task sequences, rather they were involved in theoretical reflection and a posteriori analysis.
Task design: principles and didactical choices
As regards the level of low secondary school, two educational goals are to be attained: from one side, fostering the development of argumentative and linguistic competences (thus, seeing argumentation as strictly linked to proof, see Durand-Guerrier et al., 2012), from the other side, promoting the first encounter with mathematical proof. Stylianides (2007) proposes the following definition of proof that can be applied in the context of a classroom community at a given time:
"Proof is a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics: it uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justification; it employs forms of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom community; an it is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of, the classroom community". (Stylianides, 2007, p. 291).
Accordingly, the team believes that a smooth and meaningful approach to proof requires the students' progressive acquisition of basic content knowledge, but also the ability to manage (from a logical and linguistic point of view) the reasoning steps and their enchaining (modes of argumentation) and the ability to communicate the arguments in an understandable way. It is important to develop a sort of "argumentative attitude", that is to say being aware of the fact that each choice, opinion, affirmation should be justified by means of a discourse that must be understood and accepted by peers. This is also in line with the idea that learning proof is approaching a form of rationality, as expressed by Morselli & Boero (2009), who Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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proposed an adaptation of Habermas' construct of rationality to the special case of proving, showing that the discursive practice of proving may be seen as made up of three interrelated components:
"- an epistemic aspect, consisting in the conscious validation of statements according to shared premises and legitimate ways of reasoning […];
- a teleological aspect, inherent in the problem solving character of proving, and the conscious choices to be made in order to obtain the aimed product;
- a communicative aspect: the conscious adhering to rules that ensure both the possibility of communicating steps of reasoning, and the conformity of the products (proofs) to standards in a given mathematical culture". (Morselli & Boero, 2009, p. 100)
Starting from the theoretical assumptions that were previously sketched, a list of methodological principles were derived and specific task design principles followed. It is important to underline that argumentation is a major educational goal, but also a means to achieve other educational goals, i.e. a better understanding of specific contents: argumentation is the goal and also the means. Hence, the task sequences are conceived with argumentation as a pervasive activity. We may distinguish between a core task, setting the problem to be worked on (a property to be discovered and justified), and the further tasks. Core tasks are usually proposed as open-ended questions (What can you tell about…?) where, according to the sociomathematical norms of the class, each answer must be justified. The further tasks are conceived so as to foster students' awareness of the epistemic (this is true because…), teleological (I have this goal…) and communicative (how could I communicate it in a proper way?) requirements inherent in the conjecturing and proving process. More specifically, tasks encompass: formulation of conjectures; comparison between different conjectures; justification of conjectures; comparison between individual processes and between individual final products. Didactical methodologies such as group work and mathematical discussions (Bartolini Bussi, 1996) are widely used.
The team also explored the importance of having students analyse students' written individual solutions, as it is advocated within the theoretical framework of the fields of experience didactics (Boero & Douek, 2008). We point out that the methodological choices are also in line with the principles listed by Lin et al. (2012a).
In this way, two types of argumentation are fostered: argumentation at content level, as a part of the proving process, and argumentation at meta-level, as a means for fostering reflection on the practices of mathematical proof related to the three components of rationality. Within the task design process, a crucial goal was to create occasions for meta-level argumentations aimed at promoting students' awareness of the epistemic, teleological and communicative requirements of proving. To this aim, specific tasks were created. Some examples are illustrated in the subsequent section.
Some examples
Example 1 – Isoperimetric rectangles
The task sequence "Isoperimetric rectangles" was conceived for grade 7 (age of the students: 13-14) and encompassed about 20 hours. The core activity is the conjecture and explanation of the fact that, among all the rectangles with fixed perimeter, the

square has the maximum area. That is the task sequence that underwent the most
evident changes and refinements throughout the years of experimentation. The first
version of the task sequence started with an explorative task in paper and pencil ("Draw four rectangles with a perimeter of 20 cm. What can you tell about the areas
of the rectangles?"), followed by a more direct question
area. The students conjectured that the square is the rectangle with maximum area and
in some cases they even tried to provide some numeric justifications. Afterwards, the
conjecture was proved by m
starting point of the proof was drawing a generic rectangle and a square (with t
same perimeter) and superposing
rectangle HGDA is part of both the rectangle and the square, the areas of th
figures GFDE and BCGH were expressed algebraically in order to show that the area
of the square is greater than that of the rectangle.
Figure 1 –
During the first experimentation, o
(see figure 1), raised her hand and proposed to "cut" the rectangle GFDE and place it
over the rectangle BCGH, so as to "see" that one surface is bigger than the other one.
This fact suggested to the teache
manipulate (and even "cut") could be another step before algebraic proof. A crucial
point was the place (and role) of the "cardboard" part with respect to the conjecture
about the maximum area: was it t
conjecturing phase), or after the conjecture of the property of the square (so as to
promote the idea of superposing figures and pave the way to the algebraic proof)?
Further cycles of experimentation allow
the different roles that the "cardboard phase" may have within the sequence. During
the third experimentation, there was also a very rich discussion on the way of drawing
rectangles with a given perimeter (
another rectangle from a given one; is it always possible to draw "couples" of
rectangles, that differ only from a rotation of 90 degrees
the same experimentation
individual reflection, so as to foster the connection between geometrical facts and
numerical properties. . The students were asked to write down their reflections about
three questions: "Why is it that adding and s
perimeter doesn't change? Is it the same if we add and subtract two units? Why is it
that changing the basis with the height the perimeter doesn't change?
At the end of the sequence, there was a "balance" task, con
foster students' reflection on the value (and limit) of different approaches: drawing on
paper, cutting cardboards, using algebra. A key point is that each approach offers a
different perspective on the problem, helps to grasp some aspec
the whole comprehension. The students were
following questions:
Going back to the previous tasks, you may note that we tackled the problem of
isoperimetric rectangles by means of different approaches:
paper & pencil, cutting rectangles on cardboard, using letters.
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Draw four rectangles with a perimeter of 20 cm. What can you tell about the areas
"), followed by a more direct question concerning the maximum
area. The students conjectured that the square is the rectangle with maximum area and
in some cases they even tried to provide some numeric justifications. Afterwards, the
conjecture was proved by means of algebra, under the guidance of the teacher. The
starting point of the proof was drawing a generic rectangle and a square (with t
same perimeter) and superposing them (see figure 1). Once observed that the
rectangle HGDA is part of both the rectangle and the square, the areas of th
figures GFDE and BCGH were expressed algebraically in order to show that the area
of the square is greater than that of the rectangle.
Isoperimetric rectangles: the starting point for algebraic proof
During the first experimentation, one student, looking at the two superposed figures
(see figure 1), raised her hand and proposed to "cut" the rectangle GFDE and place it
over the rectangle BCGH, so as to "see" that one surface is bigger than the other one.
the teachers of the team that giving students some cardboard to
manipulate (and even "cut") could be another step before algebraic proof. A crucial
point was the place (and role) of the "cardboard" part with respect to the conjecture
about the maximum area: was it to be proposed after the drawing phase (within the
conjecturing phase), or after the conjecture of the property of the square (so as to
promote the idea of superposing figures and pave the way to the algebraic proof)?
es of experimentation allowed for exploring the issue, shedding light on
the different roles that the "cardboard phase" may have within the sequence. During
the third experimentation, there was also a very rich discussion on the way of drawing
rectangles with a given perimeter (how much are they; how it is possible to create
another rectangle from a given one; is it always possible to draw "couples" of
rectangles, that differ only from a rotation of 90 degrees). This suggested
the same experimentation, after the ingoing analysis of the session) another task of
individual reflection, so as to foster the connection between geometrical facts and
numerical properties. . The students were asked to write down their reflections about
Why is it that adding and subtracting one unit (to the sides) the
perimeter doesn't change? Is it the same if we add and subtract two units? Why is it
that changing the basis with the height the perimeter doesn't change?".
At the end of the sequence, there was a "balance" task, conceived in order to
foster students' reflection on the value (and limit) of different approaches: drawing on
paper, cutting cardboards, using algebra. A key point is that each approach offers a
different perspective on the problem, helps to grasp some aspects and contributes to
the whole comprehension. The students were asked to answer individually
Going back to the previous tasks, you may note that we tackled the problem of
isoperimetric rectangles by means of different approaches: drawing rectangles in
paper & pencil, cutting rectangles on cardboard, using letters.
Draw four rectangles with a perimeter of 20 cm. What can you tell about the areas
concerning the maximum
area. The students conjectured that the square is the rectangle with maximum area and
in some cases they even tried to provide some numeric justifications. Afterwards, the
of the teacher. The
starting point of the proof was drawing a generic rectangle and a square (with the
them (see figure 1). Once observed that the
rectangle HGDA is part of both the rectangle and the square, the areas of the two
figures GFDE and BCGH were expressed algebraically in order to show that the area
rectangles: the starting point for algebraic proof
ne student, looking at the two superposed figures
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the issue, shedding light on
the different roles that the "cardboard phase" may have within the sequence. During
the third experimentation, there was also a very rich discussion on the way of drawing
much are they; how it is possible to create
another rectangle from a given one; is it always possible to draw "couples" of
). This suggested inserting (in
analysis of the session) another task of
individual reflection, so as to foster the connection between geometrical facts and
numerical properties. . The students were asked to write down their reflections about
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perimeter doesn't change? Is it the same if we add and subtract two units? Why is it
".
ceived in order to
foster students' reflection on the value (and limit) of different approaches: drawing on
paper, cutting cardboards, using algebra. A key point is that each approach offers a
ts and contributes to
asked to answer individually the
Going back to the previous tasks, you may note that we tackled the problem of
drawing rectangles in Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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What may you say of the different approaches?
Did the different approaches allow you to understand the same things?
Were they equally easy to understand?
The "balance" task was aimed at promoting the argumentation at a meta-level on the
potentialities and limits of each approach (for instance, drawing may only help the
conjecturing phase, but it is not a real "proof"), and on the power of algebra as a
proving tool. It encouraged the reflection not only on the correctness (epistemic
rationality), but also on the comprehensibility (communicative rationality) and
usefulness in relation to the final goal of proving (teleological rationality).
Furthermore, the task gave the teachers data for an a posteriori evaluation of the task
sequence.
Example 2: sum of consecutive numbers
The task sequence "Sum of consecutive numbers", conceived for grade 7,
encompassed exploration, conjecturing and proving in elementary number theory. The
whole sequence lasted about 10 hours. The students were proposed a first task ("What
can you tell about the sum of three consecutive numbers?"). They worked in small
groups and shared and compared the group solutions within a mathematical
discussion. Afterwards, the students were given three connected tasks to be solved
individually: "What can you tell about the sum of two consecutive numbers? What can
you tell about the sum of four consecutive numbers? What can you tell about the sum
of five consecutive numbers?". As usual for the norms of the class, each answer was
accompanied by a justification. The teacher and the researcher analysed all the
individual productions and for each task (sum of 2, 4 and 5 numbers respectively)
selected three productions to be compared and commented by the students
themselves, according to the following task:
Read the following answers provided by some of your classmates. Compare them
and write your reflections. What about the properties they found? What about the
explanations they provided?
A mathematical discussion followed. This task fostered a reflection on two connected
issues: the truth and comprehensibility of the conjectures, and the validity and
comprehensibility of the related explanations (epistemic and communicative
components). Students could reflect on the value of numeric examples (for discovery
of the conjecture and communication of the property: teleological and communicative
component), but also on their limits for justification (epistemic and teleological
component). They could also compare justifications in natural language with
justifications in algebraic language. In this ways, an argumentation at a meta-level
was promoted.
In our opinion this task is an application of the principles listed by Lin et al.
(2012) concerning having students produce their own justifications and evaluate
justifications presented by others. Furthermore, this task paves the way to the concept
of proof as presented by Stylianides (2007) and brings to the fore the importance of
three dimensions of rationality, communicative included. We also point out that
creating new tasks of reflection starting from students' own productions makes the
task sequence very "dynamic", since each implementation must take into account new
students' productions. Theme D – F. Morselli
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Discussion and conclusions
The short examples that were described in the previous session show how the
model of rationality guided the team in the task design process. The core tasks, aimed
at introducing the mathematical content (a property to be discovered and justified),
are accompanied by further tasks, aimed at fostering the reflection on the proving
process as a rational activity. For instance, the "balance" task (example 1) was aimed
at bringing to the fore the role of different methods at epistemic, teleological and
communicative level. The "comparison" task (example 2) was aimed at making
students reflect on the fact that the same conjecture and proof may be presented in
different ways (communicative component) and that different justifications are
possible (epistemic and teleological component).
Example 1 also illustrates the cycles of design, experimentation, analysis and
refinement that characterize the teamwork. One key feature is that the task sequences
are always under refinement. The team analysis may lead to a change in the task
formulation or in the sequencing of the tasks.
Furthermore, as evidenced in example 1, additional tasks, especially tasks
fostering reflection, may be inserted. Each teacher may suggest modifications to the
task sequence.
Finally, it is worth noting that some tasks are "open" and must be set up
during the experimentation. Any task sequence cannot be completely set up a priori,
because it depends on the students' processes and products. The teacher, with the
cooperation of the team, must be able to evaluate "on the spot" the emergence of
issues to be deepened (as in example 1), and to analyse students' products and
promote students' own reflection on productions (as in example 2).
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Designing and using exploratory tasks

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This paper addresses the role of exploratory tasks in mathematics learning. We present the rationale and the main characteristics of these tasks and provide three examples to show how they may support students' reasoning at different school levels. We argue that these tasks, requiring students to model situations and to design their own strategies, drawing on previous mathematical and nonmathematical knowledge, have an important potential for mathematics learning.

Keywords: Exploratory tasks, Generalization, Justification

Introduction

In Portugal, the new mathematics curriculum for basic education (Ponte, Serrazina, Guimarães, Breda, Guimarães, Martins, Menezes, Oliveira, & Sousa, 2007) indicates that teachers must use a variety of tasks in the classroom, stressing in a particular way the value of exploratory tasks. This promoted an important movement of study around such tasks and how to use them in the classroom. At the Institute of Education of the University of Lisbon several teaching experiments have been conducted to suggest ways of teaching specific topics of this new curriculum using exploratory tasks as well as to investigate the possibilities of such tasks for other school levels, with a special attention in the development of students' mathematical reasoning (e.g., Azevedo, 2009; Branco, 2008; Henriques, 2011; Quaresma, 2010). In addition, several collections of tasks, most of which are exploratory, together with teachers' supporting materials, have been designed and made available in the internet (e.g., Ponte, Matos, & Branco, 2009; Ponte, Oliveira, & Candeias, 2009; Ponte, Silvestre, Garcia, & Costa, 2010).

Worthwhile mathematics tasks are central in mathematics teaching (NCTM, 1991). However, tasks appropriate for some teaching purpose may not be for another. In addition, tasks need to be suitable for the students to whom they are proposed. Thus, the teacher needs to decide the purposes regarding students' learning and, taking into account the working conditions of the school, choose suitable tasks for his/her students, regarding mathematical challenge (Potari & Jaworski, 2002), underlying context (Skovsmose, 2001), task structure, classroom organization and Theme D – J. P. da Ponte et al.
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time required (Ponte, 2005). Classrooms in which the students do some extended work on tasks, in an autonomous way, solving problems, modelling situations and devising strategies to solve the questions proposed are becoming more common in our country. In some cases, the questions require considerable interpretation and even some reframing to be tackled in a mathematical way. After this initial work, often done in pairs or in small groups, during which students record their representations and solutions, they are asked to present their strategies and solutions to the whole class, to justify them, and to discuss the work of other students. This view of the mathematics classroom fits with what some documents refer to as "reform mathematics teaching" (e.g., NCTM, 2000).

The teaching experiments that we carry out are framed as design research (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003), with specific hypothesis relating exploratory tasks, use of different representations, and a mode of classroom work that provides opportunities for students' autonomous work and whole class discussions. Data is collected by observing classrooms (usually with video recording), gathering students' written productions, and doing clinical interviews with individual students (see further information on these experiments in the on-line appendix: http://www.ie.ul.pt/portal/page?_pageid=406,1590938&_dad=portal&_schema=PORTAL).

The present paper illustrates this work. Our aim, as a team that cut across diverse communities, is to discuss the characteristics of exploratory tasks, to indicate how they may be used in the classroom, their potential to promote students' learning, their feasibility for regular classroom use, and the critical features that make them suitable for mathematics learning. We begin by presenting the institutional context, definitions and research and design principles of our work and base our argument with snapshot of three examples at different school levels.

Context, definitions and principles

Role of authors, institutional, systemic, and resourcing context of the work

Three of the authors of this paper are classroom teachers involved in research who conducted the implementation of teaching units (Quaresma, grades 5-6, Mata-Pereira, grades 7-9, Henriques, 2
nd
year university). The first author (Ponte) has mostly a research profile with the role of supervision of the design process. All the four authors participate in the design of tasks and of its classroom use. Usually, the first idea for a task is provided by the classroom teacher and the subsequent refinement is done in interaction with the first author and with other researchers at the Institute who also occasionally discuss particular tasks and transversal issues. This developmental work has been conducted at Institute of Education of the University of Lisbon, within teaching experiments carried out as part of academic degrees. It is based in the close collaboration of two different kinds of expertise – mathematics education research expertise and classroom teaching expertise – that is shared in different levels by all authors. The resourcing (printing copies, audio-visual material for recording classes, etc.) is provided by the Institute, sometimes with external grants or contracts, and also by the schools of the teachers involved in the experimentation.

Mathematical and epistemological perspectives

We regard mathematics as an activity in which students solve problems and produce and justify mathematics statements. In this activity students may use mathematical representations, draw on mathematical concepts, known facts, properties and Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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procedures to design strategies, implement them, and arrive to conclusions. Such strategies include moments of problem posing, conjecturing, making generalizations, testing conjectures (analysing cases, doing computations, experimenting with technological tools, etc.), making connections and establishing relationships. An important part of the mathematical activity is also explaining and justifying solutions and statements. Our epistemological perspective is that mathematics knowledge develops by the process of carrying out mathematical activity and reflecting on such mathematical activity (i.e., carrying out meta-mathematical activity) which includes problem solving and systematization of ideas that is achieved by individual work and social interaction (with colleagues and with the teacher).

Tasks and teaching units

Following Christiansen and Walther (1986), we take a task as a statement of a situation to be solved by the students. It is presented to students both in written and oral form, together with indications about the time available to work on the task, the kind of students' production expected, and the mode of organization of the work suggested. A task may correspond to a single question or to a structured sequence of related questions (and possibly subquestions). In our work we strive to develop exploratory tasks, that is, tasks that may lead students to exploratory activity, from which they do substantial work and learn new mathematics. In exploratory activity students have to interpret situations in mathematical terms, formulate mathematics questions, and reason in an inductive way, making conjectures and generalizations. This is the basis for symbolizing and formalizing ideas and providing justifications to statements based on known facts and properties, assumed assumptions and definitions.

The key role of such tasks in mathematics teaching has been recognized by mathematics educators such as Sullivan, Bourke and Scott (1997), Boaler (1998), and Skovsmose (2001). We are aware that the kind of classroom work that we have in mind is quite demanding on the teacher. The selection of tasks involves a high level of understanding of the mathematics involved as well as in-depth knowledge about students' abilities and interests. Teachers must know how to introduce such tasks, negotiating the meanings that are critical for the work to carry out, but without providing too many clues that will solve the task for the students. Furthermore, the teachers are called to support the students' autonomous work, while maintaining the cognitive demand of the activity (Stein & Smith, 1998), and to be able to conduct productive discussions during which mathematics ideas are presented, confronted and clarified. In these discussions teachers must provide opportunities for all students to intervene, stimulating moments of controversy and argumentation as well as moments of systematization and formalization of mathematical ideas. In order to allow for a smooth integration in professional practices, we often produce tasks with parts or elements that are not so different from usual tasks, combining them with questions that points towards exploratory activity.

Design principles

Some tasks are intended to provide for the development of new forms of representation, new concepts, and new problem solving strategies. Others are intended to lead students to mobilize and clarify mathematical notions that they already learnt and/or to make connections among different ideas. The structure of a task may vary from seemly simple statements that require considerable interpretation and Theme D – J. P. da Ponte et al.
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specification of cases by students to complex statements that include a breakdown of questions in a structured way. However, in this case, often there is a combination of different kinds of questions, beginning with quite straightforward and computational questions to end up with open questions that require significant thinking from the students. The tasks will be ultimately organized in a sequence to be undertaken in several classes in order to promote students' understanding of an important mathematical idea or topic.

Tasks need to be engaging for most (if possible, all) students, not very difficult to get involved in, and lead to the formulation of important mathematical notions (representations, concepts, properties, procedures). In order for the tasks to be engaging for a student they must have an element of challenge, without being too difficult. The tasks involve different kinds of context. Some of them just have a purely mathematical context (usually numbers or geometric figures) whereas others are close to the daily experience of students. However, contexts must not be artificial, must not pose problems for students' understanding, and, of course, must not promote prejudices or stereotypes. It is assumed that, after tasks are introduced, students will spend an extended time working on them, in an autonomous way, and afterwards there is another extended period of whole-class discussion.

In our work, we start with an overall planning of the teaching unit, which includes the formulation of the learning objectives, assumed previous knowledge of students (often based on diagnostic assessment), time available and organization of the schedule (as there are variations among schools). Tasks are later selected to fit the overall planning of the teaching unit, and then there is a dialectic movement of adjustment between the macro level of the unit and the specific level of the task. Next, we present three tasks constructed according to these design principles (See a summary in Table 1).

Design principles	Folding and folding again	Multiplication of integers	Working with intervals
1. The objectives support the development of new representations, concepts, and strategies, lead students to mobilize and clarify mathematical notions, and/or to make connections.	i. To understand and use rational numbers in different meanings. ii. To recognize that they may have different representations, and understand the associated language.	i. To develop new knowledge regarding integers (definition of multiplication). ii. To develop the ability to generalize.	

i. To learn concepts
and procedures of
interval arithmetic
(definitions of
operations and image
of a function).
ii. To develop the
ability to generalize
and to justify.
2. The structure varies
from simple to complex
statements with different
questions (straightforward,
computational, open…).
Some questions are quite
simple and others require
interpreting complex
commands ("represent",
"compare").
Some questions are
simple and others
lead to connections
and reasoning
(generalization and
justification).
Open questions,
requiring reasoning
processes
(formulation, testing,
generalization and
justification).
3. Engaging with an
element of challenge.
Challenge in interpreting
"represent" and
"compare".
Challenge in
question 4,
especially 4c).
Challenge in both
questions.
4. Different kinds of
context (purely
mathematical and close to
the students' experience).
Use of manipulative
materials (paper strips).
Mathematical
context.
Mathematical context.
Table 1 – Application of design principles to the production of tasks Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Example 1: "Folding and folding again"
This exploratory task is included in the supporting materials of the mathematics
curriculum (Menezes, Robinson, Tavares & Gomes, 2008). It was proposed in the
first class devoted to the study of rational numbers in a teaching experiment at grade 5
focused on comparing and ordering rational numbers and equivalent fractions
(Quaresma, 2010). The class, with 22 students, enjoys working in new problems and
maintains a productive pace of work. All students show commitment regarding the
mathematics work, most of them have good achievement but a smaller group has
difficulties in this subject. The task, with two questions, intends to lead students to (i)
understand and use rational numbers in the part-whole and measurement meanings,
(ii) recognize that simple non-negative rational numbers may represented in the form
of fractions, decimals and percent, and (iii) recognize and use the language associated
with rational numbers in different representations and meanings. An important aspect
of the task is the use of manipulative materials (paper strips) and of written
representations, creating many opportunities for oral interactions among the students
and the teacher. Question 1 is quite explicit as it asks to do several folds, but then it
makes a request to represent them that most students have difficulty in interpreting as
an indication to use the decimal representation, favouring an exploratory activity.
Question 2 asks students to compare different parts of the strips and to "draw
conclusions", which is a quite open indication, that may lead to various solutions.
Task. 1. Find three pieces of paper geometrically equal. Fold them into equal
parts: a) the first in two parts; b) the second in four; and c) the third in eight parts.
After folding each strip represent the parts that you got indifferent forms.
2. Compare the three parts of the strips that you got. Record your conclusions.
We present some episodes from collective discussions after the students did
some extended work on the task. The students work in 6 groups (3 or 4 elements) in
an autonomous way, with discreet support of the teacher for about 20 minutes on each
question, after which begins the whole class discussion. To support this discussion
each group posts their work on the blackboard. Presenting the work of the first group,
Diana says: "In [item] b) we wrote: fourth-part, 1 of 4 [1/4]; 1 divided by 4, 25% and
0.4." The other students do not realize the mistake and the teacher decides to return to
question later and continues by asking the group of Tiago to present their work:
Tiago: So we have: fourth-part, one of four ,

.
], one divided by four, 25 and
0.25%.
Teacher: (…) Do you agree Diana?
Diana: Yes…?
Class: No! That is wrong…
Teacher: What is wrong?
Rui: It's 0.25…
Teacher: Why?
Rui: Because it is the fourth part.
Daniel: It is 0.25 because it is a half of the first. The first was 50, if we make half
is 25.
André: Oh teacher! I think it is 0.25 because it is the fourth part of 100. Because
25 times 4 gives 100.
The direct questioning of the teacher to Diana suggests that there is
something that needs attention. This student becomes confused and her colleagues
strive to clarify the situation. After the teacher question "why?", several students
provide successively more refined explanations to indicate that the part shall not be
represented by 0.4 but by 0.25. The style of questioning the teacher is punctuated by
inquiry questions ("What is wrong?", "Why?" …). The culture of the classroom Theme D – J. P. da Ponte et al.
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includes the notion that students can contribute to different responses to disagree and
argue with each other.
Working on this task led the students to understand the meaning of
representing parts of the strips using different representations of rational numbers,
developed their language associated to rational numbers and to compare various parts
of the strips. The fact that the first part of question 1 ("folding") is quite explicit and
uses manipulative materials, engaged students in the task, while the second part, more
open ("representing") prompted students to carry out exploratory activity and to
understand the meaning of "to represent". In question 2 (see examples in the
appendix) students further established different multiplicative and fractional
relationships among quantities.
Example 2: "Multiplication of integers"
This example is taken from a teaching experiment about integer numbers carried out
in a grade 7 class with 20 students. This is an exploratory task aiming students to
develop new knowledge about properties and concepts regarding integers, namely the
definition of multiplication, based on previous knowledge of properties of natural
numbers and inciting generalization. The task has four questions with a purely
mathematic context. Some of these questions require simple procedures, while others
lead students to establish connections and reasoning processes such as generalization
and justification. The last question seeks to identify properties of the multiplication of
integers, particularly regarding the sign of the product of two integers, based on the
connection between multiplication and successive addition of integers:
Task. 1. Find the result of the expressions 3+3+3+3+3+3+3+3+3+3 and
7+7+7+7. Explain how you found your answer.
2. Try to formulate a rule that indicates the same result without using addition.
3. Represent the expressions (-3)+(-3)+(-3) and (-4)+(-4)+(-4)+(-4)+(-4)+(-4) using
multiplication. Find the result of these expressions.
4. Try to formulate a rule that indicates the sign of the product of two integers if
they are: a) Both positive; b) One negative and one positive; c) Both negative.
The task was included in a set proposed to students on paper in a 90-minute
class and discussed in the first 20 minutes of the next class. The students, who sat in
pairs or groups of three, were asked to read all questions before starting to solve it.
There was a small whole class discussion, widely participated in by students,
supporting their interpretation of the task. As students were working, the teacher
moved around the classroom, asking them questions to clarify or probe their answers.
In the final whole class discussion, in order to construct common meanings, the
students were responsible for presenting and justifying their responses and
formulating questions regarding the responses of their colleagues.
The students solved questions 1, 2 and 3 with no difficulty. For the last
question, most students obtained the intended rules, particularly as regards the sign of
the product. Pedro, a student in this class, uses natural language in his written solution
to present his generalizations and uses symbolic language (without variables) to
present an example for each item. When questioned to justify his answers to items
4.a) and 4.b), he states:
Pedro: The first one was easy, because we had already done it long ago, both
positive. I wrote an example, four times four, sixteen. Therefore, if we multiply a
positive with a positive would be positive.
Teacher: And one positive and one negative? Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Pedro: Negative. I made an example, but then I also thought in the sum. A
negative with a positive would be negative. Therefore, minus three minus three
would be minus six.
In this question the students justify their answers based on prior knowledge
and mathematical properties used in the previous questions. This exploratory task led
them to establish generalizations, developing knowledge about the properties of
multiplication for integer numbers, based on known properties of natural numbers. As
the students still do not know the algebraic language, they justify their responses
exemplifying with particular cases. Some of the initial questions required some
interpretation but the students easily engaged in the task since it essentially involved
known properties or properties used in previous questions, as the relation between
multiplication and successive addition of equal addends. Therefore, the structure of
the task provided students with the possibility to begin with simple questions and end
up dealing with highly demanding questions as question 4. (Other situations in
appendix.)
Example 3: "Working with intervals"
This example is taken from another teaching experiment (Henriques, 2011),
supported by exploratory tasks conducted during one semester in a numerical analysis
course, in order to promote students' experience of doing mathematics and the
learning of concepts and procedures. The participants were 36 second year university
students. The task has two open questions in a pure mathematical context. Question 1
intends to lead students to deduce and justify the rules of interval arithmetic, which
were unknown to them. Students were challenged to formulate, test and justify
conjectures involving the exploration of particular cases of known elementary
operations (addition, subtraction, multiplication and division) using intervals of real
numbers. The second question has similar features and aims to extend the interval
arithmetic to functions, requiring the use of prior knowledge on functions and their
properties as well different representations to get a solution.
Task. 1. Look at the following situations
[1, 2] + [5, 7] = [6, 9] [0, 1] + [-5, 2] = [-5, 3] [-3, -1] + [1, 3] = [-2, 2]
a) Find a rule for adding real number intervals? Do all real number intervals
follow that rule? Investigate.
b) Investigate what are the rules for subtraction (X-Y), multiplication (X×Y) and
division (X/Y).
2. Consider a function f: D⊂ IR→ IR, given by f (X) = X + X and X = [x1, x2] ⊂
D, a real number interval of its domain.
a) What's the image of X = [2, 7]? Explain clearly how do you find your answer.
b) What's the image of a real number interval if now the function f is given by
f(X) = X
2
or f(X) = e
X
?
During the exploration of this task, which lasted about 100 minutes, the
students worked in small groups of 3 or 4. Afterwards, they presented their work
orally to the class, explaining their ideas and strategies, and, prompted by their
colleagues and the teacher's questioning, sought for justifications. Outside the
classroom, they also wrote a report (WR) aiming their reflection, since it requires
students to articulate ideas, at explaining procedures and reviewing the processes used
and the results obtained. The work of a student (Gonçalo) is chosen to illustrate a
variety of aspects of students' mathematical activity and learning.
Gonçalo begins exploring the task by observing the examples. That
observation raises in him some questions as he tries to find a pattern and to understand
what was really happening. He is led to a first conjecture about the rule for adding real Theme D – J. P. da Ponte et al.
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number intervals: "We concluded that by adding the lower bounds and the upper

bounds of the intervals, based on examples" (WR). Later, in an interview, he explains
the constructed meaning for the interval obtained from adding two other intervals:
Before the beginning of this task, I had never seen an addition of intervals and I
was really unaware of what it was. But then I realized that considering [a1, a2] +
[b1, b2] = [c1, c2], any value between a1 and a2, added with any other value
between b1 and b2, will be into the interval c. (E)
Gonçalo continues the exploration of the task formulating and testing
conjectures about the rules for other elementary operations (subtraction,
multiplication and division), leading him to further generalizations. In question 2, the
student feels the need to use graphical representations to extend the interval arithmetic
to functions (We show these aspects of the student reasoning in the appendix.).
In summary, the students were challenged to carry out exploratory tasks very
different from the usual ones done at university level. This kind of task, in connection
with the classroom work organization, led students to experience mathematics
processes such as looking for regularities, posing questions, formulating and testing
conjectures, generalizing and justifying them. Authentic mathematics activity was
stimulated by exploratory tasks. This work also provided students opportunities to
develop their understanding of the concept of interval, the rules of interval arithmetic
and to make connections with other mathematical topics, in particular with previous
knowledge about properties of numbers and functions. Thus, the results highlight the
potential of exploratory tasks to learn numerical analysis concepts and procedures,
suggesting that they may be used in university mathematics courses.
Conclusions
The three exploratory tasks reported in this paper demand considerable
interpretation by the students and their involvement in mathematical work. The three
tasks require students to formulate generalizations and justifications, in one way or
another, involving them in mathematical activity. They also were effective in
supporting students to achieve a variety of learning objectives related to rational
numbers, integer numbers and interval arithmetic. The critical features of these tasks
include a combination of structured elements with explicit indications with open
questions that require interpretation and the design of strategies. Another critical
aspect is the settings that enabled interaction among students and with the teacher in
the role of a facilitator. However, the tasks proved to be feasible for a qualified
teacher, requiring no special effort. The particular combination of authors in this work
– given their complementary roles and their use of a common research framework –
seems to be fruitful for the production and experimentation of this kind of tasks and
their trial in the classroom. In addition, the experience in Portugal shows that
commercial publishers draw on tasks produced in this kind of setting and introduce
them in textbooks. Of course, being demanding of teachers' expertise such tasks
require a combined effort of task design, classroom implementation studies,
curriculum development, and teacher education as we had in Portugal in the last five
years.
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Developing Theory for Design of Mathematical Task
Sequences: Conceptual Learning as Abstraction
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This paper describes an emerging approach to the design of task
sequences and the theory that undergirds it. The approach aims at
promoting particular mathematical understandings. Central to this
approach is the identification of available student activities from which
students can abstract the intended ideas. The approach differs from
approaches in which learning to solve the problem posed is the intended
learning. The paper illustrates the approach through data from a teaching
experiment on division of fractions.
Keywords: mathematical tasks, reflective abstraction, learning theory,
didactical engineering
Introduction
The "Task Design in Mathematics Education Discussion Document" stated, "We
would like to encourage an interest in tasks that have more limited but valid
intentions, such as tasks that have a change in conceptual understanding as an aim."
That is the specific aim of this paper and the research program from which it derives.
The research and theoretical work build on several core ideas advanced by
Piaget and therefore could be considered constructivist in orientation. As such the
paper could be categorized, according to the Discussion Document, as delineating
principles and frameworks for task design within a singular design community
researchers working from a constructivist perspective). However, our view is that
multiple theoretical perspectives are needed to do the work of design of mathematical
tasks. Consistent with this view, we see our work as complementary to the work of
many other research programs that work from a different core set of theoretical
constructs. Further, in our work we use principles from Cobb and his colleagues'
(Cobb & Yackel, 1996) emergent perspective and Davydov and his colleagues'
Russian activity theory (Davydov, 1990). However, it is beyond the scope of this
short paper to discuss our use of these different theories.
The emerging task design theory that I present here is narrowly focused on
the learning of mathematical concepts and as such offers an approach to addressing
difficult to learn concepts and to working with students who are struggling to learn
specific concepts. This task design approach does not address other important areas of
learning mathematics, particularly the important area of mathematical problem
solving. Therefore, the approach is meant to complement existing approaches, not
replace them. The emerging task design theory is a product of a research program
aimed at understanding conceptual learning, particularly the development of Theme D – M. Simon
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abstractions from one's own mathematical activity (activity that occurs in the context
of designed sequences of mathematical tasks). Thus our research program involves a
spiral approach in which we design task sequences to study learning through student
activity, and we use what we come to understand about learning to improve our
understanding of task design, and so forth.
We work with a broad definition of task, consistent with the one embraced
by the Study Group, as either a question that a student is asked or an objective that
they are given to accomplish (e.g., Why is a triangle the only rigid polygon? Measure
the length of line segment BC.).
Theoretical basis
The theoretical basis of our research program derives from Piaget's (2001) work on reflective
abstraction. DiSessa & Cobb (2004) pointed out,
Piaget's theory is powerful and continues to be an important source of insight.
However, it was not developed with the intention of informing design and is
inadequate, by itself, to do so deeply and effectively. (p. 81)
Our research program is aimed at building theory that can inform
instructional design.
Two Piagetian-based constructs that are foundational to our work are goaldirected activity and reflection. Goal-directed activity includes both physical and
mental activity. The notion of goal-directed is important, because the learners' goals
partially determine both what knowledge they call upon and what learners pay
attention to and can notice. Reflection, following von Glasersfeld (1995), refers to an
innate tendency (often not conscious) to distinguish (associate) commonalities in
one's experience.
In our research and theoretical work, consistent with Piaget and others, (c.f.,
Hershkowitz, Schwarz, & Dreyfus, 2001; Mitchelmore & White, 2008), we consider
mathematics conceptual learning as the process of developing new and more powerful
abstractions, in particular reflective abstractions. Again following Piaget, we
understand abstractions to be learned anticipations and reflective abstractions to be
those abstractions that result from reflection on one's activity and result in knowledge
of logical necessity. (See Simon, 2006) for a discussion of the distinction between
reflective abstraction and empirical learning processes.)
Our task design approach builds on this theoretical base and involves
specifying hypothetical learning trajectories (Simon, 1995) at multiple levels. In this
paper, I focus on the level of design of particular understandings, not the planning of
trajectories for larger mathematical topics. A hypothetical learning trajectory consists
of three components, (1) a learning goal. (2) a set of mathematical tasks, and (3) a
hypothesized learning process. Whereas the specification of the learning goal
generally precedes the specification of the tasks and hypothesized learning process,
these latter two components necessarily co-emerge. The learning process is at least
partially determined by the tasks used and the tasks used must reflect conjectures
about the possible learning processes. The design approach outlined here provides a
conceptualization of the design process with respect to these two components.
I briefly summarize the theory behind our approach to task design in the
following way. (For greater detail, see Simon, et al, 2010; Simon & Tzur, 2004;
Simon, Tzur, Heinz, & Kinzel, 2004; Tzur & Simon, 2004.) The pedagogical goal is
to promote particular mathematical concepts. These concepts are understood as
learned anticipations. Given an appropriately designed series of tasks, students can Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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call on available activities, attend to the effects of their activity, and come to an
anticipation of the effects of that activity – the development of a new abstraction. In
the next section, I provide an example to concretize these ideas and follow that with a
discussion of the key steps in the design process.
Instructional Example
In this section, I present an instructional example from a one-on-one teaching
experiment. The purpose of the teaching experiment was to promote and analyse
students learning of new concepts (making of an abstraction) as they engaged in
mathematical activity. Towards this end, we engaged her in a sequence of tasks and
restricted the researcher's role to probing thinking and initiating subsequent tasks.
The student, Erin, (one of three that participated in the study) was an
undergraduate prospective elementary teacher. Pre-assessment showed she lacked
understanding of both the meaning of fraction division and the invert-and-multiply
algorithm with which she was familiar. In sessions prior to the one described here, we
worked on the meaning of fraction division.
The goal of the session described here can be articulated at two levels of
specificity. First, we wanted her to reinvent a common denominator algorithm for
division of fractions. More specific in terms of the understanding involved, we
wanted her to understand the invariance of division with respect to variation in the
common units of the dividend and divisor.
The task sequence, developed by the author, began with division-of-fraction
word problems whose dividend and divisors had common denominators. Erin was
asked to solve them by drawing a diagram. She was able to solve the first task without
difficulty ("I have 7/8 of a gallon of ice cream and I want to give each of my friends a
1/8-gallon portion. To how many friends can I give ice cream?"). The task sequence
progressed to word problems in which the dividend and the divisor still had common
denominators, but the divisor did not divide the dividend equally and then to similar
tasks presented using only as number expressions (e.g., 8/5 ÷ 3/5 =). Erin's solution
process can be summarized as follows (actions for 8/5 ÷ 3/5 in parentheses):
1) Draws the dividend (draws 2 whole rectangles divided into fifths, shades
2/5 of one rectangle leaving 8/5 unshaded) (Figure 2).
2) Identifies groups the size of the divisor (circles each 3/5).
4) Counts those groups (counts 2 groups).
5) Identifies the remainder – the ungrouped part of the dividend (r = 2/5).
6) Identifies the fractional part of the quotient by determining the fraction of
the divisor represented by the remainder (2/5 is 2/3 of 3/5).
After a number of such tasks, the researcher changed the nature of the task.
He gave Erin a task and announced that the numbers were too "messy," that she
would not want to draw a diagram. The first such task was 23/25 ÷ 7/25. Erin made it
clear that she did not know the answer and the researcher encouraged her to talk
through a diagram solution without actually drawing. This she was able to do without

difficulty and arrived at the answer. The following task was 7/167 + 2/167. Again, Erin did not know the answer and solved the task by doing a mental run of her diagram drawing strategy. Finally, the researcher gives her 7/103 + 2/103 (same numerators as the previous task
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). Erin responded with the answer "three and a half"

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The fact, that this problem had the same numerators as the previous one, was not pointed out to Erin.Theme D – M. Simon
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and explained that she did not need to use the denominator. When pushed to justify her claim, she drew a rectangle divided into seven boxes and argued that she could group them into groups of two parts regardless of the size of the parts that were represented. An additional task verified that Erin did not over-generalize to tasks that did not have common denominators and that she could convert to common denominators to use her reinvented algorithm (make common denominators and divide the numerators). Assessment the following week convinced us that she retained her algorithm, she could justify it, and she could use the same reasoning to think about the invariance when composite units were involved (e.g., why 2400 divided by 400 is equal to 24 divided by 4).
Simon et al (2010) gives a more detailed look at the data, analysis, and explanation of Erin's learning. My focus here is the abstraction that Erin made. That abstraction became evident when Erin quickly gave the answer. So what did Erin abstract? Erin abstracted that regardless of the size of the parts (assuming common denominators), the quotient would always be the same and would be equal to the quotient of the numerators.
Readers at this point might be tempted to argue that Erin just saw a number pattern. However, remember that up to this point in the teaching experiment, Erin had not seen two tasks with the same pair of numerators. Thus, there was no pattern to observe. The first time she encountered the situation she anticipated the result.
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Furthermore, her conclusion was not only about the task in question. She now knew that she could find the quotient of any division of fractions involving common denominators solely by dividing the numerators. And, she was able to justify her conclusions using an improvised diagram related to her diagram drawing. These three indications make it clear that the anticipation in question was conceptual in nature and not due to noticing a pattern in the numbers.
The Design Approach
The first two steps in our design approach are the first two steps in most instructional design that is aimed at conceptual learning. We assess student understanding and articulate a learning goal
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for the students relative to their current knowledge. It is
after these first two steps that our approach diverges.
Our third step is to specify an activity or activity sequence that students currently have available that can be the basis for the abstraction specified in the learning goal. We attempt to identify an activity that can result in an anticipation that is the particular abstraction intended.
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The fourth step is to complete the hypothetical
learning trajectory, that is, to design a task sequence and related hypothesized learning process. The task sequence must both elicit the intended student activity and lead to the intended anticipation on the part of the students. The hypothesized learning

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Simon (2006) presents a distinction between reflective abstraction and empirical learning processes. I have argument that Erin's learning was the result of reflective abstraction as there was no empirical pattern to observe.
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Articulation of conceptual learning goals is a problematic issue not covered here. It is a theoretical and empirical challenge to specify learning goals in a way and level of specificity that it adequately guides instructional design (as well as instruction and assessment).
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Articulation of conceptual learning goals is a problematic issue not covered here. It is a theoretical and empirical challenge to specify learning goals in a way and level of specificity that it adequately guides instructional design (as well as instruction and assessment). Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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process must account for not only the overt activity of the students, but the mental activities that are expected to accompany those overt activities. I will not focus on steps beyond step 4 (e.g., symbolizing, introducing vocabulary, discussing justification), because again they are common to many approaches.
I will now use the example of the division-of-fractions task sequence to illustrate steps three and four of our task design approach. The consideration of what activity we might elicit begins in a way that is similar to Realistic Mathematics Education (Gravemeijer, 1994), a consideration of students' informal strategies. Whereas RME focuses on developing progressively more formal solution strategies, our approach is focused on developing concepts by developing anticipations from those activities.
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In many of our design situations, we have a specified learning goal and then endeavor to identify an available activity that could be used. Our example of division of fractions illustrates a design in which there was an interaction between step two, setting the learning goal, and step three, identifying a useful activity. When considering students' informal strategies and considering diagram drawing, we realized that students' informal diagram solutions would lead more naturally to (support development of anticipations related to) a common-denominator algorithm for division of fractions than an invert-and-multiply algorithm. Thus, the goal not only affected the identification of the activity, but the activity available affected the specific goal towards which the design was oriented.
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The challenge then was to construct a hypothetical learning trajectory. We began with division of fractions word problems and followed them up with numerical tasks (no context) to elicit a diagram drawing activity sequence like the one that Erin used (outlined above). Once the student is using the intended activity sequence, the task sequence must be designed to provoke the particular anticipation (abstraction). For this purpose we used larger numbers for the denominators and invited mental runs of the diagram drawing activity (which was a change in the activity from the researchers' perspective). The use of mental runs was intended to provoke two types of changes in the student's thinking. First, it is intended to help the student foreground the key quantitative relationships. For example, in the diagram solutions, the student first focused on how to create the number of equal parts in the rectangular unit. At least as much attention was likely to have been paid to that phase as to counting parts in order to circle a divisor shaped group or counting the number of groups. In the mental runs, the student can dispense with the first step by simply announcing, "I would draw twenty-three twenty-fifths." The second intended consequence of shifting to mental runs was to create a need on the part of the student to invoke concepts and mental operations that are critical to the concept being developed. In Erin's mental run she had to use whole number division to find out how many divisor-sized groups are in the dividend, whereas in her diagram solutions, she could simply count the number of groups she had in front of her. At the point at which Erin was doing the mental runs, she was engaging in the activity that could lead to the intended

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Although there are often overlaps in what is learned by students using these two approaches, I emphasize here the differences in the primary aim and the theory built to achieve that aim. We definitely build on aspects of RME, particularly their use of model of becoming model for (Gravemeijer, 1994).
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The target algorithm is not itself a conceptual learning goal. We still needed to specify the understanding involved ("to understand the invariance of division with respect to variation in the common units of the dividend and divisor"). However, I wanted to use the process of coming to the goals to illustrate how the knowledge/activities available can influence the goals that are set. Theme D – M. Simon
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anticipation, she was comparing multiplicatively the number of parts in the dividend with the number of parts in the divisor. Thus, the diagram drawing activity provided an important step towards this activity, but was unlikely to lead directly to the intended anticipation, because the student did not need to conceptualize the multiplicative comparison.
Doing a couple mental runs did not in itself lead to the intended anticipation. The researcher at that point posed a task with the same numerators as the previous task. The purpose of this particular task was to increase the possibility that the student would see the commonality in her activity and, as a result, anticipate that the answer had to be the same and realize that it was not dependent on the value of the common denominators. So how do we explain the resulting change in thinking?
As mentioned earlier, in doing the mental runs, Erin was able to give minimal attention to establishing the dividend – she merely needed to state that she would draw it. This initial statement established the size of the parts, so that (just as in the diagram drawing) she could focus on the number of parts of that size available and the number of parts of that size in a group. Thus, she naturally, without necessarily making a conscious decision, paid particular attention to how many total parts and how many parts in a group. This focus of attention (Simon, et al, 2010) was not because she considered the denominator unimportant, but because she had established the size of the parts and was anticipating her next step, a computation involving these two quantities. In the third task of this set, when Erin was faced with a second task with the same pair of numerators and different common denominators, she realized that she was about to enact the same activity as in the previous task. At that moment, she also realized why the size of the common denominators did not change the quotient. This was an example of Erin's reflection on her (mental) activity. That is, she perceived the commonality in her activity in the two cases that led to an abstraction. This abstraction was an anticipation about fraction division (and division more broadly). She now anticipated that the division of fractions with like denominators is equivalent to the division of the numerators.
Conclusions
Our task design approach is an emerging one. Three are aspects of it that we highlight here and others whose articulation will require additional analyses. I highlight here two features of our task design approach that can be seen in the example provided above. First, the approach provides a theoretically based strategy for promoting specific mathematical understandings. It contrasts with strategies in which students must solve novel problems to progress (or hear the solutions of more able peers). Although mathematics teaching cannot cause learning, it is an approach that involves engineering task sequences so that participating students predictably develop the ability to make the new abstraction. Second, the learning goal is not to learn to solve the tasks, as it is in many approaches. The tasks are made to elicit activities that the students already have available. Erin was able to solve all of the tasks prior to making the intended abstraction. Further, she was not consciously trying to find an easier way or to invent an algorithm. Her learning was a product of reflection on her activity across a sequence of tasks.
Let us examine some of the implications of this approach to task design.
1. By using a design experiment (teaching experiment) methodology, involving
design and implementations cycles using this design approach, there is an opportunity Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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to understand in greater depth the interplay among goal-directed activity, reflection, and conceptual learning. (This is discussed in depth in Simon, et al, 2010.)
2. This approach potentially provides a way to design task sequences for concepts that students traditionally do not learn well. These tend to be concepts that many students do not spontaneously reinvent in problem solving situations and concepts of which they do not develop deep understanding by being part of a class discussion with more knowledgeable students. The approach focuses the instructional designers on identifying key activities that are likely to afford the intended abstractions for abstractions that were not readily made.
3. Small group work using task sequences of the kind discussed here can lead to somewhat different class discussions. If students are making the new abstraction as a result of their engagement with the task sequence, discussions can focus more on articulation of the new idea, justification, and establishing the idea as taken-as-shared knowledge.
4. The approach has potential to address issues of equity in two ways. First, many students who have conceptual gaps early on seem to never recover. This design approach provides a methoddology for building up the specific experience, based on students' currently available activities, needed to make particular abstractions. Second, success in promoting the new abstractions during small group engagement with the tasks can lead to a greater number of students participating in and benefiting from the class discussions that follows. The underlying hypothesis here is that students who abstract ideas themselves, based on their work with the mathematical tasks, tend to learn the concepts in a more powerful way than those who follow the explanation of their more able peers offered in class discussions.
One final point that was discussed briefly at the beginning of this paper is the relationship of our approach with mathematical problem solving. The approach that I have described and exemplified does not focus on students developing their problem solving abilities. Rather it focuses narrowly on the development of mathematical concepts. Developing problem solving abilities is a key part of mathematics education. One could argue that conceptual understanding and problem solving are the two wings of mathematics education – students cannot fly without effective use of the two together. Students do learn concepts through problem solving lessons. Our approach is in no way intended to minimize the importance of lessons in which that is the case. Rather, our approach provides an additional tool that has the potential for success in areas where mathematics education has been less successful. One open question is how to use this tool in conjunction with the powerful tool of problem solving lessons to maximize the learning of students.
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Appendix
Complete list of instructional task sequence:
Solving Word Problems Using Rectangular Drawings
1. I have seven-eighths of a gallon of ice cream, and I want to give each of my friends a one eighth portion. How many friends can I give ice cream to?
2. A scuba diver has two hours worth of air in her tank. If each dive to the bottom of the bay takes three-eighths of an hour, how many dives can she make with the air she has?
3. Each ticket at an amusement park in France is worth four fifths of a Euro. If a pack of tickets costs four Euros, how many tickets are in a pack?
Solving Context-Free Problems Using Rectangular Drawings
4. 3/4 + ½ =
5. 7/3 ÷ 2/3 =
6. 8/5 ÷ 3/5 =
7. 5/6 ÷ 4/6 =
Solving Context-Free Problems Using Mental Runs of Rectangular Drawings
8. 23/25 ÷ 7/25 =
9. 7/167 ÷ 2/167 =
10. 7/103 ÷ 2/103 = Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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An Instructional Design Collaborative in One Middle School
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This article presents the principles for instructional design within collaborative communities in one middle school. During the study, members of the Collaborative engaged in creating and implementing a hypothetical learning trajectory and associated sequence of instructional tasks to teach integers in a middle grades classroom. During implementation of the trajectory, the members of the Collaborative played different roles in task design depending on their background. In this paper, we expand on the varying contributions that teachers, researchers and other educators make during design and implementation of rigorously designed instruction. We also document the artifacts and practices of design that led to increased teacher engagement and learning.
Keywords: Realistic Mathematics Education, Emergent Perspective, Integers, Teacher Research, Hypothetical Learning Trajectory
In 2009 an Instructional Design Collaborative was formed at a middle school (12-14 years old) in suburban Florida, USA. This Collaborative was comprised of a middle school teacher who was a former university professor, a doctoral student from a local university, two middle school mathematics teachers, and one special education teacher. The primary purpose of the Collaborative was to design and implement an instructional sequence for integers that was inspired by Realistic Mathematics Education Design Theory. All five Collaborative members participated daily in the design, implementation, and revision of all instructional tasks.
Design principles and task definition
All tasks designed by the Collaborative used the instructional design theory of Realistic Mathematics Education (RME) as the basis for creation. In RME, the focus is on the development of a sequence of tasks, not stand alone mathematical problems. Therefore, tasks are defined as problematic situations that are experientially real for students in that the dilemma that they encounter in the problem can be experienced as real by them. Furthermore, each task must be phenomenologically rich in that it lends itself to be mathematized (organized mathematically) by students. RME instructional sequences are guided by the following three design principles:
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Heuristic one: Guided reinvention
The sequence of instructional tasks should be designed to encourage students' reinvention of key mathematical concepts. To start developing an instructional sequence, the designer first engages in a thought experiment to envision a learning route the class might invent with guidance of a teacher (Gravemeijer, 2004). It is here that knowledge of the history of mathematics as well as prior research concerning students' invented mathematical strategies can be used. Since students do not have the time to invent the intended mathematics in the same way as mathematicians originally did, the teacher must help students re-invent these ideas in shortened time periods using carefully sequenced problems and tools.
Heuristic two: Sequences should be experientially real for students
The starting points of instructional sequences should be experientially real in that the students are able to engage in personally meaningful activity (Gravemeijer, 2004). Often, this means grounding students' initial mathematical activity in experientially real scenarios (which can include mathematical situations). While many textbook authors see the value of using realworld problems in instruction, RME goes beyond simply situating mathematics in the real world. Rather, instructional tasks draw on realistic situations as a semantic grounding for students' mathematizations and activities are sequenced so that students will organize their activity within the realistic context to re-invent important mathematics.
Heuristic three: Emergent models
Instructional activities should encourage students to transition from reasoning with models of their informal mathematical activity to modeling their formal mathematical activity, also called emergent modeling (Gravemeijer & Stephan, 2002). During the transition from informal to formal, the designer/teacher supports students' modeling by introducing or using student created tools, such as physical devices, inscriptions, and symbols that can be shared by students to explain their mathematical reasoning.
Guided by the three heuristics described above, the designer creates an instructional sequence while at the same time envisioning a path that the class may follow as they engage in the tasks (Gravemeijer & Stephan, 2002). The anticipated path has been labelled a hypothetical learning trajectory in that the designer makes conjectures about the mathematical route the class, as a community, will travel, including the mathematical goals and tool-use, as they engage with the instructional tasks and anticipates the means by which the teacher can support that route. After implementation, the designer analyzes the collective learning of the class and revises the instructional sequence accordingly. Since the former professor had multiple experiences designing instruction based upon RME, she created an integer HLT for the Collaborative.
Epistemological theory
The authors use a version of social constructivism, called the emergent perspective (Cobb and Yackel, 1996), to situate their interpretation of classroom events. The emergent perspective claims that learning is both an individual, psychological process and a social process, with neither taking primacy over another. In other words, a student's mathematical development occurs as he participates in and contributes to the mathematical practices of the classroom. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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In particular, to document how the HLT was realized in implementation, the two authors conducted an analysis of the collective learning of the classroom (Stephan & Akyuz, 2012). They found five mathematical practices that emerged as students interacted with the sequence of instructional tasks designed by the Collaborative. T-tests indicated that the students in the first author's class made statistically significant gains in their understanding of integer subtraction, a finding that is rare in integer research.
The Collaborative experiment
Integer concepts and operations is an extremely difficult content area for students to learn with meaning. Additionally, research indicates there is no agreed upon method for teaching integers conceptually. The textbook that had been adopted by the teachers' school did not support students learning integer operations with meaning, so Collaborative members decided to create their own instructional sequence. Their motivation was primarily pragmatic in that their prior students merely memorized the rules for operating with integers. Based on the tenets of RME, they created a first draft of a sequence that used net worth, assets and debts as the realistic context for exploration.
In the first phase of instruction the teachers introduced students to the notion of net worth and posed problems that encouraged students to think about possessing assets and debts, and the effect it has on one's financial situation. Here it is intended that students construct the idea that a net worth is an abstract quantity, not a concrete entity that can be counted like money, but rather the "status" of one's financial value. Net worth is contextually rich for exploring integers in that not only can net worth represent an object with positive and negative properties and can be ordered on a number line, but net worth can also be transformed by actions (add or subtract) to produce new net worths. These actions and their results can also be inscribed on a vertical number line.
Phase Two envisions students using statements about assets and debts to solve problems in which they find and compare net worths to each other. For example, students are given a listing of the assets and debts of two people, and asked to find and compare their net worths. The goal that the teachers conjecture will become taken-as-shared is to see which person is worth more than the other. Students typically create different strategies for finding net worths, such as 1) finding the total assets then total debts, and the difference between them, and 2) adding assets and debts one at a time until finished. Students' reasoning as they compare two net worths provides the opportunity to introduce a vertical, empty number line (VNL). For example, the teachers pose tasks that asked students to place the net worth of Jackson ($10,000) and Hayden (-$45,000) on a VNL and to find how much they differ by.
Up to this point, the tasks use the words "asset" and "debt" to represent the positive and negative nature of integers, respectively. Phase Three involves vertical mathematization, scaffolding students' symbolizing from reasoning with unsigned to signed integers. To this end, activities are posed that introduce assets and debts with the + and - signs rather than the words "asset" and "debt." Some net worth statements are designed intentionally so that a third strategy might emerge, that of cancelling assets and debts that are equal, and working with the remaining assets and debts.
The Fourth Phase seeks to confront the conceptual stumbling block involving why two negatives make a positive. At this point, transaction activities are enacted to help students build on their intuitions to construct an imagistic basis for operations. Students are to judge the effect that various transactions will have on a net worth. These tasks are written to encourage students
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to create negative and positive signs as both a process and object where the first sign signifies an action/transformation and the second represents the status of the quantity (negative/debt or positive/asset). Thus, - (-100) is read as "taking away a debt of $100" and - (+90) as taking away an asset of $90. Students are asked to judge whether or not the symbols signify what they called a "good decision", one that makes the net worth get better or a "bad decision", one that decreases the net worth.
The Fifth Phase incorporates tasks that encourage students to find the effect that various transactions have on a person's net worth. These tasks are called transaction tasks because they involve taking a person's original net worth, performing a transaction on it (like adding a debt) and determining his new net worth. While some students might perform erroneous calculations, teachers can expect "going through zero" to become taken-as-shared. For example, if the task is to find out what happens to Chris' net worth ($1000) if she takes away an asset of $1500, the teachers can re-introduce the empty number line (VNL) to make students' going through zero strategy more visual for the other students (Chris will have to pay off $1000 to get to $0 and she will still be in debt $500). In this way the VNL re-emerges in the class from student thinking as a model of transactions on net worths. Later, it might evolve to become a model for the formal integer operations of addition and subtraction in less context-dependent problem situations.
In the Sixth Phase, some activities require students to determine the results of various transactions including multiple ones like 100 - 2(-50). At first the problems are posed in context and move towards number sentences. To formalize the rules for integer operations, students are asked to list various transactions that could have taken place to make Chad's net worth go from $10,000 to $12,000. Students will write + (+2000) as the unknown transaction, - (-2000) or others like - (-1000) - (-1000).
Analysis of the varying Collaborative members' roles in task design
The Collaborative is comprised of five members: two mathematics teachers (McManus and Dickey), one special educator (Smith), one doctoral student (Akyuz) and a mathematics teacher with RME experience (Stephan). The Collaborative met one week before instruction began and then almost daily throughout the implementation.
Role 1: Anticipating supportive mathematical imagery
In the first Collaborative planning meeting, the HLT was introduced by Stephan. Since teachers did not have previous experience related with HLTs, the overarching sketch of the intent of the sequence was discussed by Stephan. In doing so, she elaborated the imagery, tool-use and potential discourse that could be supported over time. Even though the teachers learned about the HLT in the first meeting, they made important contributions based on their previous experiences. For example, the imagery of "pay off" came out in the meeting from one of the mathematics teachers, McManus:
McManus: When you say she has lower net worth, it means net worth is worse because she has assets, she can cash them all and pay off all debts and be still be in debt...I wonder if there is anything important, mathematically speaking,in paying it off?
Stephan: Two things I can come up with. One is when we get to integer operations and let's say we got $600,000 and pretend there is a transaction $900,000 ...she goes into debt 300, 000

dollars.
McManus: Because 600,000 goes to zero on the number line, you only have enough to go to
zero but there is still more left over. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Smith: Or I only have this much to pay off but I still owe this!
McManus: That is all related with the number line idea we try to develop later.
Stephan: That whole going under zero that took the math community so long to get.
In this excerpt, the mathematics and special education teacher contributed to the HLT by
suggesting that students might employ a "pay off" imagery as they make sense of going below
zero. Stephan connected their imagery with the difficulty mathematicians had with negative
numbers. The mathematics teacher illustrated his understanding of the intent of the instruction by
relating his imagery to the inscriptional device (VNL). This example shows that the mathematics
and special education teachers played an important role in suggesting student imagery that might
emerge during the course of instruction. Stephan played the role of tying their contributions to
the intent of the sequence as well as grounding their notions in the history of the discipline.
Role 2: Creating challenging formative assessments
Another way in which the members of the Collaborative participated in task design
involved creating tasks that would cause cognitive conflict in their students as well as cause
students to analyze the thinking of others. In one of their planning meetings, the teachers
reported that their students were having difficulties with the notation used with the VNL. Ms.
Smith was the first to suggest creating what they typically called a "Sam and Sue problem." This
type of problem usually listed at least two fictitious students' solutions and the class had to
decide which one they agreed with and why. Mr. McManus seized on Smith' suggestion.
McManus: What if we had a couple of number lines with the stuff on it incorrectly and had a
story and say "which number line tells the story?"
The teachers worked collaboratively and created the task pictured in Figure 1. The story
focused on three imaginary students named Larry, Curly and Mo, who had used a VNL to solve
a problem in which a person originally had a net worth of $4000 but added a debt of $8000. The
students were asked to decide which students' VNL was the most accurate.
Figure 1: The formative assessment task
This example illustrates the role that the teachers took in planning meetings. In other
words, for them, planning meetings consisted of bringing in student data from class that day to
discuss next moves in instruction and possibly creating a new task to address issues that had
arisen. The task described above was completely motivated by the classroom teachers and
incorporated the heuristics of RME. Theme D – M. Stephan & D. Akyuz
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Role 3. Using their mathematical knowledge to alter the instructional sequence
Sometimes members' contributions were of a matheamtical nature and questioned gaps
in the HLT. In a different planning meeting, Mr. McManus shared a concern that he would like
to assess his students' current understanding of the meaning of a negative net worth and that 0 is
not a negative number. In the excerpt below, the Collaborative discusses this issue which then
provokes a change in the instructional sequence.
McManus: I want to be sure that when I put a problem up there, all the kids realize that zero is
not same as negative numbers. Then they understand the abstract value; you can go below zero
and it makes you go in more debt.
Akyuz: Are you going to make them compare negative numbers also?
McManus: Yes… I think you cannot compare negative numbers until they conceptualize net
worth as abstract value.
Stephan: Because if you cannot conceptualize anything below zero [as objects in their own
right], you will have difficulty with that [ordering negatives]. I am really rethinking the order
of sequence now. I think the page after this…we need a number line here. I really think so
because even with concept of integer, forget operations right now, it is a big deal to order
those numbers on the number line.
Dickey: I want to see order on the number line. I want to see that - 4000 is below - 1000 and I
want them to see that, too.
Stephan: I think we need to have a page asking students to order a bunch of positive and
negative numbers. And then we may ask how much Juli's worth more than Deanna? And then
operations. This way, they can start making objects out of distances from zero and they might
also start thinking about those pay off ideas again.
During the discussion McManus stated that he wanted to know if his students were able
to understand the difference between negative numbers and zero as well as other negative
numbers. He stated that he wanted the students to be able to conceptualize negative numbers as
abstract objects. In this case, the role that the mathematics teacher played was to bring in his
knowledge of mathematics to question a gap in the HLT. For her part, Akyuz suggested that the
order of negative integers in relation to positives and zero ought to be assessed. Stephan agreed
and contemplated one of the Collaborative's first changes in the HLT. As a consequence of this
interaction, the Collaborative created tasks that asked students to show two people's net worths
on a number line and find out how much more one net worth was than the other (see Figure 2).
Paris' Net Worth is in the red -$20,000.
Nicole's Net Worth is in the red -$22,000.
Who is worth more? By how much?
Use the number line to show your solution.
Figure 2: Ordering and comparing net worthsMargolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Role 4. Working and revising already created tasks or the sequence of the instruction
Besides creating new tasks, the members of the Collaborative also contributed to the
instruction by revising the tasks that had already been created by Stephan before the study
started. In other instances, their contributions served as a catalyst for re-ordering certain tasks
within the sequence. For example, in one of the meetings we changed the order of the
instructional sequence as necessary based on the teachers' reported interactions with the
students. Originally, after the "good and bad decision" where the students decided whether a
given transaction was good or bad, the instructional sequence followed with the net worth
trackers that included vertical number lines and the transactions below them without linking
them to the context (see Figure 3).
However, since students were not naturally using the VNL to compare net worths, we
decided to pose transaction word problems before the Net Worth Trackers (e.g., Donald has a net
worth of -$5000. A debt of $3000 is taken away. Is this good or bad? What is his net worth
now?). We changed the order because we expected students to get incorrect answers to the word
problems and that this would motivate them to want to structure their reasoning using a VNL.
Then, students would naturally see the need for reasoning with a more structured tool (VNL)
when they worked the word problems.
In summary, the Collaborative members' roles emerged in a variety of ways throughout
the implementation of the instructional sequence. We have pointed to at least four different ways
that they participated including anticipating supportive mathematical imagery, creating
challenging formative assessments, using their mathematical knowledge to alter the instructional
sequence, and working and revising already created tasks or the sequence of the instruction.
Additionally, one of the teachers contributed her knowledge of students with disabilities to the
creation of the tasks. The former professor's role included preparing the HLT and supporting
instructional tasks. The doctoral student also contributed in these ways, but brought more RME
and mathematics education literature into the discussions. It is interesting to note that during the
implementation of the instruction, the practitioners began to discuss more theoretical issues (e.g.,
imagery and abstractness of negative numbers) while the researchers began to think more about
teaching practices (e.g., were state standards covered, formative and summative assessments).
The work of creating and implementing a sophisticated sequence of mathematical tasks provided
them the opportunity to marry theory and practice in a very genuine way.
Figure3: Net Worth Trackers
Artifacts that supported teacher engagement and learning
There are many artifacts that are used or produced in the process of conducting
classroom-based research: an HLT (hypothetical learning trajectory), research articles, an Theme D – M. Stephan & D. Akyuz
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instructional sequence, formal and informal assessments, interviews with students, state
standards, and instructional frameworks. The artifact that was reported by teachers as most
useful in shaping their actions and learning during the collaborative was the instructional
sequence itself. They argued that whenever they felt confused about the trajectory of their class,
they always had the sequence to go back to as a tool that could re-orient them to their
mathematical goals and to the expected reasoning of students. Often, our team meetings would
begin by working through the activity sheet that would be used in the next lesson. This activity
always led us to discuss students' current and possible future reasoning and how we might
support that with discourse, tools and gestures. Interestingly, teachers argued that the absence of
a teacher's manual was instrumental in their learning since they were forced to create a lesson
image for themselves rather than rely on an image from a book.
Of middle importance to teachers was the HLT (the conjectured learning route of
students) and the assessments. The HLT was important to them at the outset as a way to orient
themselves to the route that we expected students to take; however, teachers did not refer back to
the HLT as the experiment progressed. Rather, their conversations always folded back to the
activity sheets from the sequence, probably because the sequence was a more concrete
manifestation of the trajectory, whereas the HLT is more abstract. On the other hand, both design
researchers rated the HLT highly, probably because they were in charge of designing the
instructional sequence and utilized the HLT more before and during the experiment as a way to
keep in mind the goals of the instruction. With regard to assessments, teachers relied heavily
upon assessments, including student classwork and homework, but not as much on the more
formal unit test at the end of the sequence. Teachers commented that assessing student thinking
and discussing it with peers on a daily basis was an essential practice in supporting the
implementation of the instructional sequence.
Teachers rated the state standards, research articles, student interviews, and instructional
plans, as having little impact on their implementation of instruction and their learning. Since the
instructional sequence was already written, the teachers said they trusted that the designer,
Stephan, had incorporated the state and county requirements within the sequence. The articles
were not useful for teachers in this instance primarily because there is very little research on how
students conceptualize and learn integers. The researchers agreed with this and rated the research
on integers very low. In other cases, when the research focuses on how students think about the
content or how teachers implement sound instruction, teachers acknowledged that the articles
might be more valuable.
Practices that supported teacher engagement and learning
The practice of gathering data on students' conceptions, analyzing it, and using this data
to determine the task to be used the next day became a central practice of the teachers in this
collaborative. In addition, teachers engaged in the activity of lesson imaging in which we chose
the task(s) to be used in class, anticipated the diversity of ways in which students would engage,
and imagined how the whole class discussion would ensue, a practice we call lesson imaging
(Schoenfeld, 2000). In fact, the teachers rated lesson imaging as the second most important
activity in which they engaged during the project. They made a distinction between lesson
imaging and lesson planning, with the latter being the act of scheduling which activities to do
throughout the week. The teachers argued that lesson imaging is a practice that helped them
envision how students would be solving problems and how to capitalize on that in order to have
powerful mathematical discussions in which objectives were met. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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A third practice that emerged in a profound way among our team was collaboration.
Teachers placed the influence of this practice as number one in their own learning and success in
their classroom. They remarked that as powerful as lesson imaging is, it is not as effective in a
social vacuum; other teachers are sometimes able to anticipate student strategies that they had
not thought of or can think of better ways to engineer the whole class discussion than they would
have alone. Additionally, the presence of the experienced researchers in the collaboration was
critical to their success according to them. They recognized that they probably would not have
engaged in lesson imaging or daily reflection if not for the researchers lending their expertise and
leading the meetings.
The effects of participating in the collaborative were sustainable for these teachers as
they continued to reflect daily, lesson image and collaborate in years after Stephan's support was
directed elsewhere. In addition, the teachers involved in this project have become model
mathematics teachers for our school.
Implications of the work on different communities
The work of the Design Collaborative reported in this paper impacted a variety of
communities including students, teachers, educators, and administrators. First, their daily
reflection, planning and designing led to an increase in both students' state test scores as well as
their knowledge of integers more specifically (Stephan & Akyuz, 2012). Designing and testing
instructional sequences requires daily formative assessments in order to revise and strengthen the
tasks in real time. Thus, with input from a variety of classroom teachers, students received
instruction that addressed their immediate needs rather than waiting until the end of the unit and
re-teaching.
A second community that was influenced by their work was the teacher population at
both the Collaborative's school as well as teachers from around the state. Other mathematics
teachers at the Collaborative's school had the opportunity to work as a part of the Collaborative
if they desired. Other math teachers joined the team during the last two years as full participants
and learned about instructional design processes, analyzing student learning on a daily basis to
inform instruction and assessment. Members of the Collaborative were called on by their
administrators to lead Lesson Studies with their math colleagues.
Administrators, who held district level positions with supervisory responsibilities across
10 other middle schools, learned of this Collaborative and invited them to present their integer
work and collaboration practices to various school groups. In this way, the instructional design
and planning practices (including cognitive interviews and daily planning) were made available
to district level administrators, principals, assistant principals and teachers from other schools.
Another community that was affected by the Collaborative's work was the teacher
education population. At the time that the Collaborative was testing and revising the integer
instructional sequence, several mathematics education professors visited the classroom to
observe students as they participated in the instruction. In addition, one professor stayed after
class occasionally to question one teacher about her rationale for decisions she made during
instruction.
Finally, this design work has the potential to impact both teacher educators and
researcher/designers by the fact that several articles have been and will continue to be published
about their work. Already, the teachers from the Collaborative have published an article that
describes the circumstances that supported and constrained their work as a community of teacher
learners (Stephan, Akyuz, McManus & Smith, 2010). Akyuz documented the planning and Theme D – M. Stephan & D. Akyuz
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classroom practices of student-centered teachers (Akyuz, 2010). Stephan and Akyuz have
published the instructional sequence and the learning of one classroom as they engaged in the
instruction in a prominent mathematics education journal (Stephan & Akyuz, 2012). Finally, the
work of a mentor teacher in guiding novice student-centered teachers to incorporate inquiry
practices into their teaching repertoire will be published within the next year. These articles have
the potential to inform researchers, designers, and teacher educators.
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Mind the gap : Task design principles to achieve conceptual
change in rational number understanding
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In this paper we focus on the problem of conceptual change in
the shift from natural to non-natural numbers. We discuss students'
difficulties in this area and present a mathematics textbook analysis to
show that this problem is not taken into consideration in instruction. We
discuss a number of principles for instruction stemming from the
conceptual change perspective to learning and present a number of
experimentally tested tasks, designed on the basis of these principles.
These tasks were used to investigate and/or to induce conceptual change
in the number concept. We argue that such tasks are of value from the
point of view of instruction.
Keywords: Rational number understanding, conceptual change
Difficulties in understanding rational numbers
Mastery of the rational numbers represents an important aspect of mathematical
literacy. However, learning about rational numbers presents students with many
difficulties, mostly when the required reasoning is not in line with their prior knowledge
and experience about natural numbers (see Ni & Zhou, 2005, for a review).
In comparing decimals, students judge for instance that longer decimals are
larger, thus responding that 2.12 > 2.2 (Resnick et al., 1989); in comparing fractions
they think that a fraction gets larger when one of its parts gets larger, resulting in
errors such as 2/5 < 2/7 (e.g., Ni & Zhou, 2005). Students also extend the meaning of
operations from natural to non-natural numbers. For instance, seeing multiplication of
natural numbers as repeated addition leads to the idea that multiplication makes
bigger, which has been shown difficult to overcome (Greer, 1994), even in adults
(Vamvakoussi, Van Dooren, & Verschaffel, in press). Finally, the dense ordering of
rational (and real) numbers is difficult for students to grasp (e.g., Vamvakoussi,
Christou, Mertens, & Van Dooren, 2011, Vamvakoussi & Vosniadou, 2004, 2010;)
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An order ≤ on a set X is dense if, for all x and y in X for which x < y, there is a z in X such
that x < z < y. Unlike the integers, the rational numbers as well as the real number are densely ordered.
The real numbers are, in addition, continuous. Theme D – W. Van Dooren, X. Vamvakoussi, & L. Verschaffel
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Students initially respond that there are no other numbers between two given
pseudosuccessive numbers (e.g., 0.005 and 0.006 or 1/2 and 1/3). Later, they refer to
some intermediate numbers, but still do not accept that there are infinitely many.
Theoretical framework
Several researchers have argued that many of students' difficulties with
rational numbers can be explained from a conceptual change perspective on learning
(e.g., Ni & Zhou, 2005; Smith, Solomon, & Carey, 2005). Within this perspective, we
adopt a specific theoretical frame, namely the framework theory approach to
conceptual change (FTatCC, Vamvakoussi & Vosniadou, 2010), because it proposes a
number of specific and testable principles for the design of instruction and tasks
(Greer & Verschaffel, 2007; Tsamir & Tirosh, 2007). Originally developed to account
for the challenges that students face in regard to science learning, the FTatCC has
been fruitfully applied in the past few years in the domain of mathematics learning
(e.g., Greer & Verschaffel, 2007; Verschaffel & Vosniadou, 2004). Based on
evidence from cognitive-developmental research, the FTatCC assumes (Vosniadou et
al., 2008) that young children organize their everyday experiences in the context of
lay culture from an early age in domain-specific conceptual structures, termed
framework theories. These initial theories constitute explanatory frameworks that are
generative: They underlie children's predictions and explanations regarding
unfamiliar situations in a relatively coherent way. The incompatibility between the
background assumptions of students' initial theories and the scientific ideas to which
they are exposed mainly via instruction, is assumed to be a major source of
misunderstandings and errors for students. Regarding the development of the number
concept, the FTatCC assumes that, before they are exposed to rational number
instruction, students have formed a rather coherent domain-specific, naïve theory of
number—i.e., a complex system of interrelated ideas and beliefs—based on their
extensive experiences in the natural number domain. This theory shapes their
expectations about what counts as a number and how numbers are supposed to
behave. From the students' point of view, numbers are essentially discrete counting
numbers and are grounded in additive reasoning (Vamvakoussi & Vosniadou, 2011;
see also Ni & Zhou, 2005; Smith et al., 2005).
The FTatCC assumption about the structure and content of students'
knowledge of numbers—before they are exposed to non-natural number instruction—
implies that the shift from natural to non-natural numbers is a slow and gradual
process that is difficult to accomplish and requires substantial instructional support. A
factor contributing to this difficulty is that students are typically unaware of the
background assumptions of their framework theories and thus do not perceive the
necessity to re-evaluate or revise them. The FTatCC predicts that, instead, students
enrich via the use of additive learning mechanisms their knowledge base with new
incompatible information about numbers provided through instruction, thus creating
misconceptions such as the ones described above.
Instruction design principles
The conceptual change perspective on learning and instruction has been traditionally
associated with the cognitive conflict teaching strategy. This strategy has been subject
to criticisms and is now acknowledged as a potentially useful approach provided that
it is used with caution and only as one out of several other alternatives (Vosniadou,
Ioannides, Dimitrakopoulou, & Papademetriou, 2001). Indeed, there is a number of Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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different principles for the design of instruction stemming from the conceptual change
perspective on learning (Greer & Verschaffel, 2007; Vosniadou et al., 2001;
Vosniadou & Vamvakoussi, 2006; Vosniadou, Vamvakoussi, & Skopeliti, 2008). We
will refer here to the ones that are mostly relevant for the purposes of the present
paper. We note that, as such, these principles are not unique to the conceptual change
perspective on learning, in particular to the FTatCC. We stress, however, that the
principles have a very specific focus, namely to address the problems arising from the
incompatibility of prior knowledge with the intended new mathematical knowledge.
This particular focus has implications also for the implementation of principles in the
design of tasks, as we will illustrate in the following sections.
Take students' prior knowledge into consideration
There are several ways for prior knowledge to be taken into consideration in
instruction. This principle refers to the necessity to acknowledge the potentially
adverse effect of prior knowledge, in cases when it is not compatible with new
information coming from instruction. This requires that teachers, curriculum
designers, and textbook authors can identify the points where conceptual change is
necessary, and that they are informed about students' potential initial understandings.
Let us present an example that is related to our discussion in the next section. Prior
knowledge and experience about natural numbers can be used to introduce nonnatural numbers. In fact, it is commonly used when, for instance, fractions are
introduced via their part-whole aspect, or when decimals are presented as whole
numbers with a change of units. On the contrary, the differences between natural and
non-natural numbers are not explicitly addressed. However, downgrading the
differences and focusing on the similarities between natural and non-natural numbers-
-with a view to build on students' prior knowledge--creates many problems in the
long run, as discussed in the first section.
Facilitate students' metaconceptual awareness
As pointed above, students are typically not aware of the background assumptions of
their framework theories of numbers (e.g., that numbers are essentially discrete) and
this hinders conceptual change learning. It is thus important to create opportunities for
students to externalize their ideas, compare them with their peers', and reflect on
them. This can be done in learning environments that foster group discussions,
particularly when students are engaged in tasks that involve modelling a situation or
dealing with external representations. This brings us to the next principle.
Use models and external representations
Again, this principle is not unique to the FTatCC. However, it is important to note
that from the FTatCC perspective, taking into consideration students' prior knowledge
and how it may influence their interpretations of a situation also applies in the case of
models and external representation introduced in instruction. Consider, for example,
the (real) number line. It is a powerful representation for numbers, but it is also
known to be difficult and even misleading for students. For instance, conceptualizing
the number line as a ruler may lead students to believe that there is a finite number of
numbers in a given interval.
Foster analogical reasoning
Analogical reasoning, in particular cross-domain mapping, is considered an important
mechanism for conceptual restructuring. This is because the comparison between two Theme D – W. Van Dooren, X. Vamvakoussi, & L. Verschaffel
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domains may highlight their common features, reveal unnoticed commonalities, and
allow for the projection of inferences from one domain to the other. In the process, representation of one or of both domains may occur to improve the match, which may
lead to conceptual restructuring. Consider, for example, the complex interplay
between the domain of continuous magnitudes and the domain of number that, in the
course of the historical development, resulted in the re-conceptualization of the notion
of number, as well as of continuity (Vamvakoussi & Vosniadou, 2012).
In the following, we present a textbook analysis showing that the problem of
conceptual change in the number concept is not taken into consideration in
instruction. Then we present examples of tasks that are grounded on the above
principles and have been experimentally tested with respect to their potential to
induce conceptual change learning.
A textbook analysis
A central theme in the task design principles mentioned above is that students need to
be pointed explicitly to differences between natural and rational numbers. To see to
what extent this currently happens, we did an analysis of the three most frequently
used primary school mathematics textbook series in Flanders, Belgium. More
specifically, the teachers' manuals from year 2 to 6 were analysed, as these included
student materials and several additional clarifications and background information.
The units of analysis were the lines in the teachers' manual that in some way
dealt with rational numbers (i.e. fractions, decimals, negative numbers). For every
line, it was determined whether and to what extent it made reference to differences
between natural and rational numbers, or to similarities between them. In cases when
such a difference or similarity was pointed out, it was moreover coded whether this
happened in an implicit or in an explicit way. Finally, it was also coded for what
aspect (the way to determine the size of a rational number, the effect of operations
with rational numbers, the representation of rational numbers, or the density of the
rational number system) the difference or similarity was referred to.
The results showed that the textbooks were very comparable in their
treatment of rational numbers. With respect to the size of rational numbers, none of
the textbooks explicitly referred to the fact that rules that are valid to determine the
size of natural numbers do not hold for rational numbers. While most observations
referred to differences between both kinds of numbers, they were all implicit. Such an
implicit reference is for instance a number line showing the location numbers 0.6,
0.75, and 0.8. Students can derive that even though 75 is larger than 6 and 8, 0.75 is
still between 0.6 and 0.8, but it is not explicitly pointed out.
With respect to representations, all textbooks referred to differences merely
in implicit ways (such as pointing out that 2/4 = 1/2 without explicitly pointing out
that any rational number can have infinitely many different representations).
For the domain of operations, both similarities and differences between
natural and rational numbers are pointed out, about two thirds are similarities. An
example is that a decimal number like 0.72 is written as 72 tenths in order to do
operations with it (such as halving or doubling). Only one textbook explicitly
mentions that teachers should explicitly address the idea in students that division will
lead to a smaller result, and this happens only at one moment in the fifth year.
Finally, the aspect of the density of the rational numbers is hardly dealt with
at all in the three textbook series. In the few cases where it is addressed, this happens Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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in an implicit way, mostly by pointing out that an interval between two given numbers
on a number line can be "stretched" after which more numbers can be found, as
illustrated in Figure 1. It is however not explicitly pointed out that infinitely many
numbers can be found in any interval or that the stretching can be infinitely repeated.
Figure 1. Implicit reference to the density of the number line in a fifth grade textbook
Tasks to investigate and induce conceptual change
In the following we present specific tasks that were employed in
experimental settings with a view to investigate secondary students' understandings
of the density property of rational (and real) numbers, and to explore possibilities for
effective teaching of this counter-intuitive notion. The design of these tasks drew on a
series of studies from the FTatCC that focused on students' understanding of density
as a paradigmatic case of the problem of conceptual change in the development of the
rational number concept (Vamvakoussi & Vosniadou, 2004, 2010; Vamvakoussi et
al., 2011). In line with the FTatCC principles, the design of the tasks was informed by
empirical evidence about students' pre-existing ideas and typical misconceptions
regarding the notion of density in arithmetical, as well as geometrical contexts. This
evidence can be summarized as follows: a) the idea of discreteness is robust both in
arithmetical and in geometrical contexts, b) students are more inclined to accept that

there are infinitely many points on a segment, than that there are infinitely many
numbers in an interval, c) accepting that there are infinitely many intermediates
(numbers or points) does not imply that one understands that these can never be found
one immediately after the other, d) students do not "see" the rational numbers set as a
unified system of numbers but rather as consisting of unrelated sets of numbers (e.g.,
integers, decimals, fractions), which has implications on their judgments about the
number, as well as the type of numbers in an interval. In line with the FTatCC
principles, the tasks refer to the cross-domain mapping between continuous
magnitudes, in particular the straight line, and numbers. This cross-domain mapping
is deemed crucial for instruction-induced conceptual change in the number concept
(Vamvakoussi & Vosniadou, 2012). It also underlies a representation of numbers,
namely the (real) number line that is commonly used in schools settings.
In line with the FTatCC principles, using, evaluating, comparing and
constructing representations of numbers and the number line, lie at the heart of the
sequence of tasks designed by Vamvakoussi, Kargiotakis, Kollias, Mamalougos, and
Vosniadou (2003, 2004) (Table 1). These tasks were experimentally tested in two
different settings, both allowing for expressing one's ideas, and discussing and
evaluating others' ideas. Specifically, 30 9
th
graders were split in two groups who
worked on the tasks in pairs. Each pair presented their results to their fellow students
and they were discussed. One 45-minute session was devoted to each task.
Meanwhile, the control group (14 9
th
graders) worked in their classroom, with paper
and pencil, and the results were presented orally and then written on the blackboard Theme D – W. Van Dooren, X. Vamvakoussi, & L. Verschaffel
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by the researcher. The experimental group (16 9
th
graders) worked in Synergeia, a
software designed to support collaborative knowledge building that provides a
structured, web-based work space in which documents and ideas can be shared and
discussions can be stored. These participants had constant access to their peers'
answers, could write comments on them, and respond to comments. Both groups
received the same pre- and a post-test with tasks on the density of numbers. They
were also interviewed after the intervention. The experimental group improved
significantly more in its performance on density tasks than the control group.
Moreover, the experimental students displayed greater metaconceptual awareness of
the change in their ideas about numbers before and after the intervention. It appears
that exchanging ideas on the particular tasks in a structured environment with the
features of Synergeia was more profitable for students than the whole class discussion
(Vamvakoussi et al., 2003, 2004).

Task Goal	
1. What do you know about the real number line? Describe as good as you can. Read and comment upon the answers of your fellow students.	Express prior knowledge about the real number line
2. We often use the term "the set of real numbers". Suppose someone tries to understand what we mean by that. Could you draw a picture to help him/her understand?	Construct a representation for real numbers
3. We have been talking about two different representations of real numbers: A "formal" one, which we usually use at school, and a second one, which was proposed in our discussion and you seem to find adequate. Could you find a solid reason why we should prefer one over the other?	Compare two different representations
4. Imagine that you can become as small as a point of the number line. Then you could see the other points up close. Suppose that you are on the point that stands for the number 2.3. Can you define what point is the one closest to you? Describe in words or by drawing a picture.	Construct a representation for the number line

Table 1: Working with representations of numbers and the number line: A sequence of tasks
Vamvakoussi and Vosniadou (2012) further elaborated on the cross-domain
mapping between numbers and the line. In line with the FTatCC, they designed a text
that a) provided explicit information about the infinity of numbers in an interval, b)
made explicit reference to the numbers-to-points correspondence, and c) used a
bridging analogy (the number line as a rubber line) to convey the idea that points (and
numbers) can never be found one immediately after the other. The excerpt regarding
the bridging analogy reads:
The mathematical number line is a strange object. You can imagine it as a rubber
band that never breaks, no matter how much you stretch it. Place numbers
between 0 and 1, until it looks like you have used all the available points. If you
stretch the rubber band, then you will find out that between the points that looked
as if there were the one next to the other, there are more available points,
corresponding to more numbers. This procedure can be repeated infinitely many
times- don't forget that your imaginary rubber band never breaks!
Vamvakoussi and Vosniadou tested experimentally the value of the "rubberline" text as compared to two other texts that contained the explicit information and Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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examples of intermediate numbers, or figures illustrating the examples. Six classes of
8
th
and 11
th
graders (one experimental class per grade), in total 149 students received
a pre-test with density tasks in a arithmetical and a geometrical context, were
administered the corresponding text, and then received a post-test containing all the
tasks of the pre-test, and 5 additional tasks that examined whether students were able
to deal with the no-successor aspect of density (Figure 2). All groups profited from
the explicit information about the infinity of numbers presented in the text. However,
the experimental group (8
th
and 11
th
graders) outperformed the other groups in the "no
successor" items of the posttest, and were more consistent in providing correct
answers and justifications for their answers.
Figure 2. Example of a "no-successor" task in a geometrical context

Concluding thoughts

We stress that these tasks come from experimental studies aiming at testing very
specific hypotheses, and not in the first place at creating optimal learning
environments. As researchers in (the psychology of) mathematics education, being
mostly funded to conduct fundamental research, we are not primarily concerned with
the development of tasks that can be directly used in classroom teaching, as we
mainly aim to analyze and understand students' difficulties in learning particular
concepts starting from a certain theoretical stance. Still, we are convinced that our
perspective may have an added value for task design. We consider task design an
important part of the design of instruction. The tasks presented here were designed on
the basis of specific theoretical principles stemming from a conceptual change
perspective to learning and instruction. Furthermore, these tasks are empirically
tested, also with respect to the conditions under which they can be useful for teachers
as well as students.
We suggest that using tasks that prompt students to evaluate, compare, and
construct representations (in this case, of numbers) is informative for teachers, in the
sense that it provides valuable information on students' thinking. In particular, tasks
that were presented as "thought experiments" (e.g., task 4 in Table 1; see also Figure
1) were extremely informative for students' ideas on counter-intuitive notions, which,
were not easily accessible via verbal descriptions. On the other hand, the examples
presented here indicate that, if such tasks are embedded in a learning environment that
is supportive of structured interaction among students, then they may facilitate
conceptual change learning. We also provided evidence that even a typical school
task, namely extracting information from a text to answer related questions, can lead
to conceptual change learning gains, depending on the kind of information that is
provided in the text. Specifically, information that bridges between students' initial Theme D – W. Van Dooren, X. Vamvakoussi, & L. Verschaffel
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ideas (e.g., the segment as a "necklace of beads") and the intended mathematical
notion (e.g., the segment as a dense array of points) appears to facilitate the grasping
of counter-intuitive ideas.
We believe, therefore, that the principles and tasks as elaborated above are
interesting from an instructional point of view, and may eventually inspire the
development of learning environments. Our textbook analysis – which shows a very
large gap with the tasks and principles elaborated above – even strengthens this claim.
Finally, we must stress that the FTatCC is in first instance a cognitivelyoriented theory. Of course, students' affect, motivation and beliefs also play an
important role in the learning processes to obtain conceptual change, but they are
beyond the scope of this paper.

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Theme E

Features of task design informing teachers' decisions about goals and pedagogies

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Based on their mathematical goals for their students, teachers choose or

design tasks and sequences of tasks, select media for presenting tasks to students and for students to communicate results, plan pedagogies associated with realising opportunities in tasks, determine the level of complexity of tasks for their students including ways of adapting for them, and anticipate processes for assessing student learning. Each of these decisions is influenced by teachers' understanding of the relevant mathematics, by earlier assessments of the readiness of their students, by the teacher's experience or creativity or access to resources, by their expectations for student engagement, by their commitment to connecting learning with students' lives, and informed by teachers' awareness and willingness to enact the relevant pedagogies. This working group invites contributions from researchers and teachers who have considered such issues from the perspective of task design. The intention is to synthesise what is known about teachers' decision making about tasks, and to offer suggestions about task design for teachers, teacher educators, task designers, text and resource authors, and curriculum developers.
Among the questions that might be considered by authors contributing to the working group and which can be addressed by submitted papers are:
• How do features of design influence teachers' decisions to use particular tasks/sequences, or adapt them, or create their own?
• How do features of tasks/sequences influence teachers' choices about their potential for their class, including the media used for communication about the task?
• How does the design process influence teacher decisions about tasks within sequences?
• How do design considerations facilitate teacher adaptation of tasks/sequences to their students' experiences?
• How does feedback from classroom implementation of tasks/sequences inform future decisions on task design and use?
• How does collaboration between teachers, or between researchers and teachers, influence design of tasks/sequences?
• What are the implications for initial teacher education in task design?
• What is the effect of different cultural backgrounds on teachers' knowledge or belief on tasks and task design? Theme E – Introduction
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Teachers and researchers collaborating to develop teaching through problem solving in primary mathematics
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In this paper we focus on collaborative research developed between the staff of a primary school and a university researcher/task developer. The paper draws on findings from the first cycle of research which used a collection of tasks (word problems) and examined how teachers worked with and developed the tasks as springboards for developing students' reasoning in mathematics, as applied to the modelling of additive situations. The tasks were presenting in ways that introduced teachers to the broad principles underpinning the design of the tasks but not the specific intentions behind any one particular problem. This, we argue, encouraged the coming together of the teachers and researcher as a community of learners with teachers adapting and developing the tasks in ways that they saw fit to work with their particular learners. This approach also led, however, to tasks being appropriated in ways that may not have matched with the intentions of the designer: rather than this being an obstacle, it led to rich discussions around the nature of teaching and learning. We argue for the need to consider how tasks are mediated both between task designers and the teachers and between the teachers and learners and to accept that the appropriation at each of these 'interfaces' is open to variation in interpretations, and that working with such variation can lead both the development of pedagogical content knowledge and to student learning. Hence, rather than trying to construct tasks that are 'teacher-proof', working with the fuzziness of appropriation can be a strength.
Keywords: word problem solving, pedagogy, mediation, community of inquiry
Introduction
Problems are key tasks in mathematics education and there is growing support and evidence for the effectiveness of teaching mathematics through problem solving (National Research Council, 2001). This, however, poses challenges to teachers, particularly in primary (elementary) classes. One particular challenge is how teachers can support students to create representations that will help them find problem solutions and the pedagogical decisions involved in using and building upon these representations to make explicit the mathematics inherent in particular problems. Theme E – M. Askew & L. Canty
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We report here on a teaching experiment in a primary school that involved working with teachers to use problem solving as a starting point through which to help students, aged five to eleven, understand additive reasoning. We examine a double loop of appropriation: the first loop of appropriation of tasks (problems) and representations by teachers when working with a University developer/researcher and the second loop of appropriations that students made in working with the tasks and representations as introduced by the teachers. Through our examination of this double loop of appropriation we aim to answer the key question of:
How were the design features of the tasks appropriated by teachers and in the light of this appropriation what pedagogical decisions were the teachers prompted to make?
In answering this question we seek to examine not only how this type of task design rests on the interplay between artefacts, tools and mathematical knowledge but also on the complex relationship and interface between the task designer/researcher and teachers and between teachers and learners.
In our initial working together (teachers and designer/researcher) several themes emerged that we explore in this paper and that we are looking at in more depth as the work moves into study of developing multiplicative reasoning. The bulk of this paper is jointly constructed, but there are specific observations included from the perspective of the designer/researcher (Mike, first named author) and the teacher initiating the work in the school (Lisa, second named author): where this is the case we present these observations as quotations from one or other of the authors.
Background to the project and the roles of the authors in the design and implementation of the tasks
The work reported here arises from collaboration between a University researcher and the staff of a local primary school. The teaching and learning leader in the school (Lisa) approached the university researcher (Mike) seeking professional development for the school. Although the school's results for mathematics were relatively high, students' scores on assessments of problem solving were consistently lower than scores on items assessing 'pure' computation. The staff had put in place various pedagogies to try and improve students' attitudes towards problem solving and their ability to solve problems, including the explicit use of heuristics but this did not appear to be impacting on standards.
Lisa: Until recently, teachers have presented learners with word problems and asked them to identify the relevant information and then use some strategy, (draw a picture, make a list, work backwards, etc.) to determine the appropriate calculation to use to solve the question, by being asked, for example, do we add or subtract to determine the answer? We observed that students were getting confused by focusing on the numbers in the problem rather than the context, which in turn was leading to overgeneralisations, such as if the numbers in the problem were, say, 9 and 27, then the answer would be 3 regardless of the question. Students could confidently identify key words such as "more" but again a focus on the numbers given was leading to the assumption to always, say, add. Analysis of assessment results pointed to lower success rates in word problem solving which was in direct contrast to the generally high results in pure number algorithms without context.
The university researcher agreed to work with the school but to treat the approach as one of building a community of joint inquiry into teaching and learning about problem solving, rather than 'delivering' professional development to the staff.
At the time of writing one initial cycle of inquiry into learning about problems Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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involving additive reasoning has been completed. The approach taken was relatively informal in terms of data collection as the initial emphasis was on building trust and ways of working together. The cycle of working together has comprised:
• Staff and researcher meeting for a two-hour session after school during which models of working with addition and subtraction problems were introduced and jointly explored and the principles behind the design of the problems (as set out below) shared and discussed.
• Teachers devoting at least one lesson a week to these problems over six weeks.
• Researcher and teaching and learning leader meeting mid-cycle to discuss the progress and to plan the next meeting.
• A half-day second meeting with all the staff to which teachers brought artefacts produced by the children when solving problems and shared how they had incorporated the problems into their teaching.
• The designer/researcher visiting and observing the problem solving lessons.
Task design features
In line with others (Christiansen & Walter, 1986) we use 'task' to be what learners are asked to do and 'activity' as the subsequent mathematical objects that emerge from engagement in the tasks and through the interaction between participants and resources/representations. Drawing on variation theory (Runesson, 2005) the mathematical object(s) which learner activity enable to emerge can be both direct and indirect. The direct object here was to solve particular word problems, with a first level indirect object of the classification of problems (and thus improve learners' problem solving) and a second level indirect object of developing understanding of the nature of additive reasoning (and thus improve learners' understanding of mathematical structure). To promote the emergence of these different mathematical objects the task design features co-ordinated several different aspects, two of which are focused on here: the structure of additive problems, models and representations for solving problems. A third main feature was the choice of contexts through which to frame the word problems. Extensive use of humour and fantasy was made of here, for reasons similar to those set out by Zazkis and Liljedahl (2009) but it is beyond the scope of this paper to discuss this in detail.
Classification of word problems
The word problems were initially designed by adapting the Cognitively Guided Instruction framework (Carpenter, Fennema, Franke, Levi, & Empson, 1999).
Problems were written (by the first author) with the intent of exemplifying one of three types of addition/subtraction root problems:
• Change – problems where an initial quantity is increased or decreased;
• Compare – problems with two distinct quantities to compare, one of which is larger or smaller than the other;
• Part-part-whole – problems based around a set comprising two distinct subsets.
All teachers in the school were provided with a collection of these problems, loosely organised into year levels, in a variety of forms, including resources that could be projected onto a whiteboard, or printed off, and versions of the problems that adapted to make similar problems but involving different numbers.
The problems had been previously published (Askew, 2005) and as such were accompanied with booklets providing explicit guidance on the classification and Theme E – M. Askew & L. Canty
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suggested teaching approaches. However it had become clear that although the problems were designed with the above framework in mind, in practice, when discussing problems with teachers and students and discussing where they might fit in the framework did not always result in a particular problem being matched back onto the category that was originally designated. In the light of this inherent ambiguity in the classification of the problems less specific guidance was provided for the teachers in this project: they were left to decide on what the problems exemplified either through discussions with other colleagues or with the students. Thus the intent was that the framework would (a) help teachers make pedagogical decisions about the choice of problems to present to learners and (b) provide a common language for teachers to talk about problems, both amongst themselves and with learners, and also provide a common focus within and across the years.
Models and representations
The work of Carpenter and colleagues has also demonstrated that within this framework for additive reasoning problems the structure of particular problems affects the level of difficult (for example 'change' problem where the 'start' is unknown being cognitively more challenging than those where the final result is unknown). In the light of this a second task design feature was to introduce particular representations to scaffold the move from a word problem into a mathematical model. Rather than have different models for the different types of problems (which would presuppose that the classification of a problem into one of the three types needed to precede the setting up of the model) the use of a 'bar' diagram was chosen as a core representation that could used in all three types of problem. The choice of this model was based in the consideration of not only what artefacts and tools would classroom help learners to solve the specific problems but also, through reasoning about the types of problems, develop awareness of mathematical structure. In other words, the classification of the problems and the setting up of a model were not seen as two distinct, sequential stages but each supporting the other. Artefacts and tools are typically interpreted as having a concrete or virtual 'embodiment', with language regarded as something supporting and developing artefact and tool use. In this project we used two types of artefacts/tools used to support teachers and students. Using a "bar" diagram where, given three numerical elements in a problem, one of which is unknown, helps the problem solver to decide which is the unknown and to use this not only as a bridge into a solution method but also to think about the structure of the problem and thus became a tool for thinking with (Gravemeijer & Stephan, 2002). For example, given a problem such as:
Hamsa had 12 muffins. After Hamsa got some more muffins, she had 21 muffins.
How many muffins did Hamsa get?
A model could be set up thus:
Figure 1: Representing an additive word problem Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.

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Emergent findings
Our interest is in examining a 'double loop' of appropriation: first, how
teachers initially appropriated the classification and representation and the influence
of this on the way these were turned into pedagogical tools and second, the
subsequent appropriation of the classification and representation by the students. Here
we focus in the main on the first of these - teachers' appropriation.
Teachers' appropriation of and confidence in working with the tasks.
Our epistemological position is that mathematical understanding and knowledge
building arises from meaning making activity - with the emphasis on both the
meaning and the making. Meaning does not inhere in words - "compare problems' do
not 'exist' outside the collection of problems to which this label is attached - and the
meaning making comes about through the use of artefacts (classification labels in this
case) to think about the problems. Thus the classification of the problems was not
seen as a definitive or definite but as a means of provoking discussion about the
underlying structure and treating problem solving as a meaning making activity.
Lisa: The classification of additive problems into the three types was beneficial in
two clear ways. The first was in the initial process of identifying the fact that a
problem required additive reasoning. Second it enabled students to have a
stronger framework upon which to represent the known pieces of information.
Representing the second known piece of information in a problem and knowing
how this related to the first known piece of information was the challenging step
and could lead to some confusion. The framework provided a tool that assisted in
the completion of this sense making.
The teaching and learning leader also reported that this approach led to a
wealth of discussion between teachers and, in her experience, some of the most
extended professional dialogue about the nature and content of the mathematics
teaching that she had experienced during her time at the school.
The problems as originally published were organised in groups of three
around one of the problem types, with the level of difficulty of the actual calculation
being within what might be expected for the age of the pupils. Having solved three
problems the expectation was that teachers would discuss with the pupils what types
of problems these were. Thus variation (Runesson 2005) in the design of the problems
was deliberately based around keeping, initially, the underlying structure of the
problems constant and varying the quantities. In the project the teachers became
confident in adapting this proposed approach: as they and the students became
familiar with thinking about the types of problem, so the teachers felt confident to be
flexible:
Lisa: Initially teachers were using all three of the same type problems from the
one page, (all change for example) and students successfully solved these together
with teacher direction and also independently in pairs. When instruction moved to
another type of problem, this success pattern was continuing. When teachers
observed that students were becoming competent in solving the three "types" of
problems an assessment was given to review application of this new process. We
found, however, that some students tended to apply the process needed to
successfully solve the first question to all following problems even if these were
of different types rather than selecting appropriate processes for each one. A
successful experience in the first question where students knew they had "got it"
inadvertently led to using the same reasoning for all following questions. This
tendency to "overgeneralise" a known process to different types of problems is
typical of what we had noticed previously of our students' learn one "rule" and Theme E - M. Askew & L. Canty
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apply it all contexts. And, as before, students weren't always attending to the
contextual situation, again manipulating the numbers to get a desired or expected
answer regardless of what the question was asking. After much discussion and
error analysis of student work in our professional learning teams staff decided to
"pick and mix" to encourage the identification or classification of the problem
type as well guiding more attention to the context.
Teachers' appropriation of and work with the representations
Assumptions of 'transparency' in the use of representations
After introducing the 'block' diagrams to the teachers, as Lisa explains, these were
not initially found to be helpful by all students.
Lisa: The introduction of the block (bar diagram) to students had the intention of
providing a means to arrange the relevant information in a relational manner.
This visual, as we initially interpreted it, was intended to scaffold student thinking
of what we know and how that information related to what we were looking to
find out. Teachers modelled the representation by drawing the complete diagram
and then inserting given values in appropriate places on the diagram. Students, in
pairs, attempted new problems using the bars as a bridging tool to determine the
appropriate calculation. This was not totally successful with all students. Some
students had difficulty identifying the relationships between the numerical
information and using this to place the numbers on the diagram appropriately.
What was tending to happen was for them to use the bars as simply another way
to lay the numbers out, much as one might be told to write the answer in the box.
Thus some learners would write the two given numbers in the short bars and the
answer in the third, longest bar, irrespective of the relationship between the
numbers. If the underlying calculation was to find the difference between, say, 15
and 40 they would record this as
15 40
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Figure 2: Representing (incorrectly) the difference between 15 and 40
Even correct representations were non-proportional making it more challenging to
conceptualise the relationships. (For example a problem involving, say, 6 and 24,
would show these represented by the same size bar because of the pre-drawing of
the whole diagram.)
As the staff discussed what was happening here it became clear that the learners
interpretation of the diagram in a 'fill in the boxes' sense could be linked to
modelling of the diagram by (some) teachers as a whole rather than building it up
sequentially. Following rich professional dialogue between staff in a formal and
informal manner, sharing the work samples, (moderation) and further dialogue
with Mike, teachers went back to the classrooms to integrate the process of
building the diagram one part at a time. Students now were challenged to think
about how we could adjust the process of building the visual framework to better
represent the problem. Giving the control back to the students to see if they could
improve our process enabled them to review and reflect on their own thinking and
innovate to assist success rates. Such ownership of the learning enhanced creation
of new and shared knowledge. Teachers also noted a significant improvement in
student's meta-language used to discuss their thinking processes, thus allowing
self recognition of mistakes/successes and the reasoning to support their answers.
Mike: Looking back I can see how the way I introduced the diagrams to the staff
would have affected their appropriation of them and in turn how this played out in
classroom. I had prepared a 'sorting' activity based on a collection of problems.
The collection was based on the Carpenter and Fennema classification with
variation in what was unknown creating a dozen or so problems. After sorting and
classifying the problems, I had pre-prepared a set of block diagrams to fit with the Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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problems. Although we had a rich dialogue around which diagram might go with
which problem, I did not engage the teachers with constructing the diagrams and
so the points that subsequently emerged in classrooms were not anticipated or
addressed.
Thus assumptions of 'transparency' of representations were made in both
aspects of the double appropriation loop: assumptions of transparency by the
researcher and (both similar and different) assumptions made by the teachers. We
argue, however, that rather than this being a difficulty that could have been 'smoothed
out' by more attention to detail in setting up the work, that working through the
differences resulted in a deeper understanding of the role of the representations by
both the teachers and by the students.
Lisa: The diagram became even more valuable in that it now provided a process
through which to present known information in such a way as to illuminate what
is required in order to calculate what is not known. The process of drawing these
bars, one bar at a time was also valuable as it pushed the students to discuss and
to analyse where the second bar of information needed to be placed relative to the
first (next to it or below/above and so forth) as well as representing this
information in correct proportion to the first piece of unknown information.
But it also went beyond this in promoting a focus on the proportional relationship
between the two pieces of information that were known as well as the relationship
between them and what was unknown. For example if one was faced with a
problem where 6 apples were combined with 5 apples, the bar that represented the
6 apples needed to be slightly longer than (6/5 of) the bar that represented the 5
apples. And the bar representing the 6 and 5 apples combined together would be
as long as the two other bars combined. Thus the representation suggested that
this value was going to be in the vicinity of double 5 (10) and double 6 (12). The
diagram came to also act as a de-facto 'checking' tool.
Conversions between representations as a linear process
In the discussion in the reporting back meeting several issues emerged but one in
particular concerned the interplay between the three aspects of context, representation
and numerical equation. It became clear that a popular interpretation was that these
were three stages to move through, rather than complementary representations that
might be moved back and forth between. Thus once a model had been set up the
problem context could be 'set aside' and the model treated as the primary
representation that could then be worked with to set up the symbolic. This is
consistent with a traditional view of word problems and the contexts therein being
regarded as 'immaterial' and 'hollow' vehicles for mathematical calculation: that the
context had little to do with the abstract mathematics. In fact, in designing the
contexts for the word problems considerable care and effort had been taken to choose
contexts that might support mathematising, for example throwing competitions to
provoke comparing, or homework being destroyed as a context for change.
As mentioned, a further design feature was that the problems were not
necessarily intended to be 'realistic' or 'relevant' (the realistic-ness or of contexts to
individual learners being problematic) but rather to be engaging, particularly through
the use of humour and fantasy contexts. Feedback from the staff and children was that
everyone had enjoyed the humour thus embodied and a 'playful' approach had been
invoked, but this may also have inadvertently reinforced the perception of the
contexts only being there as window dressing to be discarded as soon as possible.
Mike: During my visit to the school one lesson was observed that reinforced this
impression of a linear movement from context to diagram to symbols. The context
of the problem was two characters involved in a domino stacking competition
with questions around the differences in the numbers of dominoes successfully Theme E - M. Askew & L. Canty
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stacked. Some of the Grade 5 students working on this were experiencing
difficulties over producing an appropriate model for the problem: several children
were putting the two quantities together end to end and representing the missing
amount as the total rather than setting up a compare model. In the discussion that
the teacher had with the class about which of these was the appropriate model, no
reference back to the context of actually stacking things up and comparing the
heights was made: the discussion was focused around the models and the numbers
- the context was not used as a way into setting up the context.
Again this has been treated as a focus of discussion in the meetings between
researcher and staff and as the work moves into multiplicative reasoning this is an
aspect that explicit attention is being paid to.
Implications of the work reported in the paper on different communities
The first implication is that word problems can provide tasks that teacher and students
not only find engaging but also which can give rise to purposeful mathematical
dialogue and meaning making.
Lisa: Initial responses from students have been most positive - they, informally,
have reported engagement with the problems, a delight in the humorous nature of
them and pleasure in discussing and exploring the possible underlying structures.
Whether this has resulted in gains on solving word problems in other situations,
most notably national tests, will become clearer as the study unfolds. Teachers
involved so far reported a shift in perspectives on the teaching of word problems
and an awareness of the meaning making processes.
Second, the introduction of a framework for working on and with the tasks, and
treating this as a joint endeavour has promoted teacher inquiry and collegiality
Lisa: As an integral part of the performance and development culture of the
school, staff observe each other's lessons within and across the levels. This direct
observation leads to a structured discussion between the staff members about the
learning taking place: as all staff have responded to the professional development
and focus on problem solving so positively along with a growing relationship of
trust, in this context observation and feedback should prove highly beneficial.
A further question is how the publication and presentation of such tasks
might be done in ways that provoke similar responses without the presence of a
researcher. While publishers might take note of the decision not to supply teachers
with the 'key' to what type of problem each word problem was meant to exemplify, in
the absence of the discussion around the design principles and the revisiting of these
to refine the meaning, then the tasks alone may simply result in a set of 'dull' word
problems.
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Conflicts in designing tasks at collaborative groups
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This is a theoretical paper in which some categories are proposed
to analyse conflicts in designing tasks in collaborative groups composed
of academics and schoolteachers. Based on insights from our own
experience in a collaborative group that designs tasks for other teachers,
and by using concepts of Basil Bernstein's theory, we propose five arenas
of conflicts between academics' and teachers' in collaborative groups
while they are producing tasks.
Keywords: Teachers, Academics, Collaboration, Tasks, Conflicts
Introduction
Task design in teacher education is a visible theme in literature (e.g.,
Zaslavsky & Sullivan, 2011). Let us follow the wide definition proposed in the ICMI
Study 22 (ICMI, 2012): "anything that a teacher uses to demonstrate mathematics, to
pursue interactively with students, or to ask students to do something" (p. 10). Some
studies suggest tasks do not determine teachers' actions (Remillard, 2005), but tasks
provide opportunities and limitations for teachers' and learners' actions (Remillard,
2005; Watson & Mason, 2007; Sullivan, Clarke & Clarke, 2009).
In this paper, we are interested in tasks designed by collaborative groups. This
collaborative group may be defined as one which brings together school teachers and
academics (Levine & Marcus, 2010). In this case we use the term academics to denote
those who work at university institutions conducting research, teaching or
developmental programmes. We were convinced by Jaworski's (2011) argument that
"teacher educator" is not the best term since we are interested in characterising the
partnership between academics and teachers.
Evidence suggests collaborative groups have positive effects on teachers' and
students' learning (Levine & Marcus, 2010; Jaworski, 2005, 2011; Ferreira & Miorim,
2011). Despite different understandings of teacher learning (Borko, 2004; Adler &
Davis, 2006; Ball, Thames & Phelps, 2008), we wish to be explicit about our point of
view: teachers' learning is related to changes in patterns of participation in the
classrooms (Borko, 2004). There is not enough space here to develop this topic, so we
ask the reader to consider our point of view when we talk about teacher learning
throughout the paper.
In collaborative groups, teachers have an active role in choosing what to do.
Every participant has an opportunity for mutual learning, i.e., academics may learn
from teachers, and vice-versa. Despite the term "collaborative", Jaworski (2005, Theme E - J. Cerqueira Barbosa & A. M. Pereira de Oliveira
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2011) and Zaslavsky (2008) identified tensions between teachers' and academics'
knowledge in designing tasks as they may present opposing points of view. This
finding suggests that collaborative groups are also a terrain for conflict (for now, let
us consider the term "conflict" intuitively). The same insight came to us from our own
experience in a collaborative group that we discuss below.
Conflicts are likely to emerge in designing tasks in collaborative groups, since
it involves the partners making decisions from different perspectives. For instance,
researchers' opinions may reflect their theoretical preferences whereas teachers may
express their opinions in terms of their own experiences in school contexts. In this
paper, we will focus on the role of conflict in designing tasks in collaborative groups.
As part of an ongoing research project, our theoretical contribution is based on
concepts from Basil Bernstein's theory (Bernstein, 1990, 1996). In order to build our
argumentation we mention studies reported in literature and our experience in a
collaborative group.
Initially we present a conceptualisation of collaborative groups. Subsequently,
we introduce the collaborative group that we were part of and an exemplary task
designed by the members of the group. Finally, we suggest five arenas of conflicts in
designing tasks inside collaborative groups.
Collaborative groups
Collaborative groups may work in different forms, and also represent
different types of learning (Levine & Marcus, 2010). Although not using the term
"collaborative", Jaworski (2005) explains the key characteristic of an inquiry
community composed of academics and teachers, which is that it requires us all to
trust and have confidence in working together. It suggests that members recognise
their different expertise and they share them in order to achieve a goal, which is in
line with what Wagner (1997) calls co-learning agreement. Ferreira and Miorim
(2011) make a distinguishing point for collaborative groups:
[It is] a group in which participation is voluntary, in which all individuals
involved search for professional growth, share trust and respect, support the group
work, engage in a common purpose, all the while creating and sharing meanings
about what they are doing, about their lives and professional practices (p. 138).
These authors state that not every group should be recognised as collaborative.
The authors call into question the nature of participation, which is expected to be
voluntary. As a result, a compulsory developmental programme should not be
understood as collaborative at all, since teachers cannot choose whether they want to
take part in or not.
In some sense, collaborative groups are also regulative on what is moved or
not to classrooms. Following this view, collaborative groups may be considered part
of what Basil Bernstein (1990, 1996) calls pedagogic recontextualizing field. The
function of this field is to move texts produced in the scientific areas (for instance,
Mathematics, Mathematics Education, and so on) and relocate them for instance into
classrooms. As collaborative groups focus on ways of improving teaching and
learning mathematics, they have a regulative action onto classrooms.
In the above paragraph, we mentioned the term "texts", which is a key concept
in Bernsteinian theory. Here, texts refer to all communicative actions, which may be
oral, written, gestural, and so on (Bernstein, 1990). A task is itself a written text,
whereas designing tasks requires participants to use different types of texts. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Collaborative groups establish a pedagogic relationship, but not with fixed
positions between who teaches and who learns (using Bernsteinian terminology,
between transmitters and acquirers). Both teachers and academics may interchange
the position of transmitters and acquirers at different times since the relationship is
based on mutual learning. We name this as floating positions in collaborative groups.
This does not mean that the pedagogic relationship is unstable, but floating positions
is its way of being, its nature, if we ensure the principle of mutual learning in the
group.
As pedagogic practice, every collaborative group has its own principles that
regulate its functioning. Bernstein (1990, 1996) points out two main features of
pedagogic practices: classification and framing. The first one regulates what is
legitimate to communicate, the second regulates how to communicate. Then by the
co-learning agreement, and as long as the collaborative group is going on, participants
address the principles that regulate the relationship between them.
As a pedagogic relationship, collaborative groups may also experience
conflicts. Jaworski (2005), for instance, reports a conflict between teachers and
academics in a collaborative group. While teachers were reporting they use inquirybased tasks in classrooms as additional to the curriculum, the academics were
emphasizing them as being curriculum-related. Piazza, McNeill, and Hittinger (2009)
refer to two sources of conflict in teacher communities: beliefs about subject matter
and the purpose of education. Conflict is not negative, with some authors viewing it
as a source of learning (Jaworski, 2005; Zalavsky, 2008; Piazza, McNeill & Hittinger,
2009). Moreover, conflicts may tell us about the nature of collaborative groups and
their internal dynamics.
Through Bernsteinian lenses, we see conflicts in collaborative groups as an
expression of the space that insulates original contexts of academics and teachers.
Even engaged in mutual learning, the participants are likely to make public their
views grounded in either university communities or schools, respectively, be they
academics or teachers.
Mutual learning does not remove horizontal hierarchies among the
participants. Let us imagine a collaborative group in which a well-respected academic
takes part in that. In such a case, the voice of the academic may be stronger in the
group than other participants. The same may happen to an experienced teacher who
has a higher expertise about anticipating students' actions. Their respective arguments
may be more vocal than others, and so reducing the opportunity for others to engage
in dialogue.
An example
At the end of 2010, we invited Master and PhD students, prospective and
experienced teachers to join with us to form a collaborative group with the purpose of
designing mathematical tasks for school students' and teachers' use. Participation was
voluntary, and the group was composed of 25 participants in all. It was called
Mathematics Education Watch (Observatório da Educação Matemática, in Brazilian
Portuguese) and is known by the acronym OEM.
The main aim of the OEM was to design a kind of written task referred to as
educative curriculum materials by Remillard (2005). Educative curriculum materials
are those that aim at both students' and teachers' learning (Remillard, 2005).
Remillard in turn uses the term "curriculum materials" for those designed only for
student learning. The adjective "educative" added to the term means that there is Theme E - J. Cerqueira Barbosa & A. M. Pereira de Oliveira
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some support for teachers to use the task. Then, the main motivation in organising the
OEM was to bring together teachers' and academics' expertise to assure that the
materials would be informed by teachers' know-how.
The group focused on the geometry competences recommended by Brazilian
official documents (BRASIL, 2008). The whole group was organised in seven
subgroups, and each one included, at least one academics (Master and PhD students
were also considered academics), one experienced and one prospective teacher.
Figure 1. Group picture taken after a regular meeting.
The group decided on the following phases to design educative curriculum
materials:
(1) to review literature on teaching and learning of geometry;
(2) to draft curriculum materials;
(3) to review and refine the curriculum materials;
(4) to use the materials in the OEM teachers' classrooms and to record the
implementations;
(5) to design the educative curriculum materials;
(6) to review and to refine them;
(7) to make them available on the Internet for other teachers.
At the time this paper was written, the group was starting phase 5.
Collaborative task design and the associated conflicts
Zaslavsky (2008) has shown that the cycle of designing, implementing and
modifying is an effective way of supporting teacher learning. Also this is a rich
opportunity for us as academics to learn more about school mathematics and how
teachers position themselves, which may ultimately help us to improve the ways of
supporting teachers.
Alro and Skovsmose's book (2002) was influential within the group, and we
decided to design investigation-based tasks. Figure 2 is an example of a task designed
by a subgroup, and refined by the whole group. This task is to be used with the
software Geogebra, which is familiar to the teachers who are involved in the OEM.
The task in Figure 2 can be described as curriculum materials, because at this stage it
is only focused on supporting student learning. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Figure 2. Task designed at the OEM (translated from Brazilian Portuguese).
The process of producing tasks provides an opportunity for many discussions
within subgroups, as well as the group as a whole. From this experience, we noted
some conflicting points of views about decisions to be made for the task. In
Bernsteinian terms, our interpretation was the conflicting messages were aligned to
the different settings participants originally belong to.
In a Bernsteinian analysis, the interest is upon discursive control. In his words:
the focus is on how power and control are translated in principles for pedagogic
communication (Bernstein, 1996). As previously noted, collaborative groups also
represent pedagogic practice based on mutual learning. As a consequence, there are
principles that regulate what the participants communicate to each other. It is possible
that the control is weaker than other contexts in which transmitter and acquirer have
fixed positions. However, it does not mean absence of control, which is operated
among and by the participants.
Conflicts about the features of a task should be viewed in terms of agents
addressing different principles of pedagogic practice. For instance, the academics
might share their theoretical views in the group. By doing this, they select what to
say, and also how to build their argumentation to be effective in that context. This is a
sort of recontextualization (Bernstein, 1996) operated by academics from scientific
field to a setting - in case, a collaborative group - in the pedagogic recontextualizing
field. In turn, teachers' arguments might be based on their experience in schools. Also,
they select what to say and how to do it as they recognise the setting has a particular
set of principles. In Barbosa (2013), it was called reverse recontextualization, where
legitimate texts move from classrooms to the pedagogic recontextualizing field.
In some sense conflicts in collaborative groups may be framed as the
encounters between texts that evoke principles from other two fields (scientific field
and classrooms). The same sort of conflict could arise with relation to official
pedagogic recontextualizing field, that one represented by state (Bernstein, 1996).
Generally conflicts in designing tasks may refer to different dimensions, all of
them related to what to select to be approached in the task and how to do that (a very Theme E - J. Cerqueira Barbosa & A. M. Pereira de Oliveira
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Bernsteinian point of view). Let us use the term arena to refer to the scope of a
conflict, a disagreement, i.e., the subject of the conflict. From our experience in the
OEM, and based on Bernsteinian lenses on discursive control, we shall suggest five
possible arenas of conflicts for task design in collaborative groups: context, use of
language, structure, distribution and subjects. These arenas of conflicts are analytic
categories proposed to analyse conflicts in collaborative groups when they are
engaged in decision making in the design tasks. The categories may be considered
part of what Bernstein (1996) calls languages of description. The function of the
languages of description is "to produce a specific text and translate these referential

relations into theoretical objects or potential theoretical objects" (p. 133).
The arena of context refers to the mathematical context of tasks. Alro and
Skovsmose (2002) theorise three possibilities for mathematical tasks: reference to
pure mathematics, semi-reality, and reality. We may consider two extremes – pure
mathematics-based and reality-based tasks – and many possibilities between. In this
case, the conflict of context arises when participants are discussing the adequacy of
the context for a task to achieve its learning goal. In producing the task in Figure 2,
for instance, the argument that a reality-based task would be more motivating for
students came into conflict with the argument that this sort of context, in the case,
could cause difficulties for learners.
The arena of language refers to the level of rigour tied to the task. In Figure 2,
the rigour is expressed through the use of some terms and, in particular, at question 6,
an algebraic representation is required. The level of rigour could be weaker or
stronger than that, and it also gives room for conflicting arguments while designing
tasks.
The arena of structure refers to the degree of openness in tasks. If we have a
look at task in Figure 2, we may note that the auxiliary questions follow a sequence
that scaffolds students' actions. The task could be much closer, if more auxiliary
questions were posed to guide students' actions. Also, the task could be much open,
for instance, if the question was only put in terms of investigating the relationship
between the number of sides of a polygon and its number of diagonals. Then, those
conflicts may be positioned between two extremes: closed-end and opened-end tasks.
The arena of distribution refers to what is expected to teach in a task. It refers
to selecting content to be focused on tasks. For instance, in the task of Figure 2, a
conflict could happen between two possibilities: the task should focus on the
relationship between number of sides and diagonals, or on other regularities. The
control on what is explored by students while they are approaching an open-ended
task is unpredictable. Stein et al. (2000) classified them in high or low levels, as they
require memorization or investigations and explorations, respectively.
Finally, the arena of subjects refers the way the task positions students and
teachers. In a closed-end task, the teacher is expected to keep themselves far from
students' doings on the task, once this is very structured. On the other hand, in an
opened-end task, teachers are expected to interact more with students, as the task
itself does not have many scaffolds. So every task suggests a level of insulation
between students and teacher (at least, an expectation).
The five arenas of conflicts suggested above are expected to work as tools to
analyse conflicts in decision making in the design of tasks within collaborative
groups. Note that conflicting arguments in each arena is characterised in terms of a
continuum, which means arguments may be positioned at any point of the segment
(Figure 3). Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22

Figure 3. Diagram representing arenas of conflicts in designing tasks inside
collaborative groups.
As a result, the resulting task holds the markers of the conflicts
inferred from an analysis on the task (Barbosa, 2013). Then the diagram in Figure 3
may be useful to analyse conflicts in a setting of task design, and also to analyse tasks
themselves.
Final remarks and implications
In this paper, we attempt
conflicts in collaborative groups while they are designing tasks. Instead of
only for agreements within col
conflicts represented by opponent po
2011). We suggest conflicts are results of encounters of texts that follow the logic of
the original settings of academics and teachers.
As theoretical description
academics and teachers while they make decisions about designing tasks. Conflicts
must not be understood as
groups. On the contrary, conflicts are likely to take shape in
and/or harmonious conversations, such as expected for this kind of group (Ferreira &
Miorim, 2011).
Whether we are interested in supporting collaborative groups and teacher
learning, we need to develop more studies on conflicts in those groups. Particularly,
is necessary to shed light on how academics and teachers deal with and solve the
conflicts. Further insights on possible strategies for
developed.
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conflicts represented by opponent points of view, which is line with Jaworski (2005,
We suggest conflicts are results of encounters of texts that follow the logic of
the original settings of academics and teachers.
As theoretical description, we propose five arenas of conflicts betwee
academics and teachers while they make decisions about designing tasks. Conflicts
must not be understood as violent disputes, which is uncommon for collaborative
groups. On the contrary, conflicts are likely to take shape in dialogical
or harmonious conversations, such as expected for this kind of group (Ferreira &
we are interested in supporting collaborative groups and teacher
learning, we need to develop more studies on conflicts in those groups. Particularly,
is necessary to shed light on how academics and teachers deal with and solve the
Further insights on possible strategies for supporting those groups may be
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A dialogue between cultures about task design for primary
school
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The aim of this paper is to exploit intercultural dialogue to
analyze two cases of task design (or, better, re-design) for word problems
in different cultural traditions (the Eastern one, within the Confucian
Heritage Culture, CHC, represented here by China and the Western one,
represented here by Italy). By means of two paradigmatic examples, one
developed in Italy and one in China, we aim at showing, on the one hand,
the effects and advantages of intercultural dialogue and, on the other hand,
the need to take into account and to respect culturally rooted pedagogies,
avoiding uncritical transfer from one culture to another.
Keywords: word problems, problems with variation, primary school,
intercultural dialogue, addition/subtraction
Introduction
Word problems are a special kind of tasks, presented all over the world, in

mathematics textbooks to link numbers to real life objects and situations. They are the heirs to ancient traditions of mathematical texts, existing in all cultures. In this paper, we consider, as a paradigmatic case, problems for early childhood classroom to be solved by addition and/or subtraction, although some observations and examples are more general.
After reviewing some literature about word problems in mathematics education in China and in the West, we present a special kind of word problem taken from Chinese textbooks, i.e. problems with variation (biànshì problems).
Then we illustrate how the meeting with the Chinese tradition has suggested to a group of Italian researchers (including primary school teachers) to re-design (and test in the classrooms) a rich system of problems with variation. On the one hand this redesign is in order to meet needs emerging from Italian school practice , and to tailor classroom activity to their system of beliefs, on the other hand. Conversely, the meeting with the Western tradition has modified some principles in the Chinese approach to problems with variation, enriching the original concept based structure and offering new criteria for task re-design.
Closing remarks summarize authors' contributions to the following questions (concerning themes C and E of the Study): Theme E – M. G. Bartolini Bussi, Xuhua Sun & A. Ramploud
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C1) How do curriculum expectations influence authors' design principles?
C2) How does an intention to promote change influence design?
C3) How do cultural considerations about instruction and pedagogy influence design?
E1) How does collaboration between teachers, or between researchers and teachers, influence design of tasks/sequences?
E2) What is the effect of different cultural backgrounds on teachers' knowledge or belief on tasks and task design?
Some literature about word problems
A short review of Chinese literature
Taoism has had profound influence on Chinese culture. The central Taoism idea of the evolution of events as a change process and the acceptance of the inevitability of change reveal the ideologies of "grasping ways beyond categories"; "categorize in order to unite categories" (以法遇释, 以类相比). In China, for 5000 years, mathematics knowledge was elicited by word problems, which stems from the "Shu" (术) spirit (similar to "general methods") in the problem-oriented tradition from Oriental mathematics , "… to produce new methods from word problems, promote them up to the level of general method, generalize them into 'Shu,' and deploy these 'Shu' to solve various similar problems which are more complicated, more important, and more abstruse" (Wu & Li, 1998). Impacted by the idea of "grasping ways beyond categories"; "categorize in order to unite categories", word problems in ancient China were organized into different categories in terms of situations or algorithms. For example, Jiǔzhāng Suànshù (九章算术) , the most classic literature of Chinese mathematics, used 246 word problems to spread mathematical knowledge.
The tradition of categorizing word problems did not go on when Chinese curriculum on mathematics knowledge was totally borrowed from the West in 1878, where word problems, labelled as "application problems" ("应用题"), played a role of knowledge application, not introducting original knowledge. In 1929, the first formal curriculum standard, since new China was founded, primary school arithmetic syllabus (draft) ((小学算术教学大纲(草案)), claimed that "application problems are the important part and should be covered about half of the arithmetic contents" (Wang, 1996), which has remained unchanged as basic requirements of elementary curriculum for 80 years at least. In 1952, application problems in the first official textbook for whole country were categorized into simple application problems and complex application problems. Since 1958, application problems have not been organized by their categories but by their application content for the reason that knowledge system is required as a curriculum framework. Although application problems in textbooks were not organized by their categories again, their teaching organization stressed not a single problem but a group of problems with variation. Following the thousands-years-traditions "categorize in order to unite categories", one distinctive instruction feature of word problems is to develop the ability to identify the category of word problems (识类) belonging to and discern different categories (归类), namely, discern the invariant elements from the variant elements between problems and recognize the "class" every problem belong to. This pedagogy is generally called as biànshì (变式) in Chinese, where "biàn" stands for "changing" and "shì" means "form", can be translated loosely as "variation" in English (Sun, 2011). Some categories of biànshì are the following (examples follow): Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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OPMS (One Problem Multiple Solutions), where, for instance, the operation to solve the problem is carried out in different ways, with different grouping and ungrouping: 8 + 9 = (8+2) + 7; 8 + 9 = 7 + (1 + 9) and so on.
OPMC (One Problem Multiple Changes), where in the same situation some changes are introduced.
MPOS (*M*ultiple Problem One Solution), where the same operation can be used to solve different problems, as in summary exercises (Sun, 2011).
To sum up, in Chinese mathematics education problem variations aim to discern, compare the invariant feature of the relationship among concepts and solutions and provide opportunities for making connections, since comparison is considered a pre-condition to perceive the structures, dependencies, and relationships that may lead to mathematical abstraction (Sun, Wong, & Lam, 2007).
A short review of Western literature
Different strands (and a different pedagogy) emerge in the literature on word problems. In this paper we mention only some strands.
The cognitive analysis of word problems (see different contribution in Carpenter, Moser, & Romberg, 1982) focuses on the difficulties met by students in understanding the problem and looking for effective solution strategies and has produced well established categorizations likely to be employed in educational setting (e.g., the well agreed categorization of additive problems into combine, change and compare).
The didactical analysis of word problems (see different contributions in the recent volume by Verschaffel et al., 2009) criticizes the stereotyped and not realistic features, highlights the distance with modelling activity and focuses on the negative effects of word problems on students' sense making capabilities. Actually, there are examples of word problems which suggest uncritical applications of rules. There is the famous case (ibidem, p. xii) concerning the age of captain: "On a boat there are 20 sheep and 6 goats. How old is the captain?" The findings show that many students would respond to such question by adding the numbers of sheep and goats.
Some literature on word problems is strongly related to the early development of algebraic reasoning that may build a bridge between the two cultures (e.g. Ofir & Arcavi, 1992, Cai & Moyer, 2008).
Examples of Chinese problems with variation
The introduction of subtraction in the first grade by problems with variation
Fig 1. Mathematics Textbook Developer Group for Elementary School, 2005, vol. 1
Chinese textbook authors never separated the subtraction concept from addition. Theme E – M. G. Bartolini Bussi, Xuhua Sun & A. Ramploud
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Whenever there is addition there is subtraction (Yang Hui, 1274, in Siu, 2004, p. 164).
Figure 1 shows a paradigmatic example of problem variation: Xiao Ming folds a pink paper crane; xiao li and xiao hua fold two blue paper cranes. How many paper cranes do they fold? There are 3 paper cranes. Xiao ming takes a paper crane. How many paper cranes does he leave? The answers are: 1+2=3, 2+1=3, and 3–1=2. The drawing intends to help learners to recapitulate the relationship of addition and subtraction, and the meaning of "equal" from the problem set 1+2=3, 2+1=3, 3–1=2.
The problem sets hinges on exemplifying relationships rather than objects and reflects the mathematical structure underlying the problems in this respect. The addition concept is different from that of subtraction, which belongs to a different category. Yet in the sense of part-part-whole we can combine these two concepts into one category and understand the ancient idea of "grasping ways beyond categories"; "categorize in order to unite categories.
A summary system of problems with variation in second grade
First solve the nine problems below. Then explain why they have been arranged in rows and columns in this way, finding relationships
(1)In the river there are 45 white ducks and 30 black ducks. All together how many ducks are there?
(2)In the river there are white ducks and black ducks. All together there are 75 ducks. 45 are white ducks. How many black ducks are there?
(3)In the river there are white ducks and black ducks. All together there are 75 ducks. 30 are black ducks. How many white ducks are there?
(1) In the river there is a group of ducks. 30 ducks swim away. 45 ducks are still there. How many ducks are in the group (at the beginning)?
(2) In the river there are 75 ducks. Some ducks swim away. There are still 45 ducks. How many ducks have swum away?
(3) In the river there are 75 ducks. 30 ducks swim away. How many ducks are still there?
(1) In the river there are 30 black ducks. White ducks are 15 more than black ducks (black ducks are 15 less than white ducks). How many white ducks are there?
(2) In the river there are 30 black ducks and 45 white ducks. How many white ducks more than black ducks (How many black ducks less than white ducks)?
(3) In the river there are 45 white ducks. Black ducks are 15 less than white ducks (white ducks are 15 more than black ducks). How many black ducks are there?
Table 1. Beijing education science research institute and Beijing instruction research center for basic education (1996), vol. 4, p. 88. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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This is a system of nine problems concerning addition and subtraction, where the organization in rows refers to the already mentioned combine, change, compare categorization and the organization in column refers to the same arithmetic operation (either addition or subtraction, see MPOS above). In each row there is a problem (in the shaded cell) and two variations (see OPMC above). It is taken from a Chinese second grade textbook. The task is very complex and requires the students not only to solve each problem but also to explain why the nine problems have been arranged in this way. Each problem is associated with a graphic scheme, that models on one or two lines the relationship between quantities. Such graphic schemes for both additive and multiplicative problems had been introduced systematically by Davydov (1966) in his algebraic approach to quantitative reasoning and relationships and are used in some countries of the CHC area, such as mainland China, Japan, Singapore. In a study carried out in Italy, we first used it in teacher education as a prompt to challenge teachers' beliefs (Bartolini Bussi et al., 2011; Bartolini Bussi et al., 2012); later a redesigned task was proposed to Italian students (several experiments from grade 2 to grade 5) in order to foster the approach to algebraic reasoning as soon as possible.
Task re-design in Italy and Hong Kong: hint at two case studies
A transposition of problems with variation in Italy
Needs in school practice
In Italy, in spite of the different suggestions of the Standards and Programs (http://www.mathunion.org/icmi/other-activities/database-project/introduction/italy/) where the focus is rather on the sense making of a situation and on modelling (mathematization), it is very popular strategy among teachers to suggest "cues" in the problem text in order to detect the operation to be used. For instance, in a popular website (http://www.lamaronca.it/), for additive problems, teachers are instructed to invite students to underline words like "aggiungere" (adding), "in tutto" (in all), for addition, and words like "togliere" (remove), "restare" (remain), "in più" (more), "in meno" (less) and so on for subtraction. Without careful control, this cue might lead randomly to either right or wrong choices. Consider for instance the two texts with the same implicit question: "how many candies does John have?".
"John has 5 candies and Anna gives him 2 more candies"
"Anna has 7 candies, 2 more than John".
In spite of the same question and of the same cue, in the former case the direct reference is to addition and in the latter to subtraction. The confusion is reinforced by the fact that in Italian "più" is also the wording of "+" (plus), that is the special sign for addition. The two situations might be related to each other highlighting unknowns: 5 + 2 = ? and 7 = ? + 2.
but teachers are not encouraged to link addition and subtraction. They are rather encouraged to follow the principle one-thing-at-a-time and to practice addition for months before introducing subtraction. A very popular series of exercise books (http://www.erickson.it/Ricerca/Pagine/Results.aspx?k=matematicaimparo&start1=1, Matematicaimparo) contains even two different booklets (with different authors), one for addition and one for subtraction problems. The above practice has the effect to produce very poor performances in the solution of arithmetic problems in the national

assessment carried out at the end of the second and fifth grades. Theme E - M. G. Bartolini Bussi, Xuhua Sun & A. Ramploud
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Task re-design
Re-design concerns the task of the Table 1. Teachers-researchers who have
collaborated in the pilot study have not implemented the same Chinese task, but have
re-designed it to tailor it to the Italian tradition and to their individual teaching styles
and systems of beliefs.
Three main changes were introduced:
1) the single task has been transformed into a set of several tasks;
2) classroom work was organized according to a sequence inspired by the
theoretical framework of semiotic mediation after a Vygotskian approach (Bartolini
Bussi & Mariotti, 2008):
a) individual or small group solution of each row of problems followed by the
invention of three problems similar to the given ones, to foster the awareness
of the problem structure;
b) collective discussion of the findings, with teacher's orchestration.
Moreover the solving graphic schemes (at the beginning) were removed and
introduced later, after thorough exploration and solution of the problems, as students
were not familiar with such schemes. In this way the use of a graphic scheme was
acknowledged by students as meaningful and not perceived as an automatic answer to
a given task. In this way the task (originally developed in China within a teacher
centered and textbook centered tradition) was modified to fit a dialogic approach
where teaching and learning are considered the two sides of the same collaborative
process.
In the same way, multiplicative problems with variation were introduced as
from the third grade in several classrooms. (Bartolini Bussi et al., 2011). The project
is in progress and involves nearly 100 teachers from September 2012.
A modified approach to problems with variation in Hong Kong
Needs in school practice
Inspired by the Western tradition of phenomenography and pedagogy of variation
(Marton & Booth, 1997) and theory on the process of mathematization (National
Council of Teachers of Mathematics, 1989), Chinese researchers found that a focus
on dealing with concepts and deriving a structure among these concepts is not
sufficient; there is also a need to pay attention to two important processes of
mathematization, induction from real life contexts and application to different
contexts, that were introduced into the tradition of bianshi problems.
Task re-design
In a structured set of bianshi problems two types of "contextual variation"
were proposed "inductive bianshi" and "application bianshi". These two types of
variation should play a role to extend and enrich the use of mathematic (structural)
variation by appropriating relevant real life contexts. The use of inductive bianshi
begins with a real life context in which the established concepts are carefully
embedded and unfolded in a set of problems that leads to the new concept to be
established (the deepening bianshi); and using application bianshi provides new
contexts (even created by students) for students to connect (or apply) different
acquired concepts (the widening bianshi). Examples associated with this approach to
the division of fractions are reported by Sun (2007) and Wong, Lam, Sun, & Chan
(2009).Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Closing remarks
The two cases of the last section have been developed independently by the authors in
Italy and Hong Kong before having the occasion to discuss with each other. In both
cases a task re-design had been started to answer needs coming from school practice
and from theoretical and pedagogical backgrounds, on the one hand, and prompts
from the other cultural tradition, on the other hand. In particular, in both cases the
needs for change (questions C1 and C2) were in the foreground, because of the
awareness that existing practice did not meet the intended curriculum expectations. In
both countries (question E1) collaborations between researchers and teachers were
realized, although with different modalities (see below).
In general, the Chinese literature on word problems as tasks is focused on
teaching, e.g. the preparation of good textbooks according to the intended curriculum
and the study of effective lessons which are structured around textbooks, whilst the
Western literature on word problems as tasks is focused on learning, i. e. the analysis
of difficulties met by students to interpret the task or to make sense of the problem
situations in mathematics. These differences have effects on the researchers' and
teachers' conception of task design (and re-design). For instance, Chinese teachers
generally have limited space to re-design tasks due to the fact that Chinese curriculum
evaluation (exam) is unified by government and curriculum content is required to
follow strictly the unified standards and the unified textbooks. In Italy teacherreasearchers have more freedom to design tasks and to devote some sessions to pilot
teaching experiments like the ones mentioned in this paper. In both cases, the
culturally rooted pedagogies were essential to re-design tasks: in Italy teachers
disassembled Chinese tasks to introduce collective discussions orchestrated by the
teacher; in China induction from real life contexts was put within the variation
scheme.
The questions C3 and E2 are surely the trickiest to be addressed. We give
some elements to support this claim, although it is not the scope of this paper to
address the general comparison of cultural traditions that encompasses the issue of
word problems as tasks (for a general discussion, see Xie & Carspecken, 2008).
Word problems have been part of the historical development of mathematics
in China and in other parts of the word as well. As Gerofsky (2009) claims, nonrealistic word problems are a written and pedagogical genre that expressed generality
through exemplification and have played an important role in all the pre-algebraic
societies. From these shared roots two cultural traditions developed in different ways.
In China the superb development of Algebra (maybe the main contribution of Chinese
scholars to mathematics) was not accompanied by a parallel theoretical approach like
Euclid's one; in the West the assumption of Euclid's Elements as the paradigm for
mathematics development gave for many centuries a different status to practical
mathematics (e.g. commercial arithmetic) from where the Western tradition of algebra
was later nurtured.
Mathematization became a key point in both traditions, although in different
periods. The focus on mathematization started in the West at the beginning of 19
th
century when the teaching of natural sciences was introduced systematically in
schools and gave rise in 1983 to the International Community of Teachers of
Mathematical Modelling and Applications that addressed criticisms of stereotyped
word problems and influenced the development of mathematics curricula,
(http://www.icmihistory.unito.it/ictma.php). The expansion to China is more recent Theme E - M. G. Bartolini Bussi, Xuhua Sun & A. Ramploud
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although Chinese educators were exposed to Western influence for decades (e.g.
Dewey, Smith, Freudenthal, see Wang, 2013, p. 38 ff.).
Hence, task re-design considered in this paper took place in different cultural
traditions. It had in both cultures the advantage of respect the local pedagogical and
theoretical roots and to exploit perspectives from the other culture. Our attitude was,
in both cases, similar to the one described by Jullien (2008) in the case of
philosophical dialogue between China and the West:
This is not about comparative philosophy, about paralleling different conceptions,
but about a philosophical dialogue in which every thought, when coming towards
the other, questions itself about its own unthought (Jullien, 2008, p. iii, our
translation).
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Scaffolding Tasks for the Professional Development of
Mathematics Teachers of English Language Learners
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This article outlines a design framework for classroom exemplars
to be used in the professional development of mathematics teachers of
English Language Learners. This framework shapes activities built
around mathematical practices to scaffold student engagement in
interactive tasks that foster their emerging autonomy. Empirical results
from applying this framework to design teacher apprenticeship is
reported. Data includes both professional development institutes and
instructional coaching cycles. Results suggest trajectories for teachers'
shifting understanding of conceptual, academic, and linguistic goals as
they appropriate a pedagogy of promise that fully develops the potential
of all ELLs.
Keywords: English language learners, scaffolding, interaction, lesson
design
Introduction: Challenging and Supporting English Language Learners
In the United States, English Language Learners (ELLs) are a rapidly growing
population, increasing by 51% in the past decade. Policy at the federal level has
positioned ELLs through a deficit lens as "Limited English Proficient". Mainstream
pedagogical approaches have remained simplified and simplistic, emphasizing
vocabulary terms taught atomistically. Within the high-stakes environment of
standardized assessments, simplification and accommodation for ELLs might be
necessary and appropriate (e.g., Sato, Rabinowitz, Gallagher, & Huang, 2009). The
classroom environment, however, offers broader and more varied opportunities for
students to learn important mathematics with use value beyond the classroom and to
interact with teachers and their peers. Just as ELLs need support in meeting the
challenges of conceptual understanding, procedural fluency, and language
proficiency, their teachers also must navigate national shifts in academic needs,
emphases, and practices. Rather than lowering the cognitive demands of tasks
(Henningsen & Stein, 1997), mathematics teachers must find multiple approaches to
provide ELLs with temporary support as they engage with mathematical ideas and
with their classmates and they develop their autonomy.
As the Quality Teaching for English Learners (QTEL) intiative at WestEd,
we engage teachers in professional development workshops and cycles of
instructional coaching through a whole-school model. QTEL employs a pedagogical
design framework across multiple disciplines. Our work entails the design of fully
articulated lessons which use scaffolding tasks, activities that invite and structure peer Theme E - Haiwen Chu
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support to develop students' independent abilities. Lessons form the basis of sitebased workshops for teachers. Teachers then apprentice in cycles of disciplinary
coaching. We nurture teachers' growing expertise in setting conceptual, academic,
and linguistic goals and their effective planning and implementation of lessons that
challenge and support ELLs.
This article is organized as follows. First, I define classroom-based,
interactive scaffolding tasks. I then describe the interlocking sets of principles that
guide the design and sequencing of these tasks into coherent lessons. Third, I report
empirical findings from practitioner research conducted during the first of a three-year
partnership at two secondary schools. I conclude with directions for further research
and development based upon these design principles.

Defining classroom-based, interactive scaffolding tasks
Classroom-based, interactive scaffolding tasks are drawn from research on second language acquisition. Although these tasks are compatible with mathematical tasks (e.g., Henningsen & Stein, 1997), I focus on meeting the specific needs of ELLs for pedagogical scaffolding and the authentic use of language in interaction.
Ellis (2003) frames the analysis of tasks along five dimensions: 1) goals, 2) input, 3) conditions, 4) procedures, and 5) predicted outcomes as product and process. Goals define general purposes and target competencies. Input and conditions are linked: input is the information given including the modality (e.g., oral or written descriptions, or mathematical representations), while conditions are how information is either split or shared among students. When information is split there is an information gap (Gibbons, 2009). Students possess or are given pieces of information which they must put together through communicating with one another in order to complete the task. Procedures give students discourse moves and participation formats, such as working in pairs, taking explicit turns, or using specified language. Predicted outcomes include products such as materials students will write or draw and the linguistic or cognitive processes the task is intended to engender in students.
The QTEL approach emphasizes not just individual tasks but repeatable tasktypes with similar structures. Through regular participation, ELLs gain familiarity with the structure of a task-type, and therefore shift focus away from following instructions toward understanding new concepts (Walqui & van Lier, 2010). For example, the Compare and Contrast task-type has the goal of identifying similarities and differences between two mathematical objects or situations. As inputs, students are given a matrix as a graphic organizer. The two columns are headed by descriptive titles and the three to six rows are labelled with focus questions. The conditions are split or shared. One way to split information is to have one student report to another as an expert on a particular case. Alternatively, students could share information, going back and forth as they fill out cells of the matrix. Procedurally, students take turns filling out the matrix, orally stating what they are writing down. Once the matrix is complete, students take turns orally pointing out similarities and differences, using the appropriate formulaic expressions such as, "One difference between these two functions is..." Finally, students write a summary of key similarities and differences. The predicted product includes a completed matrix and summary statements of key similarities and differences. The predicted process includes noticing similarities and differences and expressing them orally as well as in written form.
These five analytic dimensions more fully specify the task-types described by Swan (2007), which emphasize goals and predicted processes. These task-types Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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include: classifying mathematical objects, interpreting multiple representations, evaluating mathematical statements, creating problems, and analyzing reasoning and solutions. A "classifying mathematical objects" task provides students with multiple mathematical objects to sort, either by excluding an "odd one out" or by placing cards containing the objects into a table with given headings for rows and columns. A "multiple representations" task has students match cards containing tables, graphs, and equations. These tasks specify inputs.
Focusing attention on the conditions and procedure ensures that English language learners receive peer support through structured interaction. For example, we use a task-type called "Sort and Label". We stipulate the condition that knowledge of what is written on different cards, while ultimately shared publically, is rationed out sequentially. Information is gradually revealed to the whole group as students take turns drawing one card at a time from a stack, reading out loud or describing what is on that card. Students use targeted formulaic expressions to offer and justify tentative groupings. This procedure distributes participation more evenly. Patterns emerge as cards are placed on the table in an order that is not predetermined. Without this condition, both students and teachers will shuffle around cards wordlessly, with only a few stating reasons after the fact. Further, this task-type can be more cognitively demanding than "classifying mathematical objects" because the categories are not given in advance, and must instead by devised, discussed, and agreed upon by students working as groups.
These five analytic dimensions provide explicit invitations for ELLs to engage with mathematical concepts and procedures as they participate fully in classroom interactions. For the remainder of this article, I use "task" as shorthand for these classroom-based scaffolding tasks that require, specify, and support peer interaction.
Design principles for tasks, lessons, and units
The design of tasks is guided by a framework on three different levels. First, five Principles for Quality Teaching of English Learners address general pedagogical features of the classroom environment. Backwards design emphasizes that all planning must begin by articulating conceptual, academic, and linguistic goals. Finally, an architecture of three moments assists teachers in deconstructing broad goals into connected intermediate objectives that flow together smoothly. This framework thus provides nested layers, including outcomes with the Principles, a process with backwards design, and an architecture with three moments.
Principles of Quality Teaching for English Learners
Five Principles guide the design of instructional experiences for students: 1) academic rigor; 2) high expectations, high support; 3) quality interactions; 4) language focus; 5) quality curriculum (Walqui & van Lier, 2010). In work designed to last three years, each year focuses on different Principles. During the first year, our whole-school coaching model highlighted three Principles: academic rigor, quality interactions, and language focus.
Academic rigor considers the extent to which students acquire deep disciplinary knowledge, use higher order thinking skils, and develop central and generative concepts and skills. This aspect maps well to the construct of the cognitive demand of a mathematical task (Henningsen and Stein, 1997). Within workshops, Theme E - Haiwen Chu
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teachers have engaged in sorting tasks by cognitive demand and then constructing variations that amplify the level of academic rigor.
Quality interactions include both interactions between the teacher and students as well as those between students as peers. These interactions must be sustained and reciprocal so that the teacher is not the sole authority who asks questions and then evaluates students' responses. Rather, students should respond to each other directly, elaborating on their own ideas, qualifying or extending them, and sharing responsibility for negotiating validity. As they co-construct new understandings, students generalize, evaluate, and connect their ideas to each other, reflecting and revising each others' ideas. These quality interactions take place not only in whole-class discussions when students or groups share work, but are infused into students' participation in tasks working in small groups or pairs. This Principle is embedded in the specification of the inputs, conditions, and procedures associated with tasks.
Further, teachers must sustain a language focus by providing students with opportunities to use disciplinary language authentically. Therefore, teachers need to have pedagogical content knowledge of language to provide students with clear and purposeful explanations of the metalinguistic knowledge that will assist them in completing a task, such as false cognates or mathematical language functions. Further, language can be viewed as performance, including disciplinary subgenres and language functions such as proving, providing counterexamples, and generalizing. Building on an approach of message redundancy, teachers should not simplify the language associated with a task, but rather amplify through extralinguistic and paralinguistic cues. Finally, in terms of correctness, teachers should judiciously select feedback focusing not on perfect usage or grammar, but on language production that meets the goals of the task (cf Moschkovich, 2012).
Backwards design
Consistent with the Understanding by Design framework (Wiggins & McTighe, 2005), the design of scaffolding tasks begins with identifying goals at the level of the lesson or unit. To meet the needs of English language learners, it is essential to clearly identify not only disciplinary or conceptual goals, but also academic and linguistic goals (Walqui & van Lier, 2010).
Conceptual goals emerge from the discipline of mathematics, and are often associated with the conceptual understanding and procedural fluency that underpin teaching mathematics for understanding (e.g., Kilpatrick, Swafford, & Findell, 2001). Academic goals are generative and span multiple school disciplines. These usually require higher order thinking: generalizing, synthesizing, and comparing and contrasting. These academic goals are aligned with both the National Council of Teachers of Mathematics Process Standards as well as the Standards for Mathematical Practice from the Common Core State Standards in Mathematics. For instance, to "model with mathematics" students need to engage in generating, applying, testing, and revising mathematical representations as they relate to real-world scenarios.
Linguistic goals can be considered on two levels. At a broad level, each unit, lesson, and task has specific language functions or genres as its objectives. For example, comparison and contrast is a language function that applies not only to mathematics but any academic discipline. Providing counterexamples is a language function that is more specific to mathematics. This approach to language views proof, for instance, as a specific genre with its own rules, conventions, and structures Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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about which students need explicit instruction. Further, the genre of proof itself has subgenres: a proof by contradiction reads differently than a constructive proof or an existence proof. These differences can be understood in terms of language functions.
On a narrower level, specific swatches of language are necessary to accomplish various language functions. These formulaic expressions are used not as individual words but flexible grammatical structures. For example, "9 is odd, but not prime" is an instance of a formulaic expression useful for giving counterexamples. The expression is like a mathematical formula in that different objects or predicates can be substituted into the positions marked in italics. ELLs in particular need explicit instruction about the formulaic expressions appropriate to mathematical language functions. It would be difficult to devise counterexamples and to communicate them to the classroom community without these linguistic structures.
While it is possible to articulate and define these different goals separately in developing teachers' expertise, in well-designed instruction the goals converge and support each other. Academic goals are the generalizations and transfer of disciplinary goals, and language is a medium for both. Focusing on goals and increasing their challenge is essential in expanding teacher expertise. Setting these goals and objectives for units and lessons is a means of specifying the more general Principles of academic rigor, quality interactions, and language focus.
Lesson architecture in three moments
A typical "traditional" lesson sequence is explanation, example, exercise (Swan, 2007). The teacher gives a general explanation, demonstrates a worked example, and then students engage in repetitive practice of the target procedure. Curricula based upon NCTM Standards such as the Connected Mathematics Project focus on a single, central problem set in a real-world context and follow a "three phase" model: Launch, Explore, Summarize (Lappan et al., 2009). In the Launch phase, students are introduced to the problem and certain key contextual features or mathematical relationships can be explained. Students work in small groups to solve the problem using their own methods in the Explore phase. In the Summarize phase, the teacher orchestrates a whole-class discussion in which different solution methods are publically shared, compared, and contrasted.
By contrast, an architecture in three moments provides a more flexible structure: 1) preparing learners, 2) interacting with the concept, and 3) extending understanding (Walqui & van Lier, 2010). When directed toward a mathematical problem set in a real-world context, Preparing, Interacting, and Extending are compatible with Launch-Explore-Summarize. The more flexible architecture of three moments, however, offers three additional benefits: broader notions of prior knowledge and explicit attention to transitions from everyday to academic language, more flexibility in terms of building students' procedural fluency with embedded opportunities for reflection and interaction, and a clearly delineated, more varied set of options for extending understanding.
The Preparing Learners moment has three possible functions in the lesson. First, it articulates a focus on key understandings for the lesson. Second, the tasks bring to the surface students' prior experiences and knowledge with the objective of narrowing these contributions toward the lesson objective. Finally, the teacher can introduce essential understandings as reflected by key vocabulary terms, presented in context. For example, students can engage in a Think-Pair-Share. The prompt is kept as general as possible to appeal to students' personal experiences rather than their Theme E - Haiwen Chu
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mathematical opinions. For example, students might be asked to tell a story about how they had to balance something, in preparation for a lesson about the arithmetic mean as a balancing point. Students have a few minutes to think individually, before they take turns with a partner sharing responses. The teacher then leads a whole-class discussion, calling individuals to share what their partners said. The teacher then summarizes these experiences and connects them explicitly to the mathematical topic of the lesson. In contrast with the Launch phase's focus on a a single mathematical problem set in a real-world context, other tasks, such as ones that involve sorting mathematical objects or representations, function well in the Preparing moment.
As students are interacting with the Concept, they engage in three processes: deconstructing, reassembling, and connecting different aspects of the central concept of the lesson. Although compatible with the "Explore" phase of investigating a central mathematical problem, this moment can address procedural fluency. For example, students work in groups of four to carry out an Algorithm in Four Steps. In this task-type, students play the roles of different steps of a procedure, such as finding the slope of the line between two points. After completing a case or problem, students rotate roles so that each gets a chance to play each of the four steps. In contrast with the typical "exercise" part of a lesson, students are collaboratively engaged in interaction structured to heighten awareness of the interdependence of steps.
Another suitable Interacting task-type is a Jigsaw Project. Students convene in expert groups to learn about a particular case or solve a problem, becoming experts and answering common focus questions that cut across the different cases. These focus questions cannot just be factual and isolated, and should cohere to require students to negotiate, discuss, and select in expert groups. Students then return to base groups to report their findings. Base groups use a graphic organizer similar to a Compare and Contrast matrix. The focus questions allow the base group to see connections, such as different proofs of the Pythagorean theorem or different realworld instances of unit rate.
Finally, as the lesson moves toward Extending Understanding, students are invited to work in three ways: to apply the concept to novel real-world applications, to connect to other concepts or algorithms previously studied, or to re-present their

understanding in new genres and formats. This moment also includes having students
create their own problems and solve those created by their peers (cf Swan, 2007).
Reflecting on one's own process of thinking and the relative usefulness of different
representations is also appropriate in this moment. The Collaborative Poster task-type
has students work in groups of four to create a poster, with the condition that each
student uses a different color marker. A good prompt requires students to make a
choice as a group, such as only choosing one type of representation from among
tables, equations, or graphs, in order to compare two different linear functions.
Tracing teacher engagement and growth trajectories
After the first of three years of whole-school coaching and professional development
across the disciplines at two secondary schools, three phases in growth among
mathematics teachers have begun to emerge. First, are shifts in teachers' professed
beliefs, priorities, and approaches. Next, teachers adopt tasks wholesale during
coaching. Third, teachers have begun to adapt tasks in planning units and lessons.
When first working with the design framework and tasks, teachers respond
most frequently and extensively to three features. First, teachers express appreciation
for how tasks specify clearly outlined roles for students and how the interaction is Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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structured. They contrast this approach with the "bare" problems provided in
textbooks or other curriculum resources, or generic roles for collaborative work (e.g.
recorder, materials manager, etc). Second, teachers respond positively to notions of
language which look beyond vocabulary toward language functions and linguistic
goals in lesson planning. Many teachers say that they had not been given other tools
beyond generic state-based language proficiency standards, or that they have
previously focused only on definitions-centered vocabulary. Finally, within the
context of planning, teachers focus on the Extending moment and developing
flexibility in selecting from the multiple tasks appropriate to that moment. This focus
on Extending is particularly important given the lack of closure that is often
characteristic of many mathematics lessons in the United States.
Within coaching cycles, initially teachers often place undue focus on tasktypes as the end goal rather than as a means for achieving outcomes such as quality
interactions. This emphasis is perhaps a consequence of previous district-wide
mandates which evaluate teachers based upon implementation of specified strategies.
Modeling specific tasks in coherent instructional sequences by the coach facilitates
both teachers building belief in their students and their technical knowledge for
implementing specific tasks and transitions between tasks. A key insight that many
teacher reach through practice is that language development is not spontaneous but
occurs within the context of planned scaffolding.
Simultaneously, teachers see how the language modeled for students
facilitates their conceptual development, and they begin to select and model
appropriate formulaic expressions and genres for their students. Teachers begin
develop their capacity to enact the Principle of language focus as they become aware
of how many mathematical tasks need to be further unpacked for ELLs. A common
example is around the prompt to "summarize". Students typically produce a narrative
recount of the procedure, or a laundry list of responses to specific questions, rather
than a coherent summary oriented toward goals and methods that generalize. Once
they have unpacked the complex processes and structures involved in summarizing,
teachers can apply the process to other common but complicated commands, such as
"explain" and "justify".
Teachers' initial misfires reflect their emerging understanding of the
rationale for procedures in task-types as connected to more general goals. Often,
teachers create opening prompts that are too narrow, or after students share responses
do not efficiently focus students' contributions toward the key ideas that connect
directly with the mathematical topic. Successful openings require both pedagogical
content knowledge and implementation skills. For example, in a lesson on solving
equations by undoing, one teacher gave an example response to a Think-Pair-Share
prompt a story of making a mistake with a baking recipe. Because this modelled
example would require redoing rather than undoing, many examples students
subsequently provided did not move toward the idea of inverse operations, and the
intended question of the order in which operations would need to be undone. By
engaging in reflection during coaching, teachers produce prompts that start more
broadly and focus more narrowly. Through coaching, teachers have the chance to
revise their lessons the same day. They thus can examine and reflect upon how
changes in the clarity of directions or the inputs or conditions of the task affect
student outcomes.
The refinement of teachers' choices can be traced in the quality of the focus
questions that they generate for Jigsaw Projects and Compare and Contrast Matrices.
Generic graphic organizers for comparing and contrasting two cases may be Theme E – Haiwen Chu
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organized like a Venn diagram and do not have focus questions. Similarly, teachers
often initially misunderstand the rationale for focus questions, omitting these
questions, asking questions that are too general or do not apply to individual cases
(e.g., "How are they the same?"), or barraging students with recall questions that do
draw focus toward key ideas. Over time, teachers have developed questions that are
both better phrased individually as well as coherent and well-sequenced as a whole.
Indeed, with more experience with this design framework, teachers begin to
engage in a form of "task problematization" (Sierpinska, 2004). Not only are there
possible variations on the mathematical question but the other aspects of the design of
the task, including inputs, conditions, and procedures. Teachers begin to reflect on
the flexibility in choosing similar but subtly different task-types as appropriate to
different moments in the lesson and trade-offs between slightly different goals. In
particular, teachers gravitate toward the tasks that involve algorithms, whether in the
format of a group task as an Algorithm in Four Steps or in Comparing and
Contrasting two different algorithms, such as for computing the median. Reflective
coaching discussions with teachers about algorithms probe the dual demand for
procedural fluency and conceptual understanding. For many teachers, the explicit
modeling provided by scaffolding tasks in which students interact as they carry out
different steps of procedures is an accessible entry point to providing ELLs with
multiple algorithms which they will eventually be able to select from strategically. In
this manner, the scaffolding embedded in well-designed tasks allows teachers to
increase the academic rigor, or cognitive demand, of classroom activity.
Conclusion: Future directions for design and research
These emerging trajectories for teacher growth suggest three areas for further research
and efforts in task design, starting from the level of individual teachers and extending,
through coaching relationships, to the level of groups of math teachers working at the
same school. 1) How do individual teachers engage with different aspects of the
design framework and make connections across different components? 2) How does
this design framework function as a coaching tool to foster teachers' development? 3)
To what extent can this design framework serve as a common language as teachers
collaborate with one another? While the design framework has so far served
primarily as a means to design lessons for the purpose of professional development,
handover would suggest that teacher-created lessons could also be eventually used for
this purpose. Further in depth observational studies of student-to-student interactions
would also be appropriate on the way to evaluating the extent to which shifts in
teachers' practices around task design and implementation affect student outcomes.
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An experience of teacher education on task design in Colombia
GEMAD
Universidad de los Andes, Colombia
Gemad is a working group of researchers, educators and in-service teachers of mathematics affiliated
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We describe an experience in task design within an in-service secondary
mathematics teacher education program in Colombia. Following a model known
as didactic analysis, a team of researchers, educators, mentors and practicing
teachers worked together in designing, implementing, assessing and
reformulating secondary school mathematics tasks. We present here the main
features of the framework on which the program is based, identify some of the
characteristics of the experience lived by trainees, educators and researchers on
task design during the first implementation of the program, and analyse the
trainees' assessment on their own proposals of tasks and on the contribution of
the program on their task design competencies.
Keywords: teacher training; task design; task implementation; crosscommunities
Law sets curriculum autonomy in Colombia. Schools and teachers are fully responsible for
curriculum design and development in all areas. Schools are expected to produce curriculum
planning for each course and academic period and teachers are usually autonomous for
designing and implementing the lessons they are in charge of. They often do so by producing
what is known as "teaching guides": sets of tasks that they design or copy from different
resources, and propose to students. Most pre-service teacher education programs in Colombia
do not prepare future teachers in task design nor other practical questions; instead, they are
based on theoretical approaches to education.
In this paper, we describe an experience in task design that emerges from an inservice teacher education program in Colombia, known as MAD (Master in Didactic
Analysis). It is based on a model—didactic analysis— that enables trainees to design,
implement and assess sequences of tasks on specific topics for which a constructivist view of
students' learning is assumed (Gómez, 2007). Based on this model, a group of researchers,
educators, mentors and in-service mathematics teachers have worked together in MAD. We
use the term task as "anything that a teacher uses to demonstrate mathematics, to pursue
interactively with students, or to ask students to do something" (ICMI Study 22, 2012, p. 10). Theme E – GEMAD
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In what follows, we describe the main features of the framework on which the
program is based, identify some of the characteristics of the experience lived by trainees,
educators and researchers on task design during the first implementation of MAD, and
analyze the trainees assessment on their own proposals of tasks and on the contribution of
MAD on their task design competencies. In the final section, we reflect on the role of the
different agents in the program.
Framework
MAD is a master's degree in mathematics education for in-service secondary mathematics
teachers. We assume a functional view of school mathematics in MAD. This vision puts the
focus on the usefulness of the mathematical concepts for solving problems in a variety of
contexts. Students are expected to use their mathematical knowledge for this purpose. They
are expected to develop their own cognitive strategies, manage different representations of
the mathematical concepts, choose the best solution strategies, argue about their decisions
and communicate fluently their thinking processes. This functional view of school
mathematics is coherent with a constructivist approach to students' learning and can be
implemented with different pedagogies. MAD does not explicitly promote any of them, since
each trainee and his context impose their own restrictions. Nonetheless, there are some
implicit methodological principles: it is considered that the good tasks are those that promote
the active implication of students, imply the development of strategic knowledge for problem
solving in a diversity of contexts, and require that students make decisions and justify them.
MAD is based on a model known as didactic analysis (Gómez, 2007; Lupiáñez,
2009). This model is a conceptualization of the activities that the teacher has to do in order to
design tasks that seek to promote students' learning on a mathematics topic. It is organized
around four interrelated analyses: subject matter, cognitive, instruction, and performance
analysis. The didactic analysis begins with the identification of the student's knowledge for
the topic at hand (see Figure 1). With this information, and taking into account the global
planning of his course, the teacher determines the mathematics content he wants to work on
and the goals he wants to achieve (Box 1 in Figure 1). The next step involves the subject
matter analysis (Box 2), in which the teacher stresses the relationship among concepts,
highlights its multiple representations, and distinguishes the phenomena from which they
emerge. This information is used in the cognitive analysis, in which the teacher describes his
hypothesis about how students construct their knowledge. The cognitive analysis involves the
establishment of learning expectations, and the identification of the skills, reasoning, and
strategies necessary to achieve those expectations, and of the difficulties, mistakes and
obstacles students might face. This information allows the teacher to carry out an instruction
analysis: the identification and description of the tasks that can be used in the design of the
teaching and learning activities that will compose the instruction in class (Box 3). During the
implementation, these tasks should mobilize students' knowledge in order to generate
cognitive conflicts and promote the construction of meaning using the materials and
resources available (Box 4). In the performance analysis the teacher observes, describes, and
analyzes students' performance in order to produce better descriptions of their current
knowledge (Box 5). After this process, the teacher can review the planning in order to
improve the sequence of tasks for future implementations. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Figure 1. Diagram of a didactic analysis cycle
In this approach, trainees are expected to develop a deep enough knowledge of the

topic so that they can support the choices and decisions they make for their lesson plan (Charalambous, 2008). This is a topic-specific knowledge that trainees are expected to develop by performing a series of activities during their training and contributes to the development of their didactic knowledge (Box 6).
Didactic analysis is a cyclic process in which trainees analyze a school mathematics topic with the purpose of designing tasks that provide the learning opportunities required for students to achieve the learning expectations. Trainees make decisions in different moments and with different purposes. When they describe in detail the topic from the mathematics point of view (the concepts and procedures involved, the forms of representing those concepts and procedures and the ways in which the topic organizes the phenomena that give sense to it), they produce and organize information about the topic that allows them to make decisions about those aspects that they consider relevant, about how to formulate and specify the learning expectations, about the capacities that can be used for characterizing those learning expectations, and about the mistakes that students can make when solving tasks related to the topic. The information that trainees produce form those decisions are the basis for further decisions about their anticipations about how students' learning can develop when they solve the tasks (Gómez & González, 2009). This process is based on a procedure that allows trainees to produce the learning paths of the tasks. Tasks' learning paths are a useful tool for assessing the effects of reformulating or extend the original tasks. For instance, trainees can make decisions about the material and resources that can be more effective for achieving the learning expectations, when they analyze the implications of their use in the learning paths of the new tasks. Based on the information about the capacities that characterize the learning expectations, trainees can establish the complexity of the tasks proposed and make decisions about how those tasks align with students' previous knowledge and about how to sequence the tasks. Trainees can also make decisions about the most
Identification of student's previous
knowledge
Establishment of contents and
goals
Performance
analysis
1
2
5
6
Subject matter
analysis
Cognitive
analysis
Instruction
analysis
Tasks design
3
Tasks implemetation
4
Teachers'
knowledgeTheme E - GEMAD
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effective ways of grouping the students and about how to foresight the teachers' performance when students begin solving the tasks and encounter difficulties. In summary, didactic analysis provides trainees with a systematic procedure for analyzing a school mathematics topic and sequentially making decisions that enable them to deepen in the different aspects of the topic and design and assess the tasks with which they pretend to contribute to the achievement of the learning expectations.
Since the information that trainees produce with the didactic analysis is complex and plentiful, when they make decisions, trainees might give priority to some decisions over others. For instance, they might focus on the treatment of the students' mistakes, so that their decisions will focus on ways of facing students with those mistakes and helping them to overcome them. For instance, they might concentrate on the search of representations, resources and possible teachers interventions that can help students overcome their mistakes and difficulties. In the empirical experience that we describe below, we show the concrete decisions that were made by a group of trainees in MAD.
MAD is set up from a social perspective of the trainees' learning (Gómez & González, in press). Trainees are organized in groups of 4 or 5 teachers. Each group has a mentor that accompanies it during the program's two years duration. Each group selects a school mathematics topic on which it will work during the program. The program is composed of 8 modules, two modules per semester. Each module begins with one week of face-to-face instruction in which the educator in charge of the module presents its theoretical basis and introduces the four activities that the groups have to carry out. Each activity spans over two weeks and requires the groups of trainees to analyze or produce information on their topic from a given perspective or with a given purpose. For instance, in an early activity, the groups produce the information concerning the representation systems of their topic. Later in the program, they analyze, in another activity, the role that the teacher had during the implementation of the tasks. For each activity, the groups produce a draft of their work at the end of the first week. They then receive comments on this draft from their mentors, and produce a final version of their work that they present to their pairs, the educators and mentors at the end of the second week.
Experience
In this paper, we report on MAD's first implementation that took place during 2010 and 2011. The 26 teachers that participated in MAD were working in public and private schools of Bogotá (Colombia) and its surroundings. They were organized in 6 groups that worked on the following topics: integers, linear equations (2 groups), straight lines in the plane, and trigonometric ratios (2 groups).
The design, analysis and selection of tasks are processes that span over the whole program. It begins with a first selection of tasks that is refined and improved with new ideas and analysis proposed by educators and mentors. The different activities of the modules structure this process. Once the groups have a tasks sequence that is ready to be taken to class, the groups implement, collect information on its implementation, analyze that information in order to assess the sequence's design and implementation, and make improvement proposals for future implementations. In what follows, we show a summary of the process for the specific case of Group 5, whose topic was trigonometric ratios. This group designed a sequence of five tasks distributed in 12 lessons. The first selection of tasks was Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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guided by activities in which the group set up the learning goals, the contents they expected to cover, the materials they wanted to use and the context (personal, scientific, etc.) in which they wanted to place their proposal. This group focused the design of their tasks in the use of materials and resources, some of which were elaborated by the group itself. The group used the materials and resources for bringing together the tasks characteristics that they considered important for students' learning:
Materials and resources play an important role in our task sequence because they allow us to motivate students in working with mathematics; they facilitate the achievement of the learning expectations; they promote mathematical communication and the construction of arguments; and they put into play different systems of representation.
Their pedagogical decisions focused in two aspects: (a) the grouping of students and (b) the communication in class. They proposed to use the tasks with different types of students' groupings: heterogeneous groups of three students (with high, medium and low achievement), big group, and, less frequently, individual work. In order to promote the classroom communication, they decided to use the following strategy: at the beginning of each task, the teacher shares its goals; then he induces students to create their solving strategies in small groups; the groups present their strategies and argue in favor of them to the whole group; once the task is finished, the teacher gives students follow-up and feedback on their performance. These decisions were guided and founded on their functional view of school mathematics and on the group's aims of contributing to the development of students' argumentation and justification competencies.
In what follows, we show in detail the design process of one of the tasks, named The streetlight height. In the following excerpt, Group 5 describes the task's features based on the subject matter analysis they have previously realized.
We expect students to find the streetlight height by using trigonometric ratios, without direct measures. The task covers a conceptual content that includes elements and properties of right triangles and trigonometric ratios. It involves also some procedures: (a) identifying regularities and patterns, (b) formulating equations, (c) using the functional language trigonometric ratios, and (d) situations solving. The task design includes working guides, goniometers made of set squares of 45° and 60°, protractor, calculator and a metric strip. The task refers to a personal situation.
In MAD, once the groups make the first task proposal, educators and mentors introduce new elements of analysis. That is the case, for instance, of considering the concrete capacities that can be activated with the task or the mistakes that students can make when solving the task. The groups characterize the task in terms of these new elements. For example, once Group 5 produced a list of 35 capacities and 12 mistakes for their two learning goals (that we do not have space to include here), they produced a table in which they related the learning goals, the tasks in their sequence, the capacities that each task could activate and the mistakes that students could make when solving each task. Table 1 shows an excerpt of this analysis for Group 5.
Goal Task Capacities Mistakes
2 Streetlight 1, 7, 3, 8, 12, 14, 17, 33, 35 2, 5, 7.3
Table 1. Relationships among learning goals, task, capacities and mistakes Theme E - GEMAD
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This information led them to modify the task's wording and to determine how to implement the task in class:
Before starting working with the task, the teacher explains the use of the goniometer. In the first phase, the teacher asks students to go the location in the school where the streetlight is and to calculate its height with the instrument. The goals of this phase of the tasks are twofold: (a) that students recognize the use trigonometric ratios for measuring unreachable distances and (b) that they connect the elements of the instrument with the particular situation at hand in order to represent it and calculate the distance. The students are asked to record their observations in the working sheets provided, including the steps that they took for calculating the length and a graphical representation of the situation. They produce a poster to share their work with the class group.
Once they produced the final design of the tasks sequence, the groups implemented them in class. The following are two excerpts of the balance that Group 5 made of this phase of the process.
One of the minor changes that we made during the implementation of the tasks sequence concerned the time foreseen for each session. For most tasks, the time required was greater than we expected. We can claim that the Streetlight height task was effective as it was designed because we verified that students activated the capacities that we expected. Furthermore, we observed that they also activated capacity 17 that was not expected by us. On the other hand, we established that capacities 12 and 35 were explicitly activated on a given moment of the task's development, but that they were also present along the task, since students permanently used the trigonometric vocabulary and verified the relevance of their results, when measuring and making calculations.
On the basis of this kind of analysis, Group 5 decided the following improvements for the Streetlight height task:
(a) to include instructions for the construction of the goniometers; (b) to incorporate an activity for measuring lengths that can be found directly and to use the trigonometric ratios to corroborate the results; (c) to show the diversity of theodolites that can be found, with a brief explanation of each one; (d) to ask students to select the theodolite that they think is best suited for each situation, and (e) to ask questions that can lead students to look for tools that are different to those proposed and to include new strategies of solution.
Trainees' assessment of the program
In the final part of the program, the groups of trainees that participated in MAD performed a SWOT analysis (Strengths, Weaknesses, Opportunities, Threats) of their work and made a personal assessment of the program and their participation in it. We reviewed these analysis looking for the common themes that characterized what the groups of trainees appraised as the most salient features of their work and the most important influences of the program in their capabilities for designing, implementing and assessing tasks. We consider that the trainees' claims represent those features of MAD and their work that highlight the differences between what they usually do in class and what they actually did when planning, implementing and assessing their tasks during the program. We found several themes that were mentioned by most groups. In what follows, we identify and exemplify the most frequent ones. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Features of the work
All groups centred the assessment of their sequence of tasks in terms of its contribution to the achievement of the learning expectations and the overcoming of the students' errors and difficulties. They also claimed that the designed tasks were adapted to the context. The Colombian curriculum guidelines underline the importance of problem solving in context and, hence, of tasks that lead students to solve and interpret mathematical problems in a variety of situations. Trainees claimed that one of the salient features of their tasks was the fact that they were mathematical problems set in varied non-mathematical contexts that contributed to their students' proficiency in problem solving. As Group 1 explained: "the use of tasks in context favored the achievement of the learning expectations proposed because the situations in the tasks were close to the everyday life of the students." Similarly, all group of trainees mentioned the importance of the use of materials and resources (i.e., Cabri, Geogebra, Hands on Equations) in their tasks design and implementation. Their statements make us think that they do not usually introduce those resources in their teaching: "when designing the materials with which we developed our task sequence, we found that they have a great potential for other topics and school grades" (Group 5).
All groups of trainees mentioned the importance in their tasks' design and implementation of using multiple ways of promoting collaborative work in class. They recognized the benefits of having students working in pair or groups and of generating class discussion among them. Some groups of trainees also mentioned the relevance of foreseeing the teachers' reactions to the students' performance in class, particularly to students' mistakes. The assessment made by Group 1 sustains such claim: "We acknowledge the benefits of the groupings proposed for students' work in class. For instance, as a consequence of the interactions produced, the students were able to strengthen their argumentative capacities for validating their results." Similarly, Group 2 claimed that "the tasks sequence involved a methodology that supports constructive learning of individuals and groups because it contributed to create a ZPD and strengthen the establishment of agreements when taking decisions concerning the challenges that were proposed."
The groups of trainees also mentioned some of the problems and deficiencies of their tasks. The most common shortfall referred to their mistakes when foreseeing the time required for implementing the tasks as mentioned by Group 5 above. On the other hand, some

groups recognized that their students did not understand properly the wording of some of
their tasks or that the tasks did not generate the student's performance that they were
expecting. They recognized that, in some cases, they incorrectly assumed that the students
had the previous knowledge required to face the tasks. For example, Group 3 acknowledged
that "the wording of the instructions in one of the tasks was another weak point [of our task
sequence]. This situation affected the time required for the task and the understanding that
students developed when solving it." This assessment led them to propose new or modified
tasks for a future implementation of the sequence.
Influences of the program
The groups of trainees highlighted the impact of MAD in their competencies for designing,
implementing and assessing sequences of tasks. Trainees stated that the program provided
them with tools for assessing how the tasks' design could achieve the planned learning
expectations and how the tasks' implementation did in fact achieved them. Group 5 claimed Theme E - GEMAD
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that "the didactic analysis procedure lead us to be really conscious of the importance of
planning a task sequence and assessing its relevance and effectiveness. We know better about
the mathematics that our students should learn and how they should learn them. We
recognize now that students' learning depends on the tasks they solve and that the teacher is
the main responsible of that learning." In other words, the groups recognized that the
program (a) provided them with a better preparation for developing teaching innovations
based on a structured method; (b) encouraged them to reflect on their own practice; (c)
showed them how to track the results of their lessons; (d) questioned their meanings of
knowing and learning mathematics; and (e) motivated them to modify their role as teachers.
The groups also reported that the program led them to introduce several data gathering
instruments (some of them previously unknown to them--like the students' dairies and
observation tables) that allowed them to properly assess the students' performance when
solving the tasks. They also recognized that they did not have enough time for performing
those procedures and analyze the information gathered.
Across-communities
The design and implementation of the program was a joint venture among researchers,
educators, mentors and in-service teachers. Researchers have been working for several years
on the development of the didactic analysis model and on a model for conceptualizing the
trainees' learning of it. Some of the researchers were also educators in the program.
Educators as a team have worked on the design of the program following the models
proposed by researchers. They also implemented the program working hand in hand with
trainees. This collaborative work was set up through a mentoring process in which mentors
(the educators) interacted with their group of trainees weekly. Trainees' work informed
researchers on the framework and principles, educators on their teacher education program
design and implementation, and mentors on their performance. This joint venture has evolved
beyond the program, as this paper shows. Teachers, educators and researchers have created a
working group (that signs this paper), which continues working on mathematics task design
an implementation. Some of the near-future results of this collaboration is a book with the
reports of the groups' work in the program (Gómez, in press), the support of an international
publisher for the publication of a set of teaching guides based on the groups' program's work,
and the teaching of a course in the Colombian mathematics education congress.
MAD has generated several research studies in which researchers, educators and
teachers have also collaborated. That is the case of studies in which we have explored, for
instance, the teachers' learning of specific aspects of the program (Gómez & Cañadas, 2012;
Suavita, 2012), the role of mentors (Arias, 2011), or the impact of the program in institutional
planning (Gómez & Restrepo, 2012).
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Designing Professional Tasks for Didactical Analysis as a research process

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In the present paper, we show a process of designing, assessing
and re-designing (following DBR methodology) professional tasks for
preservice mathematics teacher training for Secondary School, based on
Ontosemiotic perspective for cognition and mathematical instruction
(OSA) and the correspondent reflective analysis about associated
professional practices. Such a process has been carried out during three
consecutive years in the context of the Master's degree for training
teachers of mathematics in Spain. The study shows how the successive
revisions promote growing depth analysis in the teaching school practices
of the future mathematics teachers.
Keywords: Professional tasks, design based research

1. Presentation

In this paper, we show a part of a wider investigation in which we analyze
the design of professional tasks in the teachers' formation of Secondary Mathematics
Teachers. We focus on the role of design based research (DBR) and teaching
experiment (Gravemeijer, 1998) analyzing the planning cycle and redesign of our
training process in successive phases, aiming the growing and building knowledge for
teaching (Zaviaswski & Sullivan, 2011) by future teachers. We explicitly focus
ourselves in recognizing factors that promote the feedback in the design of
professional tasks for development of the of didactic analysis competencies of the
future mathematics secondary teachers. Our intention is that they can develop
sequences of suitable tasks and to be able to re-plan their own designs of school tasks.
This work has been carried in a funded Research Project (Assessing and developing
professional competencies in mathematics and didactics during initial Secondary
Mathematics Teacher Training courses) being the first two authors of this work
members of the team who implemented the course. The professional tasks have been
evaluated and re-designed by the whole research team during the period 2009-2012.
In our study we call professional task those tasks that we propose to the
future teachers in order that they realize didactic analysis and develop their didactic
analysis competencies understood as the ability for designing, applying and Theme E - J. Giménez; V. Font; Y. Marsela Vanegas
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evaluating sequences of learning by means of didactic analysis techniques and quality
criteria. The aim is to establish cycles of planning, implementation, evaluation and
proposals for improvement. It is also assumed that one could identify criteria and
indicators regarding the development of this competence and how it relates to the
other professional competencies required by future secondary school mathematics
teachers. This assumption is related to the question: How might professional
mathematical tasks being designed in order to make best use of the opportunities for
being a teacher as teacher enquirer? (Mason & Johnston-Wilder, 2004).
Our main aim is to investigate how the process of building a sequence of
professional tasks (so called formative cycle from now) promotes and generate
feedback in the development of the didactic analysis competence of the future
teachers within the context of teacher training courses. Such above mentioned
development, it is stated when future teachers incorporate and use tools for the
description, explanation and process valuation of mathematical school
teacher/learning practices. By using our professional tasks design as a design based
research cycle, we also want to improve a teacher as teacher-researcher of their own
practice. We will show later, examples of professional tasks and reflections of the
future teachers on having solved them, that have served to provoke successive
feedback that have allowed to improve the sequence of tasks, and to make it
increasingly effective.

2. First year design. Building a formative cycle

In the frame of the Project of investigation mentioned, we design and
implement diverse formative cycles as teaching experiments for developing
transversal competencies as cittizenship, digital competency, didactical analysis, and
others. In particular, in this presentation we discuss a part of a cycle of formation,
named of "Didactic Analysis "articulated across diverse subjects of the courses. The
development of the cycle has been based from the beginning in considering six big
types of professional tasks:
a) Analysis of practices, objects and mathematical processes.
b) Analysis of didactic interactions, conflicts and norms.
c) Valuation of tasks and classroom episodes using criteria of didactic
suitability or quality.
d) Planning and implementation of a didactic unit in their period of
practices.
e) Analysis and valuation of the suitability of the didactic implemented
unit.
f) Offer of a well-taken improvement of his didactic unit, for a future
implementation. This proposal is realized in Master's final Work.
During the first two types of tasks (a - b) it's expected to appear and discuss
tools for a descriptive and explanatory analysis that serves to answer "what happens
in the classroom and why?" (Font, Planas y Godino, 2010). The analysis and
description of the mathematical activity is realized using the theoretical constructs
proposed for OSA. In this perspective (Godino, Batanero y Font, 2007), mathematical
activity plays a central role and is modelled in terms of systems of operative and
discursive practices. From these practices the different types of related mathematical
objects emerge building cognitive or epistemic configurations among them (see two
internal hexagons in Figure 1). Problem-situations promote and contextualize the
activity; languages (symbols, notations, and graphics) represent the other entities and Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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serve as tools for action; arguments justify the procedures and propositions that relate
the concepts. Lastly, the objects that appear in mathematical practices and those
which emerge from these practices might be considered from the five facets of dual
dimensions. Both the dualities and objects can be analyzed from a process-product
perspective, a kind of analysis that leads us to the processes shown (decagons in
Figure 1).
During the following type of tasks (c - f), we present theoretical tools
(suitability criteria, according Godino, Batanero y Font (2007) for a valuated analysis
serving to answer "what could we improve?" These criteria are as follows: Epistemic
suitability refers to the extent to which the mathematics taught are 'good
mathematics'. Thus, in addition to the specific content of the curriculum the
institutional mathematics on which it is based are also used as a reference. Cognitive
suitability reflects the degree to which the teaching objectives and what is actually
taught are consistent with the students' developmental potential, as well as the match
between what is eventually learnt and the original targets. Interactional suitability
relates to the extent to which the forms of interaction enable students to identify and
resolve conflicts of meaning, and promote independent learning. Mediational
suitability refers to the availability and adequacy of the material and temporal
resources required by the teaching/learning process. Affective suitability reflects the
students' degree of involvement (interest, motivation, etc.) in the study process.
Ecological suitability refers to the degree of compatibility between the study process
and the school's educational policies, the curricular guidelines and the characteristics
of the social context, etc.
Figure 1. Onto-semiotic representation of mathematical knowledge (Godino, Batanero y
Font 2007)
We understand that the study of descriptive and explanatory analysis for a
didactical situation is necessary to argue based valuations (Pochulu y Font, 2011).
Methodologically, the research is mainly qualitative in nature as the purpose
is to describe the development of competence in didactic analysis among aspiring
secondary school mathematics teachers, from the University of Barcelona (Spain).
The data were collected from the video recorded observations, sorting sheets
produced by the teacher trainers and their reflections at the end of the workshops and
using the documentation housed in the Moodle platform (slides, reading material,

tasks and the students' responses to them, and questionnaires and the students' responses to them) and printed material. The samples were intentional. During all these academic years, in general, these students vary in the amount of mathematical

knowledge they have, and they hold certain conceptual biases regarding the teaching and learning of mathematics.
It was selected an initial task in which students confront a short case study about proportional reasoning, using transcripts of a classroom situation. Such an initial tasks (type a; non-theoretical), introduce the students for reading and analysing the classroom example, by using their previous knowledge and beliefs of didactic analysis. During the first task, future teachers did naif comments about a proportion class. It's easy for them to identify mathematical objects but it's difficult for them to recognize all the processes involved in the task. When they analyze interactions, they focus on leadership and teacher interventions (Task b). It's difficult for the future teachers to identify epistemic conflicts and norms. In such first analysis each group of future participant teachers used just implicitly some of the levels of analysis proposed by OSA (described in Font, Planas y Godino, 2010): Analysis of mathematical practices; Analysis of objects and mathematical processes activated by these practices; Analysis of didactical interactions and conflicts; Identification of systems of norms conditioning and making possible instructional process; Valuating didactical suitability of instruction (Font, et al., 2012). In the class debate it was observed that, though every group did not use all these levels of didactic analysis, it is possible to see how the student group as a whole has contemplated the five levels of analysis. The tasks (a-b) were considered fruitful, so they were conserved for new implementations, but they should be completed by means of the 'other voices' technique (Garuti & Boero, 2002).
Later, in the different subjects of the Master, the students realize other analysis of practices, objects activated in the above mentioned professional practices (problem, definition, proposition, representation and argument) and mathematical processes (type a). Observing the analysis realized by the future teachers some difficulties appear: (1) Difficulties to distinguish between concepts and definitions, (2) Duplicity between definitions, propositions and procedures; (3) Duplicity between propositions and thesis of arguments; (4) The description of practices is overlapped by the configuration of objects and by the description of processes, (5) Difficulties to observe and to catalogue mathematical processes, etc.
It also had been designed and implemented tasks (type b), with protocols served to show constructs as cognitive and semiotic conflicts, epistemic obstacle, types of norms, interactive, patterns of models of management, etc. After first year of experience we found that protocols were statics. For the next year it was decided to use videos and corresponding transcripts. After that it was analyzed a class about equations by applying suitability criteria (task type c). The students star by analyzing mathematical practices, objects and processes. Then the teacher develops an example in which it was revised suitability construct. After that the future teachers reflect, improve and refined their analysis by using the notion of epistemic suitability. Nevertheless, it's still difficult for the students to identify some semiotic conflicts.
Next it was proposed a task of planning and further implementation of a didactic unit in their period of practices (task type d). When doing the analysis and valuation of the didactic implemented unit (task type e), future teachers found that their planning was conditioned by the school plans in which they did the practices. As a consequence it was difficult for them to identify the epistemic consideration implicit by the school teacher proposal.
The future teachers had a few autonomy to apply in the design and implementation many learned knowledge. This aspect was considered a difficult problem to solve during redesign process because of institutional framework for the

proposal, which did not deal a selection of schools. The tasks type (e) and (f) are considered activities producing the feedback for future teachers and trainers.
According to task type e, it was found a superficial use of theoretical tools for valuing teaching practices, due because we had short time to discuss after school practices and a need for more discussion about suitability criteria, and analysis based upon previous experiences. This aspect should be promoted in a redesign.
Positive results and some feedback from the first year are explicitly observed (when analyzing task type e). It was decided to redesign task type a, by emphasizing the analysis of processes. Another aspect to consider when redesign is to find so enough rich episodes which serving to propose different typologies to profit a short time available, instead of using different episodes in each task. It was also observed that some of the final practices' works (task type e) and master's thesis (task type f) were found so rich to be considered as episodes to be incorporated in a later redesign processes.
3. Second year redesign. Improving process analysis
It was decided not to do important changes of the cycle itself for the second year Project. As an important example for the redesign, we consider enlarging task type (a) and (b) by using a new video source. In such a new task (type a) it was proposed the observation of three short ways of introducing perpendicular bisector with 12-13 years old students, by observing three different teachers. The main idea is to present a discussion about the different practices, objects and mathematics processes and to introduce a reflection associated to how each of these classes contribute to introduce different kind of epistemic configurations and objects (see hexagons in figure 2) associated to three different definitions.
Figure 2. Epistemic Configuration of classroms for teachin g perpendicular bisector
It was observed that both first and second teachers did classical proposals and management about the content and the classroom. The third teacher proposal is innovative not only because of the management but mathematically as a way of changing the regular use of mathematical content as a change of configuration of practices, objects and mathematical processes by using a non-routine task (Tzur, Sullivan, & Zaslavsky, 2008). The class started by presenting a contextualized problem, driving to the division of a desert in a set of regions. Future teachers observed interpretation processes, communication of didactical and mathematical meanings, etc. Furthermore it appears a reflection about distinguishing complex

processes from simple processes and also a general reflection about the idea of processes itself.
During the second year, the tasks designed had achieved the effect of improving their analysis of practices, objects and mathematical processes and mainly about processes (Font, et al. 2012). In this improvement, it was judged a crucial role of dynamic videotapes to analyze the visualization of professional didactical processes. On the other hand, they were introduced selected episodes of students' from previous years that were considered as a short distance from prospective teachers' perspectives. We still detect that the future teachers applied epistemic suitability criteria, by means of superficial explanations, short justifications, etc. Therefore, it's needed to improve future teachers' justifications about mathematical and didactical quality of their practices as a basis of the second redesign.
4. Third year redesign. Conectness and representativeness
Epistemic suitability criteria explained for years 1 and 2 were basically sustained in the idea of representativeness, understood as a degree, of representation of learned meanings representing relations to referenced meanings. Due to the superficiality of some students' works during the moment to apply such criteria, it was decided to do an extensive study about how the students have been applied epistemic suitability criteria in their final masters' thesis (to see if they have been used the representativeness criteria, introduced some personal proposals, etc). As a consequence, the changes proposed for the third year were the following: (1) To join the categories for epistemic suitability from OSA with categories from the quality for mathematics instruction given by Hill (2010). In such a way, it was introduced new criteria for valuing mathematical quality as it is: mathematical richness, coherence, errors, etc. (2) To select new case studies from previous years students with more wide and complex explanations than the previous case studies used en year 1 and 2. The aim was to connect echoes and voices to produce more consistent arguments (Garuti & Boero, 2002) to justify mathematical quality of didactical sequences.
A prototypical example of this new task (type c) is a case based analysis upon a student that planned a sequence with 7
th
grade (13-14 years old students) for Thales theorem. The main idea is to use as a new task, a voice of a previous future teacher M that analyzed her own practice about Thales Theorem after the school practice during the course 2010-2011. It was observed that M did a personal final analysis in which she said "...Additionally, we have tried to establish connections either with the concepts of the unit (relating as an example, Thales with similar triangles; similar triangles with similar figures, and so on) as with other subjects (for example, to compute the measure of a columns with mirrors, Snell's law of refraction, relating phisical concepts to mathematical concepts)... So, in conclusion...my epistemic configuration was right" (St. M; final report of practice and master's thesis, 2011).
The student M did not really a real good mathematical connection (Figure 3).
We use such mistake to introduce our new professional task. In such task (type c) we presented three documents: (1) tasks proposed by M to explain Thales theorem in her proposal for school practice; (2) the analysis of epistemic suitability about M proposal, and (3) a textbook in which it was ensured the representativeness of epistemic configurations for Thales Theorem having a coherent connection (see figure 3). When doing the task it was promoted a discussion to understand the idea of representativeness (by using epistemic configurations 1 and 2) and the idea of

coherent connection by using triangles in Thales position. The textbook sequence follows the order: Thales, triangles in Thales position, and after that, similarity triangles.
Epistemic Configuration EC1
Thales Theorem
Epistemic Configuration
EC2
Similar triangles
Concepts: point projection; segment projection; Triangles in Thales position
Properties:
Version 1. Segments determined by parallel rights over two secants are propotionnal.
Version 2: Two triangles in Thales position have proportional sides and equal angles.
Procedures Find fourth proportion
Argument: Comprovation of Thales Theorem; justification that triangles in Thales position have proportional sides, by using Thales theorem twice.
Problems
(1) Find one out of four segments determined by parallel rights over two secants (contextualized / decontextualized)
(2) Indirect computation of unknown measures, using Thales Theorem
Concepts/definitions:
similar triangles (similar triangles have proportional sides (def) and equal angles.)
Properties: similarity criteria
Procedures: Find fourth proportion
Arguments:
Comprovation of four proportional sides and equal angles
Figure 3 Left. Two connected configurations in the textbook.
Figure 3 Right. Comparing representativeness and connectness for M and the textbook presented by the trainer.
The aim of this professional task is to recognize a deep level of analysis from such previous prospective teacher's practices (Choppin, 2011). Thus, the future teachers learn from this analysis, the idea of connecting two epistemic configurations. At the end of the third year experiment, we found the students being more carefully presenting their didactical unit as a result of such deeper analysis.
More aspects were observed in this third year, when we analyze final work of future teachers and we found better results than previous years. Just some aspects had been presented in this paper because a lack of space.
5. Conclusions. Perspectives
As a result of our study, we have analyzed in depth what we nominated professional tasks to promote competency of didactical analysis. It was useful for such analysis the levels of didactical analysis proposed by OSA. We assume the power of analyzing case studies based on texts from previous years' students. In fact, it explains the complexity of analysis that the teacher should realize to value his/her own practice to go beyond from narratives and descriptions. We are centered specially in how successive redesign contribute to have better feedback about the analysis of

processes and valuing quality using notions of representativeness, connection and coherence. One of our conclusions is that to reflect about mathematical quality it's needed that the future teachers use theoretical powerful elements (Krainer, 1993). We value that some students explicit that by doing master's degree work, "we had been developed our competence of didactical analysis". On the other hand, we recognized the final master degree as the starting point for developing research competency for future teachers. In fact, it gives opportunities for students learning and recognizing problems of their professional context (Giménez, Font, Vanegas y Ferreres, 2012).

Following our perspective we intend to see didactical analysis beyond the banality, considering classroom situation as an integral but dynamic system evolving in time, promoting autonomous mathematical thinking and independent validation of its results as future teacher (Laborde, Perrin-Glorian, Sierpinska, 2005).
Our major conjecture in terms of designing didactical sequences of professional tasks for prospective teachers, is that we need epistemic and cognitive analysis not only to criticize each task itself, but to adapt its connections as best as possible to the didactic analysis results. In fact, suitability criteria used for redesigning the tasks (considered as teaching experiments and corresponding case studies) has anticipatory purposes as hypothetical trajectories, but also helps to improve didactic training trajectories. It's important for our task analysis to identify difficulty factors providing frameworks for hypothesizing instructional designs inspired by levels of suitability.
The relevant aspects for our task design proposal are: (1) To understand the redesign process as a teaching experiment, assuming the noticing process (Mason & Wilder 2002) when doing didactic analysis. (2) To use suitability criteria for building and analyzing professional tasks and sequences; (3) To have in mind ethical perspectives of hearing the voice of the prospective teachers as self regulating process. (4) To consider the need for a collaborative research team for redesigning process.
After three years of experience, we assume the difficulties of the future teachers for having a deep reflection upon their proposals (Leikin, 2009) but we consider that our training cycle give opportunities to improve such issues.
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Designing Rich Numeracy Tasks

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In Australia, numeracy is regarded as a general capability to be developed across the whole school curriculum, not just mathematics. This paper draws on a research study that aimed to help teachers in ten schools design numeracy tasks and implement investigative numeracy pedagogies across the middle school (Grades 6-9) curriculum. Teachers were introduced to a rich model of numeracy that gives attention to real-life contexts, application of mathematical knowledge, use of representational, physical, and digital tools, and positive dispositions towards mathematics. These elements are grounded in a critical orientation to the use of mathematics. The paper identifies ways in which collaboration between the researchers and teachers influenced the design of numeracy tasks.
Keywords: Numeracy; Teacher development; Influences on task design

Numeracy is a term used in many English-speaking countries to denote the capacity to deal with quantitative aspects of life. It is often considered to have a similar meaning to terms such as quantitative literacy (Steen, 2001) or mathematical literacy (OECD, 2004). For example, Steen proposed that the elements of quantitative literacy include: confidence with mathematics; appreciation of the nature and history of mathematics and its significance for understanding issues in the public realm; logical thinking and decision-making; use of mathematics to solve practical everyday problems in different contexts; number sense and symbol sense; reasoning with data; and the ability to draw on a range of prerequisite mathematical knowledge and tools. Some of these elements are visible in the PISA definition of mathematical literacy as: an individual's capacity to identify and understand the role mathematics plays in the world, to make well-founded judgments, and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen. (OECD, p. 15)
Steen (2001) argues that, for numeracy to be useful to students, it must be learned in multiple contexts and in all school subjects, not just mathematics. In Australia, support for this challenging notion has come from several sources. A recent national review of numeracy education undertaken by the Australian government recommended that numeracy be recognised as "an across the curriculum commitment" (Council of Australian Governments, 2008, p. 7). In addition, the newly developed Australian Curriculum this country - identifies numeracy as one of seven general
across all discipline content (Australian Curriculum, Assessment and Reporting Authority, 2012). However, the new curriculum documents do not provide teachers with detailed guidance in recognising the numeracy demands of the subjec
teach, in designing tasks and learning sequences that embed numeracy across the curriculum, or in making decisions about pedagogies that support numeracy learning. This paper draws on data from a one year project that helped teachers to design numeracy tasks and implement numeracy pedagogies across the school curriculum in Grades 6-9. It was conducted before the release of the Australian Curriculum, with the intention of developing and testing a theoretically informed model of numeracy that could guide teachers' curriculum planning, task design, and pedagogical decision making in the context of an existing state
Australian Department of Education and Children's Services, 2005). For the purposes of this project, a "numeracy task" is considered to be an activity that engages students in learning and/or applying some mathematics within a given curriculum context.
This paper aligns with Theme E: Features of task design informing teachers' decisions about goals and pedagog
• How does collaboration between researchers and teachers influence the design of rich numeracy tasks?

Numeracy Model

We developed the model shown in Figure 1 to affirm the value of definitions of numeracy widely accepted in Australia (e.g., Department of Education, Training and Youth Affairs, 2000), while introducing
of mathematical thinking and action
orientation to the ways mathematics
opinions (Jablonka, 2003
characteristics of a numerate person
realised the model also provides principles for the d
Thus the elements of the model, elaborated below, were the task design principles that we encouraged teachers to use.
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developed Australian Curriculum - the first ever nationally mandated curriculum in
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ide teachers' curriculum planning, task design, and pedagogical decision
making in the context of an existing state-based curriculum framework (South Australian Department of Education and Children's Services, 2005). For the purposes
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in learning and/or applying some mathematics within a given curriculum context.
This paper aligns with Theme E: Features of task design informing teachers' decisions about goals and pedagogies. It addresses the following question:
How does collaboration between researchers and teachers influence
the design of rich numeracy tasks?

Numeracy Model

We developed the model shown in Figure 1 to affirm the value of definitions
epted in Australia (e.g., Department of Education, Training
and Youth Affairs, 2000), while introducing a greater emphasis on tools as mediators of mathematical thinking and action (Sfard & McClain, 2002) and a critical orientation to the ways mathematics is used to support arguments and influence Jablonka, 2003). The original purpose of the model was to describe the numerate person. However, when we worked with teachers we realised the model also provides principles for the design of rich numeracy tasks
Thus the elements of the model, elaborated below, were the task design principles that we encouraged teachers to use.
Figure 1. Numeracy model
the first ever nationally mandated curriculum in
capabilities that apply
across all discipline content (Australian Curriculum, Assessment and Reporting Authority, 2012). However, the new curriculum documents do not provide teachers with detailed guidance in recognising the numeracy demands of the subjects they teach, in designing tasks and learning sequences that embed numeracy across the curriculum, or in making decisions about pedagogies that support numeracy learning. This paper draws on data from a one year project that helped teachers to design
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Thus the elements of the model, elaborated below, were the task design principles that Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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A numeracy task should require application of mathematical knowledge. In a
numeracy context, mathematical knowledge includes not only fluency with accessing
concepts and skills, but also problem solving strategies and the ability to make
sensible estimations (Zevenbergen, 2004).
A numeracy task should promote positive dispositions – such as confidence,
initiative, and a willingness to apply mathematical knowledge flexibly and adaptively.
Affective issues have long been held to play a central role in mathematics learning
and teaching (Leder & Forgasz, 2006), and the importance of developing positive
attitudes towards mathematics is emphasised in national and international curriculum
documents (e.g., National Council of Teachers of Mathematics, 2000; National
Curriculum Board, 2009).
A numeracy task should involve using tools. Sfard and McClain (2002)
discuss ways in which symbolic tools and other specially designed artefacts "enable,
mediate, and shape mathematical thinking" (p. 154). In school and workplace
contexts, tools may be representational (symbol systems, graphs, maps, diagrams,
drawings, tables), physical (models, measuring instruments), and digital (computers,
software, calculators, internet) (Noss, Hoyles, & Pozzi, 2000; Zevenbergen, 2004).
Because numeracy is about using mathematics to act in and on the world,
numeracy tasks should be embedded in a range of contexts (Steen, 2001). These
contexts may be drawn from real life or curriculum areas other than mathematics.
Numeracy tasks should develop a critical orientation in students since
numerate people not only know and use efficient methods, they also evaluate the
reasonableness of the results obtained and are aware of appropriate and inappropriate
uses of mathematical thinking. Numeracy tasks could ask students to evaluate
quantitative, spatial or probabilistic information used to support claims made in the
media or other contexts. They could also encourage students to consider how
mathematical information can be used to manipulate, disadvantage or shape opinions
about social or political issues (Jablonka, 2003).
The design of rich numeracy tasks according to the principles outlined above
is not sufficient to enable learning. We argue that teachers also need to adopt
investigative pedagogies to fully realise the numeracy opportunities that such tasks
afford. Diezmann, Watters, and English (2001) define mathematical investigations as
"contextualized problem solving tasks through which students can speculate, test
ideas and argue with others to defend their solutions" (p. 170). We consider this
definition applies equally well to numeracy investigations.
The numeracy model was used in three ways: (1) to analyse the numeracy
demands of the South Australian school curriculum (Goos, Geiger, & Dole, 2010); (2)
to support teachers' curriculum planning (Goos, Dole, & Geiger, 2011); and (3) to
trace changes in teachers' understanding of numeracy (Goos, Geiger, & Dole, 2011).
This paper is primarily concerned with (2), and it extends our previously published
analyses by focusing on the design of numeracy tasks and implications for pedagogy.
Project Overview
Teachers were recruited from ten schools with diverse demographic
characteristics: four primary schools (Kindergarten-Grade 7), one secondary school
(Grades 8-12), four small schools in rural areas (Grades 1-12), and one school that
combined middle and secondary grades (Grades 6-12). Each school nominated two
teachers, thus ensuring that participants could collaborate with a colleague in their
own school as well as teachers from the other schools. They included generalist Theme E – M. Goos, V. Geiger, S. Dole
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primary school teachers who taught across all curriculum areas as well as secondary
teachers qualified to teach specific subjects (mathematics, English, science, social
education, health and physical education).
There were three elements to the research plan: (1) an audit of the middle
years curriculum to identify the numeracy demands inherent in all curriculum areas;
(2) three whole-day professional development workshops at the beginning, middle,
and end of the project; and (3) two daylong visits to each school for lesson
observations, discussion of planning documents and teaching approaches, and audiorecorded interviews with teachers and students. The overall project design is
summarised in Table 1. More details on data collection and analysis methods can be
found in Goos, Dole and Geiger (2011).
Time Researcher activity Teacher activity
February:
Curriculum
audit
Identify numeracy demands in all
curriculum areas
March:
Workshop #1
Introduce numeracy model; present
findings from curriculum audit; provide
sample numeracy tasks
Analyse numeracy task design via
reference to model; plan for
implementation
June: School
visits
Observe lessons; provide feedback on
planning, task design and pedagogies
Incorporate feedback into planning for
further implementation
August:
Workshop #2
Provide feedback on first round of school
visits; present stimulus materials for task
design
Share tasks and strategies tried so far;
practise task design with emphasis on
critical orientation
October:
School visits
Observe lessons; provide feedback on
planning, task design and pedagogies
Incorporate feedback into planning for
further implementation
November:
Workshop #3
Report on student perceptions of numeracy;
present stimulus materials for task design
Practise task design; reflect on
professional learning trajectories
Table 1. Project Design
Influencing the Design of Numeracy Tasks
We claim that there are several aspects of the project that influenced the
design of rich numeracy tasks and enactment of associated investigative pedagogies.
The first was the numeracy model itself, the elements of which provided a set of
principles for task design. The second was our numeracy audit of the South Australian
Curriculum Framework, which identified distinctive numeracy demands for each
school subject taught in Grades 6-9. For example, the subject called Society and
Environment is organised into four strands: time, continuity and change; place, space
and environment; societies and cultures; and social systems. The numeracy audit
found that data analysis and spatial sense are the most relevant elements of
mathematical knowledge for this subject. Contexts for numeracy development
included the study of social, economic, political and ecological systems. Students
were expected to develop dispositions enabling them to "to be active citizens who can
make informed and reasoned decisions and act on these" (DECS, 2005, p. 291). The
use of tools such as maps, measuring instruments, online data sources, and
spreadsheets for collecting and analysing information was vital to learning in this
subject. The goal of enabling students to participate as ethical, active and informed
citizens, requires development of a critical orientation to viewing information and
interpreting data. Presenting the audit findings to teachers was intended to raise
awareness of their subject's numeracy demands. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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The third influence on task design was the way in which the research team
used the three workshops to (1) immerse teachers in numeracy tasks we had created,
(2) model investigative numeracy pedagogies, (3) guide the analysis of tasks using the
design principles provided by the numeracy model, and (4) invite the teachers to
practise designing their own numeracy tasks. For example, in the first workshop we
engaged teachers in cross-curricular numeracy investigations suitable for use with
middle years students. These included investigations of Barbie dolls' physical
proportion (with links to the health and physical education curriculum), the
occurrence of the Golden Rectangle in art, design and nature (linked to the arts and
design studies curricula), and planning for participation in the Tour Down Under, a
bicycle race similar to the Tour de France (linked to the society and environment
curriculum). In the first round of school visits we found little evidence of a critical
orientation in the lessons we observed. In interviews with teachers it emerged that
they were unsure about how to embed this element of the numeracy model into their
planning and practice. Therefore, at the second workshop we presented a range of
stimulus materials drawn from print and digital media sources and asked teachers to
work together to develop these into tasks that would promote a critical orientation in
their students, without losing sight of the other elements of the numeracy model. In
both workshops we provided a task design/analysis template that listed each element
of the numeracy model, asked how the task developed numeracy with respect to each
element, and invited teachers to identify school subjects that could provide a context
for using the task.
The fifth influence on task design was the researchers' provision of in situ
feedback to teachers during school visits. We were able to suggest ways of modifying
tasks used in the lessons we observed to give greater prominence to elements of the
numeracy model that appeared to be under-represented. The final influence on task
design was the structured sharing of practice by teachers at the second and third
workshops. At the second workshop all teachers were asked to bring evidence of one
task or lesson sequence they had tried with their class, to describe to the whole group
how the task had been implemented and how well (or not) it had worked, explain
what they learned from this experience and how they would use this evaluation in
their subsequent planning. This workshop provided an opportunity for teachers to see
how colleagues in other schools went about designing rich numeracy tasks. At the
third and final workshop, and as a result of our analysis of school visit data, we
invited four teachers who exemplified different types of professional learning
trajectories to report on their experiences. One of these is the teacher whose
abbreviated case study is presented below to illustrate how she made decisions about
the design of numeracy tasks with an investigative flavour. Although this project
emphasised numeracy across the whole school curriculum, the example below shows
how numeracy tasks can be designed within mathematics.
Teacher Decision Making about Task Design
Maggie taught mathematics and science at a large secondary school in a rural
town. She was in only her second year of teaching. The class with which she worked
for this project was a Grade 8 mathematics class.
First school visit
Initially Maggie struggled to come to grips with how to highlight the
numeracy within mathematics, but she decided to focus on teaching mathematics in Theme E – M. Goos, V. Geiger, S. Dole
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real life contexts that would be of interest to her students. She planned a task based on
the television program The Amazing Race. Students worked for 3 weeks in the
computer laboratory to complete the task, which involved organising an adventure
holiday around the world, given an itinerary and a budget of $10,000. They also had
to complete a number of challenges for which they earned an additional $2000 each.
The challenges, which included Diving with Sharks in Cairns, Skiing in Switzerland,
and visiting The Roman Colosseum, had a focus on using directed number in context.
In The Roman Colosseum challenge students were also required to use formulas in the
context of comparing areas of the Colosseum and the Melbourne Cricket Ground, as
well as looking at exchange rates and converting between currencies.
Members of the research team observed the second lesson of this unit.
Students appeared motivated and well prepared, and they were able to explain the task
to us when we questioned them. Maggie noted that some previously disengaged
students were interested in the task, while a few others remained aloof. Some students
seemed so engaged that they acted as though the task was real; for example, when
Maggie asked one boy "Where are you up to?", he replied "I'm on my way to Paris!".
This task placed mathematics in the real life context of an adventure holiday.
It targeted mathematical knowledge of directed numbers and operations with integers
(money calculations), using digital (internet) and representational (charts, tables)
tools. We did not observe teacher actions that promoted positive dispositions towards
numeracy, but students were clearly motivated and confident in tackling the task and
trying out different combinations of flights and accommodation bookings that would
fit within their budget. A critical orientation does not seem to have been built into
this task. However, after the lesson we suggested to Maggie that this orientation could
be promoted via questioning, such as that we observed when Maggie helped a student
compare advantages and disadvantages of booking cheap backpackers'
accommodation. In this way we attempted to show how the numeracy model
informed not only task design but also teachers' pedagogies.
Second school visit
When interviewed before the lesson observation, Maggie said she had given
a lot of consideration to the types of tasks she wanted to design for the second
research cycle. She was dissatisfied with the length of the Amazing Race
investigation, as this tended to discourage some students and to make it difficult to
complete for any who missed some lessons. As a result of our emphasis in the second
workshop on developing a critical orientation, she also decided to give more attention
to this element of the numeracy model in designing the next investigation,
Approaches to a Healthy Lifestyle, which comprised a number of smaller tasks.
In one task, students investigated the relationship between the heights and
walking speeds of everyone in the class. The mathematics embedded in the
investigation included elements of collecting, representing, reducing and analysing
data. In the lesson we observed, students were to make scatter plots using Excel in

order to determine whether there was a pattern in the data they had collected. In
earlier lessons they had collected height data and calculated the mean, median and
mode. In another lesson students had marked out a 40 metre section of a 100 metre
running track and then found the time it took to walk this distance. With this
information students had calculated their walking speeds in metres per second, metres
per minute and kilometers per hour. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Students all appeared engaged with the task and each group or individual
produced a scatter plot, although the appearance of the graphs varied depending on
the scales chosen or on the choice of variable for the x and y axes. Most students were
able to describe a general trend in the data and use this to make a prediction about
what Maggie's walking speed might be, based on her height. Interestingly, many
students gave most attention to their own data point within the scatter plot with
comments such as "This is me (pointing at the appropriate data point) and "This is
how tall I am and how fast I walk". Using personal data seemed to be effective for
engaging students with the task. From a student's perspective, the activity was about
them and how they compared to the rest of the class.
Students expressed surprise that the scatter plot was not linear, so that taller
people did not necessarily walk faster. Maggie challenged them to explain why this
should be the case. Some groups suggested that alternative variables – with associated
alternative hypotheses – should be explored, including, for example, the relationship
between walking speed and leg length or between walking speed and stride rate. One
group suggested there might be a stronger relationship between a person's height and
their maximum walking pace rather than their natural walking pace.
Maggie chose an engaging context that made use of students' personal details
to introduce the mathematical knowledge that was used in this lesson. The use of
personal data encouraged positive dispositions towards involvement in and
completion of the task. This task required knowledge of how to produce a scatter plot
from a data set using Excel and the capacity to make predictions from trends in the
data. Maggie asked students to use representational tools such as scatter plots and
digital tools in the form of computers and Excel. By challenging students to explain
the variance in their data from the anticipated linear relationship, Maggie introduced a
critical orientation to the task. A critical orientation was evident in most tasks in the
Approaches to a Healthy Lifestyle investigation. For example, in the culminating task
students were to compare data on the number of overweight and obese Australians in
various age groups, and make an argument for whether or not the government needed
to introduce a healthy eating policy for South Australian schools.
Implications and Concluding Comments
The work we reported here has implications for teachers, researchers, and
curriculum developers. First, it provides some evidence that a focus on task design,
when supported by a theoretical model of numeracy that is readily accessible to
teachers, can influence teacher learning and development. For example, when Maggie
reflected on what she had learned during the course of the project, she identified her
readiness to make use of more extended tasks when teaching mathematics. However,
she tempered this view by arguing that tasks needed to be made up of self-contained
sub-tasks that allowed students to move towards smaller achievable goals. For her, the
level of engagement she observed while students were working on numeracy
investigations was a compelling case for their inclusion within mathematics classes.
Nevertheless, other teachers in the project found it more difficult to decide "how
long" a numeracy investigation should be to allow students enough time to explore all
aspects of a task without losing interest, and how much guidance to give students in
structuring the investigation.
A challenge for researchers is to design larger scale studies that do not
simply rely on recruiting more schools or teachers. This is the case for most
educational research and is not unique to our project. To tackle these challenges we Theme E – M. Goos, V. Geiger, S. Dole
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are currently conducting a follow up study that aims for both scale and sustainability
by developing numeracy curriculum leadership within schools, so that the numeracy
model and associated task design principles become integrated into schools' planning
processes for implementation of the new Australian Curriculum.
The current version of the Australian Curriculum offers some support for
recognising the numeracy demands of different school subjects, for example, by
providing a numeracy learning continuum together with icons and filters that link
numeracy capabilities to relevant curriculum content. However, additional
opportunities for developing students' numeracy capabilities are invisible unless one
knows how to "see" them and how to design and implement tasks for classroom use.
The numeracy model and ways of working with teachers outlined in this paper may
prove useful in supporting teachers to fulfil the numeracy intentions of the Australian
Curriculum.
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Task design: Supporting teachers to independently create rich tasks

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Teachers want to be able to create opportunities for generalizing
and justifying for their students; however, they often lack the skills and
the time needed. We attempt to investigate this issue by addressing the
question: How can we help teachers use their textbooks to create tasks
that provide opportunities for students to justify, and afford teachers
opportunities to push students to consider different cases and ultimately
engage in generalizing? We define task design to include the activities
engaged in with teachers that enable them to independently create tasks
that afford students these opportunities. We engage in a design-based
methodology with cycles to refine this process.
Keywords: Rich tasks, task design, design cycle, generalizing,
justifying

Introduction

Intellectually, teachers, mathematics educators, and mathematicians would
agree that it is important for students to generalize and justify. In recent years,
teachers have begun to ask students to explain their thinking. Teachers and students
often understand this request to mean, What steps did you use? This is one type of
explanation, but it is not sufficient for helping students to gain mathematical insights
that lead to deep understanding. Driscoll (1999) suggested that teachers need to ask
questions that prompt students to reflect on the mathematical ideas used in their
solution strategies and consider whether the strategies can be generalized. Stein,
Smith, Henningsen, and Silver (2000) suggest that teachers need to use rich tasks that
maintain cognitive demand by pressing students to make generalizations and justify
their solutions. Both of these suggestions assume that teachers have tasks that lend
themselves to acts of generalizing and justifying that go beyond finding the solution
of a problem. Lampert (1990) highlights the importance of the task itself, "it must be
a problem that will engage all students in making and testing hypotheses ... and push
them to think about when and whether it holds true in a larger domain." (p. 40). Theme E – L. Knott and al.
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Taking time for students to make hypotheses and explore them is a tension for
teachers when they are concerned with covering the curriculum and loss of control
(Zaslavsky, 2008). The question arises, how can we help teachers use their textbooks
to create tasks that provide opportunities to students to justify, and afford teachers the
opportunity to push students to consider different cases and ultimately make
generalizations?
Making Mathematical Reasoning Explicit (MMRE) is a five-year NSF funded
project in which teachers attend three summer institutes and participate in school year
professional development. In this context, we examine the transformation of
instruction from a model where the goal of learning is to arrive at a correct answer, to
instruction with the goal of engaging students in the process of doing mathematics.
Lakatos (as cited in Lampert, 1990) described doing math as "the zig-zag path of
discovery between conjecture, justification, and revision of conjecture or modification
of assumptions that one must necessarily negotiate when engaging in this kind of
mathematical activity." (p. 40). Our first task was to work with teachers so that they
better understood the process of doing mathematics where generalizing and justifying
were the norms. Second, we gave them two simple routines to use that provided them
opportunities to elicit students' observations, explanations, and justifications. Third,
we asked them to create a task that would prompt students to generalize or justify.
This paper examines the fourth phase of our work, which helped them to use
their textbook as a source for developing rich tasks that engaged students in this zigzag path of discovery. Our research question is: What are the essential features of task
design that teachers need to attend to when modifying their existing curriculum to
engage students in learning new content in ways that require generalization and
justification? In this paper, we define task design to include the activities that we
engaged in with teachers, so that ultimately, they are able to independently create
tasks that afford students the experience of doing mathematics.

Literature Review

Literature on teacher change suggests that for teachers to enact new
pedagogy with different mathematical goals, they must rethink and revise their beliefs
and practices in light of research on teaching and learning (Cross, 2009). In addition,
to make this change, teachers need to consider the instructional goals demanded by
the new Common Core Mathematics Standards, create a different learning
environment that encourages students to engage in a different kind of learning,
establish new classroom norms where argumentation is central to the lesson, and use
discourse in which teachers no longer control the discussion through recitation. An
additional component of such change is task design.
A review of the literature on mathematics tasks revealed research on
characteristics of rich tasks, ways to maintain the cognitive demand of the task, and
examples of rich tasks that could be used in classrooms. However, we found scant
research that described how to help teachers design tasks. We define a rich task to be
one that is complex, non-algorithmic, and non-routine, allowing for multiple
strategies and representations and no single pathway to a solution. Any solution to a
rich task is not just an answer to be circled, not even simply a description of the
strategy or reasoning used to arrive at such an answer. It affords students
opportunities to generalize and justify that go beyond finding the answer. It includes
the justification for the strategy or reasoning used to arrive at an answer; an
explanation of why this particular approach is valid. Our literature review begins with Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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a discussion of the importance of generalizing and justifying and ends with additional comments about rich tasks.
NCTM (2000, 2009) and Common Core State Mathematics Standards (2010) emphasize the importance of reasoning and justification in learning mathematics. Generalization can be thought of as the heart of mathematics (Mason, 1996). There is consensus among mathematicians that generalization is a process that serves to create new knowledge for an individual and for the mathematics community at large (e.g., Polya, 1957; Harel & Tall, 1991;Dreyfus, 1991; Ellis, 2007). In much of reform mathematics, there is an inherent tension between this process of doing mathematics and the actual product of the mathematical activity. Polya describes the process of doing math as making a tentative generalization from an observation that is tested and revised for different cases. Justification is the tool by which a generalization is verified. Justifications validate claims, provide explanations, create a context for insights or discoveries, and develop structures to systematize knowledge (de Villiers, 1999; Hanna 2000). Clearly, the acts of generalizing and justifying are intertwined and essential to developing mathematics.
The ICME 11 Topic Study Group 34 (2008) describes the nature of tasks and task design as "not merely … 'things to be done' but as crucial in framing subsequent mathematical activity. Identifying relations between task structure, tool use, and mathematical activity can inform the design and analysis of tasks, and also gives insight into the nature of engagement and learning which takes place while tasks are being carried out." (ICME 11 TSG 34, p. 1). Rich tasks are essential in providing opportunities for students to generalize and justify.
Solving a rich task includes the processes of making conjectures, arguments, justifying and generalizing. Only if a teacher establishes these processes as the acceptable socio-mathematical norms (Yackel & Cobb, 1996) for what it means to be doing mathematics in the classroom will the task provide real opportunities for students to engage in these processes. Such tasks require higher-level cognitive demand (Stein et al. 1998) and strategic reasoning to solve (Kilpatrick, Swafford, & Findell, 2001). This high cognitive demand is necessary, but not sufficient, for fostering opportunities for justification, generalization, and conjecturing in the classroom. Teachers must treat the solution to a task not as a signal to stop thinking, but as an occasion to generalize, conjecture, and ask new questions. Doing mathematics in this way requires both students and their teachers to redefine what they believe mathematics to be (Lampert, 1990).
Theoretical Framework
We draw upon sociocultural and psychological frameworks to examine the features of task design that support teachers to reconsider what it means to teach and learn and do mathematics. These perspectives allow us to work with teachers in ways that create opportunities for them to reconsider their deeply held beliefs about the nature of teaching and learning mathematics, while simultaneously providing opportunities to engage students with tasks. Deeply held beliefs about mathematics develop over a lifetime of experiences and are resistant to change. To support teacher change, we rely on concerns based research (Hall & Hord, 2010) to structure a community of practice.
A community of practice is a group of people who share a similar goal; that learning occurs through interactions among the participants (Lave & Wenger, 1991). From this perspective, the learning environment is critical. It is the place where Theme E - L. Knott and al.
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participants share ideas, take risks, and revise their notions. In the MMRE community of practice, participants experience math in new ways, discuss the implications for teaching with different goals, and consider the impact on students' learning. New ideas are developed through dialogue that includes sharing observations, asking questions, and clarifying ideas (Lave & Wenger). This learning is a product of the group and is available for each participant to examine and compare with his/her personal beliefs. When teachers verbalize them, their beliefs are placed in the public domain where they can be examined within the community of practice. The inner and public conversations interact, creating an opportunity for change. Critical to the conversation is the reflection on new experiences and information that may conflict with existing beliefs. With repeated experiences, reflection, and dialogue, we theorize that change is supported.
To make the work of doing mathematics in new ways meaningful, teachers need to examine their beliefs about what is important for teaching and what it means to "do mathematics." Unless they are willing to undertake such change, or at least acknowledge its necessity, it will be difficult if not impossible for them to embrace the type of task design we are advocating. Further, they will be unable to implement the tasks successfully in the classroom, even when such tasks are provided for them. For teachers to teach mathematics in new ways, they need to recognize a new instructional goal, namely, traveling along the zig-zag path that leads to making generalizations and justifications. If is from this perspective that we work with teachers in the MMRE project. We found it was necessary to embark on this preliminary work prior to or concurrently with the task design that we would engage in with teachers.
Methodology
Design-based methodology with cycles frames our work with teachers. We collected baseline data through observations of teachers working in their classrooms to characterize the pedagogy that they used. Data was analyzed using constant comparative methods. We created a rubric from research on types of generalizations and justifications. It was modified as we coded observations and noted classroom examples that contained types of justifications that did not fit into our existing rubric. When we made changes to the rubric, we went back and re-coded all of the observations.
From our research on generalization, justification, pedagogy, and teacher change, we created a blueprint for what we hoped to see if teachers engaged students in generalizing and justifying as tools for learning. Using the concept of backwards design, we created learning trajectories that would move them from their current practice to a new vision of teaching. We theorized that they needed experiences for themselves in order to be able to provide opportunities for their students to generalize and justify, before they would be ready to engage in creating rich tasks themselves.
Context
In our professional development project with teachers of grades 4 through 12, in rural schools in the Pacific Northwest of the United States, our first task was to work with teachers so that they better understood the process of justifying and what was involved in creating and supporting a classroom environment where student justification was the accepted norm. We were supported in this approach by the recent adoption by 45 US states of the Common Core State Standards Initiative (CCSS, Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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2010). The Standards for Mathematical Practice are arguably the most important part of the CCSS in mathematics. Additionally, they provided the motivation teachers needed in order to embrace our focus on what it would look like to see students "engaged in doing mathematics" in this way. Of the eight Standards for Mathematical Practice, the ones most relevant to our work are that students should: Reason abstractly and quantitatively; construct viable arguments and critique the reasoning of others; and, look for and make use of structure.
Design cycles
We engaged in four cycles in our work to date (see Figure 1). In the first two cycles, we provided teachers with short, easily implementable ways to engage students in justifying: choral counts and strings (Chapin, O'Connor, & Anderson, 2009). Choral counts and strings are examples of pedagogical routines that support students' learning (Lambert, Beasley, Ghousseini, Kazemi, & Franke, 2010). A choral count is some number pattern, often presented as an array, which is gradually revealed to the class by the teacher. The students are encouraged to say in unison what the next number is in the array. As they begin to see patterns, both recursive and explicit, they discuss what they observe, and make conjectures about what numbers might appear in certain spots in the array, and explain why. A string is a sequence of small problems that leads to the observation of a common property or generalization. These routines were readily understood by teachers and easily incorporated into existing classroom structure. They provided a way for teachers to encourage students to begin making their reasoning explicit. We engaged the teachers in choral count tasks and strings tasks during a professional development session, and then asked them to devise their own choral counts and strings to use in their own classrooms.
Once teachers began to see for themselves that their students were not only capable of justifying, but were enthusiastic about engaging in it, they wanted our help to adapt their existing curriculum materials to provide justifying and generalizing opportunities for their students. During the third cycle, we provided examples for teachers to engage in justifying with; then we discussed how they might use these tasks to move their students towards generalizing and justifying.
Figure 1: Design Cycles with learning trajectories illustrating the connections between theory and action
In working through these tasks with the teachers as students, we modeled and made explicit for them appropriate classroom strategies - "What do you notice?"; "Tell me why you used that strategy"; not giving confirmation of the correctness of their answers but instead suggesting they discuss it with a classmate - they might use to move their students in this direction. The next step was to work with them to create their own tasks that would lead their students to generalizing and justifying. Not Theme E - L. Knott and al.
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surprisingly, most teachers chose to develop either strings or choral count examples based on their textbook materials, replicating or imitating the work we had done with them.
We devised a strategy we termed "turning a lesson upside down", which would help teachers to start with their existing textbook and modify a task or set of tasks so that the "key mathematical idea" target of the lesson emerged from the students' own engagement with the task. This is in contrast to how their curriculum typically approaches a new concept, by explicitly stating it as an objective of the lesson, and then providing practice problems for students. The focus in this paper is on this fourth cycle - turning a lesson upside down.
Realizing that they needed our support to modify their curriculum, we worked with them in groups during the Summer Institute on "turning a lesson upside down." We made explicit for them the steps involved in turning a lesson upside down which are summarized here:
1. Select a lesson from the textbook and determine the key understanding, method, or concept for the lesson.
2. Write the mathematical idea as a generalization, or a way of thinking that allows one to see general patterns or relationships. Note: Some mathematical idea may not be treated as a generalization.
3. Determine whether the key understanding entails justification or provides a person with a tool for justifying patterns and relationships. Identify representations that can be used to justify the key understanding.
4. Find a task(s) or a sequence of problems that can be used to illustrate or develop the key understandings. Note opportunities for generalization and/or justification that could be pursued, if any.
5. Write questions that you can ask students to make their generalizations or justifications explicit.
Results and discussion
We began cycle four by providing a simple lesson on averaging that was presented in two ways. The first was following a textbook structure in which the procedure of adding and then dividing was defined. In the upside down lesson how to find the idea of averaging emerged from student investigation. The typical textbook lesson would define the average of a set of numbers to be calculated by forming the sum of all the numbers and then dividing by the number of numbers in the set. In contrast, an upside down lesson might contextualize this target idea of finding an average in the following way: Suppose that Bonita has 5 beads, Irv has 14 beads, Jacob has 13 beads and Maria has 8 beads. How could the children redistribute the beads amongst themselves so that each child has the same number of beads?
Later we showed a video that contrasted two model lessons from an elementary school classroom. The first followed a textbook structure and the second a lesson that was turned upside down. The textbook structure dictated the format: the target procedure was made explicit and then students practiced it by solving similar problems. The second lesson was turned upside down: students were given problems that were carefully scaffolded 'finding' the mathematical target, supporting student work towards providing claims, arguments, conjectures and justifications that would help to solve the problems. Both lessons used Cuisenaire rods in which a fractional part was given (the red rod is 1/3) and the problem was to find the whole. Both Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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lessons focused on the relationship between the part and whole. The mathematical target of the lessons was finding the whole when a fractional part was given. We watched the two lessons with the teachers and discussed the actions, observations, and explanations of the students. Not surprisingly, the students in the upside down lesson provided more explanations and seemed more engaged. We intended to compare the structure of the two lessons so that teachers could see the differences in how to construct an upside down lesson with one that followed the structure found in their textbooks. However, we did not examine the lessons' structure because of time constraints and we thought that the differences were obvious. Yes, they were obvious to us, but not necessarily to the teachers.
In addition, one group of the teachers experienced doing mathematics by actually engaging in a lesson that we had turned upside down. We scaffolded the task with a sequence of questions designed to help them find a way through the problem. Once they had spent a considerable amount of time wrestling with the ideas of the problem and sharing their solution processes and strategies with each other, we debriefed the session and made explicit the moves that we had engaged in with them as they investigated the problem. We felt we had prepared them well to begin replicating this process for themselves. But it did not work as well as we had hoped.
Analysis of the upside down lessons written by the teachers revealed that they had difficulty using the process that we had outlined. Many of them wrote key understandings that were too broad. For example, a high school teacher wanted to review division of fractions with her algebra students. The key understanding of the lesson was, "Students will find an algorithm for dividing rational fractions. They will understand why it works." This is a too big a target idea and requires more than one class period to develop. It involves understanding the process of unitizing, partitioning, and different ways to represent fractions. Clearly, we need to help teachers understand the importance of breaking down key understandings into smaller, target components and to focus on one component rather than to try and

address all of them at once.
The teacher who was videotaped had anticipated students' responses in her
two lessons. She used these responses to help plan questions and interventions that
would support students' learning. In the lessons that teachers wrote, this critical piece
of working through the lesson and anticipating responses from the student perspective
was largely overlooked. The teachers worked collaboratively to create their upside
down lessons and may have anticipated students' responses without recording them
on the lessons that they turned in. However, we noticed that in some cases the context
of the problem interfered with concepts that the teacher wanted students to investigate
and working through the problem as a student would have illuminated this disconnect.
A sixth grade teacher identified three key understanding for her lesson: (a)
Any two numbers have at least 1 number in common; (b) The greatest common factor
is the greatest divisor they share; and (c) You can find the greatest common factor by
listing and organizing factors. While all of these understandings are important, there
are too many to address in one lesson. The problem required students to construct
pages in a scrapbook:
Annika is placing photos in a scrapbook. She has eight large photos, twelve
medium photos, and sixteen small photos. Each page will have only one size of
photo. She also wants to place the same amount of photos on each page. What is
the greatest number of photos that could be on each page? Theme E - L. Knott and al.
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She anticipated student responses by describing different ways that they could
solve the problem using manipulative (rectangles to represent the photographs),
arrays, factor trees, or organized list. However, the problem context does not make
sense. Why would anyone want to have the same number of photographs on one
page? For example, one could put two pictures on each page, but it would look funny
in a scrapbook for one page to have two large photographs and another page to have
two small ones. Here, the context of the problem did not help students make sense of
the key understandings.
Learning trajectory
To support teachers' learning we need to respect what they know and can do
well, and build on it. Teachers were able to identify lessons that had the potential to
be turned upside down and they were able to write questions to prompt generalizing
and justifying. They will use their upside down lessons with students before we meet
again. Thus, the first opportunity to help them focus on anticipating students'
responses and articulating a component of the key understanding is through reflection.
We will frame our reflection questions around these two ideas.
Summary
This is clearly a work in progress and there will be many more cycles, and
many more revisions of these cycles. It is an ongoing, lengthy process. We are excited
about sharing the next cycles in this working group as we refine and redesign tasks
that support teachers to independently create tasks that provide opportunities for
students to justify, and afford teachers the opportunity to press students towards
justifying and generalizing.
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The same task? - different learning possibilities
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In this chapter we focus on variation of the design and the
implementation of a specific task during three mathematics lesson in the 8
th
grade
in a learning study (Marton & Tsui, 2004; Runesson, 2008). The theme of the
lesson was division, with a denominator between 0 and 1. The teachers wanted
their students to understand that when dividing with a denominator <1, the
quotient is larger than the numerator. Four teachers collaboratively planned,
analyzed and revised three lessons in a cyclic process. The study shows that the
implementation of the task changed between the lessons. Although the same task
was used in the lessons the way it was enacted provided different possibilities to
learn.
Keywords: Tasks, variation theory, learning study, mathematics, division
Introduction
In mathematics education tasks play a critical role in the teaching and learning
process (e.g., Hiebert & Wearne, 1997; Watson & Mason, 2006; Zaslavsky & Sullivan, 2011).
Tasks can mediate important mathematical ideas for the students. In this chapter we address
the issue about the implementation of a task. What is made possible to learn from a specific
task in different lessons? The task chosen is a sequence of items of non-contextualised
arithmetic (see Figure 1). Certainly, implementation of a task can be very different in different
classrooms. What is possible for students to learn from a task, may be affected by for
instance, how the task is enacted, students' response, students' knowledge, but also what
mathematics is made explicit from the task.
It has been suggested that tasks and teaching can be designed with variation in
certain dimensions to enhance student learning. For instance, Watson and Mason (2006), who
talks about using variation as a tool for designing tasks and the role of the teacher in the this
process:
Constructing tasks that use variation and change optimally is a design project in which
reflection about learner response leads to further refinement and precision of example
choice and sequence…This process cannot be done by textbook authors working alone
under tight publication deadlines but it can be done by teachers for themselves (Watson &
Mason, 2006, p. 100). Theme E - A. Kullberg, U. Runesson & P. Mårtensson
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In this chapter we describe such a design project; how one task was designed,
enacted and successively refined by four teachers teaching in the 8
th
grade. The task was one
of several tasks planned collaboratively by the teachers in an iterative process of planning,
analysis and revision of a single lesson about division. This form of collaboration, called
learning study, have shown that teachers become sensitive to their students' learning and that
the way they teach the topic changes due to the insights gained by student learning (Runesson,
Kullberg, & Maunula, 2011). The aim is to discuss students' possibilities to learn from the
same task enacted in three different ways. How did the teachers implement the task and what
was made possible to experience from the task by the learners?
Background of the study
The data used comes from a learning study about division. Learning study (Pang &
Marton, 2005) is a version of the Japanese lesson study model (Fernandez, 2005; Lewis,
Perry, & Murata, 2006; Stigler & Hiebert, 1999) informed by a learning theory; variation
theory (Marton, Runesson, & Tsui, 2004). The aim is to improve students' learning by a
careful and systematic inquiry into teaching and learning. The lessons are video recorded and
students' learning is mapped by analysing how they solve certain tasks before and after the
lessons. The current study lasted one semester with several pre- and post-lesson meetings
where the three teachers planned, analysed and revised the lesson plan. After planning the
first lesson together, one of the teachers conducted the lesson in his/her class. The videorecorded lesson was analysed and, from what was observed in the lesson and analysis of how
the students solved the tasks on the post-tests, changes for the next lesson were agreed about.
After revising the lesson, the next teacher taught the lesson to his/her (new) class, and after a
second revision, the third lesson was enacted by the third teacher. One of the researchers
(third author) had a supportive role in the learning study; she took part in the discussions,
recorded the lessons, conducted the tests and informed the teachers about the results.
The teachers wanted their students to learn that in division, when the denominator is
a decimal number between 0 and 1 (e.g. 24/0.8=?) the quotient is larger than the numerator.
Previous research shows that students' overgeneralisations of rules valid in the domain of
natural numbers lead to ideas like 'multiplication always makes bigger and division always
makes smaller' (e.g. 45/0.9 equals a quotient <45) (Verschaffel, Greer, & De Corte, 2007, p.
569). The teachers were aware of that the students often had this idea and planned the lesson
to overcome this difficulty. Furthermore, they wanted their students to be able to solve
divisions like 40/0.2= without having to multiply by ten (400/2=). They believed that this
strategy would help the students to solve the problem, however, without understanding the
underlying structure of division with decimals. The ideas of 'partitive division' (e.g. 100/20
seen as 100 partitioned into 20 groups) and 'measurement division' (how many groups of 20
go into 100) was introduced in all classes before the specific task was discussed (cf., Greer,
1992). The task chosen to discuss, is a sequence of items of non-contextualised arithmetic
designed by the teachers (see Figure 1). The lesson as a whole consisted of several other
tasks.
Analysing what is made possible to learn
In order to analyse the implementation of the task from the point of view of what was
made possible to learn, variation theory is used. An important idea within variation theory is
that learning implies seeing something in a new way by experiencing aspects that you have
not experienced previously. To make it possible to notice these aspects it is necessary to
experience variation (cf., Dienes, 1960; Gibson & Gibson, 1955; Watson & Mason, 2006).
The way we experience something, or how we learn to see an object in a particular way, is a Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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function of those aspects we notice or discern at the same time. If different individuals
experience 'the same thing' differently, they discern different aspects of the object in
question. In order to understand or see a phenomenon or a situation in a particular way one
must discern all the critical aspects of the object in question simultaneously. Since an aspect is
noticeable only if it varies against a back-ground in invariance, the experience of variation is
a necessary condition for learning something in a specific way. Four possible patterns of
variation has been identified; contrast, generalisation, fusion, and separation (Marton,
Runesson, & Tsui, 2004). In previous studies it has been demonstrated that variation theory
can be used to analyze lessons from the point of view of what patterns of variation that are
inherent in the lesson. It has also been found that this is reflected in what students actually
learn (Kullberg, 2010; Marton & Pang, 2006; Marton & Tsui, 2004). So, identifying the
pattern of variation and invariance in the lesson implies to identify what is made possible to
learn.
Here we make use of the notion of contrast, since it was shown this pattern of
variation was frequently used by the teachers. Marton et al. (ibid.) points to the importance of
experiencing contrast for learning;
"in order to experience something a person must experience something else to compare it
with. In order to understand what "three" is, for instance, a person must experience
something that is not three: "two" or "four", for example". This illustrates how a value
(three, for instance) is experienced within a certain dimension of variation, which
corresponds to an aspect (numerosity or "manyness") (p. 16).
Teaching division with a denominator between zero and one
As can be seen from Figure 1, there is a pattern of variation and invariance built into
the task. We can see that the operations vary, since there are multiplication as well as division
items. This variation makes it possible to discern differences between multiplication and
division. Furthermore, the numbers within each column vary in a certain way, starting with
larger to smaller number, keeping the number 100 invariant, making it possible to discern
what happen to the answers when different numbers are multiplied to, or divided from 100.
The numbers varied are positive integers with one or two digits and decimal numbers between
zero and one with one decimal. The numbers were chosen to make it possible to experience
differences between divisions with numbers <1. The same numbers are used in the two

columns, except for in the answers to the items. With this design the teachers wanted to make
it possible for the students to see what happened with the answer when one factor or the
denominator changed.
Figure 1. Items in the planned task. Theme E – A. Kullberg, U. Runesson & P. Mårtensson
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In the following we show how the design of the task changed as a result of analysis
of the revised lessons, the different ways they were implemented in the lessons, and from that
draw conclusions about what was made possible to learn from the enacted task.
Differences in implementation of the task
When we analyze how the same task was enacted in the three lessons we can see that
different aspects of division are made salient for the learner. Our suggestion is that seemingly
the same task offered different learning possibilities for the students. Note, the task was
successively written on the whiteboard by the teacher, one item at a time, starting with the
first multiplication item, followed by other multiplication items, and thereafter the division
items. The order in which the items were discussed in the lesson and what the teacher made
explicit by visually pointing out is shown in Figures 2 to 4.
Lesson 1
In lesson 1 the task was enacted in a way, that from the theoretical framework taken,
only made it possible for the learners to experience that multiplication can be used to "check"
division. The teacher started the discussion about the task in lesson 1 by asking for the answer
to the item 100·20. The teacher continued with another item, multiplying one hundred with a
smaller factor, 100·4, followed by two division items, 100/20 and 100/4 (Figure 2, see 1). The
teacher pointed out that she used the same numbers in the multiplication and division items,
however nothing else was discussed. From the set of items it was visually possible to discern
that multiplying with a smaller number give a smaller answer, and that dividing with a smaller
number give a larger answer, but this was not discussed. The teacher continued with another
set of items in multiplication (2) and thereafter division (3). The items were solved, however,
with no specific discussion about what happened with divisions with numbers < 1, although
this was the intention of the task. After calculating all items the teacher said "How can we
connect these rules of arithmetic". The class came to the conclusion that multiplying the
quotient with the denominator, for example 20·5 will give the same answer as the numerator,
100 (see A). This relationship was pointed out by the teacher for all division items, namely
that the answer to a division can be checked by a multiplication.
Figure 2. The enacted task in lesson 1. The items are grouped into sections (1 to 3) to show the
sequence of how the items were presented on the whiteboard. The lines (A) show what the teacher directed the
students' attention to.
As stated earlier the enacted task only made it possible to experience a contrast
between multiplication and division by using multiplication to check a division task. At one
point the teacher said "the answers get bigger the further down one gets" in the division Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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column. However, we infer that this most likely was insufficient for learning what was
intended. Even if the task was designed with the intention to make it possible for the students
to discern that the quotient is larger than the numerator when dividing with a number <1, this
was not focused upon. Instead, calculations were discussed (A), and no other relationships,
patterns or numbers were discussed.
Lesson 2
In contrast to lesson 1, in the second lesson of the cycle, the designed task was
handled in a way that made it possible for the learners to experience that when the
denominator is a decimal number >0 but <1, the quotient is larger than the numerator.
Furthermore, it was made possible to experience that the product become smaller than the
larger factor when multiplying by a number <1.
In lesson 2 the teacher started the discussion together with the students about the task
by calculating the items in both sections 1 and 2 (See figure 3), starting with the
multiplication items (section1) and continuing with the division items in section 2 (Figure 3).
After doing the calculations, the teacher asked if the students could see any 'patterns' between
the items. The students identified two 'patterns. These were summarized by the teacher:
- The smaller number multiplied with, the smaller the product and the smaller number
divided by, the larger the quotient is.
The teacher explicitly the pointed at the two columns (1 and 2) comparing the
products and quotients with one another. Thereafter the teacher pointed at the quotient and the
numerator in 100/20=5 and 100/4=25 (See figure 3) and said:
-Here the quotient is a smaller number than the numerator, is it always like that?
This question made it possible for the students to experience a contrast between, on
the one hand division with numbers < 1, and on the other hand division with 1 ≥. One student
said that after zero there was a difference. Other students said that it was not the same for
numbers <1, then the quotient became larger than the numerator. The different answers "zero"
and "one" made a contrast as to where "it turned". By comparing 0 and 1, the significant
turning point was elucidated. The teacher drew a line under items with a denominator smaller
than one (F) so the contrast became visible. Thereafter, the teacher first pointed to the
denominators 1 and then to 0.5 and said:
-When the denominator is smaller than one, the quotient (pointed to the quotient) is larger
than the numerator (pointed to the numerator) (F).
Next, the teacher made a contrast between the division item 100/0.5 and the
multiplication item 100·0.5, she said "What happens with the multiplication item then?" and
the episode ended with the conclusion that the product, 50, is <100 (H). By comparing the
items 100/0.5 and 100·0.5 it was made possible to discern that quotient became larger than
100, and the the product smaller. Theme E – A. Kullberg, U. Runesson & P. Mårtensson
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Figure 3. The enacted task in lesson 2. The items are grouped into sections (1 to 2) to show
the sequence of how the items were presented on the whiteboard. The lines (B-H) show
what the teacher pointed to and hence, directed the students' attention to.
The analysis demonstrates that, in lesson 2, relationships between the quotient,
denominator and numerator were made possible to learn by means of a pattern of variation
that elucidated how changes in the denominator affected the quotient and the numerator.
Lesson 3
In the last lesson, the task was, in the same way as in lesson 2, handled in a way that
made it possible for the learners to experience that when dividing with a number <1, the
quotient is larger than the numerator. However, more attention was paid to the items with the
denominator <1. This was done by grouping the discussion into three sequences (See Figure
4). One difference was the sequence the items (see Figure 4). The difference in sequence was
that in both multiplication and division, items with the number 0.1, was systematically
compared with the items with 0.5. This comparison made it possible to generalise both
multiplication and division items with a denominator <1.
In lesson 3 initially the discussion was about the multiplication items in section 1
(Figure 4) starting with the item 100·50 followed by 100·5, 100·1 and 100·0.5, followed by
the items in section 2; the divisions (2). After that, multiplication and division was discussed
separately. The teachers said:
-Look at the multiplication column first, do you see any pattern, anything that is the same
or different?
The students said that the smaller number we calculate with the less zeros there are
[in the product], and the smaller number there is [the product]. The teacher said:
-What are we used to get as an answer when we calculate a multiplication?
The class said that usually it is a larger number. The teacher continued:
-Is it always like that? When do we not get a larger answer?
The students answered "After one". The teacher pointed at the decimal number 0.5
and the product, and again at the product and 100 to show that the product in this case was
smaller than 100 (I). The teacher asked if that was true for all decimals (<1), and introduced
the item 100·0.1, as another example with multiplication with a number <1 (3). Thereafter the
teacher asked if the students could see patterns in the division column. The teacher said: Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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-What are we used to get as an answer when we calculate a division?
The students said that they were used to get a smaller answer when they divided. The
teacher asked the students if it always is like that. The teacher said:
- Do you see where it turns? Here we get a larger answer (quotient) [the teacher
pointed at 100/0.5=200 (J)], when do we not get a larger number (quotient)? Is this
true for all decimals? (J).
The teacher introduced another division item, 100/0.1. (J), to compare with 0.5.
Figure 4. The enacted task in lesson 3. The items are grouped into sections (1 to 3) to show the
sequence of how the items were presented on the whiteboard. The lines (I-J) show what the teacher pointed to
and hence, directed the students' attention to.
In the end of the lesson, not in the discussion of the specific task analyzed here,
division with two and three decimals were also introduced (10/0.02=500, 10/0.002= 5000), as
well as multiplication with two decimals (0.02·500=10).
Different possibilities to learn from the same task
The case study reported is an example of a design project where teachers' reflection
on their teaching and the learners' responses can lead to a refinement of the task design (cf.,
Watson & Mason, 2006), but also to a greater accuracy and clarity about what to point out and
make discernible to the learners. The task used in the lessons was designed by the teachers.
The teachers decided a specific pattern of variation in the task to elicit aspects of division with
a denominator <1. In the task the same number (100) was divided or multiplied systematically
with a smaller number, so the learner easier would see how the quotient or the product
changed. This was the principle behind how the task was designed. We have shown that,
when the (same) task was enacted in classrooms, the items were handled differently as regards
to sequencing, as well as aspects of the content made explicit by the teacher, and aspects
juxtaposed and contrasted. This was a result of the difference of that which was varied and
kept invariant between the three lessons. We will suggest that these, rather subtle, differences
affected what was made possible for the students to learn. So, what might be worth
considering from this example is whether it is possible to say that the learners encountered the
same task in the three different lessons or not. In one sense it was the same task, but what the
learners encountered in the lesson was different. Our interpretation is that, only in lesson 2
and 3 the teachers' intentions were brought out. As researchers and mathematics teachers we
ask ourselves if we sufficiently make use of the possibilities that are inherent in the task
(regardless of who the designer is). A lesson is indeed interactive in nature; for instance could
the teacher's and the learners' ways of questioning and responding affect how the content is
handled. So, a lesson might not turn out to be what we planned. Even if we think we have Theme E – A. Kullberg, U. Runesson & P. Mårtensson
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clearly brought out that which we intended to, this may not be the case. One thing we think
could be learned from this study is that, if students' fail to learn, one reason could be that it
was not made possible to learn this in the lesson. What we assume the task will mediate,
might not be possible for the learners to see. Implications of this study on teachers' pedagogy,
we suggest, is that teachers need to take an active role in how the task is enacted in the
classroom. Since, we cannot take for granted that the learner discerns from the task what the
teachers have in mind. In this study the teacher in lesson 1 wanted the students to see
connections between, for instance the quotient and the numerator, however this was never
discussed or made possible for students to become aware of. In lesson 2 and 3, on the other
hand, the teachers directed their students' attention towards this. We suggest that participation
in learning study gave the teachers time for reflection over the content and tasks used, and
contributed to the teachers' awareness of the implementation of the task.
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On doing the same problem – first lessons and relentless
consistency
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In this paper, we illustrate, through stories from practice, the
view that in designing and implementing tasks, teachers have, as a base
for decision-making, the classroom cultures they have already established
with their students. These cultures are developed over time from the first
lessons with a new group. Teachers, and curriculum developers and
researchers working with teachers could be seen as leaders of change, the
teachers of the learning of their students. In the leadership of change
literature, Fullan (2008) has developed what he calls 'six secrets of
change' that shift the focus away from detailed planning to learning
through reflective action. In Secret 4, 'Learning is the work', he talks

about addressing 'core goals and tasks with relentless consistency' (p. 76).
So, once a culture in a classroom has been established, through the
relentless consistency of practices, and children know what to do to
support their learning, this, according to Fullan, frees 'up energy for
working on innovative practices' (p. 79). Evidence for establishing
classroom practices through relentless consistency that support the
continued learning of both students and teachers is given through
observations of and interviews with teachers discussing the first lessons of
a topic or with a new group of students. For these first lessons, the
teachers often use tasks that they are familiar with and have used over
many years for their first lesson with a new group to establish ways of
working in their classrooms.
Keywords: Teacher change, mathematics education, task design,
teacher decision-making, first lessons
Introduction and Background
After studying for a mathematics degree, Laurinda completed a one-year postgraduate teacher education course (PGCE) and began teaching at a secondary school
(11-18 year-old students) where, 14 years later, she was head of the mathematics
department. An opportunity arose to have a year's sabbatical leave working as a
curriculum developer (in fact the job title was Mathematics Editor at the Resources
for Learning Development Unit (RLDU)). Her job was working with groups of
practising teachers to develop resources either in response to government initiatives (a
top-down approach) or in response to suggestions of need from teachers (a bottom-up Theme E – L. Brown and A. Coles
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approach). One such working group was focused on the algebraic activities that would
be useful for students to experience before they used them in the new examination
courses (GCSEs), where extended investigations were going to be used for the first
time. What was important for the editor to be aware of whilst working with the group
was the publication that would emerge that would be distributed throughout the local
education authority's schools and be on sale nationally and internationally. However,
the real task of the editor was supporting the professional development of the
teachers. Nevertheless, the resources produced, whilst being quite old now and out of
print, are still considered to be some of the best resources available (for example
Laurinda has recently been approached by the National STEM Centre in the UK with
a view to publishing Addendum to Cockcroft on-line and 6
th
dimension, Developing
Teaching Styles in A Level Mathematics (Brown, 1988) is cited in an e-book (Risps:
Rich Starting Points for A Level Core Mathematics http://webfronter.com/walthamforest/CFSMaths/mnu5/Rich_Starting_Points/images/risps_ebook_apr_07.pdf) as "A
wealth of rich and open possibilities for inquisitive mathematicians.').
Following on from the RLDU work, Laurinda became a teacher educator and
still works with the one-year PGCE course for prospective teachers. She and Alf
Coles have researched and taught together since 1995, Alf having recently completed
his PhD and joined Laurinda teaching on the PGCE course at the University of
Bristol.
Focus on professional learning or change
The job of editor at the RLDU was illustrative of a subset of Gee's (2004)
principles for task design listed in Sullivan (2011, p. 32):
- learners to take roles as 'active agents' with control over goals and strategies
- skills to be developed as strategies for doing something else rather than as goals
in themselves.
As editor working with the group of teachers, the role was to keep the focus
of the teachers on the publication. It was important that we shared ideas and also that
we tried out suggestions for tasks in the different classroom cultures of members of
the group (typically not more than 10 teachers and six meetings over eighteen
months) since the tasks that made it through to publication were able to be adapted
into different classroom cultures. For instance, for the 'function game' activity
(shared as part of the algebraic activities group mentioned above, see
http://nrich.maths.org/5531 for a short video of Alf introducing this activity to a class
as a first lesson for a block of work on functions and graphs), in the final publication
Developing Algebra, each member of the working group wrote up how they had used
the task and these write-ups were printed on a non-white paper and spread throughout
the publication. The teachers were 'active agents' in this process because in their
discussions they appreciated the power of the activity in being able to be used in
many different ways and wanted the activity to be a major part of the final
publication. Laurinda coined the phrase 'spurious purposes' (Brown, 1995) to
describe the strategy, consonant with Gee's second principle above, of having an
agreed endpoint, for example, a publication, when, for the group leader, the work is
actually in the process. Spurious purposes work as well in the classroom and are a
powerful design principle (e.g., we are going into a primary school to help the
students with their mathematics, as a purpose to get slightly older students reflecting
on what they have learnt or as motivation for low achieving older students). Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Working as editor at the RLDU taught Laurinda about the importance of
teacher decision-making in adapting tasks for their own classroom. She had the
privilege of visiting the classrooms of the teachers she worked with and got into the
habit of interviewing them to gain a sense of how their different classroom cultures
had developed. The technique was honed down to interviewing teachers after they had
taught their first lessons of the year. How did they choose the tasks to do? How did
they set up their classroom cultures? The rest of this paper will initially share some
stories of practice, including ones from time Laurinda spent at Monash University,
Australia (August, 2012). Rather than searching for new and exciting problems, what
seems to be important for establishing cultures is that the teachers worked with their
students on familiar tasks that were designed to introduce their students to ways of
working that they would continue to use in future lessons. As Alf commented on the
NRich website in relation to working with the problem functions and graphs,
In the end, what I like most about this problem [function game/functions and
graphs] is its familiarity; I have worked on it with every class I have ever taught.
Rather than this becoming monotonous, it seems to mean that each new class goes
further than the one before, as I become more and more attuned to what students
say and to what possibilities there are for exploration. And I imagine this is true
for everyone's 'favourite problems' – in the end it does not matter so much what
they are as what use you have made of them. (http://nrich.maths.org/5531, 2006)
So, Alf's pedagogical decision-making, focused on the meaning making of his
students within mathematics, is supported by his history of experiences of using, in
this case, the function game. His attention is freed to work contingently with what the
students bring.
After some more stories, there will be an interlude discussing Fullan's (2008)
'six secrets of change'. If we want our teachers to develop and the children to learn
mathematics we are leaders of change and insights from the leadership of change
literature can support our awarenesses of decision making in planning and in action.
Fullan's idea of relentless consistency will then be used to reflect on the stories and,
finally, some insights into classroom practices from first-lesson interviews will be
presented.
Laurinda's Stories – narratives of practice
Story 1 (Australia): I asked a group of master's students to come to the session
prepared to tell me what they thought 'curriculum' meant and what they thought
'mathematics' was. After a discussion of how she planned what she did in the
classroom, one woman, the first to speak, said 'People Maths'. She had found Mike
Ollerton's book 'Getting the buggers to add up' and now, whatever the system or
structure in her school or in the new Australian National Curriculum, she turned her
lessons into what she called 'People Maths'. She did a couple of examples actively
with the group and I followed with a couple of active loci (stand same distance from
me; with two fixed points, stand twice as far from one point as to the other). I then
asked the group what 'People Maths' was and they came up with some thoughts that
were effectively design principles for any content.
Comment: What 'People Maths' as a design principle does is focus the beginning of
the lesson on students being active. Again, the teacher has then to work contingently
with what the children do. In working actively on the second locus, stand twice as far
from one fixed point as another, questions emerge from the group, is the locus an
ellipse? A circle? These questions, force the teacher to make decisions of how to Theme E – L. Brown and A. Coles
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respond, given the particular group being worked with. The more familiar the actual
problem and the more familiar, in this case, the teacher is with doing 'People Maths',
the more extensive the range of options that lead to a decision being taken. The
consistency of offer of 'People Maths' also serves to set up a culture in the classroom
where the students expect to do something active and make meaning from those
actions. In such a case, within an established classroom culture, the students might set
their own goals, the teacher simply acting as facilitator.
Story 2 (Australia): In a primary classroom that I visited to observe, the teacher
picked out one of the problems from Peter Sullivan's (2004, second edition) Openended Maths Activities book, to work with her students on. It was from a section
entitled 'area':
My granny bought a square rug and each side measured 1 m. When she got it
home it would not fit into the hallway so she cut the rug up and joined the pieces
together again to make a shape that would fit. What might her rug look like now?
(p. 64)
- Who can read that problem out for us? [A volunteer reads.]
- OK, so what are the key words?
- 1 metre length
- Square
- What are we trying to do?
- Cut up the square to fill the hallway. (from my field notes, August, 2012)
At this point all the students started to work, beginning in different ways.
Nobody seemed upset by not being given dimensions for the hallway. There were
some heated arguments. For example, between two boys about the solutions they had.
Both started with Granny's rug being one square on their paper. One boy filled a
hallway that was one half of a square by two squares showing this drawing in his
book:
The other boy filled a hallway that was one half of a square by one and a half squares.
The first boy had actually cut up Granny's rug. The other boy argued theoretically,
'Granny's rug has 4 1 m sides so I need a shape with 8 half metre length sides. What
would you say next?
During sharing time, the children were used to saying what they had done
and others listened or asked questions. One boy lost us all:
- I doubled the first and halved the second .4 times 2.5, .8 times 1.25.
The teacher, myself and the other children look a bit bemused. .8 x 1.25?
Surely that can't be 1? How can we check using what they know?
It was natural for me to offer to work on this with the children. I write .2 x 5
on the board and someone says, 'that's 2 tenths'.
- What can I do with 2 tenths?
- 1 fifth. [Lots of the students look confused.]
- What's a half of 2? [I invite them to chant, all reply together.] – 1
- What's a third of 3? – 1
- What's a quarter of 4? – 1
- What's a fifth of five? – 1
- What's a millionth of a million? –1
We're all smiling now. Those of us who knew the answer began chanting
and, gradually, more of the children joined in. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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- What about .4 by 2.5?
- .4 is 2 fifths.
- 2.5 is 2 and a half.
- How many halves in one? – 2 [chanting again. I did not ask this time]
- How many halves in 2? – 4
- How many halves in two and a half?
This seems OK. I write one half of two over one on the board without
speaking. They will be working on fractions, decimals and percentages soon. I don't
want to get into algorithms. They see a pattern and recognize it.
- 2 fifths of 5 halves? – 1
There are giggles and the energy release that I recognize of having learnt something.
Not everyone is there. However, it's the end of the lesson. The teacher says she will
use doubling and halving as the introduction to fractions and decimals. We would do
.8 x 1.25 next. The boy who started all this comes up to me before I go for lunch with
the teacher and says he's done .8 times 1.25 that way and it works.
Comment: I can remember being amazed by the beginning of this lesson. The
question was, indeed open, and the children were well used to this way of working
because no child made a comment like "the question doesn't tell us the shape of the
corridor". There was an established process by which the children made meaning of
the question, reading it out and then identifying keywords, they then all started
exploring possibilities. The activity allowed the children to work in groups and we, as
teachers simply circulated, learning the children's responses. This led the primary
teacher to her decision, when the boy lost us, to invite me to work directly with the
group. As a secondary teacher, I was aware that these students would not have
algorithms to cope with these difficult multiplications so had to make my offerings
contingent upon what they offered me. The offer, by one child, of the fraction for the
decimal was key to me realising that the pattern could be used without the need for
multiplication of fractions in that moment. The class teacher then realised that this
experience could be motivation for a focus on fraction and decimals so that their
awarenesses could be applied. It would have been impossible for me to work in this
way with a group of students unused to tackling open questions.

Story 3 (UK, after return from travels, August 30
th
, 2012): I had a tutorial with a
Master's student who is submitting her dissertation in early September. Sitting with
her, talking, I am aware that, here it is again. In the Epilogue to her work she is
discussing her development through doing the project and articulates what are now
design principles for her teaching. I ask her if I can quote from this writing and she
agrees:
In the year since completing my research project I have further grown as a
teacher, and I am sure I will continue to do so. There are many ways in which this
research has affected my teaching directly and consciously, and I believe many
ways indirectly and possibly subconsciously. […]

With all the tasks I made sure I gave clear instructions, but only enough
instructions to understand the task. What I found crucial, was not telling the
students what to do! My year 7 class this year always wanted me to tell them if
they had finished, I never did! Each time a student asked I would ask them back
"Well, is there anything else you could do?" whatever answer they gave I said
"OK". Even when they said "no" they somehow knew they had something left to
finish so would ask their partner, or would just figure it out. Something this
simple, transformed the class, they were engaged and took ownership of their Theme E - L. Brown and A. Coles
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work. They were completing tasks for themselves, not under my instruction. They
had independence! […]
[T]he progress I have made this year, with all my classes, and in particular one
boy has amazed me! I do not think I have done anything miraculous, mainly
consistency and praise. I have always known these two things, however, the
students eventually wore me down. This year, I have remained positive with my
students and myself within an inclusive classroom. Simple things like saying
hello to the student who I had to send out of the lesson just an hour before, meant
I was building that bridge faster and more effectively. […]
I will continue to challenge students to approach problems, systematically and
logically, in order to help prepare them for life. (McCarthy, 2012)
Comment: The Masters in Mathematics Education course at the University of Bristol
is predicated on students (who are all teachers) developing their own practice through
researching an issue in their teaching. Jen, above, describes how the process of
exploration has impacted on her own pedagogy. There seem to be some design
principles emerging, one of which, common to the primary classroom in Australia and
the secondary classroom in the UK, is 'only enough instructions to understand the
task' and 'don't tell them what to do'. It would seem also that, for the secondary
teacher in Australia , in getting her class to do 'People Maths' there would be minimal
instructions to set up the task. Given the engagement of the students, however, the
teachers know that they are setting out to challenge their students in various ways and
achieve this through sharing time and not simply giving instructions, supporting their
students' independence.
At a different level, from my experience as a curriculum developer and
teacher, I was able to offer, without prior planning, activities linked to 'People
Maths'; chanting; and, support for Master's students and PGCE student teachers.
Chanting fractions in this way, for instance, was a familiar activity. Gee's (2004,
quoted in Sullivan, 2011) principles are in operation here since, in chanting, students
are placed as 'active agents' and it is clear that the purpose of doing the work is to
support students' engagement in a wider activity. Although I had never been in
exactly these situations before, what was "the same" was working contingently with
the ideas of the group and individuals and collecting strategies linked to a label. The
teacher participants are supported in becoming more aware of their own pedagogical
principles and in recognising the power of such principles.
Fullan's secrets of change and relentless consistency
The ideas within Fullan's leadership of change literature have been developed through
the process of implementing often large-scale change. Having been a consultant on
the less-than-successful implementation of the National Numeracy Strategy in the
UK, he has since been instrumental in leading teachers in Ontario, Canada to embrace
change, raising performance in literacy in the process. The purpose of 'improving
literacy in schools in Ontario' was identified, by top managers, as a focus for
development in the whole region. The challenge was, how could teachers take
forward that agenda as their own. A structure for redoing and rethinking was centred
around a meeting, once a year, called the 'Learning Fair'. In the first year of the fair,
when teachers from schools were encouraged to share what they had learnt about
literacy, many presented something from a book, say. By the third year of the Fair,
having picked up ideas from other schools and networked, there was a noticeable
growth in the motivation of the teachers whose confidence had built and they were
creatively sharing their practices. In schools, headteachers were encouraged to form Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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learning communities for sharing practice where they worked alongside the teachers,
not as expert but as learner.
Fullan's six secrets of change, distilled from these experiences, place
attention not on the creation of a body of knowledge about teaching and learning, but
on processes related to supporting change. There is an acceptance that we learn
through doing when this is related to a purpose or purposes, such as implementing
change through new materials, new behaviours/practices or/and new
beliefs/understandings, in a cyclical manner. Findings are at a meta-level to the actual
content of teaching and learning. For instance, one of the secrets is 'Learning is the
work'. In Fullan's theory-in-action, evaluations of effectiveness are built in to a
process of redoing and rethinking to implement change. Learning is a process and the
system can be set up to learn through the interaction of doing and thinking, because
"you are more likely to behave your way into new ways of thinking than you are to
think your way into new ways of behaving" (2008, see web-page). Fullan discusses
the importance of 'relentless consistency" within the system, not to dampen creativity
but to allow the rethinking and redoing cycle that seems so important. In his work
with teachers, 'snapshot views" are used to support them becoming aware of their
own learning. The system supports the teachers in observing themselves.
Reflections
In neither Story 1 nor Story 2 had Laurinda planned to use the tasks that she offered
(in Story 1, the People Maths and in Story 2, the multiplications). She had, however,
as Mike Askew said in a recent conversation with Laurinda during a seminar, faith in
them that they would be productive. In Story 1, the teacher does not start with the task
when planning, she starts with an intention, People Maths. She then adapts any given
suggested task, from the syllabus or the national curriculum, to include the practice
she has faith in. As a teacher educator working with a group of Master's students,
Laurinda did not prepare a set of slides but created a space for discussion. She has
used the invitation to share what is meant by curriculum before and is always struck
by how experienced teachers have these intentions and their students know what to do
in their classroom. In Jen McCarthy's case, she is able to reflect on her journey to be
able to create the classroom culture she wants. She is now in her fifth year of teaching
and not only does she have an intention that her students will be independent, but she
also, over time, illustrates how she has made learning her task and now has the skills
to achieve this. She is clear that she will continue the process of learning. Jen is
relentlessly consistent now in not telling the students what to do.
In Story 2, Laurinda and the teacher had not planned for a lot of the
mathematical ideas that arose. The focus in the book of activities had been area,
particularly conservation of area. The level of difficulty was beyond what was
expected and the content was not focused only on area, but particularly on fractions
and decimals. The children were 'active agents' in this process. The teacher, similarly
to Jen McCarthy, did not go beyond supporting her children's understanding of the
problem when introducing it to the group. The children were well used to the
sequence of prompts: the invitation for someone to read the text; identification of
keywords; and finally, identification of the problem. Here is the relentless attention to
consistency of practice that seems to hasve opened the way for creativity. It is as if,
not having to pay much attention to the routine whilst introducing the activity, a first
lesson of a sequence frees the attention of the teacher to learn her students and the
attention of the students to work on the activity. Theme E - L. Brown and A. Coles
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In any curriculum innovation there is the issue of interpretation. In Peter
Sullivan's book, the Granny's rug problem is defined as an open one. Given the
variety of responses, this seems clear. However, another definition of an open task
might be that it will extend people's thinking. Was it the task itself that did this in this
teacher's classroom, or the procedure established through which the children began to
engage with the activity? Another example provided by Barbara Clarke in a seminar
discussing these ideas at Monash University was that an activity was open if the
children have choices. Having watched a video of the beginning of the lesson where
Alf was teaching using the function game, there were at two contrasting views of the
task. One that it was closed because the teacher had an answer in mind and secondly
that it was open because the children had choice, they decided what the next starting
number would be, for instance. At the start of Granny's rug problem there was no
defined task to do, in that no measurements for the hallway are given. In this teacher's
classroom there were no signs of protest from the children that they did not know
what to do. So, the task remained open for the children and we, as teachers, had to
adapt to what was offered by them. What we seemed to be doing was closely
observing what the children were doing, not necessarily intervening. In the sharing
time, the move into chanting was comfortable, because it is a process that Laurinda is
able to use creatively, adapting to what is offered. In this case, the focus is on
multiplication of fractions. Laurinda could also have turned the fraction focus into a
People Maths idea. How do we make decisions about what to do as teachers? In the
moment we simply act out of all of our past experiences. So, you give me a task, no
matter how well designed and, as a teacher, I will work with it. Similarly, when the
culture of the classroom is established, even a page of calculations from a textbook
will be tackled by children such as those in Story 2, through opening up the questions
and discussing alternative strategies.
In conclusion, intentions and first lesson tasks by way of illustration
Laurinda has spent a lot of her life interviewing teachers about or running
sessions related to first lessons. First lessons both as the first mathematics lessons
taught at the start of the academic year and first lessons as starting a new area or topic
of work. Experienced teachers 'do the same problem' (see Brown, Reid and Zack,
1997) with their groups to establish a culture in which they and their students are
comfortable to act. In teaching the first lesson with a new group, established teaching
practices of teachers, which are often invisible and implicit in classrooms later in the
year because they have become part of the culture, are made visible both by the
teacher commenting on how they want the students to behave explicitly, or in telling
what happened afterwards in interviews.
Intention - story - Teacher C: I like to give things a story because I like to give the
children a natural language as a parallel to the mathematical language. I think it
allows enabling people to enter the world of maths you are talking about, then, if you
have got a story, if it's amusing or catchy in any way they might get interested in the
first place, but it does provide short simple language with which they can converse
with one another. So, it allows for group work, which is something else I think.
First lesson - activity - Teacher B: The calculation side of it is sort of trivial in a way
but the maths of it is looking at the structure, looking for a particular result, trying to
understand how the numbers arrange themselves to give you the answer that you Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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want. It's called difference of 3 although I'll probably run it as difference of 4. What
you do is put 4 numbers into a two by two grid. And you multiply the two top
numbers together and the two bottom numbers together. Add them together and write
down the total. Do the same with the columns. Add them to get a total. What's the
difference between the two totals? Can they get a difference of 4? 7? Whatever. I'd
like to see them asking each other "What if you try this?' type questions. It's all part
of modelling, it's what happens when you change something.
It seems to be the case that when a teacher has an intention such as 'People
Maths' or 'Story', then the actual activities used are not so important. There is a
relentless consistency in the style of teaching. Not every teacher would be
comfortable using these intentions all the time, but what does seem important is that,
over time, in the journey from prospective teacher to experienced, not only do
teachers find how they can work that allows their children to become enthusiastic
mathematicians but also that they share with other teachers so that they extend their
range of possible intentions, for example, not only story but also group work. In the
moment, any teacher simply acts contingently in response to their students' actions
within the culture that has been developed. Any task, no matter how well designed,
cannot be teacher proof, but is open to interpretation through the individual teacher's
and their students' cultural lens.
There is relentless consistency that allows the cultures to be established, but
also the creative space for both the children and teachers to continue to learn. The
teachers have a faith that these ways of working will productive. In talking about the
first lessons, the behaviours of the students that are valued by the teacher are
explicitly mentioned and would be mentioned in the classroom, as metacomments.
The ease of the arithmetic and the teacher's familiarity with the problem (adapting a
first offer of 3 to 4 suggesting consistency of problem with creative adaptation)
allowed the teacher to focus their attention on structures within doing mathematics
that they could stress during this first experience of students working with them
problem solving.
So, to sum up, a couple of design principles for teacher decision making in
relation to tasks:
- Do something that you are comfortable with that you have done many times
before to allow you to adapt and set up the rules of the community with relentless
consistency. You can then begin to learn the students. This can be comfortable either
at the level of intention or using a familiar problem, making decisions about how to
introduce the task, adapting or learning from previous experiences.
- Keep attention on the process and meta-comment about routines.
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Reflections of Nigerian teachers on a learning experience designed to
develop pupils' multiplicative reasoning

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Nigeria
This paper presents the reflections of Nigerian teachers on a learning
experience which was designed to develop pupils' multiplicative reasoning. The
teachers' views indicate that the learning experience was of benefit to the pupils and
teachers.
Keywords: International education, mathematics learning, instructional aids,
mathematics tasks
The purpose of this paper is to report the reflections of a few Nigerian teachers on a
learning experience which was designed to develop pupils' multiplicative reasoning. This paper
addresses how features of design influence teachers' decisions.
Description of the tasks
An outline of the tasks which were given to the teachers to implement is shown below. The
tasks were designed by a mathematics teacher (the author). The purpose of the learning experience
was to help develop pupils' multiplicative reasoning through an approach which would be enjoyable
and interesting for young children. The tasks were written in a directed manner, in part because of
the costs involved in contacting the teachers if further clarification was required.
Exploring proportional reasoning with pineapples
Targeted year groups – Year 2&3
Duration: 30-40 minutes
Group the children in teams of 4 or 6
Materials: A4/A3 paper, scissors, ruler, and a pineapple
Prepare circles on A4 paper, about 12cm radius; divide each into eight equal sectors.
Key words: double, half, quarter, triple, third, eighth, equivalent fraction, etc. Theme E – K. I. Omoregie
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Opener: Display one circle and present question 1
Q.1. Tami, Usman and Abby share a circular pineapple slice. It's cut into eight equal
pieces, through the middle, like this (show the picture) Tami gets one piece, Usman three and Abby
four. Danladi says that Abby got half of the pineapple slice. Show your group whether he is right or
wrong, and why.
Give one circle to each group of children. Guide children to cut out the sectors, and arrange
the four pieces Abby got – it should be evident, even without cutting, that four is half of eight, and
that the ratio of Abby's portion to the whole slice is 1:2, otherwise expressed as "Abby gets half of
the pineapple slice" Point out to the pupils that four eighths and one half are equivalent fractions.
If quite a large number of pupils are struggling at this initial point, it might be necessary to do a
whole class demonstration at the board, with a group of pupils as helpers to arrange the pieces over
a full circle.
Emphasize that the words in bold below are important to preserve the definition of the
word fraction – Equal portions are necessary. A pineapple slice representation that isn't perfectly
circular wouldn't fit the definition- hence the choice of paper circles to represent the slices. We can
also ignore thickness of the slice, since the paper is uniformly thick. Explain that other shapes can
be split into equal portions, e.g., a Kit Kat chocolate bar. Draw a rectangle, split into eight equal
sections, shade four of them. Draw one of exactly the same size and shade half of it. Again, point
out to the pupils that four eighths and one half are equivalent fractions.
Main activity
Guide the groups of children similarly, by arranging pieces, to compare Tami's portion to the whole;
and then, each portion with the others. For each portion, draw a rectangular representation of the
fraction, so that the children get used to multiple representations of the same fraction. Encourage
the children to discuss in groups, to write their completed statements, and draw pictures for each
fraction on a shared A4/A3 piece of paper. As the children get used to working with less direction,
introduce some open questions.
Examples of statements for pupils to complete – (You could modify these and add more
statements & open questions)
T:W
1:8
Tami's portion is one eighth of the whole slice.
Tami gets one eighth of the pineapple slice
The whole slice is eight times as large as T's portion
A:W
4:8
A's portion is . . . . .of the whole slice.
A gets ….. of the pineapple slice
The whole slice is 2 times/twice as large as/double . . A.'s portion
These are examples of the solutions one would expect from the pupils' comparison of each child's
portion to the others:
A:T 4:1 A's portion is four times as large as T's,
T:A 1:4 T's portion is one quarter of A's
U:T 3:1 U's portion is thrice/ three times as large as T's Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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T:U 1:3 T's portion is one third of U's
A:U 4:3 (slightly more difficult) U's portion is three quarters of A's
Plenary
Have each group come up to the board for a few minutes to share their pictures and
completed statements with the whole class as a small demonstration of what they have learned.
Recap the important points, check for understanding of the keywords and clear up any
misconceptions.
Snack time
Have the pineapple cut into even slices. Have one slice for each group, and guide each
group of children to share their assigned slice equally, with a grown-up at hand to help with cutting
the slices into equal sectors.
Extra Notes
Q. 2 below might be used in a follow-on lesson to reinforce what has been learned. Crosscurricular learning opportunities in science can also be explored. It might be beneficial to plan a
linked science lesson for use alongside this activity. Health science-nutritional value of fruits &
Agricultural science-types of plants are possible linked-lesson ideas.
Q2. Kore, Gbenga and Titi share a circular pineapple slice cut into twelve equal pieces. They
receive two, four, and six pieces respectively. Guide the children to do similar comparisons, using
circles split into twelfths.
A definition of tasks which encouraged looking beyond tasks to the type of mathematical
thinking and activity stimulated in the pupils (Watson et al, 2008) was adopted. It seems that a
teacher's natural intuition, influenced to some degree by the different texts read and previous
training received, informed most, if not all of the task design decisions, and that the decisions are
supported by the theories in the texts below. The following literature regarding the sorts of
mathematical thinking and activity that good mathematics tasks are aimed at generating were
consulted: (Piggott, 2004) (Piggott & Back 2004) (Watson & Mason, 2004) (Ingram & WardPenny, 2010) (Breen & O'Shea 2010).
Retrospectively, the links between the literature and the design of the tasks can be
identified: Ingram and Ward-Penny (2010) explain that it is possible for students to be 'engaged'
with a task, busy and seemingly enthused, without actually engaging with the mathematics. A
feature of this task that guards against the pitfall described is the requirement for pupils to articulate
their mathematical thinking both in writing and in speech, so that the teacher is able to check for
engagement with the mathematics.
Piggott (2004) identifies mathematical thinking strategies, some of which apply to the
tasks. They include- conjecturing/theorising, testing ideas- guessing and testing (hypothesizing),
representing information and reflecting on experience. Piggott and Back (2004) explain a concept
termed 'teaching for problem solving' – involving children in solving real problems. In the tasks,
the sharing of the pineapple slice enables pupils to apply mathematics to a real-life situation which
is relevant to children their age. Watson and Mason (2004) explain the importance of explicitly
requiring learners to reflect on their learning experience, noting that developing reflective practices
among learners is key. In the tasks, both verbal and written reflection are encouraged.
Another definition of tasks which encouraged looking beyond tasks to learning experiences
-with tasks considered as just one element of design- was also adopted (Swan, 2008a). One aim was
to create a learning experience that modelled a collaborative orientation of learning (Swan, 2008b).
Hence for the tasks, organising the children in groups of 4 or 6 was instructed. Williams (2008)
suggests that an optimal group size for interventions is a carefully chosen pair or three, and suggests
that larger groups might allow group members to get by without actively engaging with the Theme E – K. I. Omoregie
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learning. Thus retrospectively, groups of 2-3 may have been better, depending on practicalities such
as total class size.
The learning experience was structured to encourage group discussion and foster active
student engagement, based on recommendations from studies on mathematics teaching in Nigeria
(Salman, 2009) and beyond (Krainer, 1993; Henningsen & Stein,1997). The improvement of
mathematics teaching and learning in the country has been noted as essential by researchers
(Eniayeju, 2008; Popoola, Ogini & Ojo 2011). The use of paper circles to model fractions are in
line with recommendations to encourage the use of locally available concrete materials and other
learning aids in mathematics classrooms (Afolabi and Adeleke, 2010). The use of circle sectors to
represent fractions of a whole is not new: Thompson and Saldanha (2003), for instance, in their
discussion of fractions and multiplicative reasoning, give a pizza fraction example, but the context
and the approach used in the tasks were original.
The learning experience aided the pupils in developing mental images for concepts and
encouraged the use of multiple representations (Swan, 2008b). It enabled testing by pupils of
conjectures (Pratt and Noss 2010). It also aligned with the definitions of purpose and utility
proposed by Ainley and Pratt (2005), in terms of a meaningful outcome for the pupils, and helping
them gain an appreciation of situations in which it might be appropriate to apply the skills they were
developing.
Excerpts of the reflections of teachers from four schools are outlined below. The
reflections were sent in writing-the teachers were not interviewed. In retrospect, it would have been
useful to speak to the teachers and confirm whether they carried out all the suggestions outlined in
the task.
Stone pupil) said that R(another pupil) has one out of the eight divisions of the apple she brought.
Relating fractions to actual experience or to the use of what goes on around us has really helped in
making the topic easier to explain to the pupils.
At first it looked strange to them but after proper explanation, they really understood and enjoyed
it. . . The method helps the children to understand fraction(s) better as they were responding well
to the questions asked. I used three pupils to represent Tami, Usman and Abby. One represented
Tami and I gave him one sector of the pineapple. One represented Usman and I gave him three
sectors of the pineapple. One represented Abby and I gave him four sectors of the pineapple
which meant half of the whole pineapple.
Learning fraction(s) at this level is difficult for children but with cutting and sharing, the pupils
make fun with it and enjoyed the lesson. Using resources at hand and engaging the pupils to
contribute to the lesson facilitated the class exercise and it enable(d) the pupils to understand the
topic without) stress; the method is good but expensive to implement.
Demonstration and practical aspect can be introduced in teaching maths for effective learning
because not all of the pupils have interest in mathematics. The lesson was quite interesting.
The lesson was interesting. They (the children) enjoyed the lesson because the pineapple was a
motivational material for the pupil's learning. The cutting of the circle into eight part(s) and giving
the pupils (the) puzzle to tackle on the board was quite interesting and new. It was quite an
interesting class. The class was more interesting seeing the children chanting and cheering
themselves after answering questions. It was even more fun after the whole learning period when
they had to share the pineapple amongst themselves.
From the teachers' feedback, the following link between task design features and
pedagogical choices of the teachers can be inferred (Fig 3): Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Figure 3
The task type closely relates to one referred to as 'manipulating-getting-a-sense-ofarticulating' (Mason and Johnston-Wilder 2004). A pedagogical decision of a teacher was to listen
(Freudenthal 1991, Davis 1996) to the children articulating their understanding of the mathematics,
while another decision of a teacher was to increase opportunities for the pupils to actively use the
concrete objects, manipulating these to get a sense of the mathematics that they were learning
(Salman 2009).
From the reflections, it appears that the learning experience was useful to the pupils and
teachers.
It is hoped that the process outlined has the potential to contribute to the improvement of
mathematics teaching and learning in Nigerian classrooms.
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Designed to facilitate learning: Simple problems that run deep
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The article focuses on the design and implementation of simple
tasks in a teacher education course for mathematics teachers. It discusses
the rationale for this design including prospective teachers' difficulty in
task analysis and in-depth understanding of task specifications. Then the
effect of simple tasks is tested and exhibited through two detailed case
studies in which the implementation of two tasks is observed. The tasks
are different in nature, one is an algebraic traditional task and the other is
a geometric modeling task. Still, both share design similarity, afford many
learning opportunities, and can facilitate task design by these future
teachers.
Keywords: Task design, teacher education, modeling
Introduction
The design of good effective tasks is crucial in teacher education for two related
reasons. The teacher educator wants to use effective tasks to convey the many goals
of teacher education, and the prospective teachers need to experience good task
examples that would facilitate their own future task design.
In this article we suggest that effective tasks do not have to be complex. We
support this claim with data from the second author's doctoral research supervised by
the first author. The study informs us of the power of simple tasks through
observations and analysis of class implementation of tasks that were designed and
presented by the first author in a mathematics teacher education course. The simple
design of these tasks is intended to serve as a relatively easy model supporting teacher
pedagogical decision making in class,
Although the design of tasks is going to be a major part in their practice,
prospective teachers cannot be expected to be good tasks designers without getting
some support through specially designed instruction. This has become evident to us
when we asked our students in a teacher education course to compare two tasks. One
of the tasks involved cutting greeting cards (this task is described further on in the
first case study) and the second involved pouring beer from a container into cans. The
main and relevant difference between the problems involves the rigidity of the
cardboard versus the "flexibility" of liquid. This feature results in different types of
"remainders", as the rigid material does not allow remaining scraps to be put together
(unlike a situation such as cutting cookies with "flexible" dough). This difference
leads to fitting very different mathematical models. Yet, instead of identifying this Theme E – I. Peled & A. Suzan
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feature, the students suggested some minor characteristics such as the fact that one
problem is two dimensional and the other is in three dimensions.
This task analysis difficulty strengthened our motivation to design simple
tasks and to make our design principles explicit to our students. It can be said that the
nature of our tasks became a part of the didactical contract in the course.
Further support for the need to stay simple involves the type of change we try
to achieve through the tasks. Often, as teacher educators, we aim at changing
prospective teachers' conceptions. With some of our goals such changes can be more
easily acquired when task context involves simple situations. Peled and Balacheff
(2011) discuss characteristics of task examples that aim to promote a shift from a
problem solving perspective to a modeling perspective. They use simple word
problems in an effort to promote teacher understanding of the meaning of fitting
mathematical models in different types of situations. The authors claim that the use of
these simple problems instead of complex modeling problems helps avoid teacher
resistance to problems that are very different from what they use traditionally, and
that it helps focus on the main current goal.
Theoretical background
As mentioned in the introduction, the design of tasks in teacher education has to take
into account prospective teacher resistance to change. This is not surprising when we
become aware of the strength of teacher beliefs investigated in many studies in
teaching mathematic and other subjects as well (Stuart & Thurlow, 2000; Kagan,
1992; Brown, Cooney, & Jones, 1990).
According to Piaget (1985), a change in knowledge and new learning can
occur through the creation of discomfort (disequilibrium, in Piaget's terms) with
existing knowledge or beliefs, leading to an effort to make some adjustments that
would restore the state of equilibrium. Following this recognition, Zaslavsky (2005)
designed tasks that were aimed to create uncertainty. Similarly, in our own work
(Peled & suzan, 2011), we describe the effect of a task that was designed to create a
cognitive conflict, a discomfort that was supposed to lead to abandon intuitive
knowledge for the sake of using formal knowledge. However, we found that creating
a conflict is one thing and resolving it is something else. That is, as suggested and
identified by Limon (2001), the design of good tasks has to obey certain conditions in
order to create a meaningful conflict leading to conflict resolution.
When teacher educators use tasks that create cognitive conflict to facilitate
desired changes in prospective teachers' conceptions, they can, at the same time,
acquire two other important goals. The students, the prospective teachers, can learn
that tasks that lead to a cognitive conflict can be effective means for facilitating their
own future students' learning. In addition, as described by Peled and Suzan (ibid),
becoming aware of the power of intuitive knowledge over their own performance,
teachers might become more sensitive to their students' difficulties. As analyzed and
discussed by Jaworski (1992), developing this sensitivity is a central teacher
education goal.
As can be seen, cognitive conflict tasks can aim at different types of
knowledge. They might have mathematical, pedagogical or psychological goals.
Peled and Balacheff (2011) discuss the use of such tasks to acquire epistemological
goals in the case of learning and understanding issues of modeling.
Modeling, as defined by Peled (2010) in more relaxed terms, is the process of
fitting a mathematical model to a situation described in a given problem. This process Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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includes the analysis and organization of the given situation before any
mathematization is done. Most of the studies on modeling (for example: English &
Watters, 2005; Lesh & Doerr, 2003) involve quite complex problems some of which
are termed MEA (model eliciting activities). These problems involve an extensive
situation analysis and their main purpose is developing modeling skills, i.e. the ability
to analyze, represent, and make autonomous decisions and choices of representations
and mathematical tools.
Following a theoretical analysis of modeling (Peled and Bassan, 2005) and
with the goal of developing teacher insights about the process of fitting a
mathematical model rather than developing skills, Peled and Balacheff (2011)
discuss (as mentioned in the introduction) the rationale for using problems that are not
as complex as MEA.
In this study we observe the effect of simple tasks that have been designed to
make a change in different types of teacher knowledge.
Procedure
The study was conducted in a university teacher education course taken as a part of a
mathematics teacher certificate program. The 24 students participating in the course
were in the third and last year of their undergraduate studies towards an academic
degree in mathematics.
The first author, a researcher and teacher educator, was the course instructor.
She designed the course with the purpose of promoting student meta-cognitive
perspective of teaching and learning and with the pedagogical approach of "not
telling" in the spirit of Lampert (1990). Using a design experiment methodology, as
described and used by Cobb et al. (2003), she has kept reflecting on the effect of tasks
through several years of conducting the course, redesigning new tasks or modifying
existing tasks accordingly.
The second author participated in all class sessions, recorded them and took
notes. Student homework and their final course assignment were also collected and
analyzed. She also met with the instructor and took notes of the instructor's
explanations for task goals and task design.
The implementation of each of the course tasks included a sequence of
activities. Each task started with a problem that the students were asked to solve in
class, and sometimes solve again at home. This activity was followed by class
discussion and by individual reflection that was given as a homework activity.
Results
The findings presented in this article are a part of the findings reported in the doctoral
dissertation. We focus on two of the course tasks, where each task includes a
sequence of activities, as mentioned above. The research questions refer to the nature
of the tasks and to their effect. We analysed the learning opportunities created by each
of the tasks, and within each task we identified the added value of the reflection
activities that followed the problem. We also analyzed the nature of the tasks and
identified some general design similarities.
The two problems that start each activity sequence are a geometric problem,
involving a print-shop where the workers are cutting greeting cards from a cardboard,
and an algebraic problem where show tickets of different prices are purchased.
The first problem, the greeting cards problem presented in Table 1,
originated in a graduate course where students interviewed print-shop workers on the Theme E – I. Peled & A. Suzan
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situation described in the problem. While the students thought that they should divide
the area of the cardboard by the area of the greeting card, the print-shop workers used
a different mathematical model depicted in Figure 1.
How many small 6 cm X 8 cm greeting cards can be cut
from a 65 cm X 85 cm piece of cardboard?
Table 1: The Greeting Cards Problem
Using a symbolic formulation, the mathematical model used by the workers
involved taking the maximum of the value of [L/l]x[W/w] and the value of ,
[L/w]x[W/l], while the students used an area division model, [S/s], which is much less
appropriate in this situation.
Figure 1. Print-shop worker options for cutting greeting cards.
The goal of this relatively simple modeling situation was to develop
understanding of the idea of fitting a mathematical model in a given situation. The
implementation of the problem resulted in the construction of a wide range of models.
About 40% of the students suggested a model that was close or identical to the area
division model. A solution of this type is, most likely, an indication of an automatic
identification of the problem as an area problem without making an effort to represent
and analyze the situation by, for example, making some drawing of the cardboard and
cards.
In the following activity students were asked to generalize their models in
order to increase their awareness of the different parameters that were involved. As a
result about a quarter of the students shifted to a better model. The discussion that
followed these two activities, problem solution and problem generalization, created
many learning opportunities that cannot be detailed in this short paper. They include
awareness of problem features (simple and yet eliciting different models that can be
compared), mathematical knowledge (use of mathematical representations, concept of
area), pedagogical and psychological knowledge (misconception of over-linearity),
and epistemological (developing meta-understanding of fitting a mathematical
model).
The second problem, the show tickets problem presented in Table 2,
originated as a result of an error made by very few students when asked to solve a
similar problem without a request for multiple solutions. Most of the students chose x Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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to stand for the value that was the described reference, and solved the problem
correctly. The few who failed chose x to stand for the less convenient value. As a
result of their choice, they had to convert the given relationship between the values. In
making this conversion they wrongly asserted that: If A is by K percent smaller than
B then B is by K percent bigger than A. Obviously, this is a wrong generalization of
"regular" number relations.
Mary took her daughter and her neighbour's daughter to a show. Her
neighbour wanted to pay for the show's ticket but Mary did not remember the
exact price.
She knew the total sum was 143 IS and that a child's ticket was 20% cheaper
than an adult's ticket.
Help Mary figure out the child's ticket price. You are requested to do it in
two ways, each time using a different value (adult's ticket price, child's ticket
price) to be represented by the variable x.
Table 2: The Show Tickets Problem.
The original purpose of the modified task was to "not tell" students of their
mistake but give them a chance to become aware of it on their own. However, further
experience with the multiple solutions version made us aware of the many goals this
problem together with its following activities can achieve.
While a conventional problem version resulted in avoidance of the need to
make a percentage relation conversion, the multiple solutions version forced all the
students to face the conversion difficulty. As a result, 20 out of 24 (83%) solved the

non-routine and less convenient alternative incorrectly, asserting that if the child's ticket is 20% cheaper than the adult's ticket, then the adult's ticket is 20% more than the child's ticket. Since this alternative led to an incorrect solution while the convenient route resulted in a correct solution, these students realized that they have two different solutions and therefore something must be wrong.
Thus, most of the class experienced a cognitive conflict situation. However, as it turned out, conflict resolution did not occur spontaneously. A class discussion that immediately followed resulted in partial improvement. As depicted in Figure 2, only about a third (6 out of 20) of those who faced a conflict, managed to resolve it.
Figure 2: Conflict and (very partial) conflict resolution. Theme E – I. Peled & A. Suzan
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The analysis of learning opportunities created during the activity sequence and especially during the discussion reveals the many goals that can be promoted through this one simple task. Similarly to the first task, the learning opportunities consisted of mathematical, psychological, and pedagogical goals. The teacher educator seized the opportunity to talk about cognitive conflict in general, making the students aware of what they experienced. Since they also experienced difficulty in conflict resolution, this pedagogical and problematic issue could be demonstrated through their own work and thus better understood.
The discussion brought up student realization of the strong power of intuitive knowledge. Some students explicitly expressed their increased sensitivity and understanding towards their future students' mistakes in view of the fact that in spite of their mathematical knowledge they themselves failed to resist incorrect intuitions. As stated by one of the students:
I personally felt uneasy because I am supposed to know this subject very well and solve this problem easily. Suddenly I got stuck and the solution wasn't so easy and fluent as I expected. Now I realize that if I, who master this subject, got confused, it can certainly happen to my own students too. We, as teachers, shouldn't be upset with them when it happens.
In addition to discussing the task as an example of a cognitive conflict task, students were also made aware of its simple design. The modification of a conventional algebraic problem into an effective interesting task required only a simple addition of a request to solve the problem in two ways.
Conclusions
Our study suggests that prospective teachers can benefit from experiencing simple tasks. This assertion is tested and confirmed in the article through the observations made in a teacher education course conducted in the spirit of a "not telling" approach. Through specially designed tasks, prospective teachers were expected to undergo knowledge change in their mathematical, pedagogical, psychological and epistemological knowledge.
The two case studies presented in the study involve two different tasks. The analysis of learning opportunities in both tasks has shown that not only did each task achieve its main goal, but also that in spite of the relatively simple formulation of the problems, many additional goals could be achieved. This fact was explicitly discussed with the prospective teachers. The intention was that they would experience examples of pedagogical decisions in task design and would become more confident that the task of task construction is within their limits. Of further help was the extensive discussion of the use of cognitive conflict to promote learning. The self experience of change and learning as a result of working on a simple task that triggers the emergence and resolution of a cognitive conflict is expected to make these task design principles a part of their teaching repertoire.
Thus, we believe that the exposure to simple problems and the explicit discussion of problem features created an opportunity for developing task design skills. Although this skill was not tested, students' awareness of task design characteristics was exhibited in their discussions. As mentioned in the introduction, the design of course tasks using explicit design principles became a part of the course didactical contract. As a result, when a new task was presented, students could tell that even though it looked simple, they could expect it to run deep leading to interesting and significant learning. Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Engaging teachers in the web of considerations underlying the design of tasks that foster the need for new mathematical concepts or tools: The case of calculus

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This paper reports on a design experiment, within which the design principles emerged in an iterative way, through examples, expert comments, reflections, and revisions. The focus for this part is on the design considerations of tasks that problematize the use of students' existing mathematical tools and set the grounds for learning about new (for the students) tools. Having gained insights ourselves, we built on our experiences to involve teachers in similar considerations. It turned out to be a way of thinking that teachers had not experienced before, nor were they aware of the possible web of considerations that underlie design issues related to specific choices of examples for tasks that aim at preparing the grounds for a new mathematical concept. We argue that this kind of thinking is not sufficiently attended to, yet should be part of teachers' knowledge for teaching mathematics.
Keywords: task design, teacher education, calculus.

Introduction

Typically, the process of task design is often a 3-stage hierarchical process that includes: 1. Stating goal(s) and connecting the task to the goal(s); 2. Designing a generic task that addresses these goals; and then (when applicable) 3. Carefully choosing the specific examples to "plug in" the generic task. Teachers usually are not engaged in this full process, and mostly make choices about existing tasks in textbooks, rarely with some (minor) modifications.
In this process, the specific examples can change dramatically the quality of the task, its connection to the goals, its target audience, how it may unfold in the class, etc. Analysing the subtle anticipated differences between various versions of the same generic task that only differ along the specifics chosen, and making judicious choices of the specifics of a task, is an important element of desired teacher knowledge and practice that our study aims at developing. In this paper we focus on a particular goal for teaching mathematics, namely, preparing the grounds for introducing a new mathematical concept or tool. More specifically, we illustrate this Theme E – G. Ron, O. Zaslavsky & I. Zodik
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approach and way of working with teachers through the concept of the derivative as a tool for detecting the behaviour of a graph of a function.
Throughout the school years, teachers are often faced with the task of presenting mathematical concepts that are new to their students. To do this meaningfully, it is important to create the need for the new concepts that are introduced. We base our stand on the necessity-based approach to learning (Harel, 2007). That is, it is important to help teachers learn to problematize the situation for their students. One way to do this is with the use of carefully designed tasks.
This constitutes a challenge for teachers and teacher-educators, because on the one hand, we often want to point to the limitation of existing tools for a particular purpose, while we need to maintain the usefulness and merits of the existing tools for other purposes (otherwise students will believe that much of what they learn will need to be abandoned in the future). We do not want students to feel that what they have learned so far becomes useless as they progress in the study of mathematics. Thus, the progression from existing tools/concepts to new tools/concepts should lead to an extended mathematical 'toolbox'.
From a design perspective, with respect to tools, we would like teachers to be able to design tasks that foster discussion of the merits and limitation of existing as well as new tools. For example, using a function table to sketch graphs (of functions) is useful for certain functions (e.g., linear functions) and certain properties. However, when we move to the study of more advanced functions (e.g., higher degree polynomial functions), we need to introduce the concept of derivative. But even then, tables still have a significant role. Clearly, derivatives can help detect significant turning points in a graph, which could get lost using an arbitrarily constructed table. However, derivatives may also serve as a tool to construct a more useful function table, thus, they support the value of function tables, so that students do not have to abandon this tool (function table) when more advanced tools (derivatives) are introduced.
In our study, the design principles emerged in an iterative way, through examples, expert comments, reflection, and revisions. Interestingly, we experienced a certain degree of tension in attempting to satisfy some guiding principles. Having gained insights ourselves, we built on our experiences to involve teachers in similar considerations. It turned out to be a way of thinking that they had not experienced before, nor were they aware of the possible web of considerations that underlie design issues related to specific choices of examples for tasks that aim at preparing the grounds for a new mathematical concept/tool. We argue that this kind of thinking is not sufficiently attended to, yet should be part of teachers' knowledge for teaching mathematics (Ball & Hill, 2009; Hill, Ball, & Schilling, 2008; Silverman & Thompson, 2008).
We turn to a detailed example from calculus, that illustrates the web of considerations that teachers should be able to employ – some more general, and some very specific and detailed. We then move to a report on teachers' actual thinking and experiences related to this example.

An analysis of the design process of a task for students

Initial Assumptions and Decisions

As the first step of the design process, aimed at introducing the concept of derivative, we adapted a generic task that relies on students' prior experiences with Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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function value tables. Variations of this type of task can be found in some textbooks. The structure of the task that we developed is presented in Figure 1. Note that we build on students' prior experience in using symmetrical tables for sketching graphs. To turn this into a task for students, a teacher would need to choose the specific function f.
The next level of design requires a choice of the type of function f that would be used for this task. It makes sense to use a cubic function, assuming students do not have any prior image of how the graph of a cubic function should look. Choosing a quadratic function for this purpose would not be equally appropriate for the stated goal, as the behaviour of its graph can be deduced by other considerations (e.g., symmetry). Actually, it is possible to sketch a rather accurate graph of any quadratic function with the use of a table and the special symmetrical properties of the function (that is one of the reasons that many secondary mathematics curricula include the quadratic function as part of algebra while the cubic function is part of calculus). We assume that after dealing with quadratic functions, students are familiar with the concept of an extreme point of a function, at least at an intuitive level. At this stage, by an accurate sketch of a graph of a function we mean a graph that captures all its extreme points, as well as its increasing and decreasing domains (clearly, moving further on, we can increase the accuracy, by detecting inflection points and concave and convex domains as well. At that stage, a similar approach can be applied to the introduction of the second derivative).
Figure 1: A generic task for students aimed at evoking some limitations of a function value table
The teachers with whom we worked, who had not experienced careful design considerations before, maintained that any cubic function could serve equally well for this task. However, this level of design calls for more refined considerations. In other words, once we decided to use a cubic function of the form: f(x)=ax
3
+bx
2
+cx+d, a≠0,
the question remained: what specific values would be good to choose for the

parameters a,b,c,d given the goals of the task? This level of detail is not commonly Theme E – G. Ron, O. Zaslavsky & I. Zodik
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addressed and was completely unfamiliar to all the teachers in our study. We turn to
the description of some of these refined considerations.
What specific function should we choose?
For brevity, we present only part of the web of considerations that could and
should be employed. After several deliberations and iterations, we presented teachers
with the following three possible functions that could serve for the task (Figure 2).
(1) (2) (3)
Figure 2: Three cubic functions to consider for the generic task for students
All three functions belong to the following sub-set of cubic functions:
f(x)=ax
3
+cx+d, a≠0. This choice was guided by our knowledge of the properties of
such functions. In particular, we considered the following two properties significant
from a design standpoint:
1. Although the function is not linear, its specific parameters should give
students, at first, the (wrong) impression that it is linear, when they
use familiar tools.
All three functions have an inflection point on the y-axis.
For all of them, any choice of symmetrical values for the table, of the form
x=-k, 0, k (k≠0), results in 3 co-linear points on the graph that could create
an illusion of linearity for students who complete the first function-value
table in Figure 1.
2. The graph of the function should be 'manageable' for students to
sketch, in the sense that the parts containing critical points can fit
easily in a regular notebook page.
For all three functions, the critical values to which students need to pay
attention are small enough so that the graphs can be sketched in students'
notebooks in a homogenous coordinate system; no need to worry about
scaling. The possibility of sketching the graphs in a homogenous
coordinate system eliminates distractions by technical aspects of graphing.
f x)(x 3x
3
= −)(2 1
3
f x = x − x +)(2 5.1 1
3
f x = x − x +Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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Moreover, being able to sketch the graphs in a homogenous coordinate
system is important in the context of developing the initial notion of
derivative, so that examining the slopes at different points on the graph can
be done visually without the need for any calculations.
Figure 3 presents an analysis of the three functions, from mathematical and
pedagogical perspectives. It conveys the complexity of considerations surrounding the
choice of a specific function for the task (in Figure 1). It raises the difficulty to
recommend one answer to the question: Which function (of the three) should we use
for the above task? The answer depends amongst others on the level of the students,
and on the teacher's goals and intentions beyond this task.
(1) (2) (3)
The function has 3
intersection points with the xaxis, all of which can easily be
found algebraically.
• For strong students, this may
not be a good choice, since
they could immediately see a
way to detect the entire
behavior of the graph by
solving the equation: x
3
-3x=0
and making intuitive
inferences.
• For weaker students, who are
not likely to be able to solve
this equation, it could be
better to begin by asking
them to find where the
function is positive and
where it is negative. This
builds on what they are able
to do, and the need for new
tools arises when asked about
extreme points of the
function.
• Choosing this function for
the task can be an advantage
if the teacher intends to use it
later to illustrate a full
analysis of the graph of a
function that includes finding
the (exact) intersection points
with the axes.
• In terms of the goal of the
task, the fact that the extreme
points are detected by the
function value table may
leave students with the
impression that there is no
real need for new tools.
At this stage, students have
no way to find the
intersection point of the
function with the x-axis.
• An advantage for choosing
this function is that students
are not likely to anticipate
that there are extreme points,
thus, this finding will
surprise them and reinforce
the need for new tools.
• Although this function can
serve to raise the need for
new tools, it would not be a
good one for illustrating later
a full investigation of the
graph of the function that
includes finding the extreme
points of the function.
• An advantage of this function
is that students are not likely
to find the extreme points by
checking various (arbitrarily
chosen) values in the table,
as the x-coordinates of the
extreme points are irrational.
Using the values -1,0,1 (or
actually any table with just
integers) gives the impression
that the function is an
increasing function.
• This may help achieve the
goal of the task.
• This function lends itself to a
gradual realization of its
properties: first students see
that it is not linear, then -
that it is not an increasing
function, and later - that
points that seem as extreme
points are not.
At this stage, students have
no way to find the
intersection point of the
function with the x-axis.
• An advantage for choosing
this function is that students
are not likely to anticipate
that there are extreme points,
thus, this finding will
surprise them and reinforce
the need for new tools.
The x-coordinates of the
extreme points are rational
numbers that can easily be
detected when extending the
values in the table (to halves).
• An advantage of this function
is that the values in the
extended table are all
integers (or other easy-tohandle numbers); this makes
the calculations easier and
reduces distractions that
complicated calculations may
cause.
• In terms of the goal of the
task, the fact that the extreme
points are detected by the
function value table may
leave students with the
impression that there is no
real need for new tools.
Using the values -1,0,1 (or
actually any table with just
integers) gives the impression
that the function is a
decreasing function.
• This may help achieve the
goal of the task.
f x)(x 3x
3
= −)(2 1
3
f x = x − x +)(2 5.1 1
3
f x = x − x +Theme E – G. Ron, O. Zaslavsky & I. Zodik
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Figure 3: An analysis of the three cubic functions under consideration
In terms of teacher education and teacher knowledge, it is not about finding
the right function, but about helping teachers make judicious decisions that are based
on careful analysis of affordances and limitations that are entailed in the choice of a
specific function. Yet, weighing all the considerations for each of the three functions
(Figure 3), it seems that all round function (2) may be more advantageous than the
other two.
Teachers thinking about the design of such tasks
The setting and research instrument
A group of experienced secondary mathematics teachers were given a written
questionnaire, in two parts. Some were also interviewed about their responses. The
first part of the questionnaire presented the task (Figure 1) with function (2), and
prompted teachers to examine the merits of this task, analyse its advantages and
disadvantages, and reflect on their own practice with respect to the use of similar
tasks that have the same goals. This part of the questionnaire also included a question
about desirable considerations the teachers would (or actually do) take into account
when choosing a function for this or for a similar task.
For the second part of the questionnaire we collected the considerations
teachers wrote in the first part, and prepared a list of these considerations; for each
consideration the teachers were asked to state whether they agree that it is desirable
and why. In addition, we had conversations with some of the teachers, to better
understand their thinking.
Main findings
Most of the teachers' responses indicated that they had not come across such
subtle considerations prior to our study. The questions evoked their awareness to the
need to address the issue of limitation of a table for sketching graphs of functions in a
more sophisticated and rich way than they encountered both in their practice and in
professional development frameworks.
The considerations the teachers suggested included properties of a function
that they thought would make it a good choice for the task, e.g.: 'a function that is
unfamiliar to the students'; 'a function with more than one extreme point'; 'a function
that is not symmetrical'; 'a function that is not special'; 'a function that the
coordinates of its extreme points do not appear in the table'; 'a function for which it is
hard to detect immediately its intersection points with the x-axis and thus its extreme

points cannot be inferred instantly'. Interestingly, none of the teachers mentioned at
all consideration 1 and 2 above, which could be viewed as critical.
There were teachers who stated that they do not use such tasks (as the one in
Figure 1 above). The main claim was that although this is a significant activity,
curriculum constraints do not permit spending time on such a task. Some maintained
that it is sufficient to build on students' (drill and) practice, because when you
practice enough problems with derivatives, then students gain understanding that the
derivative is zero at an extreme point. Clearly, this implies that for some teachers the
necessity principle is not part of their thinking.
Several teachers claimed that they rely on textbooks and that it does not
matter what function is chosen. The three specific examples (of the cubic functions)
in our study appeared to most teachers the same, for the purpose of the task; analysing Margolinas, C. (Ed.). (2013). Task Design in Mathematics Education. Proceedings of ICMI Study 22 . Oxford.
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the subtle differences and how they impact the task was totally new to them. Through
drawing their attention to the differences between the three functions in terms of their
specific affordances in addressing the goal of the task, and to the merits and
limitations of each example in terms of its learning potential for their students, they
felt they were able to make more judicious pedagogical decisions/choices, while
before - they were not aware of or able to see such distinctions.
Conclusion
The mathematical and pedagogical analysis presented above point to the
wealth and complexity of considerations underlying what teachers' decisions with
respect to task design for their students could and should be, in general, and for tasks
that address the need for advanced tools for sketching graphs, in particular. Based on
some empirical evidence, our study establishes the need to work with teachers in
ways that develop and expand their pedagogical toolbox and encourage them to build
on it when they design and make choices regarding specific tasks for certain purposes.
In this paper we focused on the purpose to establish the need to expand new tools.
This approach can be applied to other purposes and topics. While the scope of the
study may seem narrow, it can be generalized, to other similar situations (e.g., other
tools/concepts that need to be introduced) as well as to a more general level that deals
with working with teachers on the significant of choice of the specific examples for a
task, or in other words - how going into details could make a huge difference.
One of the goals of teacher education is to prepare teachers to consider
alternatives (Zaslavsky and Sullivan, 2011). We view task design as a fruitful site for
considering alternatives. It is not enough to support a specific choice. There is value
in considering other possible choices, and by that - crystalizing and refining the
criteria for such choices.
The approach we describe moves from finding the 'right' choice to realizing
that it is a give and take situation. Each choice has its merits and its limitations.
Articulating these merits and limitations is part of what teachers should experience
and what teacher education programs should foster.
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learning from and adaPting
the theory of realiStic
mathematiCS education
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Abstract: This article focuses on the critical role of design theory in our work as mathematics educators. We give
particular attention to a specific design theory, Realistic Mathematics Education (RME). We first clarify the enduring
contributions of RME to design in mathematics education and then discuss three adaptations that we made to RME
theory while conducting a series of classroom design experiments. The first of these adaptations involves taking a broader
perspective on the means of supporting students' mathematical learning to include both the organization of classroom
activities and the nature of classroom discourse. The second adaptation involves a change in orientation that acknowledges
the mediating role of the teacher. The goal of instructional design then becomes to develop resources that teachers can
use to achieve their instructional agendas rather than to support students' learning directly. The third adaptation again
centers on the teacher and concerns the potential contribution of designed instructional resources as a means of supporting
teachers' as well as students' learning.
Keywords: design research, instructional design, design theory, Realistic Mathematics Education, mathematics teaching
and learning.
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reflect the views ofthe Foundation.
Our focus in this article is on the critical role that
design theory has played in our work as mathematics
educators. To ground the discussion, we give particular attention to a specific design theory that we found
especially useful in our work over the past 15 years,
Realistic Mathematics Education (RME) developed
at the Freudenthal Institute in the Netherlands. We
first outline the type of research in which we engaged
when we first began to appropriate ideas from RME
design theory. Against this background, we clarify
the significance that we attributed to RME theory at
that time and then move to the present in order to
discuss our current views of the contributions of this
design theory. In the second part of the paper, we
focus on three adaptations we have made to RME
theory as our theoretical position and research interests have evolved over time. The first of these adaptations involves a broadening of our perspective on the
means of supporting students'mathematical learning
to include both the organization of classroom activities and the nature of the classroom discourse. The
second adaptation involves a shift in our orientation
as instructional designers such that our goal when
developing sequences of instructional activities is
no longer to support individual students' learning
directly. We now consider the mediating role of the
teacher as crucial and view ourselves as developing
resources for teachers to use to achieve their instructional agendas. The third adaptation again centers on
the teacher and concerns the potential contribution
of instructional sequences as a means of supporting
teachers' as well as students' learning.
the contributions of rme
RME is rooted in Freudenthal's (191 ;
193) interpretation of mathematics as a human
activity. In Freudenthal's view, students should be
given the opportunity to reinvent mathematics by
organizing or mathematizing either real world situations or mathematical relationships and processes
that have substance for them. In developing this
position, Freudenthal emphasized that the material
students are to mathematize should be real for them.
It is for this reason that approach is called Realistic
Mathematics Education. Freudenthal considered 106
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mathematizing to be the key process in mathematics
education for three reasons. First, mathematizing is
a major activity of mathematicians. Second, mathematizing fosters applicability by familiarizing
students with a mathematical approach to everyday
settings. Third, mathematizing relates directly to
the idea of reinvention, a process in which students
formalize their informal understandings and intuitions. Freudenthal argued with considerable force
that mature, conventional symbolizations should
not be taken as the instructional starting point. He
was particularly critical of this practice and termed
it an anti-didactic inversion because the process by
which the mathematicians developed mathematics
is turned upside down (Freudenthal, 193). For
Freudenthal and for researchers and instructional
developers who subsequently elaborated his ideas,
the goal of mathematics education should be to
support students' mathematical learning as a process
of guided reinvention.
Positive Heuristics for Instructional Design
To provide a context for the subsequent discussion, we first outline the type of research we were
conducting at the time that we became aware of RME
and began to focus explicitly on issues of instructional design. This research involved conducting a
series of year-long design experiments in US secondgrade and third-grade classrooms with seven- and
eight-year-old children. One of our goals in conducting these experiments was to extend the constructivist teaching experiment methodology developed
by Steffe and his colleagues to the classroom (Cobb
& Steffe, 1983; Steffe, 1983; Steffe & Kieren, 1994;
Steffe & Thompson, 2000). In a constructivist
teaching experiment, the researcher typically interacts with a small number of students one-on-one
and attempts to precipitate their learning by posing
judiciously chosen tasks and by asking follow up
questions, often with the intention of encouraging
the student to reflect on his or her mathematical activity. The immediate purpose when interacting with
the participating students is to study the process by
which they reorganize their mathematical reasoning.
The larger purpose when conducting a retrospective
analysis of the teaching sessions is to develop conceptual models composed of theoretical constructs that
can be used to account for the mathematical learning of other students (P. W. Thompson & Saldanha,
2000). As a consequence, although the researcher
acts as a teacher when conducting an experiment
of this type, the primary emphasis is psychological
and centers on the analysis of students'mathematical
reasoning. The development of instructional designs
is, in contrast, of secondary importance.
Our primary focus when we extended this methodology to the classroom remained consistent with
Steffe's emphasis on developing explanatory theoretical constructs rather than on formulating instructional designs. In particular, we gave priority to the
development of an interpretive framework that would
enable us to situate students' mathematical learning
within the social context of the classroom (cf. Cobb
& Yackel, 1996). This was the case even though

we had developed a complete set of instructional
activities during a design experiment conducted
in a second-grade classroom. Constructivism and
the related theories on which we drew provided us
with a number of negative heuristics that ruled out
a range of approaches to instructional design. For
example, we rejected what Treffers (198) termed
structuralist approaches to design in which physical
materials and graphics are developed that, for the
adult, embody the mathematical relationships that
are the target of instruction. However, the positive
heuristics that guided our development of instructional activities were global in nature. For example,
the constructivist perspective to which we subscribed
at that time oriented us to view mathematical learning as an activity in which students reorganize their
activity to resolve situations that they experience
as problematic. The instructional activities that we
developed were therefore designed to give rise to a
range of mathematical problems as students interpreted the activities at a variety of levels of sophistication. We also drew on neo-Piagetian analyses
of social interactions (Doise & Mugny, 199) and
viewed classroom social interactions as a potential
means of supporting students' progressive reorganization of their mathematical reasoning. In particular,
we conjectured that interactions in which conflicts in
students' interpretations became apparent would give
rise to learning opportunities for them, and that they
might reorganize their reasoning as they resolved
these interpersonal conflicts. As a consequence, the
structure of classroom activities in the second- and
third-grade design experiments involved the students leArning FroM And AdAPting the theory oF reAlistic MAtheMAtics educAtion
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first working in pairs to complete instructional activities and then participating in whole-class discussions
of their interpretations and solutions.
It was not until we first began to learn about RME
theory that we came to realize that our attempts at
instructional design were, in comparison, inadequate and that we needed to develop a more systematic approach. Our interest in RME design theory
was first inspired by Adrian Treffers's (198) book,
Three dimensions: A model ofgoal and theory description in mathematics instruction - The Wiskobas Project.
Treffers's intent is this book was primarily theoretical
in that he schematized two decades of instructional
design work and classroom experimentation, in the
process teasing out a number of positive heuristics
for instructional design in mathematics. Against the
background of our research at the time, the design
heuristics that Treffers illustrated by describing
several instructional sequences were a revelation.
It was apparent from Treffers's account that RME
was a detailed, empirically grounded design theory
that was compatible with our constructivist perspective on mathematical learning. For example,
RME's basic tenets of mathematics as a human activity and of mathematical learning as the progressive
reorganization of activity were both consistent with
our general viewpoint. In addition, we valued the
manner in which RME placed students' mathematical reasoning at the center of the design process
while simultaneously proposing the specific means
by which the development of their reasoning could
be systemically supported. Furthermore, Treffers's
account of RME clarified that the purpose for
conducting design experiments was not limited to
developing explanatory constructs, but could also
include developing, testing, and revising instructional sequences.
These insights led us to realize that we had
attempted to study students' mathematical learning in situations in which we had not necessarily
provided adequate supports for learning. The design
decisions that Treffers illustrated served to emphasize a relatively fine-grained level of detail that
had been absent in our prior work. The types of
design decisions that we especially noted included
both the careful selection of the problem situations
that were used during the first part of an instructional sequence and the explicit attention given to
the design of non-standard notation schemes as a
means of supporting students' reorganization of
their mathematical activity. In addition to taking
account of specific aspects of RME theory, we also
learned an important methodological lesson from
Treffers's presentation. The heuristics on which we
had relied were relatively global in nature because
we had derived them from a general background
theory. In contrast, the specific design heuristics
that Treffers outlined had emerged from and yet
remained grounded in the activities of designing
and experimenting in classrooms.
As these reflections make clear, there was much
that we could learn by becoming familiar with the
general orientation to design inherent in RME.
However, we were also aware that RME cannot be
reduced to a set of heuristics. It is a sophisticated set
of practices that have been developed by a community of designers and researchers over an extended
period of time. Gravemeijer (1994) clarifies why the
process of developing and revising designs in the
RME tradition cannot, in principle, be completely
codified. In reflecting on his activity as an instructional designer, Gravemeijer argues that designer's
activity resembles that of a bricoleur:
A bricoleur is a handy man who invents pragmatic
solutions in practical situations…[T]he bricoleur
has become adept at using whatever is available. The
bricoleur's tools and materials are very heterogeneous:
Some remain from earlier jobs, others have been
collected with a certain project in mind. (p. 44)
From the sociocultural perspective, this is an
excellent characterization of creativity and invention
(cf. Cole, 1996). Novelty and originality are seen to
be historically and culturally situated from this perspective and to involve the adaptation and reconfiguring of resources from a range of disparate practices
(cf. Emirbayer & Mische, 1998 ; Holland, Skinner,
Lachicotte, & Cain, 1998 ; Varenne & McDermott,
1998). Inspired by Gravemeijer's characterization
of design, we have in fact described our own efforts
to develop theoretical constructs that enable us to
situate students' mathematical learning within the
social context of the classroom as a process of bricolage in which we drew on and adapted ideas from
a range of theoretical sources for pragmatic ends
(Cobb, Stephan, McClain, & Gravemeijer, 2001).108
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This portrayal of RME as an evolving set of
interrelated practices implies that it is one thing to
attempt to understand a particular approach to design
by reading about it and quite another to engage
competently in the process of developing designs. As
a consequence of reading books by leading contributors to RME (e.g., Gravemeijer, 1994 ; Streefland,
1991 ; Treffers, 198), we became reasonably proficient at commentating on the design theory. However,
we had to co-participate in the process of formulating, testing, and revising designs before we became
able to develop adequate designs for supporting
students' mathematical learning. In this regard, an
ongoing collaboration with Koeno Gravemeijer in the
course of which we developed several instructional
sequences proved to be crucial. With hindsight, we
view our collaboration with Gravemeijer as a process
of apprenticeship in which we were legitimate peripheral participants in the RME research community
(cf. Lave & Wenger, 1991). In the course of this
apprenticeship, we consciously adapted several of the
ideas that we appropriated from RME while pursuing
a number of other interests. In the remainder of this
article, we first step back from our personal involvement in RME to consider its enduring contributions
to design in mathematics education and then discuss
three of these adaptations.
enduring Contributions
In our view, RME theory makes an enduring
contribution to mathematics education by offering
a viable solution to a long-standing problem that
was first delineated by John Dewey. In contemporary accounts, Dewey is frequently characterized
as a strong advocate of open-ended, project-based
approaches to instruction. It is true that Dewey
condemned the predominant instructional practice of
his day that promoted rote learning at the expense of
a deep understanding of disciplinary ideas. However,
as Prawat (1995) documents, Dewey was equally
critical of unstructured project-based approaches
in which students are encouraged to pursue their
own interests (Westbrook, 1991). He argued that
these two general instructional approaches constitute opposite poles of a dichotomy between disciplinary content on the one hand and students' current
understandings and interests on the other. A primary
goal of both his educational philosophy and his work
at the Laboratory School that he established at the
University of Chicago was to transcend this dichotomy. In this regard, he anticipated a central issue
with which mathematics educators continue to
struggle. Ball (1993) articulated this issue succinctly
when she asked:
How do I [as a mathematics teacher] create
experiences for my students that connect with what
they now know and care about but that also transcend
the present? How do I value their interests and also
connect them to ideas and traditions growing out of
centuries of mathematical exploration and invention?
(p. 35, italics in the original)
The originality of the solution proposed by
RME stems from the manner in which this perennial problem is reframed. For RME designers, the
challenge is not that of directly connecting students'
understandings and interests with the ideas of the
discipline. Instead, it is to support the progressive
development of students' mathematical reasoning so
that they can eventually participate in established
mathematical practices that have grown out of
centuries of exploration and invention. The metaphor inherent in the RME approach is that of continually building up towards substantial participation in
established mathematical practices rather than that
of attempting to directly connect or bridge between
students' current understandings and the established
mathematical ideas and traditions that constitute
students' intellectual inheritance.
This orientation has become routine for RME
designers and is captured by three central tenets of
the design theory. The first of these tenets is that the
starting points of an instructional sequence should
be experientially real to students in the sense that
they can immediately engage in personally meaningful mathematical activity (Gravemeijer, 1990 ;
Streefland, 1991). For the designer, the immediate
goal is that students' interpretations and solutions
should lead to the development of informal ways of
speaking, symbolizing, and reasoning across a range
of initial instructional activities. Treffers (198) calls
this process of negotiating and generalizing informal
solution processes horizontal mathematization.
Thompson (1992) indicates the significance of this
first tenet from a constructivist perspective when he
observes that "if students do not become engaged leArning FroM And AdAPting the theory oF reAlistic MAtheMAtics educAtion
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imaginistically in the ways that relate mathematical
reasoning to principled experience, then we have
little reason to believe that they will come to see
their worlds outside of school as in any way mathematical" (p. 10). This tenet is also consistent with
recommendations derived from investigations that
have compared and contrasted mathematical activity
in the classroom with that in out-of-school situations (e.g., de Abreu, 2000; Masingila, 1994; Nunes,
Schliemann, & Carraher, 1993; Saxe, 1994).
The second central tenet is that in addition to
being experientially real to students, the starting
points should also be justifiable in terms of the
potential end points of the learning sequence. This
implies that the informal ways of speaking, symbolizing, and reasoning established during the initial
phase of an instructional sequence should constitute
a basis for a progressive process of vertical mathematization. However, as Treffers makes clear, students'
increasingly sophisticated mathematical reasoning
does not become decoupled from the starting point
situations in this process. Instead, these situations

should continue to function as paradigm cases to
which students can fold back (Pirie & Kieren, 1994),
thereby anchoring their reasoning. This latter requirement is consistent with analyses that emphasize
the important role that analogies (Clement & Brown,
1989) and metaphors (Dörfler, 2000; Presmeg, 199)
play in mathematical activity.
The third tenet focuses on the means of supporting the process of vertical mathematization and it is
here that the general approach of building up towards
substantial participation in established mathematical
practices becomes evident. One of the primary means
of support involves activities in which students create
and elaborate symbolic models of their informal
mathematical activity. This modeling activity might
involve making drawings, diagrams, or tables, or
it could involve developing informal notations or
using conventional mathematical notations. This
third tenet is based on the conjecture that, with the
teacher's guidance, students' models oftheir informal
mathematical activity can evolve through use into
models for more general mathematical reasoning
(Gravemeijer, 1999; Gravemeijer, Cobb, Bowers, &
Whitenack, 2000). This transition involves a shift
such that ways of symbolizing initially developed to
mathematize informal activity subsequently come
to support more general mathematical activity in a
range of situations. In practice, the approach of building up from students' initial informal activities to
conventional ways of symbolizing involves both the
judicious selection of instructional activities and the
negotiation of successive ways of symbolizing. As
part of this process, the teacher attempts to achieve
his or her instructional agenda by capitalizing on
students' contributions and by introducing ways of
symbolizing that fit with their reasoning at particular
points in an instructional sequence.
This third tenet of RME is consistent with Sfard's
(1991 ; 1994) historical analysis of several mathematical concepts including number and function
(cf. Gravemeijer et al., 2000). Sfard contends that
the historical development of mathematics can be
seen as a long sequence of reifications, each of which
involves the transformation of operational or process
conceptions into object-like structural conceptions.
She argues that the development of ways of symbolizing has been integral to the reification process.
However, she is also careful to clarify that it is the
process of reasoning with symbolizations, not the
symbolizations themselves, that are reified during a
model-of to model-for transition. This emphasis on
activity serves to differentiate approaches based on
RME from approaches that involve a so-called modeling point of view (e.g., Lesh & Doerr, 2000). In these
latter approaches, a model is considered to capture
mathematical structures or relationships inherent in
starting point situations. In contrast, models as they
are characterized in RME originate from students'
ways of acting and reasoning with tools and symbols
in the starting point situations. Although this distinction is subtle, it has important implications for design
in that the focus is on students' anticipated interpretations and solutions rather than on the features of
instructional activities per se.
The attention that RME designers give to the development of ways of symbolizing suggests a possible
point of contact with Vygotsky's cultural-historical
theory of development. Semiotic mediation and the
use of cultural tools such as mathematical symbols
constitutes one of the two mechanisms that Vygotsky
contended drives conceptual development, the other
being interpersonal relations (Davydov, 1995; van
der Veer & Valsiner, 1991 ; Vygotsky, 198). The
differences between RME and Vygotskian approa-110
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ches center on the contrast between the metaphors
of building up towards substantial participation in
established mathematical practices and of connecting
students' current understandings with established
mathematical ideas. In Vygotskian theory, learning
in instructional situations is viewed as a process of
transmitting mathematical meaning from one generation to the next. Within this perspective, symbols are
sometimes called carriers of meaning and are treated
as primary vehicles of the enculturation process (van
Oers, 1996). This formulation does not, of course,
imply a crude transmission view of communication. Instead, the fundamental claim is that students
develop particular mathematical conceptions as they
learn to use conventional symbols while engaging in
particular sociocultural activities (Davydov, 1988). In
this scheme, the teacher's role is frequently characterized as that of relating students' personal meanings
to the cultural meanings inherent in the appropriate
use of conventional symbols. The teacher's role
might therefore be characterized as that of introducing conventional means of symbolizing and relating
them to students' mathematical activity (cf. Davydov
& Radzikhovskii, 1985; Tharp & Gallimore, 1988).
In contrast to the sociocultural framing of
instructional design as the transmission of mathematical meaning from one generation to the next, RME
designers frame the fundamental design challenge
as that of supporting the emergence of mathematical meaning in the classroom. This orientation is
apparent in both the means of symbolizing developed
in the classroom and in the characterization of the
teacher's role. As is the case in Vygotskian approaches, RME acknowledges that the teacher is an institutionalized authority in the classroom. Further, the
teacher might, on occasion, express this authority in
action by introducing means of symbolizing as he or
she redescribes students' contributions. However, it
is apparent from the third tenet of RME that these
means of symbolizing are not restricted to conventional mathematical symbols. Instead, the designer
draws on both historical analyses and analyses of
students' informal mathematical reasoning to invent
means of symbolizing that students, at a particular
point in their development, might see as reasonable to
use to achieve their mathematical goals (Gravemeijer,
1994). The resulting instructional sequences therefore involve the establishment of non-standard
means of symbolizing that are designed both to fit
with students' informal activity and to support their
development of more sophisticated forms of mathematical reasoning. These means are, in effect, offered
to students as resources that they might use as they
solve problems and communicate their thinking. In
this approach to design, symbolic means are developed to support students' progressive reconstruction
of cultural meanings (Gravemeijer, 1994).
adaptations
We now consider three adaptations we have made
that build on the RME approach to design. In doing
so, we draw on a recently completed classroom design
experiment that focused on the teaching and learning
of statistical data analysis at the middle school level.
We give an overview of the design experiment as we
discuss the first adaptation that involves broadening
our perspective on the means of supporting students'
mathematical learning. As we illustrate, this broader
perspective enables us to view classrooms as activity
systems that are designed to support the participating
students' learning of significant mathematical ideas.
Classrooms as Activity systems
Background to the design experiment
The design experiment on which we will focus
was conducted in a US seventh-grade classroom with
29 12-year-old students and focused on the analysis
of univariate data. The experiment, which was
conducted in collaboration with Koeno Gravemeijer
and Erna Yackel, lasted 12 weeks and involved 34
classroom sessions of approximately 40 minutes in
duration. A member of the research team1
served
as the teacher throughout the experiment. As part
of the process of preparing for the experiment, we
identified both the instructional starting points and
prospective endpoints. The interviews and whole
class performance assessments that we conducted
to determine the starting points indicated that, for
most of the students, data analysis involved "doing
something with the numbers" by manipulating
them in a relatively procedural manner (McGatha,
Cobb, & McClain, 2002). For example, many of
the students simply calculated mean of every data leArning FroM And AdAPting the theory oF reAlistic MAtheMAtics educAtion
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set when attempting to solve each task posed irrespective of whether the mean would give them useful
insights into the problem. Our analysis of these interviews indicated that the students did not view data
as measures of aspects or features of a situation that
had been generated in order to understand a phenomenon or make a decision. We concluded from these
assessments that our immediate goal should be to
influence the students' beliefs about what it means to
do statistics in school such that they would begin to
analyze data in order to address a significant question
rather than simply perform calculations and follow
conventions for drawing specific types of graphs.
The identification of prospective instructional
endpoints involved delineating what Wiggins and
McTighe (1998) term the "big ideas" that are at the
heart of a discipline, that have enduring value beyond
the classroom, and that offer potential for engaging students. The overarching statistical idea that
emerged from our synthesis of the research literature
and our analysis of the interviews and classroom
performance assessments was that of distribution.
One of primary goals for the design experiment was
therefore that the students would come to view data
sets as entities that are distributed within a space
of possible values (Hancock, Kaput, & Goldsmith,
1992 ; Konold & Higgins, 2002; Konold, Pollatsek,
Well, & Gagnon, 199 ; Wilensky, 199). In the
approach that we planned to take, notions such as
center, spread-outness, skewness, and relative density
would then emerge as ways of characterizing how
specific data sets are distributed within this space of
values (Bakker & Gravemeijer, 2004; Cobb, 1999 ;
McClain, Cobb, & Gravemeijer, 2000). Furthermore,
various statistical graphs or inscriptions would
emerge as ways of structuring data distributions in
order to identify relevant trends or patterns. Thus,
consistent with the third tenet of RME, we viewed
the students' development of increasingly sophisticated ways of reasoning about data as inextricably
bound up with their development of increasingly
sophisticated ways of inscribing data (Biehler, 1993;
deLange, van Reeuwijk, Burrill, & Romberg, 1993 ;
Lehrer & Romberg, 1996).
Interviews that we conducted with the 29 students
shortly after the design experiment was completed
indicate that we had some success in achieving
these goals. The interviews included tasks in which
the students were asked to compare graphs of two
unequal data sets that corresponded to histograms
and to box-and-whiskers plots. A significant majority
of the students did so by developing relatively sophisticated arguments that involved reasoning about the
data in terms of relative rather than absolute frequencies. In this regard, Konold et al. (199) argue that
a focus on the rate of occurrence (i.e., the relative
frequencyl of data within a range of values is at the
heart of what they term a statistical perspective. As
the arguments that most of the students formulated
in the interviews involved comparing the graphs
in terms of the proportion of data within various
ranges of values, it would seem that they were in the
process of developing this statistical perspective. It
is also worth noting that when we began a follow-up
design experiment with some of the same students
nine months later, there was no regression in their
statistical reasoning (Cobb, McClain, & Gravemeijer,
2003). The students' progress at the beginning of this
follow-up experiment was in fact such that they could
all interpret graphs that corresponded to histograms

and to box-and-whiskers plots in these relatively
sophisticated ways within the first three or four class
sessions of this follow-up experiment.
The retrospective analyses that we have conducted
of the design experiment to account for the process
of the students' learning and the means by which
it was supported highlighted the important role of
the instructional activities and the tools the students
used to analyze data sets. However, these analyses
also indicate the importance of two means of support
that are not typically considered by instructional designers. These additional means of support concern the
organization of classroom activities and the nature of
the classroom discourse (Cobb, 1999; McClain et al.,
2000).
Instructional activities
One of the primary commitments that we made
when we developed the instructional activities
was that students' activity in the classroom should
involve the investigative spirit of data analysis from
the outset. This implied that the instructional activities should involve analyzing data sets that the
students viewed as realistic for purposes that they
considered legitimate. As a consequence, most of the 112
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instructional activities that we developed involved
comparing two data sets in order to make a decision
or judgment (e.g., analyze the T-cell counts of AIDS
patients who had enrolled in two different treatment
protocols). From the midpoint of the experiment,
the students were also required to write reports of
their analyses for a specified audience (e.g., the chief
medical officer of a hospital in the case of the AIDS
data). This requirement reflected the observation that
data are typically analyzed with a particular audience
in mind almost everywhere except in school (cf.
Noss, Pozzi, & Hoyles, 1999).
Tools
The students used two computer tools that were
introduced sequentially during the experiment to
analyze data sets and thus complete the instructional
activities. As these tools have been described extensively elsewhere (e.g., Bakker & Gravemeijer, 2003;
Cobb, 2002; McClain, 2002), it suffices to note that
they provided the students with a variety of options
for organizing graphical inscriptions of data sets. Our
intent in designing the tools was that they should
fit with the students' reasoning when they were first
introduced and that they should support the reorganization of that reasoning as the students used them.
Consistent with RME's focus on students' activity,
we did not attempt to build the statistical ideas we
wanted the students to learn into either the instructional activities or the computer tools in the hope that
they might come to see them. Instead, we focused
squarely on how the students' use of the tools would
change the nature of their activity as they analyzed
data and thus the types of statistical reasoning that
they might develop.
Organization ofclassroom activities
In considering the third means of support, the
organization of classroom activities, we began to
broaden our purview beyond the issues that typically
concern instructional designers. One of goals when
planning the organization of classroom activities was
to ensure that the students would come to view data
as measures of an aspect of a phenomenon rather
than merely as numbers relatively early in the design
experiment. To this end, the teacher introduced each
instructional activity by talking through the data
generation process with the students. These conversations often involved protracted discussions during
which the teacher and students together framed the
particular phenomenon under investigation (e.g.,
AIDS), clarified its significance (e.g., the importance
of developing more effective treatments), delineated
relevant aspects of the situation that should be
measured (e.g., patients' T-cell counts), and considered how they might be measured (e.g., taking
blood samples). The teacher then introduced the data
the students were to analyze as being generated by
this process. The resulting organization of classroom
activities, which often spanned two or more class
sessions, therefore involved (a) a whole-class discussion of the data generation process, (b) individual or
small-group activity in which the students usually
worked at computers to analyze data, and (c) a
whole-class discussion of the students' analyses.
In organizing the classroom activities in this
manner, we conjectured that as a consequence of
participating in the discussions of the data generation process, data sets would come to have a history
for the students such that they reflected the interests
and purposes for which they were generated (cf.
Latour, 198 ; Lehrer & Romberg, 1996; Roth, 199).
As it transpired, this conjecture proved to be well
founded. On the first instructional activity in which
the students used a computer tool to analyze data,
approximately half of the students calculated means
by hand and selected the data set with the larger
mean. However, in the third instructional activity,
all the students used the computer tool to identify
differences in the data sets that gave insight into the
question they were investigating. We interpreted this
observation as indicating that doing statistics had
come to involve actually analyzing data within the
first week of the design experiment (Cobb, 1999 ;
McClain et al., 2000).
In accounting for the effectiveness of the data
creation discussions, it is important to note that the
teacher did not attempt to teach the students how
to generate sound data directly. Instead, she guided
the development of a classroom culture in which a
premium was placed on the development of databased arguments. This observation indicates that
the concluding whole class discussions in which the
students explained and justified their analyses also leArning FroM And AdAPting the theory oF reAlistic MAtheMAtics educAtion
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played a crucial role. It was as the students participated in these latter discussions that they first became
aware of the implications of the data generation
process for the conclusions that could legitimately
be drawn from data.
Our overriding concern as we prepared for these
discussions in the classroom was that mathematically
significant issues that advanced the instructional
agenda would became explicit topics of conversation.
To this end, the teacher and a second member of the
research team circulated around the classroom while
the students were working at the computers to gain
a sense of the various ways in which they were organizing and reasoning about the data. Towards the end
of the small-group work, they then conferred briefly
to develop conjectures about mathematically significant issues that might emerge as topics of conversation in the subsequent whole-class discussion. Their
intent was to capitalize on the students' reasoning by
identifying data analyses that, when compared and
contrasted, might give rise to substantive mathematical conversations. To the extent that the teacher
succeeded, students' participation in the discussions
would serve as primary means of supporting their
progressive reorganization of their reasoning and
thus their gradual induction into the values, beliefs,
and ways of knowing of the discipline.
We should clarify that our intent in giving these
illustrations from the statistics design experiment has
not been to argue that this particular organization of
classroom activities should be applied more generally.
The crucial contribution of the initial data generation
discussions would, for example, appear to be specific
to our focus on statistical data analysis. Instead, our
purpose has been to demonstrate the importance of
attending explicitly to the organization of classroom
activities as a primary means of supporting (or inhibiting) students' mathematical learning. In our view,
the organization of classroom activities is an integral
aspect of an instructional design.
Classroom discourse
In discussing the potential contribution of whole
class discussions, we stressed the importance of
ensuring that significant mathematical issues that
advance the instructional agenda emerge as topics
of conversation. The final means of support that we
identified when analyzing the statistics experiment
focuses on the nature of classroom discourse. To
clarify this means of support, we extend a distinction that Thompson and Thompson (1996) make
between calculational and conceptual orientations
in mathematics teaching by differentiating between
calculational and conceptual discourse. We should
stress at the outset that calculational discourse
does not refer to conversations that focus on the
procedural manipulation of conventional tools
and symbols whose use is a rule-following activity
for students. The solution methods that students
explain as they contribute to calculational discourse
might in fact be self-generated and involve relatively sophisticated mathematical understandings.
The contrast between calculational and conceptual
discourse should therefore not be confused with
Skemp's (196) well-known distinction between
instrumental and relational understandings. Instead,
the distinction concerns the norms or standards for
what counts as an acceptable mathematical argument. In calculational discourse, contributions are
acceptable if students describe how they produced
a result and they are not obliged to explain why
they used a particular method. In contrast to this
exclusive focus on methods or solution strategies,
the issues that emerge as topics of conversation in
conceptual discourse also include the interpretations
of instructional activities that underlie those ways of
calculating and that constitute their rationale.
As an illustration, the computer tool that the
students used to analyze the data on the two AIDS
treatment programs provided students with a variety
of options for organizing axis plot inscriptions of
the patients' T-cell counts (i.e., the T-cell counts for
each treatment were inscribed as dots located on
an axis of values). The least sophisticated of these
options involved dragging a bar to a chosen location
on the axis, thereby partitioning the data set into
two groups. As shown in Figure 1, the number of
points in each group was shown on the screen and
adjusted automatically as the bar was dragged along
the axis. A calculational explanation of an analysis
conducted to determine which treatment program
was more effective involves describing the specific
steps taken when conducting the analysis. For a relatively unsophisticated analysis in which the bar is
used to partition each data set, the students might 114
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simply explain that they placed the bar at a particular value and then report the number of points
above and below this value in both data sets. As it so
happened, three of 14 groups of students conducted
analyses of this type. In each case, they placed the
bar so that what they called the "hill" in one of the
data sets was mostly below the bar, and the "hill" in
the other data set was mostly above the bar.
Experimental Treatment
Traditional Treatment
Figure 1. The AIDS Data Partitioned at T-cell counts of525leArning FroM And AdAPting the theory oF reAlistic MAtheMAtics educAtion
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A conceptual explanation of these solutions would
involve describing not merely the steps of the analysis
but also the reasons for carrying them given the issue
under investigation, that of judging the effectiveness of

the two treatment programs. In giving explanations of
this type, the students who participated in the design
experiment clarified that they placed the bar at a particular location in order to highlight and quantify a qualitative difference between the two data sets, the location
of the "hills." A retrospective analysis of video-recordings of all classroom sessions indicates that the interventions the teacher made to support conceptual explanations of this type also contributed to the students'
development of relatively sophisticated explanations
in which they compared data sets in terms of relative
rather than absolute frequencies (Cobb, 1999).
Our experiences in both this and a number of other
classroom design experiments lead us to conclude that
discussions in which the teacher judiciously supports
students' attempts to articulate their task interpretations
can be extremely productive settings for mathematical
learning. As these articulations focus on the reasoning
that lies behind solution procedures, students' participation in such discussions increases the likelihood that
they might come to understand each other's reasoning.
Had the discussion in the design experiment classroom
remained calculational, students could only have
understood each other's explanations by creating a task
interpretation that lay behind their use of the computer
tool entirely on their own. In contrast, the students'
participation in conceptual discourse provided them
with resources that supported their understanding of
other's explanations and thus their reorganization of
their initial interpretations of tasks. These resources are
not limited to what is said but also include inscriptions
and notations that are pointed to and spoken about (cf.
A. G. Thompson, Philipp, Thompson, & Boyd, 1994).
In the statistics design experiment, for example, the
graphs that the students developed as they used the
computer tools were integral to communication as well
as to their individual reasoning. This interdependency
of tools and discourse indicates the systemic nature of
the various means of support that we have discussed.
The classroom activity system
As the case of the statistics design experiment
indicates, the four means of support that we have
discussed are strongly interrelated. For example, the
statistics instructional activities as they were actually
realized in the classroom depended on talking through
the data generation process, the computer tools that
the students used to conduct analyses, and the nature
of the subsequent whole class discussions. It is easy
to imagine how the instructional activities might be
realized differently if the options for organizing data
on the computer tool had consisted only of conventional statistical graphs, or if there had been no whole
class discussions and the teacher had simply graded
the students' reports of their analyses.
In light of these interdependencies, it is reasonable to view the various means of support as
constituting a single classroom activity system. This
perspective is compatible with Stigler and Hiebert's
(1999) contention that teaching should be viewed as
a system. In making this claim, Stigler and Hiebert
directly challenge analyses that decompose teachers'
instructional practices into a number of independent
moves or competencies. They instead propose that
the meaning and significance of any particular facet
of a teacher's instructional practice becomes apparent
only when it is analyzed within the context of the
entire practice. In a similar manner, the four means
of support that we have discussed should be viewed
as aspects of a single classroom activity system (Cobb
& McClain, 2004). Instructional design from this
point of view therefore involves designing classroom
activity systems such that students develop significant mathematical ideas as they participate in them
and contribute to their evolution.
This systemic perspective sits uncomfortably
with approaches to design that focus exclusively on
instructional activities and tools. These formulations
focus on selected aspects of the classroom activity
system in isolation, in the process casting instructional activities as the cause and learning as the
effect. From the systemic perspective, conjectures
about what students might learn as they use tools to
complete instructional activities are, at best, metonymies for more encompassing conjectures about
what students might learn as they participate in an
envisioned classroom activity system. This broader
perspective brings other aspects of the classroom
activity system to the fore as an explicit focus of
design, thereby enabling the designer to consider
how proposed instructional activities and tools might 116
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be realized in the classroom. In addition, this perspective highlights the central role of the teacher in
orchestrating the organization of classroom activities
and in guiding the negotiation of norms of mathematical argumentation. A second adaptation that
we have made to the RME approach to instructional
design emphasizes the crucial contribution of the
teacher.
Designing Resources for Classroom teaching
Instructional designers typically assume that
they are developing instructional activities and associated resources to support the learning of individual students. In doing so, they justify instructional
sequences in terms of a hypothetical learning trajectory that focuses on the development of individual
student's mathematical reasoning. As a consequence
of our experience of developing and refining instructional sequences while conducting classroom design
experiments, we came to view this exclusive focus
on individual students' reasoning as problematic for
two reasons. First, our work in classrooms led us to
question justifications cast exclusively in terms of
individual students' mathematical reasoning for the
straightforward reason that, in any classroom, there are
significant qualitative differences in students' thinking
at any point in time (Cobb et al., 2001). To capture this
diversity in students' reasoning, it would be necessary
to formulate multiple learning trajectories. However,
an approach of this type leads to our second concern
about an exclusively individual perspective, namely
that it is unmanageable for the teachers. It is unrealistic to expect that teachers will be able to formulate
and continually update learning trajectories for every
student, or even for several groups of students, and use
these multiple trajectories to inform instruction. As an
alternative to an exclusively individualistic focus, we
have found it useful while working in classrooms to
view a hypothetical learning trajectory as consisting
of conjectures about the collective mathematical development of the classroom community. This proposal
constitutes the second adaptation that we made to the
RME design theory.
This adaptation has methodological implications
for the analysis of classroom data.
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The adaptation is
also pragmatically significant and brings the teacher
into the picture. In particular, the purpose of instructional design becomes that of developing resources
for the teacher to use to support students' learning,
rather than to develop instructional activities that
are intended to support students' learning directly.
We illustrate this point by returning to the statistics
design experiment. We noted that one of the commitments we made when we developed instructional
activities during this experiment was that students'
activity in the classroom should involve the investigative spirit of data analysis from the outset. A second
commitment was that significant mathematical issues
that advanced the instructional agenda should emerge
as a focus of conversation during the whole-class
discussions of the students' analyses. The challenge
for us as instructional designers was therefore to
transcend what Dewey (1951/1981) termed the dichotomy between process and content by systematically
supporting the emergence of key statistical ideas while
simultaneously ensuring that the analyses the students
conducted involved an investigative orientation. This
is a non-trivial issue in that inquiry-based instructional approaches have sometimes been criticized for
emphasizing the process of inquiry at the expense of
substantive disciplinary ideas. In approaching this
challenge, we viewed the various data-based arguments that the students produced as they completed
the instructional activities as a primary resource on
which the teacher could draw to initiate and guide
whole-class discussions that focused on significant
statistical ideas. As a consequence, we did not merely
attempt to design instructional activities that would
be accessible to multiple individual students who may
differ in terms of level of statistical sophistication. Our
goal when developing specific instructional activities
was also to ensure that the diverse ways in which the
students analyzed data would constitute an instructional resource on which the teacher could capitalize
to support the learning of entire class.
The achievement of this design goal required
extremely detailed instructional planning. However,
rather than attempting to influence each individual
student's reasoning in a specified manner by developing particular instructional activities, we attempted
to anticipate the range of data-based arguments that a
group of students might produce as they completed
specific instructional activities. Our discussion of
seemingly inconsequential features of task scenarios
and of the particular characteristics of data sets were
therefore quite lengthy as minor modifications to an leArning FroM And AdAPting the theory oF reAlistic MAtheMAtics educAtion
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instructional activity could significantly influence
the types of analyses the students would produce and
thus the resources on which the teacher could draw
to further her instructional agenda. For example, in
developing the AIDS instructional activity, we purposefully constructed data sets with a significantly different number of data points so that the teacher could
initiate a comparison of analyses that were based on
absolute and on relative frequency. In doing so, we
paid particular attention to the characteristics of the
task scenario in which the inequality in the size of
the data sets would seem reasonable to the students.
As a result of this detailed preparation, a number of
mathematically significant issues emerged during the
whole-class discussion of the AIDS data set, including
the contrast between absolute and relative frequency,
the interpretation of data graphs corresponding to
box-and-whiskers plots, and the use of percentages to
quantify the proportions of a data set located in particular intervals (Cobb, 1999; McClain et al., 2000).
It is apparent from this illustration that our instructional agenda included a concerted effort to support
the students' transition from additive to multiplicative
reasoning about data. However, we did not attempt to
achieve this goal by somehow causing the individual
students to make this transition as they completed
certain instructional activities. Instead, we attempted
to develop instructional activities that would result in
a range of solutions on which the teacher could capitalize as she planned whole class discussions. We therefore viewed ourselves as developing resources that the
teacher could use to achieve her instructional agenda.
Looking beyond the statistics design experiment, we
attribute a central, mediating role to teachers more
generally and in fact view them as co-designers of the
classroom activity systems that constitute the immediate social situations of their students' mathematical
development. It should be clear from the illustrations
we have given that the role we envision for the teacher
is relatively demanding and requires considerable
knowledge and judgment. The third adaptation that we
discuss takes these demands seriously and focuses on
the possible means of supporting teachers' learning.
supports for teachers' Learning
Our characterization of the teachers as co-designers implies that high-quality mathematics teaching
is a generative, knowledge building process in
which teachers learn about their students' mathematical reasoning and the means of supporting its
development as they analyze, adapt, test, and refine
instructional sequences that have proved effective
elsewhere (cf. Franke, Carpenter, Levi, & Fennema,
2001). As Gamoran et al. (2003) observe, this form
of instructional practice is complex, demanding,
uncertain, and not reducible to predictable routines
(cf. Ball & Cohen, 1999 ; Clark, 1988 ; Lampert,
2001; McClain, 2002; Schifter, 1995; Smith, 1996).

Teachers need considerable support as they learn to
adapt instructional sequences to the contingencies
of their classroom while at the same time placing
students' reasoning at the center of their instructional decision-making. The third adaptation that we
propose focuses on what is involved when designing
instructional sequences so that they support teachers'
as well as students' learning.
In our view, teachers' implementation of an
instructional sequence is necessarily a process of
conjecture-driven adaptation. In taking this stance,
we follow de Certeau (1984) in contending that an
artifact such as an instructional sequence developed
by one group is necessarily reshaped and transformed
when others use it. Wertsch (1998) extends this
argument by contending that this is the case even
when the users attempt to comply with the designers' intentions. To paraphrase Wertsch, teachers
necessarily adjust an instructional sequence to the
actual circumstances that they encounter even when
they attempt to remain faithful to the designers'
intentions. Aspects of these circumstances that
significantly influence teachers' instructional practices include the institutional constraints that they
attempt to satisfy, the formal and informal sources
of assistance on which they can draw, their students'
prior instructional histories, and the instructional
materials and resources that they use as the basis
for instruction (Ball & Cohen, 1996 ; Blumenfeld,
Fishman, Krajcik, Marx, & Soloway, 2000 ; Brown,
Stein, & Forman, 1996 ; Confrey, Bell, & Carrejo,
2001; Feiman-Nemser & Remillard, 1996; Gamoran,
Secada, & Marrett, 2000 ; Nelson, 1999 ; Rowan,
1990; Senger, 1999; Stein & Brown, 199 ; Tharp &
Gallimore, 1988).
de Certeau (1984) and Wertsch (1998) take
their argument one step further by proposing that 118
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when teachers attempt to use instructional materials
developed elsewhere, a second process of creation or
production is involved. In our view, this perspective is
potentially productive for teacher professional development because it orients us to focus our efforts on
supporting teachers' development of the personal and
social resources that would enable them to adapt the
designed instructional sequences in ways to support
their students' learning of significant mathematical
ideas. This perspective is also useful for research on
teacher professional development in that it orients
us to explain why teachers adapt instructional
sequences in particular ways by understanding their
evolving instructional practices as they are situated
in particular institutional settings.
We can best illustrate the third adaptation that we
made to RME theory by referring to a teacher professional development experiment
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conducted with a
group of middle-school mathematics teachers who
taught 12- and 13-year-old students. The statistical
instructional sequence that we developed in the prior
classroom design experiment served as one of the
primary tools that we used to support the teachers'
learning. It is important to note that the retrospective
analyses that we conducted of the classroom design
experiment justify the instructional sequence in terms
of (a) a substantiated trajectory for students' mathematical learning, and (b) the documented means of
supporting learning along that trajectory. If we had
justified the instructional sequence solely with traditional experimental data, the teachers would know
that this sequence had proved effective elsewhere but
would not have access to the underlying rationale that
would enable them to adapt it effectively to their own
instructional settings. In contrast, the type of rationale
that we developed constitutes a potentially important
resource for teachers as they adapt, test, and modify
the instructional sequence in their classrooms. Our
intent in the professional development experiment
was therefore to engage teachers in activities that
would enable them to reconstruct the rationale for
the instructional sequence.
As an illustration, we engaged the teachers in a
variety of activities that focused on how to conduct
data generation discussions. As we noted when
discussing the statistics design experiment, data
generation discussions play a significant role in
shaping the ways in which students interpret data.
Early in our collaboration, the teachers recognized
that data generation discussions were an important
aspect of instruction. However, it became apparent
that, from their perspective, effective instructional
activities involved a scenario that was immediately
interesting and personally relevant to students. For
example, the teachers considered that instructional
activities that involve soft drinks or roller coasters
were instructionally more promising than those that
focused on issues of broader social significance (e.g.,
AIDS). They therefore understood the importance
of data generation discussions primarily in terms
of capturing students' interest so that they would
engage in instructional activities. It was also apparent
that, from the teachers' perspective, data generation
discussions made little if any contribution to the
ways that students interpreted and analyzed data.
During our ensuing collaboration with the
teachers, we engaged them in a number of activities that focused on data generation discussions.
This approach was reasonably successful in that the
teachers came to view a broader range of problem
scenarios as potentially productive and saw it as their
responsibility to develop the significance and relevance of problem situations with students. Towards
the end of our collaboration with the teachers, there
were strong indications that the teachers had become
aware that the students' understanding of the process
by which the data were generated influenced how
they interpreted and analyzed the data. In particular,
they explicitly linked the issues that they addressed
while conducting data creation discussions to
students' subsequent analyses. This development
would have been unlikely had we not made the rationale for the instructional sequence an explicit focus
of professional development activities.
Methodologically, the perspective we have illustrated on teacher learning and implementation
offers the prospect that research on teacher professional development might become a design science
characterized by tightly integrated cycles in which
designs for supporting teachers' learning are developed, tested, analyzed, and modified. The products
of teacher development research of this type might
include prototypical sequences of activities and
resources for teacher-researcher collaboration
together with a rationale that is cast in terms of (a) the
actual learning trajectory for a professional teaching leArning FroM And AdAPting the theory oF reAlistic MAtheMAtics educAtion
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community, and (b) means by which that learning
can be supported. We speculate that the rationales
of this type will enable other researchers and teacher
educators to adapt the prototypical sequences to the
specific settings in which they are collaborating with
teachers in a conjecture-driven manner.
conclusion
In the first part of this article, we emphasized the
value of the positive heuristics for design proposed
by RME. These heuristics have not been derived from
a general background theory but instead they are
empirically grounded in the activity of developing,
testing, and revising specific designs in classrooms.
The enduring contribution of RME resides in the
solution that it proposes to the perennial question of
how to induct students into established mathematical
practices while simultaneously taking their current
understandings and interests seriously. The approach
proposed by RME involves continually building up
towards substantial participation in established
mathematical practices rather than attempting to
directly connect students' current understandings to
established mathematical ideas and traditions.
Against this background, we discussed three
adaptations that we have made to the RME approach
to design. The first adaptation involves broadening the
means of supporting students' mathematical learning
beyond instructional activities and tools to include
the organization of classroom activities and the
nature of classroom discourse. These various means
of support are strongly interrelated and can be viewed
as aspects of an encompassing classroom activity
system. The goal for instructional development is
therefore to design classroom activity systems such
that students develop significant mathematical ideas
as they participate in them and contribute to their
evolution. The second adaptation emphasized the
critical role of teachers as co-designers of classroom
activity systems. The purpose in developing
instructional activities and tools is not to support
individual students' mathematical learning directly.
Instead, it is to develop resources that teachers
can use to achieve their instructional agendas by
capitalizing on students' diverse interpretations and
solutions. The third adaptation took account of the
demands of the envisioned role for the teacher and
concerned supports of teachers' learning. We argued
that the implementation of instructional sequences
necessarily involves a process of adaptation. The
goal for teacher professional development should
therefore be to support teachers' development
of the personal and social resources that enable
teachers to adapt instructional sequences in ways
that enhance their students'mathematical learning.
This orientation offers the prospect that research
on teacher professional development might become
a design science that involves developing, testing,
and modifying designs for supporting the learning
of professional teaching communities and the
participating teachers.120
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endnotes5
1. In addition to the first author, the members of the research team for the design experiment were Kay
McClain, Koeno Gravemeijer, Maggie McGatha, Jose Cortina, Lynn Hodge, Carrie Tzou, Kazu Nunokawa, Nora
Stuart, and Carla Richards.
2. We have argued elsewhere that the appropriate unit of analysis comprises a collective classroom mathematical practice and students'diverse ways of participating in and contributing to its constitution (Cobb et
al., 2001).
3. In addition to the authors, the members of the research team for the teacher professional development
experiment were Kay McClain, Chrystal Dean, Teruni Lamberg, Lori Tyler, Melissa Gresalfi, and Jose Cortina.
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Memos (0)

Document body:

The teacher and the tool: instrumental orchestrations
in the technology-rich mathematics classroom
Paul Drijvers & Michiel Doorman & Peter Boon &
Helen Reed & Koeno Gravemeijer
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Abstract The availability oftechnology in the mathematics classroom challenges the way
teachers orchestrate student learning. Using the theory of instrumental orchestration as the
main interpretative framework, this study investigates which types of orchestrations
teachers develop when using technology and to what extent these are related to teachers'
views on mathematics education and the role of technology therein. Data consisted of
videotapes of 38 lessons taught by three teachers, who also provided information on their
views through questionnaires and interviews. Qualitative analysis of these data led to the
identification of orchestration types and teacher profiles. The orchestration preferences of
the three teachers proved to be related to their views. A detailed analysis of one exemplary
episode suggests how other theoretical perspectives might complement the theory of
instrumental orchestration.
Keywords Digital technology integration . Classroomteaching practice .
Instrumental orchestration . Technology. Applet
. Teachers' views
1 Introduction
In spite of technology's recognised potential for teaching and learning, its integration into
secondary mathematics education lags behind the high expectations that many researchers
and educators had some decades ago (Lagrange, Artigue, Laborde & Trouche, 2003). The
role ofthe teacher has been acknowledged as both a critical and a problematic factor in this
integrative process (Artigue, Drijvers, Lagrange, Mariotti & Ruthven, 2009; Doerr &
Zangor, 2000; Lagrange & Ozdemir Erdogan, 2009; Monaghan, 2004; Ruthven &
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Eindhoven School ofEducation, Eindhoven University ofTechnology,
Eindhoven, The Netherlandsthennessy, 2002). It is critical in the sense that the way in which teachers approach the use
ofthechnology has major consequences for the effects ofits use in the classroom (Kendal &
Stacey, 2002). It is problematic, as teachers who do not perceive the use of technology in
their teaching as valuable for their educational goals are able to avoid it, unless explicitly
required to do so by institutional or curriculum constraints. Also, teachers often experience
difficulties in adapting their teaching techniques (Sensevy, Schubauer-Leoni, Mercier,
Ligozat & Perrot, 2005) to situations in which technology plays a role (Monaghan, 2004).
What is difficult from a teacher's perspective in integrating technology into his or her
mathematics teaching? Robert and Rogalski (2005) point out that teachers' practices are
both complex and stable. Building on this, Lagrange and Monaghan (2009) argue that the
availability of technology amplifies the complexity and, as a consequence, challenges the

stability ofteaching practices: techniques that are used in 'traditional' settings can no longer
be applied in a routine-like manner when technology is available. A new repertoire of
teaching techniques, instrumented by the available tools, has to be developed. These new
techniques are likely to be related to already existing ones as well as to the teachers'
underlying views on mathematics education (Pierce & Ball, 2009).
In order to help teachers to benefit from technology in everyday mathematics teaching,
therefore, it is important to have more knowledge about the new teaching techniques that
emerge in the technology-rich classroom and how these relate to teachers' views on
mathematics education and the role of technology therein.
2 Theoretical framework and research questions
The main theoretical perspective that informs our investigation of teacher behaviour in a
technology-rich environment is the instrumental approach to tool use, and the notion of
instrumental orchestration in particular.
The instrumental approach acknowledges the complexity of using technology within
mathematics education (Artigue, 2002). According to this approach, the use of a
technological tool involves a process of instrumental genesis, during which the object or
artefact is turned into an instrument. This instrument is a psychological construct, which
combines the artefact and the schemes (in the sense ofVergnaud, 1996) the user develops to
use it for specific types of tasks. In such instrumentation schemes, technical knowledge
about the artefact and domain-specific knowledge (in this case, mathematical knowledge)
are intertwined. Instrumental genesis, therefore, is essentially the co-emergence of schemes
and techniques for using the artefact.
Many studies focus on students' instrumental genesis and its possible benefits for
learning (e.g. see Kieran & Drijvers, 2006). However, it was acknowledged that students'
instrumental geneses need to be guided by the teacher through the orchestration
(McKenzie, 2001) of mathematical situations (Mariotti, 2002). For example, Kendal and
colleagues (Kendal & Stacey, 2002; Kendal, Stacey, & Pierce, 2004) showed that teachers
privilege certain techniques for using technological tools over others and, in this way, guide
the students' acquisition of tool mastery and their learning processes. To describe how the
teacher can fine-tune the students' individual instruments and compose coherent sets of
instruments within the classroom, thus enhancing both individual and collective geneses,
Trouche (2004) introduced the metaphor of instrumental orchestration.
An instrumental orchestration is defined as the teacher's intentional and systematic
organisation and use of the various artefacts available in a—in this case computerised—
learning environment in a given mathematical task situation, in order to guide students'
214 P. Drijvers et al.instrumental genesis (Trouche, 2004). We distinguish three elements within an instrumental
orchestration: a didactic configuration, an exploitation mode and a didactical performance.
These three elements are now described in more detail.
A didactical configuration is an arrangement ofartefacts in the environment, or, in other
words, a configuration ofthe teaching setting and the artefacts involved in it. In the musical
metaphor of orchestration, setting up the didactical configuration can be compared with
choosing musical instruments to be included in the band, and arranging them in space so
that the different sounds result in polyphonic music, which in the mathematics classroom
might come down to a sound and converging mathematical discourse.
An exploitation mode is the way the teacher decides to exploit a didactical configuration
for the benefit ofhis or her didactical intentions. This includes decisions on the way a task
is introduced and worked through, on the possible roles ofthe artefacts to be played, and on
the schemes and techniques to be developed and established by the students. Decisions on
the exploitation mode can be seen as part of the design of a Hypothetical Learning
Trajectory (Simon, 1995). In terms of the metaphor of orchestration, setting up the
exploitation mode can be compared with determining the partition for each of the musical
instruments involved, bearing in mind the anticipated harmonies that will emerge.
A didacticalperformance involves the ad hoc decisions taken while teaching on how to
actually perform in the chosen didactic configuration and exploitation mode: what question
to pose now, how to do justice to (or to set aside) any particular student input, how to deal
with an unexpected aspect of the mathematical task or the technological tool, or other
emerging goals.
In the metaphor of orchestration, the didactical performance can be compared to a
musical performance, in which the actual interplay between conductor and musicians
reveals the feasibility of the intentions and the success of their realisation.
Didactical configurations and exploitation modes were introduced by Trouche (2004).
As an instrumental orchestration is partially prepared beforehand and partially created 'on
the spot' while teaching, we felt the need to add the actual didactical performance as a third
component. Establishing the didactical configuration has a strong preparatory aspect: often,
didactical configurations need to be thought of before the lesson and cannot easily be
changed during it. Exploitation modes may be more flexible, while didactical performances
have a strong ad hoc aspect. As such, the threefold model has a time dimension.
The model also has a global–local dimension. An instrumental orchestration has a global
component, in that it is part ofthe teacher's repertoire ofteaching techniques. It also has an
incidental, local actualisation appropriate for the specific didactical context and adapted to
the target group and the didactical intentions.
Even ifthe metaphor ofinstrumental orchestration is appealing, it has its limitations like
every metaphor. Ifwe think ofa teacher as a conductor ofa symphony orchestra consisting
of highly skilled musicians, who enters the concert hall with a clear idea on how to make
the musicians play Mahler the way he himselfreads the century-old partition, we might feel
uneasy. However, ifwe think ofthe class as ajazz band, consisting ofboth novice and more
advanced musicians, and the teacher being the band leader who prepared a global partition
but is open for improvisation and interpretation by the students, and for doing justice to
input at different levels, the metaphor makes more sense. It is in the latter way that we
suggest to understand it.
Despite the potential of the instrumental orchestration model to help understand
teachers' practices in the technology-rich classroom, the number of elaborated examples of
instrumental orchestrations described in the research literature to date is limited (Drijvers &
Trouche, 2008; Trouche, 2004). Meanwhile, our goal formulated above is to find out which
The teacher and the tool 215new teaching techniques emerge in the technology-rich classroom. In terms of the
instrumental orchestration framework, one aim ofthis study is to identify different types of
orchestrations. Therefore, the first research question is:
(1) Which types of instrumental orchestration emerge in technology-rich classroom
teaching?
As a means to make this research question feasible to deal with, we limit ourselves here
to orchestrations in whole-class teaching. Of course, other organisations of the learning,
such as individual student work and work in pairs or groups, are also important, but we
address them elsewhere (Drijvers, submitted).
The process of a teacher developing instrumental orchestrations is guided by his or her
knowledge of, experience with and views onmathematics education andthe role oftechnology
herein (Pierce & Ball, 2009). This includes the teachers' own knowledge and skills concerning
the integration of technology, and their concerns about time constraints and behavioural
control. Gueudet and Trouche (2009) refer to these often implicit guidelines which drive a
teacher's choices as operational invariants. Such operational invariants can be very general,
such as 'learning takes place through interaction', or more specific to the role oftechnology,
for example 'technology offers means to improve classroom interaction'. These operational
invariants work out in types of invariant teacher behaviours, which are instrumented by the
available tools. The notions of operational invariants and teachers' views are used in this
study to interpret the information from teachers in relation to their observed orchestrational
behaviour. The second aim of this study, phrased in the introduction as the investigation of
how teaching techniques relate to teachers' views on mathematics education and the role of
technology therein, is reformulated in terms of the instrumental orchestration framework as
the following second research question:
(2) To what extent are teachers' repertoires of orchestrations related to their expressed
views on mathematics education and the role of technology therein?
The way inwhichwe answerthese questions with instrumental orchestration as a framework
canbe considered as atestregarding the suitability andvalue ofinstrumental orchestration as an
interpretative lens for research into teaching with technological tools. As such, our 'hidden
agenda' is to contribute to the development ofthis model, its evaluation in terms ofaffordances
and limitations, and its articulation with other theoretical approaches.
3 Research context
The research questions are addressed in the context ofa research project on a technology-rich
learning arrangement for the concept offunction
1
. In this project, a learning arrangement for
eighth-grade students as designed. The design was guided by Realistic Mathematics Education
principles (Gravemeijer, 1994). The arrangement, which consisted of eight 50-min lessons,
aimed at the development ofa rich function concept, whereby functions are conceptualised as
input–output assignments, as dynamic processes of co-variation and as mathematical objects
with different representations (Drijvers, Doorman, Boon, Van Gisbergen & Gravemeijer,
2007; Slavit, 1997). The design of the learning materials was directed by a hypothetical
learning trajectory, in which the expected conceptual development was guided and facilitated
1
For further information on the project, see the project website www.fi.uu.nl/tooluse/en/.
216 P. Drijvers et al.by means of activities carried out with a technological tool. An accompanying teacher guide
contained suggestions and working formats for appropriate teaching techniques and included a
planning scheme.
The main technological component of the learning arrangement is a Java applet called
Algebra Arrows. The applet was embedded in an electronic learning environment called the
Digital Mathematics Environment (DME). The applet allows for the construction and use of
chains of operations (so-called arrow chains) and provides options for creating tables,
graphs and formulae and for scrolling and tracing. The DME allows students to access their
work from any location, and the teacher to access this work in order to monitor progress
and track the conceptual development of the students. Figure 1 shows these digital tools.
The right part is the Algebra Arrows window, in which students create and use their
representations. At the left, the DME provides tasks and boxes in which the students type
their answers. The numbered circles at the bottom of the screen allow for navigation
through the tasks, the underlined one currently indicating task 2.
Besides teacher orchestration, other foci of this research project are students' learning
(Doorman, Boon, Drijvers, Van Gisbergen, Gravemeijer & Reed, 2009) and the relationships between learning outcomes and student attitudes and behaviours (Reed, Drijvers &
Kirschner, 2010).
4 Method
The intervention consisted of the implementation of the above described learning
arrangement in 29 eighth-grade classrooms in one Belgian and nine Dutch schools in
three research cycles, each consisting of a (re)design phase, a teaching experiment phase
Fig. 1 Algebra Arrows applet embedded in the Digital Mathematics Environment
The teacher and the tool 217and a phase of data analysis (Gravemeijer, 1994). In terms ofthe orchestration model, this
learning arrangement offers tools and tasks, which are part of the didactical configuration,
as well as suggestions in the teacher guide concerning exploitation modes. The teachers'
freedom, therefore, mainly concerned the configuration of the classroom setting, the
exploitation mode and the didactical performance. The resulting orchestrations are the
outcome of both the design by the researchers and the choices of the teachers themselves.
To investigate the first research question concerning the types of orchestration which
emerge in technology-rich classroom teaching, video tapes were analysed of38 lessons taught
in five classes by three female mathematics teachers who volunteered to participate in the
research project. The three were experienced teachers; however, the use oftechnology in this
specific learning arrangement was new to them. One ofthese teachers taught the arrangement
each ofthe three cycles, the others each taught it once. The three teachers were chosen so as
to have some variety in their educational level (degree for upper secondary or lower
secondary education, respectively) and school type (gymnasium or general secondary school).
For reasons mentioned above, data analysis focused on whole-class episodes in which
technologywasused. Theunitofanalysis was an episode, whichwas definedas thewhole-class
treatment of a single DME-task. If this treatment contained different orchestration types, the
episode was cut into sub-episodes, each with a unique orchestration type. In this way, a corpus
of 83 (sub-)episodes was identified. These were organised and analysed with the help of
software for qualitative data analysis
2
. The analysis was carried out in a bottom-up manner,
though partly driven by theory (e.g. Kendal & Stacey, 2002; Pierce & Ball, 2009; Trouche,
2004). Six orchestration types were identified and the corpus was coded according to this
categorization. A second coding of24 episodes (29%) was done by a co-researcher and led to
a good inter-rater reliability (a Cohen's kappa of .72).
To investigate the second research question on the relationships between the types of
orchestrations and teachers' views on mathematical learning and teaching and the role
oftechnology therein, each ofthe three teachers' orchestrations were analysed on the basis
of their observed teaching behaviours, and possible operational invariants were identified.
Then, the teachers' answers to an online questionnaire were used to obtain insight into their
self-reported views. The questionnaire consisted of 40 items on a five-point rating scale,
with the range of answers from 'strongly agree' to 'strongly disagree'. Items were derived
from previous research in this area and addressed the influence of technology on students'
learning and attitudes, on the lesson organisation, and on the role ofthe teacher (Drijvers et
al., 2007; Reed et al., 2010). Teachers' self-reported views and their observed orchestrations
were then integrated to produce a profile for each ofthe three teachers, which was validated
through a semi-structured post-intervention interview with the teacher concerned. To further
explore the value of the instrumental orchestration framework, transcripts of a limited
number of episodes were analysed in more detail.
5 Results
The results presentedbelow consist of(1) aglobal inventory ofobserved orchestrationtypes, (2)
their frequency distribution over the three teachers involved, (3) profiles ofthree participating
teachers with respect to their orchestrations and their views on mathematics education and the
role oftechnology therein, and (4) an extended analysis ofone exemplary episode.
2
We use Atlas ti software, www.atlasti.com
218 P. Drijvers et al.5.1 A global inventory of orchestration types
Six orchestration types were identified through a combination oftheory-driven and data-driven
analysis; these were termed Technical-demo, Explain-the-screen, Link-screen-board, Discussthe-screen, Spot-and-show, and Sherpa-at-work. Each of these orchestration types is briefly
described in terms of the first two elements of orchestration theory—i.e., didactical
configuration and exploitation mode—and is illustrated by a short example. The localised,
ad hoc nature of the third element of an instrumental orchestration—the didactical
performance—makes it unsuitable for inclusion in this global inventory.
The Technical-demo orchestration concerns the demonstration of tool techniques by the
teacher. It is recognised as an important aspect of technology-rich teaching (Monaghan,
2001, 2004). A didactical configuration for this orchestration includes access to the applet
and the DME, facilities for projecting the computer screen, and a classroom arrangement
that allows the students to follow the demonstration. As exploitation modes, teachers can
demonstrate a technique in a new situation or task, or use student work to show new
techniques in anticipation of what will follow. As an example, teachers exploited this

orchestration type to demonstrate the technique ofmaking an arrow chain of operations, or to draw a graph and scale it with the buttons above the graph window (see Fig. 1).
Demonstration-like orchestrations are also well-known in non-technological teaching settings; however, the type of demonstration seen here is instrumented by the technical tools and also specific to the tool's opportunities and constraints.
The Explain-the-screen orchestration concerns whole-class explanation by the teacher, guided by what happens on the computer screen. The explanation goes beyond techniques, and involves mathematical content. Didactical configurations can be similar to the Technicaldemo ones. As exploitation modes, teachers may take student work as a point ofdeparture for the explanation, or start with their own solution to a task. As an example, one teacher used the inverse arrow chain ofthe square chain to explain that square roots are always non-negative numbers. Explanatory orchestrations, of course, are also used in non-technological teaching settings, where they might be called Explain-the-board. The phenomena encountered in the technological environment, however, are often different from the ones faced at the traditional blackboard.
In the Link-screen-board orchestration, the teacher stresses the relationship between what happens in the technological environment and how this is represented in conventional mathematics of paper, book and blackboard. In addition to DME access and projection facilities, a didactical configuration includes a blackboard and a classroom setting such that both screen and board are visible. Similarly to the previously mentioned orchestration types, teachers' exploitation modes may take student work as a point of departure or start with a task or problem situation they set themselves. As an example, teachers copied arrow chains from the screen to the board, often in abbreviated and more conventional mathematical forms. This orchestration type is specific to the technology used. Because transfer to paper-and-pencil is known to be a problematic issue (Billington, 2009; Bretscher, 2009; Kieran & Drijvers, 2006), this kind of orchestration was one of our a priori foci.
The Discuss-the-screen orchestration concerns a whole-class discussion about what happens on the computer screen. The goal is to enhance collective instrumental genesis. A didactical configuration includes DME access and projecting facilities, preferably access to student work, and a classroom setting favourable for discussion. As exploitation modes, student work, a task, or a problem or approach set by the teacher can serve as the point of departure for student reactions. As an example, one teacher used a screen similar to the one in Fig. 1 to discuss different ways offinding a break-even point. Ofcourse, discussing results on The teacher and the tool 219the blackboard is a common teaching strategy. The advantage ofdiscussion in a technological environment is that suggestions for different representations and techniques, as they emerge in the discussion, can be easily tried out, with fast and dynamic feedback on obtained outcomes.
In the Spot-and-show orchestration, student reasoning is brought to the fore through the identification of interesting DME student work during preparation of the lesson, and its deliberate use in a classroom discussion. Besides previously mentioned features, a didactical configuration includes access to the DME during lesson preparation. As exploitation modes, teachers may have the students whose work is shown explain their reasoning, and ask other students for reactions, or themselves provide feedback on the student work. An example of Spot-and-show is given in the episode elaborated below. In that episode the teacher spots an inappropriate answer in a student solution while preparing the lesson, and discusses it with the whole class during the lesson. As this orchestration depends on the facilities offered to the teacher to browse through the DME student work, it is technology-specific and instrumented by the DME.
In the Sherpa-at-work orchestration, a so-called Sherpa-student uses the technology to present his or her work, or to carry out actions the teacher requests. Didactical configurations are similar to the Discuss-the-screen orchestration type. The classroom setting should be such that the Sherpa-student can be in control of using the technology, with all students able to follow the actions of both Sherpa-student and teacher easily. As exploitation modes, teachers may have work presented or explained by the Sherpa-student, or may pose questions to the Sherpa-student and ask him/her to carry out specific actions in the technological environment. As an example, one teacher had students using the computer and data projector to show a family ofgraphs, explain their reasoning and change the screen as requested by the teacher. This orchestration type is described in the literature (Trouche, 2004) and was therefore derived a priori from theory rather than emerging from the data.
A reflection on this set ofsix identified orchestration types led to the distinction ofa teacherstudent dimension. In Technical-demo, Explain-the-screen and Link-screen-board, the teacher dominates the communication. Studentinputisrestrictedandtheteacherguides the interactions in an Initiation-Response-Evaluation pattern. These orchestrations can be seen as teacher-centred. In Discuss-the-screen, Spot-and-show and Sherpa-at-work orchestrations, students have the opportunity to react and have more input. Even though the teacher manages the orchestration, there is more interaction and students have more voice than in the first three orchestration types. These can therefore be seen as student-centred orchestrations.
Another remark on this inventory is that the orchestrations are not isolated, but part of orchestrational sequences, and play particular roles in such a sequence. For instance, Technical-demo was often observed to precede and prepare for a phase of individual student work, to improve the efficiency of the latter phase. Spot-and-show orchestrations, and to a lesser extent Sherpa-at-work and Discuss-the-screen orchestrations, often exploit previous student work, individually or in pairs.
In summary, six orchestration types were identified, which differ in their focus on the teacher or the student as driving the orchestration and in the degree of technological specificity: some can be seen as technological variants of regular teaching practices most teachers are familiar with, whereas others are more specific to the use of technology or to the use of the tools used in this study in particular.

5.2 Frequency distribution of types of orchestrations over three teachers

Table 1 shows the frequencies ofthe orchestration types for each ofthe three teachers and, for teacher A, for each ofthe three times she taught the lesson series. Technical-demo is the most 220 P. Drijvers et al.frequently used orchestration type. In the post-intervention interviews, the three teachers said they feltthe needto familiarise students withbasic tooltechniques, in orderto preventtechnical obstacles hindering the subsequent mathematical activities: some basic technical knowledge is needed anddemonstrations canbe an efficientmeans toprovide it. This also avoidedthe teacher having to do too much "techno trouble shooting" (Monaghan, 2004, p. 346) or acting as "technical assistant" (Lagrange & Caliskan-Dedeoglu, 2009) during the work in pairs. In the interview, Teacher C explained the importance of Technical-demo as follows:
I want to prevent students from encountering technical difficulties while they are at work withthe computer…andhaveto waitformyhelp, whereas itoften concerns aminorissue.
The Spot-and-show orchestration also occurs frequently, but is unevenly spread over the three teachers. Although all three indicated in the post-intervention interview that they appreciated the facility to browse through the DME student work while preparing the next lesson, this orchestration type was mainly exploited by teacher B, who said:
The DME is practical for seeingwhat students do –you canuse itto fine-tune yourlesson.
The teacher guide, which the teachers received before the start of the teaching sequence, contained suggestions for teaching strategies which, in retrospect, match some of the identified orchestration types. A comparison ofthe suggested orchestrations in the teacher guide with the observed frequencies in Table 1 reveals that Link-screen-board and Discuss-the-screen were not suggested in the teacher guide, but were nevertheless quite frequently observed. Sherpa-at-work, on the contrary, was suggested several times in the teacher guide but was not popular with two ofthe three teachers. This shows that teachers felt free to adapt the suggestions in the teacher guide and make their own choices independently from it. We will come back to this in the discussion section that concludes this paper.

5.3 Teacher profiles of orchestrations and views on mathematics education and technology

Table 1 reveals a number ofdifferences between the three teachers. In order to explain them, we related them to teachers' answers to the questionnaire and the post-intervention interviews.

5.3.1 Teacher A

Table 1 shows that teacher A has slightly lower overall frequencies than teachers B and C. This corresponds to the observation that she takes time for her orchestrations to play out,

Table 1 Orchestration type frequencies by teachers

Orchestration type	Teacher A			Teacher B	Teacher C		Total
	cycle 1	cycle 2	cycle 3	cycle 2	cycle 3		
Technical demo	5	3	5	2	7		22
Explain-the-screen	0	0	1	0	7	8	
Link-screen-board	3	0	3	6	0	12	
Discuss-the-screen	4	4	2	3	1	14	
Spot-and-show	0	1	2	12	2	17	
Sherpa-at-work	2	7	1	0	0	10	
Total	14	15	14	23	17	83	

The teacher and the tool Z21and does not switch between orchestrations as fast as the other teachers. Apart from Technical-demo, which was discussed above, teacher A has relatively high frequencies for Discuss-the-screen and for Sherpa-at-work orchestrations. These are student-centred orchestration types, which teacher A used to evoke interaction between the students and herself. The underlying operational invariant is that students should be given a voice and that teaching should enable students to learn through interaction. In her classroom discussions, elements such as idea generation, comparison, evaluation and filtering (Sherin, 2002) can be traced.
Teacher A's views on mathematics education and the role of technology, as she expressed it in the questionnaire and the post-intervention interview, stress the importance of interaction in the classroom. She sees technology as a means to stimulate this:
…so you can have discussions with the students using the images that you saw on the screen, … that makes it more lively.
Her observed orchestration behaviour can be related to these views. For example, her agreement with the questionnaire statement 'As a teacher, one has to tell students clearly what they should do with ICT' matches her use of Technical-demo orchestrations. She also agreed with the statements 'The use of ICT gives the teacher more possibilities to build on students' ideas during whole-class discussions' and 'The use of ICT creates shared images and experiences which one can discuss with the students', which explains her preference for student-centred orchestrations, in which discussion and interaction play important roles.
In the post-intervention interview, teacher A mentioned time constraints as the main reason not to use Sherpa-at-work in the third cycle. Technical constraints (e.g. slow internet connections during the first cycle or inappropriate classroom settings in the computer lab) also drove her choices for orchestrations, as well as practical issues such as the features of the classroom in which the lesson takes place: how are the tables arranged, how are the computers set up, is Internet access available, and so forth.
In short, the profile of teacher A is that of a teacher who prioritises student-centred orchestrations. This relates to her view on learning as an interactive process in which students should have a voice. Technology is a means to achieve this, and offers new possibilities for teaching mathematics in an interactive way.

5.3.2 Teacher B

Table 1 shows that teacher B has relatively high frequencies for Link-screen-board, and particularly for Spot-and-show. Technical-demo is not often used, possibly due to the fact that her class consisted ofhigh-achieving students who are used to working independently and therefore might not need so much technical instruction. The Link-screen-board is used to establish links between the mathematics as it is represented in the technological environment and conventional mathematical representations and techniques. The Spot-andshow orchestration is used mainly to discuss either student inappropriate answers or original and insightful solutions.
Teacher B's views on mathematics education and the role oftechnology, as expressed in the questionnaire and the post-intervention interview, stress the importance of establishing links between the DME-work and paper-and-pencil mathematics. She sees the use of technology as an effective means to achieve her mathematical teaching goals, while she recognises that knowledge thus obtained also needs to be transferred to the paper-and-222 P. Drijvers et al.pencil environment. In addition, she believes that discussing mistakes and original approaches is fruitful for learning, and that both of these are more visible within a technological environment.
Her observed orchestration behaviour can be related to these views. Her strong agreement with the statement 'Compared with paper-and-pencil, ICT makes student work more visible for me' matches with her frequent use ofthe DME's facility to browse student work while preparing the Spot-and-show orchestrations. She also strongly agreed with the statement 'ICT can help students to develop understanding', which explains her interest in the use of ICT.
In the post-intervention interview, teacher B explained that she uses the Link-screenboard orchestration to … take distance from the specific ICT-environment; otherwise the experience remains too much linked to the ICT.
Concerning Spot-and-show, Teacher B explained that many students in her class were eager to have their work shown, which was an extra incentive for her to use it. She did not use the other student-centred orchestration types, as she considered them too time-consuming and also thought they might bring about some loss of control.
In short, the profile of teacher B is a teacher who finds the mathematical content ofthe lesson to be paramount and uses technology as a means to teach this. This results in attention to the links between the DME-work and paper-and-pencil or blackboard mathematics. She likes to be in control of the situation, which made her choose mainly teacher-centred orchestration types. Even if her frequent use of Spot-and-show reflects the importance she accords to discussing misconceptions and original approaches with the students, she usually quite strongly guides the Spot-and-show orchestrations. This aspect of Teacher B's profile is illustrated in the extended extract presented in the next section, which took place in teacher B's class.

5.3.3 Teacher C

Table 1 shows that teacher C has relatively high frequencies for Technical-demo and Explain-the-screen, which are considered as teacher-centred orchestration types. The Technical-demo orchestration is used to show the students what is possible with the tool and how to use it. The Explain-the-screen orchestration is in some cases used to provide students with a good starting position for new tasks, and in some cases to explain how tasks could have been solved. The underlying operational invariants seem to include the desire to avoid technical obstacles during student computer work, and the beliefthat tasks should be clearly explained to the students, in order to foster more efficient learning activities.
In the post-intervention interview, teacher C described herself as a 'typical teacher for mid-ability students' who strongly believes that such students benefit from clear demonstrations and explanations in a structured and stepwise approach. Technology is a suitable and useful means to provide this kind of scaffolding. At the same time, she was aware that students may encounter difficulties while using technology, and wished to

prevent these as much as possible. Furthermore, she wanted to be in control of what is
happening in the classroom, and believed that more teacher-centred orchestrations support
this.
Her observed orchestration behaviour can be related to these views. She strongly
agreed with the statement 'As a teacher, one has to tell students clearly what they should
The teacher and the tool 223do with ICT', which matches with her prioritising Technical-demo orchestrations.
Teacher C was neutral on the statements 'The use of ICT provides the teacher more
possibilities to build on students' ideas during whole-class discussions' and 'The use of
ICT creates shared images and experiences which one can discuss with the students'. This
might explain her preference for teacher-centred orchestrations. Her preference for a
stepwise approach was reflected in her strong agreement with the statement 'Students
become more systematic in their approach through the work with ICT'. She explained
that she focused Technical-demo orchestrations on obstacles she experienced herself
when using the DME, and that time constraints made her avoid orchestrations such as
Sherpa-at-work.
In short, teacher C's profile shows a concern for difficulties which students may
encounter when learning mathematics, particularly when using technology, and the
conviction that clear explanations and instructions are suitable means to deal with this.
This guides her choice for teacher-centred orchestrations. She wants technology to
support a stepwise problem solving approach. Furthermore, time constraints and control
are important factors for her in making her orchestrational choices.
From the profiles described here, we conclude that the three teachers differ in their
preferences for orchestration types and that their preferences can be related to their views
on mathematics education and the role of technology therein. In fact, their views on
mathematics education and the opportunities technology offers can be seen as their
theoretical discourse that justifies their orchestrational choices and guides their operational
invariants. These choices were shown to be partially based on time constraints and concerns
about control, which is in line with findings in other studies (Pierce & Ball, 2009). Finally,
it is interesting to note that the profiles of teachers A and C reported here are close to
similar profiles from two case studies (of teacher P1 and N, respectively) described by
Ruthven, Hennessy, and Deaney (2008).
5.4 Extended analysis of one episode
So far, orchestrations have been described on a global level in order to draw up an
inventory of orchestration types. However, more detailed issues, such as the teacherstudent interaction during these orchestrations, remained unaddressed. To further
investigate the explanatory power of the notion of instrumental orchestration and of
its third element of didactical performance in particular, a number of episodes were
analysed more extensively; one exemplary episode is presented here. As the Spotand-show orchestration is highly instrumented by technology and offers interesting
interaction potential, we chose a Spot-and-show episode, taught by teacher B.
During the first two lessons ofthe learning arrangement, the notion ofarrow chains was
introduced to the students after group work on diverse problem situations involving
dependency and co-variation (see Doorman et al., 2009). In the third and fourth lessons,
students worked in pairs with arrow chains in the DME. In task 8, one of the tasks of the
fourth lesson, the designers filled the applet window with the start ofa square and a square
root chain and an empty graph window (see Fig. 2).
While preparing the fifth lesson, teacher B came upon the work of Florence (a
pseudonym) and her classmate (see Fig. 3). These two students used a Table-Graph
technique to solve this task: they added both tables and graphs in the applet window. As the
tables provide integer input values and the students did not add an input variable in the
input box, the graphs that appear are dot graphs. Then the students clicked on different
224 P. Drijvers et al.input values in the table, and observed the effects on the two corresponding dots in the
graph. This technique was used more often by these students as a means ofinvestigating the
function. In this case, this led them to think that it is 'special' that two points reflecting
function values for the same input value are vertically aligned. Their reasoning, therefore,
reads: "for the square they are all whole numbers, and for the square root they are whole
numbers and fractions. And the square of a number is always right above the root?'. The
question mark at the end indicates that the students were not sure about their findings.
Teacher B decided to treat this task, and this inappropriate answer in particular, in a whole
class discussion. The text frame below contains the verbatim transcript of this episode.
Using a data projector, the DME with the list of student pairs is projected on the wall above the blackboard.
The teacher T navigates within this list to Tim and Kay and opens their solution of task 8.
T: Here it says [points on the screen to question c using the mouse]: what do you notice? Oh yes, I
actually wanted to see quite a different one, because they had …
T quickly navigates to Florence (F) and her classmate's work. In their work, the students checked on the
Table option. This led to 'dot graphs' on the screen (see figure 2 left screen). The answer to question c,
which the students had typed in, reads:
"for the square they are all whole numbers, and for the square root they are whole numbers and fractions.
And the square of a number is always right above the root.?"
T: Look here, what this says [points at the students' answer to question c on the screen with the
mouse and follows it while reading]. For the square they are all whole numbers, okay, and for the
square root they aren't whole numbers, we agree with that too, and the square of a number is
always right above the square root.
F: Was that right?
T: I'm not saying.
St1
:
: Yes, I had that too.
T: What they say, then, is that every time there is…if I've got something here, there is something
above it, and if I've got something there, there is also something above it. [points with the mouse
at pairs of vertically aligned points in the graph] Why is it, that these things are right above each
other?
F: Well, because it…the square root is just…no the square is just, um, twice the root, or something.
St2: No.
T: Kay?
Kay: That's because the line underneath, that's got a number on it, which you take the square root of
and square, so on the same line anyway.
T: What are those numbers called that are on the horizontal line then?
St3: The input numbers.
T: The input numbers.
T: Ehm, Florence, did you follow what Kay said?
F: No, but I […]. It was about numbers and about square roots and about…
Sts: [laugh]
St: It was about numbers!
The teacher and the tool 225T: Kay said: these are the input numbers, here on the horizontal line. [moves the mouse over the
horizontal axis] And for an input number you get an output number. And that is right above it. So
if you take the same input number for two functions… [indicates the two arrow chains with the
mouse]
F: Oh yes.
T: … then you also get…then you get points above it [points at a pair of vertically aligned dots in the
graph]. So that's got nothing at all to do with the functions. It's just got to do with from which
number you are going to calculate the output value. Now, if for both of them you calculate what
the output value is for 10 [indicates 10 at the horizontal axis], they both get a point above the 10
[indicates two dots above x=10]. Do you understand that?
F: Oh yes, I didn't know that.
T navigates back to the list of student pairs.

"St1, St2, .. stands for one of the students
In analysing this episode from an instrumental orchestrationperspective, itwas recognised as
a Spot-and-show orchestration: 'Spot' because the teacher, while preparing the lesson, spotted
the inappropriate answer in the students' DME-work, and 'show' referring to the teacher's
decision to display these students' results as a starting point for whole-class discussion. The
Fig. 2 Computer task 8
226 P. Drijvers et al.teacher's phrase 'Ohyes, I actuallywantedto see quite a different one' andherdirectnavigation
to Florence's work reveal her deliberate intention to act the way she does.
The didactical configuration for the preparatory phase consists of the DME's option for
teachers to look at the students' work at any time. As a result, the teacher notices the
inappropriate answer and decides to deal with it in her lesson. Apparently, the teacher likes to
use technology for "supporting learning through analysis of mathematical discrepancies"
(Ruthven, Hennessy & Deaney, 2008, p. 312). The monitoring of students' work is
instrumented by DME-facilities that are not available to students. As the teacher finds the
computer lab not to be appropriate for whole-class teaching, the lesson takes place in a regular
classroom with a PC with DME access, connected to a data projector. The configuration
includes putting the computer with the data projector in the centre of the classroom. This
choice is driven by the constraints ofone ofthe artefacts: ifthe projector was at the front, the
projection would get too small for the students to read. The screen is projected on the wall
above the blackboard, thus enabling the teacher to write on the board, which she regularly
does—though not in the episode presented here. Both this way ofpreparing the lessons and
the setting in the classroom are observed frequently in this teacher's lessons.
The exploitation mode of this configuration includes the teacher's choice to operate the
PC herself. This, in combination with the computer being in the centre of the classroom,
results in the teacher standing in the centre of the classroom, with the students closely
around her, all focused on the projection on the wall. This exploitation mode enables
classroom discussion and student involvement.
The didacticalperformance starts with the teacher displaying the student work she had
selected to show and reading out their reasoning, while adding some minor comments
('Look here, …'). Then she reformulates the answer and asks Florence for an explanation
Fig. 3 Florence's applet screen presented during whole-class discussion
The teacher and the tool 227('What they say…'). When the explanation turns out to be inappropriate, she invites
another student, Kay, to give his explanation, and checks whether Florence understands it.
When this is not the case, the teacher rephrases Kay's explanation and once more checks
with Florence, who now says she understands, although she does not actually give evidence
ofthis. Throughout the didactical performance, the teacher uses the mouse to relate her oral
expressions to the phenomena on the screen.
The didactical performance is now considered in more detail. The starting point for
the episode is Florence's artefact-related notion: a dot graph in the applet window of
the DME suggests to her a 'special' feature of dots being aligned vertically. This is
related to the tool technique of creating a table of function values and a corresponding
graph. An understanding of the underlying mathematical concepts and meaning seems
to be lacking.
The mathematical meaning the teacher is aiming for is that the coordinates of a point of
the graph of a function reflect the values of independent and dependent variables,
conventionally represented on the horizontal and vertical axes, respectively. As a
consequence, different functions with the same input value will result in points that are
vertically aligned in the graph.
What does the teacher do to guide the learning process towards this meaning? Firstly,
she reads the students' written answer and approves parts of it that are not crucial to the
targeted understanding. However, she does not comment on the final part ofthe answer and
later says 'I am not saying' when asked if it is correct.
As a next step, she rephrases the students' answer and illustrates it with mouse
movements in the applet window, so that all students understand the conjecture made by
Florence. Then the teacher brings the relevant question to the fore: 'Why is it that these
things are right above each other?' This leads to a vague reaction from Florence. The
teacher invites Kay to explain his reasoning, which indicates that his interpretation of the
representation in the artefact is linked to his understanding ofthe meaning ofthe graph as a
representation ofa mathematical idea/concept. The teacher subsequently evokes a precision
ofthe intended mathematical meaning by asking for the type ofnumbers that is represented
on the horizontal axis. This raises awareness of the link between the arrow chain and the
graph. As a final step, the teacher checks ifFlorence understands Kay's explanation. As this
does not seem to be the case, the teacher repeats and expands Kay's reasoning in her own
words, again supporting her explanation with mouse movements on the screen. Once more
she checks this with Florence, who now says she understands. As a result, Florence and her
classmate remove the remark on the vertically aligned dots after this classroom discussion
and replace it by a remark on the increasing behaviour of the square function: "for the
square they are always whole numbers, and for the square root they are whole numbers and
fractions. The squares get higher with much bigger steps."
This analysis reveals some important strategies this teacher applies in her didactical
performance. Firstly, she evokes mathematical meaning in several ways, namely by posing
open questions, by inviting students to explain their reasoning, and by giving her own
explanations. Secondly, she uses revoicing techniques and thus 'orchestrates multiple
voices' (Forman & Ansell, 2002), while she in some cases also improves the students'
formulations and privileges specific input. Thirdly, she gives mathematical meaning to tool
techniques and mathematical representations by using the mouse to highlight parts of the
screen during her explanations. Finally, she regularly checks students' understanding. This
approach is typical for teacher B, and is consistent with the profile presented above. She
used Spot-and-show orchestration quite frequently, perceiving it as a means to enhance
student involvement and discussion, which she believes contributes to the development of
228 P. Drijvers et al.the mathematical meaning that she finds so important. Evoking mathematical meaning in
classroom discussions based on student results seems to be one of her operational
invariants.
But what about the link between instrumental orchestration and instrumental genesis? As
the episode only shows the result of students using the artefact, we do not see direct traces
of the students' instrumental genesis. We argue, however, that Florence's idea of two
vertically aligned points being special is part of her scheme of using the Table-Graph
technique to produce dot graphs. Even though her knowledge linked to this technique—the
theorem-in-action, as it is called in instrumentation theory—is inappropriate, the episode
indicates that the teacher can exploit the students' experiences—in this case those of
Florence and Kay in particular—for the purpose of attaching mathematical meaning to the
technique used. The teacher privileges the epistemic value of the Table-Graph technique
above its pragmatic value (Artigue, 2002). In her didactical performance of the Spot-andshow orchestration she put into action, she highlights the mathematical meaning of the
concept of function as it emerges from the technique. This is exactly what instrumental
genesis is about. In this sense, the lens ofinstrumentational orchestration has a great deal to
offer in this analysis. However, our analysis also reveals the importance of the interplay
between representations in the technological environment and mathematical meaning. This
suggests that the analysis could benefit from notions from semiotic mediation theory, which
stresses the mediation potential of artefacts between the learner and the mathematical
knowledge (Bartolini Bussi & Mariotti, 2008). Analyses such as those by Maracci and
Mariotti (2009) show that the semiotic mediation framework can complement theories on
instrumentation and orchestration (Drijvers, Kieran & Mariotti, 2010).
6 Conclusions and discussion
6.1 Conclusions
The first research question of this study concerns the different types of instrumental
orchestrations which emerge in technology-rich classroom teaching. The analysis revealed
six orchestration types, which are different in their technological specificity. Discuss-thescreen and Explain-the-screen can be seen as technological variants of regular teaching

practices most teachers are familiar with. More specific for the use of technology are
Technical-demo, Sherpa-at-work and Link-screen-board. Specific for technological tools
that provide the teacher with online monitoring of digital student work is the Spot-andshow orchestration. It can be concluded that the repertoire of orchestrations is diverse, that
they differ in respect to whether the focus is on the teacher or the student, and that the
orchestrations have different degrees of technological specificity. Technological and time
constraints as well as control issues may influence the choice and exploitation of the
orchestrations. Furthermore, a discrepancy was noticed between orchestrations that were
suggested in the teacher guide and orchestrations that were actually observed.
The second research question concerns the relationships between these types of
orchestrations and teachers' expressed views on mathematical learning and teaching and
the role of technology therein. The matching of observed orchestrations with data from
questionnaires and interviews reveals considerable links between the three teachers'
preferences for orchestrations and their views on what is important to achieve during
teaching and how technology can support this. We conclude that these three teachers'
choices for orchestrations are related to their expressed views.
The teacher and the tool 2296.2 Discussion
The above conclusions may inform mathematics teachers and teacher educators who are
engaged in professional development on the issue of the integration of technological tools
in teaching. With respect to these conclusions, three issues deserve further discussion: the
inventory of the six orchestration types, including its generalizability and researchers'
expectations, the relationship between orchestrational choices and teachers' expressed
views, and the explanatory power ofthe instrumental orchestration framework, including its
limitations and its articulation with other theoretical approaches.
With respect to the inventory ofthe six orchestration types, it is interesting to see that the
identified orchestrations can to a certain extent be related to 'traditional' teaching
techniques. This suggests an evolution of teaching techniques rather than a revolution.
Also, the researchers' expectations for the orchestrational opportunities the digital tools
provide, as they were sketched in the teacher guide, were to a certain extent ignored by the
teachers. They choose their orchestrations themselves, which led to discrepancies between
the researchers' prior expectations and the actual findings from observations, discrepancies
that are also reported by Lagrange and Caliskan-Dedeoglu (2009). The data suggest that the
teachers make their choices in line with their regular habits and their views on mathematics
teaching, which underpins the remark on teachers' practices as being stable, made by
Robert and Rogalski (2005) and referred to in the introduction.
For the case of spreadsheet software, Lagrange and Ozdemir Erdogan (2009) report on
the differences between teachers stressing the epistemological value oftool techniques, and
students who primarily see the pragmatic value. Similar struggles were observed in this
study, where teachers felt obliged to use technical-demo orchestrations relatively often,
whereas their primary concern was in the mathematical meaning.
The method for identifying orchestration types has its limitations, which impact the
generalizability of the results. First, the orchestration types were sometimes difficult to
distinguish. Even ifthe inter-rater reliability ofthe codings was good, it was not always easy to
decide, for example, ifa Spot-and-show orchestration gradually turned into Discuss-the-screen
or Explain-the-screen. Second, it is possible that other types of orchestrations might have
emerged in a different learning arrangement or while observing other teachers. The
observations took place during a specific intervention, consisting ofa complex and innovative
learning arrangement with which the three teachers were not familiar. This lack offamiliarity,
along with start-up problems, may have influenced their orchestration choices. Also, the time
schedule for the learning arrangement was tight, which may also have affected the didactical
performance and the decisions involved. And finally, the specific tool characteristics may have
invited (or inhibited) specific types of orchestration. For example, Spot-and-show orchestrations clearly depend on the opportunity for the teacher to access student work while preparing
the lesson.
Despite these limitations, we conjecture that the identified orchestration types also exist,
maybe in adapted forms, outside the scope ofthis study. Therefore, it would be worthwhile
investigating whether variants of the identified orchestration types can be recognised in
other settings, such as group work or individual student-teacher interactions (Drijvers,
submitted), and in learning arrangements with other types of technological tools.
Concerning the relationship between orchestrational choices and teachers' expressed
views, which was the second research focus, the data suggest that the teachers' views on
mathematics education and the opportunities technology offers can be seen as their theoretical
discourse thatjustifies their orchestrational choices and guide their operational invariants. It is
interesting to note that the teachers' views as identified in this study correspond to findings
230 P. Drijvers et al.reported elsewhere. We already mentioned the similarities between the profiles ofteachers A
and C and those of two teachers described by Ruthven, Hennessy & Deaney (2008).
Lagrange and Caliskan-Dedeoglu (2009) also mention the relationship between teachers'
expectations towards technology and their general ideas on teaching and learning.
As a limitation of the study on the relationships between orchestrational choices and
expressed views, however, we point out that the data in the study presented here are too
limited to justify claims on the teachers' operational invariants across other classes and
other kinds oftechnological tools. Furthermore, the notion ofoperational invariants in itself
might benefit from further elaboration and concretisation.
As a final point of discussion, we also wished to investigate what the framework of
instrumental orchestration offers to the analysis of technology-rich classroom teaching. If
we interpret the orchestration metaphor in terms of a jazz band with its band leader, as we
suggested earlier, we experience this framework as a productive lens through which to view
teacher behaviour. On the whole, it appears to do justice to the subtleties in the preparation
and performance ofteaching techniques. The first two dimensions ofthe model helped us to
identify global patterns in teachers' behaviours, while the third dimension of the model—
the didactical performance—turned out to be more fruitful in the detailed analysis of a
teaching episode. On a more critical note, however, one can wonder if similar results could
have been obtained in our study without the somewhat difficult vocabulary of the
theoretical framework. Furthermore, as with every metaphor, the musical metaphor has only
a limited application to educational settings. Also, we perceived a danger of sticking too
much to superficial aspects of teaching while describing didactical configurations and
exploitation modes. That is why the extension with didactical performance as a third
component was useful and allowed for more in-depth and domain-specific didactical
analysis.
As an aside, it should also be noted that, while designing orchestrations, the teacher is
also engaged in a process of instrumental genesis for accomplishing the intended teaching
task (Bueno-Ravel & Gueudet, 2007). However, the data available here do not allow us to
proceed beyond developing a state-of-the art inventory of orchestration types to observing
this process of instrumental genesis over time.
At the global level of analysis, it is worthwhile to consider the integration of the
orchestrational model with other theoretical approaches. As a first approach, Ruthven and
colleagues (Ruthven, 2007, 2009; Ruthven, Deaney & Hennessy, 2009) designed a model
with a focus on key structuring features that shape technology integration into classroom
practice. Applications of this model (e.g. see Lagrange & Caliskan-Dedeoglu, 2009) show
that it can complement and supplement the instrumental orchestration model, and may
thereby contribute to further theoretical development. A second additional approach is
Saxe's (1991) four parameter model, which centres on emergent goals under the influence
of activity structures, social interactions, prior understandings, and conventions and
artefacts. Even if it was designed for completely different purposes, several studies
successfully applied this model to frame teachers' activities in technology-based
mathematics lessons (Lagrange & Monaghan, 2009; Lagrange & Ozdemir Erdogan,
2009; Monaghan, 2004). In our own work, we used it as complementary to instrumentation
and orchestration, to capture the changing meaning students attach to existing techniques
(Doorman et al., 2009). Third, the recently developed notion of teacher resources and
documents (Gueudet & Trouche, 2009) matches well with notions on instrumental genesis
and orchestration and may be worth exploring further. Fourth and final, at a more detailed
level of analysis—for example in the exemplary Spot-and-show episode—the interplay of
tool techniques and mathematical meaning became more salient. It became clear that the
The teacher and the tool 231analysis could benefit from the semiotic mediation perspective (Bartolini Bussi & Mariotti,
2008) as an additional theoretical lens. The articulation, integration and confrontation of
these different models is a challenge for research in this domain and deserves further
investigation.
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O RI GIN AL ARTI CL E
MobileMath: exploring mathematics outside the classroom
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Abstract Computer games seem to have a potential for
engaging students in meaningful learning, inside as well as
outside of school. With the growing availability of mobile
handheld technology (HHT), a number of location-based
games for handheld mobile phones with GPS have been
designed for educational use. The exploitation of this
potential for engaging students into meaningful learning,
however, so far remains unexplored. In an explorative
design research, we investigated whether a location-based
game with HHT provides opportunities for engaging in
mathematical activities through the design of a geometry
game called MobileMath. Its usability and opportunities for
learning were tested in a pilot on three different secondary
schools with 60 12-14-year-old students. Data were gathered by means of participatory observation, online storage
of game data, an online survey and interviews with students
and teachers. The results suggest that students were highly
motivated, and enjoyed playing the game. Students indicated they learned to use the GPS, to read a map and to
construct quadrilaterals. The study suggests learning
opportunities that MobileMath provides and that need
further investigation.
1 Introduction
So far, the use of handheld technology (HHT) in mathematics education has mainly focused on traditional mathematical content, such as algebra and graphing, and took
place in regular teaching settings (Trouche and Drijvers
2010). Nowadays, however, HHT has more to offer:
handheld devices offer options for other topics, such as
geometry, and can be used outside the classroom, thanks to
wireless communication. Such extended mobile devices
also provide opportunities for game play in ways that are
new for mathematics students and their teachers. These
new features are supposed to engage students in meaningful learning, inside as well as outside of school (Gee
2003; Shaffer 2006; DeVane et al. 2008), fit into the digital
lifestyle of the new generation and are expected to have
great motivational potential. Computer games may enable
setting up rich learning environments that appeal for peer
collaboration, knowledge constructions, and new roles for
both teacher and students (Egenfeldt-Nielsen 2005).
Until today, little is known about the ways in which
these assumed potentials can be exploited for the sake of
engagement in and quality of learning. Therefore, the
overarching aim of the study that we report upon in this
paper is to explore the potential of mobile HHT for the
engagement in mathematics education, and the learning
opportunities of a geometrical out-of-school game in
particular.
2 Theoretical background
The backbone of the study's theoretical framework is
formed by notions on engagement, and on authentic and
realistic mathematics learning.
This article is based on the paper 'MobileMath: the phone, the game
and the math' (Wijers, Jonker, & Kerstens, 2008) presented at
ECGBL2008, Barcelona, Spain, 16-17 October 2008.
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ZDM Mathematics Education (2010) 42:789-799
DOI 10.1007/s11858-010-0276-3Mobile gaming is a general term that covers a large
range of activities including playing casual games on
handhelds as well as playing highly interactive locationbased games on mobile devices, where real and virtual
worlds are mixed. Digital games seem to be tools for
supporting meaningful learning and engagement of students, both inside and out of school (Prensky 2001; Gee
2003; Shaffer 2006; DeVane et al. 2008). Gee (2003)
connects engagement in good video games to players being
'enticed to try' (p. 65), put in lots of effort and spend lots of
time on task. Prensky (2001) connects engagement to 'total
involvement', no off-task behavior and continuous paying
attention. He formulates the following elements that need
to be carefully designed and combined in order to create
engaging games:
• rules: provide structure and organize the game play;
• goals: motivate players;
• outcome andfeedback: inform players on the progress,
provide opportunities to learn;
• conflict/challenge/competition: make games exciting;
• interaction: social aspect.
Mobile devices provide the opportunity to actually
situate learning outside school and make it possible to
integrate characteristics of effective learning such as situated authentic learning, peer collaboration and motivational power. In the definition of O'Malley et al. (2003),
mobile learning is any sort of learning that happens when
the learner is not at a fixed, predetermined location, or
that happens when the learner takes advantage of learning
opportunities offered by mobile technologies. A key
characteristic of mobile learning is that it enables
knowledge building and constructing understanding by
learners in different contexts (Winters 2007). Research on
game-based learning suggests that the combination of
reality and virtual elements leads to a mixed reality
experience which can contribute to the students'
engagement (Schwabe & Go¨th 2005). A term used to
stress this combination is hybrid reality game (HRG), as
this type of games augments the reality by adding virtual
elements to it.
Hybrid reality games (HRGs) employ mobile handheld technologies and GPS devices as tools for
transforming physical spaces into interactive game
boards (De Souza e Silva & Delacruz 2006, p. 231).
HRGs' intrinsic motivational properties transform an
otherwise dry curriculum into something entertaining
and fun, using affordable and ubiquitous technology.
However, these types of games are more than a new
snazzy delivery vehicle for the existing content.
HRGs force players to look at familiar spaces from
unfamiliar perspectives and at content learned in the
classroom from a different viewpoint, using learning
principles such as elements of social, experiential,
and situated learning. (De Souza e Silva & Delacruz
2006, p. 246)
Of course, in serious gaming, we are interested not only
in engagement, but also in learning. To enhance the
learning effects, games need to be embedded in learning
activities (Lave & Wenger 1991). The effectiveness of
these learning activities can be stimulated if the tasks are
authentic and realistic, and can be worked on in collaboration (O'Donnell et al. 2005). Recent research has shown
that the use of mobile location-aware games can contribute
to meaningful learning on several school and academic
subjects such as history (Admiraal et al. 2007) and science
(Squire & Klopfer 2007; Squire 2008). For the case of
mathematics, the theory of Realistic Mathematics Education (RME) may help us to design meaningful learning
activities.
The theory of RME stresses that problem situations
presented in learning activities should be 'experientially'
real to students (Gravemeijer 1994) and have meaningful,
authentic problem situations as starting points. Mathematics is seen as a constructive human activity that is
driven by the act of mathematizing (Freudenthal 1991).
Freudenthal (1991) stresses the importance of guided
reinvention, which underlies the construction of knowledge as if a student reinvented it. Students' own productions and constructions play an important part in the
learning process also because they form input for classroom discussions and interactive reflection. This is in line
with a socio-constructivist perspective on learning, in
which mathematics is seen as both an individual, constructive activity and as a social practice (Cobb, Yackel, &
Wood 1992). Social interaction is seen as a necessary
condition for learning mathematics (Freudenthal 1991;
Treffers 1987, 1991).
With notions of engagement and authentic, RME in
mind, we can now phrase the central question of this pilot
study:
Can a social mobile game, based on the geographical
reality, its virtual map representation, and locationsensitive handheld technology engage 12-14 year old
students in meaningful mathematical activities?
The words 'engage' and 'meaningful' evoke the following sub-questions:
• Game play: does the game engage the students? Do
they appreciate the mixed reality experience?
• Learning opportunities: Do the students recognize the
mathematics embedded in the mobile game? Do
students feel they learned something?

1233 Methods
To address the research questions, a game called MobileMath was designed, with corresponding learning activities,
and a sequence of pilots was set up. A first pilot was held
with the teachers of the three pilot schools, followed by
three pilots with students at different schools. In all pilots,
researchers observed the team activities during the game
and debriefed the teams afterward. Observations were
made either by accompanying a team outside as a participating researcher or by watching all teams in the game in
real time on the website. Notes were made and the game
data were stored online. Players completed an online
questionnaire immediately after playing.
3.1 Design of the MobileMath game
With the design of the game MobileMath (Demeyer et al.
2008), the design team consisting of researchers on mathematics education
1
and developers of creative technology
for social innovation
2
has tried to combine principles of
RME with principles of mobile game-based learning in
order to design a mobile mathematical game, i.e. engaging
and provides opportunities for mathematics learning.

As a first design heuristic, the elements described by
Prensky (2001) guided the design of the game structure.
The players' position is to be involved into the game play
and thus should support situative, authentic and collaborative learning of mathematics. Collaboration and interaction, including both social interaction and interaction with
the tools, were realized by having students play in eight
competing teams of two students. The game rules were
designed to promote both types of interaction.
As a second design principle, the RME theory led us to
search for an authentic and realistic problem situation—
where we take the word 'realistic' as meaningful rather
than as 'from everyday life'. By incorporating students'
own mathematical constructions in the heart of the game
play of MobileMath, we wished to integrate an important
tenet of RME with principles of situated and game-based
learning.
MobileMath is designed to be a HRG. In MobileMath,
we wanted the players to be immersed in a mixed reality
game environment, in which they create virtual elements,
in this case mathematical shapes, by interacting with the
real world.
We wanted to design a geometric game. By playing the
game, students were expected to deepen their experiential
knowledge of geometrical concepts related to shapes and
orientation/navigation. This includes: properties of angles
and edges of parallelograms (including rectangles and
squares); geometrical concepts such as parallel and perpendicular; and on the boundary of geography and math:
orientation and navigation in 2D (map) and 3D (real
world). These are all topics addressed in the math curriculum in lower secondary education in grade 7 or 8. The
objectives are to learn the names of the shapes and gain (at
least implicit) knowledge of some fundamental properties
of parallelograms regarding sides being in pairs parallel
and of equal length; and angles being straight or not. In the
game, they create parallelograms and thus must use and
make explicit the knowledge of these properties.
We identified the starting point for the design of MobileMath to be the game play. The mathematical content to
be learnt would be integrated into the game in an intrinsic
way. Previous research on game-based learning has shown
that the content of a game can be integrated into the game
in an intrinsic or an extrinsic way (Kafai, 1998, 2001;
Malone and Lepper, 1987). Intrinsic integration of content
(mathematics) in a game provides students with greater
opportunities to construct new relationships with knowledge in the process of playing. In this type of games, the
educational and game components are inseparable (Klopfer,
2005). A large number of the available games for
mathematics do not reflect this design principle. By realizing this intrinsic integration of game and mathematics,
MobileMath should give students the feeling of playing a
game, and at the same time have them experiencing
another way of doing mathematics.
A second design issue concerns the location-based
character of MobileMath. It was seen as important to make
sure that every school could use the game in its own direct
environment. Players can thus use their knowledge of the
environment as well as expand that knowledge by playing
the game. Although the game is location-based in the sense
that it makes use of elements in the real world, MobileMath
is not based on one specific location, but can be played
anywhere.
3.2 Pilot with the teachers
The pilot was carried out with one geography teacher, three
mathematics teachers and one computer science teacher.
This pilot had three objectives: to test the game play with a
group of 'new' players, to prepare for the pilots with students and to discuss and co-design the educational setting.
The game test was performed with five teams, each consisting of one teacher and one researcher as participating
observer. Afterward, the teachers filled in a questionnaire
and the game was discussed with all participants. The
closing discussion, in which the tracks were viewed and the
strategies of the teams were discussed, provided opportunities for reflecting and making connections between game
1
Freudenthal Institute, Utrecht University.
2
Waag Society, Amsterdam.
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123strategies and aspects of mathematics (geometry of navigation and orientation and shapes) and geography (map
skills).
3.3 Pilots with students
MobileMath was piloted in grades 7 and 8. By the end of
grade 7, all students have worked on mathematical content
related to MobileMath. This includes: navigation and orientation; measurements such as length (distance) and area;
2D shapes and their properties, especially those of specific
quadrilaterals. The construction of 2D shapes is not a core
topic in the mathematics curriculum for grade 7.
Three secondary schools (from the researchers' networks) participated in the pilot with a total of 60 students,
54 of whom filled in the questionnaire afterward (see
Table 1). All pilots were observed by at least three
researchers, notes were taken and in two of the three pilots
video recordings were made.
The pilots consisted of a whole-class introduction, one
round of game play of 1 h and a debriefing session. The
whole-class introduction focused on map reading skills,
characteristics of quadrilaterals, the game rules and the
technology of the phone. In schools 1 and 2, both a math
teacher and a geography teacher were involved. Introduction in school 1 took about 50 min, whereas in school 2
only 30 min was used. In school 3, the game was introduced by a mathematics teacher the day before the game
was played. On all schools, a designer-researcher demonstrated how to use the phone. After the introduction students played a 1-h game of MobileMath in a playing field
located around school, with a radius of 1 km. Because of
the large number of students who participated in school 1,
two rounds of game play were needed: half of the students
played the game outside, while the other half watched the
game play online in the computer room at school and kept
a log of what (they thought) was happening.
At school 1, for safety reasons, all outside teams were
accompanied by a teacher or researcher acting as a participating observer. At the other schools, the teams went
outside on their own and the teachers and researchers
observed the game play online on the game website. In a
debriefing session, the stored game data and the observation notes were used to have the teams reflect on their game
play. Two forms of debriefing were used: a debriefing
session with each team immediately upon its arrival back at
school or a whole-class debriefing session. At schools 1
and 2, both forms were used, at school 3 only individual
debriefings were held. Notes were taken during the
debriefing sessions, and in two schools debriefing was
videotaped.
4 The MobileMath game
The MobileMath game, as it resulted from the design
process, is a location-based game that can be played on a
mobile phone with a GPS receiver and Windows Mobile as
Operating System. Geometry was chosen by the team to
provide the mathematical content to be addressed in the
game.
The game can be played by two to eight teams. Each
team creates geometrical shapes on a previously defined
playing field (in the real world), using a mobile phone with
GPS functionality and an on-screen map. The size of the
playing field and the duration of the game can be set by the
first player starting the game from a mobile phone. Other
teams then subscribe to this game by choosing a team
name. Players are connected with the game space and each
other through the map on their screens. During the game,
all teams can see themselves and the others as colored dots
moving in real time in the playing field on the underlying
map. The goal is for a team to gain points by covering as
much area as possible with virtually constructed parallelograms ((including squares and rectangles). Players do this
outside by walking to locations where they place, and
virtually connect, vertices. To support the process of constructing and deconstructing quadrilaterals, auxiliary lines
are visible on screen. If the fourth vertex finishes a shape
correctly (which is evaluated by the game engine, within a
margin of about 10 m), it appears on the playing field in the
color of the team and is visible for all teams. If the fourth
vertex does not finish a correct shape, it will disappear (see
Fig. 1).
Since the shapes cannot overlap, it is possible to 'hinder'
a team by trying to create a shape within a shape under
construction.
Based on a conversion rule, each shape is awarded with
a score equivalent to its area; this score is multiplied by a
factor based on the difficulty of the construction of the
shape. The difficulty of the construction is defined by the
number of constraints. Squares having the most constraints
Table 1 Overview of the pilots
# of students Grade Male Female Voluntary? Accompanied?
School 1 28 7 17 11 No Yes
School 2 12 8 4 8 No No
School 3 16 8 11 5 Yes No
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123(exact right angles and equal sides) are most difficult to
construct: the factors are 29 for a square and 1.59 for a
rectangle and 1 for any other parallelogram. The created
shapes are virtual elements added to the real world as a
kind of 'overlay' (see Fig. 2): the physical world has thus
become an interactive game board.
Deconstructing quadrilaterals of other teams is also part
of the game play and teams are rewarded points for it.
Deconstruction brings extra challenge and competition in
the game. Both construction and deconstruction require
that players use their knowledge of characteristics of specific types of parallelograms and their constructions.
Figure 3 shows the locations of the four points of
deconstruction for the parallelogram in the center. Determining the location of one of these deconstruction points
on the map as well as in the real world requires a mathematical 'construction'.
As the game proceeds, teams occupy territory with their
shapes and thus the free playing space gets smaller, which
provokes interaction between competing teams. Territory
occupied cannot be used anymore, unless the shape covering it is destroyed. Deconstruction of shapes clears space
and the deconstructing team 'steals' half the points connected to that shape from the team that created it. The
game ends after a set duration, the team with the highest
score being the winner.
The game is supported by a website on which each game
can be observed in real time online and on which the game
data are stored and can be viewed back later. These data
include the tracks of all teams, which are not visible on the
phones during game play, as well as the quadrilaterals that
remain at the end of the game.
What does playing MobileMath come down to in terms
of student actions? Students look at the map and imagine
where they want to make a shape. They walk to the location for the first vertex and then enter this location using the
phone. On the map, they see the location marked with a dot
in the color of the team. When they start walking again, to
the location of the second vertex of their imagined shape, a
line occurs on the screen connecting the first vertex with
the current (moving) team location. Once the team is at the
location they see fit as the second vertex of their shape,
they enter it and a second colored dot appears; the line
connecting the dots is now solid and is an edge of the
shape. The team now plans the location for the third vertex;
they use the map on their screen in combination with
properties of the real world. There are several constraints:
the type of shape they want to make determines whether
they will go for a straight angle or not; properties of the
terrain pose constraints as well: a vertex cannot be in water
or within a building because the students must enter the
vertex on the location itself. As soon as they start walking,
Fig. 1 Finding the proper
location of the fourth vertex
Fig. 2 View of a finished MobileMath game showing the created
quadrilaterals and the playing field
Fig. 3 The four deconstruction points of the parallelogram in the
center
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123two dotted lines connecting the two 'set' vertices with the
current location appear on the screen (see Fig. 1).
Note that it is not necessary to walk along an imaginary
edge; the connecting line will become the edge once the
location of the vertex is 'entered'; this edge can run

through buildings or over water. After entering the location
for the third vertex, the location of the last one is 'determined': students now must go to exactly that position that
finishes the shape in a correct way. Until now, there was
always the opportunity to change the strategy. For the last
vertex, accuracy is very important. It is in this phase that
we see students using the zoom options of the game:
zooming out to see their whole intended shape, zooming in
to accurately place the last vertex and, e.g., make sure the
angle is indeed a straight one. Students walk back and forth
to fine-tune the location of the last vertex.
5 Results
In this section, we present the results based on the analysis
of the observation data, the stored game data and the data
from the online questionnaire.
The first pilot with the teachers showed that the game
play proved to be engaging, that the technology worked
well and that the rules and goals were clear. Based on the
observations during game play, the group discussion and
the data of the questionnaire, it was decided that some
elements needed extra attention when introducing the game
to the students. These included: the mathematics involved
in finding the deconstruction points, the inaccuracy of the
GPS, the 'mechanics' of the grid and the use of the map,
which is not a conventional street map. As a result, the
educational embedding was designed in the form of
teaching suggestions for a whole-class introduction focusing on the mathematics, the geography, the actual handheld
phone (HTC)—provided by the researchers—and the game
play. Also, the online evaluation questionnaire for the
students was designed based on these pilot experiences.
We present the results of the pilots with students in five
sections: (1) engagement, (2) game play, (3) technical
issues (all related to sub-question 1), (4) mathematics: the
construction process, and (5) learning opportunities (the
latter two related to sub-question 2).
5.1 Engagement
Students easily engaged in MobileMath as was observed
during all of the four games that were played: no large
technical difficulties were observed and all teams played
for the full hour.
On the questionnaire, the game was rated as being 'fun'
(see Table 2). This is supported by data from the
observations made by the accompanying researchers.
Typically, students reacted very exuberant when they
completed or destroyed a shape. This was visible in their
moves as well as in their verbal outings: ''yes, we did it''.
In an open-ended follow-up question, students were
asked to motivate their 'fun score'. The reasons most frequently given are summarized in Table 3.
The most frequently mentioned positive reason refers to
the mathematical content of the game. A sample student's
response for this is: ''The game is fun because, you learn to
use GPS and you learn to construct different shapes:
quadrilaterals, rectangles, etcetera, and at the same time
you play a game''.
5.2 Game play
Most students easily understood the goal and the rules of
MobileMath. Most of the 30 teams constructed or deconstructed at least one correct quadrilateral and scored points.
There are large differences in scores, both between the four
games as well as between the teams within one game, as
can be seen in Table 4.
Table 4 shows that the students in school 1 who played
the game in the second round did better than the students
who played in the first round. The students who played
game 2 had watched and discussed the game being played
Table 2 Results on the question: 'Did you enjoy playing the game?'
1 (not at all) 2 (no) 3 (neutral) 4 (yes) 5 (yes a lot)
School 1 - - 2 10 16
School 2 - - 3 6 3
School 3 - - - 9 5
Total - - 5 25 24
Table 3 Results on the question 'Why or why not, did you enjoy
playing the game?'
Reasons Frequency
Positive
Constructing or deconstructing (digital) shapes 13
Just fun 10
Game play exciting (e.g. destroying shapes) 8
Scoring points 3
Use GPS 3
Learn something 3
Walking and being outside 3
Collaborate with peers 3
Other 3
Negative
Fun but: tiring/bad weather/guided/no points 13
No reason 5
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123by their classmates on the game website. Observations
revealed that the teams in this second round were more
aware of each others locations and actions.
Six out of the 30 teams that played MobileMath scored
no points at all, which means they did not succeed in
constructing or deconstructing any shape. In game 3, more
than half of the teams (4 out of 7) did not score any points.
During this game, the weather conditions were very bad:
storm and rain. An analysis of the tracks of these teams
reveals that two of them did not seem to attempt to create a
correct shape but instead wandered of to a dry and warm
place (the nearby shopping center), while the other two
teams tried but were not accurate enough.
5.3 Technical issues
None of the students had problems using the phone.
Observations showed that most students easily interacted
within the mixed reality environment, and were able to
combine the information from the map with the information from the physical world. One team, e.g., placed the
edge they just made on screen, parallel to the street to
decide whether to go left or right.
A small number of students, however, were seen to have
problems reading and interpreting the map on screen. They
had a hard time figuring out how the map matched with
their environment. This may be partly due to the fact that
the map on screen was not a conventional street map (see
Figs. 1, 2).
The technical features 'zooming' and 'checking the
scores' were used by most students (see Table 5).
Almost all students checked the scores frequently, which
helped them to make tactical and strategic decisions, like
destroying a large shape of the leading team to steal half of
their points and take over the lead. Students who used the
zoom function mentioned several reasons for doing so.
They indicated to use zoom in during the (de)construction
process to check the accuracy of their construction. Using
the zoom-in and walking back and forth in the physical
world is required to find the proper location for a vertex or
for a deconstruction point.
Zooming out was used to get an overview of the playing
field where the locations of all teams and the shapes that
had been created so far could be seen. It was also used by
teams to check their own location, and plan their course
and actions. The inaccuracy of the GPS readings, combined
with the margins and the grid, proved to be frustrating for
some of the teams.
5.4 Mathematics: the construction process
The stored game data, especially the tracks of all teams,
show that for the creation of shapes different strategies
were employed. A first strategy is based on the (virtual)
reality of the map: students deploying this strategy used
elements from the map—like rectangular street patterns
and buildings—to identify possible quadrilaterals. A second observed strategy can be characterized as 'using
knowledge of the environment around school'—students
who used this strategy knew, e.g., where a rectangular
playground was located and used this knowledge of the
physical world, to identify and construct a shape. This
strategy in which the virtual and physical reality merges
can also cause conflicts. This happened in a team of students that wanted to make a rectangle around the sports
field near school. The students walked around it (on the
sidewalk) and 'entered' the vertices on their handheld. But
three times over when they entered the last vertex, the
game gave as feedback that their shape was wrong. Only
back at school when reviewing their tracks on the computer, they noticed that the streets around the sports field
did not form a proper rectangle; two of the angles were not
90. During the game, they did not zoom out in the virtual
layer to check their assumptions that the streets in the
physical world around the field formed a proper rectangle.
The third observed strategy is a mathematical one.
Students using this strategy made their quadrilaterals as
'free standing' shapes. Two girls even leaped over a ditch
to overcome the physical barriers of the location to finish
their shape. The tracks of teams deploying this strategy are
often not related to the edges of their shapes. They indentify locations for their vertices and use their maps only
to plan a route to walk from one to the next. For these
students, the characteristics ofthe physical environment are
hardly important for their constructions, they mainly focus
on the mathematical properties of the shapes under
construction.
Table 4 Range and average scores per game
Game Score
range
Average
score
Remarks
School 1, game 1 0-140 36 140 is an outlier
School 1, game 2 20-201 93
School 2, game 3 0-155 25 4 of 7 teams scored 0
School 3, game 4 11-111 51
Table 5 Results on the questions 'did you zoom?' and 'did you
check the scores?'
Yes No
Did you zoom in or out? 44 10
Did you check the scores when playing? 50 4
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1235.5 Learning opportunities
The questionnaire presented the following three open
questions related to learning and school subjects:
• Did you learn something? Illustrate.
• Is this game related to geography? If so, in what way?
• Is this game related to mathematics? If so, in what way?
Most students (78%) confirmed that they had learned
something. In the open follow-up question, the majority of
them (36 out of 42) reported what it was they had learned.
Students could list more than one example. The results are
summarized in Table 6.
An example of learning occurring during reflection-inaction is the following: Two students (a team) were creating a rectangle: after they fixated the third vertex, they
noticed that they had not created a right angle. One of the
students wanted to delete this shape because 'in this way
we cannot finish the rectangle', the other student while
looking at the screen suddenly realized that another option
existed ''we can make it into a parallelogram if we put the
fourth vertex here instead of here (pointing at the map on
the screen).''
Learning opportunities could also be identified in the
introduction and debriefing sessions as is exemplified in the
following examples:
The geography teacher introduces the map used in
MobileMath by projecting it on screen. He tells the class
that this is a very simple map on which not everything can
be recognized.
T Can you see the railway-tracks?
S1 Yes, black and white
T And the water?
S2 You see two small ferry boats on the map
T Can you name something you recognize on this map?
S3 The police station (by the icon of a flame)

T Where can I walk?
S4 On this type of lines ['roads']
The mathematics teacher introduces the shapes that can
be made in MobileMath. He uses a flexible device made of
four strips to demonstrate the shapes.
T How can you be sure if a shape is a rectangle?
Students name several properties:
• the upper side is long
• the length and width are not equal
• it has four corners
T Who can tell more about the corners?
S1 The corners are the same … same
S2 … 90 degrees
S3 … straight
T How can we make a square?
S By making the length and width equal
T What shape do you think will be easier to make in the
game: a rectangle or a square?
S1 Square: you do not have to walk that far
S2 A rectangle, the length and width don't need to be
equal
T What is easier to make: a rectangle or a parallelogram?
S Rectangle
The large majority of the students recognized the relation to geography in the use of the map or the GPS. Eight
of the 54 students saw no relation to geography. Three
students saw no relation to mathematics. Most students
specify how they observed the relation between mathematics and MobileMath: 32 students used terminology
related to constructing shapes, 4 students connected this
with the calculating of the score; 6 students (also) used
other geometrical terms such as area, angles and measurements; calculating (in general) was mentioned five
times.
Other learning opportunities were identified in the
debriefing sessions. Students explained their strategies and
constructions, referring to their tracks on the map projected
in the classroom. For example, one girl (age 13) was asked
to describe and explain her team's subsequent activities
during the game in front of the classroom. The tracks of the
team are projected as a curly line on the map (see Fig. 4).
T Where did you place your vertices to make your
shapes? [T. points to the location of the school on the
lower left of the map on the interactive whiteboard]
S [Pointing to the tracks on the map] First we made this
small rectangle here. Then we made a block around the
church. Then we made a third rectangle and a fourth
one (see Fig. 5). At the end of the game three shapes
were destroyed by other teams
Reports such as the one above led to lively discussions
between students centered around the game strategies
Table 6 Examples of topics learned, for the question 'Did you learn
something? Clarify.' (N = 42)
Yes # mentioned
Using the GPS 14
Constructing shapes 13
Collaborating 5
Using a map 4
Strategic thinking 4
Being accurate 2
Playing the game 2
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123employed, the mathematical 'constructions' and shapes
each team made, and around the geographical aspects of
the game. Students were able to reflect on their actions:
they indicated, e.g., why in some cases their shapes under
construction proved to be incorrect, how they tried to
'hinder' other teams and how they determined the location
of a deconstruction point.
6 Conclusion
The research questions phrased in Sect. 2 focus on the
engagement provoked by a social mobile game and its
opportunities for learning mathematics. The questions are
investigated through design and through field tests with
teachers and students. We now address the two foci of
engagement and learning opportunities, respectively.
6.1 The game play engagement
6.1.1 MobileMath characteristics
What characteristics of the MobileMath game were
important for the students' engagement? We first conclude
that MobileMath indeed fulfills the characteristics of a
HRG with its intrinsic motivational properties as defined
by De Souza e Silva & Delacruz (2006). It allows players
to create an imaginary layer on top of the physical reality
by drawing shapes. MobileMath adds a geometrical
dimension to the world which is transformed into a game
board. Observations show that for the players the physical
and digital space merge, which contributes to students'
engagement. Interaction between the teams mainly took
place in the virtual reality: they spotted each other more
easily on screen than in the physical reality. This matches
with similar results found by others Squire and Klopfer
(2007), who conclude that students 'within minutes were
diving into this mixed-reality environment' (p. 403).
A prerequisite for engagement is appropriate technical
functioning. We conclude that the technical environment of
MobileMath is user-friendly, robust and flexible enough to
be used in an actual (educational) playing setting. Students
easily used the functionality of the phones. However, the
inaccuracy of the GPS readings sometimes interfered with
and frustrated the game play. For example, what looked like
a parallelogram on screen was identified by the game as
being arectangle, or a seemingly correctly placed vertex was
evaluated to be wrong. This is in line with Schwabe & Go¨th
(2005) who discussed this type ofeffects of(in)accuracy and
concluded that for tasks that require very specific locations
the inaccuracy can cause frustration and loss ofengagement.
Other, more accurate location technologies may be preferred
in that case. For MobileMath, the margins and the grid size
built into the software need to be reconsidered.
Fig. 4 Tracks of one team after they finished the game saved on the
computer
Fig. 5 Shapes created by one team, pasted manually in the tracks to clarify the protocol
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123 6.1.2 Fun and engagement
Main factors that showed to be essential in enhancing
students' engagement are clear goals, objectives, and rules,
competition and interaction, which is in agreement with the
elements identified by Prensky (2001). In MobileMath, the
goal and objectives proved to be clear. Furthermore, all
teams kept setting new goals for themselves, during the full
hour. Although they were moving freely outside of school,
most students were seen to be highly motivated and
engaged during game play, almost no off-task behavior was
observed. We can also identify the other elements: MobileMath has clear rules that structure the game, and students had no problems understanding and following these
rules. The feedback in this game is immediate: every
action, its outcome and its consequences are immediately
visible on the map on the phone. Without this immediate
feedback, the game would be unplayable.
Competition provided an important contribution to the
engagement with MobileMath. There are two elements in
the game that can be described as competitive. First, the
game in itself as a whole is competitive: students are all
competing to cover as much area as possible on a restricted
playing field, and gain the highest number of points. Second, the option of destroying shapes of others brings in an
extra competitive element. This feature proved to be very
engaging: eight students explicitly mentioned it in the
questionnaire as the main reason for the game being fun.
Also during debriefing sessions team specifically and in
much detail reported their deconstruction actions and successes. Some teams during the game even specialized in
destroying instead of creating shapes. MobileMath offers
ample opportunities for interaction. Because there was
only one phone for two players in one team, the players had
to work together. In most teams this turned out to be
positive: often the two students were seen bent over the
screen together, discussing and planning their actions. In a
small number of teams, the student carrying the phone did
all the work and the other students just walked along.
Because all students were eager to play, most students did
not let this happen.
Altogether, we conclude that with the design of MobileMath we succeeded in creating an engaging game
activity. MobileMath was showed to be an accessible HRG
with engaging characteristics such as clear goals, objectives and rules, as well as competitive and interactive
elements.
6.2 Learning opportunities for mathematics
The design of MobileMath shows that mathematics can
successfully be integrated into an engaging mobile location-aware game. Almost all students recognized the
mathematics in the game and described it in mathematical
terms. The mathematical concepts build into MobileMath
(orientation and navigation, measurement, properties and
especially 'constructions' of specific quadrilaterals) were
already partly known by the students (see Sect. 5.5).
However, in the regular curriculum, creating shapes does
not receive much attention and the bodily experience of
making shapes—as is done in MobileMath—is fully new.
MobileMath invites several types of mathematical
activity, such as the (re)discovery and use of characteristics
of squares, rectangles and parallelograms, the notice of
geometrical aspects of the world, and to combine mathematical skills.
Students (re)discovered and used characteristics of
squares, rectangles and parallelograms when they
(de)constructed these shapes. In doing so, they went back
and forth between the realities of the map and the real
world. Students needed to imagine a shape before actually
creating it. They had to mentally 'see' it on the map, and
then walk to the locations of the vertices in the real world.
Students' perception of the city changed like in HRGs and
other augmented reality games. They noticed and discussed
geometrical aspects of the world, such as if streets make
right angles or not and whether they are running parallel;
they identified blocks, buildings or other structures as
having a rectangular 'top view' on the map. Combining
math skills was needed when a team studied the map and
decided on a strategy related to their position. They had to
consider aspects such as time and distance: which distances
can be covered walking? How much time is left before the
game ends? Where will a correct shape fit, can each vertex
be reached? How long will the construction process take?
Students were also observed using mathematical skills and
reasoning when reflecting on the scores, which also
involves reasoning about area.
Observations also show that during game play students
went through micro-cycles of learning and reflection in a
natural way. For each action, they needed to reflect on the
game status, which involved taking note of the playing
field and of the characteristics of the shape they were
creating in relation to the physical aspects of the terrain.
They discussed, e.g., where to place a vertex to make sure
that the angles and the length of the edge were suited to
create the intended shape. Apart from the names of the
quadrilaterals, they frequently used mathematical terms
such as parallel, (straight) angle, straight line, perpendicular, of equal length. Because they were 'acting out' these
concepts in their movements and constructions, it is likely
that their understanding has deepened and has become
experientially real.
For the further exploitation of the mathematical aspects
of the game play, debriefing sessions are important.
Although in the pilot these sessions were very short, we
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123could glimpse the opportunities these sessions have for
reflecting on the game play as well as on the math involved
and thus for promoting learning. Some work is still needed
to fully exploit this potential of MobileMath for learning.
Although, as stated before, students reported they
learned something, were able to name aspects of the

mathematics involved in the game and were seen to engage
in mathematical activities and talk, we cannot yet support
the claim that MobileMath has had a learning effect. The
fact that in MobileMath students experience mathematics
to be engaging may in itself be seen as a positive result. In
future research on MobileMath, we hope to be able to
measure learning effects, and thus show that MobileMath
can contribute to students' learning of mathematics, just as
some other digital games have proven to do (Kebritchi
et al. 2008; Rosas et al. 2003; Shin et al. 2006).
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Comment: *Examples of RME activities*

33 quotations

Codes (18)###didactical phenomenology, ###guided reinvention, #characteristics of RME, ##human activity, ##mathematization, ##real life, #RME, #RME theory, $$$approaches to teaching Mathematics, $$$modelling, $$teaching methods and strategies, $factors that influence teaching and learning, $mathematics, $mathematics content, %role players, &&integration across strands, @@value of using ICT, @Devices

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Realistic Mathematics Education Theory Meets Web Technology

Abstract
This paper about realistic mathematics education (RME) and Web technology towards
mathematics reform in Indonesia. First, reform in mathematics education is briefly
described. Second, three main components that influence the reform are discussed:
new goals, new content and new theory. Furthermore, the Dutch theory for realistic
mathematics education and its characteristics is elaborated. Finally, the ongoing study
in developing a Web-based system called CASCADE-IMEI that aims to support student
teachers to learn about RME as an innovation in mathematics education in Indonesia is
discussed.

I. INTRODUCTION

Mathematics education is changing continuously. It can be seen by a movement from traditional curricula to problem oriented curricula or application based curricula (de Lange et al. 1993). Such change or reform in mathematics education can also be seen as a shift away from the transmission of knowledge by teachers towards investigation, construction and discourse (Gravemeijer, 1997). This reform is mostly influenced by changes in ideas on goals, content and the theory for learning and teaching of mathematics.

Concerning the goals, there is a growing emphasis on the usefulness of mathematics in daily practice. This trend is fostered by societal changes. 'The coming information society poses new demands to the citizens as 'mathematical literacy'. These societal changes also have their influence on the content of mathematics education in terms of reconsideration of what it means to know and do mathematics. Mathematics is not seen as a ready-made product anymore but it is stressed on the process of doing mathematics. The notion of mastery rules and procedures of mathematics is being exchanged for the idea that students should have a deep understanding of their mathematics and should be able to explain and justify it. Finally after twenty five years of developmental research, a theory for learning and teaching of mathematics education called realistic mathematics education (RME) is evolved. RME is strongly related to the constructivist theory (Freudenthal, 1991; Gravemeijer, 1994;de Lange, 1993).

II. MATHEMATICAL LITERACY: THE NEW GOAL

Mathematics literacy is a new goal that is embedded in the new mathematics curriculum in many countries. The 'traditional goals' often only has two main elements: to prepare for the work-place and for future education; and to understand mathematics as a discipline. A new set of goals were prepared by the Commission of Standards for School Mathematics of the National Council of Teacher Mathematics. It lists ten goals, five the content goals - Number and Operations, Algebra, Geometry, Measurement, and Data Analysis and Probability - explicitly describe the content that students should learn and five the process goals - Problem Solving, Reasoning and Proof, Communication, Connections, and Representation - highlight ways of acquiring and using content knowledge (NCTM, 1989).

The mathematics literacy is further articulated by the NCTM standards by proposing five general goals for students.

• learning to value mathematics. Understanding its evolution and its role in society and the sciences.

• Becoming confident of ones own ability. Coming to trust one's own mathematical thinking and having the ability to make sense of situations and solved problems.

• Becoming a mathematical problem solver. This is essential to becoming a productive citizen and requires experience in solving a variety of extended and non-routine problems.

• Learning to communicate mathematically. Learning the signs, symbols and terms of mathematics.

• Learning to reason mathematically. Making conjectures, gathering evidence and building mathematical arguments.

These goals reflect a shift away from traditional practice; traditional skills are subsumed under more general goals for problem-solving, communication and critical attitude. III. REALISTIC MATHEMATICS EDUCATION: THE NEW THEORY

3.1 Realistic Mathematics Education

RME is a teaching and learning theory in mathematics education that was first introduced and developed by the Freudenthal Institute in the Netherlands since almost three decades. RME is mostly determined by Freudenthal's view on mathematics (Freudenthal, 1991). Two of his important points of view are: Mathematics must be connected to reality; and mathematics should be seen as human activity. First, mathematics must be close to children and be relevant to every day life situations. However, the word 'realistic', refers not just to the connection with the real-world, but also to problem situations which are real in students' mind. Second, the idea of mathematics as a human activity is stressed. Mathematics education organized as a process of guided reinvention, where students can experience a similar process compared to the proces by which mathematics was invented. In this case, the reinvention process uses concepts of mathematization as a guide.

Two types of mathematization, which were formulated explicitly in an educational context by Treffers (1991), are horizontal and vertical mathematization. In horizontal mathematization, the students come up with mathematical tools which can help to organize and solve a problem located in a real-life situation. On the other hand, vertical mathematization is the process of reorganization within the mathematical system itself. Freudenthal (1991) pointed out that horizontal mathematization involves going from the world of life into the world of symbols, while vertical mathematization means moving within the world of symbols. But he adds that the difference between these two types is not always clear cut.

Treffers (1991) classifies mathematics education into four types with regard to horizontal and vertical mathematization:

(1) Mechanistic, or 'traditional approach', is based on drill-practice and patterns, which treat the person like a computer or a machine (mechanic). In this approach, both horizontal and vertical mathematization are less or even not used.

(2) Empiristic approach, the world is a reality, in which students are provided with materials from their living world. This means that students are faced with situations in which they have to do horizontal mathematization activities. However, they are not prompted to the extended situation in order to come up with a formula or a model.

(3) Structuralist, or 'New Math approach' that is based on set theory, flowchart and games that are kinds of horizontal mathematization but they are stated from an 'ad hoc' created world, which had nothing in common with the learner's living world.

(4) Realistic approach, a real-world situation or a context problem is taken as the starting point of learning mathematics. And then it is explored by horizontal mathematization activities. This means students organize the problem, try to identify the mathematical

aspects of the problem, and discover regularities and relations. Then, by using vertical
mathematization students develop mathematical concepts.
3.2 The characteristics of RME
Historically, the characteristics of RME are related to the Van Hiele's levels of learning
mathematics. According to Van Hiele (cited in de Lange, 1996a) the process of learning
proceeds through three levels: (1) real; (2) analysis; and (3) formal.
Traditional instruction is inclined to start at the second or third level, while the realistic
approach starts from the first level. Then, in order to start at the first level; that deals with the
phenomenon that is familiar to the students, Freudenthal's didactical phenomenology that
learning should start from a contextual problem, is used. Furthermore, by guided reinvention
and progressive mathematizations (Treffer, 1991), students are guided didactically to process
as efficiently from one level to another level of thinking through mathematization.
Combining the three Van Hiele's levels, Freudenthal's didactical phenomenology and Treffers'
progressive mathematization results in the following five tenets or characteristics of RME
(Treffers, 1991):
(1) phenomenological exploration or the use of contexts;
(2) the use of models or bridging by vertical instruments;
(3) the use of students own productions and constructions or students contribution;
(4) the interactive character of the teaching process or interactivity; and
(5) the intertwining of various learning strands.
In the next sections these tenets will be elaborated. (1) Phenomenological exploration or the use of contexts
In RME, the starting point of instructional experiences should be `real' to the students;
allowing them to immediately become engaged in the situation. This means that instruction
should not start with the formal system. The phenomena by which the concepts appear in
reality should be the source of concept formation. The process of extracting the appropriate
concept from a concrete situation is stated by De Lange (1987) as 'conceptual
mathematization'. This process will force the students to explore the situation, find and
identify the relevant mathematics, schematize, and visualize to discover regularities, and
develop a 'model' resulting in a mathematical concept. By reflecting and generalizing the
students will develop a more complete concept. Then, the students can and will apply
mathematical concepts to new areas of the real world and by doing so, reinforce and
strengthen the concept. This process is called applied mathematization. The two examples of
context problems, t-shirt and soda and parent's night, are presented in the next section.
(2) The use of models or bridging by vertical instruments
The term model refers to situation models and mathematical models that are developed by
the students themselves. This means that the students develop models in solving problems.
Four levels of models in designing RME lessons can be distinguished (Gravemeijer, 1994):
(1) the situational level, where domain-specific, situational knowledge and strategies are
used within the context of the situation;
(2) a referential level or the level 'model of', where models and strategies refer to the
situation described in the problem;
(3) a general level or the level 'model for', where a mathematical focus on strategies
dominates over the reference to the context; and
(4) the level of formal mathematics, where one works with conventional procedures and
notations.
(3) The use of students own productions and constructions
Students should be asked to 'produce' more concrete things. De Lange(1987) and Streefland
(1991) stress the fact that, by making 'free production', students are forced to reflect on the
path they themselves have taken in their learning process and, at the same time, to anticipate
its continuation. Free productions can form an essential part of assessment. For example,
students may be asked to write an essay, to do an experiment, to collect data and draw
conclusions, to design exercises that can be used in a test, or to design a test for other
students in the classroom.
(4) The interactive character of the teaching process or interactivity
Interaction between students and between students and teachers is an essential part in RME
(de Lange, 1996a; Gravenmeijer, 1994). Explicit negotiation, intervention, discussion,
cooperation, and evaluation are essential elements in a constructive learning process in
which the informal methods of students are used as a lever to attain the formal ones. In this
interactive instruction students are engaged in explaining, justifying, agreeing and
disagreeing, questioning alternatives and reflecting.
(5) The intertwining of various learning strands or units
In RME (de Lange, 1996a; Gravenmeijer, 1994), the integration of mathematical strands or
units is essential. It is often called the holistic approach, which incorporates applications,
implies that learning strands can not be dealt with as separate entities; instead, an
intertwining of learning strands is exploited in problem solving. One of the reasons is that
applying mathematics is very difficult if mathematics is taught 'vertically', that is if various
subjects are taught separately, neglecting the cross-connections. In applications one usually
needs more than algebra alone or geometry alone.
For additional information about these five tenets relating to the designing lessons based on
the realistic approach see Zulkardi (1999a). IV. MEANINGFUL CONTEXT: THE NEW CONTENT
As mention earlier that the first tenet of RME is the use of meaningful context as the starting
point of instruction. The following is an example of meaningful context for concept
development that was designed for students of ages 12. This problem is very powerful due to
the its position in the curriculum and the connections to earlier as well as later activities. It is
presented in a visual way (de Lange, 1996a, 1996b):
This problem I used as an assessment problem in a workshop that followed by junior high
school teachers in Bandung. Before the problem was stated the participants were told to use
the informal method. However, it appeared that most of them (see A. Syukur's solution) solve
the problem using the formal method of linear algebra.
However, one of the teacher (see Herlina's solution) solved the problem using the informal
method.
Students of ages 12 at the school come with very ingenious and different solutions. They are
not familiar with the algebraic nature of the problem and are not hindered by this algebraic
knowledge that makes the problem so difficult to teachers. Instead of using 2a + 2b = 44 and
a + 3b = 30, they solve the problem just using common sense reasoning. Students operate
differently: These two solutions are very different in nature: the first one notices that if two t-shirts and
two cups are 44, one plus one must equal to 22. They take away one t-shirt and one cup from
the second picture which leaves two cups for 8 dollars. Done. The second solution uses
regularity as the starting point. The first picture shows 2 + 2, the second 1 + 3, so the 'third'
picture must show 0 + 4. The price belonging with the third picture must be 16, fitting in the
sequence; 44, 30, 16. The student shows good insight in where the essential part of the
reasoning takes place: after noting that 2,2 followed 1,3 leads to 0,4 he writes down: almost
ready. Exactly.
Later on students will handle in a formal way systems of equations. At that time they know
informally how to reason algebraically, to understand equations, to use the proper
mathematical notation and even to understand the concept of variable. But in this case it will
take about two years to reach that level of real mathematical understanding (de Lange,
1996b).
The second example, think about a classroom with students of 8 to 9 years of age. The
teacher introduces the 'parent night' problem to his students (see also Gravemeijer, 1994; De
Lange, 1996b):
Tonight 81 parents will visit our school. Six parents can be seated at each table
How many tables do we need?
The teacher makes a small sketch on the blackboard:
The students start to work on the problem in groups 3 and 4. The teacher walks around in the
classroom, asking small questions about the process of solving the problem. The students
were eager to engage in this process. After about ten minutes the teacher ends this part of
the lesson. Students are asked to show and explain their solution. They vary quite a bit.
Austin just copied as many as of the teacher's sketch as he needed to seat the parents: Another student, Ani, started out the same way, but after drawing two tables he shifted to
more schematic representation: a rectangle with number 6 on it. After drawing two of those
'tables' he realized that five tables would add up to 30. So, via 30 to 60 and then on to 72
and 78. And finally he added the last three chairs.
A third student, Alhariz, went a step further in mathematizing the problem. Although he also
started to draw the table from the blackboard as a model he immediately schematized the
problem and used his very recent knowledge of multiplication, by using multiplies of 6. He
wrote down: 6 x 6 = 36, then doubled the 36 to 72, and added another two tables to get a
capacity of 84.
If we look from distance at these three different solutions (and of course there were many
more) we notice a different level using 'real' mathematics in this 'real-world' problem. Many
teachers would even argue that the first solution no mathematics has been used at all. But
visualizing and schematizing are also important mathematizing tools that can be very
powerful. The third solution makes the mathematics more visible and will be considered to
have a 'higher' level.
After the whole classroom discussion without making too many explicit recommendations of
wich solution is the best, the teacher continued by presenting the next problem:
The 81 parents will be served coffee, of course. Each pot holds 7 cups of coffee. How many
pots wee need?
From a mathematical point of view exactly the same problem. Instead 81:6 we now have to
deal with 81:7. Not so for the students. In the first place the context prohibits an easy
pictorial solution: the students have a hard time drawing the coffee pots as Alhariz' s solution
shows:
Alhariz, who used the most formal solution in the table problem now goes completely to
mental arithmetic with visual support. Austin, who ad the most simple solution with the
tables, tries to use the schema again with the pots. He represents them in the same way: a
series of cups around the pot. But after two pots he seems to realize the discussion about multiplication as a means to
speed things up and he jumps to: 10 x 7 = 70 and adds: 70 + 11 = 81, which give him 12 pots.
The work of yet another student, Yan, shows a typical solution for this second round of the
parents problem:
No visualization, but multiplication. Within one lesson one can easily see the progress that the
students have made towards solving this class of real contextual problems and
mathematizing in order to develop new mathematical concepts, in this case division.
V. CASCADE-IMEI: A STUDY IN THE FIELD OF REFORM IN MATHEMATICS
EDUCATION IN INDONESIA
5.1 Mathematics Education in Indonesia need to be reformed
Based on the literature review in the preliminary study of CASCADE-MEI (see also Manan,
1998; Zulkardi, 1999b) mathematics education especially in the secondary level need to be
reformed. Nowadays, mathematics education in Indonesia is categorized into the
mechanistic-structuralist approach. While many countries already change their approach from
mechanistic-structuralistic to the new approach, realistic approach, it is clear that in Indonesia
mathematics education need to be changed as well.
As a consequence of reform, alternative curricula, alternative textbooks and teaching
materials, and alternative forms of assessment (de Lange, 1993; Van den Heuvell-Panhuizen,
1996; Fullan, 1991) are essential. Selter (1997) pointed out that these are all mediated
through the teacher, specifically through teacher's beliefs about how to organize and facilitate
children's learning of mathematics. Consequently, teachers are a key players in educational
change. In this context, teacher training (pre service as well as in service) plays a vital role.
Reform efforts will seriously be hampered, if reform in teacher training is not regarded as
being as urgent as reform at schools. This implies that reform can be started from teacher
training.
As Gravemeijer (1997) pointed out, it is not easy to support teachers in order to make our
common reform efforts successful. Teacher training should not primarily be an apprenticeship
training that provides recipes and methods that are directly applicable in the classroom, but it
should first and foremost assist prospective teachers in developing their autonomy. This
implies to support them in increasing their degree of awareness: about mathematics, about
children's mathematical learning, about the quality of teaching mathematics, and so forth
(Selter, 1997). This means that a learning environment is needed that contains supports with
which student teachers can enrich their knowledge about content as well as didactic
mathematics.
5.2 CASCADE-IMEI: Theory meets technology
Information and communication technology (ICT) tools currently permeate almost every
professional domain. Web technology is starting to get used in education as well. Teachers
can use this technology to store information such as lessons or problems, then can access
these materials from many places in the world, and discuss and share problems and
experiences after using the materials in the classroom.
CASCADE-IMEI stands for Computer Assisted Curriculum Analysis, Design and Evaluation
for Innovation in Mathematics Education in Indonesia. The main purpose of this study is to
develop a web-based system which provides resources, training opportunities and
communication tools for student teachers in Indonesia, learning RME as an innovation in mathematics education. This study builds on the previous studies about computer support
system (see Nieveen, 1997) and number of studies about realistic mathematics education
(de Lange , 1987; Gravemeijer, 1994, Streefland, 1991; Van den Heuvel-Panhuizen, 1996).
The study is guided by the following main research question:
"what role can a Web-based system play in supporting student teachers learn RME as an
innovation in mathematics education in Indonesia?"
CASCADE-MEI, a rich learning environment, will be developed based on the philosophy of
RME theory which is represented in a number of support tools such as infobase (various
topics of RME lessons, teacher guides, assessment problems and student productions),
communication tools (communication and discussion using e-mail and mailing list) and
training opportunity (a course about the introduction to the RME).
Now, this is end of the second year of a four year study of CASCADE-IMEI. A number of
mathematics lessons have been developed and researched as well as the course. These
tentative result can be seen in the Dikti homepage
(http://www.dikti.org) in the 'free space' part (see figure the Website below).
VI. CONCLUSION
In this paper the new perspective on the goals, the content and the theory for teaching and
learning of mathematics education have been discussed and illustrated. These are combined
on the ongoing study CASCADE-IMEI that uses Web technology as a tool in order to support
mathematics student teachers in teaching mathematics to the students in the school.
If the Netherlands is taken as an example of a successful country in reforming the
mathematics education in the level primary and secondary school then the following
considerations should be taken in to account:
• mathematical literacy should be added to the curriculum;
• the content of curriculum should be changed into the application based curriculum;
• mechanistic-structuralistic approach should be moved to the realistic approach; and
• teacher training (pre service and in service) should change as well.
CASCADE-IMEI is focusing on these considerations by providing a rich learning environment
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Chapter 4
Using multimedia cases for educating the primary
school mathematics teacher educator:
a design study
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Abstract
The overarching goal of this chapter is to better understand how multimedia video case
studies can support the professionalization ofprimary-school-mathematics teacher educators.
We investigate the use of multimedia cases to support teacher educators in learning to
mathematize and didactize and to learn how to use multimedia cases with their student
teachers. The research study has an exploratory character; we will present a framework for the
use of multimedia cases as a tentative answer, grounded in the researchers' experiments and
design activities.
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Keywords: Mathematics education; Primary schools; Teacher education; Multimedia cases
Over the last decades, the so-called 'domain-specific instruction theory for
realistic mathematics education (RME)' has been developed and gradually
implemented in the Netherlands. The RME approach capitalizes on students'
input. Following Freudenthal's (1991) adage of "mathematics as a human activity',
students are encouraged to reinvent mathematics. It will be clear that this puts
high demands upon teachers. Amongst other things, they must be able to give
students construction space, provide them with open investigation activities,
allow them to share and discuss their work and particularly, consider relationships between different approaches. This in turn puts high demands on teacher
educators. For instance, the teacher educators have to help student teachers develop
a constructivist approach to teaching. They have to integrate mathematics,
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PII: S 0 8 8 3 - 0 3 5 5 (0 2) 0 0 0 5 8 - 7didactics,
1
and pedagogy, and do it in such a way that student teachers are using
their knowledge in real practice. But, how does one prepare primary-schoolmathematics teacher educators for this job?
In the Netherlands, a 2-year long 16-day in-service course exists, intended to give
beginning teacher educators a jump-start. This course has been offered and revised
many times over the last 10 years. This situation proved to offer a fruitful setting for
experimenting with the use of multimedia cases in teacher education, which is the
theme of our contribution.
In this chapter, we report on what we have learnt about the use of multimedia
cases in the course of developing this in-service course for beginning mathematics
educators. In doing so, we want to address the question of: How to employ
multimedia cases in teacher education. More specifically, our research question is:
How to employ multimedia cases to support teacher educators in learning to
mathematize and didactize and to learn how to use multimedia cases with their
student teachers? Note that our research has an explorative character; we will present
a framework for the use ofmultimedia cases as a tentative answer that is grounded in
the researchers' experiments and design activities. The justification ofthis framework
is grounded in empirically based conjectures about how participants learn and how
the suggested educational investigations or multimedia cases support this learning
process. In this respect, the design-research approach fits with the general shift in the
norms of justification observed by the Research Advisory Committee (RAC) of the
NCTM (1996). Compared to traditional research, they argue, norms of justification
have shifted from assembling facts that prove one curriculum better than another, to
providing well-substantiated theories about how these curricula work. With those
new norms of justification, they go on to say, teachers (teacher educators) have some
informative theories that they can take as a point of departure. They can take the
frameworks and theories of the researchers as conjectures, to test and modify in their
own classroom. In this manner, the teachers/teacher educators can produce their
own contribution to the development of those theories, instead of being passive
consumers of knowledge produced by others. In short, we may say that the research
we present here can be seen as a case study—on the use of multimedia cases—that is
meant to function in a similar manner as the multimedia cases themselves.
The chapter is organized in the following manner. We start by framing the issue of
the use of multimedia case studies in teacher education. Next we move to our own
research by characterizing the teacher education course in which the use of
multimedia cases is embedded. Then we describe the multimedia computer
environment that is used. This will be followed by an exposition on the research
method that can be typecast as 'design research' or 'developmental research'. The
backbone of this type of research is constituted by the learning process of the
researchers, which is described subsequently. And, we conclude the chapter with a
short reflection on our findings.
1
In this paper, didactics refers to the continental European notion of didactics: The interrelation of
teaching and learning activities and processes aimed at the construction ofan understanding ofthe subject
matter by students.
 The use of multimedia case studies in the education of teacher educators
In traditional classroom observations each student typically observes a different
classroom, and the teacher educator rarely visits any of the classrooms.
Consequently what students notice varies considerably. Even if several students
observe the same classroom each student tells a different story. This leaves teacher
educators wondering whether the students really saw the same lesson. A problem is
that live classroom observations allow neither reviewing nor thinking time,
advantages that video and written cases do offer (Dolk, Faes, Goffree, Hermsen,
& Oonk, 1996; Mousley, Sullivan, & Mousley, 1996; Lampert & Ball, 1998). Having
the lesson present 'on demand' allows students and teacher educators to share
observations and interpretations and to revisit and review the lesson several times.
As anyone who has studied videotapes of lessons knows, each viewing and each
viewer bring added dimensions to what is observed. These two alternatives for
observing and studying classroom teaching differ in at least one respect: video allows
the learner a more objective view of the classroom; written cases force the learner to
look through the lens of the describer. Compared to text-based cases, real video is
closer to experiencing the act of teaching: it is multi-faceted, and allows for different
interpretations, reflections and analyses (Lin, 2001). However, we have to keep in
mind that video too has its limitations. Video, for instance, does not reflect the
ability of an observer to be aware of several groups while attending to what one
learner is saying. Case studies have been employed since the beginning of the 19th
century, first in Law Schools, later in Business Schools. Using case studies in Schools
of Education is more recent. Text-based case studies are documents describing
exemplary real life situations. The idea is that students learn to reason about the
situation, learn to generalize their thinking, developed for one case, to form the basis
for thinking about many cases. The situation becomes a tool to think with.
An educational case study can be designed so that students can easily investigate
the information, can engage in conversation about the situation. According to
Merseth (1996), to be effective, case studies need to be situated in a real world
context, rely on research, and foster the development ofmultiple perspectives. In our
perspective, case studies must allow learners to explore the richness and complexity
of genuine teaching problems because they are situated in the real classroom
(Putnam & Borko, 2000). However, in text-based cases authors often feel compelled
to reduce the complexity of the real situation, failing to portray the full complexity,
and nuances of real teaching situations. Such cases are often fine-tuned towards the
mathematical and didactical knowledge under consideration; real education is not.
How complex real-life teaching situations can be is shown by Lampert (2001), in the
opening chapter of her book Teaching problems and the problems of teaching where
she describes how an interval of a couple of minutes during one of her lessons asked
for many considerations, reflection and decisions. In general, real teaching is messy
(Duckworth, 1987). Linear video provides more information about classroom
events. For instance, it captures some facial and verbal expressions, use of time,
involvement of the learners, and the flow of the lesson. Nevertheless, it remains
unifocal, even where there is cutting between cameras. The availability of fast
computers and multimedia software expands the possible uses of video and cases in
education. Major technological issues with video, like reviewing footage and
searching and random accessing footage in video banks, have become incidentals.
Multimedia cases provide easy access to multiple sources of information: teachers'
planning, students' work, textbooks, and background reading. This development has
made written cases obsolete. Naturally, during the last decade, the development and
use of multimedia cases in teacher education has boomed.
The use of multimedia cases creates opportunities for teacher educators to have
users construct situated mathematical and didactical knowledge. However, we are
only learning to understand how multimedia cases can be used in teacher education
and how it influences a user's construction of theoretical and practical knowledge
(Galen van, Dolk, Feijs, Jonker, & Ruesink, 1993; Dolk et al., 1996; Mousley et al.,
1996; Lampert & Ball, 1998; Lin, 2001). The question of, how to employ multimedia
case studies in teacher education, is still open.
2. The mathematics-educators course
Our research question is addressed in design experiments on the use ofmultimedia
cases that were part of a larger design effort aiming at the development of a course
on the professionalization of beginning mathematics teacher educators. We will
therefore first describe this course.
The key tenets of the course can be summarized as ''mathematizing'' and
''didactizing''. Following Freudenthal (1991), we use the verbs mathematizing and
didactizing to emphasize the process rather than the product. By using a gerund
verb form, we express a dynamically emergent view of mathematics and didactics:
one more closely akin to the process of constructing meaning. The idea of
mathematizing and didactizing implies that mathematics and didactics can and
should be learned through one's own mental activities. Learners are finding ways to
explore situations mathematically (didactically), they are noticing and exploring
relationships, putting forth explanations and conjectures, and trying to convince one

another oftheir thinking. Our participants construct didactical knowledge. Didactics need not be a ''dead subject'', developed by others. Just as with mathematizing, they can explore situations didactically, notice relationships, put forward conjectures and explanations, and try to convince one another of their thinking. They can construct didactical meaning.

2.1. RME theory

One of the central tenets of RME is that the starting points of instructional sequences should be experientially real to students in the sense that they can immediately engage in personally meaningful mathematical activity (Gravemeijer, Cobb, Bowers, & Whitenack, 2000; Streefland, 1991).

A second tenet of RME is that in addition to taking account of students' current mathematical understandings, the starting points should also be justifiable in terms

of the potential mathematical end points of the learning sequence. This implies that students' initially informal mathematical activity should constitute a basis from which they can abstract and develop increasingly sophisticated mathematical concepts as they participate in classroom mathematical activities. At the same time, the situations that serve as starting points should continue to function as paradigm cases that involve rich imagery and thus anchor students' increasingly abstract mathematical activity.

These central tenets are worked out in a domain-specific instruction theory for Realistic Mathematics Education that is grounded in numerous concrete elaborations of the RME approach. This domain-specific instruction theory can be typified by three design heuristics (Gravemeijer, 1998): guided reinvention through progressive mathematization, didactical phenomenological analysis, and emergent modeling. This theory, however, cannot be transmitted to (student) teachers in a ready-made form.

Instead, we have to create a basis for discussions on RME and on how to guide the reinvention processes of the students, by having the student teachers experience reinvention and mathematizing for themselves. The same holds for teacher educators. In addition to, experiencing this process, the course participants (teacher educators) are presented with activities like anticipating solutions of primary school students, and analyzing student work and multimedia cases.

2.2. Didactizing

In teacher education, Goffree and Dolk (1995) have identified reflection, narration and construction as being particularly important. Reflection is seen as a precondition for learning, especially for learning from one's own activities. Therefore, it is important to stimulate beginning teachers to reflect on their classroom experiences and actions. We have, however, to take into account, that the students' a priori experiences and beliefs will influence what they learn of what is taught in teacher education courses. Furthermore, they construct their own understanding of educational theory and build their own practical knowledge. Often, educational theory presented to pre-service teachers is not used in practice (Wubbels, 1992; Wideen, Mayer-Smith, & Moon, 1998). Beginning teachers find it difficult to connect the theories they studied during pre-service teacher education to their images of a teacher, to their feelings of fear, disappointment, etc., and to values embedded in everyday practice in schools, values they quickly internalize. They also tend to consider theory oflittle worth in the light oftheir personal concerns about survival in the classroom.

The problem of connecting theory to practice can be analyzed from two perspectives. One perspective is that of the difference between the knowledge scientific theories offer and the kind of knowledge teachers need. The other perspective is that of the problematic character of the use of knowledge in immediate teacher behavior. We will briefly discuss both perspectives since they constitute key elements of the rationale of the experimental course, and of the use of multimedia cases in this course.

2.3. Scientific knowledge versus practical wisdom

Korthagen and Kessels (1999) analyze what they call the gap between theory and practice, and describe an approach to teacher education that might help to link theory and practice. In their article, they go back to Aristotle's concepts of episteme and phronesis to explain the difference. They use the distinction between episteme, which they translate with ''scientific knowledge'', and phronesis, which they translate with ''practical wisdom'', to make a distinction between Theory-with-a-capital-T, and theory-with-a-small-t (Korthagen & Kessels, 1999). Episteme aims at more general knowledge about types of situations, while the emphasis of phronesis is mostly on perceiving more in a particular situation and finding a helpful course of action on the basis of strengthened awareness. This strengthened awareness of concrete characteristics in specific situations helps student teachers to discover, in the specific situations occurring in everyday teaching, which methods and strategies to use.

We consider educational theory in literature as formal knowledge, as Theory-witha-capital-T. This theory is based on formal research, applicable to a wide variety of situations, and can be characterized as ''objectively reasonable belief''. We consider theory developed by learners as theory-with-a-small-t. This theory is often still situation-specific, and related to the context in which it was developed. This theory is formed on the basis of generalization across a number of situations. It is more perceptual than conceptual. It focuses the attention of the actor in the situation on certain characteristics of the situation, characteristics important to the question of how to act in the situation.

Narrative knowledge can be starting point to theorize theory-with-a-small-t. Narratives are an account of a situation, containing affective and theoretical descriptions of the observation of that situation, and of the events and actions in the situation. An educational narrative is more than a story about the situation, it is integral to communication among educators and, as a consequence, to the development and sharing ofeducational knowledge (Gudmundsdottir, 1995; Mason, 2002).

Teachers can tell about events in the classroom in a personal manner. Their story contains their interpretation of an educational situation, including the actions of the children and the teacher. Teachers express their knowledge in these stories; it is a narrative way of knowing (Gudmundsdottir, 1995; Tripp, 1993). These stories tell as much about them as they do about the event recounted. Narratives represent knowledge that is situated and remembered in stories and that is related to real situations, but may be overlaid by theories, beliefs, attitudes, expectations, assumptions and the desire to be seen as 'in the right'. This knowledge is personal, in the sense that there is a personal history to the situation and, of course, a personal interpretation of the situation. Constructing narratives from learning experiences will help teachers to reflect on those experiences. Narratives also play an important role in the relationship between teacher knowledge and teacher behavior.

In conclusion, we may argue that student teachers will benefit from constructing narrative knowledge. Our point of departure is that multimedia case studies can

function as a mediating tool in transcending the apparent dichotomy between theory and practice and can be instrumental in helping teachers develop situated practical knowledge about teaching (compare the framework put forward at the end of this chapter). Students' and teachers' practical knowledge is often embedded in their stories about classroom situations. In our view, these stories or narratives relate to observations, interpretations and analyses of classroom events, reveal essential aspects of (realistic) mathematics education, can be theoretically charged, and are easily remembered and recalled (see Tripp, 1993; Bruner, 1996). For teacher education, developing narratives through multimedia case studies can play a central role in transcending the dichotomy between theory and practice. In principle, experienced teacher educators will gradually develop their own way of working for this.

Our starting point is the thought that if student teachers are being taught to use new media, by using new media, it makes sense to prepare beginning teacher educators similarly. Most beginning primary mathematics teacher educators are new to teacher education; they might have experiences in other types of education, in other subjects, or with students of other age groups and often, in the beginning, are not able to transfer those experiences to their new position. So an in-service course also needs to address their concerns and help them to develop knowledge and experiences for teacher education. Teacher educators, on the one hand, need to develop mathematical and didactical knowledge, and, on the other, to develop knowledge of the teaching of student teachers; specifically, they need to develop insight in ways of using multimedia case studies for the education of their students (see Mason, 1998).

2.4. Immediate teaching behavior

Most Dutch primary school teachers use mathematics textbooks for planning their day-to-day lessons. Based on the last lesson, they will decide how to continue the next day. Teachers' theoretical and practical knowledge plays a crucial role in these decisions. Also, during the lesson the teacher has to make many ad hoc decisions on how to support students in constructing mathematical knowledge (see Clark & Peterson, 1986). Often, these decisions are characterized by a lack of time to reflect on what happened and to make a conscious decision (Eraut, 1995; Dolk, 1997; Mason, 2002). Dolk (1997) calls these unconscious reactions to a situation immediate behavior. Theoretical knowledge (Theory-with-a-capital-T) will not change immediate behavior. However, we need to understand that this behavior does not arise by magic; immediate behavior is the result of Gestalts (Korthagen, 1993; Dolk, 1997; Korthagen & Kessels, 1999), it is a response to a dynamic juxtaposition ofperceptual influences. This means that it will be possible to influence immediate behavior.

With modern technology, a learning environment can be designed for students to investigate and possibly acquire practical knowledge. In such an environment, students are given the opportunity to construct situated practical knowledge in the form of narratives and to generalize this knowledge to the broader teaching context.

Theory is no longer a starting point of teacher education. Here, narrative knowledge can be acquired by investigating real teaching. This environment and the way it is used in teacher education ought to allow for the construction of theory, the consideration of Theory, and the construction of new repertoire.

2.5. Multimedia environment

In our experimental courses, we used a multimedia environment, MILE, developed by the Freudenthal Institute, which allows users to search for classroom footage in a video database, and accompanying students' and teachers' materials. MILE is a Multimedia Interactive Learning Environment for prospective primary school teachers, with content for primary mathematics teachers education programs (Dolk et al., 1996; Goffree & Oonk, 2001). It is computer based and furnishes a database of real classroom teaching open for didactical investigations. It allows (prospective) teachers and teacher educators to study many aspects of mathematics education in primary schools. The environment consists of a video database of real teaching and learning practice in primary schools, communication tools for learners, a search engine, a scrapbook, and a journal for the learners. Several lessons are available for investigations by learners, creating opportunities to develop conjectures about teachers' practical knowledge and teaching behavior and the effect of that. Also, the environment allows them to analyze students' learning over time, and to investigate parts of the process by which students develop mathematical understanding. Therefore, MILE enables learners to investigate the relation between teaching and learning in primary school mathematics lessons (Fig. 1).

Each lesson is divided into short fragments. The division is based on subject matter and pedagogical and educational points of view. Each fragment is provided with an abstract, in which the story of the fragment is told. These abstracts make it possible for learners to select relevant and smaller fragments. They can for instance select a fragment in which students are discussing a developed strategy, where the teacher facilitates a conversation on a mathematical topic or where students develop a mathematical model.

Of course, in the case of a database with more than 3500 fragments, a learner needs other entry points besides an archive. Therefore, a search engine allows the learners to search the database in several different ways. Firstly, they can search the class dialogue based on full text retrieval. Secondly, they can search the abstracts based on full text retrieval and eventually, they can search the fragments on the basis of contextualized subject matter related labels. These labels, used to classify the narratives, are based on the educational, psychological, pedagogical, and subject matter pedagogical literature.

2.6. Rephrasing the research question

The MILE software does not offer a framework for investigating these episodes, to mathematize and didactize them and construct situated knowledge. Nor does the architecture of the software offer beginning teacher educators support in using

multimedia cases in their colleges. What is needed are multimedia case-based educational methodologies. This brings us to rephrasing our research question: How to employ multimedia cases to support teacher educators in learning to mathematize and didactize and to learn how to use multimedia case studies with their student teachers.

3. Research method

Within the context of the design, trial, and revision of the course for beginning mathematics educators, the question of how to employ multimedia cases became an independent topic of research. Given the interwovenness with the course design, the research naturally took the form of design research (Gravemeijer & Cobb, 2001; Edelson, 2002). With Edelson (ibid) we may argue that there are good reasons for engaging in design research. First, design research provides a productive perspective for theory development. After all, the range and completeness of the theory are immediately tested in the design process. Incompleteness will provide the designer with conflicting guidance, exposing inconsistencies. Furthermore, designers have Fig. 1. The archive, a part ofthe MILE environment, gives an overview ofthe lessons in the digital school and of the fragments in these lessons.

practical considerations to guide them. This provides a natural focus for theory development. The second reason Edelson mentions is the usefulness of the results. The prescriptive theories that design research produces provide educators with applicable research products. And thirdly, Edelson argues, design research involves researchers in the improvement of education. Researchers can withdraw from prevailing considerations driving traditional educational designers and educators; they have the opportunity to create innovative designs.

In our conception, design research, or developmental research, as it is called in the Netherlands (Gravemeijer, 1994), is driven by the cyclic process of thought experiments and education experiments that is at the heart of educational design.

2

Each teaching experiment, i.e. each enactment of the experimental course, consisted of a chain of such micro-cycles. In our case, a thought experiment would typically concern a deliberation on how the participants would think and reason while participating in an activity that involved the use of multimedia case studies. The

corresponding education experiment would then concern the trial ofthat activity and
the interpretation of the reactions of the participants in light of what was
conjectured.
3
As a consequence, we did reconsider our design regularly during the teaching
experiment. The ongoing analyses of the participants' individual and social activity
informed new anticipatory thought experiments, in the course of which conjectures
about possible learning trajectories were frequently revised. These revisions,
however, were guided by a more robust, but potentially revisable, local educational
theory.
In our research, this (conjectured) local educational theory was cast in terms of a
framework for the use of multimedia cases that aims at helping teacher educators
construct knowledge from episodes of classroom situations. A preliminary framework emerged from the finding that observing, reflecting, generalizing, and
theorizing had become essential components of the experimental course. Once
explicated, this (conjectured) framework could be used and put to the test in the
subsequent teaching experiments—which in our case encompassed three cycles of
teaching the experimental course. In those teaching experiments, the (conjectured)
framework, on the one hand, guided the thought experiments and the interpretation
of the education experiments, but on the other hand was revised and refined on the
basis of the very same thought and education experiments—which is indicated by the
double arrows in Fig. 2.
This focus on the ongoing experimentation process emphasizes that ideas and
conjectures were modified while interpreting participants' reasoning and learning
during the teaching experiment. The empirical data on the activities of the
participants was interpreted in the light of the theoretical framework of teacher
2
As an aside, we may note that to some degree, developmental or design research, action research, and
didactical engineering are comparable. In choosing the framework and terminology of developmental
research, we follow a long Dutch tradition in research in mathematics education.
3
Compare Simon's (1995) notions of 'mathematical teaching cycle', and 'hypothetical learning
trajectory'.
education described earlier. In this respect, we can speak of testable conjectures
about both the course of participants' development and the specific means of
supporting that development. These conjectures were then modified and revised
throughout the teaching experiment. Each session constituted a design micro-cycle in
that the assumptions and conjectures proposed while planning the activities were
scrutinized in the light of actual observations.
A further aspect of the methodology concerns retrospective analyses conducted on
the entire data set collected during the experiment. This data set included field notes,
recordings of individual interviews with participants, copies of the participants'
written work, and meeting notes of the designers/researchers who taught the course.
One of the primary aims of a retrospective analysis is to support the constitution
of a revised local instruction theory (or in our case framework). This is the starting
point for the next teaching experiment. Here, the cyclic nature of the methodology,
noted on the level of instructional design micro-cycles, reappears on a broader level.
An entire teaching experiment and the subsequent retrospective analysis together
constitute a larger design cycle; a design cycle that was carried out three times in our
research (Fig. 3).
CONJECTURED LOCAL FRAMEWORK DESIGN
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Fig. 2. Reflexive relation between thought experiments and educational experiments.
Fig. 3. Design macro-cycles.
The final version of the local educational theory (or the multimedia-use
framework) can be seen as the result of the learning processes of the researchers, founded in the reality of teaching. Similarly, we can characterize design
research as the learning process of the research team, and we would argue that
this learning process has to justify the outcome of the research project (Gravemeijer,
1994, 1998). In relation to this we can refer to the methodological norm
of ''trackability'' that is used as a criterion in ethnographic research (Smaling,
1992). This norm requires that the research is reported in such a manner that
an informed outsider could retrace the learning process of the research team. In
the next paragraph, we will describe our learning process as designer/researcher but,
due to page limitations not with the amount of detail the aforementioned norm
requires.
4. Findings
4.1. Designing a course for teacher educators with the use of MILE
The first thing the developers of the course encountered when a multimedia case
study was used was that the participants could not agree on what happened in the
footage (see Jaworski, 1988; Galen van et al., 1993). Nor was there one prevailing
approach to viewing classroom footage. Some ofthe participants felt they had such a
good overview of teaching that making notes or reviewing the footage were not
necessary. Others had made extensive notes and felt compelled to review the footage.
In the sharing after the first observation, most participants could only partly describe
the occurrences in the footage. Some were more inclined to value the teacher's
actions; others were focusing on the students. Some participants tried to stay close to
the observations; others were freely interpreting the footage on the basis of their
knowledge and personal beliefs.
In the second round, we designed a new case to make the participants-as-a-group
agree on the paraphrasing of the footage before they could interpret or analyze it.
This led to a cyclic process of viewing and discussing the footage. Each round of
discussion ended with a list of observation questions. After several rounds most
participants agreed upon the paraphrasing of the footage.
The next thing the developers noticed was that the participants preferred skipping
a discussion about their observation and analysis of the footage; they ''raced''
toward conclusions based on a mixture of theoretical and practical knowledge and
personal beliefs. Most participants tried to produce hard statements, containing
values and judgments. They were often not able to prove them. Hardly ever did their
conclusions lead to hypothesis or questions.
In the next round, we redesigned the multimedia case study framework, guiding
participants towards a more investigative analysis. After extensive observation and
discussion of the observation, we challenged the participants to analyze the footage
and propose alternative explanations, to prove and disprove these, and to formulate
investigative questions for further study.
In several cycles, several case studies were developed based on the four steps ofthis
framework: observation, paraphrasing, analysis and reflection. For example, after
observing and discussing a conversation between a student and the teacher, the
participants had to role-play the situation. Their different interpretations of the roles
and the underlying analysis of the two characters were discussed.
Over time, we became aware that this approach led the participants to develop
situated mathematical and didactical knowledge. However, they did not apply that
knowledge in their teaching of student teachers, nor did they develop theoretical
knowledge on using multimedia cases in teacher education. Furthermore, during the
four steps the participants switched constantly between discussion of the footage
from their point of view and from their expectations of their student teachers' point
of view.
These led to a redesign of our framework. We added steps to develop narrative
knowledge and to consciously expand the teaching repertoire of the participants. To
have participants construct knowledge about the framework we designed, we
designed the case to have them make the six steps twice, once at their level and then
at the level of the student teacher. Furthermore, we introduced a 'backburner': the
possibility for participants to reflect on the framework and on their learning. The
latter can be seen as some form of pro-flection, of preparing oneself to notice more
sensitively and/or to act differently in the future (see Mason, 2002).
4.2. Six-step design framework: from observing a situation to the construction of
educational meaning
Using this iterative process of experimenting, analyzing, and adjusting we
developed a six-step (Fig. 4) framework: an approach to working with multimedia
cases and to constructing mathematical and didactical meaning and repertoire in the
MILE environment. The six steps in this framework are:
1. Observing,
2. Sharing and Discussing Observations,
3. Analyzing,
4. Reflecting,
Observing
Discussing
Observation
Analyzing
Reflecting
Developing Narrative
Knowledge
Expanding the Personal
Repertoire and Generalizing
Fig. 4. Six-step design framework.
5. Developing Narrative Knowledge,
6. Expanding the Personal Repertoire and Generalizing the Situation into a Didactic
for Teacher Education.
4.3. Observing
Observation is a didactical principle (Freudenthal, 1991). Our participants need to
observe learning processes, to become aware of and report the many aspects of
teaching. As Freudenthal (1991) states, teaching and didactical experiences result
from observing learning processes. Teachers' practical knowledge and their behavior
are based on and justified by experience.
Observing is difficult. After viewing a fragment of a lesson once, participants are
often certain they can give an accurate description of that classroom episode. As
different participants share their observations, most participants have to admit that
other observers noticed things they missed. However, there is more.
4.4. Sharing and discussing observations
On the one hand, sharing and discussing are meant as stimulation to observation.
In sharing and discussing their observations, learners are encouraged to reflect on
their observational and reporting strategies, to see discrepancies between different
observations as a stimulus to observe more (better, deeper) and to question their own
observations. Not only do the participants become aware of the inaccuracy, and
incompleteness of their observations, they also become aware of the interaction
between their interpretation of the situation and their observation of the situation.
They become aware that their (un)conscious beliefs of teaching and learning, and
their educational knowledge take the form of a filter that allows only a part of the
situation to be observed, and that parts of their report of the situation are based on
their interpretations of the situation, not on the situation itself.
On the other hand, sharing and discussing the observation helps to overcome the
learning paradox—when you do not know much, you cannot see much; when you
know more, you see much more. Through comparing and discussing the
observations of peers, participants will be challenged to widen their observation
and therefore their analysis of a situation.
4.5. Analyzing
During an archaeological excavation, researchers dig up evidence of human
civilization. Later they try to reconstruct that civilization by analyzing their findings.
In the same manner, didacticians try to analyze their observations. Teacher
educators benefit from being able to analyze classroom observations and focus on
each of the three elements of the didactical triangle (the children, the teacher, the
subject matter within the contextual situation-environment and the interaction
between them). For instance, the analysis can focus on reconstructing the children's
construction of mathematical meaning. It can attempt to probe the children's
understanding of the situation. It can also focus on interaction between the
mathematical problem and the children's understanding.
These different foci help participants to understand what observations are
worth reporting, and to construct mathematical and didactical meaning for the
classroom situations. Learning to observe, report and analyze is a circular
learning process. Participants need to be able to make rich observations allowing
multiple interpretations, to be able to analyze the situation thoroughly; but to be
able to make such a rich observation, participants need to have constructed a
deeper understanding of classroom situations through ongoing analysis of those
situations.
4.6. Reflecting
Following Schon's (1983) suggestion of a conversation-with-the-situation,
participants are challenged to reflect on the situation and construct practical
knowledge from it. Reflection is seen as a precondition for the process of
constructing meaning and knowledge, especially for learning from one's own
observations and analysis. This process is based on the experiences and viewpoints of
the participant, underlining the cyclic character of this six-step approach.
4.7. Developing narrative knowledge
Students' and teachers' practical knowledge is often embedded in their stories
about classroom situations. These stories or narratives relate to observations of a
classroom event, reveal essential aspects of (realistic) mathematics education, can be
theoretically charged and are easily remembered and recalled. For teacher education,
such narratives can play a central role in intertwining theory and practice. These

narratives are more than classroom anecdotes. The latter will not support the
development of deeper understanding of the situation unless organized in
educational or psychological categories.
4.8. Expanding thepersonal repertoire and generalizing the situation into a didacticfor
teacher education
To construct theoretical knowledge through the development of narratives,
participants are enticed to generalize narratives of several analogous situations into
one encompassing narrative. By generalizing the situation and the narrative, the
participants build a personal didactical theory. Where conscious practical knowledge
can be put into words, it becomes open to examination by scholars who may then be
able to turn it into formal knowledge. At this moment in the process, we challenge
the participants to become that scholar and to examine their own theory.
At the same time, they work in classroom situations, interviewing children and
teachers, organizing mathematical investigations. This allows them to try new
classroom behavior and expand their repertoire.
M. Dolk et al. / Int. J. Educ. Res. 37 (2002) 161–178 1754.9. Cycling processes
These six steps are not independent. The arrows in Fig. 4 indicate that sets of steps
are related to other steps. The process is cyclic, observing and discussing the
observation forms the basis ofthe analysis and reflection ofthe situation. However, at
the same time, the analysis and reflection will fuel new questions for the observation.
5. Concluding remarks
In this chapter, we addressed the issue of the use of multimedia case studies in
teacher education, by investigating a specific setting—that of the professional
development ofbeginning mathematics education teachers in the Netherlands. Based
on our research, we propose a six-step framework that aims at helping the
participants develop narrative knowledge. In this manner we want to both transcend
the presumed dichotomy between theory and practice, and foster a sound basis for
immediate teaching behavior that fits the ideas of realistic mathematics education.
Our experiences with the emerging framework in our design research give us
confidence that it can work in this manner. However, for us, it is still a tentative
framework. Further research is needed, not only in our own experimental setting,
but also in different settings. At this moment, the framework as developed in this
study is being used in the development of CD-ROM learning environments.
We hope that other teacher educators and researchers will be inspired by our
account of what we learnt, and take our framework as a point of departure for more
formal research or more informal experiments on the use ofmultimedia case studies,
in other settings than the one described in this chapter. If that happened, our design
research would have achieved what we intended.
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Mathematics Competence and International
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ABSTRACT The article presents a review of the competence-framed concept of
compulsory mathematics education and appropriate instruction methods, in light
of the international influences on Croatia through comparative ranking in PISA
mathematical literacy. We begin by reviewing the conceptual constructs behind social
influences on the supra-national development of the literacy/competence concept
in mathematics education and assessment, and briefly examine its components and
instructional contributions. Within this context we proceed to map past and current,
and extrapolate future, changes in the conceptualisation of school mathematics and
mathematics teaching in Croatian compulsory education, including recent empirical
findings on mathematics instruction in Croatian primary schools. The aim is to examine
social and (comparative) political influences on subject-matter and teaching of 'hard'
school subjects such as mathematics, chart Croatia's current position according to
international assessment and instructional trends, and recommend possible steps in
immediate development of educational policy.
Key words: PISA testing, competence, literacy, mathematics, instruction, problem-based
learning, compulsory education in Croatia.
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1. Mathematical Knowledge, Mathematical Competence and PISA: What does
Society Want from Compulsory Mathematics Education?
When comparing the conceptual and social development of the aims of mathematics
instruction to the performance in international assessment such as OECD's PISA, the
crucial leverage is provided by the teachers' understanding of the aims of mathematics instruction and their access to and choice of practices to achieve it. As the aims
of mathematical education change under international and social exertion, can the
classroom practices, and teachers primarily responsible for it, follow? In the mid-
1990s international comparisons of educational systems and content began taking
shape (Bishop, Clements, Keitel, Kilpatrick and La-borde, 1996.), including mathematics education. In fact, 'hard subjects' such as mathematics and science education,
provided a more stable ground for comparison than the more context dependent
liberal arts education (Domazet, 2006.). Thus, PISA international assessment assumes that mathematical knowledge and skills are not culture-bound and thus are fully
internationally comparative.
In a comparative mathematics education volume from 1996. M. Niss (as cited in
Hoogland and Jablonka, 2003.:1) states that traditional goals of mathematics education had been, by then, considerably broadened to include the "essential aspects of
numeracy and "mathematics literacy" in society" (Hoogland and Jablonka, 2003.:1).
Despite that statement 'literacy' does not as yet feature strongly in the actual content
of the review. By 2003., and the next edition of the same review, the situation changes drastically. This is also the time when the international PISA assessment, which
strongly features the 'mathematical literacy' context, begins to permeate educational
development thinking in increasing number of countries. The PISA theoretical framework has many similarities with the theoretical perspective adopted for mathematics education called realistic mathematics education, which is itself rooted in H.
Freudental's ideas about mathematics teaching (Freudenthal, 1973.).
It is important to mention here that realistic mathematics education (RME) is, as a
teaching and learning theory, substantially determined by Freudenthal's view of
mathematics as a human activity connected to reality, close to children and relevant
to society (Heuvel-Panhuizen, 2003.:10). Accordingly, in addition to the use of contexts in mathematics teaching and learning, RME attached significant importance to
mathematics teaching as a process in which students develop and apply mathematical concepts and tools in a realistic problem context. As opposed to the traditional
mathematics education, such an approach presented a shift towards modernisation
of mathematics education and its current developments. Subsequently, different social influences on the broadened definition of goals of mathematics education can
be detected, from developing human capital to achieving environmental awareness
or social change (Jablonka, 2003.). Mathematical education was no longer just about
transferring the canon of 'mathematical knowledge' to the uninitiated youth, but
about satisfying broader social goals in that. And the national framework curricula
and international assessment were to reflect that aim. M. Domazet, B. Baranović, J. Matić: Mathematics competence and international...

Perhaps this trend is globally today most evident in the OECD/PISA approach (Organisation for Economic Co-operation and Development [OECD], 1999.) which includes mathematics education as one of the instruments of development of human
capital. Mathematics was not just some abstract body of knowledge, but contained
skills and a way of thinking to be applied in numerous real-life situations, professional tasks and global cooperation. Through broadening to 'literacy' applicable

in wider contexts than a classroom, mathematics education was to become more
causally relevant to raising living standards and intensification of economic activity
(development of human capital, Jablonka, 2003.).
When comparing the 'mathematical literacy' definitions of the contemporary international PISA assessment and the more traditional expectations of mathematics
education that Croatian teachers were trained in and work with, a gradual change
from abstract training in understanding of the world's deterministic regularities to
participation in a numerical-analytical society is observed. This is a reflection of the
global trend of linking mathematics education with real-life contexts, and expression of societal pressures extraneous to specific subject concept on the definition of
educational outcomes. Because existence of such pressures cannot be ignored, they
are relevant to the review of conceptual construction of a specific subject (in this
case the 'hard' subject of mathematics, cf. Domazet, 2006.) and its teaching practices,
without being used as an overall assessment of an intrinsic quality of mathematics
education in Croatia.
The most senior practicing teachers (just under 40% of overall lower-secondary
teacher population according to 2003. research, Baranović, 2006.) would have been
initially instructed that "The aim of primary mathematics education is to introduce the
students to quantitative understanding of the world, to identification of mathematical
structures that are used to describe the phenomena and the regularities in social
and natural sciences, and are applied in everyday life." (Furlan, Kaučič, Muhvić,
Podgorski, and Zadrović, 1974.:72) This definition is immediately followed by a list
of instructional tasks in mathematics which contains various forms of mathematical
knowledge and understanding but no specifics of their application in life other than
general rationality and orderly precision. At the outset of Croatian transition (1991.)
no significant content change is reported in mathematics instruction (Jakopović,
1991.), and average age of practicing maths teachers in Croatia makes this their
primary training definition (Baranović, 2006.). Some instructional changes are
advised with claims that teacher-centred lecture-style instruction cannot ensure
adequate development of youths as it 'passivises' the young person and hinders the
personality development. Although mathematical instruction is expected to prepare
students for employment and scientific and technological development of the state,
application of knowledge in students' everyday life is listed in the last place.
In a first revision of the educational content in the transitional period (Ministry of
Education and Sport of Republic of Croatia, 1999.) practical applicability of mathematical knowledge is abandoned altogether and the overall aims of mathematics
are listed purely in terms of knowledge appropriation and its application in further
learning and other natural and social sciences. Again mathematics is tasked with Sociologija i prostor, 51 (2013) 195 (1): 109-131
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developing rational thinking, orderliness and precision of expression. Although,
of course, all of those can be applied in everyday life, there is no explicit mention
of the real-life situation or specific professional tasks. Subsequent revision in 2006.
(with PISA present on the global stage) lists the general goals of education as development of individual personalities and overall knowledge society, but no particular application of knowledge is listed (Ministry of Science, Education and Sport
of Republic of Croatia, 2006.). In mathematics instruction itself, enabling students
to join various professions, economy, use contemporary technology and become
members of contemporary society is cited, but the main goal of mathematics education is appropriation of foundational mathematical knowledge, development of
basic mathematical literacy and ability to solve mathematical problems. Application
of mathematical knowledge in everyday life is now listed higher up in instructional
tasks, right after skills in mathematical expression and numerical literacy.
Finally, the most recent reform initiated in 2011. and ongoing, justifies compulsory
mathematical education with the need to equip students to successfully participate
in the society reliant on information and technology (Ministry of Science, Education
and Sports of Republic of Croatia [MZOS], 2011.). Mathematical instruction is now
repeatedly explicitly connected to its everyday and lifelong learning application in
various forms (numeracy, spatial cognition etc.). The curriculum framework document explicitly states that through mathematical instruction " students will realise
the importance of mathematics in their daily lives, gain insight into the historical
development of the discipline, and come to understand its role and importance in
the society of the past, present, and future. [...]They will gain exposure to mathematical problems found in real, everyday situations, thus linking the discipline with
their everyday lives..." (MZOS, 2011.:80). Among the explicit goals of mathematical
instruction, appropriation of basic mathematical knowledge and skills is immediately followed by the application of mathematical principles "in various contexts,
including professional life" (MZOS, 2011.:80).
The PISA assessment on the other hand, tests for literacy defined as "an individual's
capacity to identify, to understand the role that mathematics plays in the world, to
make well founded judgments and to use and engage in mathematics in ways that
meet the needs of that individual's life as a constructive, concerned and reflective
citizen" (Organisation for Economic Co-operation and Development [OECD], 2003.:
24).
1
The new concept of mathematics as mathematical literacy is of special importance for Croatia which has recently joined the PISA assessment and introduced
changes of the mathematics curriculum as a part of the broader national curriculum
reform (2011. reform mentioned above). Joining the assessment subjects the whole
of Croatian compulsory mathematics education complex (curriculum-teachers-students-schooling processes) to rough-and-ready comparisons on global scale, which
1
Though there are definitions of mathematical literacy assessed in PISA prior to and following
the given 2003. definition, this is the one most closely related to the discussions in other sources cited in this text, as well as the one used at the outset of the 2003.-2009. period for which
some comparative results are presented below. M. Domazet, B. Baranović, J. Matić: Mathematics competence and international...
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are more often than not one-dimensional in terms of ranking of means of assessment performance.
As such they ignore both the greater breadth of aims of school mathematics education than sole achievement of literacy in mathematics requires, and the variations in
conceptual construction of mathematics education between Croatian contemporary
textbooks and curricula on one side, and PISA educational outcomes definition on
the other. Moreover, the international assessment rationale itself assumes that different curricula affects across countries are entirely unproblematic and comparable by
simply choosing to adopt the trans-curricular character by fiat (Keitel and Kilpatrick,
1999.). Actual assessment of individual curriculum and instruction complex's performance must be aligned with explicit goals of the curriculum as they are prescribed
to teachers (cf. Schoenfeld, 2002.). Thus, for example Glasnović-Gracin empirically
illustrates the differences between goals of Croatian mathematics education as defined in in-use textbooks and curricula and PISA competences definition (Glasnović
Gracin, 2012.). Furthermore, there are empirical demonstrations of how these discrepancies between international assessment programmes (such as PISA) and different national mathematics education complexes vary between countries, making the
testing items more or less applicable to what the students were actually prepared for
through education (Schmidt, Wang and McKnight, 2005., Schoenfeld 2002.; cf. also
Mešić 2012. for related case of TIMSS international assessment).
Summarily put, what is important for our purposes here is that PISA should not be
used to indicate the overall quality of particular national compulsory mathematics
education, but its comparative output against a well-defined set of criteria, whilst noting the general influence such international assessment (and subsequent rankings
produced) has on the curricula and instruction within countries (see below). In that
light recent research findings on the state of the math education
2
indicate (Baranović
and Štibrić, 2011.) that the Croatian teachers have notionally adopted the requirement of applicability of mathematical knowledge, but still perceive the operational
components of this requirement in a traditional way.
Namely, when ranking the importance of students' learning outcomes
3
mathematics
teachers valued most highly application of mathematics in everyday life (56.0% of
2
The research we are referring to was carried out in 2010. using a random stratified representative sample of 625 math teachers (Grades 5-8) from elementary schools throughout
Croatia which is 31% of the total number of math teachers working in elementary schools
in Croatia during the data collection . The administered teachers' questionnaire covered the
following dimensions of math education: teachers' mathematics-related beliefs, importance
of students' learning outcomes that foster the development of mathematical literacy, students'
activities during math lessons, usage of teaching materials and tools, teachers' estimates of
the characteristics of the existing math curricula and teachers' perception of self-efficacy. The
research results were presented at the ECER conference, held in Berlin in 2011. (Baranović
and Štibrić, 2011.).
3
Out of 18 offered students' learning outcomes, the teachers were asked to choose 5 outcomes to which they attribute most attention. Sociologija i prostor, 51 (2013) 195 (1): 109-131
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teachers' responses) and adoption of a positive attitude towards mathematics (55%
of responses) while most other learning outcomes, which are characteristic of PISA
concept of mathematical literacy, were ranked very low: e.g. students' reasoning
about mathematical concepts and processes (13% of responses), critical thinking
about math concepts and procedures (15% of responses), mutual communication of
ideas and results (18% of responses) etc. At the same time, and as indicated by the
operational goals of mathematics instruction above, the teachers attributed relatively
significant attention to the specific learning outcomes which are typical of traditional approach to mathematics, e.g. students' hard and thorough work during math
lessons (35.0% of responses), students' confidence in solving standard mathematical
tasks (31.5% of responses), mastering the math syllabi (32%), etc. (Baranović and
Štibrić, 2011.).
Though aimed at the 'same animal' (Niss, 2003.b:215), different historical and present-day delineations of school mathematics instruction place different emphasis
on the everyday social and professional context of the current students and future
employees and citizens. Whereas Croatian 1970s and 1980s students were expected
to be initiated into the quantitative understanding of the world, which would in part
open up the riches of natural and social scientific description of it, the 2000s global
citizens, trained in mathematical literacy, are much more focused on the judgements
about the role of mathematics in occupational and social life. This further illustrates
the difference in construction of the desired outcomes of compulsory mathematics
education through time in Croatia and globally, which also bears on the straightforward assessment performance of the student population.
M. Niss (one of the participants of the PISA conceptual team) is explicit (Niss,
2003.b:216) about the role of mathematical literacy in maintenance of, and participation in, democratic social activities, something that mathematics as pure, fundamental science is decidedly outside of. And although maintenance and participation
in the social order and societal operations will involve many mundane activities like
checking someone else's sums, Niss is convinced of the deeper role that mathematical literacy plays in a society, a role that goes back some way to the "quantitative
[and scientific] understanding of the world" (Furlan et al., 1974.:72). He says that it
has a role to play in allowing individuals to come to grips with how mankind perceives the world, its natural, social, cultural and technological aspects. "[Mathematical
literacy is an essential component] in liberating literacy and popular enlightenment"
(Niss, 2003.b:217). Again, bearing in mind that PISA does not explicitly asses the
mathematics curriculum, whose goals should be broader than the outcomes assessed by PISA programme (cf. Schmidt et al., 2005.), we note that training solely for
literacy and 'applied knowledge' can miss out on the depth and comprehensive understanding of mathematics subject matter even when achieving high performance
in assessment (Schoenfeld, 1988.).
There is a worry that 'mathematical literacy' is just 'easy mathematics', the 'mathematics for the un-mathematical'. From the perspective of the historical and social
influences on the broadening of mathematical instruction and educational aims, it
is worth heeding a statement from apartheid-conscious South Africa (Julie, 2006.) M. Domazet, B. Baranović, J. Matić: Mathematics competence and international...
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that 'mathematical literacy', even when perceived as 'easy mathematics', is democratically enabling for a broader population. Given the stated importance of mathematics in everyday life in contemporary society it is exclusive to limit the instruction
and understanding to only those with 'mathematical potential'. Julie is adamant
that mathematical literacy, whether we like it or not, is justified by globalisation,
technological availability of access to knowledge and the "world-wide thrust for
regular testing [...] through international comparative studies". Nonetheless, even as
'literacy', school mathematics can come in empowering or disabling forms, as a tool
contributing to the participation in the spectrum of relevant global worldviews or as
just a 21st century version of basic arithmetic (Julie, 2006.). And in South Africa the
public can still remember how the latter 'lower form' of mathematics instruction was
used as a tool of discrimination and domination.
For the purposes of this paper it is worth noting the liberty to switch between the
mathematical literacy and mathematical competence. Though there might be entire
volumes dedicated to fishing out the fine details of the difference, we shall contend
here with using both to describe the mathematical ability that can be and is applied
outside the strict mathematical factual knowledge reproduction tests. So Hoogland
and Jablonka (2003.) say that various definitions of mathematical literacy relate to
an individual's capacity to understand mathematical aspects of everyday situations
and make judgements about them based on the mathematics learnt at school. As we
have seen above, PISA includes in the definition of mathematical literacy engagement in mathematics and forming of judgements of the roles mathematics plays in
current and future lives of the students. Likewise, a more socio-politically motivated
concept of competence (which in several of its European iterations explicitly includes mathematical competence) is based on the DeSeCo attempts to define 'key competences useful for a successful life for individuals and a well-functioning society',
and progresses towards, for example, "the ability to apply learning outcomes adequately in a defined context (education, work, personal or professional development)"
(Office for Official Publications of the European Communities [CEDEFOP], 2008.).
Moreover, the DeSeCo elaborates that 'competence is not limited to cognitive elements; it also encompasses functional aspects (involving technical skills) as well as
interpersonal attributes (social or organisational skills), values and ethics, attitudes,
emotions, and motivation (Rychen and Salganik, 2003.; CEDEFOP, 2008.). Indeed,
Silver (2003.:27) refers to the historical foundation of the attempts at settling the
definition of mathematical competence in the definitions of literacy, numeracy and
expertise. Silver hints at mathematical competence potentially being a wider term
than literacy, so as to include more of the academic or expert reasoning, whilst
maintaining the applicability to everyday life requirements. For the purposes of the
discussion of the innovations of teaching practices we can safely focus on the part of
competence where it fully overlaps with literacy, the part in which mathematics instruction has to focus on something more than the initiation into the academic discipline. Silver himself will warrant this with stressing the importance of examining the
application of mathematical knowledge in context of its everyday applicable use.
Niss, as one of the authors of the PISA definitions, bases the definition of mathematical competence on the supposedly more familiar notion of literacy in general (Niss, Sociologija i prostor, 51 (2013) 195 (1): 109-131
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2003.a). Just as literacy involves the mastery of written and spoken language, a use
of language, so mathematical competence consists of the ability to understand and
use mathematics in a variety of mathematics-related contexts and situations in which
mathematics plays a crucial role. This, of course, consists of the mastery over a lot of
factual knowledge and domain-specific skills but cannot be reduced solely to those.
It is also important to note for our purposes here that there are attempts to define
competence as an "attribute of participation in an activity system" (Gresalfi, Martin,
Hand, and Greeno, 2009.; Engeström, 1993.; Greeno, 2006.; Lemke, 1990.). The purpose of introducing this dimension here is to highlight the social and cooperative
aspects of mathematical competence. The point to be emphasised following Gresalfi
et al. (2009.) is that what counts as "competent" becomes construed in individual
classrooms, and can take a different appearance from one to another. Teachers and
their own implicit definition of mathematics competence are most immediately responsible for such constructions.
With all those in mind, it is important that Croatian teachers today mostly realise the
importance of teaching mathematics for its application in life, rather than initiation
in the abstract discipline and worldview. However, beneath the surface expression
of aim to 'apply mathematics in everyday life', they show low preference for some
of the more specific learning outcomes that are characteristic of the new concept of
mathematical competence. Following the introduction of those into the new National Framework Curriculum, they should be developed in greater detail in a way that
teachers could implement in the lessons. But, as we shall see, curricular design is
less than half of the story, as instructional practices and teachers general workload
contribute to the attainment of students in standardised testing and their retention of
knowledge for application in life.
2. PISA: Testing for Life?

Through the international assessment whose outcomes are most readily presented
in the ranked form, the PISA definition of mathematical competence eventually influences, through pressures external to curriculum conception, the understanding of
mathematics to be promulgated through the individual national curricula. That is if
the participating countries want to enhance their international ranking performance,
which most undoubtedly do as a matter of prestige. In fact, as Niss (2003.b:217)
puts it, in order to comply with the PISA competition for rank "it becomes a crucial
task to find and employ new ways to define and describe mathematics curricula that
focus on mathematical competence rather than on facts and techniques". The other
possible response is to highlight inadequacies in the PISA programme itself, which
might incrementally contribute to its overall improvement, but will eventually reach
the boundaries of 'aiming at the same animal' (cf. above) and reaping the benefits
of contemporary mathematics education trends. What we are primarily interested
here is the social phenomenon of influence of PISA on national instruction practices
and curricula, whilst aware of the misalignment of Croatian compulsory mathematics
education and PISA (cf. Glasnović Gracin, 2012. and above) and the general societal influence on competence construction (cf. Domazet, 2011.; Halász and Michel,
2011.). M. Domazet, B. Baranović, J. Matić: Mathematics competence and international...
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On the other hand, through Niss' own admission, (Niss, 2003.a:120) PISA definition
of the desired mathematical literacy is influenced by the work on the Danish national programme of mathematical competence definition, in addition to the already
mentioned RME development even further back (cf. Freudenthal, 1973. and above).
What is interesting for our purposes is to briefly survey the competence concepts
that were included in this definition, as well as the assessment practices based on
those. According to Niss, competence concept in mathematics divides into two major groups of more specific competencies. The first group consists of capability in
asking and answering questions in and with mathematics (Niss, 2003.a, p. 118): thinking mathematically, posing and solving mathematical problems, modelling mathematically, and reasoning mathematically. The second group represents capability to
use mathematical language and tools: representing mathematical entities, handling
mathematical symbols and formalisms, "mathematics" communication, and making
use of relevant and available aids and tools. These are the components closely shadowed even in the latest PISA assessment framework (Organisation for Economic
Co-operation and Development [OECD], 2009.:106-7).
These have been developed in greater detail so as to try to address many perspectives on what mathematically competent person should emulate. There is no room
here to enter into those, but what is more interesting is the ability to successfully
test for the mastery of the complex competence defined in this way. This has to be
achieved in a way to avoid the traditional and simple test of the initiation into the
academic discipline of mathematics, but also to provide the teaching process with
a realistic and reasonable outcome of instruction rather than some holistic property
which some students have and others do not. If on the PISA definition, the competence requires the ability to 'identify, understand and engage in mathematics' and
'make judgements about the role of mathematics', what is the adequate instruction
(and subsequent testing) to look like?
Niss understands, at least in part, the PISA definition as requiring the preparedness
to act mathematically on the basis of knowledge and insight (Niss, 2003.a:119).
And the test he envisages for that has to be founded on the range of the students'
competences in relation to the mathematical activities they are or may be involved
in as part of everyday life. Here he envisages the testing outcome to consist of the
three dimensions product of which is the total competence level, as the spatial
volume is the product of three spatial dimensions. The dimensions are the degree
of coverage, the radius of action and the technical level of competence activation.
What is traditionally mostly taught and tested for is the first dimension: the extent to
which the students master different aspects of the school subject's cognitive content.
But the complex notion such as PISA definition aims to test for requires the test to
vary and assess for the spectrum of contexts in which the student can recognise the
potential for and successfully draw on the different aspects of competence listed
above. Finally, in the contemporary global context, the technical level calls for the
assessment of the conceptual and technical complexity of the aids and tools the
students can depend on to activate the application of different aspects of competence
named above. Sociologija i prostor, 51 (2013) 195 (1): 109-131
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A proviso immediately follows, that the fully adequate assessment of the possession
of competence is more than a single test can hope to achieve. But when it tries to
come even close to this, it has to provide as broad as possible a spectrum of activities in which the degree and range of competence possession can be demonstrated.
In that respect, the PISA Assessment framework envisages several clusters of tasks
in which combinations of component competences can be assessed for in different
ways: the reproduction cluster, the connections cluster and the reflection cluster
(OECD, 2009.). But the clusters also respect the hierarchy of complexity, with reflection being the most complex and the reproduction the simplest.
In this sense, the competence-based 'innovative' international framework shows clear connections to the foundations of mathematics teaching and testing: to reproduce
the basic cognitive items and skills, to make independent connections and reason
through broad reflection on the abstract knowledge in combination with the specificities of the given context. In that respect the PISA assessment presents a fine-tuning
on the base of the traditional mathematics instruction ('introduction to quantitative
understanding') that majority of Croatian teachers have been trained in.
4
In other
words, it is a question of reorienting and not rebuilding mathematics instruction.
But the degree of reorienting required can be extensive, as will be discussed below.
Although national standardised testing takes increasing importance in quantifying
and certifying educational outcomes in Croatia, the main outcome of competencebased education should still be a successful application of the competences developed to everyday life beyond schooling and assessment contexts. As has already
been mentioned above, the research among Croatian teachers reflects as much: the
highest ranking elements of mathematical competence as outcome are 'apply mathematics in everyday life' and 'adopt a positive attitude towards mathematics' showing
the teachers are aware of the trends. However, some of the operational formulations
of those outcomes, as derived from for example PISA-related definitions, enjoy a
very low preference amongst teachers (as presented above).
5
In a discussion of the effect of teachers on the international mathematics standardised assessment performance (comparisons of the performance of the East Asian and
Western students in mathematics, such as Lapointe, Mead, and Askew, 1992.; Stevenson, Lummis, Lee, and Stigler, 1990.) teachers' own competence in mathematics
and pedagogy was assumed to be a major factor in influencing student performance. However, repeated study by Leung and Kyungmee (2002.), comparing the East
4
Only the most recent generation of the practicing mathematics teachers in Croatia will have
undergone their initial teacher-training based on the more recent mathematical concept definitions (2006. and 2011.), whilst majority of the practicing teachers for their initial training
relies on the 1974. conceptualization and its 1991. and 1999. derivatives.
5
Though PISA is not the only competence-based outcome construction, it is useful here
for both its broad application and expansive examination, whilst providing an interesting
expansive international comparison and paradigmatic societal influence on national education systems. M. Domazet, B. Baranović, J. Matić: Mathematics competence and international...
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Asian teachers from different regions (Shanghai, as presented in Ma, 1999., versus
Hong Kong and Korea) found that it was the ability to specialise in mathematics
teaching and the general workload that mostly contributed to the teachers' own
competence in fundamental mathematics and their grasp of, and subsequent ability
to freely manipulate, mathematical concepts. The advantage of Shanghai teachers,
which registered both in their students' performance in standardises testing (cf. PISA
performance overview below), but also in teachers' own test and concept manipulation performance stemmed from them teaching mathematics only and only in one
grade in a certain year, whilst bearing a lighter teaching load, allowing them to devote their full attention to preparing the particular grade's (innovative, conceptually
foundational) lessons.
The disadvantage of the other teachers assessed lead to their lower (though not
insufficient) grasp of the fundamental mathematical concepts, but more importantly,
to their reliance on traditional teacher-centred instruction that excludes the investigative navigating of students' own exploration or other instructional methods conducive to constructivist-based conception of mathematics learning and instruction.
Leung and Kyungmee (2002.) admit that there was also some attitudinal justification
for the teacher-centred procedural instruction, resting on the teachers' explicit beliefs that the problem-based learning is unsuitable for the early stages of schooling.
That basic rules and procedures have to be 'drilled' first. However, when pressed
to explain the success of the non-Shanghai East Asian students in the standardised
testing, despite the limitations their teachers face, Leung and Kyungmee (2002.) claim that fundamental conceptual understanding is not a prerequisite of a successful
practical application of knowledge. They are of the belief that successful blind task
completion through repeated practice can of its own lead to conceptual understanding at a later stage.
Be that as it may, Leung and Kyungmee (2002.) concede that only within a well
designed curriculum, such that concepts are introduced in a measured interlace
with systematically varying exercises, can such a 'practice-makes-perfect' route to
understanding and eventual competence-mastery be followed. This sees a division
of labour in mathematics instruction where the task of the conceptual innovation in
mathematics instruction is carried by the curriculum and textbook designers. The
teachers are then only asked to competently follow the curricular recommendations.
This seems to be similar to the Croatian practice, where little freedom in curricular
design is left to the teachers, and their teaching practices (as is reported below)
reflect the focus on teacher-centred instruction. Nonetheless, the international performance certified competence of the East Asian students can be attributed, at least
partly, to the competence of their teachers.
Country performances on the mathematical literacy scales can be a contested issue,
which becomes obvious from the change in the mode of presentation of the results
in PISA reports 2003. to 2009. (where bare statistical differences are replaced with
a more narrative interpretation of the individual results' importance). However, for
the purposes of this text there is no need to enter into their detailed analysis, and
it is useful to approximately focus on the mode of presentation applied in the la-Sociologija i prostor, 51 (2013) 195 (1): 109-131
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test report (Organisation for Economic Co-operation and Development [OECD],
2010.:134) where grouping of countries is provided with respect to significance
6
of difference in score. In that respect, and regarding the literature drawn on in this
text (Leung and Kyungmee, 2002.), it is illustrative to compare the mean overall
performance of Croatian students in 2006. and 2009. to that of students from selected East Asian economies and countries consistently exhibiting a top performance,
and participating countries from the former shared Yugoslavian state (Montenegro,
Slovenia and Serbia) where structural similarities in educational goals and teacher
education may exist.
We note the limitations of the PISA testing in relation to Croatian curriculum, as well
as the variations in the assessment programme itself as described by Glasnović Gracin (2012.), but are not after relying on PISA rankings here to expose the quality of
the overall mathematics curriculum or instruction. Moreover, whilst a lot of ink has
been shed over analyses of national PISA performance and many different overall
ratings constructed, it is the simple overall rankings of the type mentioned here that
present the paradigmatic case for the international ranking pressures on the construction of national educational outcomes. This text, on the other hand does not
endeavour to enter into details of analyses of comprehensive reasons for individual
countries' performance (which would go far beyond curriculum and instruction
alone), but to describe a simple link between educational outcome definitions in
Croatian documents through time and contemporary international pressures.
East Asian countries and economies exhibit consistent performance at the top of the
scale in mathematical literacy, which has sparked research interest some of which is
drawn on below. Croatia participated in PISA assessment in 2006. and 2009., where
the latter performance results slide downwards on point scale. Although its students
performed below OECD average Croatia remains on the scale above Serbia and
Montenegro. Slovenia, which is the first former Yugoslavian country that joined the
European Union
7
, shows performance above OECD average
8
, but not as high as the
East Asian countries and economies.
6
"Significance' here refers to statistical significance as presented by the official OECD PISA
publications (OECD, 2010.).
7
Slovenia introduced reforms to primary education structure in the 1990s by extending general compulsory education to 9 years (with 6 years primary education) and outcome-oriented
national curriculum.
8
In PISA 2003. results presentation there are significant differences presented including and
excluding the Bonferroni adjustment. We have no need to enter into this level of detail of
statistical presentation here, as we are concerned only with a rough overall trend. However,
as the exact rankings of countries' performances can be a sensitive issue politically we refer
all readers to official detailed presentations of performance in the official PISA results publications (available at http://www.oecd.org/pages/0,3417,en_32252351_32235731_1_1_1_1_1,00.
html). M. Domazet, B. Baranović, J. Matić: Mathematics competence and international...
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Moreover, Croatian students rate mathematics among the least interesting and most
difficult school subjects. It requires a lot of their study time and effort, even though
they show awareness of its importance in both present and future life. Very few of
them are encouraged to learn more, and report being mainly passive recipients of
knowledge in mathematics lessons. Mathematics was ranked second of least liked
school subjects, mainly for being difficult and incomprehensible (Marušić, 2006.).
As stated above, also no radical changes in content and its presentation were introduced in the Croatian context, and the mathematics instruction is still primarily
teacher-centred. Although mean ranking in international assessment programmes
should not be the primary and only goal of mathematics instruction, given the
complex intersection of students' attitudes to the subject, teaching practices and curricular aims it is worth exploring the link between curriculum and instruction, and
acquisition of competence as tested by PISA (bearing in mind all its limitations and
failures, as stated above).
3. Teaching for Competence
Not only is the development of competence a recently trendy expressed goal of
schooling, it is also a conventional human need for those earnestly engaged in a
learning process. It includes understanding when, where and how to achieve certain outcomes, and feeling efficacious in performing the actions leading to them. In
this way the notion of competence is closely tied to the theories of motivation for
studying mathematics, which in turn is closely connected to the way that the content is presented to the students. To support the acquisition of competence, rather
than mastery over set tasks, communication about mathematics between teachers
and students should be significantly altered from that used in traditional classrooms (Lampert, 1991.), the instruction that provides feedback, over and above the
evaluation of testing success, should be conducted, and methods which focus on
interactive, collaborative, and investigative teaching should be employed (Sriraman
and English, 2010.). Turner et al. (1998.) show that such practices are related to
increased intrinsic motivation to learn. Moreover, to make students themselves aware of the process of development of competence as part of motivation for learning,
teachers need to provide space for students to reflect on what they do and do not
understand, including explication of reasoning behind reaching certain correct and
incorrect answers. This in turn leads to increased conceptual thinking which provides a firmer foundation for the overall competence in mathematics beyond the set
school tasks (Kazemi and Stipek, 2001.). Facilitating students' cognitive restructuring
and conceptual reorganizations that precede the school mathematics competence
development is what constructivists see as one of the teachers' core responsibilities
(Cobb, 1988.).
Qualitative study by Wilson, Cooney and Stinson (2005.) examined mathematics
teachers' experiences and attitudes concerning how teaching for understanding should be designed. The interviewees offered some insights on their own and students'
prerequisites for competence development. Former refers to teachers' need for de-Sociologija i prostor, 51 (2013) 195 (1): 109-131

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veloping more efficient strategies for connecting and visualizing mathematics, as
well as assessing students' understanding (Wilson, Cooney and Stinson, 2005.). On
the other hand, some of the recognized strategies aimed to enhance students' comprehensiveness in learning mathematics are use of varied approaches for engaging
students in the learning process (precisely; ICT use, writing ideas, team-working,
applying mathematics, hands-on and laboratory activities etc.) (Wilson, Cooney and
Stinson, 2005.), encouraging hypothesis making plus building and defending student strategies of mathematical problem-solving (Lampert, 1991.) and discussing the
limitations of students' current methods together with finding alternatives (Cobb,
1988.). However, the overarching precondition for the successful process of competence development is a students' perception of free, safe, errors-allowing learning
environment in which they are inclined to actively participate. Alternatively, Voigt
(1985.) points out that if teachers impose methods, i.e. outline them as demands,
not suggestions, students are likely to run away from constructive learning activities
into finding whichever shorter, superficial way that leads to demonstration of mathematical behavior teachers expect (Cobb, 1988.; Schoenfeld, 1988.), not resulting in
profound understanding.
Evidently, in order to enhance the development of broadly considered mathematical
competence, various constructivist-based teaching methods should be deliberately
employed. However, if teaching strategies used in classrooms still largely fall under the scope of what is considered to be direct instruction (as it is evidenced by
numerous studies across educational settings, e.g. Malzahn, 2002; Baranović, 2006),
is it realistic to expect the development of broader, functional and 'realisitc' mathematical competence to be occurring and be apparent in the results of assessments?
As Halász and Michel report for the full spectrum of Key Competences, beyond the
normative curricular definition of the goals of mathematical instruction lies the more
difficult practical implementation resting on the change of professional behaviour
of teachers who already have deeply rooted instruction and assessment practices.
This requires classroom level innovations which still need to be implemented Europe-wide (Council of the European Union, 2010., as cited in Halász and Michel,
2011.:298). New ways of organising learning in an innovative school environment
remains a task to be completed in an implementation of the mathematical competence framework. But Halász and Michel's analysis admits that changing learning
environments is the most difficult implementation challenge. This poses system-level requirements for 'national educational innovation', but also massive investment
in teacher 'capacity development' (Halász and Michel, 2011.:303-304) so that they
become capable of adapting their classroom level practices to the educational goals
defined in the novel way.
One of the things this improvement involves, in connection with the requirement to
provide learning applicable in contexts outside classrooms, is the orientation on student-centred instruction. Depaepe, De Corte, and Verschaffel's (2007.) study particularly highlights that curricular and textbook innovation alone is insufficient to successfully implement the novel instruction approaches. Specifically, it is the
teachers
who play a very active role in the implementation of the curricular materials. They M. Domazet, B. Baranović, J. Matić: Mathematics competence and international...
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interpret, and this may be unconscious, the new instructions through their existing
and prior knowledge and set practice patterns. Even more importantly, the demands
of the aforementioned constructivist-based instruction may be too complex for the
teachers who have been trained in the traditional delivery of mathematical concepts
and standard examples for practice. Ensor (2001.) can be drawn on to highlight and
illustrate the importance of in-service training to raise awareness of the obstacles
expected based on the previous studies, as well as the specific preconceptions that
a group of teachers situated in a particular national context may have.
There is research (Erickson, 1999.) showing that problem-based learning (PBL) strategies in classroom can assist with the added value of competence outcome, as
opposed to bare knowledge and mathematical skills developed through traditional instruction. Applying mathematics to everyday life and complex interaction of
knowledge from multitude of domains requires students to readily combine the
mathematical conceptual knowledge with the problem solving and communication
skills, creative and critical thinking skills, as well as positive attitude and values
(Tamirzi et al., 2010.:4864). Innovative teaching methods, such as problem-based
learning, have on their own not been shown to have a significant influence on conventional mathematical assessment performance compared to traditional teaching,
but have made significant difference to the mental effort invested into the real-life
mathematics problems (Tarmizi, Tarmizi, Lojinina, and Mokhtara, 2010.). Needless
to say, problem based-learning is not the only instruction strategy to be applied in
competence-oriented classroom, nor is all mathematics curricular content suitable
for problem-based instruction.
Though this does not make them an ideal candidate for raising national PISA scores,
it suggests that they can play a role in producing learning outcomes more suited to
the competence-model. Other studies (Depaepe et al., 2007.; Silver and Stein, 1996.),
performed in a longer run of lessons, suggest that there is a significant difference
even in the test performance and post-test retention in favour of the problem-based
instruction. Even more importantly, problem-based learning led to more positive
beliefs about and attitudes towards learning and teaching mathematical problem
solving, and even greater readiness to apply the problem-solving skills gained in
real-life situations. These are all important factors in both students' motivation for
adoption of mathematical content and the type of educational outcome that international assessments like PISA aim for.
Problem-based or investigative learning is a teaching method in which students
are exposed to realistic, simulated real-life problems that are ill-structured and vague before they are exposed to new concepts and skills training covering the problem (Bridges and Hallinger, 1992.). Whilst a broader delineation of the scope and
structure of problem-based learning is beyond the scope of this text (cf. Hmelo-Silver, 2004.), the given definition is useful for conceptual and instructional comparisons drawn on in this text. The popularity of the teaching method grew internationally (Savin-Baden and Major, 2004.; De Corte and Verschaffel, 2006.) over the
last
few decades, as it raises students motivation to learn because the outcome is no
longer a successful completion of a school task ("for the sake of school", Tarmizi et Sociologija i prostor, 51 (2013) 195 (1): 109-131
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al., 2010.:4864) but the real life problems whose solution is meaningful outside the
school context. In that case the problem itself becomes the instrument of learning,
and through successful solution of the problem learning is inevitable (Culver, 2000.;
Kain, 2003.). This is opposed to the traditional mathematics instruction which is
focused on exercise, rules, concepts and mathematical relations to be learnt, but
which can easily turn out to be of limited use in unfamiliar real-life 'mathematical'
situations. Unlike in the problem-based learning situations, traditional instruction
does not provide much opportunity to develop one's own innate abilities and adapt
and change the methods applied to a task so as to fit new situations. Because of its
focus on group work and proving the successful solution of the problem, problembased learning fosters communication, graphical and visual presentation, modelling
and explicit group-reasoning.
Following Tarmizi et al. (2010.) we will take problem-based learning to consist of
the introduction of a real-life mathematics problem before being exposed to the
mathematical concept that was named as part of the learning outcome of the lesson.
Multiple paths to the problem solution are encouraged, and individual exploration
takes place. The students take to solving the problem through the group discussion
and the helpful notes from the teacher and examples from textbooks. This can also
include students' own mathematical formulation of the problem (as opposed to its
real-life description), predicting possible solutions, experimenting with different real-life situations and running their own student projects. Following the completion
of the task the students present their solution as a group, and the teacher in the end
summarises and reinforces the new concepts learnt.
The traditional teaching, on the other hand, is based on the teacher-centred instruction where students are led through introduction of new concepts and their use in
mathematical problem examples. The students most often copy this demonstration
from the blackboard. The students are then asked to practice similar questions as
the ones shown in examples by the teacher. They will also eventually be examined
on the questions similar to the ones practiced, through following tasks set in the
workbook or exercise book. Some of the practice can be done by individual students
on the blackboard, whilst others can follow and copy in their own notebooks. Part
of student learning may also be to complete the lesson by demonstrating ability to
publicly repeat definitions and formulas. The instruction itself is in this case seldom
interactive and most of the communication is one-way, with students as recipients.
In Croatia, recent research on the teaching methods, teaching materials and tools
shows (Baranović and Štibrić, 2011.) that the traditional approach to the mathematics teaching prevails in its many dimensions.
9
According to the teachers' own
responses, during mathematics lessons the Croatian students in compulsory edu-
9
We relate to the findings of the research described in the footnote 1. M. Domazet, B. Baranović, J. Matić: Mathematics competence and international...
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cation most often solve standard mathematical tasks. For instance, most teachers
reported
10
that their students predominantly solve tasks they have demonstrated
(93.5% of responses), practice mathematical problems similar to those in the exam
(87.3% of responses), solve standard tasks individually at their desks (84.0% of responses), work on formal tasks from workbooks or textbooks (82.0% of responses).
On the other hand, the teachers reported that their students are least engaged in
independent exploration of mathematical concepts and patterns in the classroom
(5.9% of responses), application of mathematical concepts and patterns outside of
the classroom (10.9% of responses), explanation of their ideas and reasoning about
mathematics topics in written form (13.1% of responses), performing their own student projects (14.0% of responses), articulation of their own mathematical problems
(14.8% of responses), etc.
The use of teaching methods reflects this traditional approach as the teachers most
often use traditional teaching materials and tools while they rarely use contemporary
ICT and visualisation tools. More than 90% of teachers stated that they use geometric
demonstration equipment (97%) and students' textbooks (91%) often and very often.
A great number of them also use printed task-sheets and workbooks (90%) and so
called 'mathematical manipulatives' (84%). But only one third of the teachers use the
Internet, one fifth use mathematics software (e.g. Geogebra, Geometar - Skechpad)
and 16% of teachers use the mass media (e.g. newspaper, TV) as sources or aides.
With, for example investigative or problem-based, learning's connection to the
everyday application of mathematical knowledge and post-test retention of knowledge, Croatia's predominantly traditional teaching methods and tools seem out of sync
with requirements for development of mathematical competence. Despite teachers'
appropriation of the novel goals of application of mathematics to everyday life,
their teaching methods still reflect the deeper attachment to the traditional aim of
the transmission of abstract mathematical knowledge. On the other hand, traditional
instruction and focus on 'practice-makes-perfect' drilling instruction does not on its
own lead to increase in performance at international standardized testing for a broad
spectrum of students. Given the shift to competence-oriented aims of mathematics
instruction, a more arduous task of changing teaching practices remains (Halász and
Michel, 2011. above). As Julie (2006.) warns, mathematical literacy is more difficult
to teach than standard school mathematics.
In conclusion, having shown that traditionally framed concept of mathematics was
dominant in mathematics curricula and teachers' perception until very recently, as
well as the prevalence of traditional components in the mathematics instruction in
10
For the purpose of measuring the frequency of the students' activities during the mathematics lessons a scale consisting of 21 items (students activities) was developed. The items
were set on 5-grade Likert scale with possible answers "never" (1), very rarely (2), sometimes
(3), often (4) and "very often" (5). The percentages of teachers' responses: „often" and „very
often" are presented in the text.Sociologija i prostor, 51 (2013) 195 (1): 109-131
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compulsory education in Croatia, we conclude that enacted/implemented mathematics curricula should be substantially changed if we want to improve mathematical
literacy or competence of Croatian students. In line with the international trends and
recent Croatian curricular changes it is laudable that teachers have largely adopted
the view of mathematics as applicable to students' everyday life and potential for
lifelong learning, but this remains to be developed through curricular concepts to
operational level.
Beyond the laudable aims, most teachers still see set-task completion and concept
adoption as the main forms of mathematical learning. Against the recent trends for,
and early calls in 1991. curricular description of mathematics, Croatian teachers still
largely employ teacher-centred traditional instructional methods. Though these can
lead to high testing scores, as exemplified by the East Asian economies, on their own they are insufficient to lead to the broader competence-framed outcomes
of learning. With PISA results unimpressive and the perceived difficulty of school
mathematics, changes to pre-service education and in-service teacher training are
recommended, so as to include detailed understanding of the new aims of mathematics instruction and the methods of achieving those.
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Matematička kompetencija i međunarodna testiranja matematike: hrvatsko
polazište

Sažetak
Članak donosi pregled koncipiranja obveznog matematičkog obrazovanja kroz kompetencijske
ishode te prikladnih metoda poučavanja. Takav pregled daje se u svjetlu međunarodnih
utjecaja na koncipiranje matematičkog obrazovanja u Hrvatskoj kroz rangiranje matematičke
pismenosti 15-godišnjaka u PISA testiranju. Na početku se obrazlažu konceptualni konstrukti
u pozadini društvenih utjecaja na nad-državni razvoj pojma pismenosti/kompetentnosti u
obrazovanju i testiranju iz matematike. U tom svjetlu pregledavaju se prošla i sadašnja, te
predlažu moguća buduća, određenja i promjene školskog matematičkog obrazovanja i procesa
poučavanja matematike u hrvatskim osnovnim školama. Pregled empirijskih istraživanja
praksi i stavova o matematičkom obrazovanju u posljednjih nekoliko godina komplement je
pregledu teoretskih određenja. Cilj je istražiti i prikazati društvene i (komparativne) političke
utjecaje na sadržaj i poučavanje 'čvrstih' školskih predmeta kao što je matematika, obilježiti
trenutnu hrvatsku poziciju po međunarodnim testiranjima i empirijskim istraživanjima stavova
učenika i nastavnika, te preporučiti moguće neposredne korake u razvoju obrazovne politike
u području matematičkog obrazovanja.
Ključne riječi: PISA testiranje, kompetencije, pismenost, matematika, poučavanje, učenje kroz
rješavanje problema, obvezno obrazovanje u Hrvatskoj.

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21. Design research in mathematics
education: The case of an ICT-rich
learning arrangement for the
concept of function
1
Michiel Doorman, Paul Drijvers, Koeno Gravemeijer, Peter Boon, & Helen Reed
Abstract
The concept of function is a central but difficult topic in secondary school mathematics curricula,
which encompasses a transition from an operational to a structural view. The question in this
paper is how to design and evaluate a technology-rich learning arrangement that may foster this
transition. With domain-specific pedagogical knowledge on the learning of function as a starting
point, and the notions of emergent modeling and instrumentation as design heuristics, such a
learning arrangement was designed for grade 8 students and field tested. The results suggest
that these design heuristics provide fruitful guidelines for the design of both a hypothetical
learning trajectory and concrete tasks, and can be generalized to other design processes.
1. Introduction
The function concept is a central but difficult topic in secondary school mathematics curricula
(Akkus, Hand, & Seymour, 2008; Ponce, 2007). Functions have different facets, and to make
students perceive these as facets of the same mathematical concept is a pedagogical and
didactical challenge. In lower secondary grades, functions mainly have an operational
character, and are calculation 'engines' that process input values into output values. In higher
grades, functions have the character of an object with various properties. The transition from
functions as calculation operations to functions as objects is fundamental for conceptual
understanding in mathematics.
Can computer tools help to foster this transition? Computer tools offer opportunities for
mediating the learning activities in which students engage (Sfard & McClain, 2002). The
availability of sophisticated computer tools for mathematics education, however, also raises
questions. Representations (e.g., formulas and graphs) and techniques (e.g., rewriting
equations) in such computer environments may signify mathematical concepts that are still to be
constructed by the students. As experts, we see the mathematics in the tool use, but does the
learner see this too? This inherent circularity is known as the learning paradox (Bereiter, 1985;
Gravemeijer, Lehrer, Van Oers, & Verschaffel, 2002; Van den Heuvel-Panhuizen, 2003).
To design a learning arrangement that avoids the learning paradox in using ICT-tools and
fosters the learning of the concept of function - and the transition from an operational to a
structural view in particular - is the central issue in this chapter. In a design research study we
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P., Gravemeijer, K., Boon, P., & Reed, H. (2012). Tool use and the development of the function concept:

from repeated calculations to functional thinking. International Journal of Science and Mathematics Education, 10(6), 1243-1267 428
investigate the question of how a learning arrangement with computer tools can foster the transition from an operational to a structural understanding of functions.
2. Theoretical framework
The theoretical framework that guides the design in this study includes (1) domain-specific knowledge on the concept of mathematical function; (2) the notion of emergent modeling; (3) theories on tool use and its instrumentation.
The concept of mathematical function
The teaching and learning of the concept of mathematical function is a widely researched topic (e.g., see; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Ainley, Bills, & Wilson, 2005; Oehrtman, Carlson, & Thompson, 2008). A key issue in these studies is what Sfard calls the two sided nature of functions (Sfard, 1991, 1992). Her theory on the dual nature of mathematical conceptions identifies operational and structural concepts, the first concerning mathematical processes and the latter mathematical objects. In Sfard's view, the operational and structural conceptions are complementary. Historically, the operational aspect preceded the structural aspect, and the same might hold for individual learning processes, because the structural approach is more abstract than the operational. Dichotomies related to Sfard's operational and structural aspects have been brought afore by other researchers such as Dubinsky (1991) and Tall and Thomas (1991).
Based on an analysis of the available literature on this issue of operational conceptions preceding structural conceptions, the following sequence of three interrelated aspects of the function concept are distinguished:
1. The function as an input-output assignment
The function as an input-output assignment that guides the stepwise calculation of an output value for a given input value. For example, we can think of a function that converts an amount of dollars into an amount of euros, or a temperature in degrees Fahrenheit into degrees Celsius. Often, this view on function is seen as the starting point for students. An appropriate symbolic representation for this function view is an input-calculation-output chain. Figure 1 shows such a chain for calculation temperature in degrees Celsius from temperature in degrees Fahrenheit.
Figure 1: Input-output chain for temperature conversion
2. The function as a dynamic process of co-variation
The second aspect of the function concept concerns the notion that an independent variable, while running through its domain, causes the dependent variable to move through a set of possible outcome values. The dependent variable co-varies with the independent. Helpful representations for studying co-variation are tables and graphs, which can be scrolled through or traced. Figure 2 shows such a table for the temperature conversion example. 429
Figure 2: Table for temperature conversion
3. The function as a mathematical object
A function is a mathematical object which can be represented in different ways, such as arrow chains, tables, graphs, formulas, phrases, each of which providing a different view on the same object. The more structural view on functions includes families of functions, function comparison, and later on function differentiation or integration. For the object view on function, it is important to be able to see the connections between the different function representations. Figure 3 combines different representations for the temperature conversion example.
Figure 3: The different representations of the temperature conversion function
These three aspects of function reflect the operational- structural dimension as identified by Sfard (1991): the function as an input - output assignment reflects an operational conception and the function as a mathematical object involves a structural view, whereas the function as a dynamic co-variation process reflects an intermediate conception. 430
In line with this conceptual analysis, which is further elaborated in Doorman, Drijvers, Gravemeijer, Reed, and Boon (2012, see also Gravemeijer, Doorman & Drijvers, 2010), we want to design a learning arrangement for the topic of function that fosters the transition from an input-output conception to a more versatile view including a structural conception. This is the first design criterion. In the envisioned learning arrangement we implement computer tools with representations such as arrows, tables, formulas and graphs that initially may have hardly any meaning for students. The challenge for us - as designers - is to find ways in which these representations become useful models for the students. The notion of emergent modeling offers an orientation for addressing this challenge.
Emergent modeling
Emergent modeling is a design heuristic which originated in the framework of the theory of realistic mathematics education (RME) (Gravemeijer, 1999, 2007). RME builds on Freudenthal's (1991) image of mathematics as 'a human activity': while engaging in mathematics as an activity, students should be supported in re-creating or reinventing mathematics. The emergent modeling design heuristic is a means for organizing such a reinvention process. The starting point is an activity of modeling problems that are experientially real to the students. Models initially are context-specific and refer to realistic or paradigmatic situations. Then, while the students gather more experience with similar problems, attention shifts towards the mathematical relations involved. This enables students to use the model in a different manner: it derives its meaning from mathematical relations and becomes a base for more formal mathematical reasoning: a model ofinformal mathematical activity develops into a model for more formal mathematical reasoning (e.g., Gravemeijer, 1999; Cobb, 2002; Doorman & Gravemeijer, 2009). The learning paradox described in the introduction can be circumvented by designing a chain of activities that allow for a process of emergent modeling that fosters a thinking process in which symbolizations and meaning co-evolve (Cobb, 2002; Meira, 1995).
This approach is consistent with conceptual change literature, which argues that students must first explore a domain and recognize limitations or a need for more sophisticated tools, before they are ready to learn complex concepts (Lehrer & Schauble, 2002).
In this study, the notion of emergent modeling is an important design heuristic. It stresses the importance of starting with problems that offer opportunities to develop situation-specific reasoning and tentative representations for organizing repeated calculations and that have the potential to develop into more sophisticated - mathematical - tools and concepts (Gravemeijer, 2007). Taking this process of emerging models seriously asks for a design in which models for emerge in a natural way from models of, that are rooted in suitable contexts. This is a second design challenge for the learning arrangement.
Tools and instrumentation
Emergent modeling as a design heuristic does not particularly focus on the use of computer tools and the relationship between computer techniques, paper-and-pencil techniques, and conceptual understanding. Drawing upon Vygotsky's view on the dialectic relation between tool use and cultural practices, we consider it important to understand and carefully plan the role of tools in a learning process (Vygotsky, 1986; Wertsch, 1998). As Hoyles and Noss (2003) point out, tool characteristics do matter in the sense that their visualizations and the available techniques affect student learning.
Instrumentation theory was developed to address the problems that may arise when one starts to use a ready-made computer tool and explains the importance of aligning techniques that emerge in contextual problems with the techniques available in the computer tool. The theory focuses on the mediating role of tools by stressing the co-emergence of tool techniques and 431
meaning in a process of instrumental genesis (Artigue, 2002; Trouche, 2004; Drijvers & Trouche, 2008). Instrumental genesis comprises the development of cognitive schemes containing conceptual understanding and techniques for using a tool for a specific type of task. The resulting instrument integrates the tool and mental schemes. A bilateral relationship between the tool and the user exists: while the student's knowledge guides the way the tool is used and in a sense shapes the tool--the affordances and constraints of the tool influence the student's problem-solving strategies and the corresponding emergent conceptions.
In the case of functions, computer tools offer opportunities to dynamically and flexibly deal with different representations such as tables, graphs, and formulas. This may help students to overcome the difficulty of construing and integrating the operational and structural aspects of the function concept and its different representations. Computer tools can support students in exploring dependency relationships and investigating the dynamics of co-variation. The third design challenge, therefore, is to shape these opportunities, while ensuring that the techniques in the computer tool match with the targeted conceptual development.
3. Design
The study aims at designing and evaluating an ICT-rich learning arrangement for grade 8 students (age group 13-14 year) in the Netherlands,
1. which fosters the transition from an input-output conception to a structural conception
2. in which 'models for' emerge in a natural way from 'models of', that are rooted in suitable starting points
3. and which benefits from the opportunities ICT offers, while ensuring that the techniques in the computer tool match with the targeted conceptual development.
As a first design step, a hypothetical learning trajectory (Simon, 1995) was designed based on the theoretical framework and the literature review. In this trajectory description, problem situations, models, ICT techniques and targeted conceptual development were summarized and outlined.
Next, the design research was carried out in a cyclic process of three design cycles, one pilot and two full cycles, each lasting for about one year. The learning arrangement includes (a) a computer tool called AlgebraArrows, (b) a student textbook with both paper and pencil and computer tasks, (c) a teacher guide describing the various activities and their possible orchestrations, and (d) a computer- and written test
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The computer tool is an applet called AlgebraArrows, which allows for the construction and use of chains of operations (so-called arrow chains) and provides options for creating tables, graphs and formulae and for scrolling and tracing. The applet is embedded in an electronic learning environment in which tasks are provided and through which the teacher can monitor students' responses (Boon, 2009). The AlgebraArrows window is embedded in the learning environment (see Figure 4). Figures 1, 2 and 3 were also made with this applet.
AlgebraArrows is meant to support the transition from operational understanding to structural understanding the concept of function in the following manner. First, students construct inputoutput chains of operations to carry out calculations. The chains can be applied to single numerical values as well as to variables. Gradually, the student activities focus on the investigation of dynamic input-output dependencies. The computer tool's slider bar and table
2
The trajectory description and the learning arrangement are available through http://www.fisme.science.uu.nl/tooluse/en/ . The digital part of it can be found at http://www.fisme.uu.nl/dwo/demo/en/ 432
scrolling options, in conjunction with the tasks, invite the students' development of a dynamic notion of a variable (see Figure 4). Finally, the students investigate families of functions, with their properties and representations, to develop a structural view of function. For instance, the table representation is initially understood as a tool for organizing corresponding input-output values in a contextual problem on mobile phone offers. Next, the applet's table option can be used as a tool for scrolling and zooming in on or out of these values, e.g., to analyze growth behavior or to find break-even points. This imagery of table and actions initially signifies repeated calculations and emerges into a model for a structural understanding of dependency relationships. In practice, these design considerations led to changes in the applet, such as the addition of zooming tools in the table option, similar to the zooming tools in graphs.
The tool techniques are meant to relate to paper-and-pencil techniques, and as such instrumental genesis should be natural and should contribute to the targeted cognitive development.
Figure 4: The computer tool AlgebraArrows embedded in the learning environment
The student textbook describes the activities for eight lessons. It starts with three preliminary paper-and-pencil group activities to create an exploratory orientation on the topic, to create the need for organizing series of calculations and to introduce the computer tool. In addition, these activities are designed to avoid discrepancies between tool techniques and conventional paperand-pencil techniques and to anticipate instrumental genesis through the use of representations and techniques that will be used in the computer tool. The three successive activities are:
• Calculating the area of a flexible quadrilateral, which students explore using a paper model, and which then leads to the introduction of a variable (the height of a parallelogram) and a calculation procedure or a formula for the area.
• Comparing two cell phone offers, with a focus on break-even points which creates the need for a function-concept (Ainley, Bills, & Wilson, 2005; Küchemann, 1981). 433
• Calculating the braking distance of a scooter for various speeds. It includes a tablerepresentation and is intended to create the need for graphing a trend for doing predictions.
During a classroom discussion of students' solutions orchestrated by the teacher (see below), an arrow chain is supposed to emerge as a useful way to investigate input-output relationships.
The arrangement continues with two computer lessons. During the computer activities, the focus shifts from solving specific situations towards the investigation of dependency relationships by means of break-even problems that ask for the generation and comparison of tables of input- and output values, and moving around in a space of possible values. According to the emergent modeling heuristic, a shift is expected to take place from sub-models that signify repeated input-output calculations to models signifying dependency-relationships, covariation and function object properties. The arrangement continues with a reflective lesson in which whole-class discussions and demonstrations align computer techniques with paper-andpencil techniques, a computer lesson with applications, and a closing lesson for summarizing results and creating consensus on representations and functional reasoning.
The teacher guide describes the different activities and their possible orchestrations (Drijvers, Doorman, Boon, Reed, & Gravemeijer, 2010). It suggests different classroom arrangements, such as small group work, poster presentations, computer activities in pairs, and whole class discussions on the results, supported by projections of computer work (Stein, Engle, Smith, & Hughes, 2008). This variety of classroom arrangements is expected to foster social interaction, to promote the articulation of tool use and to enhance reflection and generalization.
The final part of the learning arrangement consists of a computer test and a written test. This mixed media assessment reflects the arrangement as a whole. The written test captures transfer of learning from computer tool to conventional methods and contains questions about break-even points, and computer tasks where students could use the applet AlgebraArrows
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Test items in the two tests are comparable in length and difficulty and have similar scoring instructions.
4. Research methods
To evaluate the design and to understand why the particular instructional setting potentially supports learning, design research was carried out (Gravemeijer & Cobb, 2006, 2013). The main hypothesis to be investigated is that the learning environment indeed fosters a transition from an operational view to a structural view, as reflected in the students' reasoning and in their way of using the computer tool. We briefly now describe the teaching experiments and data collection, and the data analysis.
Teaching experiments and data collection
After a first small-scale pilot study in one class, two successive teaching experiments were conducted in grade 8, with mid and high-achieving 13-14 year old students, for investigating their learning processes in relation to the learning arrangement. Each round of experiments took eight 50-minute lessons. The first was conducted with three classes from three different schools. The second was conducted on a larger scale. In this paper, we focus on screencast video data from pairs of students in the first teaching experiment. These students were selected by their teachers based upon criteria we provided: an average level of mathematical performance, a communicative attitude, i.e., willing to explain their reasoning and solution

strategies, and audible articulation. The qualitative findings are triangulated with quantitative
3
The full computer test and written test can be found at http://www.fisme.uu.nl/tooluse/en/ 434
posttest data from 155 students from five classes from two different schools in the second
experiment, for which the most complete data were available.
In both teaching experiments, whole class teaching sessions, group work and work in pairs
were videotaped in two of the participating classes and, in each of these classes, screencastaudio videos of two pairs of students working with the computer tool were collected. In addition,
students' answers and results to the written and the computer test administered at the end of
the experiment were collected.
Between the teaching experiments, the learning arrangement was improved. The overall outline
of the three different aspects of the function concept was not changed but fine-tuning the modelofto model-fordevelopment and the relationship between tool techniques, paper-and-pencil
work and mathematical thinking was needed. In order to better investigate the development of
the function concept, we placed a stronger emphasis on break-even points as a motivation for
investigating the dynamics of relationships. In addition, to establish to what extent the students'
final performance depended on the use of the available computer tool and to assess the
transfer of learning, both a written test and a computer test were added to the end of the
learning arrangement.
Analysis
The data analyses started with organization, annotation and description of the data with
software for qualitative data analysis (ATLAS.ti; see van Nes & Doorman, 2010). Initially, the
tasks in the learning arrangement served as the unit of analysis. Factual codes (task number,
student names, et cetera) were used to organize and document the data. Events in class videos
and screencast videos that were notable from the perspective of the research question were
transcribed and discussed in the research team.
In analyzing the first teaching experiment, we constructed a storyline as a reconstruction of the
students' learning process. The qualitative data sources were discussed with two external
experts to identify illuminating examples of the resultant learning process. This discussion
resulted in recognizing a shift in the students' reasoning with functions. Next, illuminating and
representative examples for important steps in the shift were identified. A code book was set up.
Next, we distinguished solution strategies used on two similar computer tasks on break-even
points that were designed for studying the development of students' reasoning with the
computer tool. One task was situated at the beginning of the computer lessons and the other at
the end. The strategies led to the construction of codes with respect to the use of
representations in the computer tool. These codes reflected the different aspects of the function
concepts, the types of tool use, and the position in the hypothetical learning trajectory. In a later
phase, these codes were also used for the quantitative analyses of students' answers to the
paper-and-pencil activity concerning the arrow chain (the booklets of 5 students were missing),
the screenshots of the answers to the two computer tasks, and the screenshots of students'
final work on the two computer tasks. A second researcher coded 55 out of 306 items (18%).
Good inter-rater-reliability was achieved (Cohen's Kappa = 0.79). A paired t-test was used to
compare the results on the written test and the computer test.
5. Results
The results of the study are presented in the form of a storyline of students' learning process in
relation with the learning environment, as reflected in their reasoning and in their way of using
the computer tool. This storyline offers opportunities to trace the development of our
interpretations and understanding of these learning processes. This approach is linked to
Smaling's (1987) view on 'trackability' and to Freudenthal's notion of 'justification' (Freudenthal, 435
1991). The storyline is illustrated with examples of student work and is also supported by the
quantitative findings.
The initial cell phone offer task: organizing calculations
The results of the initial open-ended group activities of the first teaching experiment show a
variety of solution strategies. In Figure 5 (top) the cell phone offer task is shown. The students'
posters below are illustrative for their attempts to organize the situations mathematically, i.e., to
organize repeated calculations, construct variables and use various representations. In the
poster on the left, students organize their repeated calculations by systematically writing them in
a list, which resembles an input-output relationship. This helped them to see the pattern in the
calculation, and to apply this pattern to a new input value. In the right poster, students use
formulae to describe their repeated calculations. Although not in conventional form, the
formulae show the identification of the two variables of the dependency relationship (m for
minutes and b for costs). The repeated calculations reflect how the operational aspects precede
the structural aspects of functions (Sfard, 1991). From an emergent modelling perspective, the
context apparently provides a suitable starting point.
Figure 5: Cell phone offer task and studentsíposter
Introducing arrow chains: visualize functions
In a whole-class discussion, the posters helped the teacher to evoke the need for determining
variables, dependency relationships and more efficient notations and tools for finding breakeven points. The teacher exploited the students' strategies by linking their initial ideas to the
intended tool use. A new mathematical goal emerged: repeated calculations are timeconsuming, what is the general pattern in the calculation procedures? How can this pattern be
described to calculate results 'automatically'? How could a 'calculator' help in comparing cell
phone offers? A calculation recipe comprising a fixed chain of operations emerged. This arrow
chain is the central representation of the computer tool. In the last activity before the computer
lessons, the students were asked to draw calculation chains for the two cell phone offers.
Figure 6 shows an illustrative answer to this task. 436
Figure 6: Written calculation chains for cell phone offers (called 'Soms' and 'Vaak').
The arrow chain as a model of repeated calculations appeared to be grounded in students'
previous activities and could begin to function as a mathematical model for (in terms of the
theory of emergent modeling) reasoning about dependency relationships. This assumption is
backed up with an analysis of the results of student booklets from the second experiment. The
analysis shows that 124 out of 150 students (83%) drew a similar chain while another 8 (5%)
drew this chain, but used it only for one specific calculation and did not label input and output
boxes. From an instrumental genesis perspective, the tool techniques were prepared for by
paper-and-pencil work.
Using the computer tool to create arrow chains
The third lesson - the first computer lesson - started with some introductory activities and then
continued with the cell phone problem (Figures 5 and 6). The task was to determine, with the
computer tool at hand, when it is advisable to change from one offer to the other. Two students,
Lily and Rosy, worked together on this task. After reading the problem on the computer screen,
they started to construct arrow chains with the computer tool.
[Rosy drags an input box into the drawing area. The box is connected to the operation - 80 and
that one is connected to x 0.15.]
Lily: And that added to the fixed costs.
[R agrees and connects the chain to + 22.5. Finally, they connect the chain to an output box.]
L: Well, when you phone for 100 minutes....
[enters 100 as input in both chains. TomSeldom (Soms) is cheaper than the other offer.].
R: .... Well, maybe 50.
[L enters 50 as input for both chains. They look at the results and are still not satisfied. They try
some more input values. Finally, they enter 200 in the input box of both chains. For the first time
TomOften (Vaak) is cheaper than the other offer. They are satisfied with this result and proceed
to the next task.]
The vignette illustrates how the arrow chain has become a means to organize the situation and
the calculation procedure. Lily and Rosy built the chain from the input box, adding operations
and finally connecting an output box. This construction signifies their previous calculations.
They focused on specific cases by entering respectively 100, 50 ... and finally 200 for
comparing the differences between the two offers. The arrow chain is used for repeated
calculations. The tool supports the construction of these chains, and the chain becomes a
means for analyzing and discussing relationships and successive operations, as a sequel to
merely solving repeated calculations. The tool technique clearly is linked to the students' paperand-pencil experience and supports their thinking. 437
Whole-class discussion capitalizing on hands-on experience
The start of the second computer lesson (the fourth in the learning arrangement), consisted of a
teacher-guided classroom discussion of students' computer work with the computer tool
connected to a data projector. Topics of discussion were the possibility to label input and output
boxes and to use tables. These tool techniques were already used by some of the students, and
now became 'institutionalized' through the whole class discussion. The teacher started the
discussion with an arrow chain for one of the cell phone companies:
Teacher: I heard different ways to find out how much I have to pay, how the amount changes,
how can you demonstrate that? [Silence] For example, if I call 10 minutes. How can I find out
how much I pay for 10 minutes?
Student 1: Put 10 in the input box.
Teacher enters 10 into the input box. This gives an odd result (first 30 free minutes are
subtracted). The result is discussed and the teacher varies the input by entering some more
values.
Teacher: Suppose I want to know the output from many input values, what more can I use?
Student 1: The table.
Teacher clicks the table tool and shows how you can scroll through the input and output values.
Teacher: Does anyone know another way to show how the output values change for different
input values?
Student 2: With a graph...
After this suggestion the teacher opened the graph tool. She demonstrated how to connect an
arrow chain to the graph window. Together with the students she investigated the options to
trace a graph and to zoom in and out. In this way, while demonstrating the tool techniques, the
teacher discussed the dependency relationships and the ways tables and graphs can be used
to analyze their dynamics. In the previous computer activities, Lily and Rosy did sometimes click
for a table or graph, but had not used it for scrolling or tracing values.
This observation shows how the teacher used the computer tool to create whole-class
consensus on how to use it for investigating dependency relationships. Both the techniques and
related concepts were part of the discussion. She did not show how to use these features for
finding break-even points. That was still a task for the students.
A different view on function
During the third computer lesson (the sixth lesson of the learning arrangement), we observed
that the strategy of Lily and Rosy for using the computer tool had changed while solving a task
on two offers by contractors called Pieters and Tweehoog (Figure 7).
Figure 7: Handyman task offers
The task in Figure 7 is quite similar to the cell phone activity (Figure 5). However, the way in
which these students analyzed the problem situation, phrased the structure of the solution
procedure and used the representations in the tool changed: AlgebraArrows was now used to
To get some jobs done in the house we can choose from two contractors:
Contractor "Pieters" charges 92 start costs and an hourly rate of 30.
Contractor "Tweehoog" charges 45 start costs and an hourly rate of 32,75.
You have a job of 9 hours. How much cheaper is Tweehoog than Pieters?
After how many hours of work is contractor Pieters cheaper than Tweehoog? 438
investigate the dynamics of relationships, rather than for case-by-case calculations. Figure 8
shows some subsequent phases in these students' work.
Figure 8: Different phases in students' work
First, the students identified the variables by creating labeled input- and output boxes for the
first arrow chain, with 'uurloon' meaning hourly rate and 'Kosten' for costs (Figure 8 top). Next,
they filled in the operations and completed the chain (Figure 8 middle). Then they constructed
the second chain for the Tweehoog company in similar way. Finally, they added tables, and
scrolled through them to find the break-even point for 18 hours (Figure 8 bottom part). This
behavior illustrates a different view on, and use of, the arrow chain compared to the initial
operational left-to-right construction of a chain for repeated calculations. More specifically, the
vignette illustrates how the arrow chain had become a tool for analyzing dependency
relationships. Lily and Rosy organized the dependency relationship by identifying, positioning
and labeling boxes for input and output variables and filling the gap between them with
operations. This construction signifies an understanding of the problem as a question about two
dependency relationships. For investigating the dynamics, they successfully operated the table
tool to zoom in and out for finding the break-even point. Corresponding to this new view on
function, the technique of constructing an arrow chain changed, now starting with labeled input
and output boxes rather than with input and then operations. This suggests a structural view on
function. 439
Triangulating case study findings with computer output
The second teaching experiment was used to quantitatively verify the hypothesis on the
transformation of the view on function during the computer activities. We found that for the initial
task 130 out of 155 students (84%) used the tool only for calculating successive input-output
values, while in the task at the end, 89 out of 152 students (58%) used the tool for structuring
and investigating the dynamics of the relationships.
Whereas the students' initial technique with the applet suggested an operational view on
functions, this more advanced technique reflects a transition towards a gradually developing
structural view. In other words, the shift in the use of the tool reflects the development from
viewing calculation recipes as tools for repeated calculations (as processes) towards reasoning
with structural characteristics of dependency relationships (as objects).
To investigate whether this transition also encompassed students' written work during the eight
lessons, we compared the scores of the computer test and the written final test. A paired t-test
showed no significant difference (p=0.20) between the final scores on the written test and the
computer test. The Pearson correlation coefficient of 0.38 between these scores was moderate
(p=0.001). This suggests that students who improved their reasoning through the described shift
evoked by the use of the computer tool were able to transfer learning to a more conventional
paper-and-pencil test.
6. Conclusion and discussion
Conclusion
In the introduction, we raised the question of how a learning arrangement with computer tools
can foster the transition from an operational understanding to a structural understanding of
functions. From the data analysis, we see the following characteristics of the learning
arrangement as decisive for fostering a transition from a calculation understanding to working
with correspondence and co-variation:
• The three initial open-ended problems and the poster activity helped students in coming to
see input-output structures in problems about dependency relationships and to use
representations to explore them.
• By enabling students to design and use arrow chains as a means of support for reasoning
about calculation procedure that bear meaning in everyday-life phenomena, the computer
tool supported the students in developing the notion of a chain of operations.
• By generating the results of a series of calculations for a variety of input values, the
computer tool strengthened the students' notion of a function as a calculation procedure
that transforms input values into output values.
• By generating output values for series of input values, by generating tables of input and
output values, and by enabling the students to move up and down the values in these
tables, the computer tool supported the students in developing a dynamic notion of a
variable that can move in a space of possible values, and the corresponding idea of covariation.
• By displaying arrow chains, tables and graphs the computer tool offers representations,
which the students could construe as affordances to start treating functions as objects
before they had become objects for them. In this way, the learning paradox may be
circumvented.
• By discussing students' work with the tool and showing specific features that some of the
students discovered, the teacher appeared successful in supporting most students in using

the table representation. 440
We conclude that the learning arrangement with a computer tool helped students to overcome the difficulty of integrating operational and structural aspects of the concept function and supported explorative activities for investigating the dynamics of co-variation, even though the students didn't reach the full level of "function as a mathematical object" but just began to make the transition towards this level of understanding. Further advances in this direction would encompass, for example, more different types of functions and operations on functions such as composition and multiplication.
Discussion
Before discussing the design heuristics of this study, we first address its limitations. The sequence of eight lessons fostered conceptual development in the domain of functions. However, the final test data do not give insight in whether all students made a similar step towards a dual conceptualization, with procedural and structural views, of functions. Students' work during the computer lessons revealed that the final screenshots of an activity sometimes are only approximate representations of the students' reasoning. Students can be close to a good answer and then delete everything as a result of a sudden doubt. Final screenshots don't capture this entire reasoning process. Additional process information is needed for a full understanding of students' conceptual development in relation to their tool techniques, although a balance must be found between the extensiveness and the manageability of the data collection.
As a final issue, we mention the generalizability from design-based case studies (Yin, 2003). We agree with Plomp and Nieveen (2009) that the results of design research have an analytical generalizability and a replication logic: the findings in this case may not be directly generalizable, but the design experiment may be treated as a paradigm case (Gravemeijer & Cobb, 2006, see also this volume), which offers domain-specific design heuristics that can be extrapolated to similar design studies. The extensive description of the interventions in Doorman, Drijvers, Gravemeijer, Boon, & Reed (2012) is intended to enable researchers to reenact the experiment in other settings as a way to evaluate our findings and to contribute to the development of a more comprehensive theory.
In retrospect, how do we view the three theoretically based design heuristics described in section 2? First, we used domain-specific theories on the acquisition of the concept of function, and on the transition from an operational to a structural conception in particular. This design heuristic was fruitful for outlining the learning arrangement. It had an important impact on setting up a hypothetical learning trajectory and on the design of the tasks. This particular design heuristic can be generalized into an overarching one, which is that - before engaging in design - a deep analysis of the topic is needed to be able to identify its learning obstacles and pedagogical challenges and to outline a learning trajectory.
A second design heuristic was the notion of emergent modeling, and its focus on the shift from model-ofto model-for. This heuristic not only guided the design of the learning arrangement, but the students' cognitive development, and their ways of using the tool initially as a tool for calculations and later as a tool for reasoning with function, reflects this shift. As such, this was a suitable design heuristic.
Third, instrumentation theory focuses on the interrelated development of techniques and concepts. As a design heuristic, it fostered a careful integration of computer activities and paper-and-pencil activities, which prevented discrepancies between tool techniques and conventional methods. The process of instrumental genesis was enhanced by the alternation between classroom discussion, small group activities and computer activities.
Altogether, while outlining the hypothetical learning trajectory and designing the tasks the three design heuristics provided important and concrete guidelines. Having a set of such design 441 heuristics is indispensable, but in the meanwhile does not guarantee a successful design: it is not just knowledge of appropriate design heuristics, but their application in different domainspecific situations that forms the heart of the 'art' of the designer.
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MICHIEL DOORMAN, PAUL DRIJVERS, KOENO GRAVEMEIJER, PETER BOON
and HELEN REED
TOOL USE AND THE DEVELOPMENT OF THE FUNCTION
CONCEPT: FROM REPEATED CALCULATIONS TO FUNCTIONAL
THINKING
Received: 17 June 2011; Accepted: 30 January 2012
ABSTRACT. The concept of function is a central but difficult topic in secondary school
mathematics curricula, which encompasses a transition from an operational to a structural
view. The question in this paper is how the use of computer tools may foster this
transition. With domain-specific pedagogical knowledge on the learning of function as a
point of departure and the notions of emergent modeling and instrumentation as design
heuristics, a potentially rich technology-intensive learning arrangement for grade
8 students was designed and field-tested. The results suggest that the relationship
between tool use and conceptual development benefits from preliminary activities, from
tools offering representations that allow for progressively increasing levels of reasoning,
and from intertwinement with paper-and-pencil work.
KEY WORDS: emergent modeling, function concept, instrumentation, mathematics
education, technology
INTRODUCTION
The function concept is a central but difficult topic in secondary school
mathematics curricula (Akkus, Hand & Seymour, 2008; Ponce, 2007).
Functions have different faces, and to make students perceive these as
faces of the same mathematical concept is a pedagogical challenge. In
lower secondary grades, functions mainly have an operational character
and are seen as an input-output 'machine' that process input values into
output values. In higher grades, functions have a more structural character
with various properties (Sfard, 1991). They become mathematical objects
that are represented in different ways, are ordered into different types
according to their properties, and are submitted to higher-order processes
such as differentiation and integration. We argue that the transition from
functions as calculation operations to functions as objects is fundamental
for conceptual understanding in this domain.
Can computer tools help here? Computer tools offer opportunities for
thinking and learning as they can mediate the learning activities in which
students engage (Sfard & McClain, 2002). The availability of sophisticated tools for mathematics education, however, also raises questions.
International Journal of Science and Mathematics Education (2012) 10: 1243Y1267
The Author(s) 2012. This article is published with open access at Springerlink.comRepresentations and techniques in such computer environments often
signify mathematical concepts that are still to be constructed in students'
minds. This inherent circularity is known as the learning paradox
(Bereiter, 1985; Gravemeijer et al., 2002; Heuvel-Panhuizen, 2003). The
challenge, therefore, is to avoid these pitfalls and to integrate the use of
computer tools in such a way that it fosters the learning of the concept of
function.
The above is summarized in the research question of this study: How
can a learning arrangement with computer tools foster the transition
from a procedural calculation understanding to a structural correspondence and co-variation understanding offunctions?
The object of study in this report is a newly designed learning
arrangement in which the intertwinement of activities using both paperand-pencil and computer tools, in combination with the careful guidance
by the teacher, is meant to foster grade 8 students' conceptual
development of functions. We are particularly interested in activities
with computer tools that help students to construct mathematical
representations and actions, to explore these representations and to start
reasoning with them (Falcade, Laborde, & Mariotti, 2007).
THEORETICAL FRAMEWORK
What theoretical perspectives may help to design and use a learning
arrangement in which the use of computer tools is embedded and which
fosters the transition from an operational to a structural understanding of
function? First, we need domain-specific knowledge for analyzing the
character of this transition. Second, from the general background of a
constructivist view on learning, the notion ofemergent modeling may guide
the design. Third, theories on tool use and instrumentation may guide a
careful integration ofcomputer tools in the designed learning arrangement.
The Concept ofFunction
The teaching and learning of the concept of mathematical function is a
widely researched topic. Vinner & Dreyfuss (1989) show that students'
mental images of functions may be different from its mathematical
definition. Other studies confirm that students' function conceptions in
many cases are limited. For example, Bloch (2003) and Skaja (2003)
show that students cannot think ofdifferent functions that provide a graph
through the same pair of points. Based on the model by Pirie & Kieren
1244 MICHIEL DOORMAN ET AL.(1989), Meel (1998) shows that many ofthe participants in his study held
a rule-based interpretation of the function concept, which prevents them
from an object-view on the concept, which is needed, among others, for
the understanding of the notion of composite function. In their study on
the understanding of function, Oehrtman, Carlson & Thompson (2008)
stress the relevance of students' reasoning abilities and on a conceptual
orientation on the notion of function, which needs time:
We advocate for greater emphasis on enculturating students into using the language of
function in order to develop facility in speaking about functions as entities that accept
input and produce output, a more conceptual orientation to teaching function inverse and
composition, the inclusion of tasks requiring simultaneous judgements about entire
intervals of input or output values, and the development of student's ability to mentally
run through a continuum ofinput value while imagining the changes in the output values,
with explicit efforts to also promote, at developmentally appropriate time, the covariational reasoning abilities. (Oehrtman, Carlson & Thompson, 2008, p. 39).
In line with the previous research described above, we want to design a
learning arrangement for the topic of function that goes beyond the input-
output conception, that provides room for reasoning, and that helps to
overcome a too-limited concept image of function. In order to do so, Sfard's
theory on the dual nature ofmathematical conceptions plays a central role and
is exemplified by the transition from a calculation view to an object view on
function (Sfard, 1991). Sfard distinguishes operational and structural concepts,
the first concerning mathematical processes and the latter mathematical
objects. A function may operationally be seen as a computational process, as a
recipe to transform one number into another (or to link one number to
another), whereas, structurally, a function can be thought ofas a set ofordered
number pairs. The operational and structural conceptions are complementary.
Sfard observes that, historically, the operational aspect preceded the structural
aspect and argues that the same should be the case for the individual learning
process, because the structural approach is more abstract than the operational.
Similar dualities are described by Tall (1996) and Dubinsky (1991).
This dual nature of function appears difficult for students to
comprehend. Most remain entrenched in an operational-based conception
(Carlson, Jacobs, Coe, Larsen & Hsu, 2002; Breidenbach, Dubinsky,
Hawks & Nichols, 1992). Suggestions have been made to promote
operational and structural conceptualizations offunctions, for example, by
emphasizing the coordination of the dynamics of input-output dependency relationships (Oehrtman et al., 2008; Malle, 2000). Various viewpoints
can be chosen while analyzing the dynamics, like a correspondence view
and growth-oriented view (Slavit., 1997).
TOOL USE AND THE DEVELOPMENT OF THE FUNCTION CONCEPT 1245Different function representations afford different views on functions.
None of these, however, provides a complete description (Elia, Panaoura,
Eracleous & Gagatsis, 2007). Students will have to coordinate various
representations to come to get a handle at the phenomena they are
studying and to understand the relation between these phenomena and a
coherent conception of functions (Duval, 2006; Even, 1998; Janvier,
1987). To the most common representations, such as verbal descriptions,
tables, graphs, and formulas, Freudenthal (1983) adds the arrow chain
representation, in which subsequent operations are chained. He considers
the arrow chain as a didactically powerful representation for introducing
the operational input-output view.
These findings focused us on the transition from operational to
structural aspects of functional thinking and on the importance of using
various representations. Based on these findings, our design will be
guided by the following three interrelated aspects ofthe function concept:
Y The function as an input-output assignment
Y The function is an input-output assignment that helps to organize
and to carry out a calculation process. This initially somewhat vague
and operational notion gradually gets more nuances: How does the
output depend on the input; how does the input determine the output;
what patterns can be seen? The representation of an input-
calculation-output chain is appropriate for this view on function.
Y The function as a dynamic process ofco-variation
Y This aspect concerns the notion that the independent variable,
while running through the domain set, causes the dependent variable
to run through a range set. The dependent variable co-varies with the
independent. Initially, the linked change may be noticed in a
somewhat phenomenological way. Next, the question emerges of
how and why this process of joint dynamics takes place. Helpful
representations for studying co-variation are tables and graphs,
which can be scrolled through or traced.
Y The function as a mathematical object
Y A function is a mathematical object which can be represented in
different ways, such as arrow chains, tables, graphs, formulas, and
phrases, each providing a different view on the same object. The more
structural view on functions may be promoted by tasks focusing on a
global level, for example, on families of functions, on function
comparison, and later on function differentiation or integration.
In terms of Sfard's process-object duality, the input-output assignment
view stresses the operational character of functions: It is an action, which
1246 MICHIEL DOORMAN ET AL.takes the input and processes it until it becomes the output. The
mathematical object view clearly relates to the structural character of
the function: It is a 'thing', which can be member of a family, or be
submitted to higher-order processes such as differentiation.
Different function representations stress different aspects and uses of
the function concept. For example, one single arrow chain is suitable to
an operational, process view, while a family of graphs is more holistic in
nature and more suitable to a structural perspective.
Emergent Modeling
Emergent modeling is a design heuristic which originated in the frame ofthe
theory ofrealistic mathematics education (RME) (Gravemeijer 1999, 2007).
RME builds on Freudenthal's (1991) image of mathematics as 'a human
activity': While engaging in mathematics as an activity, students should be
supported in reinventing mathematics. The emergent modeling design
heuristic is a means for organizing such a reinventionprocess. The approach
takes its starting point in the activity ofmodeling contextual problems. In this
setup, models initially come to the fore as context-specific models that refer
to realistic or paradigmatic situations. Then, while the students gather more
experience with similar problems, attention may be shifted toward the
mathematical relations involved. This helps them to further develop those
mathematical relations, which enables them to use the model in a different
manner. The model starts to derive its meaning from these mathematical
relations and becomes a base for more formal mathematical reasoning. Or in
short: A model ofinformal mathematical activity develops into a modelfor
more formal mathematical reasoning (e.g. Gravemeijer, 1999; Cobb, 2002;
Rasmussen & Blumenfeld, 2007; Doorman & Gravemeijer, 2009). The
learning paradox, as described in the "Introduction," can be circumvented by
designing a chain of activities and a process of emergent modeling that
fosters a reflexive process in which representations and meaning co-evolve
(Cobb, 2002; Meira, 1995). This approach is consistent with conceptual
change literature, which argues that students must first explore a domain and
recognize limitations or a need for more sophisticated tools, before they are
ready to learn complex concepts (Lehrer & Schauble, 2002).
In this study, the notion of emergent modeling is used as an important
design heuristic. It stresses the importance of starting with contextual
problems that offer opportunities to develop situation-specific reasoning and
tentative representations for organizing repeated calculations and that have
the potential to evoke the need formore sophisticated—mathematical—tools
and concepts (Gravemeijer, 2007). For instance, the table representation is
TOOL USE AND THE DEVELOPMENT OF THE FUNCTION CONCEPT 1247initially understood as a tool for administrating values oftwo quantities (like
time and temperature). In a learning arrangement on functions, the table can
come to the fore as a tool for organizing corresponding input-output values
in contextual problems. Next, on the computer, this table can be used as a
tool for scrolling and zooming in or out these values for various purposes (e.
g. analyzing growth behavior or finding break-even points). This imagery—
table and actions—initially signifies repeated calculations and emerges into a
model for a structural understanding of dependency relationships. Taking
this process seriously asks for a design in which a table tool is not introduced
too quickly, but builds upon previous activities.
Tools and Instrumentation
Emergent modeling as a design heuristic does not particularly focus on
the use of computer tools and the relationship between computer
techniques, paper-and-pencil techniques, and conceptual understanding.
Drawing upon Vygotsky's views on the dialectic relation between tool
use and cultural practices, it is important to understand and carefully plan
the role of tools in a learning process (Vygotsky, 1986; Wertsch, 1998).

As a point of departure, we follow Hoyles & Noss (2003) in observing
that tool characteristics do affect student learning:
Tools matter: they stand between the user and the phenomenon to be modeled, and shape
activity structures. (Hoyles & Noss, 2003, p. 341)
Instrumentation theory focuses on the mediating role of tools by
stressing the co-emergence oftool techniques and meaning in a process of
instrumental genesis (Artigue, 2002; Trouche, 2004). Instrumental
genesis comprises the development of cognitive schemes containing
conceptual understanding and techniques for using a tool for a specific
type of task. The resulting instrument integrates the tool and mental
schemes. As a consequence, a bilateral relationship between the tool and
the user exists: While the student's knowledge guides the way the tool is
used and in a sense shapes the tool, the affordances and constraints ofthe
tool influence the student's problem-solving strategies and the
corresponding emergent conceptions.
Instrumentation theory reveals the problems that may arise when starting
to use a ready-made computer tool and explains the importance of aligning
techniques that emerge in the contextual problems with the techniques in the
learning arrangement with computer tools. Instrumental genesis can be
hindered by discrepancies between tool techniques and conventional paperand-pencil techniques (Drijvers et al., 2007). These insights help us to avoid
1248 MICHIEL DOORMAN ET AL.students carrying out computer tasks satisfactorily without mastery of the
underlying mathematical concepts and not knowing what to do in problem
situations without the tool at hand (Hennessy, Ruthven & Brindley, 2005).
For instance, we inserted preliminary paper-and-pencil activities as a
preparation on techniques with the computer tool and carefully sequenced
the use oftool options in the learning arrangement.
In general, computer tools are able to dynamically and flexibly deal with
different function representations such as tables, graphs, and formulas
(Ponte, 1992). This may help to overcome the difficulty of integrating the
operational and structural aspects of the function concept and its different
representations (Falcade et al., 2007). Computer tools can support
exploratory activities for coordinating dependency relationships and
investigating the dynamics of co-variation (Kaput & Schorr, 2007). In this
particular study, a computer applet called AlgebraArrows is used to support
the transition from a calculation understanding to an object understanding of
functions. The tool, together with the tasks that aim to exploit the tool's
affordances, will be presented in more detail in the "Methods" section.
METHODS
The design of, and teaching experiments with, the learning arrangement
not only aim at fostering students' functional thinking, but also at an
understanding of why the particular instructional setting potentially
supports learning. This requires cumulative cyclic, classroom-based
design research that allows for analyzing and improving learning
arrangements (Gravemeijer & Cobb, 2006). Design research involves
formulating, testing and refining conjectures, designing instructional
activities, and teaching experiments with these activities. The conjectures
provide for a connection between the teaching experiments and an
understanding of how students can be supported in the development of
functional thinking. Such a progression should take into account both the
cognitive development of the individual students, as well as the social
context in which the teaching experiments are to take place (Cobb &
Yackel, 1996; Gravemeijer & Cobb, 2006). We describe successively the
design phase, teaching experiments and analysis in this research.
Design Phase
An important part ofthe design phase involved both the literature study which
led to the theoretical framework described in section "Theoretical Framework,"
TOOL USE AND THE DEVELOPMENT OF THE FUNCTION CONCEPT 1249and the design of the learning arrangement. In this section, we report on the
design as it resulted from the sequence ofthree design cycles. As such, it is an
important result from the study.
The learning arrangement includes (a) a computer tool called
AlgebraArrows, (b) a student textbook with both paper-and-pencil and
computer tasks, (c) a teacher guide describing the various activities and
their possible orchestrations, and (d) a computer and written test.
1
The computer tool is an applet called AlgebraArrows, which allows for
the construction and use of chains of operations (so-called arrow chains)
and provides options for creating tables, graphs, and formulae and for
scrolling and tracing. Figure 1 shows an AlgebraArrows window, in
which students create and use their representations.
AlgebraArrows is meant to support the construction of input–output
chains of operations as a model of a dependency relationship. The applet
is embedded in an electronic learning environment, and the researchers, as
well as the teacher, can monitor and compare the students' responses with
class results. The activity of investigating this applet in pairs is expected
to foster a learning process of 'trial-and-improve' and 'negotiation-ofmeaning'. During these investigations, students can find patterns and
develop functional thinking while dealing with dynamic input–output
dependencies. As such, the computer tool is expected to provide students
Figure 1. The computer tool AlgebraArrows
1250 MICHIEL DOORMAN ET AL.with a means to structure inquiry about dependency relationships through
constructing and modifying arrow chains and through generating and
comparing argumentations.
The student textbook describes the activities for eight lessons. The
learning arrangement starts with three open-ended small-group activities
about different dependency relationships:
Y A first one on the area of a flexible quadrilateral, which students first
explore using a paper model and which then leads to choosing a
variable and finding a linear formula.
Y A second one on the comparison of two cell phone offers, with a
focus on break-even points, which have proved to be suitable
problem situations for addressing the concept of function (Ainley,
Bills & Wilson, 2005; Kalchman & Koedinger, 2005; Küchemann,
1981). As in the first activity, the functions involved are linear.
Y A third one on the braking distance ofa scooter for various speeds. It
includes a quadratic function.
These three activities are meant to create an exploratory stage that
reveals the students' current thinking, creates the need for organizing
series of calculations, and provides opportunities for the teacher to
introduce the computer tool. The students are asked to show their
reasoning on posters that are discussed with the whole class. This
discussion focuses on differences and similarities between the three
problem situations and their solutions for reaching consensus about a
model of a series of calculations (an arrow chain) as a sensible way to
proceed when investigating dependency relationships. This way, lessons
with paper-and-pencil in the regular classroom prepare for lessons in the
computer lab, which is meant to promote the integration of both
conventional and computer tool techniques.
After these three initial lessons, the arrangement continues with two
computer lessons. During the computer activities, students use the computer
tool to construct input–output chains of operations as representations of
calculation procedures. The chains can be applied to single numerical values
as well as to variables, and corresponding tables, formulas, and (dot) graphs
can be created (Boon, 2008; Drijvers et al., 2007). To shift focus towards the
inquiry of dependency relationships, students are confronted with breakeven problems that ask for the generation and comparison oftables ofinput
and output values, and moving around in a space of possible values. This
way, the computertool's sliderbar and table scrolling options, in conjunction
to the tasks, invite the students' development of a dynamic notion of a
variable (see Figure 1).
TOOL USE AND THE DEVELOPMENT OF THE FUNCTION CONCEPT 1251The computer activities are expected to offer students opportunities for
creating new mathematical goals that relate to co-variation and functions
as mathematical objects. The emergent modeling heuristic suggests a shift
from sub-models that signify repeated input–output calculations to
models for reasoning about dependency-relationships, co-variation, and
of function object properties.
After these five lessons, the arrangement continues with a reflective
lesson for aligning computer techniques with paper-and-pencil techniques, and finally a computer lesson with applications and a closing lesson
for summarizing results and creating consensus on representations and
functional reasoning. In total, the arrangement consisted of eight lessons.
The teacher guide describes the different activities and their possible
orchestrations. It suggests different classroom arrangements, such as smallgroup work, poster presentations, computer activities in pairs, and whole class
discussions on the results, supported by projections of computer work (Stein,
Engle, Smith & Hughes, 2008). The whole class discussions about students'
strategies and inventions are to provide a basis for focusing and converging
students' learning and helping them in developing their current thinking in
anticipation of future activities (Sherin, 2002). This variety of classroom
arrangements, in the intention ofthe designers, fosters social interaction, which
is expected to promote the articulation oftool use and exchange ofstrategies as
well as reflection, generalization, and preparation ofwhat is to come.
The final part of the learning arrangement consists of a computer test
and a written test. This mixed media assessment reflects the arrangement
as a whole and contains paper-and-pencil questions about break-even
points and computer tasks where students could use the applet
AlgebraArrows.
2
Test items in the two tests are comparable in length
and difficulty and have similar scoring instructions.
Teaching Experiments and Data Collection
After a first small-scale pilot study, the learning arrangement was tested
and improved in two successive teaching experiments in grade 8, with
mid- and high-achieving 13-14-year-old students. The experiments took
eight 50-min lessons. The first teaching experiment was conducted with
three classes at three different schools. The second teaching experiment
was conducted at a somewhat larger scale. In this paper's analysis, we
focus on qualitative screencast data from pairs of students in the first
teaching experiment, backed up with quantitative posttest data from 155
students from the second teaching experiment (five classes at two
different schools). The students were selected by their teachers based
1252 MICHIEL DOORMAN ET AL.upon following criteria we provided: average level of mathematical
performance, communicative attitude, and audible articulation.
In both teaching experiments, whole class teaching sessions, group work,
and work in pairs were videotaped in two ofthe participating classes, and, in
each of these classes, screencast-audio videos of two pairs of students
working with the computer tool were collected. These videos were used to
analyze the students' activities during group work and during computer
sessions. In addition, students' answers and results to the written and the
computer tests administered at the end ofthe experiment were collected.
Analysis
The data analyses started with organization, annotation, and description of
the data with software for qualitative data analysis (ATLAS.ti
3
; see van Nes
& Doorman, 2010). Initially, the tasks in the learning arrangement served as
the unit ofanalysis for clipping the videos. Codes were used to organize and
document the data. Remarkable events from the perspective ofthe research
question were transcribed and discussed in the research team.
In analyzing the first teaching experiment, we constructed a storyline as a
reconstruction ofthe students' learning process. The qualitative data sources
were discussed with two external experts to identify illuminating examples of
the resulted learning process. This discussion resulted in recognizing a shift in
the students' reasoning with functions. Next, illuminating and representative
examples for important steps in the shiftwere identified. In orderto investigate
this shift quantitatively, we revised the computer activities for the second
teaching experiment with a stronger emphasis on break-even points as a
motivation for investigating the dynamics of relationships. In addition, we
wanted to establish to what extent the students' final performance depended
on the use ofthe available computer tool. We therefore added both a written
test and a computer test to the end ofthe learning arrangement.
To allow for quantitative analyses of these hypotheses in the second
teaching experiment, we designed specific paper-and-pencil activities that
aimed at investigating these conjectures. We identified characteristics of
student answers to codify written student material.
Next, we distinguished solution strategies used on two similar
computer tasks that were designed for studying the development of
students' reasoning with the computer tool. One task was situated at the
beginning ofthe computer lessons and the other at the end. The strategies
led to the construction of codes with respect to the use of representations
in the computer tool. These codes could be used to codify screenshots of
students' final work on the two computer tasks.
TOOL USE AND THE DEVELOPMENT OF THE FUNCTION CONCEPT 1253For the quantitative analyses, we used data from 155 students in five
classes. These students concluded the experiment with both the computer test
and the written test. For 150 students, we coded their answers to the paperand-pencil activity concerning the arrow chain (the booklets of five students
were missing). We coded the screenshots ofthe answers to the two computer
tasks for all 155 students. A second researcher coded 55 out of 306 items
(18%). Good inter-rater-reliability was achieved (with a Cohen's Kappa of
0.79) on these items. For a comparison ofthe results on the written test and the
computer test at the end ofthe learning arrangement, we used a paired t test.
RESULTS
The results of the study are presented in the form of a storyline of
students' learning process. This storyline is illustrated with examples of
student work and is empirically supported by quantitative findings.
The Initial Cell Phone Offer Task: Organizing Calculations
The results of the initial open-ended group activities of the first teaching
experiment show a variety of solution strategies. Figure 2 (top) shows the

cell phone offer task. The students' posters that describe the two
cellphone offers are illustrative for their attempts to organize the
situations mathematically, i.e., to organize repeated calculations, construct
Figure 2. Cellphone offer task and students' posters
1254 MICHIEL DOORMAN ET AL.variables, and use various representations. Some students tried to organize
their repeated calculations by systematically writing them in a list (left
poster in Figure 2). This helped them to see the pattern in the calculation
and to apply this pattern to a new input value. In fact, the way in which
these students listed their calculations already resembles an input-output
relationship (the first number varying over the input values).
The right poster in Figure 2 illustrates attempts of students to use
formulas to describe their repeated calculations. Although their formulas
are not in conventional form, they show that these students identified the
two variables of the dependency relationship (m for minutes and b for
costs). The repeated calculations reflect how the operational aspects
precede the structural aspects of functions (Sfard, 1991). The operational
aspects were helpful for the teacher during the discussion ofthe posters in
evoking the need for determining variables, dependency relationships,
and more efficient notations and tools for finding break-even points.
Introducing Arrow Chains: Visualize Functions
These results provide support for the argument that rather open problems
in the beginning of a learning arrangement can result in various solutions
of students that (1) provide information about the students' current
thinking and that (2) create starting points for discussing the need for
more sophisticated tools.
These strategies were exploited by the teacher by linking the students'
initial ideas to the intended tool use during a classroom discussion. During
this discussion, new mathematical goals emerged: Repeated calculations are
time-consuming; what is the general pattern in the calculation procedures?
How can this pattern be described to calculate results "automatically"? How
could a "calculator" help in comparing cellphone offers?
A calculation recipe that exists of a fixed chain of operations emerged
from these activities. This arrow chain is the central representation of the
computer tool. In the next activity, the last one before the computer
lessons, the students had to draw calculation chains for the two cellphone
offers. Figure 3 shows an illustrative answer to the task to construct a
calculation chain.
The arrow chain as a model of repeated calculations appeared to be
grounded in students' previous activities and could begin to function as a
mathematical model for (in terms of the theory of emergent modeling)
reasoning about dependency relationships. This assumption is backed up
with an analysis of the results of student booklets from the second
experiment. The analysis shows that 124 out of 150 students (83%) drew
TOOL USE AND THE DEVELOPMENT OF THE FUNCTION CONCEPT 1255a similar chain while another eight (5%) drew this chain, but used it only
for one specific calculation and did not label input and output boxes.
Using the Computer Tool to Create Arrow Chains
The third lesson, the first computer lesson, started with some introductory
activities and then continued with the cellphone problem (see Figure 3).
The task was to determine, with the computer tool at hand, when it is
advisable to change from one offer to the other. Two students, Lily and
Rosy, worked together on this task. After reading the problem on the
computer screen, they started to construct arrow chains with the computer
tool.
[Rosy drags an input box into the drawing area. The box is connected to the operation –80
and that one is connected to x 0.15.]
Lily: And that added to the fixed costs.
[R agrees and connects the chain to+22.5. Finally, they connect the chain to an output
box.]
L: Well, when you call 100 min
[Enters 100 as input in both chains. TomSeldom (Soms) is cheaper than the other offer.].
R: .... Well, maybe 50.
[L enters 50 as input for both chains. They look at the results and are still not satisfied.
They try some more input values. Finally, they enter 200 in the input box ofboth chains.
For the first time TomOften (Vaak) is cheaper than the other offer. They are satisfied with
this result and proceed to the next task.]
The vignette illustrates how the arrow chain has become a means to
organize the situation and the calculation procedure. Lily and Rosy built
the chain from the input box, adding operations and finally connecting an
output box. This construction signifies their previous calculations. They
focused on specific cases by entering respectively 100, 50 ... and finally
200 for comparing the differences between the two offers. The arrow
chain is used for repeated calculations.
Figure 3. Written calculation chains for cellphone offers (called 'Soms' and 'Vaak')
1256 MICHIEL DOORMAN ET AL.Using the computer tool to create reverse chains
One ofthe following questions was to construct a reverse calculation chain for
calculating the number ofcalling minutes when you know the total costs. The
following vignette shows how three students (Ali, Bert, and Carl) discussed
their reverse arrow chain by using the dimensions ofthe involved variables.
A: You subtract that from that [points to total costs and the monthly fee] that results in an
amount.
B: Yes, say 10 euros.
A: And then you divide it by ... what do you get ... per minute or ...?
B: per minute.
A: Well, then you subtract 22,50 euro and divide it by 15 cent, and this results in minutes,
and then you add 80, that are the free minutes which you receive.
(...)
C: amount–amount and that is still an amount and you divide that by an amount, 15 cent
[the costs per calling minute]. The result is a number, not an amount, and that are calling
minutes.
The students had to stepwise analyze and unpack the function chain
and to reverse the operations with help of their contextual meaning. This
vignette illustrates how the contextual activity on the reverse arrow chain
afforded an understanding of the structure and the aim of the chain. The
tool technique to construct chains supports this, and the chain became a
means for analyzing and discussing relationships and successive
operations, as a sequel to merely solving repeated calculations.
Whole-class Discussion Capitalizing on Hands-on Experience
The start of the second computer lesson, the fourth in the learning
arrangement, consisted ofa teacher guided classroom discussion ofstudents'
computer work by using a beamer. Topics ofdiscussion were the possibility
to label input and output boxes and to use tables. These tool techniques were
already used by some of the students and now became "institutionalized"
through the whole class discussion. The teacher starts the discussion with an
arrow chain for one ofthe cell phone companies.
T: I heard different ways to find out how much I have to pay, how the amount changes,
how can you demonstrate that? [Silence] For example if I call 10 min. What ways do I
have to find out how much I pay for 10 min?
S1: Put 10 in the input box
T enters 10 into the input box. This gives an odd result (first 30 free minutes are subtracted).
The result is discussed and the teacher varies the input by entering some more values.
T: Suppose I want to know the output from many input values, what more can I use?
S1: The table.
TOOL USE AND THE DEVELOPMENT OF THE FUNCTION CONCEPT 1257T clicks for the table tool and shows how you can scroll through the input and output
values.
T: Does anyone know another way to show how the output values change for different
input values?
S2: With a graph ...
After this suggestion, the teacher clicks for the graph tool. She
demonstrates how to connect an arrow chain to the graph window.
Furthermore, together with the students, she investigates the possibilities to
trace a graph and to zoom in and out. In this way, togetherwith demonstrating
the tool techniques, the teacher discusses aspects ofdependency relationships
and how tables and graphs can be used to analyze their dynamics. In the
previous computer activities, Lily and Rosy did sometimes click for a table or
graph, but didn't use it for scrolling or tracing values.
This vignette shows how the teacher used the computer tool to create
whole class consensus on how to use it for investigating input-output
relationships. Both the techniques and related concepts were part of the
discussion. She did not show how to use these features for finding breakeven points. That was still a task for the students.
A Different View on Function
During the third computer lesson, the sixth lesson of the learning
arrangement, we observed that the strategy ofLily and Rosy for using the
computer tool had changed while solving a question about two handyman
offers by contractors called Pieters and Tweehoog (see Figure 4). The task
is quite similar to the cellphone activity. However, the way in which they
analyzed the problem situation, phrased the structure of the solution
procedure, and how they used the representations in the tool changed
from case-by-case calculations to investigating the dynamics of relationships with the tools at hand. In the following vignette, Lily starts the
discussion and operates the mouse and the keyboard.
L: Company Pieters charges start costs and an hour rate ... that is ...
R: Plus 92, times 30 ...
Figure 4. Handyman task
1258 MICHIEL DOORMAN ET AL.[L drags an input and an output box into the drawing area and creates a label for the in-box]
L: First company Pieters ... I write only Pieters here, eh?
R: Euro ... plus costs per hour Pieters ... and the total costs will come in here.
[The mouse moves from inbox to outbox and she creates a label and types the name ofthe
outbox. The variables are identified.]
L: Yes, here are the costs.
[The chain of operations is constructed and connected to the input and output boxes]
(...)
L: And the other is TweeHoog ...
[The inbox and outbox are positioned in the drawing area. Labels are created according to
the corresponding variables, with 'uurloon' for 'hourly rate' and interpreted as number of
hours.]
R: Strange costs per hour 32.75.
[The arrow chain of operations is constructed.]
R: Now you have to connect it to a table.
[They start to scroll in the table of the input values and quickly see that the break-even
point is at 18 h.]
(...)
L&R: From 18 h.
[They type in their answer: from 18 h.]
This vignette illustrates a different view on, and use of, the arrow chain
compared to the initial operational left-to-right construction of a chain for
TOOL USE AND THE DEVELOPMENT OF THE FUNCTION CONCEPT 1259repeated calculations. More specifically, the vignette illustrates how the arrow
chain had become a tool for analyzing dependency relationships. Lily and
Rosy organized the dependency relationship by identifying, positioning, and
labeling boxes for input and output variables and filling the gap between them
with operations. This construction signifies an understanding ofthe problem
as a question about two dependency relationships. For investigating the
dynamics, they successfully operated the table tool to zoom in and out for
finding the break-even point. The students' understanding ofthe problem and
phrasing ofthe related mathematical goals changed and new goals did emerge
(e.g. what are the involved variables?).
Backing Up Case Study Findings
The second teaching experiment was used to quantitatively verify this
conjecture concerning the shift in tool use during the computer activities.
We found that for the initial task 130 out of 155 students (84%) used the
tool for just calculating successive input-output values, while in the task
at the end, 89 out of 152 students (58%) used the tool for structuring and
investigating the dynamics of the relationships.
Whereas the students' initial technique with the applet suggested a
procedural view on functions, this more advanced technique reflects a
shift towards a structural view. In other words, the shift in the use of the
tool reflects the development from viewing calculation recipes as tools for
repeated calculations (as processes) to reasoning with structural characteristics of dependency relationships (as objects).
To investigate whether this shift also encompassed students' written work
during the eight lessons, we compared the scores ofthe computer test and the
written final test. In apaired ttest, we foundno significant difference (p = 0.20)
between the final scores on the written test and the computer test. The
correlation of 0.38 between these scores was moderate (p = 0.001). This
suggests that students who improved their reasoning through the described
shift were also shown to benefit from this during the paper-and-pencil test.
Overview ofthe Findings
To summarize, we found useful and illuminating pointers to what a
sequence on functions might help to achieve. The three preparatory openended group activities, in the beginning ofthe sequence, offer students an
opportunity to start from their own informal representations (see Figure 2).
The initial modeling processes resulted in tentative reasoning and
representations that the teacher could build on by introducing the arrow
chain as model of a calculation recipe. The arrow chain representation
1260 MICHIEL DOORMAN ET AL.might derive its meaning from the students' previous activities and might
support the emergence ofthe notion ofdependent and independent variables
and ofchains ofoperations during the subsequent activities (e.g. identifying,
positioning, connecting, and labeling input and output boxes). The notion of
arrow chains and varying input values preceded the scrolling through tables
and zooming in graphs for finding break-even points. This is a key element
of what is denoted 'imagery' in the emergent modeling approach: New
representations or sub-models signify earlier activities with earlier representations. Table 1 summarizes this envisioned development.
CONCLUSION AND DISCUSSION
Conclusion
The central question we formulated in the introduction ofthis paper is: How
can a learning arrangement with computer tools foster the transition from a

procedural calculation understanding to a structural correspondence and covariation understanding of functions? Although the arrangement had the potential to offer students opportunities to work with correspondence and covariation, in which functions representedby arrow chain and tables are starting to become mathematical objects for the students, we cannot claim that all students reached a structural understanding offunctions within the timeline of the teaching sequence. Nevertheless, we can identify the following characteristics ofa rich learning arrangement that foster a transition from a calculation understanding to working with correspondence and co-variation:

1. The three initial open-ended problems and the poster activity helped students in coming to see input-output structures in problems about dependency relationships and to use representations to explore them.

TABLE 1
Global outline of the intended learning outcomes

Start ofthe learning arrangement	End ofthe learning arrangement
Task perception	
Find value	Find relationship
Tool use	Construct chain (from left to right) and enter successive values
Construct chain (after identifying variables) and scroll table or trace graph	
Tool imagery	Repeated calculations
Dependency relationship	

TOOL USE AND THE DEVELOPMENT OF THE FUNCTION CONCEPT 12612. By enabling students to design and use arrow chains as a means of support for reasoning about calculation recipes that bear meaning in everyday-life phenomena, the computer tool supported the students in developing the notion of a chain of operations.

3. By generating the results of a series of calculations for a variety of input values, the computer tool strengthened the students' notion of a function as a calculation recipe that seizes on input values and produces output values.

4. By generating output values for series of input values, by generating tables ofinput and output values, and by enabling the students to move up and down along these tables, the computer tool supported the students in developing a dynamic notion of a variable that can move around in a space of possible values.

5. By displaying arrow chains, tables, and graphs, the computer tool offers representations, which the students could construe as affordances for treating functions as objects before they had become objects for them.

6. By discussing students' work with the tool and showing specific features that some of the students discovered, the teacher appeared successful in having most students start using the table representation.

In summary, we conclude that the learning arrangement with a computer tool helped students to overcome the difficulty of integrating operational and structural aspects of the concept function (Falcade et al., 2007; Sfard, 1991). In particular, the use ofthe tool supported explorative activities for coordinating and investigating the dynamics of co-variation (Kaput & Schorr, 2007).

Discussion

Before discussing the conclusions of this study from a theoretical perspective, we first address its limitations. The sequence of eight lessons fostered conceptual development in the domain of functions. However, the possibilities in the final tests were limited for assessing whether all students made a similar step towards a dual conceptualization, with procedural and structural views, of functions. Moreover, the validity of the strategy coding for underpinning the conjecture that emerged from analyzing the screencast-audio videos can be debated. The screen-audio videos and the observations during the computer lessons showed that the final screenshots ofan activity sometimes are only poor representations of

the students' reasoning. Students could be close to a good answer and then delete everything as a result of a sudden doubt. Final screenshots don't capture this entire reasoning process. Additional information is needed for a full understanding of students' conceptual development in relation to their tool techniques. However, the balance between a manageable data collection and as much information as possible will remain a problematic issue for this. A comparison with a control group is needed to better understand to what extent the improvement is the result of this learning arrangement and the computer tool.

Finally, the results cannot be generalized beyond this particular setting. The extensive description of the interventions is intended to enable researchers to implement the experiment in other settings as a way to evaluate our findings and to contribute to the development of a more encompassing theory. This study started from domain-specific theories on the acquisition of the concept of function and theoretical notions of emergent modeling, tool use, and instrumentation. How did the latter two perspectives work out? First, the notion of emergent modeling and its focus on the shift from "model-of" to "model-for," not only guided the design of the learning arrangement; indeed, the students' cognitive development, and their ways of using the tool initially as a tool for calculations and later as a tool for reasoning with function, reflect this shift.

Second, instrumentation theory focuses on the interrelated development of techniques and concepts. The careful integration of computer activities and paper-and-pencil activities prevented discrepancies between tool techniques and conventional paper-and-pencil techniques. The process of instrumental genesis was enhanced by this alternation between classroom discussion, small-group activities, and computer activities. The repertoire of applet techniques, however, was small and comprised rather simple technical skills. Even ifwe observed changes in how students built arrow chains, the main evolution concerned the reasoning attached to dependency relationships and was not a technical one.

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1 The learning arrangement is available through http://www.fi.uu.nl/tooluse/en/

2 The full computer and written test can be found at http://www.fi.uu.nl/tooluse/en/

3 Designed and produced by ATLAS.ti GmbH in Berlin.

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O RI GIN AL ARTI CL E
Problem solving as a challenge for mathematics education
in The Netherlands
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Abstract This paper deals with the challenge to establish
problem solving as a living domain in mathematics education in The Netherlands. While serious attempts are
made to implement a problem-oriented curriculum based
on principles of realistic mathematics education with room
for modelling and with integrated use of technology, the
PISA 2003 results suggest that this has been successful in
educational practice only to a limited extent. The main
difficulties encountered include institutional factors such as
national examinations and textbooks, and issues concerning design and training. One of the main challenges is the
design of good problem solving tasks that are original, nonroutine and new to the students. It is recommended to pay
attention to problem solving in primary education and in
textbook series, to exploit the benefits of technology for
problem solving activities and to use the schools' freedom
to organize school-based examinations for types of
assessment that are more appropriate for problem solving.
Keywords Mathematics education Problem solving
Realistic mathematics education Assessment
1 Introduction
The importance of problem solving in mathematical
thinking and thus in mathematics education is obvious to
many of us. Already in 1962, Polya described problem
solving as a characteristic of human activity:
Solving a problem means finding a way out of a
difficulty, a way around an obstacle, attaining an aim
which was not immediately attainable. Solving
problems is the specific achievement of intelligence,
and intelligence is the specific gift of mankind:
solving problems can be regarded as the most characteristically human activity. Polya 1962, p. v.
In line with the contribution by Schoenfeld (2007) in this
issue, we consider problem solving as the 'art' of dealing
with non-trivial problems which do not yet have a known,
routine solution strategy to the student, but which provide
opportunities for the student to develop new solution
strategies. This description distinguishes problem solving
from many activities with problems in mathematics courses, with or without context.
Realistic Mathematics Education is a leading view on the
learning of mathematics in The Netherlands. As a result,
problem solving in our country is often related to solving
real world problems. Still, we do want to stress that problem
solving is not at all restricted to real world problems; rather,
problems emerging from the 'world of mathematics' can be
very rich sources for problem solving activities.
The non-routine aspect of a problem is relative to the
history of the person who is confronted with it. A problem
can be a real problem-solving item for a student today,
while the same problem is part of a regular set of items
tomorrow. When a student has dealt with an item before,
the character of the activity will change.
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ZDM Mathematics Education (2007) 39:405-418
DOI 10.1007/s11858-007-0043-2Now, having defined our topic, we will first focus on
recent developments in mathematics education in the
Netherlands.
2 Problem solving in The Netherlands: a brief history
As is the case in many countries, problem solving (in the
sense of the above definition) was not an explicit part of
the mathematics curriculum in The Netherlands until the
eighties. In primary education it still is not. For upper
secondary level, however, policy makers and math educators realized in the seventies that a serious problem
was emerging. The curriculum was well suited for the
students who were preparing for science, but not at all
for those pursuing an education (or profession) in other
domains such as social studies. It was decided that the
latter category of students, who struggled with the formal
and abstract approach, would be better served with a
more problem-oriented approach focusing on applications
and mathematical modelling. This also implied, of
course, a very different approach to what should be
learned and assessed. The science-oriented streams in
upper secondary education were not affected by this: no
realistic problem solving was included in the math and
science-oriented curriculum. The national examinations
remained straightforward and traditional, so these—high
level—students could be prepared for higher education
without any skills to invent new strategies for new types
of problems.
For the students who were preparing for the humanities,
a new curriculum was developed, the mathematics A curriculum (De Lange 1987), based upon realistic mathematics education (which will be discussed in the next section).
It was introduced as a new curriculum subject in pre-university education (vwo) in The Netherlands in 1989. Since
that year, in upper secondary school, two different types of
mathematics curriculum were being taught, mathematics A
and mathematics B. Mathematics B contained the mathematics needed for technical studies and studies in science
and mathematics at university level, its core component is
calculus. Mathematics A was meant for students who
prepare for academic studies in social or economical
sciences or other related subjects.
The purpose of developing the mathematics A curriculum was described as follows:
Mathematics A is intended for students who will have
little further education in mathematics in their academic studies, but who must be able to use mathematics as an instrument to a certain extent. In
particular, we have in mind those who have to prepare themselves for the fact that subjects outside the
traditional sciences are more frequently being approached with the use of mathematics.
This means that students must learn to be able to
assess the value of a mathematically tinted presentation in their education. To do this they must become
familiar with the current mathematical use of language, with formulations in formula language, and
with divergent forms of mathematical representation.
Furthermore, they must learn to work with mathematical models and be able to assess the relevance of
these models.
(Werkgroep van advies voor de herverkaveling eindexamenprogramma's Wiskunde I en Wiskunde II vwo 2005,
p. 19)
In mathematics A, the emphasis lay more on applications
of mathematics and on mathematical modelling than on
'pure' and abstract mathematics, and also more on the
processes needed to come to an answer than on the answer
itself (the product). In its initial years, modelling and
problem solving were at the heart of this subject. The research along the introduction of mathematics A (De Lange
1987) showed examples of real world problem solving, and
of the use of mathematics to solve real world problems. It
also revealed that problem solving lends itself excellently to
group work, which brought new problems and possibilities
to the fore, especially in the area of assessment. The research
hinted already at a fact that is now almost taken for granted,
as the outcomes of PISA 2003 prove: inter-subjective
scoring can be very reliable and fair (OECD 2005, p. 227).
Maybe because of the successful introduction of this
new type of mathematics, problem solving in The Netherlands since that time seems to be identified with solving
real world problems using mathematics. Even in the
'harder' and more scientifically oriented curricula, problem
solving was introduced 'in context'. Inside-mathematics
problem solving seems not to have caught on in The
Netherlands, at least not on a large scale.
In the ongoing discussion in The Netherlands, there is a
lot of confusion about the true meaning of problem solving
in relation to the theory of Realistic Mathematics Education. So let us now address this relationship.
2.1 Realistic mathematics education
and problem solving
In the aforementioned study (De Lange 1987), the real
world problems were used for problem solving in two
ways. First, according to the theory of Realistic Mathematics Education, the real world is a source or starting
point for the development of mathematical concepts
(Freudenthal 1991). Well chosen contextual problems offer
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123opportunities for the students to develop informal, highly
context-specific solution strategies, and are used to support
mathematical concept building (Gravemeijer and Doorman
1999). In this phase, the criterion of potential for mathematical concept development dominates the criterion of
authenticity. In fact, the context may even be rather unrealistic or within mathematics, if concept development requires it. However, the contextual problem must be
experienced as a real problem by the students.
The second role of the real world is a little different. The
real world is used as the domain in which we can use our
mathematical concepts in the way we wish. The real world
now has to be truly authentic in order to let the highly
appreciated process of mathematization take full bloom.
The idea behind this application phase of the learning
process is twofold: to reinforce the concepts already
developed, and to engage in problem solving activities:
develop and deepen strategies.
If we reflect critically on the developments of the past
decades we distinguish three factors that made the implementation of problem solving only a limited success in

Dutch mathematics education. First, it became increasingly
difficult to make quality assessments for mathematics A.
The examinations became more and more predictable, so
part of the real problem-solving component was lost.
Second, many people translated the ideas behind mathematics A in a somewhat restricted way: the conceptual
mathematization part and the applied mathematization part
were confused, resulting in examination standards of long
stories in which students had to apply little mathematics.
Third, the level of authenticity went down with the increased use and popularity of 'applied problem solving'.
2.2 Today's problem solving practice
To summarize the state of the art in Dutch mathematics
education, we notice that students in upper secondary
education are not often confronted with inside-mathematical problems for which no known strategy is available.
This may be the reason for the modest scores of The
Netherlands in the mathematics Olympiads. Another matter
for concern is the lack of problem solving activities in
primary education. The focus here is on word problems and
eventually on solving real world problems. Solving insidemathematics problems is merely absent.
To assess the state of affairs concerning problem solving
in Dutch mathematics education, we will now first look at
the PISA-2003 results. Next, we will discuss a study on
problem solving in primary education. Then, a problem
solving competition for upper secondary education will be
presented. The paper continues with remarks on the
opportunities that the integration of technology offers for
problem solving, and ends with a conclusion.
3 Recent PISA-results
3.1 PISA-2003
In the 2003 version of the programme for international
student assessment (PISA) study, mathematics was the
main focus. Problem solving was an additional domain in
PISA-2003. The idea behind this domain, usually not
taught as a separate subject, was to test general problemsolving skills of students aged 15-16. Being able to apply
problem-solving skills in a variety of domains is an
important educational target in many countries and, as
indicated by the OECD (2003), these competencies are the
foundation of effective participation in society and development of talents in personal and professional life.
In order to solve these larger problem-solving tasks,
students must understand and interpret the available
information, recognize important elements that are represented and make connections to the real world situation.
They must be able to reason and communicate their arguments in writing. The solution is not obvious and the skills
needed usually do not belong to a single educational domain such as mathematics or science.
Within the domain of mathematics education, problem
solving is seen as solving problems for which standard
solution strategies are not yet available. In general, this
means the investigation of a larger problem. Often, these
problems in PISA-2003 are situated in a more or less
realistic situation. To solve these, the student first has to
'translate' the information and to create a (simplified)
mathematical model of the situation. After the problem is
solved within this mathematical model, the results are
compared to the real life problem situation. This might
result in rejecting (some of) the results, rounding the answers according to the situation or to adjusting the mathematical model (De Lange 2006).
There is a strong connection between mathematics as an
educational subject in The Netherlands and problem solving as defined by the PISA programme. Therefore, research
from the Dutch national test institute Cito and the Freudenthal Institute included the PISA problem-solving domain in a further analysis of the Dutch national
mathematics results of the PISA-2003 study (Dekker et al.
2006). We now present some of the findings of this study.
3.2 Problem solving and mathematical literacy
in the national study
Score points for both mathematical literacy and problem
solving literacy have a mean of 500 and a standard deviation of 100. This enables a comparison between the results
in both domains. In the 2003 study, the Dutch mean score
for mathematical literacy was 538 (place 4 on the interProblem solving as a challenge for mathematics education in The Netherlands 407
123national list) and the mean score for problem solving was
520 (place 12 on the international list). Figure 1 shows that
the difference between the two scores was the largest in
The Netherlands as compared to all other participating
countries; students in The Netherlands performed better in
the domain of mathematics than in problem solving (Gille
et al. 2004).
The international PISA report (OECD 2004a) claims
that the relatively high score of mathematical literacy as
compared to the problem solving results indicate that
mathematics education in The Netherlands, as measured by
PISA, is very effective. However, one could also argue that
problem-solving skills are lagging behind. Furthermore, the
researchers indicate that in no other country, apart from
The Netherlands, mathematics education is as aligned with
the competencies measured by the PISA study. Another
explanation is that the PISA problem solving tasks
emphasize general analytical argumentation skills without
building upon mathematical skills. Could Dutch students
be lacking these general skills? The research could neither
prove nor refute this.
When looking at student work on the mathematics
assignments, the researchers found that Dutch students,
when compared to their peers in neighbouring countries
(Belgium Flanders and Germany), were less persistent
when working at large tasks which took a lot of time and
energy to complete. This was not caused by a lack of
confidence; students in The Netherlands did not often skip
tasks, but almost always at least started the solution process. However, they gave up sooner, especially when
dealing with extensive and challenging tasks. The student
work on the problem-solving test was not analysed in this
study, but it may be expected that the same phenomenon
occurred there.
3.3 Are Dutch students mathematically literate?
As shown earlier, Dutch results were satisfactory when
compared to other OECD countries. But The Netherlands is
a highly developed industrial country and demands are
high for labourers in almost all professions. Which level of
mathematical literacy should this country strive for? The
Fig. 1 Differences between
mathematical literacy and
problem solving. Source: OECD
2004b, p. 54
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123PISA study discerns six levels of mathematical literacy.
Level 1 starts at a score of 358 and ends with 420, level 2
has scores 420-482 and level 3, which contains the overall
mean score of 500, runs from 482-544. Level 4 scores are
within the range of 544-606, level 5 from 606-668 and at
level 6 scores are 668 and higher. Students working at
mathematical literacy level 4 can
....work effectively with explicit models for complex
concrete situations that may involve constraints or
call for making assumptions. They can select and
integrate different representations, including symbolic ones, linking them directly to aspects of realworld situations. Students at this level can utilize
well-developed skills and reason flexibly, with some
insight, in these contexts. They can construct and
communicate explanations and arguments based on
their interpretations, arguments and actions.
The mean score for Dutch students preparing for university
(vwo) was 639 in the PISA 2003 study. Students preparing
for vocational studies at a higher level (e.g. to become a
high school teacher or highly skilled laboratory assistant)
scored 594. However, about 60% of the Dutch students at
age 15 prepare for vocational education at a medium or low
level (vmbo) and most ofthem do not have scores at level 4
or higher as we would expect them to have. When looking
at the content of the PISA mathematical literacy problems,
these seem more suited for vmbo students and less for havo
or vwo students.
There are large differences between the countries taking
part in the study and this means that very weak students in
some countries must be able to answer at least some of the
questions. And so the Dutch results may seem satisfactory
as compared to other countries, by our own standards they
do not seem good enough and much is left to be desired.
Furthemore, the largest contribution to the relatively high
score in mathematical literacy in The Netherlands came
from the Reproduction cluster and not from the clusters
Connections and Reflection.
Providing sound arguments is a diffcult task for all
students, but Dutch students have been trained in doing so
during their school career. Moreover, for lower achieving
students the language aspect is an important issue, for both
passive and active use of language. The result of the Dutch
analysis of the PISA 2003 study led to some recommendations which are, not surprisingly, different for different
groups of Dutch students. For students at upper secondary
level, more emphasis should be placed on formal and
abstract mathematics.
For students in the lower ability groups (vmbo-level)
reading and interpreting skills need attention in order to be
better prepared for solving problems that are challenging to
them. This recommendation was made since we feel it is
unacceptable that so many students aged 15 are not
mathematically literate at the PISA scale of level 4.
To summarize this brief review of the Dutch PISA 2003
results on mathematical literacy and problem solving, the
most surprising result is that the problem-solving scores are
lagging behind. As learning to solve problems starts at an
early age, we now first consider a study on problem solving
in primary education.
4 Problem solving in primary education
Although the reform in mathematics towards Realistic
Mathematics Education resulted in a new generation of
primary school textbook series, problem solving, in the
meaning of solving non-routine mathematical problems,
was hardly given any attention in these textbooks. At
most the reform brought--to a certain degree-more
complex realistic problems in which the children had to
figure out, for instance, which of the several journal
subscriptions is cheaper and what it costs to organize a
birthday party. Puzzle-like problems, in which it is not
directly clear from the beginning how to start with solving
them, are scarcely found in the new textbooks. At best,
one can come across these problems in additional materials for high achievers.
As in the textbooks, non-routine problems are also not
included in the Cito test, taken by 90% of the sixth-graders
at the end of primary school (Cito is the national institute
for assessing educational progress). This test, through
which children get access to the higher levels of secondary
school, only consists of routine problems in a multiplechoice format.
The lack of non-routine problems in textbooks and the
Cito test make that not much attention is paid to problem
solving in primary school. Teachers do not offer children
an environment in which they can develop problem-solving
skills and the attitude that is necessary for coping with nonroutine problems. A mitigating circumstance that is often
mentioned is that problem solving is something that is only
attainable by the best students and that these bright students
can help themselves and do not need much help for
learning to solve non-routine problems. However, this
assumption is in contrast with the results that were found
when in 2004 high-achieving fourth-graders in twenty
schools were given a test on problem solving (see Van den
Heuvel-Panhuizen and Bodin-Baarends 2004).
For all of us who thought that we as mathematics
developers and researchers did our job quite well and
contributed to bringing our students into the higher ranks of
international mathematics achievement, the results of this
study came as a rude awakening. In the next section, we
will give more details about the study.
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1234.1 A study on problem solving by high achievers
in grade 4
To begin with, it is quite telling that there was an external
reason for doing this study. The direct cause was a request
from Peter Pool and John Treffalt from the Assessment and
Evaluation Unit, School of Education, University of Leeds
to investigate how Dutch high-achieving fourth-graders

will perform problem-solving tasks that have been developed for the World Class Tests. In total, 15 problems were chosen which, for the number domain and the operations to be carried out, were more or less covered by the mathematics curriculum these Dutch students had dealt with in grade 4. However, the nature of the problems was somewhat special for the Dutch students. Most of the tasks were puzzle-like problems, such as number riddles. Students do not encounter this kind of problem that often in textbook series and tests in The Netherlands. The 15 problems were put in a test booklet with every problem presented on a separate page. The children were free to use the whole page as scrap paper. For some problems there was an explicit request to show how they solved them.
In total, 152 high achieving students in grade 4 (9-10 year olds) took the test. The students belonged to the top 20% ability range in mathematics. They were selected by their teachers on the basis of their mathematics score. In most cases this was the students' score on the Cito Student Monitoring Test.
4.1.1 Some revealing results
The analysis of the student responses in the test booklets showed that quite a number ofthe children did not make use of the work space when solving the problems. They did not have any other scrap paper available; this means that they worked on the problems without writing anything down.
This is remarkable, especially for problems that involve a lot of data or where you can find a solution by systematically trying out several options. For these problems, it is often practical to make notes and write down intermediate answers. Figure 2 shows an example of such a problem.
Table 1 shows the results on this item. Clearly, the problem was fairly difficult for the Dutch high-achieving fourth-graders involved in the study. Only 39 students, a quarter of the total group (25.7%), found the right number. The students in the United Kingdom did slightly better on this problem. Of the 184, year 4-5 students (8.5-9.5 years) who did this problem and who belonged to the 15-20% best students in mathematics, 34.2% came up with the correct answer (Peter Pool and John Trelfall, personal communication).
Furthermore, Table 1 reveals that 93 of the Dutch children, almost two-thirds of the total, did not make any use of the scrap paper. Even out of the 39 students who came up with the correct answer, 19 did not use the scrap paper. Since high achievers in mathematics were involved in this study, it is not really surprising that roughly half the children found the right answer mentally. Although this problem needs much calculation work, the difficulty does not lie so much in the knowledge of the tables of multiplication that is required. Bright students in grade 4 mostly know their tables. What it comes down to in this problem is the ability to take into account a number of different criteria at once. The result that gives more cause for concern is that 74 of the 113 children who did not find the right number did not write anything down.
The after-test interviews that were held with a selection of the students disclosed that the difficulties with this problem were not related to understanding the question. The children clearly understood the intention of the problem. Although the calculation itself did not always go as smoothly as one would expect in this group of students, that also was not the core of their problem. The study made it clear that the weakness of these good students lay in a different area, namely in not trying and not being able to keep up a particular approach. Moreover, this attitude might be strengthened by the fact that the children apparently were not used to writing something down to support their thought process.
In reviewing all the students' responses and the experiences from the interviews, three tendencies were found: many students did not write anything down, many students did even not start, and if they started quite a number showed lack of persistence.
Find the number
It is smaller than 100.
If you divide it by 7, there is no remainder.
If you divide it by 3, the remainder is 2.
If you divide it by 5, the remainder is 1.
Fig. 2 Problem: Find the number
Table 1 The results from high achievers in grade 4
Results for the problem "Find the number"
Did not use
scrap paper
Did use
scrap paper
Total number
of students
Correct answer 19 20 39
Wrong answer 74 39 113
Total number
of students
93 59 152
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123The tendency not to write anything down. This tendency raised many questions, but the study also revealed some ideas of how to explain this behaviour. For example, children (and teachers) might think that it is better not to use the paper, because they think that solving the problems mentally is a higher level of mathematics. The trend might also have to do with the fact that bright students hardly use (scrap) paper when they do their assignments in regular mathematics classes. A very different aspect that might play a part is that the children do not write anything down because they feel that you should not "make a mess" in a test. They learned from their teachers that it is important to work tidily.
The tendency not to start with the problem. The fact that the majority of the children who could not solve this problem did not try anything on paper either, is also significant. Except that the trend to not even start could result from the abovementioned refusal to write down anything in solving mathematics problems, another possible explanation could be that the children have not learned to use notes and organize data as a support for the solution process. In the case of this problem, one could think of listing the numbers that qualify according to a certain rule and then strike out the numbers that do not conform. Trying things out might break through the all or nothing atmosphere that now often prevails.
The tendency not to persist when attempting to solve the problem. A quite alarming experience in this study was that high-achieving students gave up after trying a few numbers. A possible explanation for this reaction may be that good students can find it difficult to persevere because they do not often come across problems that require profound mathematical inquiry. They usually do not have to think very long about the problems they normally encounter.
4.2 Some final remarks on problem solving in primary education
As was mentioned before, non-routine problem solving has a marginal place in the present Dutch mathematics curriculum in primary school. We believe that this should change and hope that the start that is given for this in the TAL learning-teaching trajectory for calculation with whole numbers in primary school will have an effect in teaching practice (Van den Heuvel-Panhuizen 2001). A complicating factor here is that up to now there has not been much support from textbooks series and the Cito test at the end of primary school. Both still stay far away from non-routine problem solving.
However, an interesting initiative to enhance problemsolving skills in primary education is the recently established Great Arithmetic Day. On this day, that is organized by the Freudenthal Institute, thousands of primary students (and their teachers) are engaged in problem solving. More research is necessary to see whether this extra-curricular activity can introduce more problem solving to the Dutch primary school curriculum.
After this short impression of the state of affairs concerning problem solving in Dutch primary education, we now turn to secondary education.
5 Problem solving at secondary level
As in primary education, problem solving in secondary mathematics education has only a marginal position. In the introduction to this paper, it has already been pointed out that even an application and modelling-oriented curriculum like the one for mathematics A tends to standardize problem-solving tasks into routine assignments. The national examination does not encourage paying much attention to problem solving skills. Textbooks usually do not address problem solving as a result of examination demands, designing teacher and student proof activities, and the time need for designing problem solving activities. An exception is the textbook series 'Wiskundelijn' (Bos et al. 1990), which contained so-called Systematic Problem Approach items. Such an 'SPA' explicitly aimed at the development of heuristics for dealing with open problems in the sense of Van Streun (1991).
Important initiatives for enhancing problem solving activities in upper secondary education are the Mathematics A-lympiad and the Mathematics B-day competitions. As we consider this to be types of competition which are exemplary to problem solving, an example of the first is discussed in more detail.
6 Mathematics A-lympiad: an experimental garden for problem solving
It seems obvious that problem solving should be an inherent part of the mathematics A curriculum. In the educational practice of mathematics A, however, less and less attention was given to really open ended problems that address higher order thinking goals, problem solving and modelling. This is mainly due to the fact that Mathematics A is assessed in a central, written, individual, final examination. The design of examination tasks that do justice to the purpose of the mathematics A curriculum turned out to be difficult. The examination questions, although presented in a seemingly realistic context, seldom asked for modelling or problem solving on a serious level. If these skills are not needed for the examinations, one cannot expect a great deal of attention to be paid to them during education. And so the circle is closed: 'poor' final exams bring about 'poor' education.
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123To break this circle, the Mathematics A-lympiad was established outside the examination system. Its intentions were to design a 'task' that encompasses the original objectives of mathematics A and that asks for teamwork as well. As an extra motivating factor this task would be presented to students as a team competition.
In the announcement that is sent to all secondary schools with pre-university education, the competition is described as follows:
The competition is intended for students in grades 11 and 12 (age 16-18) who are taking mathematics A as a subject. It involves tackling a challenging problem as a team (three or four students). The competition consists of a qualifying preliminary round at the school and a final round at the weekend (Friday and Saturday) in a conference centre.
The open nature of the task implies that the teams have to forge the entire path from defining the problem, via strategy definition, solution and argumentation, through presenting the solution found. The result is a paper containing all of these aspects.
6.1 The competition and school examination
The Mathematics A-lympiad competition has been running since 1989. It consists of two sessions, a preliminary round in the schools and a final session in an external location. The competition has seen a growing number of participating schools in the first 10-12 years, though that number seems to have stabilized over the last couple of years with about 150 schools participating, with a total of about 1,500 teams. This means that every year about 6,000 Dutch pupils work in teams on an open ended authentic assignment for mathematics. An increasing number of foreign teams (from Denmark, Germany and the Dutch Antilles) participate in an international final round.
In the early years of the Mathematics A-lympiad, it was exceptional in mathematics education to call on skills such as problem solving, reading, writing, doing research, forming arguments, reasoning, critically reviewing mathematical models, mathematization, teamwork, planning: the full range of general and mathematical skills. With the Mathematics A-lympiad these skills have gained a set place in a large number of schools in The Netherlands.

This type of mathematics is precisely the mathematics
that many students will deal with later on in society:
solving more complex problems in teams where technical
tricks alone will not do. In the next paragraph we will
present an example of student work on one of the
assignments.
6.2 The diversity example—analysis of student work
An assignment in a final round was about bio-diversity.
Various species of plants and animals are disappearing
every day. The task for the student teams was to decide on
criteria for determining which plants and animals are most
important to save, if the goal is to keep species diversity
large enough.
In the first three problems the teams had to familiarize
themselves with the concept of bio-diversity. Then a
relative value scale had to be developed for the genetic
relationships of species. This value scale was to be used
for determining which species it would be better to
protect in order to keep diversity, from a genetic point of
view, as great as possible. In brief: the teams had to
develop a decision-making model. This model should
then be used for making decisions on the preservation of
species.
6.2.1 The plants in the photos
The assignment started as follows (see also Fig. 3).
'A number of photos of different combinations of plants
(A, B, C and D) was given to a number of scientists who
then ranked them according to decreasing diversity.'
Using this information the teams had to decide what
factors were clearly seen as important for diversity. Teams
were not asked to devise a function themselves that produces this ranking, but almost every team tried to do so.
This was probably done to compare their own ideas with
the assessment of the scientists. In general, there was little
difference for the factors found by the teams. As most
important criteria for determining the level of diversity
were mentioned:
- the total number of plants;
- the number of species;
- an even distribution of the numbers per species.
The standard deviation of the numbers was calculated as
a measure for this distribution. A smaller standard deviation was synonymous with better diversity. Some teams did
not think that this was altogether correct as a photo with a
few plants would be given a greater diversity than a photo
with many plants in the same ratios. One team replaced the
absolute numbers by percentages, as did a number of other
teams (Fig. 4).
One team did not think this was correct and explained
why, using a fictitious photo containing only one species.
The standard deviation would then be zero and that would
mean that it had the greatest diversity, even though in fact
it has the minimum diversity.
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123Another team concluded that according to the above
method the diversity was greatest when all plants occurred
in equal quantities. However, nature is better served by
equilibrium than by equal quantities. And in equilibrium:
'... for example, very many more bacteria are needed
than elephants.'
A better factor for determining the level of diversity is how
much the number of each species in a photo differs from
the average number of this species (in all photos). The
thinking behind this was that the ecological equilibrium
will not be far from the average numbers. This method
meant that the photos with plants in the ratio of:
40:20:10:13.75:21.25 had the greatest diversity. When the
ratios were equal, a higher diversity would be allocated to
the photo with the greatest number of plants.
A different team compared the numbers of plants (see
Fig. 3) with the calculated averages. Checking this approach by using the plants on the photos meant that photo
3 had the greatest diversity, as it was the one that best
corresponded to the desired averages. Next is photo 2 as
one species was missing there. Then photo 4 because
species D had died out there and it was a species that
occurred more on average than species C, which was
absent in photo 2. Photo 1 was the last due to its large
differences from the averages, especially for species D
and E. This order corresponded to the order established by
the experts.
6.2.2 Genetic variation
In the second part of the assignment the teams were asked
to determine which species it would be best to protect from
the point of view of genetic diversity, by using a genealogical tree of genetic relationships.
The teams were asked to develop a value scale which,
together with the genealogical tree, could be used to
determine which species had to be protected and which
could be allowed to disappear if there was no alternative. In
the genealogical tree presented as an example in the
assignment (Fig. 5), it is specified for two combinations of
species which combination it would be better to protect
from the point of view of genetic variation.
Problems can arise here with regard to the term 'genetic
variation' as it can be defined in different ways. In general,
two approaches are recognized in the students' work. The
first approach uses the relationship between a species and
the original species. In this approach, the defined objective
is to preserve the original genetic material as much as
possible. Species that have branched off more often are
further away from the original species and will thus contain
less of the original material. With each division, the teams
assumed that a certain percentage of the material was
transferred into the new species. The teams that used this
approach conceded that these percentages were based on
their imagination. Generally the teams used a division in
equal parts, as is shown in Fig. 6.
Some teams who wanted to preserve the species that
were closest to the original species had rather strange
theories. The preservation of these species would have the
advantage that 'as many genes as possible' would be preserved. Species that were further away from this original
species contained 'fewer genes'. It was also believed that
by maintaining the original material the species that disappear could occur again more easily.
The second approach examines how the different species are mutually related. This can be done by counting the
nodes from one species to the next.
You go through the tree from one species to the next and
count along the way the number of places where new
species occur. This number is called the distance between
two species. These distances were set out by the teams in a
matrix and the columns were then summed. These totals
were then a measure of relatedness.
Fig. 3 Part of the diversity task
photo 1: 40 10 20 5 5
In percent that is:
photo 1: 50% 12.5% 25% 6.25% 6.25%
The standard deviation of these numbers is 16.49. If the diversity of the
other photos is calculated in this way, you get the following standard
deviations:
photo 1: 16.49
photo 2: 11.54
photo 3: 8.00
photo 4: 14.14.
These results also fit with the rankings of the experts.
Fig. 4 One team's result
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123Figure 7 shows this matrix and the accompanying tree,
made by one of the teams. The species with the smallest
total distance is the one most related to the other species and
can, if necessary, be the one most easily dispensed with.
When this species is left out, a new tree can be made and the
process is repeated. If, as a result of selection, at a node only
one ofthe species had remained, almost all ofthe teams left
this node out as if it never existed. This move was never
justified but is indeed consistent when you consider all of
the species that have died out whose existence is unknown
to us. It is striking that none ofthe teams thought of looking
at the minimum and maximum distances instead of the sum
of the distances. After all, a minimum distance means a
close relationship between the two species and a maximum
distance means a distant relationship. With two species that
are closely related, it is less serious if one of them dies out,
at least according to the exercise.
The teams now had to adapt their value scale in such a
way that it was not possible that one species did obtain an
extreme score. In other words: the system had to give
somewhat 'flattened' results. A number of teams combined
the approaches mentioned before. The one group added up
the values from the value scale, another multiplied them.
For example, the distances were multiplied by the percentages. Logarithmic and exponential functions were also
used to bring the values closer together. None of the teams
examined this problem critically. One team stated that
flattening the system leads to a higher reliability. However,
they did not explain.
6.2.3 Comments
Some participating teams commented on the task in
interesting ways. For example, a number of teams criticized the approach in part 2 where genetic variation was
the central issue. According to these teams, there are more
important arguments for prioritizing the preservation of
certain species, i.e. ecological. As an example, take a
family of species that acts as prey, prey-predator, and
predator within an ecological system.
Some finalists thought it was more important for stable
species (with fewer divisions) to be preserved. On the other
hand it can be argued that it might be better to protect
quickly developing branches as new genetic variations will
occur more easily through them.
The task went on with a question on including a time
component to address the genetic evolution in time. For
reasons of space, this extension is not described here; we
refer to De Haan and Wijers (2000) for a more extensive
description of the assignment.
This genealogical tree is used for deciding, for example, that A and B are more
genetically related than A and D. Thus if only two species can be saved, then for
example saving the combination of B and D is better than saving the combination
of B and A. That is why D will be somewhat higher up the value scale than A. How
much higher depends on the whole set.
Fig. 5 Continuation of the task
Fig. 6 Tree with probabilities
Fig. 7 Matrix and tree
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1236.2.4 Concluding the example
The search for an acceptable value scale by the teams was
often a wild search for appropriate formulae. Teams often
arbitrarily multiplied, divided, added and subtracted. The
underlying arguments were often forgotten in the process. It
was for instance stated that a certain parameter was important, but was not yet included in the scale, without saying
why they chose to include this specific parameter in this
specific way. This makes the allocation of values arbitrary.
This can be seen as a criticism ofthe assignment or ofthe
pupils, but it is more a criticism of the process of allocating
values. Furthermore, there was lack ofclarity with the terms.
Where for example is the separation between a breed and a
species within evolution theory? Is this part of the assignment about making a choice between different species of
finches or the choice between a dog, a cat and a squirrel?
The reader will notice that nowhere in the assignment it
was said that a formula had to be devised, but rather a
'system'. Why was it that almost all teams searched so
frenetically for a formula? Could it be caused by the use of
terms such as 'value scale' 'tree'? It is not easy and
often even impossible to convert non-mathematical considerations into formulae. It is often better to simply reason
them out. After all, one does not know how the different
considerations are linked together and how they have to be
included, and whether there are still unknown, more
important factors.
'Chance of survival' indeed seems to be a number, but

nobody knows how to calculate it. It is more a probability
derived from small talk than from theory or experiments.
That's why one of the teams stated:
We have to throw all the criteria into a large basket
and determine the value of each criterion. This presents a considerable problem. Every individual
weighs every criterion differently. You might think
that the economic aspect is important, but so also is
the social aspect. You can proceed purely rationally,
but also emotionally! In order to come to a reasonably objective opinion you must present your criteria
to a cross-section of society.
6.3 Influence of recent educational reform
Since 1998, upper secondary education in The Netherlands
has been restructured. A part of this reform is the growing
attention that is paid to general skills embedded in several
subjects. Larger, more complex open-ended tasks, which
focus on general skills as well as on subject-related skills
became a compulsory part of the school exam for every
subject. These skills are highly reminiscent of those mentioned as objectives for mathematics A in 1980. It is precisely because these skills will be assessed in the school
exams, that the interest in teaching and practicing higher
order skills has been revived. The assignments of the
Mathematics A-lympiad turned out to be perfect examples
of this kind of tasks and since 1998 its assignment is used
as part of the school examination in many schools. A
similar competition for Mathematics B, the Mathematics
B-day, is used in the same way and with equal enthusiasm.
The experiences with the B-day show that problem solving
tasks that are appropriate for team competitions can be
designed within the domains of pure mathematics and
mathematical applications in exact sciences as well.
From an organizational point of view, participating in
the Mathematics A-lympiad and B-day is very convenient
for the school: the assignment is provided by a committee
of experts; in one day, teams can produce a paper; students
are deprived of the possibility of obtaining 'expert
knowledge' from elsewhere, since everyone works for the
entire day on the same task.
There is, however, also a downside to this development.
As doing open-ended assignments, writing papers and essays
as well as working in teams became a regular aspect of the
mathematics lessons—and ofother curriculum subjects—the
initial motive for the Mathematics A-lympiad—to offer 'a
completely different type of assignment'—became less
important. This might have had a slightly negative effect on
the quality of the student work for the Mathematics Alympiad assignments. Nowadays, the mathematics is often
used on a more basic level, without the extra considerations
and argumentations that characterize real problem solving.
Other factors than getting used to this type of tasks play an
important role in the quality as well, the most obvious being
the recent reduction in math lessons ('teaching time') by
about 50% in The Netherlands. Nevertheless, the design of
the assignments and the evaluation of student work during
the finals remain an inspiring experimental garden for
investigating possibilities for problem solving in mathematics education.
7 New tools create new possibilities
As a final perspective on problem solving, we consider the
use of technological tools. New technological tools create
new possibilities for problem solving in mathematics.
These possibilities, as well as the difficulties, are now
illustrated for the case of the integration of the graphics
calculator.
As in many other countries, the graphics calculator was
introduced in the Dutch mathematics curriculum during the
nineties of the previous century. An important didactical
motivation for integration in the curriculum, as well as in
the national examination, is the way this tool supports
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123explorative activities and problem solving abilities (Drijvers and Doorman 1996). Tools like the graphics calculator
provide a new and rich variety in possible problem solving
strategies. Even weaker students appear to be supported by
this (Streun et al. 2000).
The integration of the graphics calculator also fostered
new mathematical practices that would not be possible
without technological tools. We sum up some of the most
remarkable changes:
• Thanks to its direct feedback, the graphics calculator
offers opportunities for exploratory activities. Even
during the initial phase of familiarization, a problem
can often already be investigated graphically. Inventory
and classification activities can lead to discoveries
which then, through reflection and generalization, result
in interesting mathematical theorems. This contrasts
with the traditional method, in which definitions and
theorems are stated at the beginning of the learning
path in the expectation that insight will be acquired
through repeated application.
• One can quickly and effectively follow the results of
variation in the problem. The influence of a given
parameterin the formula can easily be visualizedby using
graphics. This enables users to continually alter their
frame of reference from global to local and vice versa.
• Due to the use of technological tools, the repertoire of
techniques and skills a student must master will
remarkably change. Freehand drawing of a graph based
on a strictly prescribed analysis of functions—a much
practiced skill in the past—will hardly matter. On the
other hand, skills such as estimating numerical values,
reading graphs, setting viewing windows and improving approximations increase in significance. A shift in
emphasis occurred away from traditional techniques
and towards a more flexible solution procedure, in
which a critical attitude is developed with respect to
numerical results.
The following example illustrates the ways in which
students made use of the graphics calculator's potential for
explorative activities. In Fig. 8, two coins lie next to one
another. The coin on the left is fixed. P is the point on the
edge of the coin on the right where it touches the coin on
the left. The coin on the right is now going to roll around
the edge of the coin on the left without sliding. The
problem is: what is the locus of point P?
The first questions in the task contained a concrete
orientation, the choice of an appropriate coordinate system
and the constructing of the motion equations. So far,
questions with straightforward solution strategies for these
students. Then the students draw the curve with their
graphics calculator, creating a cardioid-shaped image. The
distance of the moving point to the centre of the coin can
be changed. This can be generalized and the students can
investigate a variety of cases. This activity is new to them,
and they do not have standard solution strategies at hand.
They are free to classify the graphs they find as they
choose. The focus is on their investigational activities.
The graphics calculator immediately revealed any error
by drawing an 'impossible' curve. This led, on the whole,
to an improvement in the initial motion equations. For
instance, a number of students entered the following:
x0tP ¼ 2 cos t 1 þ cos 02tP
y0tP ¼ 2 sin t þ sin 02tP
The drawing in Fig. 9 revealed what was wrong.
This could now be improved! The students then entered:
x0tP ¼ 2 cos t 1 þ cos 02t þ pP
y0tP ¼ 2 sin t þ sin 02t þ pP
Ruthven (1992) uses the expression 'trial and improve'
for such problem solving skills with a graphics calculator at
hand. The matter of varying the distance between P and the
centre of the rolling coin could also be dealt with. Students
create graphs as shown in Fig. 10.
Interesting drawings result from these so-called Pascal
curves. During the last five minutes of the lesson, the results of classifications of the curves were listed on the
board.
Fig. 8 The problem situation with the rotating coin
Fig. 9 The cardioid on the screen of a graphics calculator
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123a Shape
0 Circle
1 Cardioid
0 < a < 1 'dent'
a > 1 Loop
One student comments on the 'dent': ''When a is small,
the dent is also small, and when a gets bigger, the dent
grows too.''
Where the loop is concerned, the teacher asks: ''Does it
matter how much greater a is than 1?''
Student: ''When a is close to 1, it's small; if a is big, it's
big.''
Teacher: ''So does the small one ever catch up with the
big one?''
Student: ''No, they'll overlap, they'll get close together.''
The students watched the curves being created in a
dynamic manner. The graphs became a starting point for
further study, rather than the static end product of a vast
ritual of investigation into intersections of the axes, horizontal and vertical tangents, etc. The graphics calculator is
an excellent aid here. The students 'saw' the point on the
plane shift and the speed of the movement became visible.
The parametric equations that describe the movements on
the plane really came to life this way.
The user can see how a graph (or a curve or a group of
graphs) is created. When the picture is finished, TRACE
offers the opportunity to follow the graph point by point
and read the changing coordinates on the screen. The
graphics calculator's direct feedback encourages the students to reflect upon what they have done. This alternation
between experimentation and reflection is an important
aspect of problem solving and concept development.
As a consequence, the graphics calculator supports
exploratory activities. The students arrive at a classification
that they themselves have developed through their own
experience. The potential of the technological tool evidently
stimulates many students to engage in informative exploratory activities. The mathematical level of the result, however, tends to be rather varied. While such exploration can
lead to excellent discoveries and insights, the results are at
times disappointing. This has to do, on the one hand, with the
fact that drawing conclusions from investigative activities
places heavy demands on the students. Moreover, students
have to learn to deal with open problems, to appropriate new
tools and to develop an investigative attitude.
In addition to the aforementioned study on the opportunities that the graphics calculator offers to mathematics
education, recent research projects at the Freudenthal
Institute focus on tool-use in mathematics education and
concentrate on computer minitools (Bakker 2004, Doorman 2005, Drijvers et al. 2007) as well as on computer
algebra (Drijvers 2003).
To summarize the findings, we notice that the dynamics,
the direct feedback, and the possibilities to deal with complex
situations and to investigate many cases offer new opportunities for problem solving activities in mathematics education. The skills a student is expected to master are different
from those required in traditional situations. We conjectured
that the importance of mastering routine operations would
decline and the necessity for developing a flexible attitude for
working with tools would increase. Still, like the experiences
with problem solving in the mathematics A curriculum, the
difficulty for teachers and task designers is to avoid problemsolving tasks becoming routinized. If the students after their
work on the previous example on the cardioid would
encounter a task concerning the cycloid, their approach
would be much more standardized.
8 Conclusion
If we look back at recent developments in mathematics
education in The Netherlands, some remarkable observations can be made. While serious attempts have been made
to implement a problem oriented curriculum, based on
principles of realistic mathematics education with room for
modelling and with integrated use of technology, we notice
that this has been successful in educational practice only to
a limited extent. The main difficulties encountered first
involve the national examination, which strongly drives
educational practice, but by its structure does not allow for
real problem solving activities. A second obstacle is
formed by the textbook series, which are not very open to

problem solving and do not address its learning.
An important challenge is the design of good problem
solving tasks that are original, non-routine and new to the
students. Experiences with the Mathematics A-lympiad and
the Mathematics B-day suggest that such tasks best can be
developed in a team of educational designers and teachers.
This is a non-trivial process.
Fig. 10 Shamrock-shaped curve
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123To overcome the difficulties, some measures should be
taken—and to a certain extent are already in progress. First,
we recommend textbook series to pay explicit attention to
problem solving skills and heuristics. Second, schools
should take the opportunity that school assessment provides
for problem solving activities. National initiatives such as
Mathematics A-lympiad and the Mathematics B-day offer
inspiring opportunities to do so. Teacher networks for
designing problem solving tasks can be fruitful, as might be
an exchange through a virtual community. Third, we recommend that teachers and textbook authors exploit the
benefits of technological tools for the purpose of problem
solving. Through technology, new mathematical horizons
can be opened that lend themselves for exploration and
problem solving activities. Fourth and finally, we recommend problem solving to be on the 'agenda' ofmathematics
research in primary education, so that students start to develop problem-solving skills at an early age.
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Codes (31)###design of activities, ###guided reinvention, ##advantages of using RME, ##characteristics of RME, ##content where need help to relate to real life, ##disadvantages of using RME, ##human activity, ##real life, ##RME tasks, $$$approaches to teaching Mathematics, $$content related to real-life, $$mathematics education, $$meaningful learning of Maths, $$problem areas to teach/learn, $$teaching methods and strategies, $factors that influence teaching and learning, $mathematics content, %community, %context, %learner, %role players, %teacher, &&integration across strands, &social aspects, @@challenges when working with ICT, @@ICT systems: virtual learning environment (VLE), @@teacher-student ICT factors, @@technology as educational tool, @@value of using ICT, @Devices, @Tools

Memos (0)

Document body:

Interviewer: My study is about using technology, simple technology like cell phones, Whatsapp, Facebook, the interactive whiteboards, to help teachers to make the Maths more realistic for the learners. I want to work with a group of teachers, and where they have problem areas such as for example they have a problem teaching trigonometry or whatever the case may be. What I'm going to do is I'm going to design stuff using the cell phone, the Whatsapp, Facebook, interactive whiteboards to give it to the teachers and say here is an idea, try this in your classroom, see if that works, maybe this is a better approach. So that is roughly what my study is about. So I just want to ask these questions, and like I say answer what you know, you are welcome to feel free and relaxed as you answer. What sections of the curriculum are most problematic for you to teach?
Participant 1: Uhm, it's financial Maths, probability and what is it .... Geometry.
Interviewer: Geometry, okay because that's also new to the syllabus, the geometry.
Participant 1: But I was teaching geometry before.
Interviewer: Okay.
Participant 1: So, I'm learning again.
Interviewer: Ja.
Participant 1: Step by step now to teach geometry.
Interviewer: Okay, and probability? You don't really have an experience of probability either? Because that was also never in the syllabus before?
Participant 1: Ja, I'm also learning to teach probability. We have been to workshops of probability.
Interviewer: Okay.
Participant 1: Ja, but it takes time for one to master the curriculum.
Interviewer: Of course ja, and practice.
Participant 1: Ja, practice.
Interviewer: Ja, often the first year that you teach something, you don't know that content as well as the second year.
Participant 1: Exactly, the thing is, I started learning probability when I was teaching grade 12. After Papi left, then I took over. So I wanted to do it from grade 10.
Interviewer: I see, ja.
Participant 1: So this is how now I'm learning, right from grade 12. Because if you're teaching it at grade 12, there are some of the concepts that you don't know at the grade 11 and grade 10 level.
Interviewer: True, so that continuation is important?
Participant 1: It's important, ja.
Interviewer: That you start with the basics and then ...
Participant 1: Go up.
Interviewer: That an important aspect, good.
Participant 1: So this is now that I am learning that basics, I have taught last year grade 10 and grade 11, now in grade 12 I will know what to teach.
Interviewer: Good, now you have been exposed to the 3 levels.
Participant 1: Ja.
Interviewer: How often are you able to relate the content you are teaching to real-life situations? Is it possible or is it not really possible?
Participant 1: Ja, to some topics it is possible.
Interviewer: Okay, do you have the time to do it? Or is it not really realistic to expect one to relate the work to real-life?
Participant 1: Some of the topics, they come clear to real-life situations.
Interviewer: Okay, but others it's difficult to relate?
Participant 1: It's difficult to relate.
Interviewer: Ja, which sections of work do you think will be easiest to link to real-life situations?
Participant 1: Calculus, and also probability.
Interviewer: Good, ja.
Participant 1: And also financial Maths, because we are working with money everyday.
Interviewer: Good, ja, interest and stuff like that.
Participant 1: You see, most of our learners they are not, how can I say it, they are not familiar with the situations.
Interviewer: Hmm.
Participant 1: Like for instance you cannot just ask somebody from the farms about Kentucky, you see?
Interviewer: True, because they don't know what it is.
Participant 1: So if you're talking about the money in the bank, they have never been to a bank.
Interviewer: Hmm.
Participant 1: And how it operates, they don't have, what you call ....
Interviewer: An account.
Participant 1: Ja, an account. So it is difficult to relate this to somebody who doesn't know about the situation.
Interviewer: True.
Participant 1: That's my real problem.
Interviewer: So, it might link to real-life but the learners don't always know about those contexts.
Participant 1: Ja, those contexts.
Interviewer: Good. In which sections of the curriculum do you need help to link the content to real-life situations?
Participant 1: Like I said, in calculus, what is it .... ? When it has in calculus ...
Interviewer: The graphing? The application of calculus?
Participant 1: The application of calculus.
Interviewer: Okay, for example when they say the ball bounces, or determine the maximum speed or the minimum, okay the maximum, minimum.
Participant 1: Ja.
Interviewer: Do you think it is important to relate your teaching to real-life situations? Or is it not so important?
Participant 1: Definitely, it is more important to relate it to more real-life situations.
Interviewer: Hmm.
Participant 1: Like, I said, some of the learners they don't know, when you're playing football or any other sports, Maths is applicable everywhere.
Interviewer: True.
Participant 1: You see?
Interviewer: Hmm.
Participant 1: So they cannot relate what they are doing with Mathematics.
Interviewer: Ja.
Participant 1: When they are sleeping, when they get up, Maths is applicable to them, every way, they don't know that, so they take it as something that is very special, they put it aside.
Interviewer: And they think it's difficult, meantime is something that is part of their lives, and I think if they can see that as part of the lives, it will become less of an abstract subject.
Participant 1: They take it as an abstract subject.
Interviewer: Hmm.
Participant 1: Even their parents, they are worse than these learners.
Interviewer: True.
Participant 1: When we have a problem with a learner who does not do work, they always tell the learner, oh Maths, I've long done Mathematics in grade four, I don't know nothing about Mathematics, you see?
Interviewer: And they tell that to the children?
Participant 1: Ja, so it's difficult to teach Maths, very difficult.
Interviewer: It is. How do you feel about using technology in your teaching? Think of any form of technology, like I said interactive whiteboards, cell phone, computer, data projector?
Participant 1: Ja, last time when we were using the overhead projectors, but now the state of our school we cannot, we have so many projectors here, but they are not functioning.
Interviewer: Really?
Participant 1: It was easy to use them, rather than making use of the chalk.
Interviewer: Is it because you could prepare in advance?
Participant 1: Ja, we could prepare in advance. And then presently we've got only one projector in the school, so it is difficult to use the laptop when teaching. And some of the classes do not have electricity. That's another problem.
Interviewer: So, your facilities and your resources are a problem?
Participant 1: Ja, it's a problem.
Interviewer: If .... imagine it was a perfect world, and you could use technology, what would you like to use?
Participant 1: I like to use technology.
Interviewer: Do you?
Participant 1: Ja, if I'm given a chance everything is okay.
Interviewer: Any form of technology?
Participant 1: Ja, any form of technology.
Interviewer: Good. How do you feel about using mobile technology such as Whatsapp and SMS to discuss with me and other colleagues, possible ways of making the mathematics content more realistic for you learners? If you for example could send me an SMS is to say "I'm really battling to teach them the section, help me, give me some ideas", or perhaps to a colleague to say "how far are you with that section?" How do feel about that?
Participant 1: It's a good thing, it's a good thing.
Interviewer: Okay, and you would be willing to use something like that if it were available to you?
Participant 1: Ja.
Interviewer: Okay, what about a Whatsapp group? Would you be willing to join a Whatsapp group was colleagues, and with me? So that you can discuss maybe the same thing? Somebody could put on the group I've just taught the section of financial Maths and I had this good idea, try it in your class. Or I'm battling with that. How can you help me? Have you taught this already? A group? Is that okay? Would you be willing to be involved in something like that?
Participant 1: Yes.
Interviewer: Okay good. What experience do you have with Facebook, if any?
Participant 1: No, I don't use Facebook.
Interviewer: You don't use Facebook?
Participant 1: In fact I don't have time, ha ha.
Interviewer: Time is a problem.
Participant 1: Ha ha .... time as a problem.
Interviewer: True.
Participant 1: Time is a problem, because now I'm having extra classes with the grade 11's on Wednesdays and also on Saturdays.
Interviewer: So you don't have time for social media.
Participant 1: I don't have time for social media, I'm having a phone.

Interviewer: If one were to create a work group on Facebook, also to help with the Maths, do you think you would be interested? Or not actually?
Participant 1: I will be interested.
Interviewer: Okay.
Participant 1: Ja.
Interviewer: Good.
Participant 1: At least it will be for a good cause.
Interviewer: That's it, ja. Okay, so that question about how do you feel about using Facebook to discuss issues of making Maths more realistic for your learners? Rather than really social media? Then it's more like work on Facebook, you would be interested in using that?
Participant 1: Ja.
Interviewer: Good. What experience do you have regarding interactive whiteboards? You've got lots.
Participant 1: No, in fact I do not control it. Because that room is for CAT, and then the gentleman who showed you where I am, he is the one who is in charge, Mr Olifant.
Interviewer: Now, you've now noticed the learners sitting in the class at the interactive whiteboard sessions. What did you pick up there? You mention some of it in the classroom just now.
Participant 1: I always go and sit there doing my work. At least I can see there are two or three learners who are listening very attentively and follow what is going on and they ask questions. Some of them don't even bother to ask questions.
Interviewer: And maybe do they get bored? Some of them?
Participant 1: Ja, they are bored.
Interviewer: Because there is not somebody standing in front, it's like watching TV almost? And the show just carries on?
Participant 1: Ja, it carries on but it is very clear and loud. Everybody can see what is going on, even how it is worked.
Interviewer: But even then some of the learners don't take part? They sit quietly at the back?
Participant 1: At the back.
Interviewer: Okay.
Participant 1: Maybe it's those learners who are not interested in the subject.
Interviewer: Could be.
Participant 1: Hmm.
Interviewer: How do you feel about using the interactive whiteboard to discuss with me and with other colleagues issues about Maths, how to make it more realistic for your learners? How to deal with problem areas? Would you be willing to have a discussion on an interactive whiteboard?
Participant 1: In fact, I'm very happy when you are broadcasting because I said to my learners if it is somebody who is presenting, and then I am listening, so Maths is international because if I'm teaching this and somebody is teaching that, maybe you don't understand what I'm saying, but if you hear it from the second person, it comes nicely right.
Interviewer: True, sometimes they don't believe the own teacher. They want to hear it from somebody else.
Participant 1: Maybe they are ashamed to ask questions, and here they hear it for the second time, so somebody is going to ask questions about that, or maybe get clarity about that.
Interviewer: True, okay, that's interesting. What type of technology would you be most comfortable with in a project such as this one? Where you get involved with me and other colleagues to talk about Maths, to solve problems, to deal with issues? What kind of technology would suit you the best?
Participant 1: Laptop.
Interviewer: Laptop, okay.
Participant 1: I've got a laptop, but as I'm saying we only have one projector.
Interviewer: Ja.
Participant 1: I just wanted to buy myself one.
Interviewer: And for communicating with others? Email in other words? Or?
Participant 1: Ja, I could ...
Interviewer: Do you have access to email?
Participant 1: My email is not working, but I will try to fix it.
Interviewer: Okay. I'm just thinking of what can work for you, what would suit you the best?
Participant 1: Okay.
Interviewer: So, email can work for you?
Participant 1: Normally I'm using the email of the school.
Interviewer: Okay, thank you Mr Sindwa. Thank you for the valuable information.
Participant 1: It's my pleasure.

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WHY TO TEACH MATHEMATICS
SO AS TO BE USEFUL
My
first task at this moment is to welcome you who have come here from
various countries to sacrifice one week of your holidays
for the benefit of
mathematical education all over the world. I trust this meeting will be as
useful as according
to the general
theme of this conference mathematical
education should be held to be. I trust we all will learn as much from each
other's experiences and arguments as we like to do and often have done at
such opportunities. With great satisfaction I remember the meeting of
December 1964 at Utrecht and I hope
the few among you who have partici-
pated
in that conference will share my feelings of gratitude. But whenever I
shall remember those pleasant days and evenings, and
lively discussions, I
will never
forget
the man whom I met first and last on that occasion, the
liveliest among all of us,
the much regretted Wittenberg,
this
fiery nature who
died much too early as
though he had burnt himself in his own fire. Though
I admit there was none among us who shared his opinions,
I am sure
everybody was
impressed by his honest search into the truth of our education-
al philosophy. To my mind, his definitive absence overclouds the bright
sky of this day.
The present colloquium
is an activity of the ICMI sponsored by
the
government of the Netherlands and by IMU. It is not the first in this new
period of ICMI and in this year. In January we met in Lausanne with the
physicists,
in a meeting sponsored by UNESCO, which was attended by
some among you.
In my opinion
the resolutions adopted at Lausanne are a
mile-stone in the philosophy of mathematical education.* If I substitute my
wishes and hopes
for my opinion,
I would say they should be so. It is
evident that the use of mathematics has been a key criterion in all arguments
on mathematics at that meeting.
In this
introductory address I feel I have to
justify
the general
theme of the
present conference rather than to tell you about
techniques of
teaching
useful mathematics. This means that I will not
speak about how to teach
mathematics so as to be useful but about why we should teach mathematics
so as to be useful, or rather about why we should teach mathematics so as
to be more useful.
Of course this is a question of educational philosophy, and as such it will
*
See pp. 244-245.
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be answered in a different way according
to which philosophy we adhere to.
Yet educational philosophy
is not an abstract system. It depends on the real
educational system in which we live, and on our, positive or negative, attitude
with respect
to that system. Is the variety of national educational philoso-
phies really a drawback to international talks on mathematical education or
should I say that there is no better opportunity
to test them than to have
them bump against each other? Are not we too often and too readily
in-
clined, when reading or hearing about the educational experiences
in
another country, under another educational system,
to sigh:
it is just a pity,
but this does not apply
to our situation? I would say whenever this happens,
then something
is wrong either in the one system or in the other, or, most
likely,
in both.
It is generally admitted that there is a wide gap between the educational
philosophy of the U.S.A. and the Socialist European countries on one hand
and the continental Western European countries on the other hand, though
this gap has been narrowing
to a considerable extent. On the one side one has
for
long
times pursued
the ideal of one kind of education for all youth, on the
other side one has always overstressed that part of the educational system
which provides educational facilities for a small group of students selected
more on social than on intellectual grounds. I have to admit, and I do it
with shame and distress, that in the Western countries of continental Europe,
if we speak about mathematical education, we more often than not, mean the
gymnasiums and
lycees, and
tacitly forget
the about more than 90% who do
not attend this
type of schools. I agree
that a more balanced educational
system can be as bad if its highest
level is too low to do
justice
to the most
gifted students. But instead of discussing
the question which kind of in-
justice
is the least evil, I would rather try to do the most justice
to all people
and to the society they belong to.
I need not explain
to you why mathematics can be useful
though
the fact
itself is one of the most recent and most astonishing features of the history of
civilization. It would be more difficult to tell how mathematics can be useful
provided that we do not limit ourselves to counting up
instances of the
all-pervading
influence of mathematics in our culture, but ask what happens
in the individual if he applies mathematics or if he tries to. Much has been
done to
investigate

the
learning process, though
it is a fact that most of this
research has been rather
laboratory
than classroom-oriented. Very little,
if anything,
is known about how the individual manages
to apply what he
has
learned, though such a knowledge would be the key
to understanding
why most people never succeed in putting
their theoretical knowledge
to practical use.
Since mathematics has proved indispensable
for the understanding and
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the
technological control not only of the physical world but also of the social
structure, we can no
longer keep silent about
teaching mathematics so as to
be useful. In educational philosophies of the past, mathematics often
figures
as the paragon of a disinterested science. No doubt it still is, but we can no
longer afford to stress this point
if this keeps our attention off the widespread
use of mathematics and the fact that mathematics is needed not by a few
people, but virtually by everybody.
Mathematics is distinguished
from other
teaching subjects by the fact
that, even in its actual totality,
it is a comparatively small body of know-
ledge, of such a generality
that it applies
to a richer variety of situations than
any other
teaching subject. Modern mathematics can be seen as an effort to
reduce this body of knowledge even more and to enhance the
flexibility of
what remains to be
taught. At the same time this fact about mathematics is
the source of the principal dilemma in
teaching mathematics so as to be
useful. In an objective sense the most abstract mathematics is without doubt
the most flexible. In an objective sense, but not subjectively, since it is
wasted on individuals who are not able to avail themselves of this flexibility.
On the other hand, teaching applied mathematics is as bad,
if it means
mathematics in a specialized context, which does not account for the
greatest virtue of mathematics, its flexibility.
Though
it might
look different, I am still busy with the question why
mathematics has to be
taught so as to be useful, after we had agreed
that it is
useful and that students are expected
to use it. There are two extreme at-
titudes: to teach mathematics with no other relation to its use than the hope
that students will be able to apply
it whenever they need to. If anything,
this
hope has proved
idle. The huge majority of students are not able to apply
their mathematical classroom experiences, neither in the physics or chem-
istry school
laboratory nor in the most trivial situations of daily
life. The
opposite attitude would be to teach useful mathematics. It has not been tried
too often, and you understand that this is not what I mean when speaking
about mathematics being taught
to be useful. The disadvantage of useful
mathematics is that it may prove useful as
long as the context does not
change, and not a bit
longer, and this is
just
the contrary of what true
mathematics should be. Indeed it is the marvellous power of mathematics to
eliminate the context, and to put
the remainder into a mathematical form in
which it can be used time and again.
Between two extreme attitudes one may be inclined to
try compromising.
If this means
teaching pure mathematics and afterwards to show how to
apply it, I am afraid we are no better off. I think this is just the wrong order.
I have always considered it a remarkbale fact that people are able to apply
simple arithmetic, but not quadratic equations or even linear functions. Do
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not object
that arithmetic is so easy. It is not. Take such problems as:
If I have got ten marbles and I give three away, how many are left?
If I have got
ten marbles, and John has three
less, how many does he have?
If there are ten students in the room and three are girls, how many are boys?
If I am ten years old now, how old was I three years ago?
If B is between A and C, B is at a distance of 7 miles from A, and C is at a
distance of 10 miles from A, how far is B from C?
It is not so easy
to learn that in all these and a hundred other situations the
same arithmetical operation applies. It takes some time, but
finally everybody
succeeds in understanding
it. Why? I daresay, because arithmetic starts in a
concrete context and patiently returns to concrete contexts as often as
needed. The counterexample
is fractions. In its traditional
teaching
the
concrete context is no more than a ceremony which is hurried through
in a
jiffy.
If afterwards the abstract theory of fractions has to be applied,
its
comes too late, on too high a level, and is not connected to any previous
experience on a level where fractions should have been introduced. What is
the reason for this change of attitude of the teacher? Is the patience of the
schoolmaster exhausted when fractions turn up? I believe the answer is
rather that the schoolmaster himself does not know fractions in a concrete
context, and that for this reason he is not able to teach them in a more
responsible way
than he is used to do.
I am afraid this answer applies
to the greater part of our mathematics
teaching. Even the fact that a teacher applies mathematics himself, does not
necessarily imply
that he knows how he is able to do so and to use such a
knowledge
in his
teaching.
The problem, however,
is still much more serious. In the past, and mostly
even now, textbook writing has been dominated by quite other aims than
by
the goal of a mathematics that could be useful. Mathematics is a peculiar
subject. Arithmetic and geometry have sprung
from mathematizing part of
reality. But soon, at least from the Greek antiquity onwards, mathematics
itself has become the object of mathematizing. Arranging and rearranging
the subject matter, turning definitions into theorems and theorems into
definitions, looking
for more general approaches
from which all can be
derived by specialization, unifying several theories into one - this has been
a most fruitful activity of the mathematician, and no doubt our students
are entitled to enjoy
these fruits. No doubt modern mathematics is both
much more flexible and much
simpler
than the mathematics of
fifty years
ago. No doubt our students have to learn the most modern mathematics.
Teachers are more and more prepared and more and more inclined to bridge
the gap between school mathematics and grown-up mathematics which had
become wider from year to year.
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All use subject to JSTOR Terms and ConditionsWHY TO TEACH MATHEMATICS SO AS TO BE USEFUL
However, this is not the whole story. The problem
is not what kind of
mathematics, but how mathematics has to be
taught. In its first principles
mathematics means mathematizing reality, and for most of its users this is
the final aspect of mathematics, too. For a few ones this activity extends to
mathematizing mathematics itself. The result can be a paper, a treatise, a
textbook. A systematic textbook is a
thing of beauty, a
joy
for its author,
who knows the secret of its architecture and who has the right
to be proud
of it. Look how such an author would
justify his construction: Why have
you defined addition on page 10 in such a circumstantial way?
- because
this more general definition will prove useful on p. 110. Why have you
proved
this geometrical
theorem in such an unnatural manner? - because
at this stage I restrict myself
to affine notions which have to precede metric
notions. Why do not you mention forces as an instance of vectors? - because
mechanics has to be based upon vector algebra and not the other way
round.
Systematization
is a great virtue of mathematics, and if possible,
the stu-
dent has to learn this virtue, too. But then I mean the activity of systema-
tizing, not its result. Its result is a system, a beautiful closed system, closed,
with no entrance and no exit. In its highest perfection
it can even be handled
by a machine. But for what can be performed by machines, we need no
humans. What humans have to learn is not mathematics as a closed system,
but rather as an activity,
the process of mathematizing reality and if possible
even that of mathematizing mathematics.
New mathematics has been met with criticism. People who apply mathe-
matics often feel uneasy when observing
that the mathematics
they have
been used to apply
is replaced by something they judge
less suited for appli-

cations. It is a fact that biologists, economists, sociologists are better pre-
pared
to apply modern mathematics than physicists who carry the burden
of a
longer tradition. In the universities the gap between the mathematics
of mathematicians and that of physicists has become
terrifying. It is a habit
of physicists
to treat any particular subject with that kind of mathematics
which prevailed at the time when that subject
turned up
in the history of
physics. For
instance, though physicists know eigenvalues of symmetric
matrices because Laplace
introduced them in a physical context, they still
deal with orthogonal matrices with such oddities as Eulerian angles, because
Euler was not yet acquainted with eigenvalues.
It would be a disaster if this
lag would become permanent, though I hope
it will not. Time ago I eavesdropped on a talk between a physics professor
and his assistants, criticizing his course and particularly such a subject as
Lagrange multipliers: this is not physics, one of them said, this is plain
linear algebra.
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Probably we will have to wait for the next generation
to have physicists
reconciled with modern mathematics teaching.
It is a pity
that most of the criticism against modern mathematics is made
with no knowledge about what modern mathematics really
is. It is a pity,
because there is ample reason for such criticism as
long as mathematicians
care so little about how people can use mathematics. We are not entitled
to reproach physicists
for
identifying modern mathematics with a preposter-
ous educational philosophy, since this identification is of our own making.
I am convinced that, if we do not succeed in
teaching mathematics so as to
be useful, users of mathematics will decide that mathematics is too
important
a
teaching matter to be taught by the mathematics teacher. Of course this
would be the end of all mathematical education.
Mathematisch Instituut der
Rijksuniversiteit, Utrecht
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RECONSTRUCTING MATHEMATICS PEDAGOGY
FROM A CONSTRUCTIVIST PERSPECTIVE
MARTIN A. SIMON, Pennsylvania State University
Constructivist theory has been prominent in recent research on mathematics learning and has
provided a basis for recent mathematics education refom1 efforts. Although constructivism has
the potential to inform changes in mathematics teaching, it offers no particular vision of how
mathematics should be taught; models of teaching based on constructivism are needed. Data are
presented from a whole-class, constructivist teaching experiment in which problems of teaching practice required the teacher/researcher to explore the pedagogical implications of his.theoretical (constructivist) perspectives. The analysis of the data led to the development of a model
of teacher decision making with respect to mathematical rasks. Central to this model is the creative tension between the teacher's goals with regard to studeot learning and his responsibility
to be sensitive ttnd responsive to the mathematical thinking of the studeots.
Constructivist perspectives on learning have been central to much of recent empirical and theoretical work in mathematics education (Steffe & Gale, 1995; von
Glasersfeld, 1991) and as a result, have contributed to shaping mathematics reform
efforts (National Council of Teachers of Mathematics, 1989, 1991). Although constructivism has provided mathematics educators with useful ways to understand learning and learners, the task of reconstructing mathematics pedagogy on the basis of a
constructivist view of learning is a considerable challenge, one that the mathematics
education community bas only begun to tackle. Although constructivism provides a
useful framework for thinking about mathematics learning in classrooms and therefore can contribute in important ways to the effort to reform classroom mathematics
teaching, it does not tell us how to teach mathematics; that is, it does not stipulate a
particular model.
The word "pedagogy," as used above, is meant to signify all contributions to the
mathematical education of students in mathematics classrooms. As such, it includes
not only the multi-faceted work of the teacher but also the contributions to classroom
learning of curriculum designers, educational materials developers, and educational researchers. Mathematics pedagogy might be operationally defmed using the
following thought experiment. Picrure 2Sleamers in an otherwise empty classroom.
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expressed in this material are those of the author and do not necessarily reflect the views of
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Jaroie Myers. Martin A. Simon 115
The ingredient necessary in order to initiate mathematics learning is pedagogy.
This paper describes data from a classroom teaching experiment in which the
researcher served as mathematics teacher, the analysis of that data, and an emerging theoretical framework for mathematics pedagogy that derives from the analysis. The paper contributes to a dialogue on what teaching might be like ifir were
built on a consTructivist view of knowledge development. The specific focus of trus
paper is on decision making with respect to the mathematics content and mathematical
tasks for classroom learning.
This article begins with an articulation of the constructivist perspective that undergirds the research and teaching and then provides a review of the pedagogical theory development based on constructivism that preceded this study and contributed
to its theoretical foundation. The study reported here examines the pedagogical decisions that result from the accommodation of the researcher's theoretical perspectives to the problems of teaching.
A CONSTRUCTIVIST PERSPECTIVE
The widespread interest in constructivism among mathematics education theorists,
researchers, and practitioners has led to a plethora of different meanings for "constructivism." Although terms such as "radical constructivism" and "social constructivism"
provide some orientation, there is a diversity of epistemological perspectives even
within these categories (cf. Steffe & Gale, 1995). Therefore, it seems important to
describe briefly the constructivist perspective on which our research is based.
Constructivism derives from a philosophical position that we as human beings
have no access to an objective reality, that is, a reality independent of our way of
knowing it. Rather, we construct our knowledge of our world from our perceptions
and experiences, which are themselves mediated through our previous knowledge.
Learning is the process by which human beings adapt to their experiential world.
From a constructivist perspective, we have no way of knowing whether a concept matches an objective reality. Our concern is whether it works (fits with our
experiential world). Von Glasersfeld (1987, 1995) refers to this as "viability," in
keeping with the biological model of learning as adaptation developed by Piaget
(1970). To clarify, a concept works or is viable to the extent that it does what we
need it to do: to make sense of our perceptions or data, to make an accurate prediction, to solve a problem, or to accomplish a personal goal. Confrey (1995) points
out that a coroUary to the radical constructivist epistemology is its "recursive fidelityconstructivism is subject to its own claims about the limits of knowledge. Thus,
[constructivism] is only true to the extent that it is shown useful in allowing us to
make sense of our experience:' When what we experience differs from the
expected or intended, disequilibrium results and our adaptive (learning) process
is triggered. Reflection on successful adaptive operations (reflective abstraction)
leads to new or modified concepts.
Perhaps the most divisive issue in recent epistemological debates (Steffe & Gale,
I 995) is whether knowledge development (particularly relational knowledge) is 116 Reconstructing Mathematics Pedagogy
seen as fundamentally a social process or a cognitive process. The difference in the
two positions seems to depend on the focus of the observer. The radical constructivist
position focuses on the individual's construction, thus taking a cognitive or psychological perspective. AJthougb social interaction is seen as an important context for learning, the focus is on the resulting reorganization of individual cognition.
For Piaget, just as for the contemporary radical constructivist, the "others" with whom
social interaction takes place, are part of the environment, no more but also no less than
any of the relatively "permanent" objects the child constructs within the range of its lived
experience. (von Glasersfeld, 1995)
On the other hand, epistemologists with a sociocultural orientation see higher mental processes as socially determined. "Sociocultural processes are given anal]rtical
priority when understanding individual mental functioning rather than the other way
around." (Wertsch & Toma, 1995) From a social perspective, knowledge resides
in the culture, which is a system that is greater than the sum of its parts.
Our position eschews either extreme and builds on the theoretical work of Cobb,
Yackel, and Wood (Cobb, 1989; Cobb, Yackel, & Wood, 1993; Wood, Cobb, & Yaqkel,
1995) and Bauersfeld (1995), whose theories are grounded in both radical constructivism (von Glasersfeld, 1991) and symbolic interactionism (Blumer, 1969). Cobb
(1989) points out that the coordination of the two perspectives is necessary to understand learning in the classroom. The issue is not whether the social or cognitive dimension is primary, but rather what can be learned from combining analyses from these
two perspectives. I draw an analogy with physicists' theories of light. Neither a particle theory nor a wave theory of light is sufficient to characterize the physicist's data.
However, it has been useful to physicists to consider light to be a particle and to consider light to be a wave. Coordinating the findings that derive from each perspective
has led to advancements in the field. Likewise, it seems useful to coordinate analyses on the basis of psychological (cognitive) and sociological perspectives to order
to understand knowledge development in classrooms.
Psychological analysis of mathematics classroom learning focuses on individuals' knowledge of and about mathematics'. their understanding of the mathematics
of the others, and their sense of the functioning of the mathematics class. Sociological
analysis focuses on taken-as-shared knowledge and classroom social nonns (Cobb,
Yackel, & Wood, 1989). Taken-as-shared" (Cobb, Yackel, & Wood, 1992;
Streeck. 1979) indicates that members of the classroom community, having no direct
access to each other's understanding, achieve a sense that some aspects of knowledge are shared but have no way of knowing whether the ideas are in fact shared.
"Social norms" refer to that which is understood by the community as constituting
effective pruticipation in the mathematics classroom community. The social norms
'Ball (1991) defines knowledge of mathematics as conceptual and procedural knowledge of the subject and knowledge abom mathematics as "understandings about the nature of mathematical knowledge
and activity: what is entailed in doing mathematics and how truth is establ.ished in the domain. What
counts as a solution in mathematics? How are solutions justified and conjectures disproved? Which ideas
are arbitrary or conventional and which are necessary or logical?" (p. 7) Martin A. Simon 117
include the expectations that community members have of the teacher and students,
the conceptions of what it means to do mathematics in that community, and the ways
that mathematical validity is established.
Tt is useful to see mathematics as both cognitive activity constrained by social and cultural processes, and as a social and cultural phenomenon that is constituted by a community of actively cognizing individuals (Wood, Cobb, & Yackel, 1995).
We refer to this coordination of psychological and sociological analyses as
"social constructivism."
SOCIAL CONSTRUCTIVISM AND MATHEMATICS PEDAGOGY
Understanding learning as a process of individual and social constructiOn gives
teachers a conceptual framework with which to understand the learning of their students. Although the development of such understandings is extremely valuable, this
paper focuses on the question of how constructivism might contribute to a reconstruction of mathematics pedagogy. How might it inform the development of a framework for fostering and supporting learners' constructions of powerful ideas?
Wood, Cobb, and Yackel (1995) assert that
teachers must ... construct a form of practice that fits with their students- way of learning mathematics. Th.is is the fundamental challenge that faces mathematics teacher educators. We have to reconstruct what it means to know and do mathematics in school and
thus what it means to teach mathematics.
As I stated above, constructivism, as an epistemological theory, does not deftne
a particular way of teaching. It describes knowledge development whether or not
there is a teacher present or teaching is going on. Konold (1995) argues, "not that
a teacher's epistemology has no effect on how he or she teaches, rather that its effects
are neither straightforward nor deterministic." There is no simple function that maps
teaching methoddogy onto constructivist principles. A constructivist epistemology
does not determine the appropriateness or inappropriateness of teaching strategies.
Bauersfeld (1995) states,
The fundamentally constructive nature of human cognition and the processual emergence of themes, regularities, and norms for mathematizing across social interaction,
to bring the [psychological] and the social together, make it impossible to end up with
a simple prescriptive summary for teaching. There is no way towards an operationalization of the social constructivist perspective without destroying the perspective.
The commonJy used misnomer, "constructivist teaching," however. suggests to
the contrary that constructivism offers one set notion of how to teach. The question of whether teaching is "constructivist" is not a useful one and diverts anention
from the more important question of how effective it is. From a theoretical perspective,
the question that needs attention is, In what ways can constructivism contribute to
the development of useful theoretical frameworks for mathematics pedagogy?
It is overly simplistic and not useful to connect constructivism to teaching with
the romantic notion, "Leave students alone and they will construct mathematical I 18 Reconstructing Mathematics Pedagogy
understandings." Likewise, "Put students in groups and let them communicate as they
solve problems," is not much more helpful. History provides unsolicited empirical
evidence with respect to these approaches. Generations of outstanding mathematicians
who were engaged in mathematical problems, who communicated with their colleagues
about their work, required thousands of years to develop mathematics that we expect
our average elementary school students to construct (Richards, 1991). Thus, although
it is useful to have students work problems and communicate about their ideas, it does
not seem to be adequate as a prescription for mathematics teaching. The challenge
is, How can ma.lhematics teachers foster students' construction ofpoweiful mathematical
ideas that took the conunnity of mathematicians thousands of years to develop?
Richards asserts,
It is necessary [for the mathematics teacher] to provide a structure and a set of plans-
that support the development of informed exploration and reflective inquiry without taking initiative or control away from the student. The teacher must design tasks and projects that stimulate students to ask questions, pose problems, and set goals. Students will
not become active learners by accident, but by design, through the use of the plans that
we structure to guide exploration and inquiry? (p. 38 [Italics in the original])
Through empirical data and model building, this study attempts to examine the
process of constituting pedagogical designs.

RECENT THEORETICAL WORK ON PEDAGOGICAL FRAMEWORKS
Relatively little work in mathematics education has focused on the development
of theoretical frameworks for mathematics pedagogy consistent with constructivism.
This seems to be the result of several factors:
1. It is only recently that empirically based models for studying mathematics learning in classrooms have been articulated (cf. Wood, Cobb, Yackel, & Dillon,
1993). Earlier empirical work, which derived from, and contributed to, epistemological theory, focused on the cognitive development of individual learners (cf. Steffe,
von Glasersfeld, Richards, & Cobb, 1983).
2. Traditional views of mathematics, learning, and teaching have been so widespread that researchers studying teachers' thinking, beliefs, and decision making
have had little access to teachers who had well-developed constructivist perspectives and who understood and were implementing current reform ideas. As a
result there has been a Jack of connection between research on learning (which has
focused on constructivism) and research on teaching (which has focused for the most
part on traditional instruction).
3. The need for pedagogical frameworks is sometimes obscured by the tendency
to assume that constructivism defines an approach to teaching.
2
1 interpret Richards" s statement, "Students wiLJ not become active learners ... " as indicative of his interest in fostering more independent and reflective mathematical investigations and discussions among students. From a constructivist perspective. students are always active learners; however, the nature of what
is constructed in different classroom contexts may vary greatly. Martin A. Simon 119
Despite these factors, some important work has been done in recent years with
respect to rethinking mathematics teaching on the basis of a constructivist perspective
(in some cases without specific reference to constructivism). This work has focused
on identifying the roles of mathematics teachers and describing the nature of
"pedagogical deliberations" (Ball, 1993).
The Professional Standards for School Mathematics (National Council of
Teachers of Mathematics, 1991) envisions teachers' responsibilities in four key areas:
• Sening goals and selecting or creating mathematical tasks to help students achieve
these goals:
• Stimulating and managing classroom discourse so that both the students and
the teacher are clearer about what is being learned;
• Creating a classroom envirorunent to support teaching and learning mathematics;
• Analyzing student learning, the mathematical tasks, and the environment in
order to make ongoing instructional decisions. (p. 5)
Cobb, Wood, and Yackel (1993) elaborate the teacher's responsibilities in the
mathematics classroom. The teacher has the dual role of fostering the development
of conceptual knowledge among her or his students and of facilitating the constitution
of shared knowledge in the classroom community. Cobb et al. (1993) have
demonstrated that classroom conversations about mathematics, facilitated by the
teacher, result in taken-as-shared mathematical knowledge. They have also
described a second type of conversation that focuses on what constitutes appropriate and effective mathematical activity in the classroom. Such discussion contributes to the constitution and modification of social norms for mathematical acti vity, the contrat didactiqueJ (Brousseau, 1981).
Much of the teacher's responsibilities involve planning. However, the planning
of instruction based on a constructivist view of learning faces an inherent tension.
Brousseau emphasizes that students must have freedom to make a response to a situation on the basis of their past knowledge of the context and their developing mathematical understandings. If the situation leads the students to a particular response,
no real learning of the mathematical ideas underlying that response takes place. However,
" if the teacher has no intention, no plan, no problem or well-developed situation,
the child will not do and wiLJ not learn anything" (Brousseau, 1987, p. 8-my translation). Under these conditions, students learn other things, such as how to respond
appropriately to the teacher's leading questions.
Brousseau (1983). Douady (1985), Lampert (1990). and Ball (1993) have conducted
investigations into the nature of pedagogical thinking and decision making that
contribute to teacher planning. Brousseau (1987) asserts that part of the role of the
teacher is to take the noncontextualized mathematical ideas that are to be taught and
embed them in a context for student investigation. Such a context should be personally
meaningful to the students, allowing them to solve problems in that context. the solution of which might be a specific instantiation of the idea to be learned. (Ball's [1993)
l-y'he contrat didactique is also established by classroom routines that are not explicitly discussed. 120 Reconstructing Mathematics Pedagogy
notion of"representational context" seems consistent with Brous eau's siruarion.) The
teacher"s job is to propose a teaming situation within which students seek a response
to the milieu, not a response that is solely intended to please the teacher. For the problem to foster the learning of powerful mathematical ideas, the students must accept
the problem as their problem•: they must accept the responsibiHty for truth (Balacheff,
1990). Brousseau calls this the devolution of the problem.
The creation of appropriate problem contexts (situations a-didactiques) is not sufficient for learning. Brousseau points out that situations must be created for the decontexrualizing and depersonalizing of the ideas (simations didactiques). Learning involves
being able to use the ideas beyond the narrow context of the original problem situation. The teaching process should allow for this shift of pupils' interest from being
practitioners to becoming theoreticians" (Balacheff, 1990, p. 264).
Also necessary is what the French researchers call "situations for institutionalization" (Brousseau, 1987; Douady, 1985), in which ideas constructed or modified
during problem solving attain the status of knowledge in the classroom community.
This is consistent with the notion of mathematical knowledge as social knowledge,
as knowledge that is taken-as-shared by the classroom community. •
Lampert's (1990) useof"problems" corresponds with Brousseau's situations and
Ball's representational contexts. Lampert describes the pedagogical thinking in which
she engages to generate problems for her students.
At LhE beginning of a unit, when we were switching to a new topic, Lhe problem we started
with was chosen for its potential to expose a wide mnge of students' thinking about a
bit of mathematics, to make explicit and public what they could do and how they understand. Later problems were chosen based on an assessment of the results of the first and
subsequent discussions of a topic. moving the agenda along into new but related
mathematical territory. The most important criterion in pick.ing a problem was that it
be the sort of problem that would have the capacity to engage all of the students in the
class in mak.ing and testing mathematical hypotheses. These hypotheses are imbedded
in the answers srudenrs give to the problem, and so comparing answers engaged the class
in a discussion of the relative mathematical merits of various hypotheses, sening the
stage for the k.ind of zig-zag between inductive observation and deductive generalization that Lakatos and Polya see a characteristic of mathematical activity. (p. 39)
Such pedagogical thinking must be built on knowledge of mathematic and knowledge of students and how they team mathematics. Ball (1993) points out that teachers must have a "bifocal perspective-perceiving the mathematics through the mind
of the learner while perceiving the mind of the learner through the mathematics" (p.159).
Steffe (1991) stresses that the teachers' plans must be informed by the "mathematics
of students." The most basic responsibility of constructivist teachers is to team the
mathematical knowledge of their students and how to harmonjze their teaching methods with the nature of that mathematical knowledge" (Steffe & Wiegel, 1992. p. 17).
Decisions as to the nature and sequence of the mathematics to be taught are made,
according to Laborde (1989). on the basi of hypotheses about epistemology
'Although each problem solver may construct a somewhat different understanding of the problem, negotiation commonly takes place in the classroom to arrive at a raken-as-shared interpretation of the problem. MartinA. Simon 121
and learning. Ball (1993) further explicates the investigative narure of teaching: "Teaching
is essentially an ongoing inquiry into content and learners and into ways that contexts
can be structured to facilitate the development of learners' understandings" (p. 166).
Ball states that research is needed to further understand the pedagogical deliberations in reform-oriented mathematics teaching. Building on the worl< of the researchers
cited above and starting from a social constructivist perspective on knowledge development, my paper continues the discussion of pedagogical deliberations that lead
to the determination of problem contexts for student involvement. In particular, the
paper extends the notion of teaching as inquiry, examines the role of different aspects
of teachers' knowledge, and explores the ongoing and inherent challenge to integrate the teacher's goals and direction for learning with the trajectory of students '
mathematical thinking and learning.
THEORY MEETS PRACTICE IN THE CONTEXT
OF A TEACHING EXPERIMENT
This section focuses on data from a classroom teaching experiment, in order to
analyze situations in which a constructivist theoretical perspective came up against
the realities of real students in a real classroom. The nonroutine problems of
teaching require an elaboration and modification of theories of learning and teaching. When the researcher/theorist assumes the role of teacher in a research project,
he is uniquely positioned to study in a direct way the interaction of his theory and
practice. Particularly, this report focuses on the teacher/researcher's ongoing decision making with respect to the mathematical content of the course and the tasks
and questions that provided a context for the study of that content. This section begins
with some brief background on the teaching experiment.
Background
The teaching experiment was part of the Construction of Elementary Mathematics
(CEM) Project, a 3-year study of the mathematical and pedagogical development
of prospective elementary teachers. The project studied the prospective teachers in
the context of an experimental teacher preparation program designed to increase their
mathematical knowledge and to foster their development of views of mathematics,
learning, and teaching that were consistent with the views espoused in recent
reform documents (e.g., National Council of Teachers of Mathematics, 1989:
1991). Data collection with 26 prospective elementary teachers (20 of whom finished the program) proceeded throughout a mathematics course, a course on
mathematics learning and teaching. a 5-week pre-student-teaching practicum,
and a 15-week student-teaching practicum.
The research on the mathematics course and the course on mathematics learning and
teaching employed a constructivist teaching-experiment methodology, as described
by Cobb and Steffe (1983) for research with individual subjects. We adapted that methodology to research on classroom mathematics (in the manner of Cobb et al., 1993).
The author taught all classes. Classes were videotaped and field notes were taken 122 Reconsrmcting Mathematics Pedagogy
by project researchers. Videotapes of classes were transcribed for analysis. The
author kept a rctlective notebook in which he recorded his thinking immediately following teaching and planning sessions. Following each class the author met with a
second project researcher to discuss what he and his colleague inferred the conceptualizations of the students to be at that point and to plan for the next instructional
intervention. (In this section, "studems" refers to the prospective elementary teachers participating in the teaching experiment.) These meetings were audiotaped.
The teaching-experiment methodology involves "hypothesizing what the []earner]
might learn and finding ways of fostering this learning" (Steffe, 1991 , p. 177). This
research report represents an extension of the teaching-experiment methodology. Whereas
the teaching experiment was created to learn about students' developing conceptions
(our primary emphasis), analysis of the decision making of the teacher/researcher
in posing problems is potentially a rich source for learning about teaching (Cobb.
personal communication). This paper is based on such an analysis.
Class lessons generally consisted of small-group problem solving and teacherled whole-class discussions. No lectures were given. The primary mathematical goal
of the course was for students to learn to identify multiplicative relationships
(Simon & Blume, 1994a, 1994b). Previous research on a variety of populations (Hart,
198 1: Inhelder & Piaget, I 958; Karp lus, Karplus, Formisano, & Paulson, 1979) and
our pretest data with this population of students had shown that identifying ratio relationships tends to be difficult and that additive comparisons are often used where
multiplicative comparisons (ratios) are more appropriate. The mathematical content of the course began with exploration of the multiplicative relationship involved
in evaluating the area of rectangles.
Data and Analysis
The data presented focuses on three teaching situations, examining the relationship
among the teacher's decision making and the classroom activities. These data are
from the first 5 weeks of the 15-week mathematics course, the duration of the first
instructional unit. and are taken from class transcripts. the teacher/researcher's notes,
field notes from other researchers, and student journals. The ftrst unit (eight 90-minute
classes) focused on understanding the multiplicative relationship involved in evaluating the area of a rectangle. The three instructional situations described represent
the three subtopics of the instructional unit. For each situation, a description is
provided of the challenge that faced the teacher as construed by the teacher, the
decision that he made to respond to that challenge, and the subsequent classroom
interaction that was constituted by the students and teacher.
Note that fundamental to the teacher's understanding of the challenge was his contructivist perspective, whkh included the idea that students construct their understandings, they do not absorb the understandings of their teachers. Each of the three situations represents an attempt to promote and support powerful
constructions. Whereas
telling students what they should understand (a lecture approach) is relatively straightforward, developing situarioiiS a-didactiques or representational contexts is complex
and uncertain. In this latter approach, mathematics teaching is continually problematic. Martin A. Simon 123
The rectangles problem. As the instructor, r chose to begin the exploration of multiplicative relationships in the evaluation of area of rectangles. My purpo e was to
focus on the multiplicative relationships involved, not to teach about area. The
lesson I chose was one that I had used several times before with similar groups
of students. The lesson grew out of my observation that although many prospec-
tive elementary teachers respond to area-of-rectangle problems by multiplying,
their choice of multiplication is often the result of having learned a procedure or
formula rather than the result of a solid conceptual link between their understandings of multiplication and their understandings of measuring area. This lesson, which
was designed to foster the development of that I ink, was planned to be completed
in one day, although I anticipated that it might continue into the next class.
The lesson began with a smaU cardboard rectangle being given to each of the smaU
groups of students seated at the classroom's six rectangular tables. The groups were
challenged to solve the following problem.
Rectangles problem 1. Determine how many rectangles, of the size and shape
of the rectangle that you were given. could fit on the top surface of your table.
Rectangles cannot be overlapped, cannot be cut, nor can they overlap the edges
of the table. Be prepared to describe to the class how you solved this problem.
Each group of students used the given rectangle as a measure to count the number of rectangles along the length of the table and the number of rectangles along
the width of the table and then multiplied these two quantities. (For an extensive
analysis of the quantitative reasoning involved in this instructional unit, see Simon
& Blume, 1994b.
5
) However, a few of the groups raised the question of whether the
orientation of the rectangle should be maintained for the second measurement (see
Figure 1 a), or whether it should be rotated 90 degrees so that measuring is always
done using the same side of the rectangle (see Figure lb).
I

Figure la. Maintaining orientation of
the rectangle.
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..... j
:
` . .
-- ----------, ---..... -..... ---.. ---
Figure lb. Rotating the rectangle to measure
the adjacent side.
During whole-class discussion, students described how they had solved the
problem. Then, to focus discussion on the multiplicative relationships. I asked them
5
Much of the data for this report was reported earlier in Simon and Blume (1994b). The earlier article focuses on the students' quantitative reasoning. This article revisits some of the data to unpack the
pedagogical issues. 124 Reconstr11cting Mathematics Pedagogy
why they had multiplied these numbers. Some responded by saying that "it seemed
like the easiest way," or "in previous math classes you learned the formula for areas"
(Simon & Blume, 1994b). Others said it works; the product is the same as the result
of counting up all the rectangles. I asked whether there was reason to expect that
it would always work. From my perspective, it is fundamental in mathematics to
consider whether a claim could be defended that the observed phenomenon would
always occur under a particular set of conditions. Most of the students seemed perplexed by this question. However, Molly explained:
Molly: Well, it would work because, um, multiplying and adding are related, in that multiplying is, is like adding groups, and so it would always work because you add them
up to see how many is in the square and to multiply the groups that go like that, that' LJ
always work. You would get the same number, I'm saying if you added them (Jr if
you multiplied that side times that side. Because you're adding, I mean, you-re multiplying the number of groups by the number in the groups, which is the same as adding
them all -P·
Molly clarified her explanation by demonstrating on the chalkboard how each row
of rectangles was a group (see Figure 2) and the number of rectangles in a row was
the number in each group. She showed that summing the rectangles in each row (rePeated
addition) was equivalent to multiplying the number in a row by the number of rows
(Simon & Blume, 1994b).
Figure 2. One group of four rectangles.
The other students' subsequent comments suggested that only a few of them perceived that Molly's explanation had advanced the discussion in any significant way.

Situation I. What instructional situation might afford other students the opportunity
to construct understandings similar to Molly's? It wasn't that the other students were
puzzled by Molly's explanation; they seemed unaffected by it. They continued to
respond to the question, "Why multiply?" in ways that indicated that the question
did not demand for them this type of justification. Responses included " ... 'cause that's
the way we 've been taught." and· · ... it's a mathematical law." Asking the students
for explanations and justifications was not sufficient. Our classroom community had
not established what counts for mathematical justification (Simon & Blume, in press).
It did not seem that convinuing this already lenghty discussion would be fruitful. Manin A. Simon 125
I had engaged them in a problem-solving activity using a hands-on activity and
fostered communication in small and large groups. Yet only a few of my students
showed evidence of learning the mathematics that I had intended them to learn. The
challenge that I faced was a product of both cognitive and social factors. Cognitively,
the majority of my students were employing a procedure that was well-practiced
but not well-examined conceptually. Socially, they did not have a view of mathematical activity in general and of appropriate activity for our classroom in panicular
that included the type of relational thinking and development of justification in which
I was attempting to engage them.
I realized that the development of norms for classroom mathematics activity would
take some time. Such norms would result from the activities in which we engaged
as a mathematical community and the discussions that we had about that activity.
Their competence in providing justification would grow as they engaged in discussions
in which the demand for justification was consistently present (Simon & Blume,
in press). Thus, from a social perspective, I needed to continue the process that I
had begun with them. However, this process could not happen in the abstract. Particular
content and tasks were needed as the context for the constituting of appropriate mathematical activity. Thus, I returned to my role of problem poser, but the question was,
"Which problems?" The traditional approach-assigning practice problems similar to the original one-seemed inappropriate. After all, the students were already
able to generate correct answers; the real problem was understanding why multiplication was appropriate. I needed to find problems that necessitated an understanding of the link between the solution strategy, counting the number of rectangles
along the length and the width and multiplying those quantities, and the goal of
determining the total number of rectangles that could be laid out on the table.
To generate such problems, I made use of conceptual difficulties that I had previously observed among students working on Rectangles Problem 1. For example:
Rectangles problem 2. Bill said, "If the table is 13 rectangles long and 9 rectangles wide, and ifi count 1, 2, 3 . . . , 13 and then again 1, 2, 3 ... , 9, and then
I multiply, 13 x 9, then I have counted the comer rectangle twice." Respond
to Bill's comment.
Problem 2 seemed to engage students in making a conceptual link between the
goal of counting all of the rectangles and the prevalent solution strategy of
counting rectangles along the two sides and multiplying. The following excerpts
from the class transcript show the development of these connections. (Note:
"Simon" refers to me, the teacher.)
Karen: When we're multiplying thirteen times nine we're trying to see ... how many nines
there are .... So if I'm looking at one nine, two nines, three nines, four nine· , I could
fmd out how much it would be with those numbers, but if I'm looking for thirteen
nines,} would want to see how many of tho e I would have if I would add them up
or if I would multiply them thirteen times. How many· how much would that be if
I had thirteen nines or nine thirteens? I'm looking for the amount-total amount that
that would be if I was multiplying how many groups of those or how many sets of
those would I have, if I would add each one of them up to get the total amount. 126 Reconstructing Mathematics Pedagogy
Karen has made some progress in justifying the use of multiplication. However,
Toni goes back ro how one use the formula appropriately. Her explanation is based
on her identification of the problem as an area problem and her knowledge of how
to measure length times width. Once again r attempt to refocus Toni (and,! suspect,
oLher students) on the underlying conceptual issue.
Toni: When we're trying to fmd how many rect.angles would fill that rectangle, we're looking for the area And when we find area we multiply length times width, and the columns
would probably represent the length and the rows would represent the width ....
Simon: Why does that work, that when we multiply the number of columns times the number of rows we get the area?
Molly: Well, r thought again it referred back to when you're using a row to represtlnt the
uniu. in a group. and the columns to represent the number of groups, and since multiplication is the same as repeated addition, that when you multiplied the number
of units in a group by the number of groups, you would get the total number of partS
in the whole.
Simon: And how is that connected to this issue about the comer?
Molly: Because it ... the comer not only represents a one, it's just one numbering of a grpup,
or it's also numbering a part of that unit-a unit io that group--so it's not, it's two
different things, just like when they were saying it's a row and a column, well, it's
two different things. It's a unit and also representing a group.
Candy: ... It makes it confusing to try to look at the length times the width .... You should
really treat it as so many sets or so many groups, like nine groups ... , thirteen groups
of nine. That way, you're not even going to deal with the comer and you won't even
have that problem.
Karen: Have we responded to Bill's problem about him thinking that he has doublecounted?
At this point, Karen brings us back to the original problem. For her, it is not enough
to decide whether one would be double-counting the comer rectangle; it is also important to understand BiLJ's thinking. which led to his confusion.
Karen: It appears to me that Bill's thinking about ... counting by ones .... one represents two
different things, but in his mind, at least from what he said, it appears that he only
sees one as representing one thing, and that is a counting number. He thinks he- s already
counted it.
Candy's and Karen's comments seem to demonstrate an understanding of how
the counting (vertically and horizontally) and the multiplication are related. I
push for further verbalization of the ideas involved to ascertain whether others in
the class have constructed similar meanings. Many now insist that we are really
counting rows and columns. I suspect that some of the students have latched onto
the notion of rows and columns in an unexamined way. The shi.ft from counting
boxes to counting rows and columns does not in itself lead to a connection
between counting the total number and the multiplicative approach. I refocus my
questions on this connection.
Simon:
Class:
Simon:
So I'm not counting boxes at all?
No.
OK. Isn't it a little mysterious that we're never counting boxes here and we wind
up with a number of boxes? Does that bother anybody? We didn't count boxes here. Martin A. Simon
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we didn't count boxes there, and we wind up with boxes at the end. Tammy?
Tammy: If each box represents a portion of the row so we're really counting boxes but we're
just putting them in a set, instead of individuaUy.
Simon: OK, so which way are you thinking about the sets going? [Points at diagram on
the board] This way or that way? Choose one.
Tammy: VerticaUy?
Simon: So this is a set? OK. So you are saying this is nine what?
Tammy: Nine separate units inside of a set.
Simon: ... OK. So here 1 counted nine boxes in a set and then here I'm counting what?
Tammy: Thirteen.
Simon: Thirteen what?
Tammy: Boxes in a set.
[Ellen is shaldng her head]
Simon: Ellen, you don't like that.
Ellen: If you're going to do it that way, I think you have to say that you're going to take
the column as a set, nine boxes in one set ... then the thirteen in the row is the number of sets that you have. so it's not actually boxes, it's the number of that same
type of set that you have.
Ellen's final comment evoked many nods of agreement from her classmates. I considered, however, that for a student to follow an explanation might not require the same
level of understanding as would be needed to generate an explanation. Still, it
seemed clear to me from students' verbalizations that the number of students who were
seeing a connection between multiplication and counting the total number of rectangles
had increased. Perhaps Problem 2 shifted the discussion from my problem-justifying
the method that the students believed to be valid-to a community problem-how to
account for the "double- counting." Problem 2 seemed to provide a puzzlement, at least
initially for most of the students. We cannot assume, however, that aJl of the students
related to Problem 1 as my problem, nor that all of them owned Problem 2.
Situation 2. As we proceeded to explore the multiplicative relationship involved
in evaluiating the area of a rectangle, I came to believe that the context in which we
were working (area) was not well understood by many of the students. They
seemed to think about area as generated by multiplying length times width.
Although my primary focus was on multiplicative relationships, not on area, it seemed
clear that an understanding of area was necessary in order for students to think about
constituting the quantity (area) and evaluating that quantity. (See Simon & Blume,
1994b; and Thompson, 1994, for explications of the distinction between constitution and evaluation of quantities.) What action could I take as a teacher?
It seemed that, if indeed these students were unclear about what is meant by area,
the traditional response of "reviewing" the idea would be inadequate. Surely by this
time, their junior year of college, they had been present at many such reviews and
had some ideas that could be built on. I chose instead to pose a problem that would
push them to extend their understanding of area. I posed the following: 128 Reconstructing Mathematics Pedagogy
The blob problem: How can you find the area of this figure?
Students generated ideas in small groups. When we reconvened as a whole class,
they shared many ideas. Two methods stimulated a great deal of discussion. The
first was a suggestion that a string be put along the outline of the blob and then,
without changing the length of the string, reshape it into a rectangle, a figure whose
area we know how to determine. This method was based on an assumption,
eventually rejected by the class, that figures that have the same perimeter also have
the same area. The second method was to cut out cookie dough of a constant thickness so that it exactly covered the blob. Then, cut the cookie dough into squares
of a given size. Roll up the remaining dough and roll it out to the original thickness, again cutting out squares, repeating this procedure until there is not enough
dough remaining to make an additional square. This method was accepted by the
class as theoretically sound; however, they predicted that in practice it would be
difficult to carry out accurately.
The problem generated more than discussion on the validity of these methods.
I perceived that the contrast in the methods proposed by the students would enable
us to consider the issue of conservation of area, that is, under what changes in shape
the original amount of area is preserved. This was a challenging context for the students to think about the meaning of area. Considerable dialogue ensued in which
students seemed to be using the notion of area appropriately, comparing area
among different geometric shapes, and distinguishing it from the notion of perimeter that was brought up by the string strategy.
Situation 3. Following a discussion of why multiplication was used to determine
the total number of rectangles and after our work with the blob problem, I raised
again the issue that they had brought up in solving the original problem, whether
to tum the rectangle (Figure I b) or to maintain its orientation (Figure J a). I demontrated the former, rotating it 90 degrees to measure the second side (as in Figure
I b). (The quantitative reasoning brought to bear on this problem is analyzed in depth
in Simon & Blume, 1994b.)
Most of the students recognized that the method I demonstrated would not
determine the number of rectangles that could fit on the table. I then asked them whether
that method tells us anything about this particular table. (This question is referred Martin A. Simon 129
to as the Turned Rectangle [TR] Problem.) Consensus developed in the class that
the number generated was meaningless because the method generated a set of
"overlapping rectangles."
I was surprised by their response and concerned about what it indicated about their
understanding of the multiplicative (area) unjt. I had posed this extension to
encourage an understanding of the multiplicative relationship between linear measures of the rectangle and area measures. The measurement strategy in question and
subsequent computation was identical to what they previously had learned to do with
a ruler (procedurally). In prior courses, when I posed this question, several students
understood that the quantity resulting from multiplying the two linear measures represented the number of square units of area on the table. The square unit had sides
equal to the length of the rectangle used for measuring. (See Figure 3.) Generally,
the discussion that ensued led to a consensus with respect to tills point.
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to measure with. The sticks, however, are of different lengths. Louisa says, The
length is four of my sticks." Ruiz says, "The width is five of my sticks.-- What
have they found out about the area of the rectangular region?
Students worked the stick problem in groups of three. Some of the students were
able to see how the use of different size sticks could be thought of as determining
an array of nonsquare rectangles. Toni began the class discussion as time was running out. Ellen picked up the discussion in the following class.
Toni: It would pretty much be the same thing as a rectangle because ... the width of the
rectangle is smaller than the length of the rectangle. so it doesn't really matter,! mean.
the area ... would be 20 [of these small rectangles].
Elle11: I didn-t really understand until Tonj drew that diagram on the board, that I found Martin A. Simon 131
something out about area, and when she put that on the board, I realized that r had
started thinking that you were starting with the unit, and you had to start with the
unit to figure out the area. But ... in my mind she sort of went backwards, she ended
up with a unit of measurement by, urn, making the rectangle .
. . . Because the unit you're using is a rectangle that bas a length the size of the one
stick and the width the size of the other stick, so you're sort of going backwards and
ending up with a unit that means something .... (Simon & Blume, 1994b, p. 489).
Through the discussion of this problem and an extension question that asked them
to express the area ofRuiz and Louisa's rectangular area in terms of other units, I
was persuaded that most of the students understood the relationship between the
linear measures and the area measures in this context. The next step was to see if
they could use this understanding to revisit the problem involving turning the rectangle. Would the understandings that they had developed in the stick problem allow
them to question their assumption of the cardboard rectangle as the appropriate unit
of measure? The revised TR problem was an attempt to make the problem more concrete by having them actually measure out particular rectangular regions using the
method in question.
Revised TR problem. I used your [cardboard] rectangle and my method (rotating the rectangle) to measure two rectangular regions; one was 3 x 4 and the
other was 5 x 2. Draw these regions (real size). Record aU that you can determine about their areas.
Half of the small groups determined that squares were useful units for describing the areas of the rectangular regions and could explain their thinking.
Eve: First, when we started drawing it, we drew like all the rectangles, OK? So it
showed the overlapping up in the comer, but then we thought just take away the overlapping ... , and just think of it as a side, like this is a side and this is a side, like the
sticks that Louisa and Ruiz used, just to get this little edge right here as a stick and
this little edge right here as a stick, OK? ... So what it actually makes is squares, because
they're the same length ... you wouldn't talk about your rectangles, 'cause it has no
relevance. (Simon & Blume, 1994b, p. 490--491)
Some of the students who had not previously seen the usefulness of square units
in the revised TR problem did so in the course of the class discussion. However, anumber of students, who saw that measuring in squares worked for the problem, were unclear
how one would know "when to use rectangles and when to use squares."
DISCUSSION
The discussion of the teaching situations is divided into two parts. The first part
examines the teacher's role that emerges in terms of the decision making about content and task. This discussion, which focuses on the composite picture of teaching
seen across the three situations presented, leads to the articulation of a model of teacher
decision making called the Mathematics Teaching Cycle. The second part of the
discussion highlights particular aspects of each of the situations (considered separately), in an attempt to further elaborate the role of the teacher. 132 Reconstructing Mathematics Pedagogy
As teacher/researcher. I began the mathematics course with particular theoretical perspectives on teaching and learning, some of which were articulated earlier
in this paper. However. not surprisingly, these perspectives did not "tell" me what
to do as challenges arose. My responses to these challenges often were not the result
of well-articulated models. Rather, they were emerging patterns of operation in problem situations. It is only through a posteriori analysis that these patterns have been
characterized and used to enhance, modify, or develop theory. Thus, the reader should
keep in mind that the theoretical aspects of this section grew out of the data analysis and were not necessarily part of my explicit thinking as I was planning and teaching. Also. these patterns of operation were established over repeated encounters with
similar problem situations. 1 had taught essentially this way for many years. This
particular teaching experience led to further elaboration of my teaching; the teach--
ing-experiment design led to a new level of analysis of the teaching.
'In this section, I use the first person singular to refer to my actions and thinking
as the teacher. I use the third person, often referring to "the teacher" to designate
ideas that I am lifting from the particular .context in which I was the teacher. Because
I was the teacher in these epi odes, T use male pronouns in this section when referring to the teacher generically.
Unpacking the Teaching Episodes: Developing Theory
This section analyzes the teacher's role as decision maker as it emerges across
the three teaching situations.
The rectangles lesson was shaped by my understanding of the multiplicative relationship between the area of a rectangle and its linear measures. (The focus on my
personal knowledge does not discount that this knowledge can be viewed as
socially constituted and taken-as-shared in the mathematics education community.
Rather, it is expedient here to focus on my particular interpretation of these
socially accepted ideas.) My previous experience with prospective elementary teachers led me to hypothesize that my students would not share this knowledge.
Rather, I expected that their knowledge would be rule bound and that the concepts
underlying the fommla for the area of a rectangle would be unexplored. The disparity between my understanding, which I judged to be useful, and my sense of their
understanding defined my learning goal for the first segment.
Note that hypotheses of students' understandings may be based on information from
a variety of sources: experience with the students in the conceptual area, experience
with them in a related area. pretesting, experience with a similar group. and research
data. Initial hypotheses often lack data that are available as work with the students proceeds. Thus, the hypotheses are expected to improve (i.e., become more useful).
Having established my initial goal, that students would understand the relationship of multiplying length by width to the evaluation of the area of a rectangle, I
considered possible learning activities and the rypes of thinking and learning that
they might provoke. Following is a partial reconstruction of my thought process.
I suspect that for many of my students A = l x 111 is a formula that has no conceptual
roots. Concrete experience with area might be helpful, I need to keep in mind that they Martin A. Simon 133
come to the task already knowing the rote formula. A learning situation that does not
look like their previous experiences with area might preempt their resorting immediately to rote procedures.
Tiling a rectangular region would provide the concrete experience. If 1 can envision a
situation in which they form multiple units, sets of tiles, they may see the appropriateness
of multiplication, in essence "deriving" the formula. I will not mention area but just ask
them to find out bow many tiles. However, if they are to make connections with l x w,
they must do more than count each tile.
Ifl give them only one small tile, they will need to look for an efficient way of determining the number of tiles- which will encourage them to go beyond counting all of
the tiles. If I use rectangular tiles, they will not be able to measure mindlessly; they will
need to consider how their measuring relates to the placement of the tiles on the table.
Measuring with a nonsquare rectangle to determine the area encourages a level of visualization that is not required when one uses a ruler to determine square units, that is,
they will have to take into account what they are counting, the unit of measure, which
is based on how they are laying the tiles on the table.
The preceding thought process provides an example of the reflexive relationship
between the teacher's design of activities and consideration of the thinking that students might engage in as they participate in those activities. The consideration of
the learning goal, the learning activities, and the thinking and learning in which students might engage make up the hypothetical learning trajectory, a key part of the
Mathematical Learning Cycle described in the next section.
Besides the teacher's knowledge of mathematics and his hypotheses about the
students' understandings, several areas of teacher knowledge come into play,
including the teacher's theories about mathematics teaching and learning; knowledge of learning with respect to the particular mathematical content (deriving
from the research literature and/or the teacher's own experience with learners); and
knowledge of mathematical representations, materials, and activities. The
Mathematical Learning Cycle portrays the relationship of these areas of knowledge
to the design of instruction.
The only thing that 1s predictable in teaching is that classroom activities will not
go as predicted. Although the teacher creates an initial goal and plan for instruction,
it generally must be modified many times (perhaps continually) during the study of
a particular conceptual area. As students begin to engage in the planned activities,
the teacher communicates with and observes the students, which leads the teacher
to new understandings of the students' conceptions. The learning environment
evolves as a result of interaction among the teacher and students as they engage in
the mathematical content Steffe (1990) points out, "A particular modification of a mathematical concept cannot be caused by a teacher any more than nutriments can cause plants
to grow" (p. 392). A teacher may pose a task. However. it is what the students make
of that task and their experience with it that detennines the potential for learning.
Student responses to the rectangle problems led me to believe that students did
not adequately understand what is meant by area. As a result, I generated a new learning goal, understanding area. This goal temporarily superseded but did not replace
the original learning goal. Toward this end I posed the blob problem, anticipating
that the students would brainstorm some ways to frnd area, discuss those ways, and 134 Reconstructing Mathematics Pedagogy
in so doing strengthen their understanding of area. However, the specifics of what
happened resulted in additional, unanticipated learning. First, students proposed the
sLring strategy. On the basis of the understanding that I had developed of the students' conceptual ctiffculties in considering the string strategy, and on the basis of
the interesting contrast that I saw between the string strategy and the dough strategy (as a result of my own mathematical understandings), I revised my goal for instruction once again. I now saw as the (local) goal facilitating students' understanding
of conservation of area (not limited to Piaget' s assessment of the concept with children). I intended for my students to deal with the question, "Under what types of
change in shape does the area of a region remain invariant?"
My interest in their constructing answers to this question was based on three .factors: (a) I believed that it would further their understanding of area, my motivation
for posing the blob problem; (b) I saw an opportunity for learning based on the juxtaposition of the two strategies-an opportunity that I had neither planned for nor
anticipated; and (c) I believed that the concept of invariance, which Thad thought
about previously in relation to arithmet-c concepts but not area, was an important
one. The third factor also points out how my own understanding of the mathematical
connections involved is enhanced as I attend to the mathematical thinking of my
students. This evolution of the teacher's mathematical knowledge is also revealed
in the analysis of the third episode, the data involving measuring with only the long
side of the rectangle (TR problem).
My original goal that motivated the rectangles lesson was for my students to understand the evaluation of the area of a rectangle as a multiplicative relationship between
the linear measures of the sides. For me, as 1 began instruction. such an understanding
involved connecting an understanding of multiplication-as-repeated-addition with
the notion of identical rows of units of area and understanding the relationship between
linear units and area units. The latter concept was represented by the issue of turning the rectangle to measure- ! had not unpacked this understanding further.
The classroom discussion, however, pushed me to reexamine these understandings and to further elaborate my map of the conceptual terrain. (The use of the term
"map" in this context is meant to emphasize that the teacher's understandings serve
as a map as he engages in making sense of students' understandings and identifies
potential learnings.) The students' roisanticipation of the area unit (assumption that
the area would necessarily be measured in terms of cardboard rectangles) led me
to explore the importance of anticipating an appropriate unit. Anticipating the area
unit seemed to involve both an anticipation of the organization of the units, a rectangular array, and an understanding that the linear units define the size and shape
of the units within that array. (For a fuller discussion, see Simon & Blume, 1994b.)
The multiplicative relationship, therefore, involved the coordination of the linear units
to determine an area unit within an anticipated rectangular array. What I had
observed in my students had changed both my perspective on my students' knowledge and my perspective on the mathematical concepts involved (my internal
map). This reorganization of my perspectives led to a modification of my goals, my
plans for learning activities, and the students' learning and thinking that I anticipated. Martin A. Simon 135
The Mathematics Teaching Cycle
The analysis of these teaching episodes has led to the development of the
Mathematics Teaching Cycle (Figure 4) as a schematic model of the cyclical
interrelationship of aspects of teacher knowledge, thinking, decision making, and
activity that seems to be demonstrated by the data.
The three episodes create a picture of a teacher whose teaching is directed by his
conceptual goals for his students, goals that are constantly being modified. The original lesson involving the rectangles on the table was not a random choice, nor was
it Chapter 1 in someone' s textbook. The goal for and design of the lesson were based
on relating two factors: the teacher's mathematical understanding and the teacher's
hypotheses about the students' knowledge. I refer to "hypotheses" about students
knowledge to emphasize that the teacher has no direct access to students' knowl-
edge. He must infer the nature of the students' understandings from his interpretations of his students' behaviors, based on his own schemata with respect to
mathematics, learning. students, and so on. It is implied that the teacher can compare his understanding of a particular concept to his construction of the students'
understandings, not to the students' "actual" understandings.
As the teacher, my perception of students' mathematical w1derstandings is structured by my understandings ofd1e mailiematics in question. Conversely, what I observe
in the students' mathematical thinking affects my understanding of the mathematical ideas involved and their interconnections. These two factors are interactive spheres
of a teacher's thinking (Ball's, 1993, "bifocal perspective" discussed earlier).
Steffe (1990) states,
Using their own mathematical knowledge, mathematics teachers must interpret the language and actions of their students and then make decisions about possible mathematical
knowledge their students might learn. (p. 395)
The teacher's learning goal provides a direction for a hypothetical learning
trajectory.
6
I use the term "hypoilieticallearning trajectory" to refer to the teacher's
prediction as to the path by which learning might proceed. It is hypothetical
because ilie actual learning trajectory is not knowable in advance. It characterizes an
expected tendency. Individual students' learning proceeds along idiosyncratic,
although often similar, paths. This assumes iliat an individual's learning has some regularity to it (cf. Steffe, et al., J 983, p. 118), that ilie classroom community constrains
mathematical activity often in predictable ways, and iliat many of the students in ilie
same class can benefit from the same mathematical task. A hypoteticallearning trajectory provides ilie teacher with a rationale for choosing a particular instructional design;
thus, I make my design decisions based on my best guess of how learning might proceed. This can be seen in the thinking and planning that preceded my instructional interventions in each of the teaching situations described as well as the spontaneous decisions that 1 made in response to students' thinking.
6
1 choose to use "hypothetical learning trajectory." rather than traditional terminology, to emphasire
aspects of teacher thinking that are grounded in a constructivist perspective and that are common to both
advanced planning and spontaneous decision making. 136 Reconstructing Mathematics Pedagogy
The hypothetical learning trajectory is made up of three components: the learning goal that defines the direction, the learning activities, and the hypothetical learning process-a prediction of how the students' thinking and understanding will evolve
in the context of the learning activities. The creation and ongoing modification of
the hypothetical learning trajectory is the central piece of the model that is diagrammed
in Figure 4. The notion of a hypothetical learning trajectory is not meant to suggest
that the teacher always pursues one goal at a time or that only one trajectory is considered. Rather, it is meant to underscore the importance of having a goal and rationale for teaching decisions and the hypothetical nature of such thinking. Note that
the development of a hypothetical learning process and the development of the learning activities have a symbiotic relationship; the generation of ideas for learning activities is dependent on the teacher's hypotheses about the development of students'
thinking and learning; further generation of hypotheses of student conceptual
development depends on the nature of anticipated activities.
Teacher's
knowledge
Hypothetical
learning trajectory
Teacher's plan for
learning
activities
Teacher's
hypothesis of
learning process
Interactive
constitution
of classroom
activities
Figure 4. Mathematics teaching cycle (abbreviated).
The choice of the word "trajectory" is meant to refer to a path, the nature of which
can perhaps be clarified by the following analogy. Consider that you have decided
to sail around the world in order to visit places that you have never seen. One does
not do this randomly (e.g., go to France, then Hawaii, then England), but neither is
there one set itinerary to follow. Rather, you acquire as much knowledge relevant to
planning your journey as possible. You then make a plan. You may initially plan the
whole trip or only part of it. You set out sailing according to your plan. However, you

must constantly adjust because of the conditions that you encounter. You continue Martin A. Simon 137
to acquire knowledge about sailing, about the current conditions, and about the areas
that you wish to visit. You change your plans with respect to the order of your destinations. You modify the length and nature of your visits as a result of interactions
with people along the way. You add destinations that prior to your trip were
unknown to you. The path that you travel is your "trajectory." The path that you
anticipate at any point in time is your "hypothetical trajectory."
The generation of a hypothetical learning trajectory prior to classroom instruction
is the process by which (according to this model) the teacher develops a plan for classroom activity. However, as the teacher interacts with and observes the students, the
teacher and students collectively constitute an experience. This experience by the nature
of its social constitution is different from the one anticipated by the teacher.
Simultaneous with and in interaction with the social constitution of classroom activity is a modification in the teacher's ideas and knowledge as he makes sense of what
is happening and what has happened in the classroom. The diagram in Figure 4 indicates that the assessment of student thinking (which goes on continually in the
teaching model presented) can bring about adaptations in the teacher's knowledge that,
in turn, lead to a new or moctified hypothetical learning trajectory.
Figure 5 describes the relationship among various domains of teacher knowledge, the hypothetical learning trajectory, and the interactions with students.
Teacher's
knowledge of
mathematics
Teacher's
knowledge of
mathematical
activities and
representations
Hypothetical
learning
trajectory
Teacher's
hypothesis
of students'
knowledge
Teacher's theories
about mathematics
learning
and teaching
Teacher's
knowledge of
student learning
of particular
content
Figure 5. Mathematics teaching cycle. (The domains of teacher knowledge also inform -assessment
of students' knowledge" directly. However, because this was not the emphasis of the model, and in order
to simplify the diagram, those arrows are not included,) 138 Reconstructing Mathematics Pedagogy
Beginnjng at the top of the diagram, the teacher's knowledge of mathematics in interaction with the teacher's hypotheses about the students' mathematical knowledge
contribute to the identification of a learning goal. These domains of knowledge, the
learning goal, and the teacher's knowledge of mathematical activities and representation,
his knowledge of students' learning of particular content, as weU as the teacher's conceptions of learning and teaching (both within mathematics and in general) contribute
to the development of learning activities and a hypothetical learning process.
The modification of the hypothetical learning trajectory is not something that only
occurs during planning between classes. The teacher is continuaUy engaged in adjusting the learning trajectory that he has hypothesized to bener reflect his enhanced
knowledge. Sometimes fine tuning is in order, while at other times the whole thrust
of the lesson must be discarded in favor of a more appropriate one. Regardless of
the extent of modification, changes may be made at any or all of the three components
of the hypothetical learning trajectory: the goal, the activities, or the hypothetical
learning process.
Other Aspects of the Teacher's Role
Each of the three teaching situations portrays particular aspects of what teaching. which embodies reform principles. might be like. I discuss a few of these in
this section.
The original rectangles problem was planned for one or two class periods;
instead, eight periods were spent on the mathematics that was generated. Experienced
teachers might affirm that it is difficult to determine in advance exactly how long
it will take to teach a particular concept However, the discrepancy between the amount
of time anticipated and the amount of time spent in this case is well beyond the imprecision of planning. This discrepancy points at the experimental nature of mathematics
teaching. "Experimental" denotes the ongoing cycle of hypothesis generation (or
modification) and data collection that characterizes the teaching portrayed.
In the first situation, involving the tiling of the tables. I, as the teacher. perceived
a lack of understanding among a majority of the students of the relationship between
length-times-width and the counting of all the rectangles on the table. My response
was to pose additional problems based on students' conceptual difficulties that I had
witnessed in the past. r selected thought processes that I thought students could determine as not viable, but that would likely be problematic initially for them to
invalidate. (Rectangles problem 2 is an example.) My rationale was that previous
student 'conceptual difficulties (from the teacher/researcher's perspective) are potential difficulties for roy current students and represent useful hurdles for them to encounter
in the development of more powerful ideas.
This approach represents a sharp contrast to the approach to instruction characteristic of traditional mathematics instruction and represented by mathematics textbooks. Traditional instruction tends to focus on one skill or idea at a time and then
provide considerable routine practice to "reinforce" that learning. The mathematics
is subdivided into small segments for instruction so that students can experience success on a regular basis. In contrast, Situation L demonstrates a view of learning as Martin A. Simon 139
one involving a complex network of connections. Learning is likely to be fostered
by challenging the learner's conceptions using a variety of contexts. The teacher
can be compared to an athletic coach who employs a variety of practice activities
that challenge the athletes' strength and skill, often beyond what is required of the
athlete in competition (dribbling two basketballs while blindfolded, playing a
soccer game where each player may not touch the ball two consecutive times, per-
forming a figure skating program three times in a row with only a 2-minute rest in
between). These activities are not aimed at constant success, but rather at increased
competence. Growth is a result of challenge to body and mind. Conceptual difficulties that I have previously observed in students are not to be avoided; rather. they
provide particular challenges, which if surmounted by the students, result in conceptual growth. This fits with French researchers- notion of "obstacles epistemologiques"
(Bachelard, 1938, cited in Brousseau, 1983), that overcoming certain obstacles is
a natural and essential part of conceptual development. These obstacles are a
result of prior concepts that, although adaptive in earlier contexts, are maladaptive
given the demands of the current problem situation.
A second feature of the approach seen in Situation 1 is subtler. As a teacher, I often
do not have a well-developed map of the mathematical conceptual area in which
I am engaging my students; that is, I may not have fuJy articulated for myself (or
found in the literature) the specific connections that constitute understanding or the
nature of development of understanding in that area. Rather, as was the case when
I started the rectangles instructional unit, my knowledge of what it means to
understand the particular concept may be carried in part by particular problem sit-
uations. The kinds of difficulties that students encounter provide me with key pieces
of what it means to understand. Thus, in such cases, my operational definition of
understanding is the ability to overcome these particular difficulties; I may not
have unpacked the difficulties in order to understand the conceptual issues that
are implicated. Thus, even if I do not have a thorough knowledge of what constitutes mathematical understanding in a particular domain, having a rich set of
problem situations that challenge students and having knowledge of conceptual
difficulties that they typically encounter provide me with an approximation that lets
me be reasonably effective in promoting learning in the absence of more elaborated
knowledge. (This is not to suggest that the more elaborated understanding would not
be more powerful.) Indeed, engaging students in these problem situations and with
these conceptual difficulties gives me an opportunity to learn more about what it means
to understand the concepts involved.
Underlying Situation 2 is an idea that highlights a difference between teaching
based on a traditional view of learning and teaching based on a constructivist perspective. Rather than "review" what is meant by area or assign "practice problems,"
my approach was to challenge the students in a way that might push them to extend
their conceptions of area. The review-and-practice approach is based on learning
as improving storage and retrieval. of received information. (Although I am not negating the importance of memory, I contend that it is not what is most important, most
interesting, and most problematic for educators in the domain of mathematics.) My 140 Reconsrructing Mathematics Pedagogy
approach in Situation 2 reflected a view of understanding as a network of connected
ideas that is further elaborated as the understanding is used to solve novel problems.
Situation 3 was the most difficult to analyze. I brought to the teaching situations
a view that learning is triggered by disequilibrium. When the students were convinced that the rotated rectangle method of measuring and calculating provided no
useful information about the table, I 'lied in every way I could to provoke disequilibriwn,
but to no avail. In-depth analysis of the data suggests that my interpretations of the
students' thinking that led to their conclusion was not adequate. Whereas I had thought
that they saw my method as counting the number of overlapping rectangles, I now
beJieve that they were saying that the method involving turning the rectangle
counted nothing because in the process I was overlapping the rectangles. This subtle difference in thinking may account for my inability to foster disequilibrium.
Having failed to promote disequilibrium, I embarked intuitively on another strategy. I backed away from the particular problem to try to focus on a part of the understanding demanded by the problem. A posteriori analysis suggests that what I was doing
was fostering the development of know-edge that, when the students returned to that
problem, might contribute to the students' experiencing cognitive conflict In this case,
if I could help students build an understanding of the relationship between linear and
area measures of a rectangle, they would then experience a conflict between those understandings and the expectation that measuring with the cardboard rectangle resulted
in a measurement where that rectangle was the unit of area. This teaching episode seems
to emphasize that disequilibrium is nor created by the teacher. He can try to promote
disequilibrium. However, the success of such efforts is in part determined by the adequacy of his model of students' understanding. It also seems to support the notion that
learning does not proceed linearly. Rather, there seem to be multiple sites in one's web
of understandings on which learning can build.
IN SUMMARY
Constructivist views of learning have provided a theoretical foundation for mathematics education research and a framework within which teachers can understand their
students. However, constructivism also poses a challenge to the mathematics education
community to develop models of teaching that build on, and are consistent with, this
theoretical perspective. Small-group interaction, nonroutine problem solving, and manipulative materials can be valuable tools in the hands of mathematics teachers. Yet the
ability to use these tools is not sufficient to allow teachers to be the architects of productive learning situations resulting in conceptual growth. Theoretically based
frameworks for teaching bave the potential to guide the use of these tools.
By what means can a teacher help students to develop new, more powerful
mathematical concepts? Novice teachers. who want their students to "construct" a
particular idea, often ask for the idea from their students, consciously or unconsciously
hoping that at least one student will be able to explain it to the others (Simon, 1991).
Such an approach does not deal with a key question: If a group of students do not
have a particular concept, how does a teacher work with them to foster their development of that concept? Marrin A. Simon
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The principal currencies of the mathematics teacher (iflecturing is rejected as an
effective means of promoting concept development) are the posing of problems or
tasks and the encouragement of reflection. The data analysis described in this paper
and the resulting Mathematics Teaching Cycle address the issue of the process by
which a teacher can make decisions as to the content, design, and sequence of mathematical tasks. The model emphasizes the important interplay between the teacher's
plans and the teacher and students' collective constitution of classroom activities.
The former involves creation of instructional goals and hypotheses about how students might move towards those goals as a result of their collective engagement in
particular mathematical tasks. However, the teacher's goals, hypotheses about learning, and design of activities change continually as the teacher's own knowledge changes
as a result of being involved in the culture of the mathematics classroom.
A goal structure for mathematics education such as the one elaborated by Treffers (1987)
is needed in specifying possible learning environments by teachers. But this element
of possible learning environments is just as dependent on the experiential fields that constitute learning environments as the latter are dependent on the former. Mathematics
educators should not take their goals for mathematics education as fixed ideals that stand
uninfluenced by their teaching experiences. Goal structures that are establjshed prior
to experience are only starting points and must undergo experiential transformation in
actual learning and teaching episodes. (Steffe, 1991, p. 192)
Steffe's comments seem to underscore the cyclical nature of this teaching process.
The Mathematics Teaching Cycle portrays a view of teacher decision making with
respect to content and tasks that has been shaped by the meeting of a social constructivist perspective with the challenges of the mathematics classroom. Several
themes are particularly important in the approach to decision making represented
by this model.
1. Students' thinking and understanding is taken seriously and given a central place
in the design and implementation of instruction (consistent with Steffe, 1991).
Understanding students' thinking is a continual process of data collection and
hypothesis generation.
2. The teacher's knowledge evolves simultaneously with the growth in the students'
knowledge. As the students are learning mathematics, the teacher is learning about
mathematics, learning, teaching, and about the mathematical thinking of his students.
3. Planning for instruction is seen as including the generation of a hypothetical
learning trajectory. This view acknowledges and values the goals of the teacher for
instruction and the importance of hypotheses about students' learning processes (ideas
that I hope I have demonstrated are not in conflJct with constructivism).
4. The continually changing knowledge of the teacher (see #2) creates continual
change in the teacher's hypothetical learning trajectory.
These last two points address directly the question raised earlier in the paper of
balance between direction (some may call this "structure") and responsiveness to
students, a creative tension that shapes mathematics teaching. The model suggests
that, as mathematics teachers, we strive to be purposeful in our planning and
actions, yet flexible in our goals and expectations. 142 Reconstructing Mathematics Pedagogy
The mathematics education literature is strong on the importance of Jistening to
students and assessing their understanding. However, the emphasis on anticipating
students' learning processes is not developed by most current descriptions of
reform in mathematics teaching. Research on how students develop particular mathematical knowledge (cf. Steffe, et al., 1983, see pp. 118 & 135; Thompson, 1994)
informs such anticipation. Perhaps one explanation for the success of Cognitively
Guided Instruction (Carpenter, Fennema, Peterson, & Carey, 1988), in which
teachers learned about research on children' s thinking (Carpenter & Moser, 1983),
is that it increased reachers' ability to anticipate children's learning processes. BaU J
(1993) articulates a similar position emphasizing the role of teachers • hypotheses
about student learrung. "Selection of representational contexts involves conjectures
abom teaching and learning, founded on the evolving insights about the children's
thinking and [the teacher's] deepening understanding of the mathematics" (p. 166).
The data from this study must be seen in its particular context The teaching practice was embedded in a teacher education program; the mathematics students
were prospective elementary students. As the teacher. I felt no pressure to teach from
a preset curriculum nor to cover particular mathematical content, a condition that
is probably the exception rather than the rule for mathematics teachers. Mathematics
teaching with other populations involves a set of different constraints. Research in other
contexts wi ll inform us about the degree of context dependence of the ideas generated.
A possible contribution that can be made by the analysis of data and the resulting model reported on in this paper is to encourage other researchers to examine
teachers' "theorems in action" and to make teachers' assumptions, beliefs, and emerging theories about teaching explicit At the minimum, the paper should serve to emphasize the need for models of mathematics teaching that are consistent with. and built
on, emerging theories of learning. Much research remains to be done to understand
the implications for practice of teachers holding constructivist perspectives. It is the
recognition that constructivism does not tell us how to teach that will motivate increased
work in this area.
A well-developed conception of mathematics teaching is as vital to mathematics

teacher educators as well-developed conceptions of mathematics are to mathematics
teachers. Informed decisions in each case are dependent on a clear sense of the
nature of the content being taught. Considering the Mathematics Teaching Cycle
as a way to think about mathematics teaching means that teachers would need to
develop abilities beyond those already currently focused on in mathematics education reform, particularly the ability to generate hypotheses about students'
understandings (which goes beyond soliciting and attending to students- thinking),
the ability to generate hypothetical Learning trajectories, and the ability to engage
in conceptual analysis related to the mathematics that they teach. This last point supports proposed reforms of mathematics for teachers (cf. Cipra, 1992; Committee
on the Mathematical Education of Teachers, 1991) and supports arguments that the
mathematical preparation of teachers is far from adequate if teachers are to engage
in pedagogical deliberations as characterized in this paper.
FinaJly. it should be noted that the role of the mathematics teacher as portrayed
in this paper is a very demanding one. Teachers will need access to relevant Martin A. Simon
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research on children's mathematical thinking, innovative curriculum materials, and
ongoing professional support in order to meet the demands of this role.
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WISKOBAS AND FREUDENTHAL
REALISTIC MATHEMATICS EDUCATION
ABSTRACr. Freudenthal was the founder of so-called realistic mathematics education. In it reality
does not only serve as application area but also as the source for learning. We take a newspaper article
as an example of a rich context problem. It contains elements from all of the important areas of leaming
for mathematics education in primary school (grades I through 6). In the following these topics of
learning are discussed. Historical comments will be indicated each time, in particular relating to the
influence of Freudenthal. Freudenthal laid the foundation for this didactical realism and determined the
development of various learning strands, but more indirectly than directly because he himself did not
design or outline themes and learning strands. Attention is devoted briefly to the integral educational
development of Wiskobas and Freudenthal's contribution to it, especially in terms of developmental
research. The summary, in conclusion, is given by way of a problem that was posed by Freudenthal -
his last one.
INTRODUCTION
Freudenthal is strongly associated with Wiskobas2. This is rightfully and wrong-
fully so. Wrongfully so because Wiskobas originated in 1968 without the involve-
ment of Freudenthal and continued after 1981 without his direct participation.
Rightfully so because of the lasting influence of his range of ideas on the work of
Wiskobas. And not only on Wiskobas, and therefore on primary education, but
equally through Wiskivon on secondary education (grades 7-12), which will not
be described in this presentation3.
It was Freudenthal who in 1971 - upon the founding of the IOWO - put
Wiskobas on the track of realistic mathematics education, away from the trodden
path of traditional arithmetic, and away from the up and coming New Math. New
in Freudenthal's views was not only that he wanted to incorporate everyday reality
emphatically in mathematics education, but especially also his fundamental idea
to let that rich context of reality serve as a source for learning mathematics.
In the initial period it was predominately the rich contexts in the form of
topics, projects, locations, stories and clippings which determined the appearance
of Wiskobas. Later it was more the new learning strands which drew the attention.
In the following there is first an example of one of those themes for the
teacher, together with some historical comments about the place of such problem-
situations in realistic mathematics education in primary school (grades 1-6) in the
period 1970-1990, and Freudenthal's contribution towards the development of
these themes. Thereafter the latter is done for the learning strands of the Wiskobas
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Fig. I. Photo: EPA.
program and in conclusion for educational development as a whole. We start by
accentuating the description of the content of the Wiskobas program. Later it
is especially Freudenthal's influence as a member of the Wiskobas team which
receives attention.
1. RICH 'THEME: VAN GOGH IN THE FIELD
1.1. Van Gogh Problem
From the outset of the Wiskobas project at the IOWO Freudenthal stimulated
'mathematics fraught with relations', or as it was later called 'mathematics in rich
contexts' - to which the influence of Decroly was not foreign (Freudenthal, 1973).
From the available categories of rich themes a newspaper clipping
is chosen here,
intended for the (initial) teacher training. To be precise the example consists of
two newspaper clippings, both about a rather unusual reproduction of one of Van
Gogh's 'Vase with Sunflowers' (see Figure 1).
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It took Andrew Scott, a horticulturist in Scotland, ten days to make a repro-

duction of Vincent van Gogh's 'Sunflowers'. With the assistance of twenty volunteers he copied the painting in a Scottish wheat field, using 250,000 colourful garden plants. The botanical 'canvas' is 14 square kilometres in size and 'very beautiful', Scott concludes after having inspected the scene from a helicopter. (Trouw 17-8-'91).
In another newspaper one could read:
Scott used various kinds of plants and shrubs to portray a picture of over 10.000 square meters. (Volkskrant 17-8-'9 1).
On the basis of these clippings we asked the teachers the following question:
> Which report is the most accurate, Trouw or Volkskrant? Explain your answer as convincingly as possibly.
When this problem had been solved and discussed the next question was raised:
> How do you suppose this discrepancy in sizes occurred?
It was added that the newspapers were given the area in square yards: 12820 square yards (De Moor and Streefland, 1991). More explicitly:
> Could the area of 'over 1 0,000 square meters' be correct?
> How do you explain the error of the 14 square kilometres?
1.2. Brief Comment
In determining which area seems the most plausible, there are first of all the arguments of geometry which the teachers raise. In making estimations of length and width tractor tracks and the height of the trees were considered: how wide is a track and how high a tree, roughly speaking! These sizes as such are the first topic of discussion and reflection. All kinds of reference points are brought up. One teacher follows up with an interesting method: the 'projection' of the side-lines of the painting to the road with trees showing at the top of the photo (Figure 2). It is obvious from looking at the trees, that this 'canvas' cannot be 3 or 4 kilometres wide, but some one hundred meters at most.
Then there is also the computational argument in answering the first question with regard to the density of the plants in the painting: 250,000 plants on 14 square kilometres or on 10,000 square meters? In the latter case there would then be 25 plants on 1 square meter - that would seem reasonable, after some sizing up. But 250,000 plants on 14,000,000 square meters is 25 plants on 1400 square meters or 1 on 56 square meters, so 1 plant on a rectangle of 7 by 8 meters, the floor of our classroom, and of course that is too far apart.
Moments of discussion are introduced when making an inventory of the various solutions. But the result of one and other is clear: 10,000 square meters seems by far the most realistic size. It could also very well concern a rectangle of 60 by 70 meters, as reported on the news on television the day before! (Length and width do not differ that much, but the area does. How is that possible?)
Then the correct area is given - 12,820 square yards. We take another look at those 10,000 square meters and 14 square kilometres and do so in that order.
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Fig. 2.
When converted does 12,820 square yards correspond with somewhere close to 10,000 square meters? One yard is 91.44 cm, let us say 0.9 meters.
There is disagreement: one group feels that the answer should have been well over 1 1,000 square meters (0.9 x 12820 = 115380) or even 1 1,500 square meters; someone else prefers close to 12,000 square meters, and there are those who can agree to the figure of some 10,000 square meters (0.8 x 12820).
Of course the issue of rounding off is discussed at length. But particularly the main issue of whether to multiply by 0.9 or by 0.8 (or divide).
A square of 10 by 10 yards can serve as a model.
This is then converted to a square of 9 by 9 meters. Therefore 100 square yards corresponds with 81, let us say 80 square meters.
9m
9m)Oy
IOy
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If 100 becomes 80 than 12820 therefore becomes ... The double number line is used:
100 12820
80
Or perhaps better the ratio table, because of the possible confusion of 'square' with 'length' on the number line:
100 10.000 12820
80 800 ?
Different methods of calculation are possible; for example via 10,000 and correspondingly 8,000, or abbreviated via 130 times as much at the top, therefore also 130 times 80 at the bottom, or via 8/10 times 12820 - some 10,000 square meters does appear to be correct!
Actually this problem is best posed separately and the solutions inventoried and discussed one by one. But even more important is to relate the conversion of square yards to square meters to the earlier conversion of square kilometres to square meters, and to draw attention to the matter of the zeros and the linear connection between kilometres and meters, and the conversion of measures of volume.
In conclusion the unravelling of the matter of the 14 square kilometre.
The teachers are helped on their way: the first mistake also occurred here in the conversion of square yards to square meters via the factor 0.9. Who remembers what else went wrong? We latch on to a smart suggestion from one teacher: indeed 12820 was not multiplied by 0.9 but divided by it. Perhaps an understandable error, just like the first one, because the idea was that the result in square meters would have to be smaller than 12820, and becoming smaller means that you must divide, right?!
Next, the result of 12820: 0.9144 will probably be worked out on the calculator: 14020.122. The decimal point is easily overlooked. Now move the decimal point six positions to the left and we have 14.020122 square kilometres, rounded off to 14 square kilometres.
That is how it might have gone, but more likely the decimal point was placed three positions to the left (from meters to kilometres as it were) from 14020.122. Whatever the case, the error has taught us that division does not always give a smaller result. And the most important thing that was learned is perhaps, that in cases like these you must always examine the answer to see if it is plausible.
By the way, one question remains unanswered: how did the Scottish arrive at 12820yd. 22 5 641?
At any rate it has become clear that many aspects of mathematics can be raised here in an integrated fashion. In the early days of Wiskobas, but also thereafter, theme-oriented examples like this, but also at the level of primary education, had an important informative and stimulating influence.
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1.3. Historical Comments
Before the founding of the IOWO, Freudenthal wrote the following about mathematising:
'The globally structuring force, as we called it, should be lived through reality. Only this way can we teach mathematics fraught with relations, can we be sure that the student integrates the mathematics he has learned, and can we guarantee the applicability of mathematics'. (Freudenthal, published in 1973, p. 132).
It is evident that the introduction of rich themes corresponds with this basic idea, i.e. was strongly stimulated by it. The IOWO snapshots, published in Educational Studies in Mathematics on the occasion of 'H. Freudenthal's Retirement from the Directorship of IOWO' (1976) attest to this: 'An Island of Geometry', 'Ship Ahoy', 'Our Earth', 'Building a Bungalow', 'Gulliver', etc. Many themes were developed in the course of time. In his farewell speech Freudenthal even went so far as to predict that the integrative aspect of mathematics was dominant to such a degree that mathematics as a separate subject would cease to exist in time:
'What will mathematics education look like in 2000? The answer is simple. There will be no more mathematics education in 2000, it will have disappeared. There will be no more subject called mathematics, no math program, no math textbook to teach from' ... 'It is there to be lived and enjoyed, just as reading, writing, handicrafts, art, music, breathing, in integrated education'. (Freudenthal, 1977, p. 294).
After 1976 mathematising was distinguished in a horizontal and vertical component (Treffers, 1987) and also in the functions which rich context can accomplish in mathematising (de Lange, 1979, 1987). The integrative themes have a specific horizontal mathematising purpose, i.e.: they are primarily directed at the application, practice and relating of learned knowledge and skills in context situations. The model-type contexts on the other hand have primarily a vertical mathematising purpose, i.e. that they are in particular directed at the progression within the subject structure, therefore on the practising of knowledge, skills and concepts within learning strands (Gravemeijer et al., 1990). By this distinction in horizontal and vertical mathematising, about which Freudenthal said 'For a long time I have hesitated to accept this distinction' (1991), the importance of the integrative themes was brought back to realistic proportions: they are important, but overemphasis leads to empiristic mathematics education with too little attention for vertical mathematising. In an evaluation discussion about the Wiskobas project4 in the IOWO period (1971-1981) Freudenthal himself came to the conclusion that Wiskobas spent too much energy in those days on the design of themes ('domains of reality disclosed to the learner in order to be 'horizontally' mathematised') and too little on model contexts (to be 'vertically mathematised') on which in the eighties the emphasis came to lie much more. Also the form and content of the theme changed in the eighties to some degree. Besides themes, projects, locations and stories there appeared the clippings. Their value was recognized by Freudenthal:
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Here is not a domain but a small piece that is cut out, although its paradigmatical value for mathematising and for acquiring a mathematical altitude may be enormous in comparison. (Freudenthal, 1991).
See 'Van Gogh in the field', we might add.
2. WITH A VIEW TO LEARNING STRANDS
In the Van Gogh problem there were elements from the main learning strands of the Wiskobas program:
- basic skills and column arithmetic
- ratio and fractions
- measurement and geometry.
In the following these learning strands will be regarded from the points touched on in 'Van Gogh' to acquire some insight into the specific characteristics of each. For each main learning strand some historical comments are made about their development in the period 1970-1990. In each instance there will be a short reference to the involvement of Freudenthal.
2. 1. Basic Skills and Column Arithmetic
Mental arithmetic is efficient calculation for which both memorized knowledge as well as insight in arithmetic rules is necessary, with furthermore the particular 'faces' of numbers and the relations between them. The Van Gogh problem offers various possibilities for mental arithmetic in combination with estimated calculation, forexampleof 1% x 12500 = 8 x 1250 = 4 x 2500 = 2 x 5000 = 10,000-
the strategy of halving and doubling. The Wiskobas program leaves much room for mental arithmetic. Column arithmetic is not introduced until the beginning of grade 4. Until that time mental arithmetic has received every opportunity to develop. In the traditional math program column arithmetic (with adding and subtracting) started one year earlier, at the beginning of grade 3, and in consequence snowed mental arithmetic under completely. There are even countries (among them the USA) where column arithmetic to one hundred often already starts in grade 2. In consequence pupils have almost no opportunity to consider numbers on their value, something which is also expressed in the achievements (for a comparison see Treffers, 1991 a). Estimation arithmetic has meaning when it concerns the rough determination of the result, the general assessment of the result in terms of magnitude, and if one must handle not precisely determined information. And that is exactly what was done in the Van Gogh problem, where experience information was used (misused), approximation, rounding off and (in)accuracies in determining the height of trees by roughly converting yards to meters and square yards to square meters and examining whether about 10,000 square meters is close... Although the Wiskobas program does devote much attention to estimation - it is both a goal and a means - this is not yet sufficiently
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768 408
360 10 blssen 360 10
408 48
360 10 bussen 36 1
48 12 (
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( ) (b.)
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1090 30
48
36 1
iz ('l)
(C.)
Fig. 3. Buses.
evident in educational practice in Holland, contrary to mental arithmetic that is practised sufficiently.
In conclusion, column arithmetic that in 'Van Gogh' could remain in the back-

ground by using the calculator. But this also applies in general: the importance
of learning the number algorithms has been strongly reduced by the introduction
of the calculator. In the Wiskobas program this is expressed in a special manner,
namely in the connection between mental arithmetic and column arithmetic. Col-
umn arithmetic procedures are developed from stylized mental arithmetic. This
occurs according to a gradual process of schematising and abbreviation. And the
end level can differ (per group or per pupil). An example of how this works for
division. The following transportation problem can for instance serve as point of
departure for long division: 1128 fans must be transported in buses that seat 36.
How many buses will be needed?
This problem is solved at different levels in the course of the learning strand
(Figure 3).
And these solutions illustrate the abc of the entire learning strand.
Something similar happens for multiplication. For example in a context prob-
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lem in which 7 x 24 occurs: (1) via repeated counting, (2) abbreviated to the
front-end method 7 x 24 = 140 + 28 = 168, and (3) the end-front standard
algorithm that starts with the units.
Historical Comments
One might even go so far as to elevate the stylized front-end methods of mental
arithmetic to (new) standard algorithm. The Wiskobas program, which serves as
informal National Curriculum (Treffers, De Moor, 1990), emphatically points out
this possibility. Currently it is not clear to what degree textbooks and teaching
practice are prepared to go with this. There is clarity about the modest importance
of column arithmetic: currently there is an obvious trend in The Netherlands to
delay it (let us say until grade 4, as proposed by Wiskobas) and to restrict it in
favour of mental arithmetic and estimation arithmetic. This shift did not take
place until in the eighties however. During the IOWO-period (1971-1980) the
mentioned parts of basic skills and arithmetic were still fairly unrelated: there was
a separate algorithm program. And column arithmetic was introduced in grade 3
in those days.
No special didactics had been developed for the learning of the tables of
multiplication or elementary mental arithmetic. That did not occur until the
eighties for addition and subtraction (Treffers, 1991b), for multiplication and
division (ter Heege, 1985), for elementary mental arithmetic with the aid of
the empty number line on the basis of an idea by Whitney (1988), and for the
combination method mental arithmetic-column arithmetic (Treffers and De Moor,
1990). Programs were developed for mental arithmetic for instance, and the writers
of math methods also gave an important impulse. In short, too many people to
mention together brought about a small revolution in the program development
for basic skills and arithmetic.
In conclusion, a few remarks about Freudenthal's influence on the outlined
development of basic skills and column arithmetic. Contrary to what one might
expect, Freudenthal was hardly involved at all with this area of development.
And as far as the above mentioned issue about the relation and shift in position
of mental arithmetic and column arithmetic is concerned, one will find nothing
on the subject in his last book (Freudenthal, 1991). He was not keen, so it
became evident in the early stages of Wiskobas, on the fundamental importance
of memorized knowledge of addition and multiplication tables. This did change
somewhat under the influence of the Wiskobas group, but the fact remains that
this area certainly did not have his special interest. It lay much more with the
other major areas, which we will now review from the Van Gogh problem.
2.2. Ratio and Fractions
The first of the 'Van Gogh' problems concerns visual ratios. Numbers barely play
a role in determining that 14 square kilometres cannot be correct: the practised
eye sees immediately that this cannot be a rectangle of 3 - by 4 kilometres,
considering the trees, the tracks, the whole scene. Ratios also appear in the
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Wiskobas program in similar fashion, therefore at first only as visual ratios from
the observation world of drawings, maps, models and illustrations, whereby it
must be determined whether the ratios are correct or have to be adjusted. Later in
the program a number element is added. At issue then is estimation or determining
the exact ratio relationship, whether ratios are equivalent, or else to compare ratio
relationships, or determine the fourth proportional. In the 'Van Gogh' item for
instance, visual ratios were concerned, to determine the plant density for the two
given measurements and the conversion of (square) yards to (square) meters.
But of course there are many more forms in which ratios can appear: mixing,
borrowing, conversion, fair distribution, and the linear relating of measures such
as unit-price, price-weight, road-time and so on, to which the mentioned questions
about ratios can be applied. The dominant think and calculation models employed
are the (empty) double number line and the ratio table (also consider the (estimated)
conversion of 12820 square yards to square meters) which can both serve on many
levels of schematising and abbreviation - of course also for the special ratios which
are called percentages. Wiskobas has from the very start devoted much attention
to this rich area.
Fractions do not play an important role in the Van Gogh problem. Only in the
relationship between yards and meters do they appear, and then only in the form of
decimal numbers. And working with them can even be avoided by size reduction
or by working with squares of ten by ten. This immediately indicates one of the
strategies with which to approach and explain the multiplication and division of
decimal numbers, namely by changing measurement units with the help of the
rectangular model or else trying to get rid of the decimal point. But fractions can
also be used to solve ratio problems: ratios and fractions can be related to each
other. Ratio (fair) division for example of pizzas, chocolate bars etc. can lead
to breaking up and hence to fractions. Sharing of objects can achieve the same.
Accurate measurement with a prescribed measure also. And thus there are many
inroads to fractions which are used by Wiskobas in the first concept-forming phase.
At issue is not so much operating with fractions in the more formal sense, but fair
sharing, distributing and measuring and then the description of these activities and
their results - the development of comprehension and language. The relationship
with ratios is expressly maintained at this stage, partly to compare fractions and
to determine the (in)equality thereof through ratios (Streefland, 1991 a). In a next
phase when operating with fractions is brought more to the fore, it is especially
the measurement aspect that is employed. Keywords here are: mediating unit,
named fractions, the rectangle model and the strip or the (empty) double number
line (Streefland 1991a, b, and Treffers, 1991b). An example from the last in
conclusion:
|
i i year
(3) (12) months
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Problems such as -y + 4y; 4y - 4y;
↑
x ly; y : ly can be solved via the
3 2 3 2 3 2 3 2
mediating unit of month, therefore the underside of the double number line, but of
course also directly with fractions on the upper side of the number line. Note the
difference in notation between adding and multiplying: -y x -y is not possible
but l x ly is right, i.e. that
|
serves as operator. In the abstract notation 3 +
and 4 x 4 one cannot read the different status of 4 in these notations, as one can in
the notations with letters. Later on in the course pupils must themselves try to find
the right divisor to be able to work away the fractions by imagining a chocolate
bar with a varied number of pieces, or a strip, or a double number line. So much
for details concerning fractions, otherwise we will digress too far from the 'Vase
with Sunflowers'.
Historical Comments
In describing Freudenthal's influence on the development of education for fractions
and ratios one must immediately list his monumental 'Didactical Phenomenology
of Mathematical Structures' (1983) of which parts were already published as
early as 1973. And it is notably the phenomenological reflections about ratios and
fractions which have had great influence on the development of learning strands
in this area. And yet there was no evidence of direct extraction of education from
this analysis. Even stronger, upon first introduction one sooner had the feeling
that one was being distracted from it. Take fractions for instance: the enormous
variety and complexity of appearances, as described by Freudenthal, had an almost
paralysing effect on anyone doing developmental work on this subject. Add to this
the emphasis that Freudenthal placed on unicity of individual learning processes
- thereto in part inspired by the 'walks with Bastiaan' his grandson - and one can
imagine what an enormous burden developmental research in this area was faced
with. The result of these efforts is visible in the fundamental work of Streefland
(1991a) which in turn adds new elements to the phenomenology, such as fair
sharing via ratios and the working with a mediated unit for named fractions. And
the notation thereof with letters, plus the (empty) double number line as master
model for operating with fractions by Treffers (1991b). In short, Freudenthal
indicated where to look and how to look (observation of learning processes) but
even he did often not know in advance what would be discovered. His influence
was and is fundamental, but also difficult to pinpoint concretely because he was
not directly involved in the design of learning courses and did not really have
clear cut suggestions for the total education structure for these complex learning
courses.
A typical and intriguing remark in this regard was the following: "Why don't
we approach fractions in the same way as initial arithmetic, namely depart from
elementary context problems and from there try to arrive at formal arithmetic?" A
remark that has remained with me until today and a major principle of Wiskobas
for the development of ideas about calculating with fractions - leading to ratios,
to mediated measures, to named fractions, to explicit notation thereof, to the
(double) number line... Therefore indeed comparable to initial arithmetic where
also initially one works with named numbers, and the empty number line plays
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an important role. But it took ten to fifteen years for this encouragement to
take concrete shape, one which Freudenthal himself had not anticipated either,
but could ultimately appreciate. And still the developers had the feeling that it
was what Freudenthal had intended all along. 'Wiskobas and Freudenthal', a
complicated relationship - difficult to separate, and certainly in terms of ideas
because Freudenthal was a member of the Wiskobas team.
2.3. Measurement and Geometry
'Van Gogh' is mainly about measuring - the measuring of area. Listed below are
six aspects of measuring in the Wiskobas program. These apply generally, but we
will illustrate them briefly using the example of area.
1. Comprehension of measurement is not automatic. What does it mean to
measure area in square meters, kilometres, yards? What is area? The
number of plants, for example, says something about the size of a field or
else the area. And through this relation to another, more concrete magnitude,
area can be made visible, and thus the development of comprehension of
measure be promoted.
2. Secondly, 'everyday' measurements must be related to the realm of experi-
ence and imagination of the children. What is a square meter, how big is
that? Can a non-square also have an area of 1 square meter? How big is a
square kilometre, imagine this in your surroundings. How big is 14 square
kilometres in that regard? What is the area of the floor of our classroom?
Pupils must allow a host of reference points about measure to sink in: how
tall are you, how fast is the car going, how many people live in a large
city, what is the temperature today... and so very many more. Items like
'Van Gogh' pertain to this. The Wiskobas program places strong emphasis
on it. Numbers of measurement can establish arithmetic in the reality of
children.
3. Development of strategies of measuring is inherent to measurement. Notably
estimation is of crucial importance, especially estimation on the basis of
personal reference points. See the trees, the tracks in 'Van Gogh', we already
mentioned them in respect to estimation. Estimating and measuring are
closely related.
4. The relationships between measures and magnitudes must be examined.
Metrics is one of these aspects - see 'Van Gogh'. But also relations between
circumference and area, between distance, time and speed, and so forth.
5. Calculating with numbers of measure sets special demands, notably when
dealing with rounded off or inaccurate numbers. Compare the indications of
250,000 and 12,820 in the Van Gogh problem, for instance. And the matter
of the rounding off of yards and meters - is that allowed? Errors can be
blown up when multiplying or dividing (compare 60 x 70 m - not such a
poor estimate but the area is quite far off). Matters to which attention must
be devoted in relation to measuring, estimating and calculating.
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6. And finally, the collection and processing of numerical data in graphs, tables,
diagrams and so forth, something not found in the 'Van Gogh' item. Drawing
graphs, interpreting them, all of this is also part of measuring.
All in all measuring offers a natural access to arithmetic, it provides models
and context problems which cover a wide area. Various skills and fields are related
through measurement - see 'Van Gogh'. The Wiskobas program reflects the great
importance of measurement.
There is very little geometry in Van Gogh... Although...? There is an element of
spatial orientation in the first problem and certainly in the perspective-problem (De
Moor, 1991). The Wiskobas program contains problems concerning projecting
and visualizing (1), orienting and localising (2), and shapes and pictures (3). The
children are encouraged to construct, calculate and reason.
In the first phase of geometrical instruction the structuring of spatial experi-

ences are mainly visual and implicit: the insights are in fact not expressed in words other than 'I see it this way'. Later intuitive notions must be more considered and better argued by means of language symbols, pictures, reasoning. Then explicit concepts are developed such as angle, perpendicular, direction etc., which possess a certain relation and which can be built up on in secondary education.
In the category of projecting and visualizing, to be somewhat more explicit, problems fit such as:
- Why do shadows become longer when you walk away from a street light and not when you walk away from the sun?
- Why does the moon walk along with you?
- Why does the thumb when held near the eyes change position if you close first one eye and then the other?
- Why do railway tracks converge in the distance?
No easy questions. They especially give an impression of the continuing line of geometry from primary to secondary education. The Wiskivon team (Goddijn, Kindt, Schoemaker and others) produced a great variety of ideas in this field on which Wiskobas also took advantage. Mathematical world orientation requires investigation, explanations of the self-evident, of own experience. There is a great deal of geometry in the Wiskobas program. And that is visible in the publication 'Five years IOWO' (Freudenthal, Janssen and Sweers, 1976) - you can tell at a glance.
Why? Because there is not another part in the program that can be as motivating, that can promote the forming of a mathematical attitude so much, that appeals so strongly to general objectives, as does geometry.
Historical Comments
Freudenthal, the great promoter of geometry education wrote:
The deductive system of geometry - the ideal for an elite school, was a farce, even for the elite pupils, and for the schools for the masses even less than that. At the end of the sixties I succeeded in formulating what I considered to be the solution: deduction yes, but then local, something the pupils can oversee, not global, as a system thought up by hobby horse riders. (Freudenthal, 1987)
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The Wiskobas group and certainly also Wiskivon, the group responsible for secondary education within the IOWO, needed only little urging in this direction: geometry was manifesting itself from reality on every side - geometry as mathematical world orientation which required no extensive didactical phenomenological analysis to develop quickly. The major part of 'Five years IOWO' is about measurement and geometry, for a part integrated with ratios. Freudenthal was also the great promoter behind measuring. He devoted a didactical phenomenological reflection to measuring (1984), to measuring length (1983) and area (1978).
Generally it can be said that the result of Freudenthal's strong fixation on phenomenological diversity of the appearance forms of mathematical structures and on the great individual variety of learning processes was that he was extremely reluctant to dictate learning paths in the steering of education.
In the earlier mentioned evaluation talks about Freudenthal's own estimate of his role in the development of mathematics education in the period 1970- 1980, also regarded against the background of recent developments in the eighties, he was very frank in putting forward that his contribution had probably lacked enough of a steering element. Steering both in regard to those giving instruction and the alignment of the instruction material. Seen in that light it is no coincidence that the keyword 'reinvention' from this first major work (1973) is called 'guided reinvention' in his last book (1991).
Summary
Realistic learning strands start with the informal context bound working methods of children, in their personal reality. From there models, schemes, symbolisations are developed which serve as intermediaries to gradually bridge the gap between these start situations and the level of the formal, more general subject related operations. The learning of the tables for instance starts with the item about the number of days in so many weeks. Visualising via the time line and the calendar leads directly to the (empty one sided and double) number line and the rectangle. Two master models which are of essential importance for insight in multiplication, also in the areas of ratios, fractions and measuring. And for learning long division a dynamic context problem about the transportation of passengers with buses serves as model context or paradigm that provides a concrete basis for formal operation. The same occurs for fractions with problems about fair sharing and measuring, whereby the language of fractions is developed and where comparing and operating are given a concrete basis of orientation.
With this specific vertical mathematising combination of the starting situation from personal reality and the subsequent model forming, schematising and symbolising, and added to the horizontal theme-approach, all of this together distinguishes the realistic approach from the mechanistic, structuralistic and empiristic domain theories and practices. (See Treffers, 1987, 1991b). Freudenthal laid the foundation for this didactical realism and determined the development of concrete education for various main learning strands, but more indirectly than directly because he himself did not design or outline themes and learning strands.
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3. INTEGRAL EDUCATION DEVELOPMENT
3.1. Historical Comments
As mentioned before, Wiskobas already existed before the founding of the IOWO in 1971. And the idea of integral education development already existed before Wiskobas became a department of the IOWO. Wiskobas pioneers Wijdeveld and Goffree more or less brought it along when they joined the IOWO, or really, acted as co-founders. The idea to approach innovation of mathematics education on a wide front - via training institutions of future teachers, via in-service training, training of counselors and instructors, research, development, via textbooks, and curriculum development - this was an idea that Freudenthal was most keen on. The more so when he established in 1970 that the Wiskobas movement was not only viable, but that is was also well organized in one national and several regional working groups. When the IOWO was founded he wholeheartedly gave his best efforts towards coordinating and realizing the integral approach of education development in the best possible manner. And it is predominantly due to Freudenthal's authority, status and influence that this integral approach did indeed bear fruits. Even though his critical attitude with regard to educational research, curriculum and test development did guarantee that the discussions were always lively - to put it mildly. The consequence of the broad approach was in any case that at the end of the eighties the development of textbooks, tests and curricula were well attuned to the national final standards and to the (informal) national curriculum (de Jong, 1986). While in many other countries (Ernest, 1991) any such attunement is lacking and the textbooks, the national curriculum and the tests explicitly lack the concrete views of progressive educators. For a description of this process of integral education development and the construction of an adequate infrastructure it would lead too far a field to go into this very complex process at this time. Freudenthal does dwell on the matter rather extensively in his last book (1991) in his outline of 'The Landscape of Mathematics Education', in which he concentrates on theory forming, research and dissemination. These three aspects become visible most explicitly in developmental research. And what Freudenthal writes about it is so important that I would briefly call attention to it here.
3.2. Developmental Research
Developmental research denotes a comprehensive program that is directed at innovation and improvement of education. It is research with an important development component: it is not merely established how things 'are' in existing education (especially research on errors is very popular in this respect) but much more how things should be, and one develops education that suits these findings (in a theoretical and practical sense). Development and research take place in an integrated cyclistic process. And the developer must report on that process: the own learning process therein, and the rational and empiric justification of the developed product must be made clear, so that others can also benefit from it in their development, research and teaching (also see Gravemeijer, 1989). Developmental
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research comprises much more therefore than empiric testing, but rather all sorts of aspects of innovation, among them that of dissemination - this as opposed to the more common research. Freudenthal observed that developmental research of this kind does not fit well in the current RD and D model, and therefore also not in regular educational research.
4. CONCLUSION
In July 1990 Freudenthal pointed 'his' associates of the OW & OC (currently the Freudenthal Institute) to the following article in a national newspaper:
THE POLISH KNOWN AS HARD WORKERS
Every year tens of thousands of Poles come to The Netherlands to work in the flower bulb industry for a few months. (...) This is the fourth time for Zygmunt. He has worked both in the fields and in greenhouses. Currently he is working in the transportation department of a company at the flower auction. "I load up the trucks, that is heavy work. On average I work 220 hours per week. That's no problem, because that's the way to make money" says Zygmunt (NRCIHandelsblad, July 1990)
> The question is: Is that possible, to work 220 hours per week?
Children of 9 and 10 (start of grade 4) appear to answer this question in many different ways:
- "I have no idea, I guessed 'yes'."
- "No, because my mother already works 180 hours per week. If he worked even more hours than he already does she would be busy all day."
Interesting answers because it proves that children do not automatically mathematise the problem, for example by coming up with the given that a day has 24 hours and a week 7 days. Children of 5 and 6 do not yet have this knowledge. And apparently neither do some nine year olds. In any case they do not already use reference numbers of this kind.
Even if they do have that knowledge, the reasonings and calculations differ substantially. Take the reasoning which departs from an 'exhausting' work week, therefore of '7 x 24' to refute the result of 220 hours. The calculations of '7 x 24' vary greatly: repeated counting 24 + 24 + 24 + 24 + 24 + 24 + 24 by various calculation methods; front-end method of stylised mental arithmetic 7 x 24 = 7 x 20 + 7 x 4, and the end-front method of the standard algorithm.
Characteristic for realistic mathematics education is that it puts such diversity to good use. For instance the front-end method can be presented as an abbreviation of repeated addition and also as basis for column arithmetic.
This does not exhaust the source of instruction possibilities: efficient arithmetic and estimation can also be used as examples:
- A day has 24 hours, so I take 7 x 25 = 175; minus 7 hours = 168 hours, the answer is 'no' (Figure 4).
- No, because 10 workdays is already 240 hours, so 220 hours would be more than 9 (Figure 5).
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Fig. 4.
"xx t0
wa
Fig. 5.
- No, because 220 7 = 31 2, but a day only has 24 hours (Figure 6).
The example shows how many learning possibilities there are in that one context problem. Not that all of these opportunities must indeed be employed, but one or several should be used, such as the discussion and analysing of the relations between the various solution methods of (stylised) mental arithmetic, estimation and column arithmetic.
Mathematics can be developed from (personal) reality in natural manner: the formal rules and procedures can be derived from the informal working methods of the pupils - that is the heart of didactic realism.
In the first part of 'Revisiting Mathematics Education' (1991) Freudenthal gives an in-depth analysis of the specific 'common sense' character of mathematics. 'Specific' because according to Freudenthal the mathematical activity distinguishes itself there from natural sciences. Elementary mathematics was invented many times over in the world in an independent manner, but there was only one Newton, Mathematics develops according to a gradual ongoing process. But primitive mathematical notions retain their original value as knowledge, skill and insight increase, while the natural science intuitions often prove not to hold after a time. In short, mathematics is excellently suited for gradual rediscovery, accord-
220"23 l4r"s ere
Fig. 6.
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ing to Freudenthal (also see Goffree, 1992). In the second part of his last book the didactical principles of 'guided reinvention' are described, and in the third part Freudenthal concentrates on theory forming, research and dissemination. A few elements from these two parts about didactics and innovation are brought forward in this article about realistic mathematics education:
to bear witness for the product to make it plausible and transferable, which demands an attitude of self-examination on the part of the developmental researcher: a state of permanent reflection. (Freudenthal, 1991).
Much about Freudenthal and his relationship with Wiskobas has not been highlighted: his ideas about levels in the learning process, his views on the issue of assessment, didactical classification methods, the educational-theoretical framework of realistic mathematics education and so on. But most of all his personableness, his intriguing off-hand remarks, his elaborations on seemingly futile observations of learning processes, his passion, his ability to use simple examples to show that mathematics is everywhere, his attention for both 'simple' and 'complex' forms of mathematics, his respect for practitioners of teaching, his admiration for a point well argued, his style of writing, his undisguised disapproval of developers and researchers who do not do justice to mathematics nor to the child, do no justice to mathematics as a human activity, to 'mathematics as an educational task' ...

One of the children answered the Zygmunt-item as follows: 'Yes, 220 hours
per week is possible, because you can load trucks and let the tulip fields grow at
the same time'.
Indeed, that is possible. Because that is also how Freudenthal worked... And
at the same time he let grow the tulip fields of realistic mathematics education. It
was a privilege working together with him in the Wiskobas team.
NOTES
I would like to thank Aad Goddijn, Fred Goffree, Rob de Jong, Edu Wijdeveld and other colleagues
for their useful comments on the first draft of this article.
2 Wiskobas is an acronym for the Dutch translation of 'mathematics in primary school. The Wiskobas
project started in 1968 and was later adopted by the IOWO (Institute for the Development of Math-
ematics Education). When this article mentions Wiskobas it refers to the project team that operated
within the IOWO and after its termination continued its work at other institutes, and in the course
of time was even expanded somewhat. When the text refers to the Wiskobas program, this refers to
the informal National Curculum for arithmetic in primary school that was compiled at the end of
the eighties at the initiative of Wiskobas staff, by a wide group of progressive educators to whom,
for this occasion, we will refer as the Wiskobas movement. Wiskobas spearheaded this movement.
Freudenthal was the most prominent member of this innovative group.
3 The department for secondary education within the IOWO was called Wiskivon. It originated
somewhat later than Wiskobas. Freudenthal's direct involvement with this group was initially less
intensive than it was with Wiskobas. But his ideas also found their way into secondary education via
Wiskivon, both through the SLO (Curriculum Development Foundation) and the research group OW
& OC (currently the Freudenthal Institute).
4 This conversation was held on 11 August 1989, between H. Freudenthal and the author.
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The Netherlands
(HF. in: Editorial letter 5 July 1978)
In the case at hand however, these so-called heuristics are dependent in a more awkward manner. I
mean, dependent on the specific problem to be solved, and therefore I call them clues - a clue for
every problem and a problem for every clue. It reminds me of teaching techniques such as: '3+2 are
...? No answer. 'Think'. No answer. "F..". "Four". "No". "Five". "Very good'. Of course this is
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an exaggeration but you will understand what I mean. I am not at all surprised that the experimental
group is superior to the control groups.
Then for the subject of 'integration'. Of course, if you train students in a subject like this you
should do it methodically. Unfortunately the first question is whether students should be trained in
a subject like this, or whether there are not more important subjects in mathematics to be taught,
regardless the career the student has in mind. Do not forget that the whole teaching subject of
calculating indefinite integrals was invented to be tested out. I wonder whether it still exists at most
universities.
As a matter of fact almost all efforts to develop diagnostic tools in teaching arithmetic show
the same features as the project you have sent me. One of the most outstanding and worst among
them is the Swedish project. Any physician who would make such a superficial diagnosis, from mere
symptoms, would rightly be labelled a quack. I have never seen any benefit in ascertaining whether
a child is not able to perform this or that partial arithmetical operation even though all of them might
have been carefully selected. It is not important to know which problems a child cannot solve if
no effort is made to ascertain why it cannot. In order to know why children fail, one must observe
them. In medicine it is called aetiology if one tries to find the causes for failure. I am sure this is the
right way to make a diagnosis because in the course of the years I have had some experience with
remedial work, and during the last few years I have cooperated with a group of psychologists, who in
fact have singled out a few important causes of failure, which in a sense are still symptoms, though
more profound than those one discovers through armchair analysis of arithmetical problems rather
than by observing the leaming processes. The tests proposed in your project do not even apply to low
achievement computers, let alone to human low achievers. I suppose that serious work on diagnosing
low achievement in arithmetics has been done and is still being done at quite a few places in the world,
but unfortunately it is lost in the flood of irresponsible work. I call it irresponsible because it seduces
good teachers, who try to understand why certain children fail, to believe that superficial testing may
supersede careful observations, and because it teaches average teachers and student teachers to suffice
by looking for mere symptoms instead of looking for more profound reasons for failure. In the research
I alluded to, it appears that more than 50% of serious cases of low achievement are due to the lack of
any decent instruction, which for that matter is the consequence of poor teacher training. Those poor
teachers, rather than learning to look for possible causes of failure, were educated to believe in blind
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GEOMETRY BETWEEN THE DEVIL AND THE DEEP SEA
PHILOSOPHIES
For many years New Mathematics has meant new subject matter, and this is
what it still means in many cases. Where people tried to teach it, it became
meaning new teaching techniques. There are few people to which it means a
conscious philosophy.
Of course, there is no educator without a teaching philosophy, or should
I say, without two teaching philosophies, an explicit philosophy he professes,
and an implicit one he acts out. The latter is the more interesting and the
more important. I do not like explicit philosophies but I cannot understand
educators unless I make their implicit philosophies explicit. This is why I
start with philosophy. It is too bad, but I cannot help it.
I will try first to answer three philosophical questions. Since they are
questions of a philosophy, that is of human attitudes, they admit a gamma of
answers, and among these answers there are a few which I like better than the
remainder.
My first question is: what is mathematics? I mean not how mathematics is
defined but what is it to the human mind? Les us ask another question: what
is language? There are two answers: language as a stock of words, prefixes,
suffixes, flections, grammar and syntactical rules, and language as an activity
of speaking, reading, writing, and expressing oneself. Linguists consider it as
a great discovery of de Saussure's distinguish these two aspects of language,
ready made and acted out language. Both aspects are operational in language
teaching. As long as children are small they will be taught a language as an
activity, whether it is their mother tongue or a foreign - this does not make
too much difference. As they mature into rational beings, the teaching philo-
sophy shifts from the activity to the ready made subject matter interpretation
of language. You know that it does not work too well. Why not? Because it
is a wrong philosophy? No, there are no wrong philosophies. Both the
stock and the activity interpretation of language are correct but they apply to
different people. The one is good for linguists who take a language as a
subject matter and the other for the users of the language, who in fact, are
the majority.
What is mathematics? Of course you know that mathematics is an activity
because you are active mathematicians. It is an activity of solving problems,
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of looking for problems, but it is also an activity of organizing a subject
matter. This can be a matter from reality which has to be organized accord-
ing to mathematical patterns if problems from reality have to be solved. It
can also be a mathematical matter, new or old results, of your own or of
others, which have to be organized according to new ideas, to be better
understood, in a broader context, or by an axiomatic approach.
A great part of mathematical activity today is organizing. We like to
offer the results of our mathematical activity in a well organized form where
no traces betray the activity by which they were created. This objectivation
is a habit of mathematicians from the oldest times. It is a good habit, and it
is a bad one. We freeze up the result of our activity into a rigid system, be-
cause this is objective, because it is rational, and because it is beautiful, and
this we teach.
Few people know that mathematics is an activity. Little children are
taught mathematics as an activity, but as they mature into rational beings,
we are prone to teach them a well organized prefabricated deductive system
of mathematics, because rational beings may be supposed to understand
deductive systems. You know that it does not work very well.
My second question is: what is education? It is good luck that the first
piece of education in history we know about, is a lesson of geometry, the
socratic lesson Menon's slave was taught on doubling the square. Socrates
taught the slave not the solution of the problem nor solving the problem, but
finding the solution by trial and error. He did not teach a ready made solu-
tion but the way of reinventing the solution. Two millenia later Comenius
said: "The best way to teach an activity is to show it." This is a socratic idea,
though it involves more than a socratic lesson. While Socrates taught his
lesson, the slave listened, whereas Comenius will show the student an activi-
ty to explain afterwards what it means and finally to have it imitated by the
student.

Modern educators are likely to subscribe to a variation of Comenius'
device. Not
"The best way to teach an activity, is to show it" but rather
"The best way to learn an activity, is to perform it."
It is a mere shift of stress though it is an important one.
The traditional answer to the question "what is education?" is "transfer of
culture". I think it still prevails. Recently I discussed the task of the univer-
sity (or rather the Task of the University) with a bunch of students, mostly
from the humanities. Its task is, they said, to provide information, and if they
challenged the university, then they did so because the university provided
not enough information, or wrong information, or biassed information, or
because it suppressed information. The university should offer its disciples
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an encyclopedia of all available information, and from this they would
choose what they liked. Maybe this was just the look of the disciplines they
studied (if they studied at all and did not refer to information about the
study). Anyhow they seriously believed that elsewhere studies were conduct-
ed according to such medieval didactics; that the future physician was told
by his professors how to cure illness, the engineer how to build bridges, the
mathematician how to calculate. In fact this was once the practice of teaching.
Today, I told them, scientific aptitudes are taught in hospital wards, labo-
ratories, and practice rooms, and for the advanced student participation in
research is a chapter of his curriculum. They agreed that cycling, swimming,
skiing cannot be learned from a textbook, but I could not convince them that
science was as well an activity that could be learned, not in lectures and from
books, but by acting it out.
No astrologuer is needed to read the writings on the wall. The students
I told of who could understand the learning process only as an information
flow, are the same who reproach the university that it provides wrong infor-
mation and who would use the information channels of the university to
funnel what they consider as true information. If this were the point, they
would even be right. The only answer they should be given is that learning is
a different thing but receiving information.
From Comenius onwards learning has been becoming more and more an
activity. This has been a consequence of the changes in science and society.
For this reason I may dismiss the question whether people learn better by
active building up the subject than by passive reception of a ready made
matter. In fact much depends on the kind of subject matter what is better. It
does not matter whether the authoritarian method of cultural transfer is bad
or wrong or inappropriate in the learning process. It is much worse: this
method is simply impossible, it does not agree with the character of modern
society. The authority of the older cannot be maintained in the learning pro-
cesses of emancipating youth. Our cultural assets are too dangerous to be
offered the youth as ready made material. The instruction we provide should
create the opportunity for youth to acquire the cultural heritage by their own
activity. They should learn that the selfreliance they claim elsewhere extends
to their own role in the learning process.
My third question is about mathematical education. Should mathematics
be taught according to a deductive system? Most mathematicians are prone
to answer in the affirmative. Maybe it is not always possible, but in principle
it should. In any case they will try.
How can you teach logarithms before a theory of real number? How can
you teach probability without an axiomatic definition of the fundamental
concepts? How can you teach angles before analytic functions? How can
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you compute the area of cylinder, cone, and sphere before superficies of
curved surfaces has been defined?
How can it be done? It can. It happened in history and didactically it is
the correct way. Logarithms were invented when the ancient theory of real
number was eclipsed by more naive approaches. Probabilities were computed
for centuries without axiomatic principles. Angles are as old as mathematics
itself. Fortunately Archimedes has never been bothered by H. A. Schwarz'
paradox on the area of curved surfaces.
Actually students can learn to work with the slide rule without any
axiomatic or any creative definition of real numbers and can discover the
logarithmic property of the area of the graphic for $y = 1/x$ before they know
integrals. They can work with probabilities as soon as they know fractions.
They can understand and work with angles in mathematics and reality.
They can find out areas and volumes by rational means with no general
theory on areas and volumes.
Of course, mathematics involves deductivity. Working with the slide rule
and the protractor is no mathematics, measuring areas and volumes is no
mathematics. But accepting a deduction is no mathematics either, unless you
adhere to the interpretation of mathematics as a ready made subject.
In which order, if not in a deductive one, should mathematics be taught?
The answer is simple: in that one in which it can be learned, which means,
the order in which it could be invented by the student. This is not at all a
revolutionary idea. It is the socratic lesson. In a thought experiment the
teacher has been reinventing the subject matter as though he himself were
the student, and this is what he teaches. Modern didacticians would require
that rather than teaching the reinvention, the teacher should have the stu-
dent reinventing himself the subject matter. This is a modern reenforcement
of the socratic idea.
Let us take the example of teaching Peano's axioms on natural numbers
to a boy who knows quite a bit of school algebra. The deductive approach
would be formulating the axiom system to derive a series of theorems from
it. The thought experiment, on the other hand, guided by historical expe-
riences, shows that nobody can invent Peano's axioms unless he has first
formulated the priciple of complete induction. Further, nobody can formu-
late complete induction unless he knows what it is, and there is no knowl-
edge about complete induction, unless it has been experienced as a proving
activity. So the student should be brought into one or more situations where
he can reinvent the trick of inductive reasoning. There are quite a lot of them,
Pascal's trinagle, number of permutations, fundamental arithmetical laws,
and so on. With virtually no help students learn to find and so use the prin-
ciple of complete induction, though it is a hard thing to formulate it. From
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complete induction one could proceed to Peano's axioms if the thought ex-
periment shows a way to do it. Of course, the student cannot hit on the idea
of framing complete induction into an axiomatic system, unless he has
already acquired a certain experience of axiomatic systems, that is of axio-
matizing. If he has not, Peano's axioms is certainly not the best example to
start axiomatics with.
The previous example shows a remarkable structure of the learning pro-
cess which completely differs from the deductive structure of mathematics.
The van Hieles called it the levels in the learning process. The student applies
certain new rules unconsciously until at a certain moment he becomes con-
scious about them. From analyzing the structure of the subject matter he
switches to analyzing his own mathematical activity. He grasps the laws and
regularities of his activity, which enables him to apply them more conscious-
ly. Maybe he even succeeds in formulating their principles in general terms.
The van Hiele levels of the learning process are often characterized by a
logical feature: the activity on one level is subjected to analysis on the next,
the operational matter on one level becomes a subject matter on the next
level.
Mathematics on the lowest level used to be allowed to start as an activity.
Experiments, in particular by Dienes, have shown that children can per-
form rather involved mathematics on this level, that is to say they can
develop an activity which on a higher level would be interpreted as mathe-
matics. This bottom level is an indispensable precondition of mathematics.
This should be stressed against people who rightly object that it is no mathe-
matics at all. It is bottom-level, it is indispensable premathematics.
Adding numbers represented by counters is no mathematics. As soon as to
add 5 to 8 the child physically or mentally divides the 5 into 2 and 3, it per-
forms an act which on a higher level can be interpreted as mathematics. As
soon as the child grasps this principle and tries to formulate it, it reaches
the first level of mathematics.
WHAT IS GEOMETRY?
For long times mathematics has been synonymous with geometry. In fact,
there existed other branches, too, algebra, trigonometry, calculus, which,
however, were not much more than collections of haphazard, badly founded
rules, whereas geometry was a perfect logical system, where everything
rigorously followed from definitions and axioms. You know that things have
changed; today mathematicians are prone to reject traditional geometry,
because it is not a rigorously deductive system.
The deductive structure of traditional geometry has never been a convincThis content downloaded from 143.160.38.146 on Tue, 24 May 2016 06:16:52 UTC
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ing didactical success. People today believe that geometry failed because it
was not deductive enough; to my opinion it failed because its deductivity
could not be reinvented by the learner but only imposed.
There have been many efforts to reform teaching geometry. In my coun-
try Tatiana Ehrenfest-Afanassjewa, the wife and collaborator of the famous
physicist Paul Ehrenfest, was much influential. Her "Ybungensammlung" of
1931 is still a treasure of geometrical exercises. Tatiana Ehrenfest did not
reject deductivity. On the contrary, some deductive system, influenced by
Helmholtz' axiomatics, was the goal of her method. But her geometry
started as a science of physical space, of the space in which the child lives and
moves, as an organization of the learner's spatial experiences.
It sounds oldfashioned to claim that geometry should be related to physical
space. In fact, Tatiana Ehrenfest knew mathematical axiomatics and culti-
vated it, but she was also a physicist who knew about the profound physical
implications of space theories. Many didacticians today would reject physi-
cal space as a field to be organized by mathematics. We are mathematicians,
or are we not? We are builders of mathematical structures, and if physicists
and other people can use them, they may take them, but no deductive
system can be built on the quicksand of reality, and mathematics may not be
contaminated by elements which are foreign to deductivity. If for instance
the system commands us to start geometry with the affine plane, then physi-
cal space with geometry with its solid bodies, its distances, deplacements and
fittings is only in obstacle to good mathematics.
This is one philosophy, and I think it is a good philosophy. As long as
people are building a deductive system they are often obliged to close their
eyes for the seductions of reality. But there are other activities, and there
are people who will never build deductive systems of their own or even
rebuild those of others though they need mathematics. We cannot expect
them to espouse this philosophy.
Finally nobody can deny that we live in space, that we move in space, that
we analyze space, to be better adapted to it. In fact, we are so much used to
space that we easily forget about its importance for ourselves and for those
we are educating. Let us ask a few haphazard questions:
Why does a piece of paper fold along a straight line?
Why does a rolled piece of paper become rigid?
Why does a tied paper ribbon show a regular pentagon?
How do shadows originate?
What is the intersection of a plane and a sphere, of two spheres?
What kind of curve is the terminator on the moon?
Why can the radius of a circle be transferred six times around periphery?
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How come a beautiful star arises by this construction?
Why is the straight line the shortest?
Why do congruent triangles fit to cover the plane and why do congruent
pentagons in general fail to do so?
How can people measure big distances on the earth, the diameter of the
earth, and distances of celestial bodies?
What does a cube look like if viewed along a spatial diagonal?
What is larger, the superficies of a sphere cap, or that of the cylinder around
it?
What is the shortest path for a light ray to travel from one point to another
while touching a mirror?
How does a kaleidoscope work?
What is the biggest sphere in a tetrahedron?
Which kind of closed curves are equally as wide in all directions?
How does the liquid level in some vessel change if a certain quantity of
liquid is added?
What is the relation between real and apparent size of a body?
If a cube is split into six square pyramids with their vertices in the center and
these pyramids are turned outside upon the corresponding faces, why
does a rhombododecahedron arise?
How can you measure the inclination of a line and a plane, or of two planes?
Is there a horizontal (a vertical) line in every plane?
What is the difference between a right and a left screw and why are not they
equivalent under rigid motions?
What is a rigid motion on the sphere?
Why is a convex polyhedron rigid?
Why can a table with four legs wobble, and what is the difference with a
table with three legs?
Why does a door need two hinges, and how can we add a third?
And finally the old question: why does a mirror interchange right and left

though not above and below?
Notice that I did not ask any questions of practical use. My task would have been much simpler if I had done so. Notice, too, that there were no "puzzles" in the list, but only questions of principal importance and which nevertheless can be answered with little ingenuity.
Many mathematicians will consider my questions as irrelevant in teaching mathematics. With no regret they would leave them to physicists or other kind of people if need be that students get acquainted with space. Indeed, almost none of those questions can be fitted into a preconceived system of mathematics. To be sure, this is not only true for such questions of geometry,
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but for almost all questions which arise in reality. In particular, probability is an instructive case. All real problems of probability are irrelevant for an axiomatic probability fitted into a preconceived system of mathematics, and axiomatic probability in turn is irrelevant for solving real problems in probability.
I espouse the philosophy of teaching mathematics related to reality. Mathematics is important for many people because it admits of multifarious applications. I do not trust teachers of other disciplines to be able to tie the bonds of mathematics with reality which have been cut by the mathematics teacher. Moreover, I do not believe that mathematics, not tied to lived reality with strong bonds, can have a lasting influence in most individuals. We mathematicians retain the mathematics we learned, because it is our business. People usually forget what is not related to the world in which they live. For most people mathematics cannot be an aim in itself; if they have learned it in an unrelated way, they will never be able to use it.
CONCRETE MATERIAL THE CHILD CAN HANDLE
Tatiana Ehrenfest developed an introductory geometry course of exercises in space. Meanwhile many others designed concrete material, in our country for instance the van Hieles and van Albada, to stimulate a geometric activity of folding, cutting, glueing, drawing, painting, measuring, fitting, paving. People generally call it experimental geometry. I do not like this word. If the word means that the student is expected to experiment, then the greater part of his mathematical activity should be experimental, as is ours as far as we are creative mathematicians. If the term is chosen to remind of experimental physics, it is wholly mistaken. "Experimental geometry" suggests procedures like determining X by measuring a string laid around a circle periphery, a silly exercise you will never meet in such collections. In former times they called it mensuration, an activity which leads nowhere. The introductory geometry exercises by Ehrenfest and others, however, are zero level activities in that they prepare for the higher levels. The attitude of the children who work with this material greatly differs from the realism of the physical laboratory. To illustrate this I will give you a few examples.
Children in Dina van Hiele's class (12 year olds) got the task to design a sidewalk pavement with square flagstones. One of the children drew stones with cracks and sandgrains in the groves between the stones. One hint was sufficient to convince the child that these were inessentials in geometry.
Dina van Hiele gave the children bags with congruent cardboard triangles, quadrangles, pentagons, hexagons, which should be used as tiles to pave the plane. The pieces were not congruent enough to give a good fitting, but
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the children never complained; they built the patterns as though they were regular and understood them as regular. In general from the beginning of geometry they accept bad fittings and imperfect congruencies and symmetries as though it were perfect ones.
As the children get the task to draw a triangular pavement, they will first use the cardboard material but as soon as they have grasped the pattern, they will finish building and start drawing. None of them will add the drawn triangles piecewise but all of them will first draw long horizontal lines between which the triangles are to be fitted. Some of them use a second set of parallel lines; it is very rare that a child starts with drawing immediately three sets of parallel lines.
DINA VAN HIELE'S EXPERIMENTS
Before Tatiana Ehrenfest it was an axiom that geometrical instruction started in the plane. More precisely it started with points, continued with lines, and since the plane were the place to locate these objects, all happened in the plane. Tatiana Ehrenfest once told when she taught classes, parents complained that she discussed spheres with children before they had learned defining straight lines. In many countries plane geometry started in the 7th grade, geometry in space in the 10th or 11th grades. Children who were good in plane geometry, often failed in space geometry. Their spatial imagination had been killed by too much and too exclusive plane geometry. In our country the usual start in geometry is now in space. To give an example, I sketch Dina van Hiele's first course, with 7th graders.
She started geometry with the cube, but of course with no definitions. Cubes of different material were shown, cardboard, wooden, meccano cubes. The discussion led to constructing cardboard cubes. In this process the children became acquainted with the geometrical instruments and with fundamental notions such as right angles (defined by folding). In the meccano cube two kinds of diagonals were identified. After face diagonals had been drawn on the selfmade cube, the question arose how to find exactly the spatial diagonals. The teacher showed a cube with its upper face open to allow a look inside. There was a cardboard diagonal partition wall in it. The wall could be removed. It was a rectangle. The children understood this hint to construct the spatial diagonals. The children made a cardboard partition rectangle and noticed that it could be fitted in several ways into the cube. In how many? Other examples of double counting were dealt with. The children made more regular bodies.
The class then proceeded to studying symmetries first in space, then with plane figures. One half of a symmetric figure was completed to a whole. A
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mirror was used to check the result; the discussion led to an analysis of the concept of symmetry. A host of symmetric figures were considered and constructed. The discussion then focussed on the rhomb as a starting point for many constructions. This trimester of geometry finished with angles. Of course no definition of angles was given. Angles were made, estimated, measured. The principle of this first stage of geometry was, as Dina van Hiele put it: Rationally guided operations with concrete material.
A detailed report on a second trimester's course has been the main subject of Dina van Hiele's thesis. It is the wealthiest treasure of didactic observation and analysis in mathematics I ever saw.
The subject of this course was "tiles". After the children had become acquainted with congruency and regular polygons, they were given bags containing congruent regular polygons as a first concrete material to study the covering of the plane by congruent shapes. The children discovered by reasoning that the angles of a triangle sum up to 1800, the analogous facts for other polygons, and the interrelation between these facts. They discovered why certain shapes of tiles did not fit to cover the plane. (Not all pentagons, but are there pentagons that fit!) Logical relations were put into arrow patterns. In the triangular pavement the children discovered parallelograms, "saws", "stairs", equality of angles, "enlargements", symmetries and regularities; areas were compared by decomposition. Logical arrow patterns were combined into logical "trees".
The third trimester subject was again congruency, mainly in space. Solids were made, and volumes were compared by decomposition. The final piece of work and the acme of the course was the construction of the rhombododecahedron.
FITTING
I could only sketch the subject matter of Dina van Hiele's first course. The method and its analysis is more relevant. Though the plane provides easier opportunities of logical analysis in geometry, space with its solids is more concrete. It better meets the children's creative wishes. Figures are drawn, solids are made.
There is one exception in plane: paving a floor with congruent tiles. Here the leading psychological idea, I mean fitting, is the same as in space and it is realized as concretely. Fitting is a motor sensation. Psychologists can tell you how strongly the motor component of the personality is marked at a young age, how important motor apprehension and memory may be.
Things fit. Do children ask why? Apart from a rare exception young children do not. All these miracles of our space do not seem to make any impression. But they grind as millstones. The highest pedagogic virtue is
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patience. One day the child will ask why, and there is no use starting systematic geometry before that day has come. Even worse: it can really do harm. The clue of geometry is the word "why". Only joy-killers will deliver the clue previously. The miracles of fitting are to prepare maturity for systematic geometry. But even if this has been reached, they will not become redundant. They are raw material for geometrical thinking. The child should recall and reconsider the treasure of old problems at every new stage.
Of course fitting as a leading idea requires that space is acknowledged as the home of solid bodies. There are no fitting problems in affine geometry. Affine geometry is a sophistication that needs a lot of deductive experiences to be invented. It can never be the start of geometry.
DEDUCTIONS
There are many ways of organizing the child's exploration of space. I quoted Dina van Hiele's because it has most profoundly been analyzed. The period covered by Dina van Hiele's course shows a steady though discontinuous growth of the child's ability to logically organize his own activity. On level 0 the child is thinking as it were with his hands, his eyes, his kinesthesis. To the viewer who knows mathematics the child's activity looks like mathematics. When the teacher shows the partition wall of the cube, the child succeeds in constructing its spatial diagonal, and this activity can be interpreted by the viewer as the same reasoning that leads himself to construct the spatial diagonal. In geometry the child can for a long time continue this 0-level activity until it starts reconsidering it consciously - there are quite a few who excellently perform on this level and never reach a higher one, but the reason that they fail often is that they are pressed too early and helped by algorithms to simulate acting on a level they did not reach.
Several times I stressed that in this approach no definitions are provided for geometrical objects when they are introduced. If the antididactical inversion is applied, a deductive exposition starts with definitions. (Traditional geometry texts even define what is a definition, which is still a higher level of learning.) In socratic didactics such an approach would be impossible since nobody can define a thing unless he knows what he should define. It is even worse in mathematics, where "definition" has a very special meaning. A definition in mathematics does not just tell people what a certain word is supposed to mean, but definitions become links in a deductive chain, and you cannot forge the link unless you know where it should fit.
If Dina van Hiele showed their students a cube, a rhomb, a parallelogram, or if she gave examples of congruent figures, the students never failed to understand which shape she meant, as well as they recognize what is a chair,
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or a bottle, or a doll though this has never been defined nor have they been confronted with all objects of these sorts. Of course there may be some uncertainties, for instance whether a square should be counted among the rhombs, and a rhomb among the parallelograms. Though the teacher can impose definitions that settle the questions, this means degrading mathematics to something like spelling, ruled by arbitrary prescriptions.
If the child knows what a rhomb or a parallelogramme is, it may visually discover properties of these shapes. There are a lot of them, and in the class discussion children will recite them. A parallelogramme has its opposite sides equal and parallel, its diagonals bisect each other, opposite angles are equal, adjacent angles are supplementary, the parallelogramme has a centre of symmetry, it can be split into congruent triangles, congruent parallelogrammes will cover the plane. There are a host of visual properties which ask for organization. Here starts deductivity; rather than being imposed it unfolds from local germs. The properties of the parallelogramme become deductively interrelated, and finally one emerges (maybe different ones for different children) from which the others can be deducted. This property can be taken as a definition, and now it can be understood why a square and a rhomb should be considered as parallelogrammes. By this activity, the child learns defining, too, and it experiences that in mathematics a definition is more than a description, that it is a means of organizing properties of a thing deductively. However, on this level the child would not yet be able to analyze the process of defining and to tell what is a definition. Indeed, this is again a higher level of the learning process.
To those who believe in a system, this method is a horror. A deduction needs a firm ground, and you cannot reason rigorously about things you have not rigorously defined before. This is a sound philosophy though it is not that of a creative mathematician. Definitions are not preconceived to derive something from them, but more often they are just the last element of the analysis, the finishing touch of organizing a subject. Children should be granted the same opportunities as the grown-up mathematician claims for himself. Telling a kid a secret he can find out himself is not only bad teaching, it is a crime. Have you ever observed how keen six year olds are to discover and reinvent things and how you can disappoint them if you betray some secret too early? Twelve year olds are different; they got used to imposed

solutions, they ask for solutions without trying. Good geometry instruction would be a means to teach them a lot of things, to teach them organizing a subject matter, and what is organization, to teach them conceptualizing, and what is a concept, to teach them deducing, and what is a deduction, to teach them defining, and what is a definition, to have them to distinguish why some organization, deduction, definition is better than others. Traditional inThis content downloaded from 143.160.38.146 on Tue, 24 May 2016 06:16:52 UTC
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struction is different. Rather than giving the child the opportunity of organizing spatial experiences, the subject matter is offered as a preorganized structure, with all concepts, definitions, deductions preconceived by the teacher who knows of every detail which aim it serves - or rather by the textbook author who has carefully built in all his secrets.

THE SYSTEM

Traditional geometry was taught as a deductive system. It was a fake system but by teaching it teachers used to indoctrinate themselves to believe in the system. In that time geometry was the only part of mathematics that was taught according to a system, while other parts were simply collections of thumb rules. Things have changed. There have been rare attempts to save the character of geometry by means of some Hilbert style axiomatics. The other extreme is, rather than mathematics, to teach a system of mathematics and to fit geometry into this system. Which pieces of geometry are really fitted into the system and on which places, does not depend on geometry but on the system. The usual frame is to begin with affine vector space on which later on an inner product is imposed. The most conspicuous aspect of space, I mean fitting, is disregarded and ignored by affine geometry. But it is still worse. In such a rigid approach no opportunity is left for children to explore space and the bodies in it, to organize the subject matter, to invent definitions and deductions. Among all the problems I quoted when I asked 'what is geometry?' there is none that can meaningfully fit into such a system. Maybe you can compute the intersection of a sphere and a plane or of two spheres by linear algebra but to discover that the result is a circle you have to know what is a real sphere in real space. Of course you can figure out that the radius of a circle is the side of the inscribed regular hexagon by means of an analytic proof if you like proofs that obscure all essentials. To discover that congruent triangles can cover the plane, while pentagons can in general not, linear algebra is entirely unfit, but is even no appropriate means to prove such facts. I would be able to continue this way with all examples I gave earlier. The geometry allowed by linear algebra is an utterly dreary product. Its highlights are to prove that two different straight lines can have none or one intersection, that for circles these numbers are 0, 1, or 2. Maybe vector algebra still provides a tasteless proof for the three bisectors of the sides of a triangle passing through one point. The only geometrical subject that could adequately be tackled by vector space, I mean barycenter and convex bodies, is generally disregarded because it does not fit into the system of mathematics. Usually such a system of mathematics is not less ramshackle than old geometry was, but today the authors learned how to make more successful
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attempts to hide its deficiencies. This, however, is not the feature for which I blame system builders. It is rather the dogmatism of the system, and the didactical inversion. Systembuildingcould be one of the aims of mathematical education and maybe even a realistic one provided that organizing a subject matter has been exercised on a lower level. An imposed system, however, prevents the student from learning organizing a subject matter on any level.

GEOMETRICAL AXIOMATICS

Some people have been anxious to save genuine geometry from denaturation by linear algebra. To do so, they tried geometrical axiomatics. In fact, it is another way to kill geometry. It is more or less Hilbert-style axiomatics, with a long list of axioms, most of them seemingly trivial, though there may be a few quite complicated ones among them. The deductional structure usually is quite sketchy; the greater part of argumentations consists of such formulas as "it is easily seen that ...", and "we omit the proof which would be lengthy though it would not offer any new insight". It is a frank confession and it is true; reasoning within such system is, indeed, a dreary occupation. What is interesting in an axiomatic system is constructing it and verifying whether it is complete, that is, checking whether its geometry can be coordinatized in the usual way. If somebody needs geometry in algebra or analysis, he never reasons within an axiomatic system, which would be much too complicated.

It is not because of its complexity that I blame such an axiomatic system. It is the way in which it is offered to the students. They should use it to perform mechanical deductions in it, an activity which the author honestly dismisses as irrelevant. The essential activities around such an axiomatic system are reserved to its author: first, organizing globally the geometric subject matter to get the axiomatic system, cutting the bonds with the organized subject matter, to afterwards restore them by checking the completeness of the system. Geometrical axiomatics cannot be meaningful as a teaching subject unless the student is allowed to perform these activities himself. Usually he is not allowed to do so. Either the author judges that axiomatizing is a business of grown up mathematicians rather than of learners, or he knows from his own axiomatizing activity or a thought experiment that it would be much too difficult for the student. Indeed, a student who never exercized organizing a subject matter on local levels will not succeed on the global one. Prefabricated axiom systems have merits of their own. They are an acceptable subject matter for people who are experienced in axiomatizing. If a student has learned axiomatizing with easier material, he will recognize in a complicated axiomatic system the same features he
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knows from his former experiences, and he will be able to disentangle this system and to understand it as though he built it himself. But if axiomatizing has never been exercized, such an axiomatic system of geometry is only one more piece of indigestible mathematics.

The method by which the axiomatician attempts to save geometry from the claws of linear algebrists, is: competing with the mathematical rigor of linear algebra. It is a mistaken idea.

RIGOR

System builders know one level of mathematical rigor and often they manage to stick to this one during a course of many years. All below this level, which is their own, they consider as fake, and all above as highbrow. Active mathematics, however, knows many levels of rigor, and good teaching should respect them. It should also respect the fact that rather than being imposed each of these kinds of rigor should naturally develop in the learning process. On a low level it can be rigorous figuring out 8 +5 by counting five more steps from eight. On the next level the learner is required to split it into (8 + 2) + 3 and to use mental addition tables up to 10. Still later it is rigorous to know 8 + 5 = 13 by heart and to use it as a technique of addition tables up to 20. On a certain level it may be rigorous to read 5-8 from the number line, on a higher level rigor would mean to prove it by a kind of induction, on a still higher level rigor should require that it is understood as a definition, and finally it can be rigorous as a deductive consequence from an axiomatic system.

Every subject matter knows, on the level on which it is tackled, its own criteria of rigor. In the learning process they arise as tacit conventions between the learner and his surroundings, which are dropped or modified as the learning process continues. Reaching a certain level implies knowing by which rules of rigor it is governed, the main deficiency of an algorithmic activity on a level which has not actually been reached is that the learner does not master the rigor of that level.

This level theory of rigor is not to excuse sloppiness. Verbalistic definitions and arguments that in fact neither define nor argue anything, are as bad on every level. Circularity of arguments shows the same features on any level, though the criteria of circularity may vary. In a highly formalized system non-circularity can be checked by a computer, but you have to educate your mathematical sensitivity to feel on any level what is a circular argument. Axiomatics is not at all the highest level of rigor. Most axiomatics are poorly formalized, in particular those of geometry. This is no blame if the degree of formalizing corresponds to the level on which the axiomatic system is presented.
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AXIOMATICS AND TRADITIONAL DEDUCTIVITY

Often the objection is raised: Traditional deductive geometry is also axiomatics though it is the bad one of the juggler who is hiding his axioms in his tophat and sleeves. Is not honesty a better policy?

The assertion that traditional deductivity and axiomatics proper are not essentially different is both right and wrong. Axiomatics from Pasch and Hilbert onwards up to our days, presupposes the syntax and semantics of of every day language. One defines "lines are parallel..." and continues speaking of "parallel lines", "parallel projection", "parallelism", and so on, one defines "intersecting lines" and continues with "lines that meet each other". Of course this is entirely justified on this level. But there are other levels. One can take offense of this linguistic unformality and try to formalize axioms and other statements. Formalizing, too, knows levels: on a very high level even deduction is being formalized. It is not true that axiomatics is the summit of honesty and rigor. Every level knows its own honesty and rigor, which cannot be enforced in teaching, if the student has not reached this level. It is again and again attempted, though with sham success. Geometrical axiomatics is particularly dangerous. High level tricks are so easy and captious to be built in. The author hopes that thanks to the automatism of the axiomatic method these built-in features would function automatically, that is, without the students' understanding of the trick. I know such policy is not unusual in teaching though conscientious educators fight it. I fear axiomatics may lead us deeper into this swamp.

If didactically viewed, axiomatics and traditional deductivity are much different. The difference can be characterized as that between global and local organization of a subject matter. Everybody knows how long it lasts to have an average student view a proof as a whole. It lasts still longer to have him oversee the connections with other propositions. If, with much trouble, he has reached this level, he is pressed upon the next, the global organization. There is still another important factor. In the course of his instruction the student has become acquainted with two kinds of defining, the descriptive definition which fixes a familiar subject by reading characteristic properties from it, and the algorithmically constructive, creative definition which produces new subjects out of old ones. Axiomatics shows another feature by which it is distinguished formally and didactically from traditional deductivity: the implicit definition, which through descriptive, is creative but not in an algorithmic way.

LOCAL ORGANIZATION

How to save geometry if axiomatics cannot do it? Nobody would ask how to
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save physics or zoology, which have never been axiomatized. Students learn to figure out the price of 3 pounds of sugar if that of one pound is given, and the area of a rectangle if the sides are known, though the greater part of the notions involved have never been fitted into an axiomatic system. In applying mathematics you never move within an axiomatic system, and if by chance somebody would ask how to save applications, he probably would mean how to protect them against the axiomatic dogmatism. Probability is a most beautiful application which is gravely endangered in teaching to be killed by axiomatics.

Geometry can be saved if it is presented as a field in which the student can be active. As a prefabricated subject it is due to die by suffocation.

I already explained how in introductory geometry the student can learn to invent organizing the shapes and phenomena in space, by means of geometrical concepts and their properties. On the next level he would organize these concepts and properties by means of relations of a logical nature. On a still higher level this relational system may be a subject of investigation.

Let us take one example. The crucial mathematical childhood experience of quite a few mathematicians who wrote autobiographies, was the perpendicular bisectors of a triangle passing through one point. It is, indeed, a beautiful theorem. Children can easily find it, provided it is formulated in a less symmetric way: "Draw the bisectors of AB and BC, which intersect at M; look where the bisector of AC passes." Let us analyze the proof in the same way as the learner should do after he found it.

The proof rests on the property of the bisector of XY being the set of all points equidistant from X and Y, which may have been recognized by symmetry arguments. M is on the bisector of AB whence

MA = MB;

M is on the bisector of BC whence

MB = MC.

From both follows

MA = MCI,

whence Mis on the bisector of AC.

The proof is a combination of a few surprises. The first and second equalities are found by applying the bisector property one way, the third is a consequence of applying it the other way, the third is a consequence of applying it the other way. First and second, M is on the bisector, thus M is equidistant, third, M is equidistant, thus M is on the bisector. I think this is the first psychologically convincing occurrence of the logical complex
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characterized by such concepts as inversion, necessary and sufficient, if and only if.
The next surprise is the transitivity of the equality of line segments. The seemingly trivial property of transitivity should be made explicit to understand the proof. Again I think it is the first possible approach to the transitivity property and the first example of its productivity.
The third surprise is that such a symmetric statement as that about the three perpendicular bisectors must be tackled in an asymmetric way to prove it. It is an important step to learn that "three lines pass through the same point" means the same as "one line passes through the intersection of the two others". It is the first example of a methodological paradigm which serves up to the highest levels of mathematics.
The fourth and maybe the greatest surprise is that an incidence theorem (three lines passing through one point) is proved by metric arguments. It requires a not so easy analysis to understand the profound reason of this feature.
Finally it may be mentioned that the theorem leads to the fascinating construction of the circumcircle.
The proof of the theorem on the perpendicular bisectors is not only a marvelous piece of geometry, and a rich source of didactical ideas, it is a good example in geometry for what I have called, local organization. It can be dealt with as soon as children have understood the perpendicular bisector as a locus of equidistance. They need not be able to prove this crucial property of the perpendicular bisector. A proof of the locus property of the bisector cannot contribute anything to the understanding of the circumcircle theorem. I even doubt whether a proof of such an obvious fact as the locus property of the bisector can be recommended at an early stage. At a more advanced stage, it would make more sense to ask why the perpendicular bisector of XY is the locus of points equidistant of X and Y; then one may even ask why equality is transitive, which comes down on understanding what equality means. One can ask why the bisectors of different sides of the triangle intersect, why straight lines have at most one intersection. Of course such an arrangement contradicts the philosophy of mathematics as a prefabricated system. It is the way of exploration, in mathematics and in any science whatsoever, the way to understand and explain phenomena.
The question why the bisectors are concurrent is much alike that why the electric bell rings or why it does not, why the stomach does digest food and why it does not digest itself; why comets have a tail, and who committed the murder. It is questions "why?" asking for a reason or for a cause. And they have also in common that no answer is definitive. You can continue asking "but why is the bisector a locus of equidistance?" "why is equality
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transitive?", "why does the current activate the electromagnet?", and so on.
The answer to any question contains the germs of fresh ones. It is an apparently infinite chain, although Aristoteles believed it could be suspended on the principles as the first reason and on God as the first cause. But does it really matter whether the chain is infinite or not, if at least I am able to avoid circularities? The practice of every day knowledge as well as of science, indeed, is, at a certain moment, to stop asking why. To repair the bell, I do not need Maxwell's theory, and if I study Maxwell's theory, I can dismiss field quantization.
I admit from this point onwards mathematics is a bit different. Mathematicians invented the axiomatic trick, that is accepting the vice of circularity as a virtue. You stop asking what are points and lines; instead of an explicit definition you tell what you are allowed to do with them. Actually every science behaves so, every science is based on implicit definition, but mathematics is the only one in which this is cultivated as the summum of deductivity, and even more, after a system has been axiomatized, its bonds with reality can be cut; by anontologization it becomes a self contained field.
We are again back at axiomatics, but I think we can now more clearly distinguish what axiomatics means. A mathematical text may start with axioms because it is ready made mathematics. Mathematics as an activity never does so. In general, what we do if we create and if we apply mathematics, is an activity of local organization. Beginners in mathematics cannot do even more than that. Every teacher knows that most students can produce and understand only short deduction chains. They cannot grasp long proofs as a whole, and still less can they view substantial part of mathematics as a deductive system. We are lucky if they can learn organizing locally a mathematizable field of reality or a piece of mathematics itself, because this is just what they will need in every day life and in their profession.
Organizing locally is not a deficient or illicit or dishonest activity in mathematics. It is a generally accepted attitude of the grown up mathematician in pure and applied mathematics even if he would never publish such exercises. None of the various axiom systems of geometry has ever been used to find or to prove geometrical statements if they are needed in algebra or analysis. It is much too complicated and the road from the axioms to important theorems is much too long. The proof is superseded by the firm conviction that it can be done but that it is hardly worthwhile. One is satisfied with the local organization up to a changing horizon of evidence. Coxeter's Introduction to Geometry is a marvellous demonstration of this attitude. The author knows in any case exactly where this horizon is lying, and which kind of rigor is adapted to the subject matter, for instance, that no recourse to the axioms is needed for Morley's theorem on the trisectrices, whereas a theoThis content downloaded from 143.160.38.146 on Tue, 24 May 2016 06:16:52 UTC
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rem like Sylvester's on collinear points which essentially depends on order properties, needs an axiomatic background.
OTHER VIRTUES OF AXIOMATICS
None of the axiomatic systems of geometry are such that they can be taught high school students as an experience, but maybe some could be offered to otherwise axiomatically experienced students (I mean students experienced in axiomatizing) as a subject to be analyzed. What would be the educational context of such an attempt?
We know that axiomatics is important, and if possible, the student should learn what axiomatics is. There are, however, so much easier subjects to be axiomatized than geometry, such as groups, measure, linear order, cyclic order, angles, and so on. I have already rejected geometric axiomatics as an expression of rigor. What is rigor, depends on the context. To justify axiomatizing geometry by arguments of rigor, one has to show a context in which local organization is not adequate. There are such contexts but usual axiomatics does not apply to them.
Another benefice of axiomatics is cutting ontological bonds. In the usual abstracting axiomatics it is too easy a business. In geometrical axiomatics, which is describing rather than abstracting, it is much more difficult. Linear algebra on the other hand, contains too little geometry to make the cut of ontological bonds effective. It would be worthwhile to show the student how geometry can be made independent of physical space but it is not easy. Of course, you can tell it with formulae like "it is easily shown", and "we omit the proof", but this is not more than verbalism.
Axiomatics is also an organizational form of foundational investigations. The axiomatical structure allows to investigate the scope of such and such an axiom by omitting or replacing it. This, too, can more easily be done with simpler axiomatic systems though it shows special aspects and charming features in geometry.
I would mention here an approach which is foundational though it has not been worked out to full fledged axiomatics. It is P. J. van Albada's:
Within spatial euclidean geometry one turns to geometry on the sphere, which by identification of the antipodes may be even transformed into the elliptic plane. Theorems known from euclidean plane geometry are investigated in elliptic geometry. Some remain valid, others cannot be transferred. To become familiar with elliptic geometry one tries to derive theorems of elliptic geometry from each other. Theorems that hold in both geometries ask for proofs that are valid in both of them. This leads to a problem which the student should have met in abstracting axiomatics: a common fundament
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of euclidean and elliptic geometry. I have not yet studied this problem sufficiently to tell you how it can be solved on high school level: I suppose the best way would be to try something like the approach of the Helmholtz-Lie space problem.
(In this proposal the elliptic geometry could not possibly be replaced by the hyperbolic one. The model of the hyperbolic plane is artificial and hardly accessible to a synthetic approach, and without a model, that is, as Bolyai and Lobacevski did it, it is hardly recommendable.)
GROUPS IN GEOMETRY
Transformations in geometry were long ago advocated by F. Klein as a consequence of his so-called Erlanger Programm. The breakthrough of transformations in geometry is of a rather recent date. How to explain this delay, even in Germany, where Klein had been the venerated master of a generation of teachers?
It is a complicated story. Foundations of geometry in the 19th century originally meant the Helmholtz-Lie foundational approach of defining a geometry by group axioms. Klein was never a foundationalist; he was even a complete stranger to the axiomatic idea. The Erlanger Programm is classifying given geometries by their groups. The slogan "geometry is the invariance theory of a group" is only intelligible within the restricted frame of a few classical geometries which may be derived from projective geometry by distinguishing certain subgroups of the projective group. Not before E. Cartan's homogeneous spaces has the Erlanger Programm been interpreted in a foundationalist sense. As to the technique Klein's geometry has always been algebraic rather than synthetic. On the other hand Pasch and Hilbert turned away from the Helmholtz-Lie group theory interpretation of foundations of geometry. Without doubt Hilbert's Grundlagen der Geometrie have contributed to lengthen the life of Euclid's methods for half a century, whereas Klein's algebraic approach was too little geometry to be interpretable in school mathematics. At their best Hilbert's and Klein's approaches were teacher's background knowledge which hardly influenced school instruction.
In a 1956 report I stated that though transformations were progressing in school geometry, the exposition was classical and there was not any textbook based on the transformation idea. Even the problem how to introduce and to teach transformations had not received due attention. Though meanwhile many textbooks have been going to teach transformations, rather than being solved, the problem has been aggravated. Wrong concretizations to visualize transformations have become quite usual in experiments and textbooks. It is an old misapprehension in textbooks, and it is continuously made in
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Piaget's work, to visualize a transformation as picking up a figure and laying it down elsewhere without changing its shape or after having applied some similarity or affinity. With the free mobility of figures no group can be formed. To get a group one has to pick up the whole plane (or space) and put it down elsewhere. Free mobility of figures is much more intuitive than geometrical transformations. The less intuitive is very likely to be blocked by the more intuitive one, especially if moving models are used.
A more recent, and even more mistaken concretization of transformations, say of the plane, is putting pawns or persons on selected spots of the plane and interpreting a transformation as a command to these figures to change their places. Of course, all is due to end up in nonsense.
Good teaching of geometrical transformation is not easy, indeed. Maybe this has been one more reason why transformations progressed so slowly in school education. In the 1956 ICMI report I stated there is still some hope left. There is one transformation that is immediately seen as a transformation of the whole plane, rather than as the movement of a figure in the plane or as a march of selected pawns. It is axial symmetry. Central symmetries and rotations are much more difficult; translations is the most difficult case. If the teacher starts with axial symmetry and if rotations and translations are introduced as products of axial symmetries, there is a real chance that the child will grasp the concept of transformation even in the introductory phase. There is another approach to geometrical transformations. It is by means of the square lattice, which is the most natural, or even the only natural infinite figure. Free mobility of the square lattice can, indeed, suggest translations and rotations of the whole plane.
There are, however, more reasons, why to start with symmetries. They are more interesting than translations and rotations. To a young child congruent figures are the same. It will not hit upon the idea that something has happened if a figure has only been carried to another place. To an unsophisticated mind translating is no transformation. In this regard rotation is somewhat better than translation. If a cube is translated, nothing has happened; if it is turned and put on a corner, something has changed. But mirror reflection gives the strongest feeling of an important event. Symmetry as a transformation is more attractive, more abundant, and more problematic than translation and rotation. In almost all national ICMI reports of 1956 its usefulness in teaching was stressed. Meanwhile symmetry has vastly penetrated into textbooks as far as they teach geometry.
There is one Dutch course of modern geometry by transformations for 12 year olds, by Kuipers, Siepelinga, Troelstra and Tromp. They start with symmetries and introduce translations and rotations as products of symmetries; equiformities and affine transformations appear later. The books are
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written in a group theory spirit though groups are not explicitely mentioned. Deductive bonds are loose in the beginning to grow gradually stronger.

TEST PAPERS
Such new methods are more readily accepted by the students than by the teachers. Traditional geometry was difficult to teach, but its level of rigor was firmly ruled by a common opinion. Teachers complain that with deductively looser methods grading of testpapers becomes more problematic. "We cannot any more tell wrong from right", is a frequent complaint. They overlook that with old geometry it was easier only within a complex of mouldy and rusty conventions.
Old style geometry will not survive for more than a few years, but if teachers, indeed, choose the easy way to tell right from wrong when grading testpapers, geometry will be lost. Problems on sets and linear algebra are more reliable provided they are dull enough.
Geometry is endangered by dogmatic ideas on mathematical rigor. They express themselves in two different ways: absorbing geometry in a system of mathematics as linear algebra, or strangulating it by rigid axiomatics. So it is not one devil menacing geometry as I suggested in the title of my paper. There are two. The escape that is left, is the deep sea. It is a safe escape if you have learned swimming. In fact, that is the way geometry should be taught, just like swimming.
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199 PREFACE

This book is a product of love and respect. If that sounds rather odd I initially apologise, but let me explain why I use those words. The original manuscript was of course Freudenthal's, but his colleagues have carried the project through to its conclusion with love for the man, and his ideas, and with a respect developed over years of communal effort. Their invitation to me to write this Preface enables me to pay my respects to the great man, although I am probably incurring his wrath for writing a Preface for his book without his permission! I just hope he understands the feelings of all colleagues engaged in this particular project.

Hans Freudenthal died on October 13th, 1990 when this book project was well in hand. In fact he wrote to me in April 1988, saying "I am thinking about a new book. I have got the sub-title (China Lectures) though I still lack a title". I was astonished. He had retired in 1975, but of course he kept working. Then in 1985 we had been helping him celebrate his 80th birthday, and although I said in an Editorial 'State1'ment in Educational Studies in Mathematics (ESM) at the time "we look forward to him enjoying many more years of non-retirement" I did not expect to see another lengthy manuscript. Most people in their 80s do not aspire to write yet another book (particularly if they have already written nearly 200previous publications in mathematics education -- see the list at the end of this book). But of course Freudenthal was not like most people, and that is one reason why this book is so important. Although it wasn't his last piece of academic writing, it was his last major contribution to our field and it offers the reader a kind of synopsis of his perspectives. It is the definitive Freudenthal.

Not that there is anything final about the book nor about its messages. As Leen Streefland said in his tribute in ESM 21/6 "he himself already prepared the first milestone of the post-Freudenthal era", namely this book. In particular the book poses the kinds of questions and issues which Freudenthal himself enjoyed grappling with, and which in another way show the love and respect he had for children. He rarely spoke overdy about this, but shines through all his writing as a beacon to all of us in mathematics education, and in education generally. His writing focuses as always on the essential concerns of mathematics education, as he saw them, and children are our principal concern.

The book is sprinkled with anecdotes and stories, with references to colleagues in the pursuit ofunderstanding and wisdom, both adult and children, with respect to the predecessors on whose shoulders he stood, and with criticisms of those whose ideas he did not approve of. The language is "Freudenthal-English" and I do not say this in any unkind way. He enjoyed languages so much, and he was so aware of the dangers of words controlling his thoughts, that he tried to be creative

craftsman with his languages. He said in his letter of May 1989, when he sent me his draft manuscript "I did it by text processor -- a time-consuming activity because the easiness of correcting makes it difficult to stop correcting".

If you have read Freudenthal before, you will know something of what to expect -- the insights, the reflections, the charming and apposite examples, the scathing criticisms, the amusing asides, the wisdom -- they are all here. I f you haven't read any of his writings before, then I have only one piece of advice -- don't try to skim-read, or to read too quickly. You need to engage with his words, in order to relate to his ideas. I f you can manage to do that, I am sure that you, like everyone else who has engaged with them, will never be quite the same person again. That is, I believe, what he would have wanted.

Alan J. Bishop

APOLOGY AND EXPLICATION

This is neither preface nor introduction but rather an apology .and a .warning: -e present book adds nothing but itself to work that I have p.ubished m the past m various places. I have renounced originality, I have reframed from offering the reader any new experiences, aspects, or ideas he might feel entitled to expect -- reason to apologise. Resuming old ideas, my aim is now one of comprehensiveness, not by compiling but by selecting and streamlining, by includin- essentials and eliminating contradictions. Views, even when supported by evidence, _are susceptible to change. *My* present viewpoint is "here -nd now", th-t of rev1-w rather than of overview. I felt prompted by two expenences. Negauve: readmg dated quotations from my own work where I did not recognise myself. Positive: a visit to China, where in lectures, seminars, and discussions I tried to display and illustrate my view on mathematics education in all its aspects. The present ?-rk somehow reflects these talks, albeit in no way textually. Therefore the subtitle, meant as an homage to my listeners who, by their critical curiosity, induced me to improve on clarity and concreteness.

Hans Freudenthal

MATHEMATICS PHENOMENOLOGICALLY

1.1 WHAT IS MATHEMATICS?

I . I.1 Sure and certain

What is mathematics? A thorny question. Don't look it up in a dictionary! Whenever I did the answer was wrong.

"Mathematics' looks like a plural as it still is in French "Les Mathimatiques".

Indeed, long ago it meant a plural: four arts (liberal ones worth being pursued by free men). Mathematics was the quadrivium, the sum of arithmetic, geometry, astronomy, and music, held in higher esteem than the (more trivial) trivium: grammar, rhetoric, and dialectic. Only when this plural was no longer understood, was its final 's' usurped by sciences, like physics and economics, that historically did not deserve it.

On closer inspection it appears that I have answered the question "What was mathematics?" , to wit in Greek-Roman era, in the Middle Ages, and even somewhat thereafter.

As far as I am familiar with languages, Dutch is the only one in which the term for mathematics is neither derived from nor resembles the internationally sanctioned Mathematica. The Dutch term was virtually coined by Simon Slevin (1548-1620): Wiskunde, the science of what is certain. Wis en zeker, sure and certain, is that which does not yield to any doubt; and kunde means knowledge, science, theory. Indeed, since Stevin, and even before, mathematicians have behaved as though this were the proper defmition of mathematics; but since it seemed to be pretentious rather than a definition it could not find favour in the eyes of dictionary authors.

Even though it does seem to be pretentious it was a lucky strike of Stevin's to name a science (and quite a few others) after what he grasped to be its most characteristic property, rather than after its subject. The most characteristic property of mathematics was certainty, or so he believed. But how certain is "certain"? Isn' t common sense the first, the nearest certainty, and is there anything as remote from common sense as mathematics? Common sense takes things for granted, for good reasons or for bad ones. Mathematics asks for good reasons, as does any science, maybe even for better ones than any other. The need for certainty in science is not satisfied by taking things for granted; certainty has to be pursued, and in mathematics this is done by a quite peculiar mental activity. It is this mental activity rather than its subject matter that characterises mathematics as the field

where this a-tI':ity -an be exercised most adequately and most efficiently. We shall keep this m mmd m order to deal with it more specifically in due course:

that ts, mathematical method as a mental activity which, for some mysterious reason, creates certainty and does so, it would seem, beyond compare.

".1athemal1cs as the field of the most adequate and efficient use of the mathematical method? So there are fields outside mathematics where mathematical method does apply, maybe less adequately and efficiently. Or are these really outside mathematics? How far does mathematics as a field extend? Let us turn back to Stevin! He did not, in fact, use the word wiskunde but wisconst, or as we would spell it nowadays, wiskunst. Kunde means science while kunst means art. At our medical schools they teach geneeskunde, the science of curing, while the medical doctor's practice is geneeskunsl, the art of curing. As a name-giver for domains ?f ;-search Stevin pref-rred the noun kunst above kunde, art above science. "Art" m . Arts and Sciences sounds different from "art" in "Arts and Crafts". With which second noun did Stevin combine "art"? Slevin was an engineer but I am sure he understood it both ways, as many before and after him had 'done and would do.

Mathematics as an -- a mental art to be sure, which for most people will be closer to crafts than to sciences, a tool rather than an aim in itself, *more* relevant beca_use it works than because it is certain. But why does it work? Because it is certam? Although many people trust mathematics more than it deserves it works only w_hen it.is rightly applied. But what is right or wrong? Is there a-y way to ve-fy 1t and if so, 1sn 't such an verification again mathematics and -- if it is -- to which degree? Once one has admitted that mathematics is an art one cannot shirk the responsi?ility of judging_ whether, in particular cases, it is being properly used or rather ?:mg abused_; whlle trying to decide, one behaves once again like a mathemat1c1an. Let us illustrate this with a well-known historical example:

In spite of precursors, probability began with the discusion of two problems. The second was the probl1me des partis. Let me present it in a simplified version:

Two _persons A and B have agreed on a series of games (with equal chance of wmning for either); the first to earn five points will collect all the stakes. It so happens that the series must be broken off with the score standing at

A:4 B: 3.

How should th- stakes be divided? People could not agree. Some said: in the ratio 4_: 3; others said: (5 - 3): (5 - 4). Indeed, there is something to be said for each view. But what does mathematics say? Or rather, what did the mathematician Pascal say when he ":"as asked his opinion? Nowadays we would say: Let us auc-11-n the state of affars! How much would A' and B' pay in order to continue? If B evens the score the stake must be divided fifty-fifty but the chance that he would do so is in fact 50%. Therefore he can claim just one-fourth of the stake and so the fair ratio of sharing is 3 : 1.

Behold -e -athe-atidan called to decide upon the right or wrong way to apply

t-J--c; :-:.J t' C.I-- fl.Tl -.'~.::nr-:.:-.! 1 -...,i:,- .. , A 11. "7 "::: -: .:-: _co 1-.1<?;? ·:::;------ · .. ;- ·:'..ff-':'- ;-_t-::s' ·•1- ..,t "-.,._, ,.-"--- --

was again one of arithmetic, that is of mathematics; and a mathematician was sorely needed to reveal this perspective.

As a useful tool, mathematics has conquered a rich variety of fast expanding areas of science and society, and as a tool it has proved indispensable for a rapidly growing host of people, who use mathematics because they cannot do without it.

Yet again: how sure and certain is this mathematics?

Voyager 2 arrived one second late at its meeting with Neptune -- after having travelled for 12 years, it is true. So sure and certain is mathematics. Or is it? Astronomers count by millions of years, and on the scale of millions of years the one second would be one day -- at least if one admits proportionality. But sure and certain? When they predicted the moment of closest approach, they added some '±' to account for the "mean error". Mathematics knows precise numbers only -

- I once heard a purist affirm. And what about error calculus? -- I asked him. That is physics, geodesy, astronomy -- he retorted; contrived, indeed, by mathematicians involved with mathematics "outside" mathematics. Judging how sure and certain some applied mathematics may be is again mathematics -- formal mathematics in error calculus; as such it is sure and certain as long as it is rightly used, that is, with insight rather than as a recipe, which unfortunately happens more often than not in its educational version. Nowadays, streamlined error theory no longer suffices for judging how sure and certain mathematics is "outside" mathematics. How about the numerous "mathematical models"? Mathematics is sure and certain to the extent that one faces the question how sure and certain it is.

Yet let us not forget mathematics as an aim in itself, which allured adepts as early as Babylonian Antiquity, and which historically has proved indispensable as a forceful motor for its own long-term development. It is an important aspect, although ofless concern to us here, since our subject of mathematics education embraces a much larger group than only future professionals of whom once again only a small minority choose mathematics as an aim in itself.

Even if the "inside" certainty of mathematics is taken for granted, there still remains something to be said about its "outside" certainty. The answer to this is predesigned: it depends on the mathematical behaviour of the one who ventures out -- a point for further discussion. Anyway, Simon Stevin may have had good reasons to characterise mathematics as what is sure and certain, and his predecessors and successors had to behave accordingly, even if they failed to reflect on it.

But what about the great majority? Are we too bold in assuming that, even to the

man in the street certainty is mathematics' most significant feature, far beyond
the limits of his ability to handle it? If this assumption is correct, one may ask
how this confidence in the certainty of mathematics has come about. The answer
is determined by the extent to which and the way in which people have become
acquainted with mathematics.
Most people have been taught mathematics as a set of rules of processing or, as
we call it in mathematics, of algorithms -- an agreeable experience where they
have learned to master them, and a disagreeable one if they have failed. One reason why teachers teach it this way is tradition: it is the way they learned it them-4 CHAPTER I
selves, while they have forgotten that it was not the way they really understood
mathematics if ever they did. The other reason is the structure of mathematics
which, unlike any other art or science, can be mounted in rules and algorithms. It
is an astonishing fact that these rules never fail or -- if they do fail -- that it is the
user who feels guilty, looks for errors of his own and tries to repair them. But to
whom is this fact astonishing? Or is it simply an unfounded belief -- subjectively
unfounded? Let me close this subsection by telling a little story:
A 75 year-old woman whose help is regularly invoked by children and students
s-uck in their ho-ewo-k, met -- perhaps for the first time in her life -- a profesmna- mathema!c-an, JUSI when she had taught a college student the (Euclidean)
algorithm for finding the greatest common divisor of two whole numbers. She
asked the mathematician why this works or rather, whether there were any reason
'.^hy. The ma1'e-atician helped her to answer the question on her own. Maybe it
is worth mentl!oming how he did this. He gave a mere hint: I f a and bare two numbers and a >b (in fact he chose a pair of such numbers), compare the greatest common divisor of a and b with that of a and a-bf
The next time she asked him a question it was about an old-fashioned algorithm
for finding the square root, in which she had believed all her life. Now the mathematician refused and instead taught her a more efficient algorithm, which she
accepted comprehendingly.
-till another time she asked him why under addition, subtraction, and multiplicallon of two numbers the sum of digits behaved in correspondence with the original numbers -- a phenomenon that was easily explained.
Never did she ask the mathematician why, performing arithmetic, the inverse operation "proves the sum", why the algorithms of the various arithmetical operations ".ork, why operations are commutative, why prime factorisation is unique.
Why did she doubt some rules (or did she?) and accepted others? Later on we will
try to answer this question.
I.I.2 Mathematics as common sense
Earlier on I aske- whether common sense isn't the primordial certainty, the most
abundant and reli-ble source of certainty. In a sense it is, although from olden
limes onwards philosophers and explorers have sounded warning notes against
that -ery common sense, which is easily deceived by cunning Nature.
Iron is colder than wood -- this is common sense, at least as long as one uses the
common sense thermometer of one's fingers. Unfortunately, sometimes this
statement ceases to be true; for instance, when exposed to the blazing sun, iron
feels hotter than wood. "Conduction of heat" helps us out of this dilemma. This
is "uncommon" sense, yet common to physicists and to people who learned and
didn't forget their physics lessons. It is the weakness of school lessons to be forgotten, and physics is not exempt from this rule. Whenever this haooens, ii is
- - - ' - r ---. -
MATHEMATICS PHENOMENOLOGICALLY 5
worthwhile. In due course we shall reconsider and discuss the poor permeability
of the membrane separating classroom and school experience from life experience and we will ask for measures on how to improve it.
My example confronted the physicist's "uncommon" sense with the "commoner's" common sense. But is it really that simple? Whether something is common
sense is witnessed by how it is verbalised in common language. According to
common sense the sun rises and sets. On authority it has been accepted that this
statement is amended, if necessary, by the adverb "apparently". Moreover, the arguments substantiating why reality is different are accepted on authority, as is apparent from the almost verbal way in which they are repeated. I wonder how
many people would be able to tell which phenomena and arguments in this particular case moved astronomers to distinguish between appearance and reality.
But many distinctions between appearances and realities have likewise moved to
the state of common sense and are verbalised in common, even daily language.
However, sunrise and sunset occur in this context of appearance and reality only
as far as astronomy is concerned, and even astronomers aren't astronomers all the
time.
According to common sense the Earth used to be flat. This common sense statement differs essentially from the preceding one beCause in (what mathematicians
call) a first approximation the Earth is indeed locally flat and allows us even now
to behave as though it were so. However, not only as astronomers are we engaged
by its roundness, but also as travellers, and as early as we are shown globes and
maps. Nevertheless, even adults sometimes wonder why the sun is the hottest at
noon rather than in the morning and evening when it is "closer to earth". Yet this
common sense conflict is nothing but a low level example of a behaviour which
is not unusual even on higher cognitive levels; in fact, it is an example, and at that
a striking one, of applying simplistic rather than readily available sophisticated
models.
If things have not changed in the mean time, it is the didactical principle in phys-
ics and chemistry to fight common sense by exorcising it: common sense ideas
obstruct scientific ones. Researchers wonder why people -- even those who had
been taught physics at school -- still adhere to common sense ideas, which should
long ago have been expelled. Take the following example:
Again and again subjects are given tests of the following kind: A cyclist. two opposite
arrows on the ground, and the question of whether the forward force is stronger than
the backward one (that of friction). Or the parabolic path of a ball tossed up, and the
question of at which point it experiences the maximum force. Researchers search
again and again for new explanations of why most people fail these tests. They forget
that more often than not force- is associated with velocity (rather than with acceleration). Descartes and Leibniz mathematised force as a function of speed (though they
disagreed about the kind of function) and even in the past century conservation of
force meant conservation of energy. Even today (and even in technical language) the
Newtonian definition of force has not at all displaced other uses. Although the physicist on the meny-go-rolllld feels the centrifugal force, the physicist outside calls it apparent. Would it not be wise to recognise this fact in instruction and to deal compara-6 CHAPTER 1
lively with various concepts of force and appreciate them according to their merits?
In. general I believe that in instruction it would be more recommendable to start
with common s::nse -d-s rather than to reject them as outdated and better being
suppressed. Tius belief is supported in any case by the fact of the more or less
spontaneous development of mathematics.
Among mathematics rooted in common sense the most striking example is, of
course, whole number. (-imilarity is another, though superficially less striking,
-xample.) <'.hlldren acquire number in the stream of their physical and mental acllvihes, which U:a_k-s it difficult for researchers to fmd out how this happens in
detail. The acqu1511tion of number is strongly supported (if not made possible) by
the corre-ponding numerals in spoken language, the acquisition of which can be
mor- easily traced through observational research. As the child acquires the syntacnal structural means to build new sentences so it occurs with the morphological structural mean- to build spoken numerals, beyond necessity and need, and
far beyond -e physical 3.'d (at least initial) mental grasp of number. Going on
-d on m this. se-uence is the first verbalised expression of the mathematical
mmd, an astomshm.g feature which can hardly be overestimated, though it has not
drawn much atten-on'. due 10 being simply a matter of fact, and experienced as
con:mon sense, which is not to be questioned. The morphology of spoken numerals is, moreover, the first algorithm of mathematical character the child acquires:
a common sense algorithm acquired by means of common language.
My stat-ent that researchers have paid little attention to children's first and spontaneous arithmetical activities would seem to be contradicted by the title of p- ag t'
work (wTh S . k) 'La ,, ' e s
. I ze- a . ge:-se d- nombre chez l'enfanl" (1941). The contradicllon, how-ver, hes only m the btle, since the work itself is not at all concerned with
the g-ne-is of -umber. In fact, the genesis of mind had been thoroughly studied by Pi-
aget - his earh-r work. This genetic approach should be sharply distinguished from
the_ ep-temolog1cal one h.e chose in his later work. In order to make clear what this
-witch mvolved, let me bnefly sketch the history of the number concept and its teachmg.
In contrast to his geometry, Euclid's arithmetic is rooted in the whole number of common -ense and, up the middle of the 19th century, the scientific approach to number
remamed common sense. As soon as infmities are to be dealt with arithmetically this
-mmon sense _approach becomes blemished by paradox. Cantor removed this b1em-
15h by abandorung the then common sense approach in favour of a more sophisticated
on-, later on extended by Frege-Russell in an even more refined way from infinities to
fimte natural number and its foundation.
-Piaget'- epistemology, epistemo means a state of knowledge as advanced as accessible to himself, which in the case of number happened to be the Frege-Russell approach, at le-- such as he interpreted it (and at a later stage, for mathematics in general, Bo-baki 'i system-. This_highl- sophisticated number concept is
Piaget's touchstone, his tool for asse-sm- children s understanding of number at various ages, in far
from common sense s1!uallons, and using an artificial language both created for the
-ake of assessment Obviously, this - -o way to trace the genesis' of number. It proved
instead to be the source of such oddities as non-conservation never observed in the
i.mfnrtrunfitely -nre -ex.:unples :f genetic lo\'..Se.1:'.T--. It is a pity th-t Pil.1,1l)'.e-s 1f".".'.,,.Dl1Ment?_!
MATHEMATICS PHENOMENOLOGICALLY
genesiology to epistemology has ever been noticed.
Later on I will return to this question.
7
Let me add the remark that the common sense roots of number extend further (or
are extended) alongside learning language, which proceeds from speaking to
reading and writing. Isn't it noteworthy that in written and printed language the
mathematical symbols for natural numbers precede their phonetically constructed alphabetical equivalents of spoken numerals ("3" precedes "three", "100" precedes "one hundred")? Anyway the mathematical symbols for nahlfal numbers
are integrated into written language. Literacy and numeracy overlap in this region, and even beyond. The place value principle, though deviating from the seriation principle prevailing in common written language, participates in the integration as do -- to a higher or lower degree in various languages -- the basic operations
on natural numbers.
I have adduced these striking examples of the overlapping and integration of
mathematical and common language as witnesses of the common sense origin of
early arithmetic. The tree of knowledge, rooted in common sense, has, as it were,
sprouted arithmetic as a branch in its own right -- historically as well as in individual life histories. In the course of life, common sense generates common habits, in particular, where arithmetic is concerned, algorithms and patterns of actions and thoughts, initially supported by paradigms, which in the long run are
superseded by abstractions. These products of common sense acquire in turn the
behavioural status of common sense, while their common sense ancestry may
have even been forgotten. Historically viewed, there have been stages, or even
levels of common sense, and the same holds for individual development what is
common sense may depend on the community that shares it.
Is common sense then something like the greatest common di visor of insight,
shared by the members of a certain community? Indeed, but it is then a quite large
common divisor with relatively small mutual divergences, at least as long as
emotion is checked by cognition. In the case of the probleme des partis the parties
quarrelled about how to divide the stakes until a judge resolved the quarrel, albeit
not on the strength of common law but by common sense. This was a sounder
common sense than that of the quarrelling parties, which was spoiled by badly
understood law-like algorithms. The judge's decision created a new paradigm of
common sense.
Whereas historians are able to trace the cognitive development of mankind as a
development of common sense, it is difficult to attempt the same with individuals, whether by observation or reflection; wrong application of rules, however,
and wrong transfer of patterns may provide indications. At any developmental
stage of common sense it may be significant how much the learner has contributed to this progress. One extreme is learning without being intentionally taught,
and the other is learning what has been bluntly imposed; and since the first is
more deeply rooted in previous common sense, it may matter some time in the
future how the development took place in the past It matters whether, for instance, an arithmetic algorithm was acquired as an abridged and streamlined ver-8 CHAPTER I
sion of former common sense activities, or whether abridging and streamlining
(or even the algorithm itself) were imposed. Sure, some, algorithmically gifted
people, learn to apply even imposed algorithms adequately; others -- perhaps the
majority -- fail 10 identify the new algorithmic procedures with the common-sensical ones from which they should have originated through abridging and streamlining. They fail because some time in the past they were asked to take mental
leaps which exceeded their mental powers. Even though they flawlessly learned
the algorithm, they will fail to use it in true life situations where common sense
counts; they will instead depend on less efficient lower level operations.
The following is a well-known example: We are taking a trip of 215 km; how
much do we still have to drive after 88 km?
Children reason: 88 + 12 = 100, +100 = 200, +15= 215. The subtraction is not
recognised (or not ventured); to such pupils the minus key on the calculator is of
no use.
Researchers have signalled this "relapse" and marvelled on it. Rarely, however,
has it been diagnosed as a consequence of instruction, since no alternative instruction was envisaged alongside the imposition of the new algorithm (which
was sometimes embroidered with explanations for conscience's sake). The new
algorithm, however, never did have the opportunity to reach the state of common
sense; afraid of applying a wrong algorithm or the right algorithm wrongly, the
learner instead relies on what has remained common sense to his mind.
Let us now answer the question posed at the end of the story of the 75 year-old
algorithmic performer! The fact that after more than half a century she still masters the various algorithms well enough to teach them, characterises her as an algorithmically gifted person. Why did she worry about one kind of rule but take
others for granted? Well, common sense is taken for granted. Rules learned by
imposition will be taught the same way, at least if the learner who became a
teacher is able to teach. But having been imposed they never had a real chance to
develop into common sense ofa higherorder. Is it a privilege of old and wise people to doubt what looks like common sense, or can you not teach this behaviour
to the young?
Why "sure and certain", I asked earlier on, and I announced that this might be a
question of behaviour. Of common sense? Yes, but also--I would add now -- of
a tendency to doubt common sense.
Where common sense as the root of mathematics is concerned, up until now I
kept silent about geometry (except for one clause in parentheses). Yet I am convinced that geometry even precedes arithmetic in individual development. One
of the earliest symptoms -- I believe -- is awareness of similarity of figures; this
is found so early that it looks like innate (or might it even be so?)
"Similarity" is even characteristic for what happens with geometry by the token
that every day language has no word for it. (This was already the case in Greek,
and Greek mathematicians had to think up a word for it, which they did by reMATHEMATICS PHENOMENOLOGICALLY 9
wards mathematics: Until the higher grades a term for the striking feature of similarity is withheld from the pupils -- obviously because on- does not dare- speak
about geometric similarity until one feels able to define it formally. THiS parsimony with regard to geometrical language, which is more difficult, indeed, heavily contrasts with the way in arithmetical instruction, where the number sequence
is handed to the learner, as it were, on a plate.
Geometry is doomed to fall behind arithmetic in the development of verbal expression. Geometrical arguing is kept on -e level of"I can- it", and when an
attempt is made to raise it to a higher one, it may be loo late m most cases. Common sense does not get the opportunity to develop into more common sense -- a
consequence of lacking instruction.
1.1.3 Why mathematics is different
Let us be clear about one thing: to the view and mind of most people, mathematics, though deeply rooted in common sense, is more remote from. it than anyth;ng
else. What made it move so far away? The deeper the root the higher the top
Mathematics is different, indeed, and we will take notice of this fact when we will
turn to education. From the very beginning it has been different It is the oldest
among the sciences, even preceding astronomy by more than two millennia.
Mathematics was more easily invented, as it was simply a question of common
sense -- only better organised. And it developed in this way, becoming more and
more organised, according to a pattern I am going to describe.
You know that 3•2=5 and the area of a rectangle by common sense. But as soon
as Nature gets involved common sense becomes misleading 3-': everybod- knows.
For common sense is even more urgently in need of expressmn and ennchment
by transfer of knowledge than is mathematics -- and ever more urgently the -ar
ther removed a science from mathematics; and even where common sense is a
nuisance it may persist alongside more educated views. Unlike the law of inertia
or Newton's theory of gravity, the elements of mathematics have been invented
independently of one another at various places in the world. While sciences underwent revolutions mathematics evolved, even during these revolutions and under their influence. How to explain this difference, what dynamic granted mathematics this apparent continuity of development?
Common sense in order to become genuine mathematics and in order to
progress, had to
0
be systematised and organised. Common sense experiences, as
it were coalescez into rules (such as the commutativity of addition), and these
rules a-ain became common sense, say of a higher order, as a basis of even h.igher

order mathematics -- a tremendous hierarchy, built thanks to a remarkable mterplay of forces. . .
I have stressed earlier and I stress again that our first verbalised mathematics --
that of number -- distinguishes itself from any other verbalised knowledge by the
fact that its mere form, detached from its contents, foreshadows [ifnot radiates) 10 CHAPTER I
its mature form and contents. Its tremendous structure -- strictly regular after a
short archaic start -stands unrivalled by any ol.her linguistic phenomenon; and in
I.his respect I.he phenomenon is interlinguistic as far as languages have been created in need of number. By I.he same token, number is even more common sense
I.han any ol.her human idea. Moreover, I.his fact itself is common sense to such a
degree I.hat it is hardly noticed and rarely made explicit, all.hough it has greatly
influenced mathematical thought as well as I.hinking about mathematics. The formal isomorphism between the mental object "number" and its numeral expression, which looks like substantial identity, has been I.he source of thinking about
the relationship between mathematics and language, of identifying mathematical
language with the language of mathematics, and of viewing mathematics as a
mere language.
Properly said, it has not been as simple as that: what is considered to be true
mathematics started by impoverishing that primordial structure of whole number,
depriving it of the positional strait-jacket (the primordial fall as it were) by which
mathematics escaped and outgrew common language. It was New Math's historical fault not to notice the depth of this fall, which eventually became a jump,
from the positionally structured numeral to the more abstract free whole number.
Number is the first realm where symbolism acquires a reality of its own, a reality
apparently independent of its creator, who in turn tries to reorganise his creation,
and by means of it his environment. But even this organisation would seem to
have existed for eternity, at least as long as the view is restricted to whole
number, which Kronecker believed to have been created by God, only to be
spoiled by human efforts to transcend it.
Structuring, whether applied to products or to processes, means emphasising
form. The first non-trivial structure as such, i.e. whole number as I.he product of
the process of counting, begot rich process and product content which, organised
by ever new structures, in tum begot new contents -- a never ending cyclic process.
Through reflecting on his own activity man discovers paradigms, which are abstracted into patterns of mental action, and made conscious as schemes by which
thought is organised on behalf of new progress -- adaptable schemes, that is,
which allow for varieties of use, as well as, in the same right, rigid single-purpose
schemes which, thanks to their rigidity, can lead a life of their own, called algorithms. These forms in tum can become subject matter, a nucleus, as it were, of
higher order contents -- again a repeating process, an interplay of form and content, which characterises mathematical thought. This interplay includes the linguistic expression by means of which mathematics is communicated as a product
and a process. New content and form require new terminology which, in order to
be efficient, has to be streamlined, both symbolically and notationally. Symbols
and notations, in the course of reflection, again become subject matter handled as
such.

Historically, discoverers incessantly switched duties. fa fact, each discoverer behaves as an organiser as soon as he sets out to _make his discov_enes known, although this is not a specific feature of n_iathemat1cs. In mathemaucs, h.owever, o'.·
ganising and reorganising is a consunumg affair, and the newly acquired organisation forms may become content in the sense of subject matter to .be exammed
as such. As far as longevity is concerned, no system of mathematics can equal
Euclid's, which came near to suffocating its subject Yet only a bnefbreathing
space was allowed to Bourbaki's: it became outmoded almost at the same pace
as it was developed; outmoded as a system, that is, while giving birth to new content. .
This is why mathematics is different and why it looks even more different to
more people. Schemes of thought can be imposed, algorithms can be taught as
rigidly as computers are programmed, and, to be sure, such -fro?' ."e not l?-t on
algorithmically gifted people; concepts can be taught by lmgmsuc defm1uons,
and this again works very well with people who are good verbalisers themselves.
Last but not least, the interplay between form and content includes what is commonly called applied mathematics, provided the environment to which mal.hematics is applied is already being seen through the spectacles of mathemau-s.
"Mathematical model" is the fashionable term for form, abstracted from parad1gmatical application, and again models and their use become the subject matter of
mathematical reflection, as though they were part of the environment. Modelling
too can be taught by impoiition -- successfully or to no avail -- depending on the
preparedness of the learner.
All in all, this explains why mathematics is different, both objectlvely and subjectively, and it also explains why people do not grasp the reason why it is different. Content must be assimilated, while form can be imitated for the purpose of
reproduction. It seems to be easier to teach and learn structured form than structureless content. It is as easy to yield to this temptation as it is difficult to resist
it This, then, is I.he familiar image of mathematics: a set of algorl.hms, as worthless as it is strict if one does not understand how and why it works. It is a misleading view, which is given the lie as soon as less strict schemes are to be applied.
We will recall the reason why mathematics is different when we discuss education. There we will explain why mathematics is learned differently and therefore
should be taught differently, that is, neither as form nor as content but while
maintaining respect for lhe interplay between them, acted out in the teaching/
learning process! Learning is progress in knowledge and ability. Their interplay
does not sound like that of instruments and voices in a concerto. It is rather a
change of viewpoint from content to form, and conversely, leading to ever higher
levels, by jumps as high as the learner can perform, and guided but not lifted by
the teacher. 12 CHAPTER I
1.1.3.I Examples
Let us dwell upon the interplay between form and content! As stressed above the
origin of the number sequence is form, even linguistic form. While used for
counting it acquires content - - a rich variety of content. When in turn abstracted
from this variety of counted phenomena, it acquires the status of lhe mental object_ th_at is: more or less formal whole number, though still laced in the formal
str.ut-J-cket of _lhe d-irnal system, and still attached by short strings to counting
something, w-ch a:ms at content. Addition and subtraction are meaningful as
c:nntent operauons m preparation for their formal practice, which for a certain
ll-e r-mams supported by models which arose as content. Commutativity of add1uo- is s--ested by what addition meant as content before it acquired the -- at
least1mphc1t-- state of a formal rule; it can then, on a higher level, become a matter of mathematical content, when studied, for instance, with its consequences, in
the context of operauve laws. The relation between addition and subtraction arises as a matter of content before it is formally applied, in order to become once
again subject matter and content in the context of algebraic structures.
Let us skip multiplication, whi-h shows similar features, and turn to divisibility.
This -ses m -he c_ontext. of distri-uting, which is content, but more rapidly than
any!hmg else m anth?euc 1t acqurres a most surprising formal status, only to recover that of content m number theory. There it becomes a matter of arithmetic
scrutiny, in particular s-mulated by the phenomenon of prime number, which
means, as far as content 1s concerned, number resisting any arrangement in rectangular patterns. The search for prime numbers and research on prime numbers
is largely one of mathematical content Yet the convention not to admit the
number 1 as a pri_me number, is formally motivated, namely by the preference for
an -3;5Y formulal:IOn of the theorem on unique prime factorisation. Extending divmb1hty to other domams. (such as polynomial rings) is a formal procedure,
where the part played by 1 1s taken over by lhe inversible elements.
"I'e swn of digits (in decimal number notation) provides a formal criterion for
d1v'.s1b1hty by 9, although its proof, at least at a low level, requires content reasonmg, for _mstance, by displacing balls on the bows of the abacus. The clock dial
s_uggests anthm-llc modnlo 12. Operating modulo a whole number is a formal actlVity, from :-'h1ch such mathematical objects arise as finite rings and fields,
:h1ch m th-1r tum become ?,1athen:iatical contents. Manipulating formally .f1.,
as th_ough its square were 2 , acqmres mathematical content in the polynomial
domam modulo !·- 2.
Positive n-mber owes its existence to the geometric context of distances. Positive
and ne?auve --umbers (together with 0) form what is called real number, beyond
which is 1magmary or complex number. In brief, historically (for more of it, see
[146], Chap. XV) as early as Babylonian Antiquity there were methods -- one
might even say, formulas - -to solve linear and quadratic equations arising in geOfffff·-_T'f Qr U1 irl!Y".1'-0.<:' W•" ,.. -01.... ·,_.., • • 1 . .
. -·-· ·_ ---- ----- ... ,_.. ..:, ·:·- ._ .an7.,.i--f-17.:!1.11c-,a-·-v1e'c:..--:-:-:-c•J1, ·"_:.i·.-z-cr:'f,.I-:c?<:

them had to be rejected as being "false", and then merely because one only knew
positive numbers? Eventually, around 1500, courageous mathematicians took the
plunge. In spite of their being "false", one conld marvellously reckon with these
numbers by preserving and extending lhe old arithmetical laws. And lo and behold, one century later, the "negative" numbers were assigned a -- geometrical -
- place, left of O on the number arrav pointing at the right Formal number acquired a content, thanks to which the whole plane could be co-ordinatised. The
formal requirement that all algebraic equations can be solved brought forth a content-rich geometric-algebraic activity on the entire plane. We hardly need to add
that soon this content again turned formal -- a new link in the long chain of form
and content. Thanks to Gauss' geometric representation complex numbers
(which had come into being as formally required solutions of algebraic equations
and had long been repudiated as ontological monsters) acquired respectability
and content status.
Common formal properties of certain algebraic structures led to define lhe field
concept through formal requirements, with the obvious preconception of adding
a new object -- the field -- to mathematical contents. Efforts to lend content to
formal divisions by O, and to formal (divergent) series such as 1-1+1-l+ . . . are
other examples of the dramatic historical interplay between form and content In
ancient history formal patterns for solving quadratic equations geometrically by
the so-called application of areas led to the discovery of curves, afterwards recognised as conic sections and redefined as such by Apollonios, only to be redefined anew by means of formal equations in Cartesian geometry.
The term "variable", as it is used today, embodies the symbiosis of form and content in the most striking way. Originally letters, whether indicating arbitrary
points as in Greek geometry or arbitrary numbers as in Vieta's algebra, were
nothing but easy generic names for these kinds of objects; yet, in the kinetic imagination of the inventors of Calcnlus and their followers, as well as in analytical
mechanics, letters became symbols for mathematical and physical objects which
varied time-dependently or due to mutual dependence. Initially the law ofthis dependence was not made explicit but implicitly given by the surrounding mathematical or physical context, which could be a curve or a mechanical event, involving these mathematical or mechanical objects. When the first literal function
symbols appeared, that is, for solving lhe differential equation of the vibrating
taut string, they were again no more than generic names of mathematical objects;
as soon as necessity required it, they then became variable objects themselves. In
the course of history as well as in mathematics acted out day by day, we may notice this incessant switch back and forth between letters as symbols for "empty
places" (indeterminates or unknowns) and as symbols for variable mathematical
objects.
As flout of necessity our exposition led us to the role oflanguage in the interplay
of content and form. Linguistic form should be well distinguished from abstract
(mental) form. Language serves to express both (abstract) form and content. Historically, addition, subtraction, and equality were formal mental objects, present-14 CHAPTER I
ed in clumsy colloquial language, long before shorthand use of the plus, minus,
and equality symbols appeared (which later even crossed the frontiers of mathematics and penetrated into colloquial language); and this is the way it still happens nowadays in individual histories. On the other hand, reasoning about such
symbols, for instance about meaning and use of the equality sign, involves dealing with them as subject matter, that is, lending them the slatus of content. The
same holds lrue for brackets as tools for slructuring formal expressions and formulas. The need to linguistically express x
2
as a function of x requires the creation of linguistic tools, which resemble content It may be a matter of content
whether in a group structure the group operation itself is called addition or multiplication; but this very choice formally decides whether the neutral element is
then called 0 or l . Interpreting (in highbrow style) a group as a quadruple, consisting of (I) a setG, (2) a mapping ofG x G on G (multiplication), (3) a (neutral)
element of G, (4) a mapping of G on G (inverse), and thereby fulfilling certain
(group axiom) conditions, means that mathematical language is being dealt with
as if it were content.
1.1.4 Mathematics as an activity
When we distinguished slages or levels of common sense we noticed that attaining them depends on the individual's own contribution, on his activity. Common
sense lakes things for granted, as I mentioned, while mathematics asks for good
reasons. Certainty has lo be pursued, and the way this is done, characterises mathematics as an activity, leading to ever improved versions of common sense.
Most of the Dutch names Simon Stevin proposed for the various sciences involve
activities rather than objects and instruments. Mathematics is a quite peculiar activity. Thinking logically? It depends on what "logic" is intended to mean. Logic
as an established science or as common sense? Let us postpone tackling this question!
Mathematics as an activity is a point of view quite distinct from mathematics as
printed in books and imprinted in minds. To be sure, mathematics is a precious
treasure-chest of tools, precious that is for those who can put them to good use.
This is true for any instrument, many of which can be handled by an inexperienced child: a switch to light a room, a key to open a door, a bell to ring; and
many more, which require more sophisticated, yet narrowly streamlined programs in order to be handled, such as algorithms in mathematics.
Every researcher, every producer of mathematics will readily admit that mathematics is an activity -- his private activity, the product of which may or may not
be published. Indeed, any author is entitled to have his privacy respected. Moreover, why should he annoy the public with the !ale of the production process as
it took place? Indeed, the author should not lead the reader of his work along all
the wrong trails and into the blind •lleys explored by him and eventually aban-
. ·.

lowed to watch the process leading to the result as it would have laken place if
the author had somehow suspected all along what he finally came to know for
sure? In the past some textbooks and even research papers were -tten in this
way, but the fraction of mathematicians who prefer this style of editmg has been
declining steadily.
Let us avoid any source of misunderstanding! If I speak about products of mathematical activity, I do not do it in the narrow sense of new propos11:1ons and th-
orems. I include proofs, even definitions and notations, as well as the layou- m
print and thought. Reading mathematics and listening to it is also mathei:iauca,
the mathematical activity of reproducing what is being offered, as though it were
the receiver's own production. Reproducing should be an easier laSk than producing. Unfortunately, the ever more common style of presenla-on frustrates ra-er
than favours this opportunity. The style of modern presenlal:lon creates the lli-
sion that what is sure and cerlain, has been so from eternity rather than smce its
discovery was completed and its presentation polished. Yet our cerlainties are incessantly menaced, if not by confulation -- which has become r- -- th-n by a
longing for more certainty, deeper insight and broader -nderstanding which .has
become quite common. In this respect at any rate there is no reasonable cerlamty
about what will be considered sure and certain in-the future.
Mathematicians, even when reproducers rather than producers, tend to restrict the
right to practic<>mathematics as a menial activity to an elite -- to those who ennch
mathematics as a subject matter by new discoveries. Those who do not belong to
this elite are required to purchase a stock of knowledge and abilities fro'. ' -e
shelves of the mathematics supermarket. If some choice is indeed granted, is this
at least left to the consumer, or does somebody else even do the shopping for
him?
Does this profusion of metaphoric language bother you? Mathematicians -e or-
ganisers albeit in other fields and with other aims than those who are active m
industry: trade, traffic and administration. A similar, if not identic- .men?11
make-up is characteristic of all of them. But in no other field does organising display itself in such purity, impose itself with such force and infil-te so profou?d-
1 y as it does in mathematics. Mathematics grows, as it were, by its self-orgamsmg
momentum.
No wonder mathematicians like to present their subject in a well-organised slate,
and textbooks are frequently judged according to their degree of organisation.
But for the great majority neither mathematics nor its organisatio- is an aim in
itself. As a useful tool mathematics has conquered and covered a rich - variety of
fast expanding fields of science and society, and as a tool it has become ndispensable to a quickly growing crowd of people. People increasingly use mathematics
more often than they are aware of. They use mathematics because they cannot do
without it.
That is mathematics as an activity of discovering and organising in an interplay
of content and form; later on we will have to ask the question of whether mathematics teaching has really caught up with this development. 16 CHAPTER 1
1.1.5 Mathematics and reality
If I were to continue in the same way, that is, by focusing on the mathematical
process, I would imprudently be neglecting the medium in which this process
takes place, which provokes this process and which is affected by it. Let us have
a closer look at it!
Common sense reveals itself in those actions -- physical and mental -- which are
common to people who share common realities. There is no reason, were it even
possible, to restrict "reality' to the mere experience of sensual impressions. Even
at the lowest level, reality is an inextricable mix-up of interpretation and of what
a purist would call sensual experience. If somebody says
when I heard the doorbell ring, I looked out the window and saw Alice and smelled
her perfume

one may believe him and take for granted that this is the report on something that,
in the reporter's eye, really happened. There is no reason why one should listen
to vulgar positivists who object that "in reality" the ear of the person calling himself " I " was struck by noise ofa certain pitch, interpreted as that of a doorbell, his
eye by a flood of light waves, interpreted as issued by some source called Alice,
and his nostrils were invaded by a stream of molecules, interpreted as Alice's
scent In fact, the " I " could have been a dog, and the above sentence could have
been part ofa story on the animal's behaviour, told in the first person's style. The
doorbell's ring, and Alice's look and scent are primitives, hardly suitable for further analysis. Questions like "how do you know that it was the doorbell's ring and
Alice's look and scent?" may or may not elicit reflection. "It is windy" may be a
primitive indoor observation, which only afterwards is justified by references IO
moving branches and clouds. "A bang" and "a short-circuit" may be descriptions
of the same event on various cognitive and linguistic levels. A number of atomic
events read by a Geiger counter, a passing K-meson, and a supernova burst are
primitives to the physicist or astronomer who observed and reported these events,
although to be reduced to more pedestrian ones, they need profound theories.
"Cheerful" and "sad" may be irreducible predications (whether pronounced with
regard to other people or oneselfi), and reducible only by reflection. To some
readers, Homer's characters may be real persons, and their adventures real facts,
as real as the world on the stage may be to the spectator. There can be little doubt
that, as early as the lowest cognitive level, the species canis is as real as any particular dog, with whom it even shares a common name. I once told the story of a
little boy to whom at an early stage "Utrecht" meant any place he spotted which
resembled the neighbourhood of his home; a few years later this became a vague
territory, later on he was able to localise it on a map, and after a few more years
it became an administrative-political entity, which finally acquired a new, and to
him even more vigorous dimension -- that of history. This all happened within an
ever expanding reality, which included such institutions as The University, The -- ;, - ,' ,-c - cMATHEMATICS PHENOMENOLOGICALLY 17
alities let us ask what a micro-computer means to various people: -commodity
with a certain price to the seller, a weight in i-ilogran;is '.'.J -e deliverer'. unexplored land to the novice, and to the exi:ert a part of his bfe s world, which has
long since been abstracted from the tangible mstrument. . .
How real concepts are depends on the conceiver, and under given crrcums_tances
cognitive grasps can be more vigorous than man- and --sual ones, which are
in fact always mixed up with cognition. Mathemaucal n:al1ues --early p-enom
ena in individual development, that is, not only geometncal realities but -th'.'et
ical ones as well. Numbers which are represented by observed and imagined
quantities, named by spoken and written numerals, and mutually_ connected by
actual, imagined and symbolised relations, belong to a real'." which can ex'.'nd
from the nucleus of everyday life experience to the far frontiers of mathemaucal
research, depending on the involvement of who is con-e:ned. In sp11: of hundreds
of years of resistance -- even on the part of mathemaucrans -- nega-v_e and complex numbers and their operations have become as real to ma-ema_11c1ans as positive ones had been for centuries, and whole numbers for millenma, and at -ey
now belong to the reality of most people who have learned some math_em-ucs.
Reality is historically, culturally, environmentally,_individually, and subiec:u-ely
determined. Lawler's "micro-worlds" are less- obiecuve and Bauersfeld s domains of subjective experience" less subjective th- these_ te-- may suggest.
Solipsism is :is artificial a construct as is all-embracmg obiecuv1sm. I pre-er to
apply the term "reality" to that which at a certain stage common sense experiences as real.
"Real" is not intended here to be understood ontologically (whatever ontology
may mean), therefore neither metaphysically (Plato) nor physically <A2stotle);
not even, I would even say, psychologically, but instead commonsens1-ly as
when one uses is meant by the one who uses the term unreflectingly. It_ 1- ?ot
bound to the space-time world. It includes mental objects a-d mental acuv111es.
What I called "expanding reality" is accounted for on ever h1g-er levels 7f common sense and witnessed by levels of everyday language or various techmcal languages.
The farther one's reality extends, the more often and more sharply one -eeds to
zero in on more or less manifest fragments, isolated by aspects of real11? .. Th-s
may tum out to be an advantage or a disadvantage_. Transfer of electnclly -s
served by insulation, yet transfer of ideas by short-crrcmts. In -twee- there is
the cellular structure of biological life, defined by membranes with a highly_ selective and effective permeability, which is the amazing result oflong evo1uuon.
The permeability of the membranes dividing individual realities may de-nd on
the active or passive origin of these fragments: whether they are due to mtemal
evolution or imposition from outside. . . .
This is the key, in my opinion, to understanding the poor permeability 7f the
membrane separating classroom and school experience from life _expenen-,
which I alluded to and promised to analyse in due cour- when I wil_ deal :-"1th
mathematics education; in spite of the fact that mathematics, as well as its various 18 CHAPTER I
domains and aspects, is already a striking example of the poor, or even the lack
of, permeability of membranes. The specific relation between content and form -
- the emphasis on form, reinforced by the existence of a particularly efficient language -- favours the growth of watertight membranes between mathematical and
"outside" contents, between mathematical language and everyday life or more
technical languages. And this takes place in spite of the fact that mathematics
could be an outstanding example of bruad-minded integration into reality. Indeed, as pointed out earlier, mathematics, unlike any other science, arises at an
early stage of development in the then "common sense reality" and its language
in the common language of everyday life. Why does it not continue in this way?
How can one strengthen this integrated existence in order to resist the temptations
of separation and isolation? "Mathematics starting and staying in reality" was the
title of a lecture I once gave on several occasions to answer these questions. In
due course I will repeat these answers, which are of an educational character.
1.1.6 Concepts or mental objects?
What is the difference between number and number concept, between ratio and
the concept of ratio, between triangle and the concept of triangle, between similarity and the concept of similarity, between X (an object) and the concept of X?
This is a meaningful question for non-mathematical objects X as well, as are:
force, mass, heat, evolution, content, form, science, government, race, humanity,
poetry, pop art. The above sequence looks like one of increasing distance between "X" and "concept of X". There is at any rate a difference between both of
them. "Concept ofX" seems to mean how one conceives of an object X in acertain perspective, say, by inspection, reflection, analysis, scrutiny, or whichever
you wish.
There would be no cause to ask that question were there not a tendency to decorate tenns for objects, for dignity's sake, with the prefix "concept of"; even this
would hardly matter if one could disregard the consequences, in particular for
teaching where cognitive learning is concerned. Unfortunately teaching concepts
looks more dignified than mere teaching. Teaching concepts is likely to create the
illusion of adding more understanding to what is learned.
There have been various number concepts -- Euclid's, Canlor's, Dedekind's,
Peano 's, Frege' s, Russell-Whiteehead's, Kleene' s -- which differ formally as well
as substantially from each other. This difference, however, is a matter of expert
sophistication, and of more concern to researchers in Foundations of Mathematics than to the great majority, even among mathematicians. There have been various concepts offorce --Aristotle's, Descartes', Leibniz's, Newton's. Physicists
decided in favour of Newton, but they have so far failed to prevent people from
nourishing less educated ideas on force.
Cognition does not ,-tart with concepts, but rather the other v;ay around: conce11ts
MATHEMATICS PHENOMENOLOGICALLY 19
an earlier stage than any other field of knowledge. Forinstance, :.'odd" and "ev-n"
can be defined on the basis of "whole number". Indeed, the first mathematical
science, that of the Pythagoreans, was about Odd and Even. But what about
"whole number"? It is generated by a process, that of counting, rather than by an
explicit definition, only to become a matter of common sense rather than a -on
cept. At least since Euclid, mathematicians have searched for deeper foundauons,
and they have succeeded. New Math adherents believed that children should not
be denied this benefit. They fought common sense number, without success.
Back to Basics eventually meant back to common sense.
Specimens of circles suffice to explain what are common sense circles, and nobody who is shown a circle doubts that it has a centre (whic? may be fou-d by
trial and error or in a more sophisticated way). But at least smce Euchd, crrcles
were defined -s locuses, and in Cartesian geometry by algebraic equations. New
Math adherents fought the common sense approach, and in a s-nse they s3cceeded, at the price of expelling geometry entirely from mathemaucs mstrucuon.
Jn the mean time it has become clear to increasingly more people that, where n_onmathematicians are concerned, teaching the concept of X is not the appropnate
way to teach X. Cautious researchers now admit that concepts are pre-eded _by
something less fonnal, by initiations, preconcepts, or whatever they call it, which
in the long run means that the proper goal is still that of teaching concepts. In my
view, the primordial and -- in most cases for most people -- the final goal of
teaching and learning is mental objects. I particularly like this tenn because 11 can
be extrapolated to a term that describes how these objects are handled, namely,
by mental operations. .
Whole number, the number line (even if drawn on the blackboard), geometncal
shapes (even if materialised); all are mental objects, so far:" It is understood_(and
it is understood) that visual images are rough representabons of mental objects.
How do mental objects develop to become concepts, and what criteria reveal
whether this has indeed happened? In the case of whole number, shedding _the
strait-jacket of the positional system may be the sym-tom. -ost often verball_sation and formalisation may indicate concept formauon. It is worth menuonmg
that historically the mental object of group preceded the group concept by about
half a century. Leibniz and John Bernoulli used the word "function" for something that was no more than a mental object, and only upon the first appearance
of a letter symbol for a function in papers of d' Alembert and Euler was the road
paved for the function concept. .
The distance between ment:lil object and concept will depend on the subject matter, but even more on the individual and his particular situation. This is the reason
why it must be respected in instruction. When we deal with the teaching/learnmg
process the question "Concepts or mental obje-ts?" will be para?elled by the d1-
dactical one: "Concept attainment or constituuon of mental objects (by mental
operations)?" 20
CHAPTER I
1.2 STRUCTURE AND STRUCTURES
Structure is form abstracted from its linguistic expression. We will focus on
structure with regard ID mathematics, although much of what we have ID say applies ID a broader context In mathematics the relationship between form and content is reflected by that between something having or being a structure. Structuring is a means of organising phenomena, physical and mathematical, and even
mathematics as a whole.
1.2.1 Structures --poor and rich
A few examples will give us a clearer view of what structure means in mathematics.
A tetrahedron may be taken as a siructure consisting of four vertices, six edges,
and four faces in their mutual relation of "this vertex lying on that edge and in
that face, this edge containing that vertex and being contained in that face, this
face containing that vertex and that edge"; it is what one calls a combinalDric
structure, which in this particular example entails restricting oneself to the relations of containing and being contained in, and forgetting about what points, edges, and faces actually are, i.e. that edges are straight and faces are flat. What is
the use of so poor a structure? Well, with sets of such combinatoric tetrahedra one
can build absiractpolyhedra by identifying faces of different telrahedra with each
other -- sticking them IDgether as it were. Tetrahedra can thus be used as building
blocks for larger structures, which may become the subject of profound research
known as combinatoric topology.
We need not, however, content ourselves with the combinatoric stucture of the
telrahedron. It is more natural IO consider it as a solid in space, with true points,
edges, and faces. This is again a structure, a geometric structure. As a geometric
structure a tetrahedron is a richer structure than combinatorically. There is more
one can say about it; within this structure one can, for instance, measure distances, edges, angles, surfaces and volume.
A tetrahedron is also called a triangular pyramid. Or is it the other way round, is
the triangular pyramid a tetrahedron? No. I f this were ttue, how could it happen
that a regular triangular pyramid need not be a regular telrahedron? Well, triangular pyramids and tetrahedra are different structures. By definition, a pyramid
has a bottDm and a top (even though one is allowed ID turn the pyramid upside
down). A telrahedron can be converted into a triangular pyramid in four different
ways, as a matter of fact, by appointing one of its vertices to its top, and the opposite face to its bottom. The triangular pyramid is a richer structure than the tetrahedron.
Let us now knead the solid tetrahedron as though it were clay or dough, let us
press and deform it but refrain from tearing apart what is connected, and connect-
• :- - `Cl • • -)' .. ,_,_. I c 1_ c.
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dumb-bell. The vertices, edges, faces have disappeared. The thing is still connected and is so in a very special manner, spherelike (and certainly not like a
ring). it is a topological structure, poorer than the previous one of the geometrical
tetrahedron: no role has been assigned to distances, angles, rectilinearity, and so
forth.
1.2.2 Structures defined by relations
So far we could have contented ourselves with calling our objects figures rather
than structures. In order to become structures they must be described structurally.
Figures can be constructed out of their parts. Something that is finished can be
described structurally. One focuses on certain relations between the elements of
the figure, and chooses among them a set that characterises the figure in mind.
The easiest case is the combinatoric tetrahedron: four arbiirary objects are called
vertices, each pair of them is assigned to be an edge, and each trio to be a face,
with the mutual relations of containing and being contained. The geometric tetrahedron, on the other hand, can be described as a metric structure, by assigning
a mutual "distance" to each pair of its points (ofvertices only). It is less easy to
explain how a to polo gica/ siructure is described: by telling what is near to each
other, with no ap,lJeal to quantitative distance, that is, in a qualitative way; indeed,
topology forbids tearing and glueing, which would affect qualitative distance. As
a topological structure the border of a disk is the same as a whimsical closed
curve, but it is different from a siraight or curved line segment, which on their
part are IDpologically both the same.
More -- as well as more stringent -- relations increase the wealth of a slructure.
Dropping relations impoverishes and appending relations enriches sttuctures in
the same medium.
1.2.3 Algebraic structures
The system of whole numbers 1,2,3,. . . can be interpreted as a slructure in various
ways. While postponing the cardinal aspect, I start with the ordinal one, the
counting sequence, which is an order structure of the kind "somehow it starts and
each has its successor", while the names of the particular things do not matter. An
addition can be derived from the order in such a structure: to each pair of things
it assigns a third, its sum. Now imagine you have forgotten how this addition has
arisen! Then all that is left is an infinite system of things, distinguished by symbols, and a table with two entries which, for every pair of these things, tells what
you want ID be their sum. The relations in this system are of the form a + b = c.
It is what one calls an addition structure. The present one has some remarkable
properties, for instance, that a+ bis always the same as b +a. Maybe, next ID this
table you allow for a second one (which may or may not be dependent upon the
first) which of every pair of things tells what you want to be its product. This en-
" /"Jtc..S [::2fi/
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riches the structure with relations of the kind u .v.=w.
Such systems are called algebraic structures. There are plenty of them with two
or even more operations, which may satisfy widely different condition-.
I.2.4 Structures --from smaller to larger
The best known among the algebraic structures are the number structures
• Ill, of the natural numbers 0,1, 2, . . .
• 21, of the integers . . . -2,-1, 0, I, 2, . . .
• <I;I, of the rational numbers, represented by fractions of integers such as 1 _i
5.33 • 3. 3.
•JR, of the real numbers, among which, for instance, ..fi., e, it
• CC, of the complex numbers a + b i where i = J:j

It is a sequenc_e of structures of increasing size, although not primarily -- and not
even necessarily -- of mcreasing richness.
The operations of addi-on and multiplication are the same in all these examples,
as are structunng relations of the kind
a+b::c, u. v::w,
whereas some of their properties may change. The equation
a+x:: c
is in general solvable for x only if negative numbers are admitted, the equation
u. x= v (u ,oQ)
only if rational numbers are involved, the equation
x.x=k (k>O)
only in the domain of non-negative real numbers. For instance, the solution of
x2=2 cannot be obtained as a fraction of integers, although it can be approximated
-y such frac-on-; --, H, -, : . are increasingly better approximations of ..fi..
Approowatmg is a topological concept. Beside with the algebraic structure,
the_domam ofreal (as _well as -t of complex numbers) is provided with a topological structure. In this sense 11 is not only larger but also richer than that of the
mtegers. In another _se_nse_ N, -e structure of natural numbers, is richer than <Q,
-'_and C. Indeed, div151bTIIIy JS a rich concept in JN, but a poor one in <Icl, JR, and
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I.2.5 Generating the number system
The system of real numbers is geometrically represented by the number line,
where each point, by its distance from the point 0, represents a number, positive
at one side and negative at the other. In this model the algebraic operations are
visualised by simple geometric mappings.
In the Greek tradition this was the common sense approach to (positive) real
number, which was inundated by the "arithmetisation" wave in the past century.
Arithmetisation meant disavowing the traditional geometrical foundation, and
founding conversely geometry on number. The real numbers were to be defined
by sequences of (approximating) rational numbers, the rationals as fractions of
integers, and the integers as natural numbers preceded by a plus or minus sign.
One long step backwards was taken when inatural number in turn was founded
on nude sets, deprived of any structure (and ultimately all sets on the empty set).
Sets were to be compared with each other by mapping, that is, by laying them
mentally elementwise along each other. If the sets A and B fitted each other they
were said to have the same cardinal; if they did not, even though A could be fitted
to a part of B, then A was said to have a smaller cardinal than B.
The common sense.counting number, and the common sense measuring number,
as visualised on the numberline ruler, were derived from one common root by
this profound analysis: the structureless set. Lack of structure may be an advantage in Foundations of Mathematics, which is high level mathematics, but much
too high to start with. Indeed, the proper beginning is or should be common sense.
At present most people will agree that New Math's contempt for common sense
has been a historical mistake. But has everybody really learned this lesson?
1.2.6 Geometric structures
The geometric structure most familiar to us from early childhood onward is the
space we live in -- euclidean geometty as it is called. Solid bodies allow us to
compare, or rather to define, distances, and according to this system of distances
the straight vision lines are experienced to be the shortest. From early childhood
onwards we are familiar with faithful pictures of objects, obtained by shrinking
and enlarging, called similar in mathematics. No doubt similarity even precedes
number in cognitive development.
During the 19th century poorer geometric structures than the euclidean one began
drawing the attention of geometers. Impoverishing a structure -- provided this
means removing what under certain conditions look like scrolls and curls -- can
lead to greater profundity. It is an art creative mathematicians have learned to
practise and master during the past century.
In euclidean space we are dealing with distances, angles, straight lines, circles,
planes, spheres. A first step towards impoverishing this structure is to forget
about the general comparability of distances and angles, while preserving recti-24 CHAPTER I
linearity and parallelism. This leads to affine geometry. In affine geometry all
parallelograms are the same: rectangles and squares cannot be distinguished from
other parallelograms, nor can circles be distinguished from ellipses.
A next step is to forget about parallelism, while preserving rec ti linearity. This
produces projective geometry, where all quadrilaterals are the same and all conics are the same.
One more step, and then even rectilinearity disappears as a structuring property.
Space is being impoverished to become what we formerly called a topological
structure, where we can still distinguish open curves from closed ones, the interior of a closed surface from its exterior, see whether or not a closed curve in
space is knotted, and whether or not a pair of closed curves is linked.
More than a century ago this kind of classification was introduced into geometry
by Felix Klein, when proclaiming his Erlanger program. This was the first modern attempt at structuring mathematics itself, or rather a part of it: geometry.
1.2.7 Structure of mathematics
It is one thing to interpret a small or large object as a structure, and quite another
one to attempt this with an entire area of scientific cognition. When compared
with creating, organising scientific cognition seems to be an inferior activity. Yet,
as stressed before, in no science are these two activities so densely interwoven as
they are in mathematics. A century ago such an organisation was attempted in geometry and for half a century the much broader task of organising the whole of
mathematics remained dormant. After the second World War this task was undertaken by the French group Bourbaki. Although their work required a large
amount of creativity, its eventual result could not in principle be anything but a
deductive codification of existing mathematics. To be sure, Bourbaki's work is a
monumental system of mathematics, which -- although now out of date on essential points -- has contributed enormously to the growth of mathematics. Even so,
it has in the mean time been overtaken by mathematics itself. I doubt whether a
fresh attempt can or will be tried in the future.
Bourbaki's hierarchy is primarily oriented from poorer to richer structures and -
- accidentally, and certainly not as a matter of principle -- from smaller to larger
ones. As a matter of fact this is indeed the most natural strategy for a systematic
build-up: from poorer to richer. It starts, as it were, with a tabula rasa, with that
which lacks any structure, i.e. the clean set. But after all, how can rich structures
be created from this poor start?
If A is a se- one may consider the set P(A) of its subsets, which in fact has a remarkable structure, effected by the structuring relation of containing and being
contained in.
Or, starting with two sets A and B, one gets their product, the set of pairs; it is
obtained by steadfastly taking one element of A together with one of B -- a con-
-.-.-;-.---:- I
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each of three boys (set A) is paired with each of four girls (set B).
j - --- -
I f:j
i t ij i-
i1Jj It
j j !8
--.i -f .-j --
Fig. 1
25
This product set is heavily structured, both horizontally and vertically. Similarly,
the whole plane can'be structured as the product of two s-aight line-, where --ch
of its points is represented by a number pair -- the way this 1s done m coordinate
geometry. .
These are mere examples of how to impose structures on given or ad hoc constructed sets -- order structures, algebraic structures, topological structures, and
combinations of structures. They can be built into a hierarchy, in order to impose
a hierarchy upon mathematics as such -- one hierarchy among many conceivable
possibilities. There is not, and never will be anything like the -erarchy of mathematics. This fact is evident from the variety of textbooks, which differremarkably from another. There is certainly an inclination, at university .level, in particular to teach mathematics as a hierarchy, although one teacher will put more water
i-to the hierarchic wine than the other, depending on the degree of immediate
utility of the mathematics taught. .
Even so consumers of mathematics should be introduced to mathemattcal structures wherever it matters, and maybe even be made aware of their structural aspects to a certain degree. This does not, however, mean 1!13t mathemati-- as such
should be presented to them as a structure, either accordmg to Bourbaki s or any
other conception. It would be a futile attempt, to be sure. .
Moreover, a system like Bourbaki's, or any variant whatsoever, d-s not do JUStice to mathematics as a servant rather than a master. The most striking shortcoming of any system of mathematics may be its treatment of natural number: most
often a disregard for the numeration structure, i.e. the decimal system.' From the
most pedestrian to the most sophisticated practice of numbers, the decimal structure is its dominating feature. From the (merely linguistic) learning of numerals
to operating efficiently with numbers, this structure is indispensable. Ye- m no
system of mathematics is it ever mentioned. Indeed, high level mathe-attcs has
been objectified and stripped of such human rudiments like the dec!mality of our 26 CHAI'IER I
fingers and toes. True mathematics behaves universally rather than anthropomorphically.
This, then, is the most striking feature of the epistemological and practical inadequacy of Bourbaki-like systems. There are more misfits. Visual geometry has
no place in them. Similarity, which in cognitive development even precedes
number, is far remote in this hierarchy, and even then is detached from its visual
origin. No place is assigned to computers, which are functional slruct!ures. Mathematical modelling of slructures for the benefit of even the simplest applications
is incompatible with the hierarchic rigidity of such systems. No attempts have
ever been made to structure mathematics as seen from reality as epistemological
source and domain of applying mathematics, although such attempts would be
extremely desirable -- psychologically, pedagogically, educationally and didactically. By their mere existence the deductive systems prevent these attempts
from being undertaken.
1.2.8 Structures as viewed from reality
This is not to belittle mathematical structures. By discovering structures within
mathematics we have learned to better understand the inherent organisation of
our knowledge. Many psychologists and educationalists still perceive cognitive
development as concept attainment. Superficially this is evident from the numerous titles of research papers which contain the word "concept", and in a more profound way from innumerable experiments where the possession or development
of concepts is said to be tested.
Knowledge as a hierarchy of concepts is a prominent feature of the Aristotelian
theory of science and epistemology. Its methodological inadequacy has become
m-nifest, mainly under the influence of mathematics. Aristotelian hierarchy is
oriented from the general to the particular, the general embracing the particular.
It is most perfectly realised in biological taxonomies, where one descends along
the lines of groups, divisions, classes, subclasses, families, species, varieties. Yet
outside botany and zoology concept formation by classification has proved unsatisfactory. Knowledge is not just more than but quite a different category from
a reference !fora or fauna, as is slructuring, when compared with classifying.
Classes contam each other, or are contained within each other; in this way generality and particularity are expressed by mere size. If, in the case of slructures, one
speaks of general versus particular -- which I do not -- the poorer struc:ture would
be the more general, and particularity would develop by enrichment.
By slructuring rather than forming concepts we get a grip on reality. We do this
with rich or with poor structures according to our needs. Impoverishing may
mean generalisation in the sense of wider applicability. Among the poorest scientific slructures, those of mathematics are distinguished by their large measure
of applicability as exemplified by numbers and geometrical objects. MathematiMATHEMATICS PHENOMENOLOGICALLY 27
As I stressed before, the identiFication of learning with concept attainment is a
superficial view, even though it is not unusual with psychologists and educationalists. It is to Piaget's merit that he stressed slructure, at least in theory, though
not as consequentially in practice.
1.2 .8.1 Structure of science, and development
I wish to stress once more the distinction between slructures within a science --
in particular mathematics -- and the s1ructure of a science. It is an old adage that
cognitive development proceeds from the particular to the general, and this direction is therefore considered compulsory for the educational process. Yet it lacks
sophistication as an adage, mainly because of the multifarious interpretation of
"general versus particular". It may be taken for granted that, for a child, being acquainted with a particular dog (or with a few of them) precedes acquaintance with
the species "dog", but classbuilding is only one aspect of cognitive development
and, for that matter, quite a modest one. Generalisation starts with situations rather than with objects and the function of a particular situ-tion is paradigmatic rather than classbuilding.
Piaget's view is more ;;ophisticated than the traditional adage, if not its direct
converse. According \o Piaget, development takes place along epistemological
lines, where epistemo is not individual cognition but, with regard to its contents,
independent of the developing individual. As a consequence, the structure that is
manifest in the present state of science is at the same time the pattern of the individual's cognitive development Experiments serve to corroborate this parallelism, and they were designed with this purpose in mind, or at least this was their
intention.
Piaget nourished this conviction before Bourbaki built his system, or at least before he could have become aware of it. He fully practised it in his work on geometry'. He chose the structure of geometry familiar to him as the canvas on which
the child's development must be recorded. This structure was Felix Klein's, although geometry had by that time already outgrown it, if ever it had fitted at all.
According to Piaget development progresses from the poorer to the richer slructures as he found them in Felix Klein's hierarchy; it goes from topology via projective and affine to euclidean geometry and, according to him, this is true as well
for the perceptive, the representative, and the cognitive aspects of development.
Piaget's view betrays a high degree of confidence in -- or should I say, obedience
to -- mathematics' hierarchies, although this has been belied by his own experiments as well as by those of others. At an early stage a child who is unable to
draw neat squares and circles, can easily distinguish neat copies of squares and
circles from bad ones.
When Piaget was confronted with Bourbaki's system of mathematics, he accepted it as an epistemology which can be interpreted genetically, or so it would
seem. In fact, I believe this to be a misinterpretation of Piaget's reaction. I rather 28 CHAPTER 1
tllink that Piaget's own conception of cognitive structure had been entirely completed by tl!at time. His acknowledgment of Bourbaki's structure of mathematics', rather than being a recognition, was an attempt to adapt the Bourbaki system
to his own conception. This has, however, often been interpreted as a support lent
by psychology to the enterprise of teaching mathematics according to the structure of this science, an enterprise which became famous under the name of New
Math and eventually proved to be a failure.
Piaget can hardly be blamed for this. Even though he believed in the genetic-epistemological value of a structure of science, tllere is no proof that he advocated
teaching according to curricula patterned on the structure of a science.
This brings us to the next point, which will anticipate later discussion.
1.2.8.2 Structure of science, and instruction
In spite of their failure in mathematics, curricula patterned on the structure of a
science, are becoming fashionable in other areas. As a pure mathematician I am
able to view more clearly the relativity of what is offered as a structure of mathematics. If I place mathematics within the larger context of knowledge and abilities, I am struck by the large gaps in what is often propagated as structure of
mathematics; for example, the lack of the numeration structure and the disdain
for visual geometry and for even the simplest applications. As a mathematician I
also feel obliged to tum against the structure of science as a means to structure
education, because my personal experience has shown me how easily mathematicians yield to this temptation.
Piaget's work supplied me with no argument from developmental psychology in
favour of a science structure curriculum, nor can this be justified by educational
theory. A science structure codifies systematically (and in the case of mathematics this means deductively) the state of that science at a given moment and, for
that matter, of a science tl!at is not even the subject of the instruction envisaged.
Moreover, there are serious arguments against science structure curricula. Systems of mathematics reveal a hierarchy oriented from poorer to richer structures
which the curriculum designer tries to imitate. "From poor to rich" is, viewed
mathematically-didactically, a questionable principle. Poor structures are utterly
abstract as is evident from the poorest of all, the structureless set. Didactically
one cannot come to grips with it, or it should be by concretising, by filling out the
abstract form, and in practice this is done by creating artificial, and often even
false concretisations. In genuine mathematics, sets as well as all of these structures serve goals; they are because they are operational. At the level where science structure curricula start, however, there is nothing mathematical one can do
with sets. So, as a curriculum designer, one arbitrarily invents things to do with
sets, which have nothing to do with the need for sets in mathematics -- false concretisation and operationalisation; nor do these have anything to do with the need
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other poor structures in elementary instruction are introduced with no otller aim
than to pay lipservice to the science structure.
If these are but unfortunate slips, there are also more profound arguments against

the precedence of poor over rich structures, even from the viewpoint of developmental psychology. What is usually called abstracting, is most often nothing but impoverishing a structure. Mathematical structures have arisen in richer contexts and be created in order to be applied where they have arisen and beyond. The orientation from poor to rich has been suggested by ready to hand mathematics. In due course we will deal with learning "mathematics in action", which is re-creating mathematics, not aimlessly of course, but under guidance. If this is agreed upon, the didactically recommendable direction will be the same as that in which mathematics arises, that is, from rich to poor.
1.2 .8.3 Structuring rich context mathematically
Didactically I have opposed rich to poor mathematical structure. This, however, is not enough. One should not be satisfied by staying within mathematics. The rich structures to be offered should also be sought for olf(side mathematics, albeit with a mathematical-didactical afterthought. After the failure of science structure curricula, teams of developers chose another way: that of offering non-mathematical rich structures in order to familiarise the learner with discovering structure, structuring, impoverishing structures and mathematising. By this means he may discover the powerful poor structures in the context of the rich ones in the hope that, by this approach, they will also function in other (mathematical as well as non-mathematical) contexts. Starting with poor mathematical structures may mean that one will never reach the rich non-mathematical ones, which are in fact the proper goal.
Let us illustrate this by an example!
Who is not familiar with the so-called logical blocks, 24 in number, available in a rich variety of models, for instance, red/blue, circle/square/ triangle, big/small, thick/thin? They are a paragon of an entirely prestructured world: one piece for each combination of the four criteria. Abstract operations on sets can be concretised marvellously by means of such a model. I have opposed to this system another one which I have called a liUle world, although I have never insisted on teaching it: it is, as it were, a toy shop containing cars, animals, dolls, buildings, and so on, of various sizes, kinds of material, colours, grades of mobility, and so on. The criteria of classification are not imposed a priori but are discovered and developed by the learner himself -- characteristics of colour, size and so on, which need not be sharply determined but may be susceptible to shades. lu this little world, combinations of characteristics may be represented by many objects as well as by none, for instance, no big black wooden house but many little red tin cars. In brief, it is not a prestructured world but rather a world to be structured.
In this case, structuring is performed by classifying. I chose this example not be-30 CHAPTER 1
cause I may believe in classifying as an important cognitive activity, but rather as an example that, thanks to its simplicity, sharply features the difference between poor and structured on the one hand, and rich and to be structured on the other. Logical blocks are a striking example of the implementation succeses which can be reaped with sharply structured material -- cheap successes obtained thanks to love of ease. Rich material, open to structuring, which provides for more didactical opportunities, is more demanding and therefore less easy to implement.
Let us finish here! A large number of rich contexts for mathematics instruction is now available, more than anybody can imagine. The main problem is that of implementation, which requires a fundamental change in teaching attitudes before it can be solved.
1.3 MATHEMATISING
1.3.1 The term
After this discussion on the context and the internal and external structure of mathematics, we tum again to mathematics as an activity and look at one of its main characteristics: mathematising. Who was the first to use this term which describes the process by which reality is trimmed to the mathematician-- needs and preferences? Such terms usually emerge during informal talk and discussion before they ent-r.the _literature, and nobody can tell who invented them. In any case, m-themausmg 1s _a process that continues as long as reality is changing, broadening and deepemng under a variety of influences, including that of mathematics, which in tum is absorbed by that changing reality.
Mathematising as a term was very likely preceded and suggested by terms such as axiomatising, formalising, schematising, among which axiomatising may have been the very first to occur in mathematical contexts. Axioms and formulas are an old heritage, although the meaning of"axiom" (or "postulate"), and the form of formulas has changed in the course of time. Euclid's Elements are no paragon of flawless deductivity, as people believed for centuries, nor were they ever intended as such, as some people still seem to believe. Axiomatics, as we now use this term, is a modem idea, and ascribing it to the ancient Greeks is, in spite of precursors, an anachronism. Nevertheless, reshuffling an area of knowledge so that end_s,are chosen as starting points, and conversely, using proven properties as defm1uons to prove what was a definition originally -- this structuring 'upside down" is a mathematical activity of long standing. It is as old as Greek mathematics, or perhaps even older, although never until modem times has it as con-
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rooming axiomatic systems are the result of attempts to reorganise fields of mathematical research. This technique is called axiomatising, and is well understood and mastered by modem mathematicians. Its first striking example was groups.
From the tum of the 18th century onwards, mathematicians were confronted with mappings of sets upon themselves, often singled out by invariance properties, and were led to compose such mappings. In this way they became acquainted with sets of transformations which, under composition, automatically satisfied the well-known postulates, required later on for groups. Cayley, in 1854, took the unifying step to define, by means of these postulates, the (finite) object he called a group; yet not before 1870 did this new conception become whole-heartedly accepted by leading creative mathematicians, and then also in infinite substrates.
Formulas, in everyday life as well as in symbolic language, are as old a feature as axioms, or even older. Improving the linguistic expression by increasingly effective symbols and symbolisms has been a long process, which first concerned mathematical subject matter and only later also affected the language in which this subject matter was expressed. This process of trimming, adjusting, and transforming language is called formalising.
To be sure, as popular as axioms and axiomatising might be in modem mathematics, they are only the highlights and the finishing touches in the course of an activity where the -tress is on form rather than on content. The same holds true with regard to forlnulas and formalising. Axioms arise from paradigms or sets of paradigms, and axiomatising means generalising experienced paradigms. It is an old human habit to make one's experiences and actions paradigmatical, to generalise them by abstracting them into laws and rules, to create schemes to fit reality. This last activity is called schematising, which is the counterpart to axiomatising and formalising insofar as contents rather than abstract form and language are concerned.
The preceding exposition serves to explain the origin of the term mathematising as an analogue to axiomatising, formalising, schematising. I pay so much attention to it since it is not unusual, in particular in education, to restrict the term to one of its components. I myself insist on including in this one term the entire organising activity of the mathematician, whether it affects mathematical content and expression, or more naive, intuitive, say lived experience, expressed in everyday language. But let us not forget about the individual and the environmental dependence of "lived" and "everyday life" on expanding reality and progressing linguistic sophistication!
1.3.2 Some aspects
Modelling
Moreover, with regard to schematising there is a tendency to identify schemas with such things as solving formulas and procedures within formalised mathematics. Nowadays the term "schema", in the broad sense, seems to have been su-32 CHAPTER 1
perseded by more topical "model" -- a valuable term, but unfortunately devaluated by thoughtless use and misuse. I have castigated this practice often enough, or so it seems to me. [87]
Mathematics has always been applied in nature and society, but for a long time it was too tightly entangled with its applications for it to stimulate thinking on the way it is applied and the reason why this works. Counting was indeed, common sense; surveyors acted as though their pegs and rods were geometrical points and lines, and money changers, merchants, and ointment mixers behaved as though proportionality were a self-evident feature of nature and society. Even the astronomers in Babylonian Antiquity stuck as long as feasible, or even longer, to linear inter- and extrapolations in their attempts to describe celestial phenomena numerically; which meant, by means of piecewise linear and by zigzag functions, which their Greek heiresses eventually changed into goniometric ones. But goniometric functions did not drop from the sky they were studying. The underlying theory was that celestial movements ought to be circular. Efforts to save both this postulate and the contradicting phenomena gave birth to what we now would call a model describing the celestial movements -- a contraption of circles, epicycles and excenters which, geometrically and numerically processed, required goniometric functions. This model survived for almost two millennia. Kepler, rather than offering a new model, formulated three general mathematical laws of planetary motion, derived afterwards by Newton as consequences of his theory of gravitation. Newton himself refused to contrive primitive mechanic models for explaining gravitation. So, as time went on, physicists had to accept, quite often grudgingly, attraction by the force of gravitation as a model in its own right which, second only to Huygens' wave theory ofllight, was the first model in modem times to transcend common sense experience. History repeats itself: When elastic models of light propagation, suggested by 19th century's common sense mechanics, to refine Huygens' wave theory failed, physicists had to accept Maxwell's electro-magnetic light model.
Modelling is a modem feature. Until modem times the application of rigorous mathematics to fuzzy nature and environment boiled down to more or less consciously ignoring all of what appeared to be inessential perturbations spoiling the ideal case. For a long period, simple geometry and arithmetic had sufficed to strip such ideals of their disguises. But what is ideal and what perturbation? The first example of a non-commonsensical split between ideal and perturbation was delivered by Galileo: uniform motion as the ideal, perturbed by resistance or, as Newton put it, by force in general. In a way this methodology has survived up to the present day. Even ifa "rigorous" theory is available, it is almost never applied as -uch but instead simplified in order to be accessible to actual processing, which afterwards may be refined by better approximations, or immediately by feedback models. The first great example of non-trivial idealisation of this kind is d' Alembert's vibrating taut string: by neglecting, as it were, the string curvatL1 .,1\;\ 1.-'.'". ;;'.'--'G-;;'.-..ds ---- fl:'.f--'";T-O:-f--'-1-Q! >"I'.'SJ';.<f.\'\1 ,,.<--.1_-c;;;, ", .. .__-_ -c; ":-:'--f-t:-_: -I'l\f':' .-._t

become a standard tool in applied mathematics.
In natural sciences the first use of the word "model" is probably related to the well-known orrery models of the solar system, where the interplay of planetary and lunar movements caused by gravitation is rendered (coarsely simplified) by means of a mechanical device; being merely a model, justice is done to the kinematics though not to the dynamics of the celestial motions, while for practical reasons the radii of the spheres representing the celestial bodies are disproportionate to each other as well as when compared with the sizes of the orbits. Familiar as the Rutherford-Bohr atom model may be, it describes the atom and its manifestations as a little solar system, with strange restrictions imposed on the possible orbits; its model character stems from the ad hoc conditions to which the orbits are subjected, and the ad hoc assumptions regarding jumps from one orbit to another which are incompatible with the laws of classical physics. A more recent model is the drop model of the nucleus, in which protons and neutrons are smeared out as a fluid -- an idea typical of a simplifying model. A particularly revealing example is the cosmological model of the expanding universe. It was originally contrived as a purely kinetic explanation of the all-sided flight of the galaxies yet, as time went on, it became enriched by numerous features of dynamics and of elementary !'article physics, although it was still understood as a grossly simplified model of the evolution of the cosmos.
All these are idealising models, which introduce mathematical precision into a coarser physical reality or simplify a reality that is tacitly agreed to be more complex than its so-called model. Strangely enough the first use of the term "model" in mathematics aims at just the contrary: concrete models of abstract geometrical shapes, in plaster or wire and cardboard. The same mathematician who had a large collection of such geometrical models made -- Felix Klein -- was also, if I am not mistaken, the frrst to apply the term "model" within mathematics itself. I mean here the image of non-Euclidean within projective geometry--Cayley's invention which was interpreted as a model by Klein in order to concretise abstract looking non-Euclidean geometry within the frame of the more concrete looking projective geometry. Although not as palpable as gypsum models, this model is indeed more easily accessible to visualisation than its pre-image. Klein's example was the root of the present model concept with regard to axiomatic systems: what is implicitly given by formal axioms is made explicit by means of a suitable mathematical object which, as it were, fills the axiomatic form with what looks like substantial content. For instance, a particular group or transformation groups in general function as models of the general, axiomatically defined, group concept Or euclidean, in particular three-dimensional space may serve as a model of axiomatically defined linear or metric spaces. On the scale of concreteness, however, one can progress even further, by stepping out of pure mathematics, and consider physical, or merely experienced space as a model of its axiomatically defined pre-image.
Only for the sake of completeness did I mention this use of the term "model", which is actually the inverse of the term we started with. As a matter of fact, in 34 CHAPTER 1
the. present context we are not concerned with models of axiomatic systems, wiech are. lav1sh.ly used in foundational research, but rather with modelling in the sen.se of 1deahsmg. By this means we are able to simplify complex situations which are dommated by mathematical theories of too great a complexity to be pracucal, or which are only accessible by ad hoc mathematical theories.
Since onr -ubject is modelling as an aspect of mathematising, I would like to stress that m the present context I should readily include tangibly concrete models such as wind-tunnels where aeroplane models arc tested and laboratory simulauons of hydrodynamic theories. In other words, models that are evaluated by observauon rather than by mathematics, even though their mere construction may require more mathematics than the processing of many less tangible models. I would not even -xclude computer simulations of such tangible models, where the factual evaluauon requrres less mathematics than does the simulating activity. On the other hand I strongly oppose the fashion of slapping the label "model" on any -ystem of algebraic, differential or integral equations (related to or arising in applied r-earch) -- "math- -ical model", as they like to call it. According to my. term mo logy, a m?-tel 1s JUSt the -- often indispensable -- intermediary by which .a complex reality or theory is idealised or simplified in order to become accessible to more formal mathematical treatment. I therefore do not like the term "-athematical mod-l" in a context where it wrongly suggests that mathematics drrectly o- almost directly applies to the environment. As a matter of fact, this only remamed true as long as mathematics was tightly entangled with the environment. I lay so much stress on the role of the model as an intermediary because peo-le are all too often unaware of its indispensability. Much too often mathemaucal. formulas are applied like recipes in a complex reality that lacks any intermediate model to justify their use.
Probabil.ty and statistics are particularly striking examples. In probability the um from which -ots are drawn is, along with other random devices, the model by means of w-1ch one attempts to mathematise everything in the world that seems to be. condluoned by c-nce: pollination of plant by another of the same species, m-ages and deaths m a populauon, which are viewed -- rightly or not -- as if maung and dymg were decided by casting lots. This model suffices for simple -pphcauon- o.f probability in statistics. As far as I know, however, no models exist for mediaung the conventional -- or should I say, ritual -- use of correlation a_nd regression. coefficients and factor analysis in a certain kind of social, in parucolar, educat10nal research: these tools have simply been copied from other sciences. where they are reasonably justified by intermediate models.
Looking back I realise that I have dwelt more on models than on modelling, and m rather general terms at that. Maybe suspicion that this might happen was the proper reason why I hesitated so long to tackle the subject. Of course, I could have overwhelmed the reader with a catalogue of streamlined models such as harm-nic oscillator-, electric networks, transition matrices, diffusion p-ocesses, ?0:f'f) "-'-.. d-V_'(:::'--.. . ,,.,.:·.·.•.-.·.·.·

within a formulation and across formulations
within a symbolisation and across symbolisations,
within an axiomatic system and across axiomatic systems.
Why this multiplicity of "and across"? Because discovering common features,
similarities, analogies, isomorphisms is the way towards
generalising
as a subconscious habit or as a more or less conscious activity. More often than
one tends to believe, generality is achieved by the aha experience of one single
• paradigm
only to be reinforced by a few (albeit not necessarily many) more of them. Now,
generalising paradigms
is the inverse of
• exemplifying general ideas
which, if imposed, is an instance of what I have tend to call the "antididactical
inversion", which I will deal with later on. However,
approaching problematic generality paradigmatically
is a valuable
• heuristic activity
to be distinguished from fashionable heuristics, which is understood as a kit of
prefabricated tools.
When stressing the efficiency of the unique paradigm, I had in mind the sudden
emergence of fresh mental objects and operations. Yet objects and operations can
become routine through daily practice, and eventually tiring enough to provoke
streamlining and short-cutting,
which may lead to
progressive
organising
schematising 36 CHAPTER 1
structuring
and, in particular, as far as clumsy language and symbolics are concerned, to
progressive
formalising
algorithmising
symbolising.
A particularly important aspect of malhematising is !hat of
" reflecting on one own's activities,
which may instigate a
• change of perspective
with the possible local result of
• turning ihings upside down
and the global one of
axiomatising
which again, if imposed, are instances of antididactical inversion.
1.3.3 Examples
(to be continued in 1.3.5)
1. Find the middle between 16 and 72 on lhe number line! Children I observed
shifted the two points uniformly towards each oilher, first by units, and then
by larger steps, in particular by steps of 10. Short-cuts led to halving the difference, lhe half being added to the smaller number. In general terms !his
means the expression
a+ 1 (b - a).
which by algebra equals the more usual one
1 (a+ b).
After I suggested that shifting away from each other lhe two numbers would
as well keep the middle, the children eventually shifted the smaller one to 0,
and consequentially lhe larger to a+ b which, as it were, proves the usual expression for the middle.
Rather than imposing a method of finding the middle between two numbers,
it is allowed to gradually develop through progressive schematising. To
achieve generality of this scheme, one paradigm seems to suffice, even beyond the domain of whole numbers. If verbalised, the general solution " I add
the two given numbers and divide by 2" may be reformulated via the stage
"half the sum of the two given numbers" in algebraic idiom and thereby contribute to motivating the creation and the use of algebraic language.
Another generalising streak may be by asking the original question with regard to more than two numbers, with the aim of establishing the mental object
of average and the scheme of averaging. Only if one is satisfied by formal
generalisations this can be done by "the sum of the given numbers dIYide<l by
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soon as contents are aimed at, one should look for situations where the envisaged addition imposes itself in a natural way or is even implicit to the situation. I.e: not the adding of ages, sizes, prices, and so on, but rather daily consumptions of a certain food, working hours, one person's receipts or expenses
during a week or a month; or, given the total consumption of some food or
other commodity by some population, ask for the consumption per capita, or
as for speed per second, as derived from speed per hour.
It would be too much to go into more detail in dealing with the concept of average only on behalf of schematising and formalising. I will, however, mention one more generalisation of the concept of "middle", namely, the "middle" of a plane figure or a solid body. Many aspects of mathematising are required for answering the
questions which may be asked in this context.
2. If one tap fills a basin in one hour and the other takes two hours, how long
will it take both of them together to fill it?
This problem of venerable age (together with venerable ones such as two labourers working together, two people surviving together on a certain quantity
of food, and so on) looks ridiculous only as long as it is not integrated into a
broader context of mathematising and is expected to .be solved according to
imposed traditional schemes. Children to whom I posed the problem divided
lhe filled basin into two parts, each of which was assumed to be filled separately by one of the taps; two-thirds of the basin by the "larger" and one-lhird
by the "smaller" one, so both of them were filled in two-lhirds of an hour.
Even when larger numbers were given children stuck to this visualised proportionality reasoning, supported, for instance, by the substitution of a
number of "slow" taps for a "fast" tap. This strongly diverges from the traditionally sanctioned scheme of reduction to one hour: If the two taps fill the
basin in a and b hours, respectively, then in one hour the first fills - and the
second i of the basin; therefore both of !hem together manage ! + ! of it, and
the whole basin in
hours. The children's reasoning, however, corresponds to dividing the whole
basin in the ratio
b: a
to be separately filled by the two taps, thus ah of the basin by the first, which
is then the factor by which the original a hours have to be reduced.
Surprisingly, as soon as this kind of problem takes the guise of two people
walking towards each other with different speeds, adults familiar with this
type of problem usually do not notice its isomorphism with the other ones but
solve it by linear path-time graphs. It seems that distances to be distributed
among two persons are more likely to elicit geometrical strategies than are
quantities, such as tap water, work and food. :!i
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Mental objects such as "speed" are basically schematised and formalised in
two -- opposite -- ways:
path per time and time per path;
the latter is particularly preferred when comparing sport achievements. Gasoline consumption is another example of the twin schematisation: the driver
figures out how far a full tank can take him, in order to know whether he can
cover a certain distance with one tank.
If one is aware of the manifold phenomena relaled to this twin schematisation
and the important interrelatedness of its components, the basin and taps problem and its companions begin to look less ridiculous. Harmonic adding and
averaging (that is, affer transformation into reciprocals) is indeed a precious
scheme which, to be acquired, requires carefully guided schematising.
3. The criterion for divisibility by 9, as learned at school, is hardly mathematics,
but challenging its validity-. The positional system, m<;>delled by the abacus,
can be a source of schematising: If the given number is represenled by balls
on the bows of the abacus, than transferring a ball from one bow to another
means changing the number by a multiple of 9; so if all of them are transferred
to the units bow, it appears that modulo 9 the number itself equals its sum of
digits. This reasoning extends by generalising to other positional systems.
4. With regard to schematising, percentages is too far-reaching an instrument to
be dealt with at length here. We will only indicale one feature as it is of paramount importance: the restructuring switch
from p percent more or less
to (l+ ff;) or (1 - -r(lo) times.
5. When do the two dial-hands of the clock cover one another? Infinite series,
simple algebra, or linear graphs can answer the question but, once answered,
a short-cut produces the very scheme: during one full turn of the short hand
the long hand turns 12 times, thus overtaking the short hand 11 times within
12 hours, at equal intervals. This is a far-reaching scheme, applying among
others to explain a few astronomic phenomena.
6. Ten children are at a birthday party; there are two more boys than girls.
A milk-can with milk weighs 10 kg. The milk left in it weighs 2 kg more than
the empty can.
Chickens and rabbits on a farm: 13 heads and 36 feet.
Starting by trial and error, which becomes less efficient with larger numbers,
children begin to use a rich variety of preferably visual schemes to answer the
related questions. Reasonings starting with If: If each girl takes a boy, . . . If
each rabbit was a chicken, . . . Again the end of generalising is reached in algebra.
7. In case you are not yet familiar with it, stop to think about the following problem: In a crowd of people among any five people there are always two of the
':::'JI'.i.c- .-, . .'.€. Sh :;-.,.:r -rW--1 t_ ':'.TIC-C:: G - 7 .::;--'r tr :'"I --- -'-" .' ---': -- \\' ---:-; _-. r- :' -:'I'- -, . - 1.A _,.:., .:.,, '>-1
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change the perspective: In fact there are no more than four year-classes.
8. Two players, and a pile of !(ij)matches, from which they shall in turn remove
at least one and at most ten. The player who takes the last matches, wins the
game. Since the winner cannot butreveal the trick to the loser, almost all people know it. So let the couple play another game:
A pile of matches, from which each may in turn remove a power of 2. Again
the player who takes the last matches, wins the game.
If the pile is 1or2, the player whose turn it is wins; if it is 3, he loses; if 4 he
wins, and so he does with 5 by removing 2 and thus putting the other in the
losing position 3. I f 6, however, whether he removes l, 2 or 4 he cannot but
yield the other a winning position, and therefore loses himself. 7 and 8 are
saved by removing l or2 respectively, but 9 is again on the wrong side. Going
on one would guess that multiples of 3, and no others are losing positions for
the player whose turn it is. How can you prove it? The result suggests arithmetic modulo 3. Powers of 2 modulo 3 equal l or 2. So all those high-brow
powers of 2 do not matter at all but instead it all comes down to removing one
match or two - a slight variant of our old game.
Another variant: only prime numbers (1 included) are allowed to be removed.
Again we draw up a list of winning and losing positions for the player whose
turn it is: e '
Clearly
1, 2, 3, 5, 6, 7, 9, 10, 11, . . . are winning positions
4, 8, 12, . . . are losing positions.
Indeed, whichever prime you subtract from 12, for instance, you grant the
other a winning position; however, subtracting 1, 2, or 3 from a number on
the upper line puts the other player into the lower line. This suggests arithmetic modulo 4, which degrades the game to the old one with simply replacing
10 by 3.
Still another version: The numbers you may remove are 1 and 4. Then the positions
1, 3, 4, 6, 8, 9, 11, . . . are winning positions,
2, 5, 7, 10, 12, . . . are losing positions.
Modulo 5,
1, 3, 4 are winning,
and 0, 2 are losing.
Indeed, the player whose turn it is in the first kind of state can answer any
move -1 by -4 and conversely, and thus preserve his stale.
The games proposed here show similar features to each other. What is the
deeper nature of their similarity? Are they mere paradigms of a more general
one and, if so, how to formulate this?
Rather than starting paradigmatically in each case as we did, the usual way of
presentation is to start with the final result in order to prove it -- the antididactical inversion. We presented an open end rather than a final result.
9. A sequence of disks 1, 2, 3, . . ., one side black, the other side white. To start 40 CHAPTER 1
with, all have the black side up. All even numbered disks are turned upside
down, then all divisible by 3, then all divisible by 4, and so on. Which ones
will in lhe long run again be black side up? People start experimentally, then
look for prime divisors and suchlike, only to finish with the short-cut that any
non-trivial divisor k of n has its counterpart divisor -,both of which coincide
only in the case where n is a square. A neat expositi-n would start just at this
point
10. Schematising experiences like the following leads to the idea of multiple
counting: three edges meeting at each of the eight corners of the cube would
seem to result in eight times three edges, whereas in fact there are no more
lhan twelve of them.
11. Draw one broken line throngh the five points (fig.2) of the Five of
• •
•
• • Fig. 2
the die, touching each point once and only once! How many different figures
can be obtained?
First, the idea of "different figures" must be schematised, which is done by
what is called congruence. Then the counting procedure must be structured
by classifying suitably, for instance, with respect to the midpoint of the Five:
take it as the start, as first stop, as second stop, and continue this way with a
comer.
12. Quite another aspect of malhematising but that of the preceding problems is
exemplified by the famous "grains on the chessboard": In order to estimate
264 , one replaces 2
10
by 10
3
• It is an example of numerical schematising.
13. So far I have neglected the linguistic features of mathematising. So in order
to make a choice, I consulted [87, IV,15]. Choosing means losing, which I do
not like. So please, do as I did!
14. I also didn't pay enough attention to change of perspective. This is a particularly rich subject as appears from the examples in [87, IV, 16]. Meanwhile, I
could add many more but I won!t. The subject deserves a more systematic
treatment, which I do not dare to undertake.
15. A barrel, closed on the upper side, except for four holes, which are arranged
in a square (fig.3). Right beneath the holes there are four disks, one side
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Fig. 3
black and the other side white, although their colour is invisible. The player is allowed to choose one or two holes and tum the corresponding disks upside down.
After this operation the barrel is rotated at random around its vertical axis into a
new position, yet in a way that the player cannot identify the holes he chose before. The operation is repeated ad libitum. As soon as all the disks show the same
colour on heir upper side, a bell rings and the game is finished.
Look for a strategy which guarantees that eventually all the disks will show the
same colour!
This problem is a rich mine of mathematising features. Since I should not like to
disappoint readers who would enjoy solving the problem by themselves, I have
relegated the solution to '111 Appendix.
I.3.4 Mathematising -- lwirzontally and vertically
Treffers, in his thesis3 of 1978, distinguished lwrizontal and vertical mathematising -- not sharply but with due reservations: Horizontal mathematising, which
makes a problem field accessible to mathematical treatment (mathematical in the
narrow formal sense) versus vertical mathematising, which effects the more or
less sophisticated mathematical processing. For a long time I have hesitated to
accept this distinction. I was concerned about the theoretical equivalence of both
kind of activities and, as a consequence, their equal status in practice, which I was
afraid would be endangered by this distinction. How often haven't I been disappointed by mathematicians interested in education who narrowed mathematising
to its vertical component, as well as by educationalists turning to mathematics instruction who restricted it to the horizontal one (to use Treffers' terminology)!
Eventually I have reconciled myself with the idea of this distinction, even to the
point of appreciating it positively; I do add certain nuances to its formulation, but
in a way that still respects Treffers' intentions, I believe. I have accepted the distinction because of its consequences for mathematics education, and in particular,
for characterising educational styles. I will explain this in detail when I will deal
with theoretic frameworks of mathematics education (3.1.2).
Let us characterise the distinction as follows: Horizontal mathematisation leads
from the world of life to the world of symbols. In the world oflife one lives, acts
(and suffers); in the other one symbols are shaped, reshaped, and manipulated, 42 CHAPTER 1
mechanically, comprehendingly, reflectingly; this is vertical mathematisation.
The world of life is what is experienced as reality (in the sense I used the word
before), as is symbol world with regard to its abstraction. To be sore, the frontiers
of these worlds are rather vaguely marked. The worlds can expand and shrink --
also at one another's expense. Something may belong in one instance to the world
oflife and in another to the world of symbols (road-systems, geographical maps,
geometrical figures, bills, tables, forms to be filled out, and so on). Natural
number can already belong to the world of life, while abstract addition still requires symbolic schemes. Abstract addition may have been incorporated into the
world oflife, while the cognition of its commutativity (or multiplication based on
it) still needs models which are processed and the equivalence of which is understood in the world of symbols. For the expert mathematician, mathematical objects can be part of his life in quite a different way but for the novice. The distinction between horizontal and vertical mathematising depends on the specific
situation, the person involved and his environment. Apart from these generalities,
examples on various levels are the best way to explain the difference between
horizontal and vertical mathematising.
1.3 .5 Examples
I . Counting: In order to be counted an unstructured set ofobjects or events must
be structored -- manually, visually, acoustically or mentally -- while in a more
or less structured set the available structure must be uncovered or reinforced.
This requires horizontal mathematising. Applying the counting sequence to
this (created or uncovered) structore, on the other hand, is vertical mathematisation, which, depending on the structure, can take place in a more or les sophisticated way: by using multiplication, for instance, to count a set presented
or interpreted in a rectangular structure.
2. More or less: Structoring two given sets simultaneously may be horizontal
mathematisation, while finding out which is part of which may be vertical.
Or, at another level, counting both of them may be horizontal mathematising,
while in this case reciting the number sequence, and listening to which
number precedes which, vertical.
3. Adding: A problem requiring the addition of five and three imagined marbles
to be added to each other may be mathematised horizontally by the "fingers
schema", while counting the fingers may still be vertical mathematisation. Or,
at another level, the preceding problem is mathematised horizontally by the
arithmetical sum 5+3, which is solved vertically by counting forth, or by the
replacement 4-M, or by memory.
4. Adding: If the number realm up to 10 belongs to the world oflife, then solving
8+5 by way of (10-2)+(5+2)=10+5 may be vertical mathematisation, while
dle structures of both S1lmmands have been obtained horizontally.
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ing if 2 and 9 are visually or mentally combined as linearly .structured sets and
their combination is read backwards. It may be vertically interpreted as soon
as the law of commutativity is generally applied.
6
. Addition: It is a sign of vertical mathematisation when additiondisfusedBin
1
slCtuations like the following: After the distances from A to B an ro- o
have been paced off, the distance from A via B to C, rather than bemg paced
off anew, is obtained by adding.
7. Multiplication: Five times eight (things) may be mathematised ho-zo-tally
by the rectangular scheme of 5 rows of 8 each. In vertical mathemattsatton it
is for instance read as the sequence 8, 16, 24, 32, 40.
8. Multiplication; In the long run, addition of equal summands is recog-ised anlld
dealt with as an operation in its own right -- a process that starts homonta Y
and finishes vertically.
9. Division: When dividing a number of objects among a number of persons
(dealing playing-cards to players around a table, for instance), one can start
by distributing the objects one by one, or by distributing an equal number of
objects to each person, continuing until the objects are exhausted; this is horizontal mathematisation of the distribution problem. Vertical mathemattsmg
can be seen in the search for increasingly larger shares (eventually as large as
is convenient) in order to shorten the process. This Pr,ocess is a conspicuous
example of progressive schematising (in the present case, progressive al?-
rithmising, eventually directed towards the standard algonthm of long division).
10. Combinatorics: If A and B are joined by 3 roads and B and C by 4 roads, then
how many ways lead from A via B to C? Horizontal mathematisation is -ec
ognising the structure of the problem, which may start with clever countmg,
in order to finish with vertical mathematisation by means of the product Applying this "roads scheme" in other situations may":" either horizontal or vertical mathematising, depending on the case. Replacing 3 and 4 by letter symbols is vertical mathematisation.
11.Ratio: Mathematising of visual similarity geometrically and arithmetically
may take place along a path where horizontal and v-rtical tra-ks alternate
with each other, starting with statements like: what 1s double m size here,
must also be double in size there.
12.Ratio: Putting the football scores 2 to 1, and 3 to 2 on a par with eac- other
may be refuted by comparing them with 4 to 3, 5 w4, and so on, wh1ch.'s a
trick of vertical mathematisation. Trying to find a farr method of companson
may require the use of horizontally introduced and vertically processed geometric schemes or proportionality tables.
13. Linearity: Ratio can be further mathematised vertically by the sch-me .and the
straight line graph of the linear function, as can many everyday s1tuattons involving ratio be mathematised horizontally. The rela-on bet-een cons-t ratio and straightness is a feat of vertical mathemattsatton, as is the relatton between the value of ratio and the steepness of a graph. The homontal mathe-i !
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matisation of commercial transactions involving both a fixed and a
proportionally detennined rate is followed by a vertical one of relating features of the transaction to features of the graph.
14.Figurate numbers: Sizes and relations between figurate numbers may be a
matter of h-rizontal mathematisation as long as they are geometrically represented. For mstance (fig. 4): the sum of the first n odd numbers equals thenth square. Or (fig. 5): the sum of the (n -1)-th and then-th triangular number
equals then-th square. As soon as such expressions and relations are put into
fonnulas to be processed, vertical mathematisation takes over which, in the
-ong run, is e-perienced horizontally. The inductional steps required for provmg suc.h relatmns '.'"e again of- vertical character, even though in the long run
they will be expenenced as bemg horizontal. Verbalising complete induction
such as what is used in the proof is again an occurrence of vertical mathematisation.
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• • • • • • • • • •
• • • • • • • • • •
Fig. 4 Fig. 5
15.P-scal t-iangle: This situation is similar to the preceding one. As long as the
tnangle ts given as such, then numerous relations between its elements are obtained .by horizo-tal mathe-atisation. The usual algebraic expression of the
bmom1al coefficients reqmres vertical mathematisation, as do well-known
combinatorial problems related to the Pascal triangle.
16.A-ea: Areas delimited by rubber strings on the geoboard are numerically obtam-d by honzontal -athematisation. In this vein the discovery that triangles
havmg the sru:ne honzon- basis and equal heights also have equal areas appears .to surpnse e-en qmte a few adults who had learned geometry at school.
Relating this expenence to the fonnula for the triangular area requires vertical
mathematising.
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Rather than being a didactical phenomenology of mathematical structures [146]
the first chapter was one of mathematics itself, although I left "didactical" out of
the title. Here it reappears. As far as the relation between teaching and learning
is concerned I like "didactics" better than "education" -- influenced as I am by
continental tenninology. I therefore chose it also for the title of the second chapter, together with "principles", which means trying to avoid details, certainly with
respect to subject matter. The principles I will deal with are my choice. In fact I
was led by one particular principle: doing didactical justice to the results of the
phenomenological analysis .
Didactics of a subject area means the organisation of the teaching/learning processes relevant to this area. Didacticians are organisers: educational developers,
textbook authors, teachers of any sort, maybe even students, who organise their
individual or group learning processes. The present work is not primarily concerned with didactics, and npt at all with ready-made didactics as opposed to didactics as an activity. Our'view on didactics will reflect the one on mathematics
presented in the first chapter: mathematics arising by mathematising is mirrored
by didactics arising from didactising. Notice that the intended parallelism even
extends to distinguishing horizontal and vertical didactising: from the didactical
reality to becoming conscious of it on the one hand and to paradigmatising it on
the other.
Yet there is a difference: the view on mathematics was the result of a phenomenological analysis, while mathematical didactics viewed as an activity is a postulate, inspired by the character of the subject area. When we discuss the practice
of mathematics education will be discussed (3.3), this parallelism will be elaborated in more detail; meanwhile it will remain implicit to the exposition of ideas,
which were conceived and are to be reborn by didactising. So, by didactising, justice shall be done to the results of the first chapter in the broader sense of this parallelism.
2. I GUIDED REINVENTION
"Problem Solving" and "Discovery Learning" have become catchwords. I never
liked them as mere slogans, and I like them even less since I first time saw them
exemplified. Problem solving: solving the teacher's or the textbook author's or
the researcher's problems according to methods they had in mind, rather than the
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learner himself grasping something as a problem. Discovery learning: i.e. uncovering what was covered up by somebody else -- hidden Easter eggs.
I shall not start an idle discussion on whether mathematics arises by discovery or
invention, or to what degree it does so as a science or as an art. With regard to the
subject matter to be organised I preferred the term "discovery"; in the context of
teaching, however, my choice oflong ago was "invention", which embraces both
content and form, fresh discovery and organisation. Inventions, as understood
here, are steps in learning processes, which is accounted for by the "re" in reinvention, while the instructional environment of the learning process is pointed to
by the adjective "guided". In the first chapter I analyzed mathematics as an activity. Indeed, mathematics in individual lives starts in this way. But is the learner
allowed to continue like this? Curious children will not ask for permission; indifferent and lazy ones prefer to be guided. So in order to explain how I imagined
mathematics would be learned I long ago' chose the term 'guided reinvention".
It didn't catch. Should I have done otherwise? Fortunately I did not
Why not "heuristic", for instance? Old Pol ya, in his marvellous books, guided
young Polya, according to the heuristic method, to reinvent mathematics. Heureka -- I found it -- is what Archimedes exclaimed after an unexpected discovery.
Some time ago, upon consulting English dictionaries, I could not locate the adjective
"heuristic" between "hetman" and "hew"; only in a Dutch-English dictionary did I
find it as the translation of its Dutch counterpart. So I went to a larger library. Neither
the adjective nor the noun "heuristic" were recorded in any English encyclopedia,
whereas they occurred in the oldest Dutch, French and German ones I consulted. With
the dictionaries I fared a little better but I had to resort to the biggest ones to encounter
at least the adjective "heuristic"; not until the nineteen-fifties and sixties of this century did either "heuristic" or "heuristics" emerge as a noun. Needless to say, in the continental vocabularies the corresponding adjectives and nouns have long been terms in
good standing. Of course, the lack of terms does not at all imply the lack of the concepts they cover. Believe it or not, up to a quarter of a century ago the geometric term
.. congruent" did not exist in standard English, although we may take it for granted that
British teachers and students knew as much about congruent triangles as did their continental peers.
No doubt "heuristic" entered the English vocabulary via mathematics education.
Unfortunately, it did so in the form "heuristics" where the additional "s" did not
reflect that of physics and economics but was indeed meant as a plural. Unfamiliar with the continental tradition, the anglophone authors did not mean by heuristics a didactical method but rather a set of tricks and wrinkles, helpful for solving
mathematical problems. When I made my choice, I could not possibly have foreseen this development. So it was mere good luck that I chose "reinvention" rather
than "heuristic method". I would not have liked to see ideas on learning mathematics which I tried to convey by "reinvention", drowned by what is now fashionably known as "heuristics". -
I also failed to choose another term of good standing -- the term "genetic method"
-- because it does not allu-i.e to any activity of the kamer; moreover, althou-h it
.

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Leaming as a genesis, as opposed to teaching as implementation would be a good
metaphor, were it not for the biological associations of the adjective "genetic" as
opposed to "societal".
On the other hand it can hardly be denied that the first appearance of mathematics
in cognitive development does look like an almost biological genesis -- I mean
here the emergence of similarity and of whole number. Al the same time, the unfolding of number is driven by the learner's dynamics, which manifests itself (as
I pointed out earlier) in a constructive linguistic activity. The learner reinvents
the number sequence more or less consciously and this reinventing activity may
even extend to arithmetical operations. It is well-known that some children reinvent arithmetic on their own, to various degrees indeed which depends on the
children's individual characteristics as well as on their environment. Is it therefore too far-fetched to assume that, with some support, every normal child might
be able to reinvent as much mathematics as needed in one's future daily life? This
does not in fact happen and it is difficult to ascertain whether it would be possible
or not, because in general, after a promising start, the child is not given the opportunity to reinvent anything whatsoever, at least in institutionalised learning.
Instead, knowledge and action patterns are more or less imposed, as is most of
the knowledge the child acquires by instruction. This is not the place to ask to
what degree this is justifiable in general, since we are concerned here with mathematics which, as I explained earlier on, is different. One.of the reasons why it is
different is history. Can the historical learning process of mankind somehow be
repeated by individual learners? A clever youth can reinvent quite a lot of mathematics on his own, I said. So why should less clever ones not be able to do so
with the help and under the guidance of others -- adults as well as their peers?
Why should they not be able to continue the way they began?
Not all the way, to be sure. No individual needs to run through the whole historical pedigree and conceptual hierarchy of knowledge and abilities which grew
and were built through the incessant interplay of form and content. But why
should people not get the chance to aspire, to climb and to dive to heights and
depths as steep and as deep as they can reach and afford? Aside from this there
must be, in whatever direction, some level within everybody's reach. I confess
that the horizon of these levels is the one thing in mathematics education that,
over the years, I have become ever more interested in -- its natural extension and
the opportunities and possibilities to extend it as far as feasible.

This is a view at variance with that of prescribing to people a priori the mathematics they should learn. Learners should be allowed to find their own levels and explore the paths leading there with as much and as little guidance as each particular case requires. There are sound pedagogical arguments in favour of this policy. First knowledge and ability, when acquired by one's own activity, stick better and are more readily available than when imposed by others. Second discovery can be enjoyable and so learning by reinvention may be motivating. Third it fosters the attitude of experiencing mathematics as a human activity.
Traditionally, mathematics is taught as a ready made subject. Students are given 48 CHAPTER2
definitions, rules and. alg?rithms, according to which they are expected to proceed. Only a small mmonty learn mathematics in this way. If you ask mathema-
11c1ans how they "'.'cl papers, most of them will answer that they try to reinvent
tllerr co-tents. I believe the young learner may claim the same priviliege.
To do JUSlice to the phenomenological analysis I include definitions into the
realm of -ings students are entitled to reinvent. A fonnal theory of ready-made
mathematics starts with definitions and notations. Once upon a time each of tllem
was coined. Gradually and in connection with each other tlley have been improve- ul.Jn. Indeed, knowl-ge (however acquired) must be organised and systemalised m order to be efficient and effective, and fit to be communicated to otllers and to oneself (that is.' to remember it). If the student is granted the same opportumty `:_e may trust hrrn to better grasp the need of definitions and notations.
But definitions and notations are only llle start of ready-made mathematics. I
chose these as particularly striking examples, but all ! said about them holds for
the entire course of the more or less deductive ready-made system.
History teaches us how mathematics was invented. I asked the question of whether the learner should repeat the learning process of mankind. Of course not.
Throughout the ages history has, as it were, corrected itself, by avoiding blind al-
-eys, by cutlling short numer?us circuitous paths, by rearranging the road-system
itself. We know nearly nothing about how thinking develops in individuals, but
we can learn a great deal from the development of mankind. Children should re-
7'3t the learning process of mankind, not as it factually took place but rather as
11 would have done if people in the past had known a bit more of what we know
now.
New generations continue what their forbears wrought but they do not step in at
the same. level reached by their elders. They are consigned to much lower levels
from which they resume the learning process of mankind, albeit in a modified
way. Ed-cators are ch-ged with the ?ask of helping them, not by prescribing but
by allowmg the-m- to rem vent the mathematics they should learn. I agree this is far
from easy, yet it 'is even -or- difficult to properly understand how far and why.
The first and for-most thmg 1s to become aware of this challenge and to prepare
on-self to meet •.t as well as prepanng those one wants to guide on the way of
grndng therr gmded ones: "Teacher training" will give us the opportunity to
resu-e this questlo- (albeit entangled with the problem ofreinventing didactics
by didacusmg) but m the mean time I can say that common sense and guided remvenllon of mathematics will again pave the road to the answer.
It will not be a simple answer since guiding reinvention means striking a subtle
bal-ce between the freedom of inventing and the force of guiding, between allowmg the learner to please himself and asking him to please the teacher. Moreover, the learner's .free choice is already restricted by the "re" of "reinvention".
The learner shall invent something that is new to him but well-known to the
gm de.
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2.1.l Guiding -- where to?
Governments, who consider themselves responsible for assessment, want educational objectives to be formulated in tenns of subject matter and neatly tailored abilities, the mastery of which can be tested "objectively" (as they call it) -- that is, by computers. It is for the teachers' own good, our objectivists stress; teachers should be free to arrange their teaching at will, as long as their students are deemed likely to attain the prescribed goals. Educational authorities are supported by educators who promise to shape these kinds of instruments of assessment, and opposed by others who doubt their validity in advance. A justifiable doubt, it seems to me, but this doubt does not discharge them of their responsibility to look for better tools of assessment. This is a serious problem, which we must tackle in due course.
Meanwhile we cannot evade the question of where to guide the reinventor. Not surprisingly, I shall not answer this question with catalogues of subject matter and neatly tailored abilities. To be sure, they are indispensable, that is, as indispensable as is the table of contents in works like the present one. But table of contents and indexes are not useful unless one is familiar with the content itself -- globally familiar, I mean.
Whatever the importance of subject matter and neatly tailored abilities, they are, they are considerably less so in mathematics than in any other teaching. Since I stressed mathematics as an activity my answer to the question "where to?" will be: "to an activity". In other words, the learner should reinvent mattlematising rather than mathematics; abstracting rather than abstractions; schematising rather than schemes; fonnalising rather than fonnulas; algorithmising rather than algorithms; verbalising rather than language -- let us stop here, now that it is obvious what is meant.
If the learner is guided to reinvent all this, then valuable knowledge and abilities will more easily be learned, retained, and transferred than if imposed.
For almost a century it has been known through experimentation that meaningful matter is more easily learned and retained than meaningless matter. Whereas those experiments primarily involved verbal matter, more recent research focuses on mathematics. on arithmetic, in particular. Multiplication tables are a striking example -- at least
as long as they are considered worth learning and memorising, which I still think they are. Didactical methods have changed. The oldest method was to have children build. say, the table of 7 in the natural order 1x7, 2x7, 3x7, ... , perhaps acoustically supported by unison recitation; a student who had memorised it in this way, was expected to answer "6x7=" by whispering the table up to the stop "6x7=42". This method has now yielded to methods of unguided incidental learning by unsystematic exercises of great frequency, much like memorising phone numbers and addresses. The newer method seems to be less successful than the old, but both of them are unacceptably wasteful.
More recently, attention has beCn paid to what is called "children's informal methods", such as doubling the better known 3x7 to get 6x7. Such examples of vertical mathematising should not only be pennitted (which is not at all a trivial matter if seen from the children's viewpoint) but indeed reinforced and encouraged, either directly or indirectly, that is, through the learners' interaction among each other5. ! i
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There are also indications that transfer is favoured by guided reinvention. in particular if word problems are to be solved by column arithmetic. We will present the evidence now available in due course.
The third member of this triplet deserves closer scrutiny. Once a more definite stage in the process of learning by reinvention has been attained, intermediate stages are likely to be wiped out of the reinventor' s memory; and as time goes on, the whole learning process may even be obfuscated as though it never took place. This is not necessarily a loss as long as its essentials can be restored in case of need. If not, there is nothing left to be transferred. What kind of measures can be taken to prevent this? We will try to answer this question when we deal with "retrospective learning" (2.4.5, 3).
When asked "where to" the reinventing learner should be guided, my answer was: to mathematising and its various aspects. The lack of more substantial objectives can be made up for by asking what the learner is expected to mathematise. This can be answered in one word: Reality. What kind of reality? The learner's own reality as laid open to him by his guide. This leads us to the next question.
2.1.2 Guiding -- where?
1. Counting is the child's frrst verbalised mathematics; verbal counting may even precede counting something. So many kinds of things are present in the child's world to be counted, so many are waiting to be counted, and so many new ones can be offered to be counted. Take the number of people around the table, the number of noses, eyes and ears, even feet, invisible under the table. Applying the sequence of numerals to such sets is horizontal mathematising. Wondering why some among these numbers are equal is asking a vertically mathematising question, which is answered by the extrapolating transfer of the "as many as" from one's own body to the bodies in a group. This is a valid answer, even if the underlying one-to-one relation is not made explicit, and asking to provide it would be schoolmarmish. The same holds if the "earlier" and "later" of the sequence of numerals is horizontally transferred to the "less" and "more" of sets counted or viewed as being counted. Transitivity may evolve from a horizontal mathematising experience into a vertical mathematising activity, only to become part of reality itself in the long run. For a long time in the long run, if not forever, the cardinal aspect may remain an implicit feature of number, which needs no verbalising, as it has been for millennia in the history of mankind; the same is true for transitivity of order.
The things counted are structured sets, rather than structureless ones. Such a set structure may be more or less distinctly visible in the given situation, or else, the task of structuring can be left to the learner.
The set of guests 1found the table or of children sitting in a circle on the floor is
·. ! ; ___ ; r_- --l _ - 1-'
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count themselves, each adding 1 to the number of his neighbour at the right. Or number one, as soon as it is his turn, may continue numbering himself, one more round, two, three, in infinitum -- unwinding, as it were, the cyclic. order. In a group of seven, which numbers does the frrst oue get? And what -d of numbers? And !he seventh, the third? And who gets number 100? Have 1t marked on the number line -- unwinding the circle as it were! This is a marvellous example of prospective learning (2.4.5, 3) of multiplication and its tables, as well as of division and, ifreflecte:l upon, of vertical mathematising. Strangely enough I have never seen this most natural example of guidance applied.
The set of eyes in a company is structured by the set of people. Or is it structured as a set of pairs? If counted by means of 2, 4, 6, . . . it is vertical mathematising, at least the frrst time it happens. How many seats in a theatre, how many occupied? How many places in the parking lot, how many cars? How many days until the holidays? How many pages left to read? To be sure, one can count all of them one by one, or more cleverly, by looking for structures or creating them. The basic arithmetic operations are likely to be applied in counting; they may even be given new meanings by the counting activity -- examples of horizontal as well as vertical mathematising.
Conversely, counting can serve to structure a situation: take !he third street comer from here! How many blocks are needed to build a tower as high as that? How many layers of briolts to build this house? A lot
How to count large quantities? Or is there any need to count them? It depends on what is worth knowing. On December 31, 1974, there were
13,599,082
people living in The Netherlands. How for heaven's sake did they count them?
Elsewhere their number is given in thousands of people,
6772 men and 6827 women.
Except for 82 people !he sum is correct. How far off could it ha-e been? .
2. Mathematics education developers, who believed m mathemaucs anmg m the reality, and in having this reality organised and mathematised by the learner, looked first for primordial sources. There are plenty of them. Make your ch0ice!
But isn't it a pity to choose one source, say of addition or subtraction, and to forget about the others? Well, the others may still get their turn. Once the said operations are mastered, they are to be applied. This is what they call word problems, disliked by students as well as by their teachers. Indeed, would you like to lea-e the safe harbour of routines and venture into the open sea? Each word problem is a new one and if you are not told how to solve it, you must then remember what addition a:.d subtraction mean and return to their sources -- which might have been clogged by the algor!hmic routines. .
There is an alternative. When mathematics education developers looked for pnmordial sources, they were nearly drowned by the variety they found. They could not make a choice or else they did not want to; if they did, they preferred not to make a single choice, but to have a rich variety. They did not want to narrow the wide real world on display, nor to degrade world problems to word problems. ! i
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So an all-round approach is needed, or so it seems: at the very beginning a broad display of situations to be mathematised with a view to learning addition and subtraction, telling and drawing little stories, played by the teacher or invented by the children themselves. Wins and losses, spending and saving, growth and decay. jumps on the numberline (there isa big one in any classroom), bus stops with people getting on and off, signposts suggesting distances to be added and subtracted, block towers to be built and compared -- let us stop here!
3. Addition and subtraction are learned as activities, and there is no need to obliterate this origin, to freeze the dynamics of adding 3 to 4 into the static interpretation, prescribed by New Math, where 4 + 3 is merely another name for 7; even having a lry makes little sense as long as no situations are available where such ideas can be reinvented rather than being imposed upon the learner.
4. When addition is incorporated into the learner's reality, it may serve as a source for multiplication: adding a sequence of equal summands. Yet it should not be the only source, as was in lraditional teaching. It should not even be a primordial source but rather an auxiliary one. Indeed, why should one add sequences of equal summands? This "why?" is readily answered by numerous situations where summands are equal, and all of them are sources of multiplication, and for that matter, are richer than the more formal source of merely adding equal numerical summands. {Notice that this procedure may, thanks to the richer approach, afterwards even resemble vertical rather than horizontal mathematising.}
5. Division is rooted both in subtraction and in multiplication, more formally in the latter, that is, as its inversion. Elementary divisions are most easily performed by making use of just this fact. But this does not matter here. The root of division in subtraction is twofold: exhausting one measure copying a smaller one, and distributing. Formally, both look the same, even with regard to the numerical procedures applied. But division, like multiplication, requires the use of quite different sources: real-world exhaustion and distribution problems. Experimental research' has proved that disguising long division problems as "word problems" consistently and for the entire length of the learning period not only solves the old didactical "word problems" problem with regard to division, but also favours the process of learning long division: much better results being obtained in a much shorter time. We will try to explain this experience when we will discuss learning algorithms in general.
6. When the wealth of sources in reality is concerned.fractions is second only to ratio, and both of them are closely connected with each other in real-world situations. In cognitive development -- as I have often s1ressed -- similarity, thanks to its visual expression in proportionality, even precedes number, which in turn is much more easily verbalised than ratio. Although situations involving ratio and proportionality are as easily and as early accessible (in particular visual and visualised ones) as are those involving number, they are hardly considered worth paying attention in lraditional instrUction. Why? Number is so much more easily
- ... ~ .\;i-.,,,: !i,,-e_...,i_ th,,'1.f'l -.r;-'i ?:"~'b:::.'U -r.'.f.:'dr'r.r.J:'_,r; -- -r.;f ···r;:-> '?:r·r'r:i _:. 3 i-.,, (-- .. !"' :.._,i t::.::~' ,·-·;-r-"-: 1
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ucation developers are aware of the educational opportunities lost by the lack of attention to ratio, and research done in this field is negligible compared with that on number and separated as it is from development, hardly to the point One of the few exceptions is L. Streefland's Thesis", in which the learning of fractions and ratio are interrelated. Although the emphasis, even in Streefland 's work, is on the one lraditional didactical access to fractions -- cake divided into equal portions -- it is distinguished by the much more abundant exploitation of this source. In fact, in S1reefland's approach not one cake -- or rather pizza -- is divided by fair sharing but a varying number of them, which moreover takes place at a varying number of tables. By this simple artifice this single source of ratio and fractions is explored and exploited to such a great depth that a whole world of fractions and ratio is displayed.
7. I must admit that one important source [at the same time application) of fractions and ratio has amply drawn the attention of developers: probability. But where it did so this source has most often been tapped on behalf of probability rather than fractions and ratio. I don't feel that much progress has been made since the first attempts. We know nearly nothing about the individual learning processes by which probability as a mental object is constituted and eventually mathematised. I would guess that by necessity they are unusually long and depend on an unusually great variety of facts being experienced in reality. In spite of this, the examples of teaching probability I saw developed, were aimed at relatively short term and rather compact learning processes. As soon as long term learning processes will are considered, probability will have to be discussed anew.
8. It seems to me that the teaching of probability suffers from the emphasis on combinatorics, which through the feedback of the teaching/learning process is likely to be transfigurated from a tool serving probability into a safe algorithmic shelter, protecting against it. Of course, this does not mean we should reject combinatorics as such as a matter to be reinvented. On the conlrary, combinatorics - - and figurate numbers in particular -- is a most appropriate matter for reinvention: Starting with numerical paradigms, guessing general relations, experiencing and satisfying needs for good definitions and convincing proofs, encountering mathematical induction thanks to these efforts, and using mathematical induction, first instinctively, then intentionally, and eventually in a more or less formally verbalised manner -- all this together appears to be a most efficient course in reinvention.
9. Let us pass to less elementary mathematics! Nowadays exponential growth is on the verge of becoming an everyday idea if it has not already. On the other hand, mathematising the idea of growth has been a rather recent feature in mathematics history. Logarithm tables and the logarithmic function {the integral of- as a function of x) preceded exponential functions. If history is allowed to be rt.
invented, emphasis must be shifted from the exponential as the inverse of the logarithm to the exponential as a growth function. As a matter of fact, compound interest as an instance of discreet exponential growth had been cultivated much 54 CHAPTER2
longer; extending it to continuous growth is a historically rather recent acquisition, which only afterwards was formalised by means of the well-known exponential functions. In teaching, however, supported by a rich variety of examples,

it should be a source for and a guide to reinventing exponential (and in its slipstream, logarithmic) functions.
10. As a matter of fact, "growth" as meant here, includes shrinking --negative
growth as it were. "As it were"? It has become a linguistic habit (even far outside
malhematics) when speaking about values, magnitudes, forces, trends, or whatever, to quasi-malhematise opposite of -- in any sense whatsoever -- by negative
of. This transfer of an arithmetical relation is a most striking example of formal
transfer which carries momentous consequences for contents. Acquired as it is
much earlier in the learning process, it certainly deserves to be made conscious,
and growth and shrinking is perhaps the best opportunity to accomplish this.
11. At a more formal level, the isomorphism:; mediated by exponential and logarithmic functions between addition and multiplication are a matter of constructive reinvention. Indeed, it starts operationally in the interpolation process leading from discreet to continuous growth, only to become paradigmatical, and if
subjected to reflection, a generally formulated feature ofreal number.
12. Another case of history which must be revised by reinvention is the sine (and
other goniometric functions).
It so happened that the 75 year-old woman I introduced earlier on once asked me
what sines were good for. Though for many years I had already cherished ideas
on this subject at variance with the traditional ones, I was for a moment in danger
of relapsing into the old habit of citing trigonometry. Fortunately I did not Instead I told her about oscillations, vibrations, and superposing sine functions.
This is a more convincing answer, in particular if trigonometry as such is no longer remembered.
The sine (as well as other goniometric functions) was once upon a time invented
for the computation of triangles -- frrst astronomical and later also survey-geodetic ones. In the law of refraction it was still related to right triangles. Not until
the cycloide was studied and its equation discovered, did the sine occur as a function. Its importance greatly increased when it reappeared in the mathematical description of the oscillations of the pendulum, and of the shape and the vibrations
of the taut string. At present, vibration phenomena of all kinds is the vast domain
of application of sine and cosine.
Triangles are not the frrst and certainly not the only source for reinventing sines.
They should rather be reinvented on the level of functions, more precisely, as
graphically represented functions of time: While a wheel is uniformly spinning
on its axle, a luminous point on its rim moves periodically up and down. The
graph of its variable height is easily drawu and can be shown on a TV screen; The
graph's dependence on the wheel's radius, on its speed, on the spinning sense, on
the point observed (with consequences for the relation between sine and cosine)
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produced by instantaneous exposures of the spinning wheel. The occurrence of
sines in the description of phenomena of oscillation and vibration should at this
stage simply be accepted as an empirical matter of fact, which nevertheless adds
much \0 their importance.
(Note that I used the plural "sines" more often lhan the singular "sine". The plural
sounds like one is speaking about exponential and logarithmic functions alongside with the exponential and the logarithmic function. This is a mere consequence of the spinning wheel approach, with all its parameters. As soon as one
has settled on the unit for both the radius and the circular speed, on a counterclockwise spin, on the luminous point at time 0 at the "right" end of the "horizontal" diameter, and its height measured "above" the "horizontal" axis, one has got
the sine. Is this mathematical slang? I believe it is rather terminological flexibility.)
13. Neither Calculus nor Analysis is included in this overview, not because I have
renounced comprehensiveness, but because both of them, if taught at all, should
be preceded didactically by something I propose to call Differential and Integral
Metlwds. This topic deserves a place in an early stage of the learning process
where algorithmisation has not yet been developed far enough as to allow teaching Calculus or even Analysis. It is an approach (in principle by graphic representations) initially merely qualitative and later on quantitatively refined (if possible). It aims at understanding and interpreting sucff ideas as the steepness of a
graph and areas co-ered by the moving ordinate segment, maybe even curvature,
in contexts where the drawing of the curve mathematises a given situation or occurrence in primordial reality -- we hardly need stress the vast opportunities here
for reinvention. In our exposition, Calculus is better discussed alongside with algorithms. Analysis was once invented as a safeguard against false Calculus. It
still waits for justifying as a subject at Highschool level. By justification I mean
guiding the learner to avoid the pitfalls of automatically applied Calculus by reinventing more critical Analysis. As long as I have yet to see any didactical attempt at repeating this very historical origin of Analysis, I think that "Differential
and Integral Methods" is a didactically more trustworthy safeguard than Analysis
can be trusted to be.
14. In my overview I did not strive for comprehensiveness. Two serious gaps,
however, are still lefc Algorithms, and Geometry. There are reasons to delay filling them.
2.1.3 Guiding -- how?
Guiding means striking a delicate balance between the force of teaching and the
freedom oflearning. It depends on such a perplexing manifold of hardly retrievable and only vaguely discernible variables that it seems inaccessible to any general approach. Observational reports on guiding may be a source of understanding and a help for teaching guidance. Unfortunately, most of the reports available 'li'
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are concerned with single lessons or short sequences, and little is known about
long term learning processes.
My own teaching experiences, on which I have reported incidentally and unsystematically, are concerned with individual children. Let me mention two more
syste-atic reports [73; – 54]. Th_e firs_t, though concerned with a 5 1 year-old boy's
expenments on a physical subject, is essentially mathematical reinvention. The
second publication is unfortunately only a short summary of a larger report on the
long-term teaching arithmetic and mathematics to a 12-14 year-old: fractions;
negative numbers and vectors; linear graphs, functions, equations and inequalities; powers. Let me quote myself from the above publication:
It was my instructional principle to put her in concrete. if possible, visual situations
and to let her work intuitively. Never did I explain anything to her. never did I formulate any rule, nor did I ask her to formulate rules. When I had the impression that she
;as s-II behaving as acting intuitively and insightfully, whereas she had already algonthmised the type of problem, I exaggerated the numerical data in the problem in order
to make the algorithmic procedure visible. I asked the question "why?" only if I was
sure she knew the answer.
The method was conditioned by the learner. With other children, for instance, I
insisted much more on reflection. With others I did on occasion suggest an answer, but when this happened with her I inserted this symptom of impatience into
the report and blamed myself for it.
Never -xplain anything -- how is it possible? Well, the situations this girl was
placed m were self-explanatory, and the only thing I added was something resembling the caption to a picture. For instance:
That is the x-axis;
that is a vector;
that is putting into brackets;
2.2.2.2.2 = 2
5
reads 2 to the power 5.
I can extract two principles from my report:
a) Choosing learning situations within the learner's current reality, appropriate
for horizontal mathematising.
b) Offering means and tools for vertical mathematising.
It may rightly be argued that individual instruction like the above cannot be normative for the reality of instruction in general. This is correct, not because of the
mere fact that it is individual instruction but because of the unlimited opportunity
the teacher for improvisation granted in this setting, by which he can reinforce
the learner's att-mpts at. reinventing. In the average classroom -- one would say
-- only pedagogical gemuses can act tllis way, but the average teacher needs ins?"ctio.nal plans, predesigned by able developers. Yes, indeed, but then predes1gned ma style that does not unnecessarily restrict the teacher's freedom to take
advantage of the class situation as it presents itself at any given moment, so that
tile sit-ation can be grasped by the teacher's intuition and experience, and fitted
according to the rec,cher' s princ-i?les. This requires a sv-tef1:1 of
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lation, but -- maybe even more so -- in that of a mutual relation between the learners, leaving the teacher in the background more room and time for efficient improvising. The teacher's apparent withdrawal is manifest in allowing and stimulating
d) The learner's own production, which includes the reinvention not only of solutions but also of problems -- a most effective kind of training. Viewed from the
curricular perspective, improvising is facilitated by intertwining teaching
strands, which should result in
e) Intertwining learning strands, an idea that shall be elucidated in connection
with long-tenn learning processes (2.4.5, 4).
The Five Tenets a-e are from A.Treffers
7
• I would not be able to formulate more
concisely what, in my view, guidance means in "guided reinvention". This brief
exposition covers the principles and at the same time leaves plenty of room for
further elaboration.
2.1.4 Algorithmising
1. Algorithmising aims at the mastery of particular algorithms to be acquired. Or
rather, let us con-ider it for a moment from this viewpoint Mastery of algorithms
is as crucial forindividual progress as it has been historically for that of mankind.
Algorithms allow us to act automatically for long stretches of time, avoiding the
perturbing or delaying interference of insightful thought. But algorithms are exacting; mastery means either complete mastery or none. Less than 100% mastery
can mean that everything is wrong. Of course, nobody is infallible -- not even
computers. Mastery includes the ability to identify and to correct one's mistakes,
casual ones such as computational slips, and fundamental ones such as applying
an algorithm where it does not fit. Mastery, moreover, includes lost mastery to be
recovered.
Algorilms can be taught. The learner is offered a paradigm to be imitated. One
paradigm may suffice. The paradigm donner suffices to teach the first conjugation of French verbs (except the irregular ones). One paradigm may suffice to
teach an algorithm. If not, a few more are offered, or as many as are needed. Usually it works. If not, the pupil is considered a failure.
Learn first, understand afterwards! For a long time this was tile adage for teaching, and maybe it still is to many teachers. Was the "first" ever justified by an "afterwards"? Anyway, instruction has proved to be a success story. Otherwise how,
for heaven's sake, would you explain the undeniable fact that so many mathematicians are ready to produce so much mathematics as is published today? Well,
there are people who can learn mathematics this way. I am one of this sort. Or
rather, I was -- up to the moment when I discovered that what I had learned was
no mathematics at all. (Let me forget geometry for a moment, which is our next
case.)
There are algorithmically gifted people who can learn a lot of algorithms in this 11"
58 CHAPTER2
way. Moreover some can apply them if needed, and a few can remember even
those they never applied. Still fewer somehow get the opportunity of experiencing mathematics to be quite another thing than they had thought it to be.
Algorithmically gifted people seem to form a small minority. Nevertheless, even
those outside this group still had to learn algorithms and to apply them. This is
still true in the age of computers and calculators, perhaps even more so. One has
to know which button to push, and hardware buttons are no easier than brain buttons.
Learn first, understand afterwards. I don't remember when I myself eventually
understood. Rather late, I guess. Too late, at any rate. I mention it, by the way, in
order to argue that the medicine I am going to prescribe is also meant for those
who will understand eventually.
As a matter of fact, "understandingfirst" isn't a new medicine. At the beginning
of our century (or even earlier) textbook authors started justifying the algorithms
they were going to teach (or have taught by the teachers) in advance, and so did
teacher trainers, or at least they pretended to. Did teachers follow suit? It is a useless question. "Justify beforehand" is not an effective medicine. "Justify beforehand" implies "forget afterwards". It is a tenet of the logic of separating form
from content which, if at all, works after rather than before understanding. Even
if argued beforehand, an algorithm is not better learned than if argued not at all.
Algorithmising means that arguing is left to the learner, even if it remains implicit
to the learning process for some time or forever. If argued, algorithms create the
illusion of simplicity because they were tailored to look simple. Reinventing algorithms, however, can be a tedious and time-consuming activity, and it sophisticated strategies are necessary to convince teachers, textbook authors, developers and researchers that the final result is worth the labour and time spent. Reinventing
algorithms involves a progression of schematisings, which is shortened
again and again by the reinventor, who is allowed to approach the standard algorithms as closely as learning needs and abilities urge and permit Let me illustrate
this by a few classic examples from teaching arithmetic/.
2. Palpable material, at present available to learners to fill the linguistically acquired formal number sequence with contents, is intended to stimulate schematising by packing units (fig. 6)
•1 Fig. 6
DJDACTICAL PRINCIPLES
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6 .z 7 6 0 I
Fig. 7
and tens of bead units are readily exchanged for higher ones (fig. 8).
Fig. 9
again schematised by the position card (fig. 10)
Fig. 10
which is the last step before the final schematisation of column addition.
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Actual reinvention is less streamlined than the above example. For instance, 60 CHAPTER2
choosing the order of processing the particular columns is left to the learner. The
latitude is even greater in reinventing column subtraction, where heterodox tools,
such as (implicit) negative numbers, are allowed:
43
-19
3Q
;24
where the underlined 6 means 3-9 ; - 6.
Column multiplication allows for an even larger number and greater variety of
reinvention paths. A problem such as
A directory of 62 pages has 45 names per page; how many names are there?
may first be interpreted as 62 times adding 45 -- a process that is gradually shortened and schematised by using (and at the same time improving) the knowledge
of multiplication tables, and by taking ever larger groups of tens in the multiplier
(fig. 11). The students are allowed to work on different levels of shortening and
schematising and to proceed at their own pace towards the standard pattern of
multiplication.
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DIDACTICAL PRINCIPLES
An even greater diversity of learning processes may be observed in
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the progressive algorithmising of long division (fig. 12), which has already been
mentioned in (1.3.5, 9). The progression of shortening consists of subtracting
ever larger chunks of the divisor from the dividend.
During the entire process of learning column multiplication and division, all exercises are formulated by way of "word problems", which aid rather than impede
learning. Even when column addition has become part of the learner's reality, the
redundant support from a reality acquired earlier is still valuable, or so it seems;
and the same is true for long division with respect to subtraction and multiplication. When we put Reality in its didactical context, we will deal with this question
once more. Using Treffers' terminology we call this teaching strategy integrated
progressive algorithmisation. Experiments' have proved it to be much superior to
the traditional "isolated progressive complication", as labelled by Treffers. Jn
spite of the integrated learning, almost all children reach the stage where the algorithm can be detached from the learning context and attains the status of having
a reality of its own, although in the case of long division, however, not everyone
gets as far as the standard form, which can hardly be considered a "must" in the
age of calculators.
3. Fractions are a case apart, at least in the eyes of L. Streefland as set out in his
Thesis'. Fractions are a subject of schematising, not only to a greater degree than
the arithmetic of whole number, but also in a more sophisticated way, even
though this is less so with regard to algorithmising. Mastering the algorithms for 62 CHAPTER2
fractions is not aspired to, and perhaps in many cases not even desirable, lest it
be degraded to blind mastery. There is nothing lost if the goal of algorithmic fractions is relegated to those parts of algebra where they are really needed, although
I judge it feasible even in arithmetic, provided the teacher is allowed to observe
the uunost restraint in guiding the learner.
This view is based on the teaching experiences [154] I mentioned above (2.1.3),
where algorithmic rules were never explicitly formulated, but where the learner
acquired them by tacit generalising. For instance, after a large sequence of exercises such as
made on the numberline,
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and suchlike were made without any visual support. One operation after the other
was tacitly detached from its visual source. When, after a time the child's teacher
noticed her progress and taught her the algorithmic rules for fractions, she needed
three weeks to recover from this intervention, which had seriously damaged her
mastery of fractions. I must confess that I did not teach her division by fractions
until it occurred automatically in algebra. Although she never explicitly learned
the famous rule on the multiplicative behaviour of the decimal point, she did apply it correctly.
4. In the same vein I had her reinventing algebra, that is, by algorithmising. Indeed, algebra offers a profusion of algorithms, less standardised than those of
arithmetic, and for each particular problem a greater choice. One may question
whether negative number properly belongs to arithmetic or to algebra. In one respect -- computations with letters -- it does not. On the other hand, the algebraic
permanence principle is didactically crucial for operating on and with negative
numbers. I applied it in the form I have called the geometric-algebraical permanence principle in [146, Chap.15), from which I also borrowed the teaching slrategy: first the "thermometer", then vectors in the coordinatised plane, in order to
step fmally with linear equations and inequalities into algebra. Let me give a typical example of progressive algorithmising: Up to a certain day the same girl had
the habit of solving equations like
by first multiplying by 3, and then dividing by 2. On that particular day it so happened that she drew the direct conclusion
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which then became a tacitly applied rule. I never intervened to suggest such
short-cuts. She had, in fact, discovered them long before but didn't dare apply
them overtly because she was unsure of my approval; indeed, sometimes when
she took short-cuts, she cast a playful glance as though asking for my permission.
(I attach great value to the above experience, although pecple usually shrug about
the fuss I make ofit. "People" -- I mean non-mathematicians. Creative mathematicians have experienced similar developments by themselves)
Another example of short-cuttings: Exercises like
3 5
a -a =
were initially dealt with by way of
until larger exponents provoked short-cuts. Never was any rule formulated on the
behaviour of exponents under multiplication and division.
Still another example: For a long time she clumsily calculated expressions like
before applying short-cuts toward the usual formulas.
How fast do algorithms learned in this way stick? I will try to answer this question later on (2.4).
5. I promised to deal with Calculus from the viewpoint of algorithmising. As a
matter of fact, Calculus was invented as an algorithm, or rather, one of its inventors, Leibniz, was obsessed by a passion for algorithms. To his regret and our satisfaction, this passion was satisfied once and only once, by the invention of Calculus. The advantage of Leibniz' notation above Newton's is that, with its
differentials, differential quotients, and integral symbols, it candidly exposes its origin.
I know very well that Leibniz' suggestive notations are in the black book of authors who fear the wrath of the rigorous mathematician and I agree that teaching
differential and integral methods while doing justice to Calculus as an algorithm
requires more sensibility than imparted in teacher training. On the other hand I
firmly believe we are not allowed to deprive the learner of such great values. In
[40, chap. 17), I lried a synthesis of Calculus as an algorithm and as Differential
and Integral Methods. Reinvention is here a bigger problem than in the domains
I have dealt with so far. Reinventing something that since Archimedes has waited
for about two millennia to be invented the first time is not that easy. It requires
slronger but nevertheless more subtle guidance. It seems to me that we are just
beginning to understand and tackle this problem.
6. At the beginning of this subsection I announced a more comprehensive view
on algorithms and algorithmising. Algorithms are, as it were, the shop-windows
of mathematics, that is, of ready-made mathematics. For many, if not for most 64
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people, the result of mathematics instruction is a view of mathematics as a basket
of formulas and recipes, which at a closer glance may be as full or as empty as
suits its owner. Algorithms are indispensable. One can get quite far with algorithms -- also far astray. Algorithms are tempting. Mastering an algorithm may
make one feel happy. Algorithms are easily taught and when taught as snch, they
are likely to be remembered in the same way. But how to rectify this picture of
mathematics as a basket of algorithms? How to cure this aberration?
An ounce of prevention is better worth a pound of cure. The treatment I propose
is: algorithmising. Reinventing algorithms may be a means of preventing them
from becoming calcified, and mathematics from being identified with algorithms. It may indeed be a means but will it suffice as such? Probably not, but
perhaps it will, provided the algorithm once acquired does not wipe out the memory of its acquisition. Keeping this alive, at least in a dormant state, is a subject
we will discuss when dealing with learning processes.
2.1.5 Reinventing geometry
Viewed developmentally, geometry is the direct opposite of arithmetic. Space
and the bodies around us are early mental objects, the results of structuring and
being structured. To what degree are they mathematical objects? To what degree
are distances viewed as lengths, chunks of matter as volumes, bodies as figures?
Name-giving is a first step toward consciousness. But a name for geometric similarity, for instance (which is one of the earliest geometric experiences), is still
far away at the time it is experienced. It has to be invented. The Greeks did this
by lending the general word for similarity a specific geometrical meaning, but
even nowadays this has hardly influenced everyday language. One says: copying,
reducing, enlarging. "They are the same, uh, this one is big, that one small, uh,
the same shape." Geometry, though omnipresent, too often lacks the linguistic
means for expression.
Arithmetic, however, starts as it were as a language, of counting, which eventually becomes the language of counting something. It is an algorithmic language,
which provides the first algorithm -- an automatism producing the number sequence. And so it goes on in arithmetic: Every idea is as quickly verbalised as it
arises, by existing as well as by fresh vocabulary. Verbalisation propels abstraction.
But abstraction starts earlier, as early as language or even earlier. It is implicit to
the processing of sensual and motor experience, and not until linguistic support
is needed, is abstraction adopted or created. I think that this is the reason for the
lack of geometric vocabulary in everyday language: it is simply not needed. In
our world there is a need for words like chair and table, and for words that help
us to distinguish diffeient chairs and tables from each other. There is a need for
topographical terms like above imd below, left aPd right, mside and 01 tside. GeDIDACTICAL PRINCIPLES
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and palpable force that delays, if not impedes, their veroalisation. Moreover, various modes of geometrical abstraction are competing with one another and, in
particular, with the topographical mode. Pieces of a jigsaw puzzle are freely
moveable; all that matters is their congruence class, or so it would seem. But the
picture they are expected to compose may not be hung upside down, and this information imparts such stigmas (edges, sky blue, feet, smoke) to some of the
pieces as may support the work of composing the picture.
Geometrical abstractions are much more stringent than those leading to chairs
and tables. They depend on contexts, which as a mere fact is trivial. They depend
on a geometrical context, which as such is not at all trivial. I say ! ? geometrical
context because there are many, although which one is usually implicitly understood.
Verbalisation propels abstraction. This cannot be less true in geometry than it is
in arithmetic. Anyway it need not, but this need has been awfully wronged by traditional instruction. Indeed, all means of verbalising geometrical ideas and operations have always been available in little children's language. They were simply
not used because teaching geometry did not start until the age of 12 to 13 when
children were judged mature enough for Euclid -- forgive me this venerable British instructional terminology! They were judged mature enough, although almost
everybody knew they were not. The need for propaedeutic courses was felt; in
[40] I reported on s6me marvellous ones, but they did not solve the genuine problem. Propaedeutics as such is a preposterous view on mathematics learning. If it
is true that learning is a discontinuous process, then it is so by jumps rather than
by cuts: jumps of reinvention rather than cuts of indoctrination. There is as much
difference between schooling and preschooling as there is between suppositions
and presuppositions, that is to say, none at all. But let us not anticipate too much
what we still have to say about the unity of learning processes.
If it is true that arithmetic and geometry develop mentally as direct opposites of
one another, this does not imply that one of them should be dropped in favour of
the other. Instruction should instead continue this development, that is, help geometry to stride at a pace comparable to that of arithmetic, while respecting their
opposite origins. Instruction should pay as much attention to geometry as to arithmetic, though a different kind of attention: broader to arithmetic and more profound to geometry. Whereas in arithmetic algorithms should be pursued watchfully so as not to loosen the bonds with reality -- the primordial one included. Indeed,
reality is omnipresent in geometry, both visually and palpably, waiting for
non-algorithmic schematising, and supported by badly needed veroalisation.
Tue variety of situations where arithmetic can start and stay, although much richer than traditionally displayed, is utterly poor if compared with those in which geometry arises -- nay, can be made to arise: rather than gazing at "Euclid", let us,

in order to find out, pass in review the impressive row of more recent proposals
and attempts. It is a precarious wealth, however. Choices have to be made, not as
in arithmetic, of sources and approaches, but of subject matter. There is so much
available, and the danger looms large of choosing too many. Or even if there are i'
ii
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not too many things to choose from, the choice may be too incidental: nice problems, one nicer than the other, and each of ready to stimulate reinvention but as
well ready to be lost in the stream of new experiences, a hapax /egomenon, a
nonce experience. Cautiously applied, the operational structure of arithmetic may
provide a global guide-line for reinvention. Geometry can be structured deductively, although this is of no use unless deductivity itself is reinvented. Little deductive steps -- why not? -- it is common sense; perhaps even a bit of local deductivity. But how to link together nice pieces of geometrical reinvention, to get
chains of long-term learning processes, rather than leaving the learners with
heaps of loose ends? I think that this is the very problem in teaching young children a kind of geometry that is likely to come too late for older children.
Posing rtice geometrical problems is particularly gratifying when little hints suffice to solve them Take, for instance, the following three:
Find the centre of a given circle!
The diagonals of a parallelogram bisect each other -- why?
The angle inscribed over the diameter of a circle is a right angle -- why?
The hint I have in mind is quite simple: cut out the figure! When cut out, a figure
can be manipulated, folded, moved, turned around. Symmetry proves a lot of
things, and as a principle it can concatenate quite a number of geometric reinventions. But this does not suffice to solve our problem. Or more pointedly, problem
solving does not solve the problem.
Let me recall the tenets d and e from (2.1.3)! One of them is the children's own
production, which includes reinventing not only solutions but also problems.
This tenet has successfully been applied in arithmetic; so why should it not be applied in geometry? The wealth of situations to be geometrised is an ever-lasting
guarantee.
The other tenet was the intertwining oflearrting strands. The strand "numberline"
is long enough, and it can be broadened to the co-ordinatised plane, which in turn
can be restricted to the geoboard. Ratio and measurement have geometrical components, which should be taken seriously: similarity and congruence.
But don't forget about it the very first key. I mean tenet b: offering means and
tools for vertical mathematising. In the case of geometry this means, as emphasised above: linguistic means and tools to geometrise spatial experience.
2.2 BONDS WITH REALITY
2.2.1 Primordial reality
Mathematics has arisen and arises through mathematising. This phenomenological fact is didactically accounted for by the principle of guided reinvention.
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DIDACTICAL PRINCIPLES 67
more perspicuous mathematising.
Mathematising is mathematising reality, pieces of reality. But reality is not just
one thing; it is as many things as there are people, and to one people it may be as
many things as there are states of internal understanding and external circumstances. Anyway, as soon as mathematising is didactically translated into reinventing, the reality to be mathematised is that of the learner, the reality into which
the learner has been guided, and mathematising is the learner's own activity.
What is reality to whom depends on many variables, as does what is mathematical, what is non-mathematical, what is mathematical enough or not enough, what
asks for more mathematising. Our world, described in a vernacular that shares
names for arithmetical and geometrical ideas with mathematics language, has already been mathematised so extensely that we are no longer aware of it unless it
is brought to our attention. Much mathematics in our reality is common sense and
taken for granted, and again, how much and how great its influence is depends
both on the individual mind and its specific environment
In the preceding section I spoke casually about primordial reality. Let us focus
on the learning process for the multiplication algorit/im! In our setting questions
like
A directory of 62 pages has 45 names per page; how many names are there?
are likely to be answered by clumsy procedures, to be shortened progressively,
such as adding 62 times the number of names per page (while adding 62 numerical summands 45 is already within the range of the learner's reality). What, then,
is the rationale of this mode of instruction? Why, if at all, should progressive algorithmisation take place in an integrated way, rather than in splendid isolation?
-- to use the former terminology. Why should clear essentials be disturbed by lot
of noise?
Simple as they are, both this particular piece of instruction and the query it provokes is no more than a didactical paradigm for a principle intended to prevail on
mathematics instruction in general. Noise is eliminated as soon as newly invented
mathematics is published, and for it has long been a didactical principle to avoid
noise as much as possible where basic abilities and concepts were ID be taught.
But when mathematics once learned is to be applied (even as schematically and
straightforwardly as it is expected to happen in word problems); then noise could
no longer be escaped, and its sound is likely ID become the more disturbing however less noisy the previous learning process had been. The world is noisy; mathematising the world means looking for essentials, sensing the message within the
noise. This, too, has to be learned, that is, reinvented by the learner, and the earlier the better; once the learner has fully been indoctrinated by ready-made
schemes and algorithms it may be too late.
This explains why mathematising should not start in a necessarily adjacent reality
but in what I called a primordial reality. In still another respect our example is
paradigmatic. Is there anything more obvious than having multiplication reinvented as the addition of equal summands? But as one stops to think about it, one 68 CHAPTER2
asks oneself why, for whatever reason should one add equal summands? The answer is obvious, that is, to whoever knows it But in fact there is not one answer,
or rather, if one, it is an abstraction from many. Primordial reality yields a rich
variety of answers to this question. Both are matters to be reinvented, the question
and its many answers.
With Treffers I distinguished horizontal and vertical mathematising. Vertical
mathematising is the most likely part of the learning process for the bonds with
reality to be loosened and eventually cut. Indeed, in the long run algorithmic habits, say in learning long division, will be formed to enfeeble the operative association of division with distributing or exhausting and to relegate it to the subconscious or even unconscious. It is this danger that should be prevented by a long
lasting insistence on formulating problems in the primordial reality. Not only the
product of the learning process but also the process itself are worth remembering
and should remain accessible to the memory if they are needed. We will raise the
question of whether this policy is indeed efficient enough when we deal with
long-term learning processes (2.4).
2.2.2 Odd worlds
Strange as it may seem, it cannot be concealed that the bonds with reality may be
endangered as early as horizontal mathematising albeit in another way than during vertical mathematising arises. The imminent danger is that of the wrong
bonds -- wrong in principle rather than due to grave or minor errors. In order to
explain what I mean let me cite a most interesting research entitled 'Que/ est
l' age du capitaine?" by the "E/ementaire" team of the IREM (Institut de la Recherche sur l' Enseignement Mathematique) of Grenoble.
The team had posed the question
On a ship there are 26 sheep and 10 goats. How old is the
captain?
to 97 CE 1-2 pupils (7-9 olds). 76 among the pupils succeeded in figuring out the captain's age on the strength of these data.
Encouraged by this "success" the team administered a battery of six similar tests to
seven classes CE and six classes CM (9-11 years olds):
I have 4 peppennints in my right pocket and 9 in my left. How old is
my father?
A shepherd has 125 sheep and 5 dogs. How old is he?
A shepherd has 360 sheep and 10 dogs. How old is he?
Jn a class are 12 girls and 13 boys. How old is the teacher?
On a ship are 36 sheep; 10 fall overboard. How old is the captain?
There are 7 rows of 4 tables in a classroom. How old is the teacher?
Every problem was followed by the question: What do you think about this problem?
I will spare the reader the statistical result. Anyway. the harvest was indeed reaped
(From: N 19, decembre 1979, or Bulletin de l'APMEP, no.323, avril 1980.)
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For illustration, the Grenoble team has added a collection of word problems gathered from old and more recent French textbooks; these will give you the creeps.
It is a welcome suggestion that instruction (and in particular instruction of acertain kind of word problems) is responsible for the non-mathematical or rather
anti-mathematical attitude adopted by the tested pupils, but I am not sure whether
one should accept it at face value. Is instruction the real culprit, or is it not too
easy to accuse instruction of having sinned, albeit only by negligence? Aren't
there more active causes that might have produced these failures, and if so which
ones? Is the culprit really those word problems that give you the creeps? I am not
that sure.
In my youth the Sunday supplements of most newspapers and most weeklies contained a kind of puzzles -- I don't know whether it still exists -- say, a drawing of
an English garden -- a foliage vault with garden furniture and tools underneath -
-- with the caption question: "Where is the gardener?" By turning the drawing or
one's head and looking closely, one could discern somewhere between the
branches, leaves and sunspots the contours of a person who looks like a gardener
hidden in the foliage. This then was the answer to the question.
Why do we accept such a solution and at the same time refuse similar answers in
the case of the captain's age? A gardener, even when hiding, does not hang himself askew, feet up and head down, in treetops. This isn'..t normal. But our question deserves to be taken seriously and tackled thoughtfully. It touches, or so it
seems to me, the fun-t'amentals of mathematics and mathematics teaching. The
word logic is likely to be dropped when discussing this matter. The problems,
parodied by that of the captain's age, are indeed vested with a kind of logic that
uniquely determines their solution, whereas problems like those of the gardener
seem to assume nothing but visual abilities. I cautiously said "seem" since there
are more sophisticated ones of this kind that do require intelligent reasoning combined with visual aptitudes. Anyway, the appeal to logic is too easy a solution.
"Logic" and "lack of logic" are bromides to lull in nagging sceptics.
I have dubbed this prob/ematique: text and context "Context" refers to (l.2.8.3),
where it is dealt with in particular in the sense of rich contexts. "Text' means a
linguistic vehicle, in particular of word problems or text problems as they are occasionally called.
The context of the gardener problem is well-defined: the Sunday puzzle page --
the same kind of problems week after week, one generation telling the next how
to solve them. The textbook problem "26 + 10 ="is first of all situated in the context of mathematics instruction, where one learns what these symbols mean, but
its most important feature is that it remains meaningful outside any context, or if
one prefers another formulation, that it fits or can be fitted into any .context This
is, indeed, what characterises mathematics, or at least its formal aspect.
Word problems are different:
John has 26 marbles; he wins 10 more; how many does he have now?
John is a boy, boys like to play marbles -- girls do too -- it is a game where you 70 CHAPTER 2
win and lose, and with each marble won or lost the number of marbles in your
pocket increases or decreases by one unit. The context is self-evident: the problem stems from the plain reality of the child's life, and whoever does not grasp it
is advised 10 stop playing marbles -- nothing prevents him from continuing to do
arithmetic.
But let us now turn to butcher Smith whom I crune across in a textbook (or was
it on an achievement test?). Here the problem is not about 26 marbles but the 26
kg of hrun Smith had in his shop and the 10 kg more he ordered, and the question
is about how much he has now. Again a context, let us guess, from the plain reality of a butcher's life. Obviously there is some isomorphism assumed between
two worlds, the child's and the butcher's world. "John" is replaced by "Smith",
"marbles" by "kg of hrun", "wins" by "orders", and so everything is settled. More
closely viewed, however, this isomorphism is all but perfect. Marbles are won
and lost, while hrun is bought and sold -- all right. But marbles won are put in
your pocket, whereas the ham ordered by phone does not fly instantly into the
shop, and when it arrives some of the 26 kg of hrun -- I don't know how much --
will have been sold. Indeed, this is why the butcher ordered more.
The context of the butcher problem, rather than the butcher's reality, is the textbook, or more precisely the vast domain of pseudo-isomorphic images of the
marble problem, and everything I have said about the butcher problem applies as
well to the whole treasure of word problems: the context is the textbook, rather
than reality proper, or in other words, it portrays a world of pseudo-isomorphisms.
In the textbook context each problem has one and only one solution: There is no
access for reality, with its unsolvable and multiply solvable problems. The pupil
is supposed to discover the pseudo-isomorphisms envisaged by the textbook author and to solve problems, which look as though they were tied to reality, by
means of these pseudo-isomorphisms. Wouldn't it be worthwhile investigating
whether and how this didactic breeds an anti-mathematical attitude and why the
children's immunity against this mental deformation is so varied?
Arithmetic applies because of true isomorphisms. In [146, p.58], besides the usual marbles added to each other in play I listed one under the other
5 steps and 3 steps (of stairs)
5 days and 3 days
5 km and 3 km
5 florins and 3 florins
5 times and 3 times
a list I could have continued ad lib. Even though the mental process of adding 3
to 5 belongs to the learner's reality, these elementary additions still have to be
performed and reflectingly understood in primordial reality in order to grasp their
isomo1phisms. Obviously this cannot be done in a mechanistic setting, which is
likely to generate pseudo-isomorphisms, nor in the New Math fashion of cardinal
'P-·•" •rc.-.i-.-nof :y-:_:n.1:--r _'PT: J -:' .. --cUtl-J71_
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is required. The tension between the contexts of reality proper and the textbook
does not suffice to explain such phenomena as those signalled by the Grenoble
team. There is still another context implicated in the evolution of that -- mathematical, or rather anti-mathematical -- attitude. Before giving it a suitable name
I will introduce it by a few exrunples.
"How old are you?" The answer -- "four" -- is accompanied by the gesture of lifting four fingers. The child does not know what "age" means, nor what the fingers
lifted have to do with the "age", with a number of years, nor with the question as
such. According to children's folklore a lady-bird is as old as one can count dots
on its wings. Counting the number of rings of a cross-section, one learns the age
of a tree, even though one cannot explain why. One asks the cuckoo the number
of years one may still expect to live. The picture of a birthday cake together with
the question "how old is she?" provokes counting the candles. A fly-agaric pictured as a gnome's home betrays its owner's age by its white dots. And, taking a
long step into the adult world, you come face to face with hosts of people, juggling the Number of the Beast of the Apocalypse and the dimensions of the
Cheops pyramid and the Stonehenge monument. This is a venerably old preoccupation, called hermeneutics if concerned with .. the Holy Scripture, but as a matter of even older than the art of reading and writing. Looking for marks on flowers and leave,,. in order to know what ills they may cure, foretelling the future
from the flight of birds and the livers of sacrificed animals, unveiling secrets of
J.S.Bach's profundity by counting beats and bars in his compositions, and -- let
us not skip it -- expecting discoveries in psychology and education by means of
regression analysis -- it is all cut from the same cloth: hermeneutics, or else the
term I would propose, the magic context. Stories about Pythagoras and his school
tell us they practised witchcraft through numbers. Is it too bold an assumption to
credit children with this kind of mentality? From my own childhood I remember
this context only too well, which I succeeded in eliminating from my mathematical activity. But what about many other people? To solve a problem they look
for secret marks, for signals hidden by accident or intention, and in particular, for
numbers that to put them on the right ttack.
How old is the captain? I gave this problem (with 18 sheep and 16 goots to make
it less easy and thus more stimulat1ng) to an 8 year-old girl who lived at that time
in a world of fairy tales and sorcerers where she played her small and large roles,
with gnomes sitting on each toodstool. She beamed with joy because she had discovered the secret, and had calculated the captain's age. Thanks to her enthusiasm she was unaware of the mockery of her two years older brother, who is as
sober-minded as she is imaginative, and so no illusions were cracked. When I
tried to explain to the boy how others reasoned when they calculated the captain's
age by adding, subtracting, multiplying, and eventually by dividing, he shook his
head: "I cannot understand what you mean; this yields at most the number of
sheep per dog."
This brings me to tell of another experience with these two children I had half a year 72 CHAPTER2
earlier. At the end of a stroll we came to a huge sandpit, bordered by a circle of thick
poles -- actually cuts of tree-trunks. I asked the children how many poles there were.
Resolutely the boy paced around the pit, counted 38 steps (each step being 7 poles according to his reckoning), and then concluded 38 x 7 = 266. (It is a nice example of
mathematising but this does not matter now.) Meanwhile his sister had effortlessly
solved the whole problem. She said "60", and when I asked .. why?", she answered: "It

looks like a clock-dial, does it not, and that is 60 minutes.
At the risk of being pronounced silly, I confess I like the girl's answer better than the
boy's. It is the most beautiful nonsense. which promises more geometrical imagination
than does her brother's solution. In her fairy tale world she did with her beaming eyes
what he had done in his prosaic world with his feet. I am sure that, had she been two
years older, she would have parcelled out at sight an arc corresponding to five minutes,
counted with her eyes the number of poles per arc and multiplied it by 12 (the direct
converse of her brother's kind of mathematisation).
How old is the captain? The 26 sheep and 10 goats on board are like the data used
by the astrologer to foretell the future. Children's realities are worth studying, in
particular the magic context as well as its influence on learning processes and the
shaping of attitudes. Framing the children's world in rich contexts may be a preventative measure. This has been tried in various ways. Plain word problems, if
disconnected, are probably not rich enough a context to prevent the pseudo-isomorphistic temptation; they should be interrelated by constructive and reflective
measures. Streefland's approach (cf. 2.1.2, 6, footnote) towards fractions, within
one single but deeply excavated context, is promising, although it has not been
continued as far as to reach formal fractions. IOWO's abundantly rich contexts
have been tried out successfully but dissemination of these ideas strongly depends on the teacher's conception of which realities are accessible to the learner.
We shall come back to this problem.
Let me illustrate it right now by an experience had by an IOWO staff member in a 4th
grade class with a worksheet which included:
Railroad fares according to the number of kilometres travelled (half-price for the under ten years olds), a railroad map of distances, a story about Mum's train trip with
John (10) and Mary (9) from Hilversum to Enkhuizen, and the question how much she
must pay?
Annette could not solve it, even after she had correctly found the distance HilversumEnkhuizen.
"What is the matter?"
"I don't know how old is the mother."
"What do you think?"
"I would guess: 39." (Her own mother's age.)
"All right. It is a good guess. Continue!"
So she added 39 + 10 + 9 = 58, looked up the price in the 58 km row, which at that
time was? 8.15.
"And if the mother were 50 years old, she would pay more?" our staff member asked.
"Of course she would."
"And the old people, the grand-dads and grannies, they pay even more, do they?"
"Sure, they do."

at all. What is the reason for this? My guess is that the reality to be conveyed by
this context is nothing but an adult prejudice. One should ask oneself at what age
the railroad trips, tickets sold at the windows, and the very thing travellers pay
for, represent a valid context. Or rather, under what condition, since it is not the
mere age that counts. Would practical experience be a better guide for answering
such questions? Our knowledge of instruction is covered by a thick layer of prej-
udices, but I am afraid that this holds as well for the so-called practice. Moreover,
people with much practical experience are prone lo premature interventions,
glossing over rather than revealing and bridging chasms between the child's and
the adult's realities.
Our colleague did not get the opportunity lo do this, or he simply didn'tdare because the chasm looked too wide. Remedial teaching starts at the roots, which
may lie unexpectedly deep. Yet let us ask what should be done in the Grenoble
case! The remedy lo prescribe, which for that matter, could have already been administered along with the tests, does seem readily available. I mean problems
where the applied methods will not work, such as:
250 sheep and 120 goats aboard -- how old is the captain?
360 sheep and one dog (or two, rather than the improbable 10) -- how old is
the shepherd?
I confess I arn not sure whether this would be of much help. The children might
object that there are not enough data available to answer the new questions while
insisting that the previous ones did provide enough indications. This is no miracle. Magic sometimes works and sometimes does not. Or do the fresh examples
provoke what is called a cognitive conflict? "Cognitive conflict" is an adult contraption. Cognitive conflicts have first to be experienced as conflicting realities.
If there are no bonds with reality, then conflicting realities cannot provoke cognitive conflicts.

2.2.3 Rich contexts

Fraught with relations was the term I chose in [40] for the mathematics I wanted
to be taught. In the mean time the term has become mathematics in rich contexts.
I used the term several times; for instance in 1.2.9.3, and again in the last subsection, I opposed rich and pool"• contexts lo one another. Let me briefly sketch the
evolution of the terminology!
As I pointed out in 1.3 mathematics, unlike any other science, has arisen and still
arises in common sense reality -- broad-minded common sense and broad-minded reality as I explained in l.1.6. Where it was once invented, mathematics
should now be reinvented. Contexts then means that domain of reality, which in
some particular learning process is disclosed to the learner in order lo be mathematised.
When in the early nineteen-seventies developers, in particular for primary education, looked within the reality they judged accessible to the learner, for sources '
!

of specific mathematics to be learned and taught, they were overwhelmed by so
rich a variety that they saw their task shifted from finding sources to choosing
among them. Not: choosing one and rejecting all others. But: allowing for a vast
space and a continuous flow, creating what was called context-rich mathematics
instruction. Language is more than mere communication; it influences thought,
whether one is aware of it or not Somehow, maybe under the influence of "rich
structures" (see 1.2.I), context-rich became rich contexts. Rather than discussing
the difference, I shall explain the situation, that is, what kind of wealth is meant
by "rich"?
Bonds with reality is the title of this section, shortened from Creating, strengthening, and maintaining bonds with reality. This, then, is what rich contexts have
to effect as domains of reality disclosed to the learner to be mathematised. We
once tried to group our experiences with contexts. Let me copy that unassuming
list!
First on the list, and at that lime the most striking item, the location: a meaningful
gathering of situations, which can be handled separately, or in more or less close
connection with each other. Our best known example
10
was "Waterland", a Disney-like island-- its large picture hung in the first grade classroom -- where lots
of things were happening: there were landings for ships to moor, bus stops with
people getting on and off, roads leading hither and thither, road signs placed and
to be placed, block towers and mountains to be climbed, networks of streets to be
walked on, play grounds, windmills -- a profusion of situations, where malhemat'
ics could be discovered and acted out -- an incentive to stimulate the imagination
of both teachers' and children.
A counterpart of this kind of rich context was the story, that is, rather than a gathering, something that, reeled off as a succession of worksheets, is structured in
lime. It may be a true story or fiction, a classic or invented ad hoc Examples':
"Gulliver in Lilliput", where one string -ratio- is played on over and over, or
"Grains on the Chessboard", with its powers of 2, or "Around the World in 80
days", where one tone -- travelling around the world -- is coloured by many overtones, or "Ralph the Buccaneer" with the area of islands as the main lheme, or
"Ship Ahoy" (geometrical orientation implicated in the rescue of a ship), and
many others.
The third example was the project, which means reality to be created, such as
building a bungalow (7th grade), or a peep-show box (kindergarten), or collecting and classifying all kinds of commercial packing material, or constructing regular bodies.
The fourth was the theme, a mathematically oriented strand of subject matter with
varying relations to reality, such as "Light and Shadow" (7th grade), "Flying"
(lhematising goniometric functions), "Exponential Functions" (wholly integrated
into the context of growth), "Matrices" (in various contexts, among which transition processes), "Two Variables" (that is, surfaces as though abstracted from
:-:;v-.-'f:':.'-)-f---.

time been tried out extensively and successfully. Let us call it: clippings -- mainly
from newspapers and weeklies, but also from books and other media. Anyone intent on it has little trouble discovering a lot of mathematics in the printed and published reality -- meaningful as well as nonsense mathematics. Thanks to this large
quantity, die small percentage that lends itself to mathematics lessons is still significant. Rethinking the author's thoughts, analysing whether they were right or
wrong, pursuing his ideas in order to review, strengthen, weaken, modify them,
may provide valuable mathematising activities in the upper grades and in teacher
training. Let me mention that each year one of the problems at on our final mathematics school examinations is of this kind. But even in the middle and lower
grades a variant -- let us call it pseudo-clippings -- may be useful. A pronouncement by somebody, or a quarrel in a group about some question, an explanation,
or pairs of contradictory explanations, in particular, if the mathematics involved
is still hidden, may be a useful context as soon as children are able to communicate. Should this happens in the classroom it deserves to be exploited, but inventing such stories can be an equally good idea. This then is what I would call pseudo-clippings.
Contexts were defmed as domains of reality disclosed to the learner in order to
be mathemalised. In the cases of location, story, project, and theme such domains
are purposefully -- and sometimes artificially -- delimited by the teacher or developer, who wants the learner to reinvent certain processes and products of
mathemalising. The case of clippings is a bit different Here it is not a domain but
a small piece that is cut out, although its paradigmalical value for mathemalising
and for acquiring a mathematical attitude may be enormous in comparison.
But in all cases it should be kepfin mind that context is not a mere garment clothing nude mathematics, and mathemalising is quite another thing than simply unbuttoning this garment. Or, to give a former metaphor a new twist: Viewing context as noise, apt to disturb the clear mathematical message, is wrong; the context
itself is the message, and mathematics a means of decoding.

2.2.3.1 Contexts versus "material"

No doubt once it was real progress when developers and teachers offered learners
tangible material in order to teach them arithmetic of whole number. (lo the case
of fractions it never did work properly.) Even now I would not like do without It
Whether unstructured, at the start of arithmetic, or structured, as on the abacus, it
is indispensable as part of the primordial reality. It is a small part, to be sure, one
source among many others of whole number arithmetic. Relying solely or too
strongly on it is dangerous, and since this is the habit in some kinds of instruction
I feel obliged to warn against it. Low achievers have great difficulty detaching
arithmetic from the palpable material -- according to remedial teachers and developers of remedial teaching. This may unfortunately be true, that is, for a kind
of inflexible instruction where from the start onwards learners have been tied too 76

tightly to palpable material. Flexibility should be allowed, and if need be, taught
rather than fought. Moreover, one may ask, what is the use of low achievers
learning an arithmetic they are judged in advance unable to detach from tangible
material.
Offering the learner tangible material was progress when drawing and writing
tools were too expensive to be handed out lavishly in the classroom, when teachers on a platform watched classes of children sitting on benches, when instruction
had to look like military drill and children's initiative and activity were suppressed and nobody even thought about Jiving contexts. Tangible material has a
tremendous value in teaching geometry, provided the learner is allowed to structure it himself. If we condemn the learner to the prison of prestuctured blocks --
whether counting ones or logical -- we should not wonder why they are not able
to put the mathematics learned to good use. The best palpable material you can
give the child is its own body.
At this moment I cannot resist quoting myself from 2.1.2, I:
The set of guests around the table or of children sitting in a circle on the floor is structured by its cyclic order -- .. didn't you forget anybody"? Or the children may count
themselves, each adding 1 to the number of his neighbour at the right. Or number one,
as soon as it is his tum, may continue numbering himself, one more rolDld, two, three,
in infinitum -- unwinding, as it were, the cyclic order. In a group of seven. which numbers does the first one get? And what kind of? And the seventh, the third? And who
gets number 100? Have it marked on the number line -- unwinding the circle as it
were! This is a marvellous example of prospective learning (2.4.5. 3) of multiplication
and its tables, as well as of division and, if reflected upon, of vertical mathematising.
Strangely enough I have never seen this most natural example of guidance applied.

2.2.4 Paradigms

"Paradigm" would have deserved a section or subsection of its own in the first
chapter, were it not that there, among the manifold occurrences of the term, it
would have looked like a needless repetition. The present chapter, however, offers an opportunity to reconsider paradigms as places where bonds are attached
to reality.
"Paradigm" means "example", although not as used in "for example", where it
properly means an after-example. If such a word existed, I would rather say that
a paradigm is afore-example. As far as I am familiar with the history of instruction the term "paradigm" originated in teaching foreign languages (including native language which may have looked or sounded to quite a few of the learners as
though it was a foreign one). In (2.1.4) I cited the French donner as a teaching
paradigm for the first conjugation, that is, for all the verbs ending in er (the irregular ones, of course, excluded). To what degree are paradigms really needed?
People acquire a great deal of their idiom -- structural paradigms included -- before, without, and even in spite of institutionalised instruction. This certainly
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learn languages by listening and by reproductive reinvention -- guided among
other things by corrective measures, and possibly reinforced by increasing
awareness -- and this process continues when listening and speaking are supplemented by reading and writing. As far as I am familiar with modem methods,
they take advantage of this experience by trying to approach an everyday life
style in language teaching. Structures, such as paradigms, rather than being imposed, are given the opportunity to be reinvented by the learner. A striking counter-example is our Dutch spelling. It is one of the easiest of the world. Its notoriously difficult part is the one governed by rules set in advance.

2.2.4.1 Paradigms imposed

Let us leave language teaching! The term "paradigm" has meanwhile conquered
wider instructional ranges, which is why it was applied frequently to mathematics
instruction in the first chapter. Why paradigms? Mathematics is a universal tool,
the most flexible instrument to be handled. But universals are no teaching matter,
or at least not in the manner of what is called concept attainment Universals are
learned by paradigms, and the most efficient paradigms arethose which allow the
easiest or the widest transfer. Improperly chosen, however, an intended paradigm
can as well block the transfer, by its singularity or by lack of flexibility, or simply
because of a wrong view on learning by means of paradigms.
In everyday life, in language instruction, in science teaching, etc., paradigms are
as abundant as they are unengaging. Mathematics can do with a relatively small
number of paradigms, which engage the learner unambiguously, or so it would
seem. Is an algorithm a paradigm? Of course not. But, traditionally, algorithms
are taught by paradigms, long division, for instance, by means of a numerical example (or if need be, by a few of them), which promises transfer by inviting imitation. Some children have a knack for algorithms and for learning them in this
exemplary way, and some among them will even, at least implicitly, understand
why they work, even though they are *not* interested in knowing why. Others will
fail to learn algorithms this way. Does it matter? Long division is now the job of
pocket calculators. But long division is only the bitter end. Let us look for the
roots of arithmetical algorithms!
Arithmetical algorithms used to be taught by numerical paradigms. I chose the
plural "algorithms", although properly understood, for one single arithmetical
operation one paradigm should be just enough, or if really more are needed, they
should virtually be one, in the sense of being somehow isomorphic. Indeed, nonisomorphic ones would instead endanger the intended uniform algorithmic-paradigmatic character and as a consequence confuse algorithmically less gifted
learners. The ever lasting trouble, however, with the strategy of teaching arithmetical algorithms by numerical paradigms, was that, a from column addition onwards, one paradigm was not enough. Neither teachers nor developers were
aware of the problems created by a multiplicity of paradigms needed for one and 78

the same operation, a multiplicity that in tum creates just as many sources of failure. I said "from column addition onwards", which would seem a bit exaggerated.
But do examples like
24
38+
24
35=
really suffice as paradigms (at least for additions below 100) or are not sums

taught in agreement wilh this strategy such as
24
. - i+
24
Jl.. +
5
32+
5
38+
very likely to suggest to learners that they are just as many fresh paradigms, and
wouldn't teachers think they are needed as such in teaching? And what about, say
24
39
19
24
3
12
24
9
19
2
24
-2
7
24
19
a list of new-looking paradigms, still far from complete yet long enough to frustrate learners, and for that matter their teachers as well? Long enough, I said, and
yet I have restricted myself to additions below 100 of two or three summands.
What about more summands, about more empty positions, multiple transfers, and
soon?
Although the other operations only involve two term, they certainly do not rank
second to addition with regard to the variety of what may be looked upon by the
learner as brand new paradigms. Think about all lhose perfidious zero digits involved in the other arithmetic algorithms, about decimal points, about comparing
decimal fractions with one another -- each new difficulty to be conquered by a
new rule, introduced by new paradigms! ·
How real are these difficulties? To tell lhe truth, I did not invent this story, but
took it from the literature; there has been plenty research to prove again and again
that something is wrong with teaching column arithmetic, while little attention
ha_s been. paid to ":hat is wrong and why, though one thing is clear: that teaching
anthmellcal algonthms by numerical paradigms heavily clashes with the essence
of what arithmetical algorithms are supposed to mean and to perform. Even if
teaching column arithmetic is not such a failnre as some researchers have concluded, learning arithmetical algorithms by numerical paradigms is a tremendous
waste of time.
I am not against numerical paradigms as such, and certainly not where they are
-eaningful -- see the numerous examples in (l.3.3 and 1.3.5). Numerical paradigms can be the key lllld stepping stone to general problems and algebraic relatJ, , .,,-,. V.:.
0
"I' r:-;-<-- ·:·;_;,"_S- r\1···,_' ·r-~" "-" ,· "- ·-·1-:-""". _,1 ,.-, f-' C:f ":..•-:'-'!". ":·,
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this principle must not be interpreted by numerically formulating ready-made
general rules. In the process of reinventing, algorilhms arise by progressive schematising, with the stress on shortening, and the very thing paradigmatical in this
process is progression and short-cuts rather than numerical examples.
Word problems are a similar case, or even worse if their context is a textbook
page, headed by one item that looks like a paradigm, that shows how to solve all
the other items on the same page. To be sure, it works. It does work until the context is extended to the textbook as a whole and the problem arises which page applies under which circumstances. Didacticians of yesteryear liked to embroider
all word problem paradigms on frames, intended as algorithms and looking like
them. Or at least they tried, albeit unsuccessfully since learning these pseudo-algorithms was even harder than understanding the word problems they were to facilitate.
Algebra looks a bit different, or does it? Cnneiform texts show how Babylonian
mathematicians solved linear and quadratic equations, or pairs of them with two
unknowns, yet all they left us is numerical examples. Did teachers use them as
paradigms, or did they explain to their students in general terms what to do? We
don't know anything about Babylonian didactics. Do we know any more about
teaching algebra today? We know textbooks -- good ones and bad ones -- but
what do we know about. teaching? What do we know about how paradigms are
used by teachers and ihterpreted by learners in instructional practice? Are they
seen as a form to be filled out or, even worse, as a sequence of acts to be imitated?
In [40, p.632] I gave an example of how problem solving is understood by students who might have been taught in this fashion. Let me repeat it!
To prove that ,,;z + 12x has a minimum and to determine the x where it is reached.
(I) 3,2+12x = 3(x+2)2 ↔ has a minimum, which is assumed for x= -2. Reason:
(x+2)2 must become 0 ---: -2 if this is multiplied by 3 ---still 0 then 4 added and
one gets the minimum which is assumed: 4.
(II) The expression 3,2 + 12x is a quadratic equation. If in both parts 3 is put outside brackets the expression does not change. Then inside the brackets is (x
2
+ 4x,l ,
that does not equal (x+2J2 for we have added 4. To make it correct, -4 must be
subtracted. The-4 must be multiplied by 3. Then we get 3(x+2)
2
-12 and this is
entirely equal to ,,2 + 12x. The expression has a minimum of -12 since a square is
always positive or O. It must be multiplied by 3, thus it is at least 0, namely if:x=
-2 one gets 3 (-2+2) -12, and this is -12. The value of the square is always positive, x 3 remains positive thus the minimum is -12.
(Examples by J.van Dormolen, reported by J.S.ten Brinke in Euclides 45(1970),
327-336.)
What is wrong here and who is guilty? Too highbrow a paradigm is sheepishly
applied in a common sense situation. Let us skip the question of whether the
whole problem is worth the trouble. Anyway, one thing is clear: that the students
acted out a paradigmatic sequence which had been imposed rather than reinvent-80 CHAPTER2
ed, or so they saw it.
2.2.4.2 Paradigms reinvented
We don't know how Babylonian mathematics teachers used their numerical paradigms. Let us take a long step. To prove the infinity of the set of prime numbers,
Euclid showed nothing more than how, given three prime numbers (indicated by
letters), one can find a fourth, but his celebrated proof is what I called quasi-general12, that is, such that everybody understands how to generalise it and then to
apply it to an arbitrary finite set of given prime numbers. For centuries general
theorems on figurate numbers were proved by quasi-general methods, that is,
paradigmatically, and even now numerical paradigms are a stepping stone to theorems and proofs, in number theory in particular. But mind, this function of the
paradigm vastly differs from the one I censured in the teaching of arithmetical algorithms and word problems, which was quasi-copying a pattern, while having
too feeble an understanding of its essentials.
Let us review a few examples from (1.3.3 and 1.3.5)! Finding the middle of 16
and 72 on the numberline was just a paradigm. So were the three roads joining A
and B and the four roads joining B and C, with the question of how many roads
there are from A via B to C. When arriving at figurate numbers, I was sick of numerical paradigms, or so it would seem, but in Fig. 4 -- 5, as illustration, the dots
represent five, rather than n odd numbers, and the fourth and fifth, rather than the
(n - 1)-th and n-th triangular number, as I would have done in teaching.
In the learner's mind such numerical examples may exert their paradigmatical influence unconsciously; it may be reinforced by consciousness, and even more by
verbalising, certainly if supported by labelling variables through the use of letters, which is quite easy with the first and second example --just say "the middle
of a and b", and "m and n roads". The other two cases aren't that easy. Euclid
didn't hesitate to denote numbers by letters (as he did with points and lengths in
geometry) but he didn't dare to speak about n prime numbers, and so he had to
proceed quasi-generally, which in a way is paradigmatically. Archimedes" didn't
do any better: where he needed an arbitrary number of line segments, he called
them A,B,C,D,E,F,G,H14
and even introduced new letters for objects derived
from them (for instance by addition) rather than create algorithmically constructed names. This then was the habit adhered to up to Vieta (16th century)". Nevertheless as late as in Hilbert's Grund/agen der Geometrie the Archimedean
style, though somewhat refined, is still witnessed in such a sentence as
Satz 6. Sind irgendeine endliche Anzahl von Punk ten einer Geraden gegeben, so lassen sich dieselben stets in der Weise mit A,B,C,D,E, .. ., K bezeichnen, daB der mitB
bezeichnete Punkt zwischen A einerseits und C,D,E, . . .,K andererseits, femer C
zwischen A und B einerseits und D,E, . . . , K andererseits, sodann D zwischen A,B,C
einerseits und E, ... K andererseits usw. liegt. . .
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in the sense of indeterminates and of variable objects), and where names for them
were required, Babylonian mathematicians used words like "length" and
"width". The Greeks did it with letters; yet there are not enough letters in the alphabet to satisfy the need for a potentially infinite number of names for variables.
Subscripts ranging through the entire infinity of whole numbers are a historically
late invention in mathematics, and letters as subscripts are even of more recent
date, as are subscripts of subscripts".
Why am I telling this story? I have repeatedly emphasised that history may be a
good advisor to teaching us that things are not as simple as they seem to habitual
users. History may be able to warn us against having the learner step in at the
present level of ready-made mathematics. In instruction too the use of letters for
variables should answer a need, and developers and teachers should create situations where this need is so urgently felt that it stimulates reinvention, and they
should this strategy as the use of letters for variables gets more and more refined.
Let me tell a story about a colleague of mine and his little son-- I don'tremember his
age, but in the mean time the son has become a distinguished mathematician himself.
The boy had calculated the number of diagonals of -- stop, one cannot tell this in English so let me use the German terms -- the number of diagonals of a Viereck, Filnfeck,
Sechseck -- I don't remember how far he had proceeded, -when his father asked him:
"What about an-Eck?" The boy answered: .. k times n divided by 2."
3 is subtracted from n-by going three steps back in the alphabet. Isn't it logical?
Even historically the trick for indicating the predecessor of n by n - l (and its successor by n + 1) had to wait for its invention, as well as for such things as indicating odd numbers by 2n - 1, but for a long while the sum of the first n odd numbers had still to be written as
l+ 3 + 5 + . . . +(2n - 3) + (2n - 1).
Even the invention of subscripts and sum symbols hardly changed this style.
Mathematicians have always been slow to change their notational habits, and
they never did so unless they felt an urgent need for change. I am pleading, as I
often have, for the same right to be granted to young learners, that is, to reinventors.
"The fifth triangular number", if understood according to its essentials, that is, as
a paradigm for any triangular number, changes from content to form, only to
change again into content in such statements as "any triangular number plus the
preceding one equals the square of the same rank". How to formalise and, in particular, to algorithmise such a clumsy statement? The learner has to be guided
along a road of alternating contents and forms, where no step may be skipped,
and none shall be longer than the learner's pace.
It would take me too long to describe this road in detail but at a later opportunity
I will try to make up for this. Combinatorics is of course not the only field (though
in my view it is the most excellent one) where subtle mathematical language can
be acquired paradigmatically, that is, as though it were one's vernacular. Teaching mathematical language is a problem worthy of our attention. Let me illustrate
this by a well-known example
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Let S be the number of students and P the number of professors at some college, where
it happens that there are 6 times as many students as there are professors. Express this
by a formula!
The percentage of subjects (even among university people) that answered
6S=P
was distressingly large. What if S and P are replaced by X and Y or some other
pair of letters? I was told this made no difference but I could not check it. But
what about the following formulation:
At some college there are 6 times as many students as there are professors. Put this
statement into a formula!
None could tell me whether this has ever been tried. Indeed, it is not the traditional way to formulate problems. But this does not matter. We don't need new investigations to know that something is wrong; we do need phenomenology based
didactical research to know what is wrong and why. In the present case choice is
between bluntly confronting learners with letters and formulasl6a, and guiding
them to experience formulas as new content derived from paradigmatical form.
Let me add a brief note on what phenomenology based didactical research may
mean in the present case!
Letters and letter combinations are put to use in mathematical and similar texts
in various ways. Let me mention a few of these:
Labelling fixed objects in a given context, for instance, to answer the question of
how many orders are possible for three buses to leave a garage.
Labelling, and thereby fixing arbitrary objects in a given context, for instance,
points on a map in order to make statements on their particular mutual distance.
Labelling, and thereby singling out arbitrary, but somehow equivalent objects in
a given context, for use in general statements; points in the plane, for instance, or
line segments, or numbers (as did Euclid).
Labelling geometrical or physical objects that vary within a given context (abscissas, ordinates, angles, and so on in a figure; times, path lengths, and so on in
a kinematic process), in order to express their mutual relations by formulas.
Labelling numbers and other mathematical, or mathematised objects, that vary in
dependence from one another, in order to express this dependence by formulas.
Labelling fixed numbers, functions, and so on, by fixed conventional symbols such as
-. e, log, .J, -.Il(number of), $ (mapping numbers on monies), cm (mapping numbers on lengths), kg (mapping numbers on weights), and so on.
Labelling logical objects, such as propositions, predicates, and so on, as occurs in symbolic logic.
Labelling both some species and its members by the same letter or letter combination,
often chosen as abbreviations, for instance F for females, M for males; A for Americans, B for British, C for Chinese, and so on.
It is worth noting that the last kind is one of the oldest (or perhaps even the oldest)
symbolic use of letters, which in traditional logic, and even in mathern.ati.cs nro[>cr (a!-
-1 ___
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totle, and formalising such propositions as "all men are mortal", are -ot yet out of use.
Half a century ago, after such a definition as A topological space R is a set such that.
. . it was decent mathematical style to casually continue Le.t S be an Rand then to map
R on S. Yet in a few other contexts, computer science, in particular, similar formulations are still not unusual.
This is hardly surprising since it is a quite natural and therefore common use of letter
symbols. In my opinion that coexistence amply explains the ·:6s = p:· phenomenon,
even in the case of adolescents and adults who should be acquainted with the now prevailing mathematical fashion; in addition to this influence, however, the unsatisfactory
formalisation of the .. number of . . . " function in school mathematics should not be
neglected either. It goes without saying that children, unacquainted with the pres-t
use of letter symbols in mathematical formulas, cannot at all be expected to react m
the way the interviewer judges to be the only acceptable one, as happened in im.
In the preceding analysis I did not include computer science, which would have required a special study.
2.2.4.3 Acts, actions and activities as paradigms
If the fifth triangular number is taken as a paradigm for anr. triangular number, it
is so as a mathematical object. When I spoke in the first chapter of paradigms, the
things to be taken paradi1p11aticallY looked rather like activities. Let us reconsider
paradigms from this vieWpoint! Indeed, the mere act of taking the fifth triangular

number as a paradigm for any triangular number may again be a paradigm for
similar actions, for instance, if numerical squares or cubes are used paradigmatically for squares and cubes of whole numbers in general. Labelling some triangular number as the n-th in a sequence, its predecessor as the (n - 1)-th and its
successor as the (n + 1)-th may be paradigms along with other sequences for labelling terms together with their predecessors and successors. Getting then-th triangular number from its predecessor by adding n may be a paradigm for similar
procedures such as getting then-th square from its predecessor by adding the nth odd number, and eventually for the strategy of viewing sequences as series.
Stepping
from (n-1)
2
to n
2
as expressions for successive squares, and calculating their difference as then-th
odd number
2n-l
may be paradigmatical for all such procedures where a given sequence is interpreted as a series arisen from the sequence of the successive differences, and this
procedure may in tum be paradigmatical for forming second, third, or any order
difference sequences from a given one.
If the (n -1)-th triangular number has been constructed as the part below the diagonal of then-th square, that is,
(n
2
-n) n(n-1)
-2 --2-
finding then-th triangular either by replacing n - 1 with n, or otherwise, by adding 84
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n may be paradigmatical. Anyway the trick
If the n-th triangular number is supposed
(n+l)n
-2- -
then by adding n +l the (••lI• triangular
number is proven to be
(n+2) (n+ 1)
2
which is the same as the result of replacing
n with n + I, and so it proves the expression
for any triangular numbers so ever
has a wide paradigmatical range. As a matter of fact, it is a paradigm for complete
induction.
The preceding exposition is not meant to be a didactical sequence. I was therefore
able to abstain from overburdening it with more details. I also renounced references to the many back and forth switches involved between content and form.
As a frame for viewing activities as paradigmatic, however, it is still too narrow
and deserves to be broadened.
"Looking for essentials" in (l.3.2) may serve very well as a check-list Situations,
problems, procedures, and so on, may be paradigmatical objects while the corresponding "across" may aim at a paradigmatical activity. Yet let us fix our attention on one special kind of activity. Inventions are edited, prior to communication, and so should reinventions as well. Redundancies are cut out, paths are
shortened, concepts and notations are smoothed out, the matter is rearranged, often to the degree of what I have called, putting things upside down. All this, as
far as needed, can better be learned paradigmatically than by filling out prescribed forms.
And, last but not least, what a paradigm is and how it works, 1s learned paradigmatically.
2.2.5 Applications
I began the first chapter with a historical explanation of the plural "mathematics".
The medieval quadrivium has now been superseded by a dualism: pure and applied. By their very names the two oldest mathematical journals claimed to cultivate both of them, although applied mathematics had still to wait one more century to get a journal of its own. Al universities there were (and in some places still
are) separate chairs for each of them. The famous mathematician G .H. Hardy
wrote a famous textbook "Pure Mathematics". He was proud to assert he had not
created m1ything mathematical that could ever be auplied. He was wro11- but he
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There are good reasons for researchers to specialise and for higher education to
be departmentalised. But pure and applied is an awkward dualism, as are theory
and practice, or (in textbooks) theory and exercises, or (in mathematics education) insight and drill, and so on. Reality is a virtual unity, where for practical
purposes at any moment some part is focused and others are out of focus. Yet
shifting focuses is no less important than focusing itself and it should not be impeded by what I formerly called isolating impermeable membranes. This applies
in particular to learning mathematics. So I propagated rich contexts, just to fight
impermeability.
The most serious trouble with dualisms like those above is the tendency to favour
one-way traffic. In instruction this means: first theory and then exercises, first insight and then drill -- the first for conscience's sake only and very likely to be
skipped. After all, the mastery of the first does not imply that of the second
whereas the second may bring about the first, or one would hope.
If history means searching for the roots of the present in the past, then history of
pure mathematics is a tree deprived of its strongest roots. Teaching is not different. In the past, most mathematics was never applied or meant to be applied by
those who learned it at school, but things have changed spectacularly. More and
more people use mathematics more often than they are aware of. When I said this
I continued: They use matllematics because they cannot do without it Indeed, the
amount they use is astonishing, but have they learned the mathematics they eventually use, only because they were once taught mathematics? Mind, I don't ask
whether they use the mathematics they were taught. One can readlily deny it, since
most of the stuff they learned is of no use at all for most of those who were supposed to learn it My proper question was whether learning the mathematics they
were taught has been conductive to educating mathematical behaviour. Or rather,
if there was some influence, could it not as well have been detrimental?
I agree that, in adaptation to social needs, views on teaching mathematics have
changed. Henceforth, one now postulates, students shall also be taught to apply
the mathematics they are to learn; so reformers and developers call and look for
more applied mathematics alongside the pure. As so happens in the course of
teaching, "alongside mathematics" is interpreted as "after", and this is true at various curricular levels: The eventual result is a pair of curricula, rather than one:
first, pure theory; second, application -- a menu of applications, that is. Or the two
curricula are shuffled into one another as packs of cards: parts or chapters alternate with one another -- theory first, applications after. Or every bit of theory is
followed by another bit of application.
It does not matter which variant is chosen. The view itself is wrong and it is even
worse if it is identified with mathematising, which is a learner's rather than a
teacher's activity, or at least should be. Applying mathematics is not learned
through teaching applications. The so-called applied mathematics lacks mathematics' greatest virtue, its flexibility. Ready-made applications are anti-didactical. Mathematising some problem situation in nature or society should not be
demonstrated by the textbook author or the teacher but left to the learner to be 86 CHAPTER2
reinvented. In this regard applications are no different from mathematics as such,
and for that matter the present section could as well not have been written, were
it not for the urgent need of using every opportunity ID propagate starting and
keeping mathematics in reality, while switching back and forth between realities
-- natural, social, and mathematical. Pure and applied is a highbrow dualism rather than a learner's concern.
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2.3 LEARNING PROCESSES
2.3. l "Learning process" as a didactical principle
What led me to deal with learning processes and their observation, in particular,
in a chapter on Didactical Principles? Aren't these rather tools and subjects of research? Unfortunately, whatever educational research may mean -- and it may be
quite a number of things -- one cannot but agree that little attention has been paid
in the past to learning processes and even less to their observation. Even at the
present state of knowledge with regard ID learning processes, didactics can hardly
draw on a substantial capital of educational research. This, then, explains and justifies the prevailing practice of short-cuts in education where, without much ado
(let alone any reflection) and trusting one's intuition, more or less conscious experiences on learning processes are applied wherever needed in didactical situations. Let me account for this fact -- and where I consider it a gap try to mend it
-- by dealing with learning processes in the present chapter! Mathematical learning processes, that is, although most of what I have to say is also true for the didactical ones.
Anyway, the use of and the emphasis on processes is a didactical principle. Indeed, didactics itself is concerned with processes. Most educational research,
however, and almost all of it that is based on or related ID empirical evidence, focuses on states (or time sequences of states when education is to be viewed as development). States are products of previous processes. As a matter of fact, products of learning are more easily accessible ID observation and analysis than are
learning processes which, on the one hand, explains why researchers prefer to
deal with states (or sequences of states), and on the other hand why much of this
educational research is didactically pointless.
Why can -- even dense -- sequences of states if appropriately inteipolated, never
be equated with processes? The reason is again a didactical principle. In [87,
p.165] I formulated and italicised the thesis:
What matters in learning processes are the discontinuities.
Inteipolation between states obviously applies only to continuous processes. The
process approach is fundamentally different, even if its scope is restricted to short
processes. Let this be enough for the moment, and let me delay justifying my thesis didactically and elaborating on it!
2.3.2 Teaching/learning processes
Since Herbart, if not earlier, views on teaching processes have influenced didactics and, though in instruction teaching and learning processes are strongly interrelated, there are few people left, even among practitioners, who profess the outright identification oflearning with teaching. Moreover, one cannot disregard the
large part of learning - - even at school -- which is independent of teaching. To 88
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my mind, the amount of what is learned by formal instruction is grossly overestimated. But since my concern is didactics, more often than not when I speak of
learning processes, this will be shorthand for "teaching/learning processes", or
"instructional processes" (which includes both learning and teaching).
l . Pure and mixed learning processes
Knowledge of pure learning processes may greatly help us to understand and organise instructional processes, but whatever pure learning processes may mean,
this field still looks like a desert: shrubs of unconscious experience, incidental or
intentional observations, and sparse efforts to organise this experience. There is
one oasis in this desert: Piaget's early worlc
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, based on continuous observations
of his own children during their first and part of their second year. Of course, active observation, too, is a kind of teaching but to a much lesser degree than the
influences of the social environment in general. It is a pity that Piaget's original
records have never been published.
Indeed, Piaget's material, as selected and offered by himself, is processed to fit into
the stream of his profound theory. It is chosen in order to describe what he calls the
sensory-motor stage. One strange feature should be mentioned: his complete neglect
of language. The subjects never say a word (as they never weep or cry). Did Piaget
reserve language for another volume that never appeared, or did he underestimate the
part language plays in the .. construction of reality"? By the quite sophisticated tasks
these little chilchen are able to perform they distinguish themselves from animals to
such a degree that one is inclined to conclude that this is due to their ability to name
things, events and acts, and by this way to imprint them more strongly in their memory. It should be noted that Piaget did not pay adequate attention to the part played by
language in cognition in his later work either.
Piaget's original approach was time-consuming. The laboratory research carried
out by his assistants represents methodologically another extreme: instantaneous
exposures, as it were, where learning is considered to be a disturbing nuisance
and therefore counteracted and eliminated as much as possible.
Let us add one remark: Rather than the population of subjects, the only constant in the
explored time sequence of states is the complex of tasks to be tested; yet, since they
are propounded at linguistic levels corresponding to the subjects' ages, one can, in
each particular case, question what has actually been tested: levels of task performance
or of linguistic proficiency.
Factual learning processes are mixtures, depending on the involvement of unintentional and intentional, informal and formal teaching. However important observations on everyday learning may be, didactics, which includes both teaching
and learning, is best served by knowledge of more formal teaching/learning processes and the interaction between the guides and those guided.
2. Participation and reservation
Another dilemma in learning processes is related to observation: Participation
":'t1 ---=<-c"---- ---- '--.,._.'1: `-.,,,, ,,,,.( `-, `- .-
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ure of his subjects in doing tasks was fed back into his strategy as an observer.
For instance, after an initial failure, the same task was offered anew, often ten to
twenty times. Piaget's assistants, however, used repetitions to make sure whether
a subject's reaction as was by chance or consistent
The teaching component increases along with the observer's participation in the
learning process, . Yet teaching/learning processes, say in the classroom, can be
attended and registered just as well by non-participating observers. Personally, I
do not like the idea of non-participatory observation. I even feel unable to refrain
from intervening, and I still regret some cases in the past where I missed the opportunity or lacked the readiness for intervention, for instance, when I failed to
ask the person observed why he did what he did, and why he did think what he
thought
3. S/wrt-term and long-term learning processes
As is the case with the interdependence of teaching and learning processes, I emphasise here that my proper aim with regard to !earning processes are long term
ones, which is in fact the title of (2.4). Short-term learning processes look as if
easy subjects, easier than they actually are. They are, indeed, easily accessible to
direct observation, which no longer holds true for their long-term effects, whether the short-term processes observed are embedded in long-term ones or not. Of
course long-term learning processes are the most difficult thing to observe.
When I objected, on the ground of the discontinuity of learning processes, to intePOlating between consecutive states as a procedure to grasp and to guide processes, I did not mean to exclude the reconstruction and the pre-construction of
long-term learning processes from sequences of short-term ones. The conditions
for realising these goals will be the main concern of this section and the following
(2.3/4).
2.3.3 Observation as a didactical principle
1. Observing
Observation is an indispensable tool for any research based on experience. Observing natural phenomena under the microscope is quite different from observing patients, or wildlife, or people in the supermarket, or children in the classroom. Inventive and influential methodologists have developed models for organising
classroom observations according to a pattern they themselves consider
to be as characteristic of the natural sciences. Assistants, preferably students,
armed with check-lists, are charged with observing teachers' and children's
classroom behaviour, and according to the minutes spent on certain activities
they are to classify the actors in a system of predesigned categories -- a methodology that is a caricature of observation as understood in the natural sciences.
Even the naturalist does not truSt check-lists to explain unexpected events, which
as a matter of fact may prove to be more important than the anticipated ones. His 90 CHAPTER2
strategy of observation is instead controlled by vast experience and a certain
amount of theory. Proceeding according to check-lists is a good thing for starting
an engine or any other predesigned and controlled process, but it is an impediment to acquiring new experience or to learning new behaviour.
All the same, educational research will be a subject of the next chapter. In the
present chapter the observation of learning processes will be dealt with as a didactical principle. For many years I have propagated observing learning processes, although in fact it is an art that is practised more often than people would admit -- the observers themselves included. What I meant to ask for was to report
on such observations, which presupposes getting conscious of them. Teaching
and didactical experiences result from observing learning processes. Teaching
and didactical behaviours are justified by experience, and more explicitly by observations inside and outside the classroom which have consciously been registered.
2. Observing and recording
Are conscious observations of learning processes as exceptional as they seem I
don't believe so. But if these are so many, then why do so few of them get a
chance to be reported? Are non-professional observers afraid that reporting observations would come across as telling classroom anecdotes? If so, they are right
to avoid this appearance. Indeed, telling classroom anecdotes will not acquire scientific respectability unless it is organised by means of all-embracing educational
or psychological categories. Or, rather, this postulate is the very reason why developmental research as practised by Piagetians is blossoming: classroom or laboratory observations are classified according to categories and hierarchies or at
any rate make a pretence of doing that. I admit there are no ready-made tools for
organising observations of learning processes but, as long as no substantial material is available, the quest for organising it is meaningless. Nevertheless, there
is one valid criterion for estimating whether observations are by chance worth being reported or not: whether they are paradigmatica/. As a matter of fact, this extends to educational research, where paradigms play a part unrivalled by statistics, but in the present chapter we are concerned with the paradigm as a didactical
principle. It was introduced here as an answer to the quest for experiences worth
being reported, though this is not at the heart of the matter. As valuable as reports
may be for those who are expected to read them, the didactical value of reporting
resides first of all in its being an incentive to observation, with the aim of becoming conscious of one's observations and having them analysed.
Learning by paradigmatising would seem to be a vicious circle: on the one hand,
the criterion for knowing what observations are worth recording resides in their
paradigmatical character while, on the other hand, learning what is paradigmatical, must take place in the course of observations worth recording. It lli circular,

yet so are learning processes in general. In this particular case we are, indeed,
-- tn.--o::[T]. --j "Vh. f 1. ?t--e-;_,r;r:: "-:'.<:::: .... "."\;;: "-;'),--.:'-: "" -v -,;:e-.1. f __ -:-.'.('-= f H '-- - tr-. "'-< .. "'. "-V. .-"".:.:--;f.-= _c!
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and didacticians; this process is guided by the experience of observed learning
processes -- those of their subjects -- and moreover, as happens or should happen
in teacher training, reinforced by exterior guidance. It is an apparent circularity
that is not vicious at all; and in the right perspective the process is a spiral rather
than circular.
Later on, we shall look for more objective criteria for what is worth being observed and recorded and at the same time fill in a gap in our argumentation.
Meanwhile, we will consider the described process of learning by paradigmatising, albeit lifted up to a higher plane.
3. Self-observation
Observing learning processes, whether formally recorded or not, is itself a learning process - a meta-process, as it were. It becomes an opportunity to learn how
to observe learning processes as soon and in the measure as the observer is aware
of details of this meta-process. What kind of details? The answer is again: details
of what is paradigmatical. (In [87, p.205] I described the paradigm in my own
learning process as an observer that led me to shaping the idea of paradigm.) But
this is too easy an answer. Observing oneself can be a difficult undertaking as
everybody knows who has ever tried it; in particular, any mathematician who has
tried to observe him)self solving a mathematical (or for that matter, any other)
problem knows that proving a hypothesis often means finding out the reason why
one believed it to be true. Let me therefore deal for a moment with self-observing
in a broader context than that of only observing learning processes; this anticipates themes to be dealt with in (2.4.2).
The difficulty of self-observation is that of being actor and spectator simultaneously. It can be learned, to be sure, and where it may mean the observation of
learning processes, teachers and didacticians should indeed learn it; but self-observation is also worth learning with regard to the acquisition of subject matter,
and particularly if this subject matter is mathematics. Anyway, it is a difficult
thing to learn, which may require considerable endurance. There is, however, a
short-cut Let me fust describe it in terms of fust line mathematics learning and
teacher training!
Observing others is definitely easier than observing oneself. In the past I have
fought individualised learning (which is fashionable today) even harder than the
traditional teacher-centred classroom variant. I will come back to this point when
dealing with "Mathematics for All" but let me say in advance that I favour small
learning groups of, say, four learners, and that this preference extends to teacher
training. Observing the learning processes of others is implicit to learning in
small groups, and what happens in such groups should be made explicit by the
learners themselves or by their supervisor, although this is hard to do and we
know little about how to do it systematically.
The counterpart for teachers and didacticians (and even for researchers) of what
the classroom and teacher training learning groups mean for pupils and student
teachers is workshops: learning to observe oneself by observing others. Indeed, t .
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observing others and oneself is a source of consciousness for one's own physical
and mental actions, and this again is an important aspect: reflecting on what one
does and thinks. Of course, we would not be able to act in everyday life if tried
to plan everything in advance and think about it afterwards. A large part of our
incentives and intuitions are unconscious and will always remain so, but making
unconscious things conscious as often as possible is a way to improve the operational value of our unconsciousness.
Was self-observation a misleading title? No, its scope has been extended quite
naturally from the individual to the collective.
4. Discontinuities in learning processes
I promised more objective criteria than having a paradigmatic character for indicating what is worth observing and recording; at the same time I would like to fill
a gap in my argumentation. To do this let me repeat my self-quotation: What matters in learning processes is the discontinuities -- or jumps, as I prefer to call
them. In what sense does this matter? First of all, it is the jump in a learning flow
that superficially looks continuous -- the learners' overt or covert aha-experiences -- that strikes the observer and challenges him to participate by reinforcing
their influence.
How to recognise jumps in the learning processes? There may be objective criteria, such as in the cases I reported in (2.1.4, 4): for instance, a clumsy procedure
for solving linear equations like "first multiplying by the denominator, then dividing by the numerator", may after a long while be spontaneously shortened to
multiplying "upside down". Was this a chance event? To make sure whether such
an occurrence is a jump, the observer should offer the learner a sequence of similar problems and after a while and as it happens resume testing once it is clear
that the occurrence was not accidental. As a matter of fact, in all the cases I mentioned above (and in many others) there was no relapse whatsoever, which proves
that the jumps -- or what I had diagnosed as such -- were real. Clearly the girl did
not dare to leave the beaten track until she was sure she would not get lost I myself was satisfied to reinforce through mere repetition, lest explicitation of rules
should harm her. Indeed, as I mentioned earlier, her teacher (by supplying her
with explicit rules for operations on fractions) had managed to undo the effects
of many weeks of reinventive learning. So I was afraid that, unless she verbalised
by herself, rules would prove harmful rather than helpful. It seems that she never
had learned to verbalise mathematics on her own, in spite of her otherwise normal
mastery of language. Attempts to rectify this shortcoming might seriously have
confused her, or so I feared. Notwithstanding the lack of verbalisation, she understood very well what she was doing - later on I will report on how I tested her
insight
Good verbalisers are easier subjects where learning processes are to be observed.
My publications contain a large number of references to a boy whom I started oba fev,r ,rr_;r;nt-~~ :.:.:fterr h2-'. <Je13r.nd bi_rtt.:1 e_v (rr.c:-~~ ;;;•-r:.'.ern.?tJ.r: 71't'('rt';; "I1" - i'.:'i:- . - -
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his intentions in advance as well as whatever plans he was going to carry out, and
to reflect on his own speech and thought, sometimes even while speaking and
thinking. On the other hand, great discoveries, or what he felt to be such, were
accompanied by heavy emotions and explosive utterances. He was a most appropriate subject for observing learning processes of all kind. Although my thesis
about discontinuities being the significant thing in learning processes dates from
long before his birth, he has taught me a lot, for which he has been credited in my
various publications.
I have just sketched two extremes of learners. From old material and as a "revisitor" I could add a few more reports". All they have in common are jumps with
paradigmatical consequences - anyway they do not provide enough indications
for attempts at categorisation. Still, let me mention what I personally learned
from these experiences: all together they have helped me to get rid of the prejudice that something learnable may be self-evident. What is now a matter of course
was once a matter of learning. It sounds trivial but then it is one of the trivialities
that one should be reminded of again and again. For this let me report a striking
example!
5. A group learning process
The two cases I just reported were learning processes of individual children. For
instance, if instead of the one girl, I had been teaching two together, I would have
followed an entirely different strategy: The need for communication between the
two of them would automatically have stimulated verbalisation, which I would
have cautiously guided.
Anyway teachers as well as developers are dealing in the classroom with collective, rather than individual learning processes. My own first experiences in this
respect, well over thirty years ago, are related to Dina van Hiele-Geldofs marvellous reports" on classroom observations. She took notes while teaching; and
I was astonished how many more -- and more important -- things she observed
than I was able to do. Since then, modem technical tools have become a great
help for observers in the learning environment. On the other hand, small groups
are a better environment for observing learning processes than large classrooms.
Let me give an example:
After measuring a few angles, it is time to draw angles of a given size. Mariska tries
to draw an angle of 200. She draws at random. She adapts the protractor to one leg,
carries the pencil to the other one, reads off: it is wrong. She erases the angle and draws
another random angle, while talting into account the first measurement. She measures
again, and again it is wrong. She erases the second angle, swearing at it Her neighbour
Monique is alarmed by her swearing. She watches the third attempt. and then she explains how to do it at one go. Anja is listening; --th-t is how I did it" she says: "So .I
have done it." Mariska draws the next angle -- 13 -- m the new way. Meanwhile, Cecil
has learned how to do it from Anja. Cecil consoles Mariska: "I didn't know it either."
The next misfortune is that a point to be marked falls outside the sheet of paper. Anja
helps. Anja and Monique always wait for the others before going on to the next task.
When I recommended observing learning processes as a treatment against unwar-94 CHAPTER2
ranted prejudices on self-evidence, the above proceedings were one among many
examples I had in mind; I found it a distressing experience, and at the same time
a warning. Thirteen years old and behaving like this just to draw an angle of 20',
isn't this sheer stupidity? Or were they really that stupid, these two among the
four 13 year-old girls? Would they have acted the same way if the task had been
to draw a line segment, say of 5.3 cm? Certainly not! Why weren't they able to
use the circular protractor in the same way as they would have used the linear
scale of the ruler? Thirteen year-olds who behave like Mariska and Cecil, what
have they missed in their most elementary education? In order to answer this
question more profoundly, it should be posed in a much broader context. What is
behind this distressing experience?
6. Change of perspective
It is not merely the switch from measuring to constructing. Il is a kind of jump,
which I have called change of perspective - an important and indispensable activity, characteristic of a mathematical attitude - in this particular case, from
measuring a given thing to making a thing with a given measure and, more generally, from examining a given thing and stating its properties to making a thing
with prescribed properties (and of course if need be, the other way around). In
traditional geometry teaching this duality was called analysis and construction.
Geometry, though not necessarily of the traditional kind, is an excellent training
field for changing perspective, although its domain of transfer is much wider,
both inside and outside mathematics. Change of perspective takes place as soon
as the child starts to conquer the so-called egocentrism which, in my experiences
happens much earlier than Piaget's laboratory experiments seem to indicate.
Learning processes are marked by a succession of changes of perspective, which
should be provoked and reinforced by those who are expected to guide them. My
diagnosis with regard to the failure of these two girls is: victims of rigid instruction that never left any room for individual initiatives. Change of perspective develops most naturally in reinventive learning and, if consciously trained, allows
for a wide area of transfer, as do all behavioural attitudes. Anyway one should
begin exercising it quite early. Is 13 too late to recover what has been missed?
Am I too pessimistic if I say it is? What a tremendous gap between the two girls
who succeed and the two who fail! Can we do anything to narrow it? Or better,
anything to avoid it? I am not at all hopeful as long as teacher trainers fail to teach
their students how to observe learning processes. Or shall I call it learning to observe when they send their students into classrooms with prefabricated questionnaires or have them observe classrooms on the monitor? But let us not say too
much about "teacher training" beforehand!
7. Communication
I almost forgot that the above story was meant as an example of a collective learnL1-fii" ;DIOCBS:'., It -S, hD -V-~.rp--, :\-;"" -<i-\P:-1" "l,. r-. --r":'t :-1;.'.f":-'-" t:--: >-->:; -1 cp,- -.(i -" ----; --,t.", ,.-.--
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the two others needed and received was too straightforward. I could repeat more
sophisticated examples from earlier publications but this would take me too far
afield, while high quality research is just now taking place in this domain. The
general feature of the observational evidence at my disposal is: four children cooperate more or less efficiently to solve a problem. An accidental remark of one
venturing may mean a hint for the other; or the one who, venturing the jump, cannot explain it herself but another can and thus enables other members of the group
to follow suit. Linguistic tools arise from the need for communication. Discussions evolve, and participants are compelled to reflect on their own utterances
and actions. The teacher, walking between the tables, whether or not requested
for by a particular working group, should not intervene actively before having listened to the - perhaps confusing - report on their previous learning process. The
mere fact of reporting may mean such great progress in the learning process of
the group itself that the teacher's intervention can be restricted to polishing language and adapting ad hoc invented terms to the conventional terminology. Or
the report may include keys to guide the group in the right direction. Or exchange
with more successful groups may be helpful.
This then is observing group learning processes - a scarcely explored field.
8. Thought-experiment
What use is the knowledge acquired in observing learning processes for the instruction developer? Certainly, there is some use. As a developer he can profit
from these pieces of knowledge, provided he views them in a broad sense as paradigms. As early as 1961, if not even earlier, I propagated the thought-experiment
as an instrument in educational development
Imagine a student, more or less distinctly defined, and have him in his mind reinvent
a mathematical idea, observe his actions and analyse them by logical methods. [14,
p.32]
That is the Socratic method, or as I would prefer to say, the method of the thoughtexperiment. The latter expression comes from Mach, who described it as a method of
theoretical physics; it was the central method from Galileo to Einstein (and maybe still
is today). In didactics, I mean by thought-experiment the attitude of a teacher or textbook author of imagining a student or a group of students and teaching them in mentally while reacting in advance to their probable reactions. The imaginary students are
active, and their activity allows the teacher to determine his way of proceeding. .In a
narrower sense I will assume, as Socrates did, that the teaching matter is reinvented or
re-discovered in the course of teaching. Rather than being presented dogmatically, the
subject matter originates before the students' eyes. Though the students' own activity
is a fiction in the Socratic method, the students should be left with the feeling that the
teaching matter arose while teaching, that it was born during the lesson, and that the
teacher was in effect only a midwife . . .
In the Socratic method "reinvention" was not understood literally; it was simulated
rather than being true reinvention. It could not have been otherwise, could it? The
teacher's authority was still dominant . . . The initiative was only on the part of the
teacher. Not only did he lead the student, he also showed him how rediscovery works,
he rediscovered on behalf of the student. (40, p.100-102]. 96 CHAPTER2
The variants I proposed were, of course, thought-experiments for real albeit
guided, reinvention. Moreover, I postulated that textbook authors and d-velop
ers, w-etheronn?t they shared my educational philosophy, would reveal the very
-sentials of -err thought-experiments (if there were any); this would; in my
view, be t!1e ng-t w-y to prove, as it were, the soundness of their approach or at
least to bring this --mt up for disc.ussion; and it would eventually lead to having
t!1- thought-expenm--ts checked m the classroom - let me stop here and not anlic1pate too much on educational development" (3.2/3).
2.3.4 Levels in the learning process
What matters in learning processes are the discontinuities - quoting myself - or,
another word that I ha-e frequently used: the jumps. I owe the conception of the
level structure of learnmg processes to my collaboration with the Van Hieles a
couple who embodied, as it were, the marriage of theory and practice. Let r:ie
quote from [40, p.121 sq.], which actually dates back to the spring of 1969, while
pieces relevant to the present exposition are mere translations from the French
[14, p.32-34], written in 1961, as I afterwards noticed to my own surprise:
When the Van Hie l es started teaching they were just as unprepared as many other
young teacher-; nobody had told them how to do it. They had, of course, passively undergone teaching, -d maybe even observed how their teachers performed, but this
-as ni't enough. As time went on, they had the opportunity to discuss their own teachmg with each other and with others. They subjected their own actions to reflection
they .obi:ervv- th-selves ".hile. teaching, recalled what they had done, and analysed
1t. Thinking is contmued a-tmg, indeed! but there are relative levels. The acting at the
lower leve- becomes an object of analysiS at the higher level. This is what the Van Hiel es reco-ruze- as a remarkable feature of a learning process, namely in the learning
process m which they learned t-ching_.They transferred this feature to the learning
process that w- the goal of -err teachmg, namely the learning processes of pupils
who were. learn.mg -athemat1cs. There they discovered similar levels. To me this
seems an nnportant discovery.
In my exposi-on -is is followed by a series of examples, which included the levels on the -- h1stoncal as well as individual -- road from the intuitive unreflected
and, as it were, incidental practice of complete induction (see 1.3.5, 14) to the
formulauon of Peano' s axiomatic system of natural number. Then it continues as
follows:

Here the levels o- the -earning process stand out in bold relief. On the lowest of the
levels under consideration complete induction is acted out. On the next level it is made
conscious as - organi- -inci-le and can become a subject matter of reflection. On
the same or -higherlevel it IS put mto a linguistic pattern. From here to the Peano system t?e path -s n? lon-er lo-ally determined. Now the data is no more a bunch of mathei:nat1c- d-1vations m whic- a -omrnon pr-ciple (like that of complete induction) is
hidden, it ,IS rather. the o-garuzauon of an entire field, in which linguistic formulations
(th-----f---om-lete n1ducuon? amon- other ma_thematical activ-tie- bet".-.0me the sub_-ect
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which, along with others, should characterize the natural nwnbers.
. . . . . .
How levels stratify this [the mathematical] activity has been shown by an example: the
means of organization of the lower level become a subject matter on the higher level.
Which phases the learning process passes through from one level to the next is apedagogical question. which should be answered in specific cases by pedagogical experience. The relation, however, between one level and the next is overwhelmingly logical
and accessible to logical analysis. Often the level-raising tools are heavy quantifiers -
- over all properties of natural numbers in the case of complete induction.
. . . . . .
The learning process is structured by levels. The activity of the lower level, that is, the
organizing activity by means available at this level, becomes an object of analysis on
the higher level; the operational matter of the lower level becomes a subject matter on
the next level. The pupil learns to organize by mathematical means, he learns to mathematise his spontaneous activities. Or rather it would be desirable to have him taught
in this way.
Let me stop quoting from what I wrote early in 1 '39 and turn to similar passages,
written in the late fifties and published in [13, p.26 sq.]:
The analysis of mathematics as a ready-made subject presents a deductive system in
which all steps are fundamentally equivalent. The analysis of mathematics as an activity, however shows a layered structure. This feature is accounted for by the level theory of the Van Hieles. The stages through which the le-gprocess passes from one
level to the next, are of pedagogical concern (which can be disregarded by the present
exposition). The rrlation between one level and the next, however, is preponderantly
logical. To discoVer it, we can use logical analysis . . . .
What it is that characterizes the level structure may be expressed in a few words by
saying that the operational matter of a lower level may become subject matter on a
higher level. If, on a certain level, complete induction has been a significant activity,
this activity may become a matter of conscious consideration and, finally. of explicit
formulation on the next level. The higher level is clearly indicated here by the heavy
quantification over all properties of natural numbers which occurs in the statement of
complete induction . . . . Operations with integers such that multiples of a certain m are
considered inessential, and operations with pairs of integers according to the equivalence relation for :fractions are unproblematic activities. On a higher level they can be
promoted to subject matter and finally described by the device of abstraction through
equivalence classes. On a still higher level the equivalence relation and the abstraction
by equivalence can become subject matter.
In geometry, space phenomena are studied on the lowest level. The properties discovered on the lowest level are subject matter on the next level, where these properties are
related to each other. The relations acquired on this level are then studied on the next
one, in order to build up what are usually called theorems. The logical connections be-
tween the theorems are subject matter on a yet higher level, which is still below that
of logic as a subject matter.
The level theory of learning is closely akin to what can be characterized in logic by the
words .. theory and meta theory". It is not by chance that the Van Hieles seized upon
this idea. To my knowledge they were the first who wrote a textbook in which the
learning process is purposefully initiated and kept up as a process of re-inventing. Traditional methods, however, often show the opposite tendency: to descend from higher
levels to lower ones instead of climbing from the lower to the higher level.
Let me fmally yield the floor to the Van Hieles themselves'' , for dealing with the 98 CHAPTER2
learning process in geometry at the secondary level:
. . . Take the example of the rhomb! Before learning geometry children possess the
"image" of the rhomb. They are acquainted with rhombs that are special concrete objects. Now they will experience what .. rhomb" means in the geometrical context. They
discover properties of this figure, which the teacher calls "rhomb". They form the
"symbol" rhomb. In this symbol they condense all internal properties of the rhomb
they have experienced . . .
Thanks to the possession of a stock of symlx>ls the pupil will be able to organize the
subject matter. Mainly as a routine the symbols become signals. From the properties
belooging to the symbol rhomb (equal sides, parallel sides, equal opposite angles, diagonals that halve each other under right angles) the rhomb as a signal (equal sides)
arises, or rather one property (equal sides) becomes a signal for the symbol rhomb as
the complex of all its known properties. The pupil is able to anticipate . . .
The pupil who has started at "level O" with undifferentiated visual structures, is now
at the first level. At level 0 concrete objects and images are the subject matter which
is organized by the use of symbols and signals. At the first level the symbols and signals of level 0 will become the subject matter. The system of properties of geometrical
figures will be organised by means of relations connecting the geometrical figures to
which the symbols and signals of level 0 refer. At the first level relations like congruency, similarity. parallelism appear. At first they have the character of symbols, later
on that of signals (e.g. congruency of two figures is a symbol for the fact that the two
figures cannot be distinguished in any geometrical aspect; a congruency theorem for
triangles may be a signal). However. at this level relations are not yet a subject matter
as are the properties of figures.
At the second level the relations which have been devices of organizing at the first level will become a subject matter. Here the organizing devices are relations between relations, mainly of a logical character. The symmetry of a relation between geometrical
figures. the interrelatedness between relations by means of implication can be used at
this level, but they can become a subject matter at the third only.
At the third level the pupil will be able, e.g. to distinguish formally between a proposition and its converse. Finally at the fourth level (hardly attainable in secondary
teaching) logical thinking itself can become a subject matter . . .
Learning is a discontinuous process. The discontinuities are, as it were, the jumps in
the learning curve. These jumps reveal the presence of levels. The learning process has
stopped. Later on it will continue, as it were, by itself. In the meantime the pupil seems
to have "matured". The teacher no longer succeeds in explaining the subject. He (and
also the other pupils who have reached the new level) seem to speak a language which
cannot be understood by the pupils who have not yet reached the new level. . .
The attentive reader will have noticed a fundamental distinction between the picture unveiled in 1969 and that of a decade earlier. There are common elements:
the discontinuity of the learning process; its level structure, discernable by a kind
of mathematical analysis; and the levels as determinants of the discontinuities.
The pictures diverge from one another with respect to the bonds by which successive levels are interrelated.
In the first version the learner's operational matter on the lower level becomes
his subject matter on the higher level.
In the second version the learner's lower level activity becomes an object of analrsis fo him on tJ"_e higb-r 1e-vel, or i,n oth.-r words: On the ne-:- leve! thh- nct_;vitv is
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The second version adds to the first the means by which operational matter on the
lower level is transformed into subject matter on the higher level:
Being made conscious and subjected to reflection is the means of level-raising,
which is applied, to the- still intuitive (operational)- activities of the lower level,
rather than to the subject matter.
When did this change of view take place and why, or is such a historical query
worth the soul-searching? There is some indication in the old exposition of a foreshadowing of the newer view, and some relapse in the newer exposition to the
older viewpoint, but I don't believe that this really matters. Was the change of
view caused by observations of learning processes made in between both events?
There are a few of these I might conjecture to have caused the change but I am
not sure whether they were really responsible for it. A great many observations
of learning processes relevant to the level theory were made after the crucial date,
although I would hardly have noticed them as such if I had not at that time been
in full possession of that theory, or so I believe. So my guess would be that the
theory was rationally acquired and only afterwards tested by observat1On.
What kind of considerations could have guided me? In the first paragraph quoted
at the start of the present subsection from [40], I related Pierre van Hiele's own
report on how he discovered levels in learning processes, that is, as a matter .of
fact, first in the process he and his wife went through as young teachers while
learning to teach, and -ow he transferred those levels and their features to their
pupils' processes of learning mathematics. The remarkable feature was that of
levels related by reflection: "At the higher level the acting at the lower level becomes an object of analysis." However, the levels as characterised in my first version, and as substantiated by themselves'' are of a quite different nature -- I will
come back to this point.
I think that the change between the two versions of level theory was influenced
by my drawing the full consequences of Pierre van Hiele's report on the discovery of the levels, which were caused by reflecting on their own learning process
and then made parallel with the learning processes of their pupils.
But there is more to it. While writing these lines, I discovered to my surprise that
the change between the two versions took place as early as 1961. Let me quote
from [14, p.33]:
Ce qui caractB!se l'hiBarchie des niveaux en general, c'est que la technique des
operations a uncertain niveau devient matifile a reflexion a un niveau supErieur - relation apparentee a celle qui, en logique, est caracterisC.e par les mots .. thC.orie et metathC.orie".
The main example illustrating this statement is what I described above as "the -
historical as well as individual - road from the intuitive, unreflected and, as it
were, incidental practice of complete induction to the formulation of Peano's axiomatic system of natural number", and with regard to which I asserted:
Here the levels of the /earning process stand out in bold relief. This, indeed,
seems to me essential for the reconstruction of the change from the first to the 100 CHAPTER2
second version. Obviously I judged the level theory important enough to be extended from geometry -- where it was discovered and in terms of which it was
defined by Pierre van Hiele -- to other domains of mathematics teaching and
learning. This, indeed, was in accordance with my main interest at that time, the
midclle and upper grades of secondary education.
For two millennia mathematicians had practised complete induction
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-- the ancient "side and diagonal numbers" being the most profound example
12
-- when
Pascal (and independently, James Bernoulli) couched it for the first time in the
now well-known principle; there is not the slightest evidence for there having
been any conscious reflection on this practice during this long historical period,
although figurate numbers would have offered an excellent opportunity. The following steps on the road to the modem theory of whole number took much less
time in the learning process of mankind, but all of them seem to be due to reflecting on previously unreflected activities.
In the first chapter I repeatecily emphasised reflection as being characteristic of
mathematical thought, in particular in the interplay of form and contents (1.1.3);
this was substantiated by some "examples" (1.1.3.1). I don'J remember when I
discovered reflective thought as a forceful motor of mathematical invention. It
certainly happened by introspection, that is, by tracing my own ways of thinking
(which is itself reflective thought), long before I tried to discern it in the thoughts
of others and in the learning process of mankind. Let me save the topic of reflective thinking for the next subsection, where it will be dealt with more systematically! For the moment let me confine myself to one example, which will explain
why I attach such importance to reflective thought as the motor of mathematical
invention -- it is an example I have adduced many times in lectures and papers.
Why do the diagonals of a parallelogram divide each other in half each other? Of
course you can prove they do by applying congruency theorems, but that isn't the answer I expect to my question, since I am sure you were aware of this property of the
parallelogram long before you had any idea of geometry. But how did you know it?
What was the source of this knowledge? After so many years of mathematical instruction you will not succeed recalling it .. except you be converted and become as little
children". But such conversion requires greater effort than most people wish to spend.
It is much easier to observe the little ones playing with such toys as sets of mosaic tiles.
They know that any parallelogram turned around fits into its own hole, "which proves
the theorem"; indeed, by this tum both diagonals are turned around as well while their
intersection remains fixed.
Try the experiment! Draw a parallelogram with its diagonals! Have the child discover
properties of the figure. and if need be pave the way to the particular property you have
in mind: the diagonals dividing each other in halves! Ask: "why do you think they
do?" "Because I can see it." "How do you see it?'fusist on an answer! Depending on
the age of the child you will eventually get one, which at that age can be considered as
a valid proof, obtained by thought-searching, by reflection.
If reflective thought is, indeed, a forceful motor of mathematical invention, it is
only natural to put it to good use in such educational design as is based on the
. . . .r ---- --1'; fir ,I)ff . ,"'"-'', ,,, ...,-'-r- q • !-.'-' -'--'.. '0.,-.: T"'-f' '--.', I:'"', '-- .,'- ,,.-:-'-' -1
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But let us return to the start of this long story, where I asked when and why I
switched from the first to the second version of the level theory! My first version
is merely a report on the Van Hiele's ideas, altho-gh illustrated by e-ampl-s beyond geometry (to which the Van Hieles had restncted themselves), m parucular
examples of complete induction. In the Van Hieles' practice I had much earlier
identified reinvention as the leading didactical principle, unless my memory deceives me. I certainly knew the part played by reflection in mathematical invention and perhaps its level-raising function (at least with regard to my own mathematical activity) and I judged it important enough to transfer it to reinvention.
This then appears in my second version of the level theory; reflecting on the higher level on one's activities of the lower one.
I always knew that my levels differed from th?se of the Van Hieles an- at many
opportunities and I stressed this at many occasions; my levels were relauve rather
than absolute ones, I said, although I gave Pierre Van Hiele the full credit for the
level idea as such. I should confess that never before have I as consciously considered that difference as I am doing now.
This brings me back to a question I have delayed discussing. If you read my quotation from Van Hieles' text", you may notice that the last paragraph does not
properly link up with the preceding text. This may be a fault of mine since- edited
that text on the basis of their publications and other data, but I think there 1s more
to it than this. Discontinuities that reveal themselves by transitory stops in the individual learning processes are certainly more numerous and diversified than can
sufficiently be explained by structuring geometry-learning according to three or
four levels- these levels are moreover said to be determined by a structure of geometrical fuought which, though significant, is however too global. When I hit
on reflection as the thing responsible for the jumps in the learning process, the
road was prepared for as many discontinuities and levels in a multitude ofearning processes as there are significant occurrences of reflection. These, then, rather than three or four levels, were what I began to observe as levels in the learning
process. As I am now looking back on my various reports on learning processes,
I cannot but state that almost all of them include examples of learning by reflection. I dealt with reflection in [97,98,137] more systematically, the contents of
which will somehow be resumed in the next subsection.
I still have to add another remark. Mathematising and reflecting are closely connected to each other. Indeed, I mentioned reflection among the various aspects of
mathematising. Of course, routine mathematising does not require reflection.
Similarly, there is routine reflection, that Jacks the level-raising jump feature.
Moreover Treffers' distinction between horizontal and vertical mathematising
should no; be confused with the relative height of level. Horizontal mathematising may just as often mean a jump from reality to fresh mathematics and vertical
mathematising a mere routine, as well as vice versa.
When I asked the question of whether this whole story is worth that much soulsearching, I did so in such a way that the reader could have anticipated my answer
to be: No! Reporting on one's own learning processes (in the present context, 102 CHAPTER2
with regard to educational research) is likely to elicit the reproach of overestimating one's own importance; though it requires some courage, this is worth being
felt as a challenge. Self-effacement is good style in scientific reports. As I have
stressed often enough, keeping silent about the process of inventing may mean a
loss to those who are expected to learn by reinvention. This is even more true in
the social sciences than it is in the mathematical and natural sciences. The lack of
hard and objective evidence can at least partially be made up for by knowledge
of the soft and subjective experiences and considerations that were crucial in the
development of some theory; thanks only to this frankness can it become accessible to understanding and criticism.
By no means can I assert that I have been as faithful to the policy of disclosing
my own learning processes (in which I firmly believe) as I have asked others to
be. In the course of the years, however, thanks to my growing insight in the value
ofreflection, I have mended my ways. I could have shortened this subsection by
keeping silent about the fact that only during this historical search did I discover
I had introduced reflection as the bond connecting one level to the pext as early
as 1961. While preparing the next subsection, I met yet another explanation of

the divergence between the first and the second version of the learning level theory. It would be unfair to conceal this fact, but let me save it for the next subsection.
Meanwhile let me add another remark, which I could not put at the place where
it belonged without interrupting the historical report, that is, after the first quotation in the present subsection: The Van Hieles' discovery is a striking example of
a didactical insight obtained through horizontal followed by vertical didactising.
2.3.5 Reflection
1. Reflection discovered
My memory includes much of what I would now call reflection. The least insignificant discovery I ever made in mathematics -- more than half a century ago --
was the work of probably less than half an hour, including theorems and sketches
of proofs. Yet ittook me at least a fortnightto fill the enormous gaps between the
apparent mile-stones along the hazy road of reasoning -- soul-searching work that
consisted in asking myself why I believed in the truth of one or another particular
statement. To be sure, at that time I didn't use the word reflection. Von Neumann
once said: "A fortnight before you prove something, you have to know it is true,
since verifying is easier than proving proper." Maybe it stands for the same feeling as I -- and certainly many other mathematicians -- have experienced many
times in the course of their mathematical life.
I don't remember when I first used the verb "to reflect" for such experiences of
mine or others. I f I may believe a conjecture uttered in [98], I even shed away
from it for a long time because it seemed too colourless a term for a mental ac
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ple, simply use it synonymously with "to think". In [98] I even suggested that the
divergence between the first and the second version of the learning level theory
was due to my aversion of the term "reflection". The story I dug up in the preceding subsection was different but, as a matter of fact, not until the sixties can I find
any instance of "reflection" in my papers and, as far as I see, up to the late seventies it occurs only in the context of the level theory.
My paper [98] starts as follows:
About a year ago I suddenly realised that I may have overlooked precious evidence in
the past when observing children. Did I look consciously enough for symptoms of reflective thinking and of the growth of reflective thinking? -- this was my problem. The
idea of a possible shortcoming occurred to me while I was preparing a paper for a conference of which the theme was 'Proof in Mathematical Instruction".
I had set out to consider the activity of proving, not as an isolated activity with roots
of its own. but as the final stage of activities that develop naturally and might be developed didactically. The stage previous to proving I imagined was constructing
which, when it happens mentally, means implicitly proving the truth of existence. But
if constructing was a stage previous to proving, there must be an intenmediate stage,
which, I decided, was reflecting.
The paper mentioned at the end of the first paragraph was [97]. As a matter of
fact, the theme of that conference was: Beweisen im Mathematikunterricht. Anyway, I postulated that one should teach proving rather than proofs, as is the habit
in traditional instruction; my e-amples were taken from geometry, which explains why I considered constr1cting as proving existence -- as a matter of fact,
this still applies outside geometry, provided "constructing" is broadly enough understood. In no other field of reinventive learning does reflection play as conspicuous a part as it does in geometry. That what is in the eye of the beholder is transferred into the mind of the thinker, and this happens by reflection: Insist on asking
yourself -- or others, as I did above -- why you see it like this, why you think like
that.
My notes include so many observations on reflection that I now believe my fear
at the time that I could have overlooked precious evidence on learning by reflection was ill-founded. Indeed, the first note of this kind dates as far back as July
12, 1943:
It concerns three brothers I will call Yed (7;7), Matthew (5;10), and Tom (3;10). John
and Matthew attend different schools. Y eel tells his father that, while playing on the
street with Matthew, he was greeted by Matthew's teacher "Hi, Yed!" "How did she
know my name?", Yed asks his father. "What do you think yourself?", Father answers.
"Maybe Matthew once told her about his brothers". Yed explained and immediately
continued "How did she know I was Yed (rather than Tom)?" "What do you think
yourself?"Shesaw that I was older than Matthew, instead of younger, as Tom would
be", said Yed answering his second question himself.
My second relevant entry is a sentence taken from a technical exposition about fishing
by aforesaid Matthew (at that time 6;9;15):
Because then the fish think that the men are mistaken thinking they are dead.
With a view on the evidence now available, I would describe my state of mind at
the time I set out to prepare my paper [97] as reflecting more consciously than 104 CHAPTER2
ever before on the idea of reflecting, and more precisely, on how reflective thinking develops. At that time
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, while wrestling with the problem of how proving
arises, I felt for the first time the need for a more compact term for thinking about
one's own [and other people's) thinking, and when I hit on "reflection", I realised
it was not as colourless as I had thought before. If I now use the term, there is an
undertone of physics in it: reflecting as mirrors do, mirroring.
I don't know whether philosophers or physicists were the first to pick up this tenn. In
any case Latin speculum for mirror was not useful since speculatio means exploring;
both. however, do have the common root specus, which means eye socket.
Indeed, when I use the word "reflection", I mean mirroring oneself in someone
else in order to look through his skin, to explore him, to take him in. And, consequently, since somebody else is like oneself --a human -- this is an experience
about human behaviour and, finally, knowledge about one's own behaviour. So
from mirroring oneself in someone else follows -- as the night the day -- the mirroring of oneself in one's own person, that is, introspection. It becomes reflecting
on oneself, on what one did, felt, imagined, thought, on what one is doing, feeling, imagining, thinking. Reflecting, once started, is an activity we perform every
moment, in order to determine our course of action, yet, as a mental exercise, it
can become an aim itself.
2. How does reflection arise?
At the age of three, children start asking "why?". It can be meant causally or purposively even though the adult may interpret it in a logical sense. The child's
"why?' may ask for arguments. If however, you wish the child to argue something, you had better ask, "how do you know?' Perhaps authorities will be cited,
parents or friends. Sometimes, in the cognitive field, the answer may be "I just
know it", or, if it is a representation "I see it". May we find fault with the child
for such a short answer? As adults, do we always know why we know something
or haven't we exercised reflection and introspection for a long time?
I coupled the question of what is reflective thinking to that of its origins. In fact we
know little about them. Lack of means of expression is one reason and the other --
more important -- is lack of attention to how we experience what we experience. Or
rather than lack of attention shouldn't I say: inability to pay attention? Nevertheless I
ventured to assert that reflecting in one's own mind is triggered by reflecting in another's mind.
I feel that this view is fortified by Piaget's research on the origin of imitation
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, which
is based on observations of his own rather than of others. Imitation starts reflectively,
or as Piaget says, circularly: initially the child imitates only those sounds and motions
of the adult that are more or less faithful reproductions of soWids and motions produced by his own, perhaps haphazard, activity. (The importlllnce of this discovery is
unfortunately obscured by such arbitrary terminology as "assimilation and accommodation", which has been overextended in its application.) Imitation starts reflectively,
that is not as self-imitation but the mirroring of one's own behaviour in someone
else's. Wh-t about ollier utter.a."l.ces, intentions. thou.-hts? For instance. ex:periencin-
i-·:.:.
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haviour may or may not lead to questions like "why did you do it?", "did you rightly
do so?", or "would other people do the same in this situation?" Is it a long step from
here to asking oneself the same question? And, even then, what is asked for may be a
reason rather than an intention.
There is one argument why reflective behaviour in general should start with mirroring someone else's mind. The argument is language, or more generally, communication. The child learns to say "I feel, I want, I think" from other people.
Well before saying it the child must interpret it How does it succeed? The analogy -ith what happens in imitation would indicate that the child succeeds interpreting " I feel, I want, I think" as his own action because he is prepared _to, because somehow he "knows" what it means. But even then, the outside stimulus
is needed to make it explicit.
Let us shelve these questions! We simply know too little to be able to answer
them. Learning how to observe children is a prerequisite to more knowledge. The
little I know are (translated) quotations from diaries.
3. Modes of reflection -- shifting standpoints
Let us turn to more easily accessible questions! Reflection unfolds itself under
many aspects. One of them is what I would call shifting one's standpoint-- shifting mentally though the standpoint itself may be physical or mental, while the
shifting may take place in,space, time, or any other, say, mental dimension.
With respect to the "points" considered let us distinguish a few possibilities of
shifting -- without any pretence of being exhaustive. One is
shifting from A to B in order to look back at A,
which I call
reciprocal shifting.
Another is
shifting from A to B while considering C,
which I call
directed shifting.
And finally
shifting A's environment to B's,
which I call
parallel shifting.
The most concrete realisation of reciprocal shifting is looking into a mirror in order to know how one appears to others. Another example is a reciprocal shift in
time, concretised by a shift to an older or younger person: when I grow older, I
will be able to climb higher than now; when I was younger I could not yet do what
I can now. The mental reservation is also a reciprocal shift of standpoint: after I
receive supplementary information I shall judge the present situation better.
An example of directed shifts of standpoint, and then in succession, is describing
a path. Well, the easiest way to do this is by pointing.
We took our walk. Bastiaan (2;8) found the top of a bike-bell. When he dropped i- it
made a sound. So he dropped it repeatedly. He threw it, he kicked it. I found it danger-106
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ous. "Let us go to New Clarenburg!" (a building with a broad terrace), I said and put
the bell in my pocket. At New Clarenburg he asked for the bell. I warned him that it
could fall down from the (one metre high) terrace, which indeed eventually happened.
Bastiaan looked down at the bell and then said 'get it, down the stairs" while pointing
the way with his hand. This took place at a distance of about 10 metres from the stairs.
Not only did he make a plan of how to recover the bell, but he also disclosed it verbally.
Describing a path is an example of a directed shift of standpoint or of a sequence
of such shifts in succession. As I said, the easiest way to do this is by showing,
as Bastiaan did to recover the lost bell, albeit complemented by the verbal explanation "down the stairs".
Monica (4;4) wants to define some place by means of the walk leading to that point.
Her forefmger raised yet motionless, she exclaims: .. and then you go so, and then you
go like this, and then you go like this, . . . ". Her wide eyes betray her intuitive vision:
she clearly sees the path in her imagination but she both lacks the mimic and verbal
resources to describe it. Bastiaan (6;4) in a similar situation shows with his hand which
direction to follow .. after crossing the sluice gates ...
Describing or explaining a real or imagined situation may require mentally shifting one's standpoint.
We passed a spot by car where an accident had happened: there was a damaged car,
and across the road something I could not identify, a tube or a pipe. Bastiaan (4;2) said
something like: "It was a streetlamp, and I could see that because I am sitting on this
side and you couldn't because you are sitting on that side."
Bastiaan shifted as it were to my seat in order to explain why I couldn't identify
the fallen streetlamp. But, in general, one may ask whether shifting one's standpoint is facilitated by assuming another standpoint or may in fact sometimes be
blocked by this. In order to draw a child's attention to some object, one may point
to it or turn the child's head in the desired direction, or move or lift the child to
overcome some obstruction. The child is well acquainted with these procedures
-- it is kind of baby language -- which may stall its learning what is relevant in
particular cases and developing more sophisticated means.
When the sun had just set, Monica (4;8) asserted she saw Venus near the crescent
moon; probably she was right, whereas my eyes' acuity failed. So she wanted me to
sit on my heels in order to see the planet from her -- lower -- viewpoint.
Her sister Daphne has always resisted having her body or her head moved as a device
for showing her an object, and she refuses to be helped in almost all physical activities.
Asking preliminary questions before tackling the main question may also be an
example of directed shifting of one's standpoint. Before interviewing someone
on a certain subject, one tries to find out where be is standing, and in order to do
so, one changes one's own standpoint. In the long run, this may be the most striking feature of reflective thinking.
Reflecting oneself in someone else may be facilitated by family relations. The
generation shift is an outstanding examnle of what I called parallel shifting.
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very yo\Dlg age he mirrored his relation to his younger sister Monica in the relation
between his parents; (this was reinforced by the relation between his grandparents,
who lived nearby). When they grew up, he said. both of them would live in a certain
now vacant house nearby and would have a little Bastiaan of their own. From the intrageneration relation the stress was soon shifted to the inter-generation relation: when
his mother was a child. Grandpa was her father and took walks with her. and when
Grandpa was a child, he had also a grandpa. Quite early, Bastiaan structured the past,
in particular the technical development. by means of the generation shift: .. what did
cars look like when you (grandpa) were as old as I am now?". "was there T.V. when
Mum was my age?", and many more and more complicated questions.
Of course in general, young children are not in Bastiaan's situation to perform
such clear double generation shifts. It is perhaps this situation that explains his
early differentiation of the past. As a contrast let me cite a teaching experiment
with third-graders, the vast majority of whom were unaware of any differentiation of the past until they were taught it (which also happened by means of generation shifts)!
4. Conclusion
I could continue in this way for many more pages, although when rereading what
I had written I asked myself repeatedly whether I should.not spare the reader
these stories. Indeed,. most of what I have recounted here about reflection does
not properly belong in a chapter titled 'Didactical Principles". Or does it?
I finally decided to include these stories (albeit shortened). I did so because of the
concluding paragraph of [98] (from which these examples have been extracted):
Reflective learning at school
Unfortunately. this section, which should be the most important, is almost empty. Observations of other children (except for Bastiaan, then 9;2) would not fit into the
present frame. I never observed Bastiaan at school, nor did he grant me more than a
few glances into the growth of his scholastic achievements. At that age he fully widerstands the meaning of the arithmetical operations, he knows how to perform them,
though his performance is not yet flawless. He accepts all he learns without any criticism, and in scholastic matters he never asks the question "why?", though he is still
asking it in any other maner. He is not interested in questions like "why does 3 x 4 =
4 x 3?". let alone in reasoning about spelling.
However, I think this is a transitional stage. School is a phenomenon he has still to
learn to cope with.
To what degree he eventually succeeded I am unable to judge, although later on
I noted splendid examples ofreflection in his diary [124, 137, 140, 141, 142, 143,
145, 148, 160, 166, 184, 185], most of them related to mathematical experience
though none of them to school mathematics, which, as a matter of fact, he managed to master as indifferently as easily. This was thanks to the character of the
curriculum and instruction, which matched his algorithmic talent on the one hand
and on the other lacked any bonds with reality, and moreover any challenge to
reflect on his own activities.
This is not exceptional. Little attention is paid to reflection in mathematics in-108
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struction, although this can hardly be blarued ou the teachers alone, since not
much more attention is paid to it in research, development, and teacher training.
Let me characterise traditional mathematics instruction by copying a sentence
from a few pages ago:

In order to draw a child's attention to some object, one may point to it or turn the
child's bead in the desired direction, or move or lift the child to overcome some obstruction. The child is well acquainted with these procedures -- it is kind of baby language -- which may stall its learning what is relevant in particular cases and developing more sophisticated means.
2.3.6 Reflection and observation
The present section started with my demand to observe learning processes.
Jumps, I claimed, are the most striking 'observables' -- blockades are perhaps
just as easy to notice but less easy to appreciate and explain -- jumps to higher
levels. Look for the level-raising reflection, concealed or manifested in the
jumps! The reflecting mirror is the observer's most powerful tool, which closes
the circle. Let me quote myself:
So from the mirroring oneself at someone else it follows as on the night the day, mirroring oneself at oneself, the introspection. It becomes reflecting on oneself, on what
one did, felt, imagined, thought, on what one is doing, feeling, imagining, thinking.
Reflecting, once st- is an activity we perform every moment, in order to determine our course of action, yet, as a mental exercise, it can become an aim itself.
Observation is the converse: mirroring the observed one in oneself. Or are things
more intricate? Who is the actor, who the observer in a learning process? Look
for the interaction in a learning group, while involved in solving a problem!
Whether involved or not, it is easier to observe others than oneself. Does the other
person understand you, does he approve or disapprove? Does he feel what you
feel, can he help you better than you can help yourself to understand what you
mean? What can a third person listening to the exchange of vague arguments
learn from and contribute to it? Why do I not understand you, why don't you understand me, why do these persons not understand each other? Is it lack of information, or what information does one of them possess and consciously or unconsciously conceal, which prevents the other from understanding? Or is the other
feigning that he does not understand? How can we know what to tell and what is
too trivial to be told?
We are driving by a shop belonging to the Jamin chain. The name "Jamin" appears
twice in Neon letters on the front of the building, horizontally and vertically, but separated. fu the horizontal version the letter Mis missing. I drew Bastiaan's (8;2) attention to it. He said: This is only possible with [an] odd [number ofletters]. I didn't understand what he meant He became angry: .. You know very well what I
mean." I told
him, as I had often done in the past. that I never lost my patience if he didn't underst:mrl m-. (He shon1d flo l.i.kiev-ii;;:e a.'l.d th5;'l\ abnut t:he reason -vhy I didn't JJT1de:rst2-nd.)
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number of letters must be odd if the words are to be put together to form to form a regular cross. -- Let me add that at that period the ideas of even and odd strongly occupied
his mind.
2.4 LONG-TERM LEARNING PROCESSES
The present title was announced in (2.3) as being my actual aim. Again long -
term learning processes means: long term teaching/learning processes. How
long? Life long! Extending beyond the limits of school life? These questions are
not lo the point. The limits, whether reached or not, are set by the teaching goals
as viewed by the designer. Well, there is no absolute limit: each goal, as definite
as it may look, can be outdone by more ambitious ones, which, however, are
meaningful only from the viewpoint of those previously viewed as definite.
Textbook series are examples of long term planning of teaching/learning processes. I mean another thing: planning long term teaching/learning processes. Later on the difference will become clear.
2.4.l Learning to forget
. .
An ambiguous heading: learning lww and what, or in order to forget? Both of
them are worth discussion. As every teacher knows, breaks in learning processes
may cause total losses of learning products. Starting a new school year often
amounts to starting with a clean slate. In my country successful final examinations are celebrated by hanging satchels at the flagstaffs. Indeed, after the examination, most of the stuff learned may be forgotten never to be remembered, and
this extends even beyond the narrow limits of what was taught with the sole aim
of the examination. It is too old a story to be told again: people who, though failures in school mathematics, succeeded in life, or so they claim. Or even worse:
people who assure you that, thanks to a mathematics they never used explicitly,
they learned a lot of valuable things (especially logical thinking) and, whether
asked for or not, substantiate this assertion by examples that are as many proofs
to the contrary.
No doubt, in every respect, discarding is no less important an issue than keeping.
Beyond a multitude of subjects and activities, one has to learn which ones are
worth forgetting and which ones remembering. Much depends on the learner, on
individual inclinations and aversions, that foreshadow the learner's future life.
This does not absolve teachers and educational developers from the task of steering and designing the process of forgetting as well as that of recalling. Mathematics can again be distinguished from other areas of learning by this fact itself and
by the way this task is to be tackled.
We viewed mathematics as an activity, and learning mathematics as guided reinvention. In contrast with traditional instruction, the stress was shifted from instructional products to instructional processes. This shift cannot but influence the 110 CHAPTER2
line to be drawn between potential remembering and forgetting. Anyway, if
learning processes are so important that they are specified by guided reinvention,
it is quite improbable that their ultimate destination should be obliteration by
their products, although this feature does seem characteristic of mathematics. I
am going to explain why this is a serious misapprehension, caused by a wrong
perspective.
Let us take an example from geomelry! A famous theorem, by folklore ascribed
to Pythagoras, is learned by the great majority of young people, and the fact that
it is famous proves that it is even remembered by a great many adults, or so it
seems. Many teachers and even more textbook authors feel obliged to provide
learners with some proof, though it is an undeniable fact that the theorem is successfully applied innumerable times by people who never learned proving it or
forgot the proof they learned. To be honest, even in the highest regions of mathematics, people often apply results obtained by others without checking the
proofs or caring at all how to prove them.
So why learn proofs if afterwards they may be forgotten? Once one of my colleagues was asked by a freshman: "Excuse me, Professor, isn't it a curious habit
of yours that in your course you prove each of your statements? Why do you do
so? Are you afraid your students don't trust you?" It is a question that can be answered in manifold ways. Mathematics is different, I claimed. It is a remarkable
fact that, in mathematics, each statement can be corroborated by a proof, although
it is not the fact that matters here where we are concerned with teaching and
learning mathematics, rather than with mathematics as a deductive system. As we
noted several times, the relation in mathematics between theorems and proofs is
not akin to that between products and processes in mathematics instruction.
What then can be the didactical value of mathematical proofs? The answer depends on how a particular proof is expected to function in a course as a whole. A
proof can be held up as an example to what proving properly means, and let us
hope that as such it is paradigmatical enough to convince the learner! Or a proof
can be an opportunity to teach a number of isolated facts or activities in an integrated way, that is by logical linking, although the course of reinvention itself
may be as forceful a linkage, or even more so. And finally, by analysing its proof,
the teacher may discover how to guide the learner to reinvent the theorem it is to
prove. Proofs refashioned for this aim are valuable in themselves as sources of
insight On the other hand, proofs with the sole aim of being forgotten after having produced the theorem they are to confirm, are didactically worthless.
This, then, is the intrinsic didactic fallacy of traditional teaching of mathematics,
as well as that of its all time high in New Math: the large extent of subject matter
learned solely in order to be forgotten, aggravated by the eventual criterion of the
examination. In addition to this there is the extrinsic problem of what school
means for life: How to adapt the relation between remembering and forgetting to
the diversity of demands of future lives?
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2.4.2 Remembering learning processes
" If learning processes are so important that they are specified by guided reinvention," I said, "it is quite improbable that their ultimate destination should be obliteration by their products". Learning processes have a value of their own, which
entitles them to be remembered. Not in detail., of course, as there are very few
things we remember in detail, if any at all. Rather than details, we record essentials, or what we take to be such. Anyway, that is all we can recall and, if need
be, put to good use.
Learning processes, or at least part of them, can be more essential than their products. As far as they fulfil this condition, and in the way they doe this, they should
remain accessible to memory. Not in detail, to be sure, but to the extent and in the
fashion that they are essential. Few people, if any, remember how they learned
arithmetic. This is a good thing if they learned it thanks or in spite of bad teaching. Fortunately there are people who can reconstruct the learning process as it
would have taken place if they had been allowed to reinvent arithmetic. It is those
people to which the old adage, cited in (2.1.4) applies: first learn, then understand. And the others? They are better served by guided reinventive learning,
duly recorded.
Written records, even the most circumstantial ones, are far from being complete,
but a good record allows others and oneself to repeat the process recorded, maybe
in a more streamlined way than it actually took place. Mental records are no different Mental records are due to reflection. This, then, is what makes processes
of learning by guided reinvention accessible to memory: built-in opportunities to
reflec- which can be reinforced by verbalisation and by communicative interaction with the guide and between the guided ones.
Mathematising is to a great degree liable to obliteration by its result, that is,
ready-made mathematics. Once schemes and forms have been consolidated, once
concepts have been attained, once short-cuts have been performed, the stages of
schematising, formalising, constituting mental objects and short-cutting may be
dismissed as dispensable nuisances. Algorithms are an utterly extreme case.
Once mastered, or believed to have been mastered, they are most likely to disavow their origin. Indeed, it is the great virtue of algorithms that they can be performed mechanically, as it is their drawback that they become useless or even
dangerous as goals in themselves: mathematics identified with performing algorithms. It is algorithms that created what looks like the fundamental antinomy
within didactics of mathematics: insight versus drill.
2.4.3 Insight"
A cherished antinomy in teaching and learning mathematics is putting on one
side of a deep gorge such noble ideas as
insight, understanding, thinking, 112 CHAPTER2
and on the other side such base things as
• rote, routine, drill, memorising, algorithms.
If I were malicious, I would add another pair of opposites
• theory versus practice,
suggesting that learning by insight is a noble theory while base practice is learning by rote and memorisation. However, it is not that simple, and it has never
been so. Even in our computerised age, children memorise tables of addition and
multiplication and acquire certain skills by rote, though one might argue that the
balance has shifted in favour of the nobler activities due to the rise of the computer.
It is not that simple, firstly because the question is not which side of the gorge to
choose but rather to bridge it by the learning process that I called schematising
and formalising. Secondly, I do believe that, at any time more mathematics has
been taught from the viewpoint of insight and more has been learned by insight
than we are aware of -- indeed, common sense is insight. Everyone agrees and
textbook writers have witnessed that elementary arithmetic cannot be learned in
any other way than by insight, whether it is taught that way or not. But it is also
true that, as things go on, as teaching proceeds to ever higher grades, to column
addition and multiplication, to long division, to fractions (ordinary and decimal),
to algebra, to learning mathematical language -- the part played by insight changes. The learner's insight tends to be superseded by the teacher's, the textbook
writer's, and fmally by that of the adult mathematician. And the same tends to
occur on the long winding road which leads from concretely understood word
problems to highly formalised and badly understood applied mathematics.
This is why people who advocate learning through insight, disagree about what
insight is. New Math's wrong perspective was to replace the learner's insight
with the adult mathematician's.
Yet this is not my main point. I have still to explain why we are not aware of how
much is learned by insight nevertheless. It is quite natural that, once an idea has
been learned, the learner forgets about the learning process, once the goal has
been reached, the trail is blotted out. Skills acquired by insight are exercised and
perfected by training, intentional and unintentional. This is a good thing. What is
bad, is
sources of insight clogged by acquired routines, never to be reopened,
though that is what usually happens. It explains why upper grade teachers so often complain about teaching habits in lower grades. If it is restricted to the first
acquisition of some idea, learning by insight does not deserve this name.
What is crucial, is
• retention of insight,
which is gravely endangered by drill, that is
• premature training,
too much training,
r\~") SJJC-'.
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how to keep the sources of insight open during the training process,
how to stimulate retention of insight, in particular in the process of schematising and formalising.
How can this goal be pursued? The solution I have proposed is
having the learner reflect on his learning process.
As I explained earlier, mathematics is to a great degree reflection on one's own
and other people's physical, mental and mathematical activity. The origin of
proving theorems is arguing things that are apparently obvious. Nobody tries to
prove a thing unless he knows it is true. This he knows by intuition, and the way
to prove it is, as I claimed, to reflect on one's intuitions. Successful learning processes, if observed, should be made conscious to the learner in order to be reinforced and in order to be recalled if needed. This, however, is not what usually
happens. Let me illustrate this by an example I have cited many times.
Many children and adults can tell you that in order to multiply by 100 "you have to
add two zeros" (which is only true for whole nwnbers) yet most of them cannot explain why. Even worse: most of them don't even understand that the matter can be argued and why this should be done. Did they learn such rules by rote? I don't believe
so. I have observed too many children applying such rules intuitively before they were
verbalised and formally taught at school. Rather than be-g taught the rules, they
should have been taught to argue their intuitions, to reflect on what appears to be obvious. But this requires more patience than teachers can afford. Indeed, compare this
with my story of the girl' who was taught the rules on fractions after she had acquired
a good working knowledge of fractions. (See 2.1.4, 3.)
2.4.3.1 Testing insight
When, in 2.1.4.3, I discussed the jumps in the learning process of the girl I had
been observing for a few years, I promised to provide an example to show how I
tested her insight, that is, to see whether its sources had not been clogged during
a period of training.
She had come to the point that she solved equations like
l l 1 1
---: :t=-:c--
2 3 2 3
according to the rules of the art, automatically as it were. I had a strange feeling:
the better she performed, the more urgently I asked myself whether she still understood anything. Finally I was terrified by her excellent performance. On the
other hand, as explained earlier, I dido 't dare to have her verbalise her behaviour.
Then I got an inspiration. I drew a horizontal numberline, that is, without numbers, having only an origin and equidistant marks. I took an interval between
thumb and forefinger, around the origin yet not symmetrical, and asked her: " If
xis here in between, show me where 2x is!" She did it with her thumb and forefinger. "Where is i• ?""Where is x ÷ 2, x - 2?" "Where is -x?" This was the first
hitch but it was soon conquered. Then I asked the converses. " If 2x is here in between, where isx? I f 1• is here in between, where isx? After a series of questions 114 CHAPTER2
like these I gave her wriuen problems. She already knew the notation -3 < x < 2
for "x between -3 and 2". Step by step I led her to sums like
if-5<-x-1<6
what can you tell about x? Or
if 1-lx<lx-1
2 3 2 3
what can you tell about x?
So I became convinced that her insight had been preserved in spite of training.
As an afterthought, the above seems to be the didactically most promising means
of formalising the mental object of variable. (Cp. [ll9a] .) It is a pity that I didn't
mention it in [146].
2.4.4 Training
Advocates of insightful learning are often accused of being soft on training. Rather than against training, my objection to drill is that it endangers retention of insight. There is, however, a way of training -- including memorisation -- where
every little step adds something to the treasure of insight: training integrated with
insightful learning. If I compare our mathematical textbooks -- from the lowest
to the highest grades -- I cannot but notice a historical trend toward this kind of

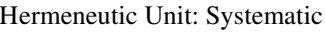
integration, In the past there was a clean separation between so-called theory (often nothing more than a model) and its so-called applications (or rather, imitations of the model), which were as numerous as the theory was meagre; and the
closer one gets to the present day the more deliberately a more vigorous theory is
broken up into insightful steps, each of which may contain both a training and a
reinventive aspect Big problems, posed in contexts, can serve large scale training
better than a flood of small applications. In other words, textbook authors have
shifted from extensive to intensive training. For teachers who learned mathematics the old way and accuse the new style of being chaotic, it is difficult to use such
textbooks adequately. Students, however, adapt to them more easily.
At the primary level there are clear indications' that reinventive learning -- learning column arithmetic by progressive schematisation, for instance -- is far less
time-consuming than extensive training. This even extends to pure memorising,
say of multiplication tables'. where children are allowed to use their own, socalled informal, methods. At the secondary level, comparisons are not so easily
made because of the interweaving of subject matter and teaching style. A more
fundamental divergence is, indeed, whether, for instance, exponential functions
are introduced as a formal subject or as the result of exponential growth in realistic contexts. The most intimate long-term integration of reinventive learning
and intensive training with which I am familiar, is L.Streefland's research
6
on
teaching fractions in grades 4 through 6; this research, moreover, include- 13 glo-
- ...
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(2.1.2, 6) (dividing a number of pizzas among a number of children sitting at a
number of tables, where the symbolic recording proceeds from a suggestive pictorial notation towards the standard one by means of formal fractions) as shown
in fig. 13-15 from 6, p.203, 269, 302. The learning process is an alternation between what Streefland calls the children's own constructions and productions,
which may roughly be distinguished by catchwords like: open end and open start.
Fig. 14
-cp
1-0Jt.f.P-X
-- ,fi 0
__1-fQC. -- '''tlr I[:,"
- +- i ''' 2 ".1..-___]
' >_.•ii];oJ);rJ.- (j
\-XI •t 1l
i--rf-------
Fig. 13
6xt+
6
1 =.
51'-t L.l.l-::::.
`-J-t f-fi.:'
l '(i-•./-
+ 6: -
Fig. 15 116 CHAPTER2
2.4.5 Organising the learning process
l . Levels
Why did we pay so much attention to the level slructure in learning processes?
Let me quote myself from (l.1.2):
Sure, some. algorithmically gifted people, learn to apply even imposed algoritluns adequate!-; others -- perhaps -e majority -- fail to identify the new algorithmic procedur-s -1th the commo---ens1cal one- from which they should have originated through
abr!dgng and streamlining. They fail because some time in the past they were asked
to take mental leaps which exceeded their mental powers. Even though they flawlessly
learned the algorithm, they will fail to use it in true life situations where common sense
counts; they will instead depend on less efficient lower level operations. Researchers
have signalled this .. relapse" and marvelled on it. Rarely, however, has it been diagnosed '.15 a con_seque1!- of instruction. since no alternative instruction was envisaged
alongside the l!ilpo51tlon of the new algorithm (which was sometimes embroidered
with explana-ions for conscience's sake). The new algoritlun, however, never did have
-e opportuni- to reac- the state of common sense; afraid of applying a wrong algonthm or the right algonthm wrongly, the learner instead relies on what has remained
common sense to his mind.
What I stated there ';ith regard to inl"_lsed algorithms applies more widely, l
daresay, lo almost all imposed mathemaucs,. Indeed, the fact that researchers signal and marvel at such relapses is shared by a variety of fields. Whatever levels
may mean theoretically, it is the duty of the organisers of learning processes to
respect them. The fact that a few -- gifted people -- can get along on levels to
which they have been lifted inconsiderately, does not justify ignoring levels in a
teaching strategy. The question of how jumps in the learning process can be diagnosed, has already been answered. Teachers (or peers in learning groups) need
to insist on the justication of what appears to be new knowledge or new procedures, thereby requiring the inventor to reflect on what he -- consciously or unconsciously -- performed. This need not necessarily be done by verbalising,
which anyway should not be enforced; a paradigmatical status, witnessed by repetitions, is enough to prove the jump in the learning process.
If levels are to be respected, than learning processes must be organised flexibly
enough to be adapted to the needs of learners or learning groups. This demand for
flexibility also applies to the goals of the learning processes.
2. Differentiation
How fast and how far learning proceeds, depends greatly on the learner and on
which way and whether the goals staked out are eventually attained. Reinventive
learning cannot possibly be planned rigidly, but lack of planning may be even
worse. in Z.1.2 both didactical extremes were presented within one examole: the
-·-·:-'" 0'•7 .',' 0f ''Pl'');:.:..'-'l'- .,,;_-:-.. -, +_'t'··C>' 'l' -·1-J<-'·:'..-l-';';'.C/ - r..-1 t---:::- 'L-.c ii_f _ - -:-l-':' ·' '. -r----. --'.': --1 ---;--- -.,, '; --.t'''':'i
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ate ones, a widely branched network of possible learning roads is then opened lo
them. It becomes even wider in the case of learning the algorithms of column
arithmetic. The standard algorithms are theoretically much less compelling than
is usually assumed; there are variants, indeed, which don't commit teachers and
learners beyond their didactical function as guidelines. Comparative research on
such variants, if viewed didactically, has never been more than a hobby, and is
unlikely to survive in the computer age. The advantage of reinventive learning,
and in the present case of progressive algorithmising, is that individual learners
or learning groups may advance at their own pace and eventually reach the best
of the goals accessible to them.
In general, this means organising inslruction so that, rather than it being differentiated in advance, the learners differentiate it themselves, and do so on levels as
high as are accessible to them: spontaneous versus imposed differentiation. In the
insiructional design, branches are preferred to dead ends and branches that may
lead back to the main stream, to branches that lead nowhere. If we may use metaphors I prefer that of the jump, if necessary produced by change of perspective,
to the not unusual one of the sharp turn, where no perspective at all is involved.
3. Prospective and retrospective learning
It is a well-known experiential fact that column aritllinetic may function both as
a motivation and an opportunity to improve one's knowledge of addition and
multiplication tables!: This experience will be viewed in a broader didactical context, as is indicated hy the above heading.
In former times, mental addition, subtraction, multiplication, and division were
taught one after the other and separately; arithmetic teaehing was divided into
stages "up to 10", "up to 100", "up to 1000", and so on; fractions, measures, proportions were dealt with as separate subjects. Things have since changed and will
change even more in the future. As everybody can observe, children who have
just mastered additive arithmetic up to 10, are proud ID tell you that 10 + 10 = 20,
or even that 20 + 20 = 40, that 3 x 3 = 9, and to recite the multiplication table of
l, of 10, even perhaps of 5. In their informal arithmetic, doubling and halving often precede systematic learning of the multiplication tables, with as next best,
squares which occupy a particularly cherished place in the tables. Rather than
suppressing such predilections by premature systematisation, mathematics inslruction should take advantage of them, which is actually happening in more and
more textbooks
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•
This is what I call prospective learning. (Other terms are "anticipatory learning"
or "learning by advance organisers".) There are many more examples. Ratio, for
instance, has profound visual roots, which can be arithmetised early on by estimate and measurement. There are many informal opportunities in contexts for
common sense ratio in everyday language before it is dealt with more systematically and formally. Long before fractioning the traditional "cake", the clock dial
is divided according to halves and quarters of an hour, which, unlike the cake segments, have an existence of their own. Functions have visual and numerical roots 118 CHAPTER2
in graphs and tables, and describe situations and processes, long before they are
focused on as such. In general, deep roots should be preferred to virgin soil, and
opportunistic to systematic learning.
I chose the term prospective for this organisation of learning as a counterpart to
what I call retrospective learning, which means recalling old learning matter
whenever it is apt to do so and worth being reviewed from a higher stance or in
a broader context. Retrospective learning serves dual purpose: it roots the new
matter in the old one, and it strengthens the old roots. Learning a new idea is often
nothing but becoming more conscious of a complex of previously less conscious
pieces of knowledge and abilities and of their interrelatedness. If experienced as
such, it is retrospective learning. When the laws concerning the arithmetical operations are formulated, their former occurrences should pass the retrospective
review in full. This means including -- besides those of an arithmetical character
-- also such geometric experiences as: when pacing a circumference to measure
it, it does not matter where I start; when measuring areas of rectangles or volumes
of beams, the length, width and height are interchangeable data. Combinatorics
is a particularly fertile domain in which single paradigms may rule broad fields
of isomorphic problems (cp. (87, chap.IV, 9]), antedating for a long stretch of
time their general expression and its verbalisation; but as soon as this goal is
reached, the old experiences should be renewed and their paradigmatical character should be understood as a general feature of isomorphism.
It has been a habit of adult mathematicians to review old ideas over and over just
as it has long been experienced that this activity leads to ever deeper understanding. Young learners should not be treated differently. Prospective learning should
not only be allowed but also stimulated, just as retrospective learning should not
only be organised by teaching but also activated as a learning habit.
4. Intertwining learning strands
Just as prospective and retrospective learning aims at an integration of past and
future learning processes, so does intertwining learning strands" locally, yet
with a view on the involved learning processes as a whole. Rather than running
on separate tracks which, except for incidental references and loans, are independent of one another, learning should be organised in strands which are mutually intertwined as early, as long and as strongly as possible. When loose ends are
inevitable, they are taken up at the first opportunity where they can be connected
to other ones in order to be continued. In a sense, examples of prospective and
retrospective learning can also serve as such for intertwining learning strands, or
at least for points where intertwining can start in order to be continued more consequently. But let us look for other examples!
Ratio and fractions can go together right from the beginning -- and I do mean the
very beginning: visual comparison of separate objects is paralleled by comparing
the parts of one object Double scales on the numberline and proportionality tah-·,--s :- -of--- e "2-1"1"- -_,.-.!"--_ ?.:- ,,.1/e-H "1.S ---;;tiI"i -"---::it.: t ---c"1--,,,, - ::.,+--) ___ _.:.;.- _Y-"'--:.
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twined in various ways. Functions, graphs, and equations, which in many textbooks form separate chapters, should be intertwined as learning strands. Thanks
to the geometrical-algebraical permanence principle (cp.(146], chap.16]), learning negative numbers, vector algebra and geometry, and linear graphs and functions can be so closely knitted that they appear to be one subject
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• The intertwining of algebra and geometry must not be restricted to traditional analytic geometry; plane and solid geometry must not be dealt with as separate subjects.
Teachers who prefer systematic instruction are likely to accuse this approach of
being chaotic. They forget that systematics is an a posteriori contraption. What
looks like chaos may be well-organised didactically, while, on the other hand the
system as such may be a subject of guided reinvention.
5. Active learning
In the first chapter I approached mathematics phenomenologically as an activity,
and the overruling principle of the second chapter was to do justice to the first,
which I tried to do by advocating guided reinvention. Modern textbooks and other recent literature are a wealthy source, providing examples of mathematics as a
learner's activity which can be stimulated and realised by guided reinvention.
Rather than quoting some of them and keeping silent about others I preferred to
look for a paradigm that displays as many features of the reinventing activity as
possible. I didn't fmd it in the literature, although I can hardly believe that it is
indeed as new as it looks to me.
Eeny, meany, miney, moe.
Catch a tiger by the toe.
If he hollers let him go.
Eeny, meany, miney, moe.
Isn 't this a way of counting? As a matter of fact, in a few languages with which
I am familiar, the term for this game of reciting such a rhyme while moving the
forefinger in the round is derived from the word for counting. It is counting; only
the words are different. But how much do words matter in counting? After all, the
children playing the game give the numbers different names, depending on their
mother tongue. In some languages I know counting rhymes that start with an ordinary number sequence; in German one even goes up to 20 and then continues
in normal language, where eventually Danzig rhymes with zwanzig.
So why not all the way count with genuine numerals? How far? One can agree
on a limit. For instance, up to 100. This "why" or "why not" may become a point
of discussion -- I mean in the classroom, before or after the lesson or sequence of
lessons I have in mind. 120 CHAPTER2
Fig. 16
Let the children be seated in a circle; let them count themselves rather than be
counted off by one of them or an outsider but to count themselves. One of them
starts with l, the neighbour says 2, then comes 3, each adding l to the number of
the predecessor, going on up to . . . No "up to", because after the last in the circle
the starter simply continues counting. One round, a second, a thin!, . . .
In the drawing in figure 16 I numbered the four stages 0, l, 2, 3: at the start 0,
after the frrst round l, after the second 2, and after the third (and a little bit more)
3. (I apologise: I drew it as though I myself had counted off the children, that is,
clockwise. I venture a bet that if they coumed themselves they would do it the
other way round. As in writing things are being done from the left to the right.
Two variants or additional activities: During the counting process a pile of
number disks (bingo cards) are passed around, and each child takes one in turn
while saying the numberout loud. Or each writes his numbers on a sheet of paper.
The sketch shows 16 children. Most classes are larger. But the circle need not include the whole class. On the contrary, it is just to the purpose to start in a small
way, preferably in pairs, speaking or writting down in turn the odd and even numbers. After that, groups of 3, 4, 5, . . . ; 10 is particularly nice.
What has this game to do with mathematics? *My* answer is: all you can make of
it. To prove it I will give you a list of questions I would put to individual children
or the whole class:
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two rounds, after three, ... ? What did you get a round ago, two rounds
ago, ... ? What in the first?
2. What numbers would you have got, if number 1 had started as ... ? What if
number ... had started as 1?
3. Who gets number 100, and after how many rounds?
4. What will your first number above 100 be, and after how many rounds? What
your last below 100, and after how many rounds?
5. How do your numbers change if counting is inverted?
6. If you did the game with ... as many mates, how much faster would they reach
100?
And a more general question:
7. What catches your eye if you look at your set of numbers? Any pattern, any

regularity?
It will be clear what I meant when I asserted that this game has as much to do with
mathematics as one can make of it. Certainly the four fundamental operations --
the multiplication-tables, and even more striking, the division with remainder
stand out in bold relief. What kind of division? N numbers divided among n children, or N numbers arranged in circular groups of n? It doesn?t matter, and this is
one more aspect that does matter.
Remember, the idea was to.start with small groups, but in the long run it is didactical wisdom to resume the game again and again, as long as the children enjoy
counting out So it can serve as meaningful training. But since even little children
can do it with large numbers, as can older children with small ones, it is a marvellous opportunity for prospective and retrospective learning. Another striking
feature is the tight intertwining of learning strands. The learning processes will
be differentiated according to how the problems are tackled on various levels,
which can be discovered as such by reflection and transmitted by group cooperation. Last but not least, it is a rich context, the richest one can imagine: a children's game.
As for the guide: an excellent opportunity to observe learning processes.
2.4.6 Acquiring a mathematical attitude
The original heading of this subsection was: Products of learning processes. Indeed, learning processes are not aims in themselves. But how to define their products? Hard products, though interminable in high level mathematics and valuable
as objectives of mathematical research, are not the motive for teaching and learning mathematics at school, nor is their number significant. The pupils are expected to develop certain abilities. Detached from the learning processes by which
they were acquired, these abilities may lead lives of their own as methods and
skills, and as such they may be exercised, studied and tested. One can try to list
and classify them, but what is the use? It would not answer the question of why 122 CHAPTER2
they should be products of learning processes. After quite a few false starts I decided to attempt a comprehensive answer under the above heading.
In 1975, in Hungary I happened to take part in a conference Evolving a Mathematical Attitude in Secondary Education -Age Range 14 -18 Years. I started my
contribution27 by taking exception to the general theme. I did so for two reasons:
First, 14 years is much too late to shape a mathematical attitude, although it is not
too late to develop it further, provided one knows on what basis one can continue
from 14 to 18 what was achieved at lower age levels. My second, and even more
important argument, was that a mathematical attitude is more easily described for
lower than higher ages if such descriptions are to be based on observations rather
than on logical analysis. Indeed, what matters is the interior rather than the exterior attitude, and the older people grow, the more their interior and exterior lives
will diverge from each other; the most extreme instance of this phenomenon is
found in publications on mathematical discoveries, where any intellectual or
emotional detail of the discovery process is carefully avoided. The only access
left to the interior life of adults, is introspection, but this is a hard thing to realise.
So, if attitudes are to be described on the basis of observations, one is very likely
to find more useful material in the lower age bracket than in the one.
How to describe a mathematical attitude? Such descriptions have often been attempted, though most often in too general terms -- thinking clearly and distinctly,
criticism, looking for problems, killing problems. It is the drawback of such descriptions that by the way they are formulated, they do not properly refer to mathematics. The opposite of this procedure can be found in Polya's repeatedly mentioned work, whose tendency, rather than whose details, can be a source of
inspiration -- though less of subject matter -- for didacticians of mathematics.
Mathematical attitude can, in my view, be described most efficiently as the mastery of big strategies, and this is the way I will use the term. I will deal with five
strategies, while referring to experiences in actual teaching, most of them related
to 3-13 year-olds, but viewed as an initial condition for developing attitudes at
higher ages.
1. Developing language above the ostensive and linguistically relative level, in
particular at the level of conventional variables and functional description21.
Although linguistic levels are reached by learning processes they should not be
confused with the levels in learning processes. Levels in learning processes are
there for staying on unless, for some particular reason (for instance as an education developer), one is required to descend. Linguistic levels, on the contrary, are
there to for moving on however it works out, that is, at each opportunity on the
level that best fits the situation.
Besides the levels, I also distinguish two modes of expression: describing something by means of an action or as a state of affairs. For instance:
To get the mirror image of P with regard to! , drop the perpendicular from P upon
l and extend it as much behind l as P is in front!
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I distinguished the following levels:
ostensive language, where pointing with the index finger or pointing mentally
may be accompanied by words like "this" and "that";
relative language, where objects are described by their relation to other objects;
conventional variables, which make relative language function more smoothly;
funcrionallanguage.
To make things clear let me quote an example from algebra: define the square
root!
Ostensive:
P = ' so 3 is the square root of 9,
s2 = 25 so 5 is the square root of 25, and so on.
Relative:
The square root of a number is found by looking for a number that squared reproduces the given number. Or, in another mode:
The square root of a number is the number, the square of which is the given
number.
Such clumsy names for variables as are "number" and "given number" are avoided by the use of
Conventional variables:-
x = Ja ?if x
2
'" a.
By means of a new concept the language becomes
Functional:
Taking the square root is the inverse of squaring.
2. Change of perspective, a complex field of strategies whose common feature is
that the positions of that which is given and that which is sought for (of data and
unknowns) in a problem or a field of knowledge are -- partially -- being interchanged; this includes the recognition of wrong changes of perspective
27
•
3. Grasping the degree of precision that is adequate to a given problem21
18
•
4. Identifying the mathematical structure within a context, if any is allowed, and
barring mathematics where it does not apply2'.
In fact, there are large fields where mathematics is more often illegally than legally applied.
5. Dealing with one's own activity as a subject matter of reflection in order to
reach a higher level. CHAPTER3
. THE LANDSCAPE OF MATHEMATICS EDUCATION
3.1 THEORY OF MATHEMATICS EDUCATION
The heading is wrong but I could contrive no better one. What is wrong is the
word "theory" that -- used by so many people on so many occasions with so many
different meanings -- needs circumstantial explanations that do not contribute
factually much to our subject I recently read a paper on theories of "theory"
where the author claims:
When a mathematics educator studies the effects of lax and restrictive learning environments on children of different anxiety levels, she preswnably has a theory that relates achievement to both anxiety and the structure of the learning envirorunent. Or,
when a cognitive psychologist examines classification tasks in the learning of early
number concepts, the psychologist most likely has a hnnch as to how these tasks are
related. Or, when a doctoral candidate designs an experiment in which children are
taught several different problem solving heuristics, she preSUmably has a theory that
predicts which of theSE', treatments will be the most effective.
The middle sentence attests a terminology where a "hunch" may have the status
of a theory. Indeed, in colloquial language the word "theory" is occasionally used
in this way, although under these circumstances most people would prefer the
word "hypothesis". However, this is only one extreme of a broad spectrum of
meanings. The other extreme (which will be described in what follows) deserves
a name as well, and I would not know any name for it if "theory" is used in too
liberal a way.
3.1.l Theory
l . Domains and theories
People familiar with the terms and with the domains they cover will agree about
number theory being a theory, and about distinguishing, if need be, between algebraic and analytic number theory. All the same, they might disagree about
some concept or proposition, for instance, whether it falls under analytic number
theory or under complex function theory. In fact, depending on the context in
which it occurs it may be attributed to both of them. The infinity of the number
sequence is neither a theory nor a hypothesis but a mere fact, which nevertheless
may be the subject of sophisticated theories. The infinity of even or odd numbers
is a fact as well, as is the existence of an infinite number of primes, which -- requiring a non-trivial proof -- has the status of a theorem in number theory.
Neither mathematics nor physics nor chemistry nor biology are theories, al-
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though there are a great many -- sometimes competing -- mathematical, physical,
chemical, biological theories, that cover various domains and sub-domains of
these sciences or deal with their foundations. Mechanics is not a theory but Galilei-Newton mechanics is, as are the theories of elasticity, relativity, and quantum
mechanics. Originally, biological evolution was not a theory but a mere tool for
organising fossil evidence, which proved successful long before the discovery of
evolutionary mechanisms. On the other hand, almost from the onset, astronomic
evolution has been a theory.
History is, of course, no theory. Historiography, rather than being a theory, is the
technology of historical research. There have been attempts to develop theories
on history. "History does not repeat itself' is a theoretical statement on history,
as is "History does repeat itself but then as a farce" (Hegel). Although theories
on history are more sophisticated than such statements, I doubt whether they have
been more successful. Or, in other words, they were topical as long as they did
not have the time to outlive themselves.
Language, or any particular language, is of course no theory, but there are a great
many theories on various aspects of language and of particular languages, that is,
on origins, structures, recording and social functions, interrelatedness, artificiality, and so on.
Economy is no theory, though economics includes a large number of -- competing -- theories. Strangely enough "education" is used in both ways, like "economy" as well as like "economics". Rather than interpreting this twin use as ates-
timony to integration, I see it as evidence of a shortage of relations between them.
2. "Theory versus practice"
The saying "that's mere theory yet practice is different" may mean various
things. It may be used if an actual state of affairs does not match the one reported
on or desired, or if some tool or recipe does not work the way it was told or expected to. Pure theories are descriptive rather than normative, even though they
might have been created under the influence of norms or with a view to establishing norms, and even in spite of a virtual interrelatedness between norm and description. For instance, the all-embracing aristotelian-scholastic theory of potentia and actus, which arose from opposing female passivity to male activity, was
both the result and the cause of social and sexual norms, although it was formulated to be understood as part of an ontology, that is, as a profound description of
the way things are.
The saying may as well be meant to express the more fundamental dissatisfaction
about a theory or something like a theory that does not really apply. This can
mean that it falls as a description of reality or that there is no reality at all to which
it can be tied. In this case, the theoretician is very likely to charge the complaining
practitioner with the crime of the so-called confusion of reality and (descriptive)
model (see [87]), which is simply a lame excuse. It can also mean that the theory
r- cc:=-- '-?V)t nr.r--,--rle ::'?"-.o::;h r--o r,?-.-,, f0-; -1-(;.):-'; 0.,.,,, 0tt !-- -?'<'-1{?- -"--r": ".\-?.- : ; '--1--"t].· ;·--- ---:: ·:·, ::·<::::: ", ,, ... ,,.--,
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The theoretician might then appeal to the restrictions inherent to the use of models -- an apology that may be accepted provided there is any use at all for the model in question.
3. Pure versus applied -- descriptive versus normative
Antinomies can be illuminating. Yet rather than pitting theory against practice, I
would like to pit the desire for cognition and understanding against that for action and change, pitting pure against applied research, descriptive against normative models. To be sure, the interaction between them may be more frequent
and fruitful than their contrast Chapter I of the present book was written under
the first aspect, chapter 2 under the second, but the result was not a clean separation. Items such as mathematising, mathematics as an activity, mathematics in a
context can be introduced as descriptive, pure concepts, even though they were
created with application» in mind, in particular in education. Levels are descriptive structural features of learning processes, which are meaningful only from a
particular didactical point of view.
In (1.1) I signalled the difference between shades of meaning of the word "arts"
in "arts and sciences" and "arts and crafts". From the performer's perspective,
"arts" in the two pairs is interrelated, as are "artist" and "artisan". If an integrating
rather than differentiating term is desired, one may choose the word technique,
as performed by technicians, which is derived from the Greek tee/me for art. Then
the interaction;between what was announced as opposites is known as technology.
Education is a technique, and so is mathematics education. A teacher is a technician, as is an engineer, a medical doctor, a lawyer, a clergyman, a social worker.
They depend on more or less advanced technologies, which in character extend
across the entire range from the mere mastery of a professional language to applications of hard science. In [87] I discussed this and many other related ideas
in abundant detail, and with as much sophistication as I was able to produce, so
Iam wary of repeating myself, even in paraphrases, which I am afraid would not
be any better or more convincing than the original. I didn't even get the opportunity to defend my ideas since they were, as far as I know, never questioned. Far
from believing that any of them was accepted, I even wonder whether they have
been noticed at all -- silence does not imply consent".
4. Theorising
The question that should occupy us here is not whether and under which conditions or circumstances a theory might be true or applicable. We should instead be
concerned with the ill-founded belief that mere theorising in itself, if profound
enough, produces theory. As valuable as theoretical statements and conceptual
tools may be, they do not necessarily, even though assembled to form an extensive collection, constitute what to my mind what is worth being called a theory.
As I see it, a theory should pursue understanding in a reasonably well-defined domain, the extent of which does not matter much. It should do so in a reasonably 128 CHAPTER3
coherent and consistent way, and it should in principle provide its adepts with
guidelines on how to tackle questions, relevant to that domain, and how to answer
them with unequivocal explanations or by proposing appropriate instruments for
action.
This restrictive definition does not intend to underestimate the merits of theorising, which I admitted may be the source of valuable theoretical statements and
conceptual tools. But by granting more latitude I would then lack a tenn for what,
in my eyes, is a full-grown theory. Indeed, much of what now rightly claims the
status of a theory was for long periods no more than a collection of theoretical
statements and conceptual tools, which in the course of time or by some lucky
stroke was eventually forged into a lheory.
This statement is relevant to our general theme in so far as education, and mathematics education, in particular, is a field of extensive and intensive theorising.
The preceding chapters are an example. They display more conceptual tools than
would be sensible to enumerate here, such as phenomenological approach, mathematising with all its aspects, context, guided reinvention, levels in learning processes, paradigm, reflection, insight, prospective and retrospective learning, intertwining learning strands, mathematical attitude, and many more. All the same, I
would not dare to claim that all together they constitute a theory.
After writing the last paragraph I happened to remember a dozen occurrences --
even in the plural -- of "learning level theory" in 2.3.2. I could plead that these
occurred in a context dating from more than a quarter of a century ago, while, in
the mean time my way of using "theory" has changed. I cannot exclude the last
possibility. Indeed, the lavish use of the tenn, in particular in education, might
have aroused an increasing suspicion with regard to the prevailing pretensions,
or at least doubts about its suitability; so I may have become more demanding in
granting something the status of a theory.
Yet this does not discharge me from the obligation to check whether learning level theory is, indeed theory in the restricted sense proposed. My answer is: yes, it
is. And although this answer may extend to other items mentioned in the same
context, which I did not label as theories, but I am wary of checking it. What I
actually denied was that they together constitute a theory of mathematics education, and I would now add: not even together with other conceptual tools that I
dealt with in the past and did not recapitulate here (see, for instance [87,
Chap.IV]).

The fact that, with regard to theorising in education, I focused too sharply on my
own results deserves an apology and an explanation: I am not nearly as well-acquainted with other people's work as I am with my own. But it seems to me that
I am not the only one who has lost his grasp on the abundant literature, although
I guess that on closer inspection (if this is possible) one would find less diversity
in results than in terminology. Long lists of references can be misleading. Theoreticians are isolated or restricted to small groups where each understands the
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Let us resume this question later on!
3.1.2 Theoretic frameworks
I should confess that I don't feel at all happy with the gulf I created between what
I called results of theorising (such as theoretical statements and conceptual tools)
and theory proper. Theorising is organising fields of experience, and the result of
this organisation, unlike a heap of sand, will be gifted with a structure, thanks to
its origin in connected experiences and the processing that took place when they
were organised.
The tenn I propose to bridge the gulf is: Theoretic framework .. I took the -om
Treffers who indeed uncovered and analysed frameworks of vanous educauonal
theories;. I will use the term here in a broader sense, that is, by also applying it to
models of educational practice, such as those exhibited by textbook series or by
products of experimental educational development, as well as by less fonnal
classroom practice.
Indeed there can be little doubt about the theorising origin of models of mathematicai instruction. From the outset they must have been framed, consciously or
unconsciously, by cogitation, reflection and argumentation. If need -- sue-
framework can be uncovered by interviewing the authors or by analysmg therr
production. Beside; and beyond this, theoretic framework can be designed rationally beforehand in order to be realised by models of instruction, but most often the frameworks will be shaped in an incessant interaction with the matter to
be framed. Their relation is comparable to that between form and content or between theoretically essential and accidental properties.
Indeed, in any model of instruction we will notice pieces that are less essential
than others, and which may be replaced by ones which are somehow eqm.valent.
How far can we go with substituting and rearranging, while at the same ume respecting the spirit of the model? What do the eventual pieces de resistance look
like? The answer to this question is an a posteriori definition of what I mean by
theoretic framework. . . .
I f! am asked an a priori definition of theoretic framework I would say it 1s a more
or less connected set of theoretic statements and conceptual tools, obtained by
theorising (and maybe even including mini-theories), and reflecting the essentials
of an actual or imagined instructional system in a way that allows the expert reader a reasonably faithful reconstruction. (In order to match the case in which Treffers used it, one has to replace "system" with "theory" in the last sentence.)
"Theoretic framework" is a weaker notion than is "theory". On the other hand,
by its mere defmition, it is bound more tightly to educational practice, which fact
need not be a drawback. As a consequence, by the first token, it applies more
broadly and by the second more narrowly than does "theory". Goal description,
when extended to all its dimensions (as understood by Treffers') certamly matches my defmition, although the expert reader can be satisfied with less detail than 130 CHAPTER3
the novice. (By an expert reader I mean someone familiar with the theoretic background, for instance, thanks to familiarity with the framework of the underlying
theory.)
3.1.3 Background Philosophy
The first question I asked in [87] was: What is science? I tried to answer it somehow by means of certain necessary criteria (relevance, consistency, publicity),
but otherwise I dido' t do much more than to oppose science to -- among others -
- technique and faith, without involving any comparative appreciation. I could
not, nor would I, restrict "technique" (or "practice" as I called it occasionally) to
inanimate nature. Since, together with biotechnique, I had to include medical
practice, I saw no reason why the term "technique" should not be applied to the
practice of teachers and other practitioners, those of the so-called humanities not
excluded.
Until quite recently, almost all technique was practised with no technological
support at all, or supported at most by some technology that for its part lacked
significant scientific support. This still holds for many techniques, in particular
for that of education and instruction -- a statement that again does not mean to
suggest any kind of appreciation.
In the past, even up to modem times, much of what is now called science, was
part of philosophy -- just think of Newton's Philosphiae Natura/is Principia
Mathematica, published by the Royal Society, whose fellows called themselves
philosophers. Step by step this all-comprehensive science has 'de-umbilicated'
the present specialised sciences -- to use a suggestive term of O. von Neurath's -
- but even today a great deal of what claims the name of science for itself still
dwells in the womb of philosophy. I dido 't make this statement as a gesture of
condescension. On the contrary, what I called background philosophy in [87] (or
else ideology) is indispensable as the rationalisation of a faith that in tum comprises the faithful one's picture of world, man and society. It is indispensable, not
only to any technique and wherever philosophy decides (implicitly or explicitly)
on the choice of technical tools and objectives, but also in "pure" science with its
rich choice of goals and methodology.
Did I start my exposition too far from mathematics and mathematics education?
Shouldn't I speak rather more specifically about the picture of mathematics and
mathematics education? In a sense this is true, since this is my proper aim. Yet
more things taught than just mathematics, and in this totality, pictures of subject
areas and their instruction are somehow related to each other by their relation to
pictures of world, man and society.
For any picture of mathematics and mathematics education one has in mind, it is
important where one localises mathematics -- inside or outside the world, bracketed (if out of bracket-. within brackets that close and open more or less easily.
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pected to acquire their mathematics, how freely or slavishly, under guidance or
bridled. They all determine the manner in which one thinks mathematics can contribute to the mutual benefits expected by society and its members from one another, and they depend, at least partially, on one's picture of society and its inherent hierarchy.
Do not misunderstand me! I say "picture of society" rather than merely "society".
In primitive societies -- as well as iii "primitivising" ones (Freyre, Mellin-Olson)
-- "society" and "picture of society" coincide, as they do with the picture of world
and man. In our society, a broad sjlectrum extends between the naive and the utopian picture of society, and even "realistic" ones can differ widely from each other.
3.1.4 Pictures of mathematics and mathematics education
Yet the picture of mathematics also influences that of mathematics education. in
a direct way. Whoever cherishes a picture of mathematics outside the world -- a
deductive system or a catalogue of formula -- is likely to systematise or to interpret mathematics instruction in the same spirit. On the other hand, whoever experiences mathematics something in the making, vibrating under the impulses of
world and society, will be inclined to teach it in tile same way -- directly or as an
educational deve,l9per. There is little need to refer to literature that shows this
correlatior. between the personal characteristics of teachers; but it is perhaps
worthwhile recalling something of a more historic-anecdotal character. It is even
great history I am referring to. The colloquium (Melon, 1952)2, which was crucial
on the one hand, for the development of mathematics education in the sixties and
seventies, and on the other hand, significant for Piaget's first meeting and confrontation with Bourbaki, personified there by Dieudonne. At his colloquium,
E.W.Beth claimed -- with far-reaching consequences for the future of mathematics education -- that
Le rOle de la formation mathCmatique dans l' enseignement secondaire consiste
presque exclusivement. me parait-il, a familiariser les e1eves avec la methode deductive.
Although this claim was disputed by no one -- implicitly or explicitly -- it nicely
characterises the logician Beth -- the only thing that interested him about mathematics was its deductive system. Dieudonne, on the other hand, confronted the
much more encompassing mathematics, exercised and represented by him, with
traditional instruction, which, in his eyes, had run aground due to its lack of deductivity (in particular in geometry -- a bas Euclide!). New shipwrecks could be
avoided, provided one trusted older and wiser people who have explored the
world to the benefit of the young, and used Bourbaki's compass to sail from the
poor to the rich structures; many believed this along with Dieudonne.
With the claim quoted above, E.W.Beth only followed his father H.J.E.Beth,
who, as a designer of a new mathematics programme, had declared (text
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ttans-132 CHAPTER3
lated):
It is the main objective of mathematics education to contribute to mental culture and
development; transferring useful knowledge is a secondary objective.
In the 19th century (and even before that in The Netherlands) the fonnal value of
mathematics learning was claimed by preference by "modern" schools in their
competition with grammar schools where Latin enjoyed a similar reputation. I
cite here the fonnal value of mathematics as an instance of a faith that usurped
the state of a theory, while in fact being a hypocrisy. Only a few people dared to
unmask it as such -- teachers as well as research mathematicians like D.van Dantzig as is shown by the mere title of one of his articles
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(translated) "The social
value of mathematics instruction", which at that time (1925) was disregarded if
not considered an insult.
All my life I have fought this faith in the fonnal value of mathematics, only -- I
confess -- to convert to in the end, although (and this is my excuse) with in mind
another kind of mathematics instruction, which has its roots in the most recent
past and where deductivity is assigned the place it rightly deserves.
I called it a conversion, and this is no metaphoric language, because it happened
in the sphere of faith, where the picture of mathematics is framed by the picture
of the world, the picture of the mathematician by the picture of man, and the picture of mathematics education by that of society.
Pictures of mathematics education are made explicit and rationalised by frameworks. Later on (3.1.6), I shall discuss how they are related to background philosophies (3.1.3). Meanwhile, the question arises of whether designing mathematics instruction presupposes making a decision on a background philosophy.
The answer is that it does not. Everything can remain implicit: it is a faith that is
not being rationalised. On the other hand, the relation between a theoretic framework of mathematics education and the philosophy behind it now looks much
closer. But beyond it one can imagine theory (or metatheory) charged with the
task of relating theoretic frameworks of mathematics education to background
philosophies. An excellent starting point here is Treffers' classification'.
3.1.5 Classifying mathematics education
I never did like classifying, and I still distrust it for what I consider are sound reasons: I suspect it as both too easy an approach to problems and as too low a bid
to buy scientific respectability. In (1.3) I explained my fonner resistance to Treffers' distinction of horizontal and vertical mathematisation. Eventually I did accept it as an effective tool to characterise various kinds of mathematics education.
The present section shall be concerned with this subject.
Treffers' four types of mathematics education are distinguished by a double dichotomy with regard to horizontal and vertical mathematisation: presence or abTHE LANDSCAPE OF MATHEMATICS EDUCATION
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a view to global orientation, yet accessible to refnement, depending on the degree to which the yes-no contrasts are toned down Treffers applied his -lassifica
tion to primary mathematics education, but this is in principle no restnction.
The lack of horizontal mathematising characterises mechanistic and structuralist
instruction: in the first case even the vertical component is lacking, whereas in
the second it is in fact cultivated. Horizontal mathematising is fostered in the empiristic and the realistic approach, in which the fonner neglects_ the vertical component and the latter does full justice in the second kind. Seen m a table:
horizontal vertical
mechanistic
- -
empiristic + -
structuralist
- +
realistic + +
Fig. 15
Let me illustrate this classification by a somewhat anecdotal report [176,180], a
Gennan educational tragedy, as it were:
Lisa Hefendehl-Hebeker" tells about a fifth-grader who could not answer the
problem
Why is 4 < 9 ? Give three arguments!
and the child's desperate family that, equally unable to do it, made an appeal to
her as a mathematics education expert. By looking for the context, that is, the
homework in which this problem was one item, she succeeded in suggesting
three possible arguments, such as "because in the number sequence 4 preced-
9" "because on the number line 4 is to the left of 9", "because the 4-arrow IS
sh--rter than the 9-arrow". One could also have suggested such answers as "because 9 > 4", "because 4 < 5 and 5 < 9", "because 9 -4 > 0",
though it is unlikely that these would have been accepted by the teacher.
Let me remark that, in Gennan, the <-symbol in "4 < 9" is read as what corresponds to "smaller than". So a smart pupil could have objected that both figures
are equally large. I mention this in order to point out that all three symbols mean
something, and in order to know what they mean one has to consult the context,
which is indeed what Lisa Hefendehl-Hebeker did.
What has this to do with classifying mathematics education?·
Well, neither the mechanist nor the empirist would ask such a "why", whereas
the structuralist and the realist possibly would. The teacher who set the problem
was obviously a structuralist; the answers as reconstructed match this view. The
empirist and the realist would notice that 4 and 9 belong to the vernacular as do,
say, 4 marbles and 9 marbles, 4 florins and 9 florins, 4 kg and 9 kg, 4 hours and 134
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9 hours, 4 floors and 9 floors, 4 metres and 9 metres, and then, depending on the
context, "4 < 9" can mean that 4 marbles are less than 9, 4 florins is cheaper than
9 llorins, 4 kg lighter than 9 kg, 4 hours shorter than 9, 4 years younger than 9, 4
floors lower than 9 floors, 4 metres shorter, smaller, shallower than 9 metres.
Mathematics taught and exercised in rich contexts would provide for all of these,
and many more, comparatives, and realistic mathematics instruction, as opposed
to the empiristic version, would account for their isomorphism by vertical mathematising. Then a question like "Why is 4 < 9?" may validly be answered by "be-
cause4 marbles are less than 9", 4 florins is cheaper than 9 florins, "4 kg is lighter
than 9 kg", which would already be three arguments out of many more possible
ones.
Due to its rough schematism Treffers' classification applies to theoretic frameworks rather than to theories. The lauer should be uncovered- by more profound
search in actual instructional systems if they are not presented in detail by the desiguers themselves. Moreover, this classification describes ideal types. In textbooks of the mechanistic trend, which boast a long tradition in teaching arithmetic, concepts and operations may in some way be related to reality but this usually
happens in a way that neither teacher nor learner are likely to take seriously. It is
often an intellectual alibi for the developer's or the teacher's conscience, a misty
cloud veiling an ideology, which in spite of its venerable age can best be described by a modern metaphor. This I will do in the next sub-section
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Meanwhile, let me add an indispensable remark. In Treffers' double dichotomy
it pays to replace "mathematising" with "didactising" in order to characterise
styles of teaching mathematical didactics (rather than mathematics), which happens, among other places, in teacher training and retraining. Philosophies of
mathematics education should be judged according to both interpretations. This
is not at all a triviality. More than once I have read and heard proponents of realistic mathematics trying to implement their ideas in a way that, didactically
viewed, appeared to be mechanistic-structuralist rather than realistic. The didactical attitude was not matched by its mathematical counterpart.
3.1.6 Philosophies of mathematics education
I . Mechanistic
According to the mechanistic philosophy man is a computer-like instrument, that
can be programmed by drill to perform, on the lowest level, arithmetic and algebraic, maybe even geometric operations, and to solve applied problems, distinguished by recognisable patterns and processed by repeatable ones. This, then, is
the lowest level, where man is placed within a hierarchy of ever more skilful
computers, which are related to one another as are programmers and programmed
subjects. Skinner has forcefully propagated this kind of human society. New
prospects ';ve n-: been opened for this ideology by computer controlled inTHE LANDSCAPE OF MATHEMATICS EDUCATION
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should be educated to perform tasks on a level where, by many orders of magnitude, computers are faster, cheaper, and more reliable than humans.
2. Structuralist
The structuralist view is also historically rooted, in particular in the traditional

teaching of geometry. A well-structured system of mathematics or a mathematical domain shall be taught. It is a human right and dignity to learn by insight and
understanding and as a rational being be is judged able to perform deductions
more efficiently the more systematically the subject matter is structured. In the
ideal case, however, of the socratic method he needs the midwife who delivers
him of his mathematical ideas -- Socrates says explicitly about the slave, "you see
he has expressed nothing but his own opinion". This, then, was mere "theory"
since, in (Socrates' and) the classroom's practice, the learner was expected to
obediently repeat the master's deductions. In order to check the quality of the repetition -- whether it was mere parroting or full of insight -- problems were set,
which in turn were deactivated by drill. In the nineteen sixties and seventies of
our century, under the name of New Math, the structuralist view was advertised
and propagated. Yet it became soon clear that this wrong perspective -- from the
poorer to the richer structures -- was an obstacle to any kind of genuine mathematising. On behalf of the prestructured mathematics to be taught, a correspondingly structured world was invented of Venn diagrams, arrow schemes, "games"
and so on, to be mathematised by the learner. This was, indeed, a kind of horizontally mathematising activity, yet it started from an ad hoc created world,
which had nothing in common with the learner's living world. It was mathematics taught in the ivory tower of the rational individual, far from world and society.
3. Empiristic
To the empirist the world is a reality, where man can acquire useful experiences
-- a respectable point of view provided reality and usefulness are broad-mindedly
interpreted. Empirism is deeply rooted in English utilitarian education. Provided
with material from their living world, learners get the opportunity to acquire useful experiences, but they are not prompted to systematise and rationalise these experiences in order to break the barriers of the environment and to expand the reality they are familiar with. This matches the picture of a society stratified into
layers that correspond to separate realities.
4. Realistic
In realistic instruction the learner is given tasks that proceed from reality, that is,
from within the learner's ever expanding living world, which in the first instance
require horizontal mathematising. The individual's and the group's progress in
the learning process -- how far and how fast -- determine the spectrum of differentiation of learning output and the position of the individual learner therein. One
example: If, in the course of progressive algorithmising, column multiplication is
first carried out as successive addition, in order to gradually be shortened (by using the tables of multiplication and the positional system), individual learners--
in the social context perhaps the whole group --- will eventually acquire the
standard algorithm. In the case of long division, the learning output of progres-136 CHAPTER3
sive algorithmising may be more differentiated -- an unavoidable differentiation,
which nonetheless deserves a positive qualification.
The realistic picture of mathematics fits without brackets into the world picture.
The picture of man is that of the reinventor, who is stimulated to put his abilities
to good use. The corresponding society is differentiated continuously, rather than
by artificial layers.
3.1.7 Use of the classification
Two criteria have been singled out for the design and the analysis of mathematics
education: horizontal and vertical mathematisation. There are many more. The
choice of these two criteria may be justified by their relevance to background philosophies. It may be useful both for creators and reviewers of instruction to localise instructional creations within a non-trivial scheme. This may help creators to
better understand themselves, and reviewers to more easily examine explicit or
implicit pretensions of the creators. Whatever the objections may be to comparing instructional creations that start from different didactical -- and, basically,
philosophical -- principles, they do not affect the submission of each of them to
the test of their own principles. Wherever in the past the mechanistic view prevailed, textbook authors often pleased themselves by embellishing their production -- implicitly or explicitly -- with realistic or structuralist touches, and the
same is likely to happen in the future as a defence of mechanism against a realistic trend that is gaining momentum. The classification can prevent educational
developers from misleading themselves and others.
3.1.8 A matter of faith
We put forth a theory that relates kinds of mathematics education to philosophies.
It is determined by the theoretical tool of (horizontal and vertical) mathematising
and thus presupposes faith in (guided) reinvention. It is open to anybody to brand
this faith as "irrealistic": "Where in the world are those reinventors and those who
are able and willing to guide them?" "How well-founded is this faith?" Answer-
ing such questions is not easy. In matters of faith tolerance is becoming. I f somebody believes in the hierarchical society of programmable human computers, that
is his business, although of course tolerance does not extend to incorrect computer programmes.
Detecting errors in computer programs is no great feat, but it is quite another
thing to evaluate instruction without any regard for its own basic principles. How
to test whether one kind of instruction is better than another? Instruction serves
goals. But what to do if the goals are formulated in terms of a philosophy, thereby
being a matteroffaith? Isn't it a vicious circle? No, it is not if the testing criteria
»l?<P: t)"'et:. ler -.1i-h thB •»n ;iP?Jiik:-i, (7:' -:"; .p!],--r ,.: Cb-e,,-t-ve:,.- r-7 th;: -rl-tr-i-ct!: 11,'1,
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the most favourable case, the term may extend up to graduation.
Is there any way to at least compare the quality of various instructional systems
under these favourable circumstances? Given an instructional system, collect all
those exam items which are judged representative for it, each with its weight
within the system; in order to compare two of them, form their intersection and
put it to the test! For the higher levels and with regard to the mechanistic and re-
alistic approach, one can safely predict that this intersection will be almost empty
or at least negligible if compared with the remainder, and one can do so with even
more certainty if the main goal of the realistic instruction is learning to mathematise. At lower levels, in particular if one restricts oneself to pure and applied elementary arithmetic, the intersection might still be considerable and accessible to
empirical comparison.
Although some of the research carried out in my country seems to strongly suggest the superiority of the realistic view, I am still cautious of generalising its conclusions. In one case' a small sample of classes and schools adhering to the realistic style were compared with those of other styles (or with the average of that
half of schools which were taking a nation-wide test). Meanwhile, a large scale
investigation" took place, which also suggested the superiority of the realistic
style. Unfortunately, because of flaws in the provisiolJJll report I must postpone
my final conclusion. Yet this is not what matters.
Realistic textbook sepes for primary education, patterned on the Wiskobas made!!• 11, are a relatively recent feature in my country; they are supported by a majority of educationalists. Realistic textbooks are gradually beginning to supplant
older ones; and it is not far-fetched to assume that this process started at "better"
schools and gradually began to affect more schools in an order corresponding to
their quality. At present, somewhat more than 50% of the schools use realistic
textbooks. If, in the course of this process of substitution, low-quality schools get
involved, the trend that now seems to prove the superiority of the realistic style
may become reversed -- a similar phenomenon has been observed in Hungary".
In the last paragraph I spoke about good and bad schools. This I did for simplicity's sake. I was not referring to their overall quality but to the measure in which
schools that switch textbooks -- in the present case most often from mechanistic
to realistic -- are prepared to correspondingly switch philosophies and teaching
habits. Indeed, one does not automatically imply the other. Working with realistic
textbooks in the mechanistic instructional style to, which one is accustomed may
even make things worse; in fact, this may explain the failures as mentioned
above.
We shall deal with implementation in due course: rather than selling textbooks,
one has to sell a faith.
3.1.9 Learning theories and philosophies
Let me first remind the reader that learning theory is meant as shorthand for 138 CHAPTER3
teaching/learning theory or instruction theory, which, for the moment, will include theoretic frameworks.
There are a great many general learning theories, where, according to the authors,
"general" means subject-area-independent; only as means or domain of exemplification and concretisation do subject areas -- by preference, mathematics, or
what generalists consider to be mathematics - come into play. Usually the view
is even held that the general learning theory leaves no room left for learning theories specific to any subject areas, other than those derived from the general one.
I strongly distrust general learning theories, even if their validity is restricted to
the cognitive domain. Mathematics is different -- as I emphasised before -- and
one of the consequences is that there is no didactical analogue to guided reinvention in other fields. I don't know about learning theories for other areas. I can imagine some general learning theory derived from learning theories specific to a
variety of subject areas. This inductive process is primarily the one by which new
theoretical knowledge arises, rather than the other way around in a pseudo-deductive way.
As I have emphasised repeatedly, I am allergic to armchair hierarchies of phases,
stages, types, levels, and so on, often illustrated by examples from mathematics
instruction. Very few of them have ever been properly and successfully applied
or empirically corroborated. In [87] I analysed the only one I knew at that time to
have been put to the test by its author (or perhaps by his ignorant assistants). It
was a most amusing job. Unfortunately, more an more textbook series and models of instruction are allegedly being based on such theories, which are arbitrarily
interpreted in order to eventually be degraded to a kind of billboards. Treffers
7
more patiently analysed such general learning theories and convincingly showed
how to justify any kind of mathematics instruction -- I mean each of the four
styles -- by each of those general learning theories. Maybe one can also do the
converse: justifying any general learning theory by any kind of mathematical instruction.
A particularly striking example is Piagetian instruction. In his early work
17 22
Piaget developed an empirically based descriptive learning theory, which has hardly received any notice
17
-- it seems that he himself even forgot about ii. A great
many normative theories and practices claim the label "piagetian", although it
seems to me that they have borrowed words rather than ideas from Piaget, by
preference terms like operation".
As promised in the heading, I will focus on the philosophical background of a few
learning theories.
3.1.9.1 Gal'perin, and materialism
I . Gal'perin's levels
The levels distinguished by Gal'perin
36
in any learning process are
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an act based on material objects, or their material representations or signs,
an act based on audible speech without direct support from objects,
an act involving external speech to oneself (with output only of the result of
each operation),
an act using internal speech.
"These levels indicate the basic transformation of an act as it becomes mental."
Tue terms require some explanations, which are easily extracted from Gal'per-
in's exposition (a few ambiguities may be ascribed to the translator). Western instruction designers who base their wmk on Gal'perin usually describe his hierarchy as follows:
orientation basis,
material act,
verbal act,
mental act,
which in no way does justice to Gal'perin's intention.
The philosophical materialism behind Gal'perin's level structure of the learning
process is unmistakable, or so it seems to me. But is it as operational as it looks?
"Material act" -- as it is called by Gal'perin's followers -- evokes associations
with manipulability as a way (particularly in arithmetic) to start with concrete,
that is solid, material, which today is available in a great variety, indeed. If Gal'perin was aware of any of this material at all, he must have considered it irrelevant.
It is clear from his description that material or -- as he later says -- materialised
has a rr.uch broader -- or even quite different-- meaning from palpability. As emphasised once more later on", p. 253, he also admits -- or even prefers -- material
representations or signs, which he equates straightforwardly with materials. To
state this in the extreme, a tree the child can touch, a picture of a tree recognisable
as such to the child, the printed word "tree" as soon as the child can read, a printed
definition of "tree" as soon as the child can understand it, all belong to the same
learning level (albeit in different learning processes), but as soon as they are
voiced audibly or inaudibly or internally, they belong to the next levels, respectively, in the corresponding learning processes.
In the same way is "verbal act" a misleading reproduction ofGal'perin's intention. According to Gal'perin, written or printed material is as "materialised" as
are pictures. What matters at the third and fourth level is speech (to others or to
oneself), and at the fifth its elimination. (This interpretation is confirmed by",
1977, p. 33-38.)
Although this sounds mad, it is absolutely rational and intelligible, as I will
promptly explain. In order to understand it, one has only to forget about one's
own ideas on learning. The impression of madness is due to the extreme example
of "tree", which doesn't fit in this context. The context that fits is that of teaching
arithmetic, in particular column arithmetic, which becomes clear from the one
example that has been elaborated in detail", p. 270-273. Or, more precisely, Soviet instruction of aritlunelic as it was familiar to Gal'perin at the time he applied
his psychological ideas and wrote the article we are concerned with. Middle-aged 140
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readers in more Western countries will perhaps be able to relate column arithmetic as they learned it themselves, to this frame -- it is a kind of instruction that is
likely to have become obsolete much earlier in Western countries than in the Soviet Union, where it is now equally outdated, or so I think. The wine of gradually
renewed instruction had better not be bottled in a dated learning theory, unless
the "bottle" is adapted to the new contents, which implies a preposterous use of
the original label if it should happen.
"Dated" is not meant to blame Gal 'perin. As a psychologist, he registered and analysed the then prevailing instruction, and did so carefully and not bent on educational additions and ambitions. Maybe he emphasised the level structure that
resulted from his analysis too strongly, and extended it too widely, but if somebody is to be blamed for this, then it is rather those who believe that such level
structure is suitable lo frame instruction of a quite different character.
2. Gal' perin, and initial arithmetic
Let us consider initial arithmetic as framed by Gal 'perin' s levels! From the very
start "material" means quite another thing than what is available as such in our
classrooms. The "orientation base for action" in "abstract counting" (without
numbers) is an instruction card with stairs picturing piles of an increasing
number of objects, and the instruction:
"Next number .. means "one more object than on the pile shown"!
"Preceding nmnber" means .. one less object than on the pile shown"!
(Notice that the number of objects on the pile "far exceeds the arithmetical
knowledge" of the pupil!)
At any assignment the pupil first consults the instruction card in order to fulfil the
task, which, when this is done, is materialised action. Then the card is turned
over and eventually removed, in order to have the pupil repeat the task interpretation orally while fulfilling it, which is the vocalised form of action. Then the
pupil is asked to "solve the problem silently, reporting only the result", which is
action at the intellectual level.
This is to say that, after the instruction "name the next (preceding) number", the
pupil acting at the materialised level reads the card and points to the next higher
(lower) pile (or does the whole class do so in unison?); at the level of audible
speech, he does so without consulting the card; next he acts the same way while
whispering before pointing, and finally he points immediately to the "number"
asked for.
It becomes clear from the above that, rather than referring to piles of objects,
Gal'perin's "material" refers to instruction cards. This is confirmed by the next
example.
3. Gal' perin, and column arithmetic
Here the procedure looks somewhat different One might expect t.hat the way of
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from "materialised" to "mentalised", but things have now become complicated
by a gradual transformation of the card itself. After a start restricted to two-digit
numbers, the pupil gets a sheet of paper divided into columns to learn numbering,
addition and

Billions Millions Thousands Sin e Units
100 10 I
~
I 100 10 I 100 10 1
Fig. 16
subtraction. He also has small ready-made numerals at his disposal, which, when
placed in the columns, create numbers to be read or formed, added and subtracted
by the pupil. After the orientation basis has been laid he enters the stage of the
materialised action.
Next, the verbal descriptions are removed from the triple columns while the pupils still have to pronounce them. Then the thick lines are.replaced by thin ones.In
the next step, the figures above the columns disappear, while still being pronounced by the pupils.,One step further, the horizonlal lines are also removed,
and eventually the entire grid disappears; the ready-made numbers are laid out on
the table or else corresponding digits are recorded directly in the notebook but vocalisation of numbers and operations is still maintained. Here the audible stage
ends, and that of external speech to oneself begins, from which, finally, that of
menial action takes over.
4. Gal' perin, and geometry
Gal'perin's interpretation of"material" also extends to teaching geometry". It is
the most traditional and the barest form of concept formation: reading and learning to apply definitions, such as those of straight line, angle, bisector. The GDR's
Lompscher" has described this kind of instruction which is aiming at concept attainment. Although the example set forth in
37
reminds one of a catechism rather
than of geometry instruction, our point is how it fits into Gal'perin's hierarchy,
rather than its intrinsic quality.
In this type of geometry instruction, geometrical figures don't play any significant part as a material element. The materialised level is represented by written
or printed orientation cards that disclose the various definitions of geometrical
objects. With these instructions before their eyes the pupils have to decide whether something is a straight line, an angle, a bisector, and so on. At the next level
they must answer such questions without using the cards, while reciting aloud the
defining criteria. Via the stage of whispering the pupils arrive at that of directly
armouncing the result; only if need be -- for instance in testing - -are they to reproduce the definitions. 142 CHAPTER3
5. Gal' perin: conclusion
Gal 'perin' s learning theory distinguishes itself by the refined elaboration of all
details by means of the given examples and by the straightforwardness of the intended learning process. Properly stated, it is a pure teaching theory, inspired by
a kind of instruction that allows the learner no initiative. If Gal'perin starts his
article, saying
Our basic hypothesis is that the formation of mental acts passes through a series of
stages. At each stage a given activity is carried out in a new form and undergoes
changes in several directions.
and subsequently specifies this, among others, by his stages or levels, he undoubtedly has in his mind a certain instructional system in which his basic hypothesis can possibly be tested. Within the limits of this system such a model can
be considered to be descriptive, but in a broader frame it is a quite restrictive normative model. Could it have been more liberal than suggested by the above examples? Probably not! To corroborate this judgment let me quote Gal'perin"
himself:
If, however, we reproduce the material activity systematically step by step, first in its
verbal and thanks to it in its ideal form, we can keep supervising it, whereas in the case
of"spontaneous" transition we don't know what has happened nor how it did. As soon
as we lose the command of the process, we can no longer understand it.
Gal'perin's main concern is the checkability of the learning process; which is
considered to be mere interiorisation. What is the use of it and what can one expect from it? The answer depends on one's picture of mathematics and mathematics education, that is, on one's philosophy. If it fits philosophical materialism
at all, then it does so according to a strange interpretation of "material", or so it
seems to me. In Treffers' classification, Gal'perin's learning theory is an extreme
example of mechanism.
For justice's sake, let me add I.hat more recently Gal'perin's level theory-- in particular his strong insistence on the material and materialised level in the teaching
process -- has experimentally and with great success been challenged, which in
fact has happened in his own school"-
3.1.9.2 Constructivism, and "Kant"
-The plenary sessions of the PME-11 meeting41
, Montreal 1987, were dedicated
to constructivism. Through personal talks during the meeting and by afterwards
studying the literature I tried to find out whether and in what respect "constructivism" were more than a new slogan. To no avail --1 must confess. What follows
is the result of my analysis.
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of I.hem. This doesn't matter as long as it has a clear meaning to whoever pronounces, writes, hears or reads it -- clear not in analytic isolation but in a clear
context. For topical words one creates a fitting context for oneself, which is not
bad as long as one is aware of this fact What is bad, is being unaware of I.he need
for a meaningful context, at least for the user, who risks being deemed a misuser.
I don't aspire to the offlce of linguistic judge, and even less of legislator. On the
contrary, flexibility is a great virtue of language, and can hardly be overestimated. Pictorial and metaphorical language can enrich both language and thought.
But flexibility does not mean weakness; pictures should be focused.
2. Language: Construction.
"Construction" was soon extended from the building trade to other manual work
and to mental work as well. It even came to signify poor work: artificial, a mere
construction.
"Construction" presupposes a maker. Who has made it? The designer, the foreman, the bricklayer slave, the drawing board slave, the assistant to the architect
or the architect himself?
"Constructing" rings like "creating", and if this is no more than a ring, then reconstruction is mere re-creation. But reconstructions caii also be of greater value
than constructions. For instance, efforts to reconstruct a dilapidated or destroyed
building, or an obscwtcourse of affairs can by far surpass the efforts that once
were needed for the original construction.
"Construction" does not reveal anything about the constructor's own contribution, nor does the fmished product do so. The "Do it yourself' package contains
every little part as well as directions on how to form I.he desired object Only by
reflecting on the own activity can the re-creator become a re-creator. If the directions are missing, the reconstruction of a decomposed jigsaw puzzle can require
more inventiveness than many constructions.
3. Constructivism --philosophically
is our world picture a picture of the world, and if so, how faithful is it to the original? Since olden times this question has been asked again and again. How many
tricks does not Nature play on us, and is not Reason the means to unmask Illusion? But how reliable is Reason? How often has the lie not been given to Reason? Yet there is one thing that cannot be denied: that in the course of history our
world picture has become widely extended -- in the macro-world of the universe,
in the sub-atomic micro-world, and towards ever and ever greater wealth in the
intermediate dimensions of our direct sensation.
Yet we have to put up with people who object that this whole picture is a mere
reconstruction of what, albeit unconsciously and unintentionally, was constructed by ourselves beforehand -- reconstruction by the very means by which it was
first constructed. Well, we need not bother about B question and an answer that
are equally meaningless. But what about poor Kant, to whom the problem and the
answer are ascribed by people, who probably never read a single word of his? My 144 CHAPTER3
answer to this question was to surround his name in the title of the present subsection with quotation marks. This, of course, does not refute what philosophically some people call constructivism. Jns1ruc1ionally it refutes itself by its irrelevance.
4. Construction and constructivism - - mathematically
In Euclid, besides theorems (quod erat demonstrandum) we find constructions
(quod eratfaciendum) - -constructions by means of the ruler and the pair of compasses. Although, in the past, definitions and theorems arose by construction as
well, they are presented here as ready-made articles; yet this does not prevent
constructive activities, such as drawing and producing lines and assuming points,
from sneaking into the proofs.
To be sure, in mathematics (and not only in mathematics), inventing and reinventing must be distinguished from reporting. As emphaised repeatedly, a variety of styles are possible. Reports can be restricted to the ready-made matter, but
they can also do more or less justice to the process of making. Less, if it is mere
phraseology; more, if the reporter tries to play the part of inventor, and thus
smooths the road for the reinventor.
If, from a given infinite set, Cantor forms the set of its subsets, is this a construction or idle verbosity? Language on behalf of itself or transmission of something
that can be transmitted in no other way?
Constructivism in the foundations of mathematics means restricting oneself to
activities that take place in lime, or that are understood as such: In an unrestricted
time, in a restricted lime, arbitrarily accelerated, or reeling off with theoretically
or practically constant speed.
The set of subsets of a set is formed in one breath, provided these words are pronounced fast enough. With the natural numbers one can progress towards eternity, or towards the endpoint of an interval halved again and again; with ever faster
computers one gets a grip on ever larger numbers, whereas number language conjures up an exhaustible number sequence.
5. Constructivism in developmental psychology
Construction starts as reconstruction, to wit by imitating oneself or others, yet the
required physical-mental tools must first be shaped and improved by ever more
purposeful efforts. At the same time, and entangled in this process, a powerful
cognitive tool is developing and maturing: interpretation. With a profundity that
can hardly be matched, let alone surpassed, Piaget, in his unnoticed early work
17
22, uncovered the roots of interpretation. Unfortunately, and for incomprehensible
reasons, he didn 'teven touch here on the role played by language, which is equally entangled in the process - • the children he observed never babble nor weep. Is
this indifference to vocal utterances somehow related to his later habit of taking
linguistic failures of his subjects for cognitive ones, as though this were obvious?
-t. L[`.:;]n. 1ce '9. Yle-1t--'ifl T"Curistr--:-. t..ic- ,1Jf t1l:.e if thi:l. (`.l-j-d •v-;--.hi"' --. `·'-r-n- d1e
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play-pen is so obvious that it provokes this cyclic interpretation. Invisibly structured playthings, in particular mechanical ones, are different: most, if not all of
them, are black boxes, where the reaction on the stimulus is a datum, inaccessible
to analytic interpretation.
We have already discussed construction in the sense of slavish obedience to assembly directions. The development of language is to be viewed quite differently: it is a reconstruction rather than imitation, which is extrapolated by the creation of words and clauses never before heard, the novelty of which is wimessed
by sins against morphology, grammar, and syntax.
Reconstruction is again most productive where it is guided by global structures.
An exatnple we mentioned already is that of the first mathematics (or is it the
first?) to emerge in development. When, by listening and speaking, the child acquires the number sequence (that is, the sequence of spoken numbers), counting
starts with a verbal imitation of a fragment or of fragments, separated by gap.
This is then superseded by imitation of the structure, introduced by interpretation
into the linguistic system, which allows the learner himself to close gaps and
playfully continue the sequence. Playfully, that is, beyond need and necessity.
Without limit? New milestones are needed; million, billion, trillion, etc. suggest
something fading away in limbo. More profound analysis is-needed to transform
" . . .illion' into a recursion exhausting the number sequence. (As a matter of fact,
even with regard to the deHmal positional system nobody bothers about the feature that the size of numbers runs far ahead of the size of the means by which they
are defined - - on a higher level, a source of paradoxes.)
The preceding is cited as an exatnple uf where re-creation stands out in bold relief against the kind of construction and reconstruction •. manual or mental· · that
is mere imitation, e.g.copying a drawing, blindly obeying assembly directions.
What does the self-styled constructivist mean here by "construction"? "Construct
the perpendicular bisector!", whether the one charged with the task, already
knows how to do it, or not?
6. Constructivism - - didactically
What matters here is teaching and learning, pedagogy and didactics. Only in order to show their irrelevance did I regard the constructivism of an empty philosophy and of a poor developmental psychology.
For shaping teaching and learning processes there is as little need to discuss
whether the world is a mere construction and the world picture a reconstruction
of what was constructed by oneself as there is for the claim that the individual in
his development constructs or reconstructs his world, that is, as long as this is a
mere claim, and as long as this claim is not substantiated by any explanation of
how the individual manages to do that· - indeed, only this knowledge could help
one to better understand and to exploit the constructivist thesis.
If "constructivism" is to mean anything didactical, it must indicate the one who
is expected to "construct". There are people who call themselves constructivists
because they allow the teacher (rather than the textbook author or some other au-146 CHAPTER3
thority) to consfruct education. Constructivism also means an artistic, say Bauhaus, style. The teacher who adheres to this necessarily creates structuralist mathematics. If I were to accept the term "constructivism", I would mean a programme having a philosophy that grants learners the freedom of their own
activity. Then it doesn't matter whether this activity is called construction or reconstruction or whatever, as long as such words do not prejudice anything that
concern the learners' freedom. Whatever is imposed upon the learner remains imposed, whether this is called consfruction or reconsfruction or something else.
There is no use pleading here that learning in freedom is more dignified than under coercion, since the hope that the first is more efficient than the second is being
strengthened daily by an increasing number of examples. Lacking a convincing
context, such terms as construction, reconstruction, and constructivism are
doomed to remain slogans. The only context that counts didactically is instruction itself, that is, instruction developed from the direction of the design onwards
towards its realisation. If a term is, indeed, needed I prefer guided reinvention,
which I discussed in (2.1).
7. Von Glasersfeld
One would expect Von Glasersfeld's name to appear in this exposition. I didn't
mention him originally because I didn't know where to place him, but a recent
publication of his has made things easier. His paper starts as follows*•:
During the last three decades faith in objective scientific knowledge. a faith that formerly served as the unquestioned basis for most of the teaching in schools and
academia. has been disrupted by unsettling movements in the very discipline of philosophy of science.
Indeed, and this is a most striking symptom of the ever broadening and, to my
mind, most deplorable gulf between the philosophy of scientists and the philosophy of philosophers of sciences, at least as preached by the noisiest among them.
Would any scientist who lacks faith in objective knowledge have ever had Voyager 2 launched for a visit of the big planets -- as far as Neptune -- to gather
knowledge about them? Would he have had big accelerators built that are to discover missing elementary particles? Would he analyse long DNA strings in order
to manipulate them? Would he excavate ancient cities and decipher ancient inscriptions? These are just a few questions out of an interminable variety, and yet
I did not even mention the mathematician searching for new knowledge.
Rather than trying to uncover the scientist's implicit philosophy and methodology, philosophers and methodologists of science, if unacquainted with real science
or frustrated by it, dream up their own fancies. To be sure, Von Glasersfeld is in
good company, but it is a company of spectators rather than of actors in science.
Yet, in his particular case, it is a third kind of philosophy that should count: philosophy of education. These three may be mutually related provided they are not
confused with each other. At any rate, I cannot see any bond between mathemat-
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exact sciences. Yet, whatever beauty may mean forits creator, it is aimless unless
there are more people to enjoy it, and somehow they will have to pay for it, either
as a buyer or as a taxpayer. Is beauty measured by its return? Few people are
ready to appreciate the beauty in low-level mathematics, and fewer, including
even experts, mathematics at the research level. Therefore, as far as mathematics
research is concerned, beauty is not a convincing answer to the question "What
is the use of it?".
There is, however, another aspect: mathematics is true. Wis en zeker -- sure and
certain. Research results can be checked, errors can be tracked down, and, if experts disagree, they may do so about good or bad, rather than about true or false.
In the so-called exact sciences, checking becomes more troublesome the farther
,'i 148 CHAPTER3
away one moves from mathematics. Anyway the ultimate judge is Nature, who
rewards good questions with good answers.
Man and Society are much less co-operative and reliable than is Nature, and this
is the very reason why, in the social sciences research is a tremendously bigger
problem than in mathematics and the natural sciences. It is a problem that, unfortunately, bothers the assiduous researcher less than it does the not so compliant
user -- as long as it is useful to the former, in one way or another.
As early as the beginning of the 17th century Thomas Mun and Edward Misselden discovered and explained the fluctuations of exchange rates by the balance
of foreign trade
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• It was true and useful; as a discovery it was beautiful and historically it is still so since it happened in a period where money meant coined
metal and paper currency was at most embodied by bills of exchange. In the 20th
century this is no longer true, let alone useful or beautiful.
Research on Man and Society badly needs and distressingly lacks criteria of
truth, which are present in all kinds of research on Nature. Just when the following lines were being written, it so happened that both professionals and the public
al large were surprised by the announcement of a new and almost trivially simple
method of nuclear fusion. Lo and behold, within a few days researchers all over
the world undertook to imitate the experiments as described and to check the alleged effects. Not even for a second did I doubt that, long before these same lines
would be printed, it will have been settled for ever what was and was not true
about it To be sure, most often it is not as simple as that: it can take more time,
greater trouble, and bigger machinery to test an alleged effect. Where Man and
Society are concerned, however, each experimental falsification can be met by
the objection that it is a fact life that two situations can never be the same; moreover, once an experiment has been terminated, later discovered mistakes in design or execution (if admitted at all), cannot be undone, because the costs of repetition would be prohibitive.
Can the lack of criteria of truth be reduced by trust in the capability and honesty
of the researcher? The question is loo rhetorical to be answered. Man and Society
are developing at a much faster pace than Nature (if something al all like the social version of evolution applies to Nature). Nature is more uniform, or so it
seems to the investigator, who can profit from centuries of experience provided
by a stable Nature. This, to my mind, justifies or even requires a more utilitarian
attitude towards social and societal research than one is inclined to grant the natural sciences if the question "what is the use of it?" is to be answered.
Keeping the problems we are concerned with here in mind, let me disregard what are
called humanities. on the one hand, are as much an art as are their subjects, and, on the
other hand, are on no score that matters here inferior to the so-called exact sciences.
There are, however, examples of psychological research I like because of their beauty.
For instance - I forgot the author's name - the experiment where subjects were given
a weight,, a string, and a nail to be fixed on the wall in order to construct a pendulwn:
whether the subjects did or did not succeed to solve the problem depended on whether

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There is a kind of inquiry that characterises itself by such attributes as listing,
stock-taking, registering, recording and fact-finding. If carried on at secretary
level these can be enormously useful. They can also be useful at research level,
provided the attributes attest to modesty or are as critically handled as they deserve. They can be useful as rough material according to the degree to which they
are still rough material. The question "what is the use of it?", however, becomes
more urgent the more theorising attitudes dominate or reshape what once was (or
never was) rough material.
Henceforth where research is concerned, my crucial question will be "what is the
use of it?" I mean here its use for the consumer rather than for the producer, albeit
in a broad-minded interpretation; so broad-minded that the mere intention of being useful is counted as high as the result
But let us now approach the actual aim of our discussion more closely!
3.2.2 Educational research
l . What is the use of it?
By education I mean a practice, though often enough this word is used as a synonym for "research on education"". For the present moment, when I speak about
educational research in general I intend it to include reSearch on mathematics education as long as, for the researcher, mathematics is no more than an easily
available and easily handled subject matter, chosen to test and apply general ideas
and methods, with no regard for the specific nature of mathematics and mathematics instruction.
The enormously rich and useful educational literature, which addresses itself
most directly to its presumed users - parents, teachers, trainers, counsellors - is to
such a degree and so straightforwardly the result and the expression of everyday
experience, craft and philosophy that however they might have been influenced
by research, these influences can hardly, if at all, be retrieved nor their impact be
evaluated in this vast and wealthy field. This is the reason why I shall restrict myself to the more professional literature on education. Let me summarise my negative feelings beforehand: as a general trend, the greater the pretention with
which something is presented as research, the less satisfactorily it comes across
as an answer to the question "what is the use of it?' Mind, I do not expect all research (or even a substantial part of it) to be somehow useful. I would simply like
the researcher, whatever his undertaking, to ask himself the question "what is the
use of it?', and then in a way as though he were one of the intended users himse~f.
To be sure, a great deal is demanded here. It is a hard thing to place oneself m
another person's environment, let alone in his mind; and, on the other hand it is
a long way from the researcher's armchair or laboratory to the user's classroom,
even though both of them may be located in the same building. Indeed, their mutual distance is properly determined by the length of the chain of mediators leading from the one to the other, provided there are any mediators at all; or otherI i i
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wise, by the depth of-a gulf that nobody is able or cares to bridge. In both cases,
mediation means interpretation, either of facts, which is the most fortunate case,
or of words, which can be preposterous -- producing a situation not unlike the one
I depicted in (3.1.9).
My statement about the long way from the researcher's workshop to that of the
user may be a source of misunderstanding. Rather than "from" I should have said
"between". It is a symmetric relation. Or rather, wherever fresh trails are to be
laid, the trail-blazing should start in the classroom; it makes a difference here
whether it is the researcher himself who starts the trail-blazing, or some delegates
who are expected to carry out his prefabricated instructions, as well as whether
the teacher is considered lo be a fellow traveller or is expected to stay home.
I failed lo mention curriculum developers among the users of educational research; this gap will be closed in 3.2.3. Nor did I mention decision makers, policy
agents and legislators. In what respect, if at all, can they be counted among the
users of educational research? I do not know about other countries but at ours two
kinds of government-advising bodies are lo be distinguished: Fact-finding and
opinion-shaping commissions on the one hand, and, on the other, councils of --
presumably scientific -- advice. When asked, they will commission groups of researchers to write reports, which, in the case of education, are no more than
sources of vocabulary; this helps them to pretend that their final advice, whatever
it may be, is not merely a product of common sense. Policy agents behave in a
similar way. Whoever their advisors are and whatever their advice may be, they
are not used for anything else than rationalising a politically based decision.
2. Methodology
Too much metaphorical language? Yes, because as far as I have been informed,
trail-blazing is not what the future researcher is told and taught is his business in
education. His business is rather: collecting data and processing them according
to ever refined standard methods, which are neither argued nor questioned, but
simply taught with the aim of being obediently applied to a variety of themes, depending on the prevailing fashion or the whims of the researcher, his director, or
his client. I don't remember when it happened but I do remember, as though it
were yesterday, the bewilderment that struck me when I first heard that the training of future educationalists includes a course on "methodology". This is at any
rate the custom in our country but, judging from the literature in general, this
brain-washing policy is an international feature. Please imagine a student of
mathematics, of physics, of -- let me be cautious, as I am not sure how far this list
extends -- impregnated, in any other way than implicitly, with the methodology
of the science he sets out lo study; in any other way than by having him act out
the methodology he has to learn! In no way do I object to methodology as such -
- I have even stimulated the cultivation of it, but it should be the result of a posteriori reflecting on one's methods, rather than as an a priori doctrine that has
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methodology no better than it ever did in mathematics, that is, where it works it
does so to the benefit of a small minority of learners only -- the future specialists
of methodology. Yet, fortunately, the intimidated majority can count on the precious assistance of this authoritative guild, the pure methodologists, whose
strength consists in knowing all about research and nothing about education.
They gladly leave the educational researcher a responsibility of his own to fill
empty methodological vessels with educational contents, and they are unconcerned with the question of whether these fit or not. Mathematics can afford a
vivid interplay between form and contents because in playing this game they can
be neatly separated from each other. In educational research, however, separating
form and content is not feasible, and, wherever this is attempted the consequence
is estrangement. This is covered up, ifneed be, by distinctions like those between
validity, which is the researcher's business, and reliability, which is the methodologist's responsibility and which by mere convention is measured by a formula
that, in fact, does not measure anything that deserves this name. Please imagine
a biologist who has defined a measure for immunity against some plant disease
that can be determined with a high degree of precision while it is most doubtful
whether it has anything to do with resistance against that disease! Who would accept such a measure, reliable up lo the third decimal if nothing is known about its
validity? This mIJY be a rhetorical question in biology but not in educational research, where it can only be answered historically: formalised testing was first
concerned with intelligence; when the query what, if anything al all, the I.Q. did
measure, had to be shelved, one became satisfied with measuring this same thing
as reliably as possible. From the I.Q. this easy answer spread to psychometric
measures in general, as did the habit of measuring first and trying lo explain later.
This is a bad habit, since it might be a short step from bad to worse: as soon as
scientific scruple and honesty are degraded to mere formalities, it is almost unavoidable that precision and accuracy will count only as far as they can be expressed in numbers of decimal positions.
This may sound exaggerated, and in a sense it is. Let me quote myself from [87,
p.145):
According to a well-worn joke there are three kind of lies: lies. damned lies, and statistics. When the joke was invented. mathematical statistics had not yet come into being. Otherwise one would have added three kinds of surreptitious credibility: by word
of honour, by oath. and by mathematical statistics. Or, three methods to assume a
learned air: footnotes, bibliography, and correlations tables.
This is a witticism, to be sure. But even so it is the gist of many years of -- sometimes disgusting -- experiences, which I gained as a reader, reviewer, referee, and
editor, and of which only a small but paradigmatical part has been published. At
no point have my detailed scrutinies of small and large pieces of educational research ever been refuted, nor, lo the best of my knowledge, has anybody ever seriously tried to challenge them on minor or major points, whether published [99,
102, 116) ornot.
Matters are even worse. The collection of poor research I criticised was nothing 152 CHAPTER3
more than a sample chosen from what I came across. My actual target was the
underlying research method, imposed by means of the authority of a misleading
and misled methodology, which is still taught today, although I feel its authority
is waning. Anyway, I don't know of any attempt ro defend this methodology or
parts of it against my criticism. Or is this a policy of killing by silence? Yet I received no positive signals either. As far as I know, up to now nobody has ever
tried IO explain the mathematical meaning and the cognitive function of correlations within the various contexts -- where educational researchers have computed
them by tens of thousands and arranged the min tables -- and their use other than
for impressing laymen and sponsors. Nobody has ever tried to vindicate the normality of distributions that, by definition, are not normal, or IO justify linear regression as such, in particular, if applied IO connections which are obviously nonlinear, as in the case of stable equilibrium. I have never seen any explanation of
how mathematical methods from biotechnics -- where their applicability is warranted by models from physics, chemistry, and biology -- can be transferred ro
education, which lacks any fundamental insights for building intermediate models.
I cannot but recognise the fact that some things have changed in the mean time.
Taxonomy, for instance, has fallen into disgrace (though not because ofits inherent weaknesses but because it is too strongly a contents-dependent tool) and is
now successfully being shipped to developing countries. Test theory and technology, on the other hand, have boasted ever new successes in fighting sound education. Qualitative research is gaining ground, but we are still far from the point
where mathematical methods can add a finishing touch IO qualitative knowledge,
and many researchers are even farther from the insight that mathematics is not
able to do more than just this. But it is hard to fight the prevailing superstition that
ready made mathematics can solve all problems.
Reluctantly I reiterated my old complaints on educational research. My intention
was actually IO review them while asking the question "what is the use ofit?',
other than cultivating professional and public relations.
There is a hard struggle for life and grants going on in the community of educational (and some social) research. One cannot do without advertising and the
headlines in newspapers, or so people think. Is this a necessary evil, and if so,
why can it not be used for good things? Or do people who are wary of this policy
suffer from allergic reactions to it?
Are good things not expensive enough to be supported? When the first IBA cycle
started it was the most pretentious project ever undertaken in educational research, and the most costly -- some 50 million dollars if corrected for inflation, I
would estimate - and when it was fmished it turned out IO be a mess: a heap of
spurious data, processed by means of nonsense mathematics, and - what is worse
-- useless IO such a degree that after a few years even in the most extensive bibliographies ignored it. What was the use of it? Was the road IO New Math paved

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was really a major factor, although years after New Math's eventual failure,
T.Husen, proud of the hundreds of headlines it had earned him in the American
newspapers, still thought it was.
In my country, IBA received even Dutch TV coverage because our Netherlands
pupils had scored so badly in a subject area they had never been taught -- a fact
unknown to the inexpert researchers -- and because they had performed so excellently in other areas, thanks to international test instruments, debased by translators, who had not been informed about the aim of the translation.
As far as I know, no international report on the second - much less pretentious -
IBA Mathematics Study has been published. In our country it made the headlines
because we finished second or third in the international field, albeit thanks to the
fact that among the various types of schools the possibly lowest scoring one had
been left out of the tested population: this was "special" education, which, according IO our (by international standards unusual) definition includes about 10%
of the school population under consideration"".
A similar mistake has marred an otherwise excellent national study of ours on the
quality of primary mathematics instruction. For our daily papers girls score much
lower than boys was the most important news it revealed. As a fact, this contradicted worldwide experience at the primary level, but rather than being a fact it
was an artifact, due to the same omission as menti<1ned above, and easily explained: only behind the desk do researchers forget that, in our "special" education, boys form the,overwhelming majority. The blunder could easily have been
avoided, but the report was not even made available under embargo IO experts,
who had to wait one more day than the press IO get information and to discover
the blunder. This in fact happened at an educational meeting the day after the
press conference. Too late, of course, since girls equal to boys or even better is
not worth the trouble of a correction.
Who does not remember the Coleman report -- the most extensive, pretentious,
expensive and impressive report, ever carried out in the USA? Or rather, who
would at present still remember it for any use whatsoever, even if it had not been
wrecked by sophisticated analysis [68] 44.
3. Comparative research
A large part of educational research is comparative. Cognitive or affective results

of education are compared with one another according to such variables as nationality, instructional system, educational philosophy, schools and school system, gender, ethnicity, race, socio-economic background (of pupils or teachers), the teaching and evaluation methods applied, the aids, instruments, textbook series used -- a list that is all but complete. I do not object to comparisons. Whoever judges a given kind of education, will relate it somehow to the experiential or ideal models he has in mind; the more consciously and systematically this is done, the more valuable the result may be, in particular, for enriching experience and improving models. But "systematically" should not mean: according IO a preconceived system, in the mechanically thoughtless way, as taught by methodologists. 154 CHAPTER3
In education, comparing is a most precarious business. What are comparable situations and how to get them? We just gave examples of rather simple and transparent situations where researchers overlooked obvious differences. How to avoid this? One way is lo hold strictly controlled instructional experiments; this
is, in fact, both the most efficient and the most useless way, since rigid instruction
is the worst instruction one can imagine. One can try to eliminate disturbing factors, and if this is done by common sense, I would not object either. But common sense is suspect to methodologists bent on employing refined mathematical
methods such as factor, variance, and regression analysis. To the best of my
knowledge they have never been rationally justified, either in general or in some
particular case, and, depending on the way they are applied, can often be used lo
prove anything and its opposite. Moreover, little if anything at all is known about
influential factors, for want of a fundamental theory on the subjects under consideration. Some people try to make up for this by the sheer number of factors,
chosen ad lib, a policy that increases the impressiveness and decreases the credibility of the applied mathematical technique. However it is done, this is merely
an attempt lo apply mathematics lo bunches of numerical data, irrespective of the
way they were gathered, with no intervention of any model based on theory. People who know that this is wrong excuse themselves by bestowing the dignity of
model on any linear regression equation they write down; they do not bother to
justify the hypothesis of linearity, even where common sense suggests that it cannot be trusled, as in case of equilibrium.
4. Tests
Rationality requires statements to be lesled for their truth instructional methods
for the benefit of education, and individuals, for their own benefit and that of so-
ciety. Tests should be trustworthy, and what is more trustworthy than numbers,
obtained by measurement, and moreover amenable to mathematical processing?
I already emphasised that, in the study of Nature, measurement is rather a finishing touch, applied lo models of physical reality. These models are there to explain
what has to be measured by what yard-sticks -- the more rigid the better -- and
for what aim. Man and Society are much less accessible lo such models, and often
enough, in particular in education, they are not accessible at all; yet in no way has
this hindered the tremendous growth of psychometry in education from its first
roots onwards, from early in our century to its present state. Let me quote
RS.Bloom & A.W.Foshay4' as witnesses for the high esteem in which rests were
(and by many still are) held.
There is one field in which a considerable sophistication has developed since 1920:
the field of achievement testing. It is possible now to study the degree and nature of a
student's understanding of school subjects with a subtlety not previously available.
Modern objective achievement tests. when properly developed and interpreted, offer
one of the most powerful tools available for educational research. Findings have been
made through tli.eir use that rise far above common sense.
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who co-operated in the same IBA-study from which this quotation slems; yet, as
far as tests are concerned neither Bloom's celebraled Taxonomy6, nor work accomplished under his guidance [cp.99] can be looked upon as paragons of sophistication. And what about the last statement of the quoted passage? Where should
one look for such "findings"?
It cannot be denied that, bit by bit, a highly sophisticaled testing technique has
been developed, and measuring sticks have been constructed, which, from the
viewpoint of rigidity, can compele fairly well with those of the natural sciences,
although this might probe to be a disadvantage rather than an advantage. In the
narrow sense of this technique, the word "test" now belongs to the common vocabulary of most languages. The attribute "objective", which is only added when
strictly necessary, nowadays means that the formerly expensive job of resting can
be left to the much cheaper computer. (In my own country, objective examiner is
a governmental-hypocritical euphemism for cheapest examiner.)
Objectivity, if opposed lo subjectivity, is indeed praiseworthy, if people are to be
judged. Yet, rather than at people to be judged, resting rechnique aims at objects
defined through tests or lest batleries and allegedly independent of human caprice, although they are given names, which suggest connections to human life
and learning. It is a process of rectification by which new objects are created,
with the sole aim" of being studied numerically, and where no regard is paid lo
their effectual stale nor their usefulness for understanding and explaining phenomena. This process is started, sustained and sleered by a quite natural tendency
to detach form from conlent, which, although it has proved enormously fruitful
in mathematics, thanks lo its, as it were, self-controlling character, is condemned
to sterility wherever it escapes control.
Although the present exposition starred with a quotation involving achievement
tests, it applies to any kind of tests in the lechnical sense, including opinion polls,
such as preposterously used for research in the affective domain.
The above criticism of formal testing lechnique with regard to research in mathematics education may appear lo be outdaled: There is an unmistakable trend
away from it among educationalists with a strong mathematical background, who
have good reasons to do so and are courageous enough to no longer be afraid of
being reprimanded by methodologists for neglect of "objective" methods. Open
testing, inlerview, and observation of learning processes are gaining ground, and
well-designed intersubjectivity has is given a chance to supersede ill-conceived
objectivity. Little attention, however, has been paid to the methodology behind
the new approach -- J. de Lange's work" is here qui le exceptional. Anyway, there
are promising perspectives on fresh approaches. Will they be successful in the
long run?
Technique and lechniques are indispensable and beyond praise in our society, at
least as long as they deserve praise. If they are blamed, it is for what nowadays is
called pollution. How far has the pollution by testing techniques spread in education? The negative influence examinations can exert on education by their mere
existence is a well-worn truth, and nobody can tell how much resting technique 156 CHAPTER3
has added to iL How much has diagnosis and how much have diagnostic abilities
suffered under the influence of badly understood diagnostic tests? And how
much profound qualitative research has never been undertaken because it could
not compete with shallow quantitative pseudo-research?
3.2.3 Developmental research
l . Change
Formally, the present subsection looks like it is one more part of the preceding
one. I could have formally separated them by simply adding the adjective "traditional" to the title of the former subsection, but I preferred to avoid this because
of its negative connotations. Moreover, what we are now accustomed to define
as developmental research is by no means a new thing. On the contrary, it is perhaps the oldest kind of educational research, though it has not been recognised as
such.
The question on educational research I started with was, "what is the use of it?"
I didn't ask the question "what is the use of education?", which is almost rhetorical as Jong as it aims at education as such rather than at a special kind, say, actual
or topical education. No doubt, education is meant to prepare one for future life,
yet neither the future of individuals nor of society are predictable well enough to
provide a solid basis for any choice of education, which, conversely, is influential
as a condition of this future. Once, when asked by an interviewer whether I
thought that attempts at innovation have improved education, I hesitated for a
short while, only to eventually stamp this as a wrong question. Pictures of education, taken at different moments in history, cannot be compared. Each society at
a given period got the education it wanted, it needed, it could afford, it deserved
and it was able to provide. Innovation can effect no more than to adapt education
to a changing society, or at best it can try to anticipate the change. This alone is
difficult enough. If any proof was needed, it can be seen in the efforts, vaccillating between childish and spasmodic, to prepare for what is commonly called the
computer age, though in fact the computer is only one of the features of this age.
Or, to take an example from the past, a past that started after World War JI: Even
now innovation has not yet succeeded in coping with the fast growing demand
for education, which is too readily answered by "more of the same", rather than
by diversification of the supply. So the gist of this paragraph is to give a the first
answer to the question "what is the use of it?": Change.
2.Errors
If I am not mistaken the first subject formally studied in mathematics education
- which means, as early as the turn of the century- was errors, that is to say, /earners' mathematical rather than teachers', developers' and researchers' didactica/
errors, and according to research published up to a few years ago this is still the
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hardly been noticed. Even now failure rather than success is the main discriminator in many research papers. Even though, the variety of investigated errors has
increased in the course of the century. It is remarkable how much attention has
been dedicated to the same kinds of errors over and over, albeit more and more
related to an increasing number of other parameters, such as age, gender, peer
groups, ethnicity, race and social-economic status of parents and teachers, -r
formance in other subjects, intelligence, affective features such as mathemattcs
love or anxiety, and so on.
Tue attention to errors is as indispensable as to physical or mental diseases, provided it is guided by the wish to cure or to prevent and is undertaken in a context
in which this wish has a chance to be satisfied. I recently attended a meettng
where a researcher presented a new diagnostic instrument for mental arithmetic
in the domain 20 to 100. He modestly added that instruments of remediation were
not the researcher's but the expert teacher's concern, and he didn't change his
view even after a teacher in the audience urgently asked him about the use of diagnosis.
Mere diagnosis of diseases is a most valuable statistical tool of epidemiolog- on
behalf of medical policy but in education the epidemiol_ogical approach to failure
testifies to a trust in educational policy that is hardly justified. Diagnosis may be
important for employers and insurers, and it cannot be ignored where the product
of education is what hlatters, but even then processual diagnosis should be given
a fair chance. The trend in medicine to focus on health rather than on illness is
almost too obvious to warrant being mentioned. There are no vaccines against
mathematical errors. On the contrary, and in particular if mathematics is to be reinvented under guidance, errors are unavoidable or even welcome provided they
are used to stimulate resistance against their own consolidation and recurrence.
Ideas on the need and possibility of retiology - to stick to the medical terminology
- are as rare in research on errors as is the attempt to recover the past of the learning process and insight into the relevance of long-term learning processes.Errors
as 62 - 45 = 23 have been signalled hundreds of times, but the strong suggestton
that they are due to a premature separation of form and content - positional system and subtraction context, in the present case - has not even been a subject of
research as interpreted by the majority of researchers. In education, it does not
yet go without saying that retiology is a condition for effective remediation and
prevention.
Tue inclination to study error is quite understandable. Normal things need no explanation, but what really strikes the beholder are deviations from the norm: -at
is from what is felt or agreed on to be normal. Education, as looked upon m its
p;esent state, is taken for granted, and against this background of certainty its actual or potential diversity can be a matter of study that handles liability to success
or failure as its criterion - comparative research, which is questionable as such -
rather than retiological.
Are educational researchers indeed as conservative as that? Or wasn't New Math
a historical counter-example? If at all, this example was research on how to put 158 CHAPTER3
new wine into old bottles -- is it Enseignement des mathimatiques modernes ou
enseignement moderne des mathimatiques? I asked in 1961 [14].
3. The cradle of developmental research
Most research is on factual education. Developmental research is different, and,
as I already claimed with some reservation, historically it even preceded formal
educational research in mathematics (and other fields). In the past, mathematics
instruction progressed by the activity of educational developers, in general documented by textbooks, which are still to dale the main agents of change. When I
claimed that educational research started developmentally, I admitted as well that
it was not recognised as such, and I meant -- not even by those who carried it out.
I didn't actually say it was the oldest kind; I said "perhaps", which means that the
statement depends somehow on what is understood by "research".
In (2.3.1) I recalled that, as early as 1961, I propagated thought- experiments as
instruments of educational development I should rather have spoken about drawing attention to their function in this activity. Indeed, it is clear from the context
that I took it for granted that thought-experiments had always functioned this way
and that the only thing lacking was consciousness of it. Observing their lhoughtexperiments and reporting on them would have transformed educational developers into educational researchers, who present their findings togelher with the
arguments, rather than dogmatically.
Although factual experiments would be more convincing, thought-experiments
will remain indispensable (at least in order to prepare factual ones) as long as educational development is being undertaken, and this will happen as long as the
need for change is felt Tradition and change, and their mutual relation, are fascinating subjects. In [40] I dealt with their role in malhematics and in education.
A good teacher, who does not blindly follow traditions dictated by bis own past
or by prescribed and prescribing textbooks, acts as an educational developer, who
most often is unaware of this role. In fact, awareness of the motives of one's own
activities can be either useful or harmful, depending on the situation, the action
and the actor. A teacher can be struck after school by the idea that "today it was
different". Was it really different, and if so, what is the "it" lhat was different and
why was it so? Soul-searching isn'teverybody's business, but change lhrough accident can be a source of search, just as search can lead to change.
Is this simply a bunch of trivialities? I wished to argue that even nowadays research can start at !he grass-root level, lhat is, under no olher conditions lhan !hat
of becoming aware of one's new experiences, as well as !he wish to understand
them in the perspective of a change for better, and that at any time it can restart
under !he same conditions. This is not to belittle high -level research but to prevent it from clogging its most abundant sources.
4.R&D
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former. This, then, characterises developmental research: it takes place wilhin !he
educational environment, which is expected to undergo and to activate change.
"Development" made its entrance as a technical term in !he nineteen-sixties,
when educationalists were trying to take advantage of the then noisy call for
change. It appeared in !he combination R&D - research and development - whlch
was soon amplified by another D, !hat of dissemination. The terminology itself
was borrowed from the natural sciences and their technology, though, as I have
often emphasised [e.g.80], this happened with little regard for its meaning in that
field, and without asking whether it adequately applies in education. In the natural sciences it alludes to the road leading from lheoretical work to technical applications: the test tube replaced by retorts of more and more industrial dimensions, the laboratory by ever bigger workshops, !he researcher's desk by the drawing board and the computer. This is not the way it takes place in the social
sciences -- except perhaps in economics. Theory may be a source of inspiration
but seldom, if ever, the place to look for germs of development towards applications. There is research for creating means of organisation and processing information, such as terminology, methodology, test theory, statistics, but all this is
applied in a descriptive and inventorising, ralher than in a developmental way.
For a discovery in chemistry or biology to be developed industrially, buildings
get designed in one way or the other, though whai really matters is not the structure but what it contains and what is intended to happen !here. In education, however, the R of R'&D is concerned with the formal element of the building rather than with its contents.
The problem with the educational R&D terminology in everyday practice are the
strange consequences, as nicely illustrated in the following example: creating
new courses of instruction is filed under "development", even though it requires
-- at least implicit -- fundamental research, whereas the mere routine of testing or
adapting a given course on behalf of, say, level differentiation, mastery learning,
and so on, falls under research, which is likely to be held in higher esteem and
correspondingly be subsidised more easily and more generously than development
The R&D mania stimulated productions like curriculum theories, general goal
description theories, taxonomies, hierarchies, model designs. In [87] I still judged
these worth punished, while nowadays it is hard to find anybody who even cares
about them; so I can only hope !hat !he same will happen to system lheory, "cognitive psychology", and other lhings, which are imposed on or proposed for
malhematics education by general educationalists. Indeed, even though the R&D
terminology has been dropped, the R&D mentality still subsists, by which I mean
the mentality of separating research and development.
5. Cyclicity
It still subsists, that is to say, in stale theory. Practice, at least in education, requires a cyclic alternation of research and developmen- which can be more efficient the shorter !he cycle. What was developed behind the desk is put to the test 160 CHAPTER3

in the classroom in order IO be analysed, and development is resumed with the
results of this analysis.
This seems IO be a trivial statement, and indeed the alternation between thinking
and doing, planning and testing is nothing more than the everyday way of life.
Developmintal research means that this alternation is understood as a macro feature, rather than a micro feature of the domain where it takes place and of the flow
of events. By this broad field and long-term feature it distinguishes itself fundamentally from what is sometimes called constructive research.
How essential is the fall-out of development to deserve the name of theory, how
essentially fertile is the development stimulated by such theory? The feature that
gives developmental research the right to the name of research is its bringing to
consciousness that what is fundamentally new and essentially fertile in research
that is arisen fro educational development.
But I should not leave it at this. Educational research as fall-out of educational
development - this again erects a dividing wall, which even if only a curtain, is
not flexible enough to do justice to what developmental research means. "Development ensuing from research" and "research as fall-out of development" is too
weak a synthesis. How can we achieve a stronger and more efficient synthesis in
order to answer the question "what is the use of it?"
6. Proof by process
Rather than R&D, I borrowed another feature of research in the natural sciences
on behalf of education: the thought-experiment. Design, even if undertaken behind the desk, is a better start than stale theory, but it also better than a long series
of thoughtless experiments. Physics, as we know it at present, originated neither
theoretically nor experimentally, but in a rational way nevertheless: from ideas,
suggested by the analysis of nature and tested later-- often even much later. The
double helix was a splendid idea for the mental reconstruction of DNA, and it became the model for numerous successful reconstructions before it was actually
proved, that is, a quarter of a century after it had been proposed the first time.
The high tide of attempts at educational change after the Sputnik shock flooded
mathematics education with new programmes, new teaching matter, new courses, all of them contrived behind the desk, discussed at symposia and, as a matter
of urgency, pushed into classrooms. Little was ever put to the test, and still less
survived even the most benevolent trial. In fact, innovation is an arduous process,
and gives the lie lo the most cautious predictions. My criticism at that time did
not aim at the "behind the desk" origin of the innovation projects -- how else
should one proceed? -- but at the lack of any report on the underlying thoughtexperiments, if any. Researchers publish products of their activity, rather than the
processes by which they were created; the knowledge of these processes is considered to be their private domain; and so did it happen in this case loo.
This policy may work in mathematics, where a proof, if sound, justifies the theO'f'm [Hl_;-- -_f-.:o,..-lf, ru-t,!-1-1-_-_-gh v-_e-v-,.'11 -t -_s l-:::'S' _.,.f'R-- ---._ _ ;":VC"'"_ 1'--f'"':-, 1
1: 1'", 18"
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scribed, can be reproduced by others more or less easily. In the social sciences,
however, such a situation is quite exceptional. How, in any case, can one reproduce thought-experiments that have been kept secret?
Knowledge can successfully be presented as a product If the process of its acquisition is reproducible - a characteristic of "hard" science. Wherever this condition
is not fulfilled, knowledge presented without any indication of the process that
brought it about, lacks all characteristics of rationality that distinguish genuine
knowledge from dogma. It is unproven, unargued, unfounded, or so it seems.
How to transfer such a knowledge, how to have others reinvent it if one keeps
silent about that which is the most essential?
This question is particularly pressing in the case of educational development.
Taking notice of the product, which allows for many interpretations, is not
enough. In order to apply the product, one must know how it came into being.
Not in detail, of course; the point is not to get stuck in blind alleys but to be
warned against them. Reproducing does not mean parroting.
7. Dissemination as an agent of synthesis
Above I claimed that "development ensuing from research" and "research as fallout of development" is too weak a synthesis. For reasons I just expounded, the
idea of dissemlllllition, which in the RDD conception is a separate component,
should be kept in mind the entire time that the developmental process lasts. In
short,
developmental research means:
experiencing the cyclic process of development and research so consciously, and
reporting on it so candidly that it justifies itself, and that this experience can be
transmitted to others to become like their own experience.
In contrast, constructive research is a more modest conception than developmental research of which transmission is an essential feature. To be useful, constructive research trusts its results to implementation. In education, or at least in mathematics education, little that can be transmitted with impunity as mere
implement. Developmental research asks for more by including the view on
dissemination.
Anyway, dissemination should not be separated from development. This requires
a continuous awareness of what is happening in the ongoing process, a state of
permanent reflection, recorded as much as possible". Dissemination is one of the
aspects under which hard and soft knowledge differ from each other. Hard, that
is wrought iron - it does not matter how it was wrought. Products of educational
development can also feel hard - loo hard to be applied. Hard science shall meet
hard criteria. Where should one look for criteria for soft products? The answer is:
among the hard facts of the developmental process. They are needed to bear witness for the product in order to make it plausible and transferable, which demands
an attitude of self-examination on the part of the developmental researcher: a
state of permanent reflection. 162 CHAPTER3
8. The Landscape
Dissemination is a misleading tenn if it suggests sowing after harvesting. In fact,
from the outset it should be closely connected to the educational development itself, while the seeded territory is expanding in step with it. Both teachers and pupils are partners on equal standing with the professional developers in the rapid
alternation of activities behind the desk and in the classroom; they are simultaneously sowers and harvesters and educational developers, like the professionals.
(Let me add that I never did like experimenting with human subjects other than
while giving them a chance to profit from the experiments themselves.)
Developmental research includes
development on behalf of
first line instruction
teacher training, guidance and retraining
guidance and retraining of teacher trainers
guidance and retraining of counsellors
testing
media
advising and supporting
educational development by other agents (schools, working groups, text
book authors) in a broad sense, with the emphasis on co-operation
creating cadre for the increasing territory of dissemination research
within the development
as fall-out.
Developmental research covers
longitudinal developments with a view on
long-tenn learning processes
and is itselfa
long-tenn learning process.
9. A bit of history
In order to do justice to a repeatedly professed philosophy by me, I ought to tell
in a few words how developmental research, born by necessity and owned by reflection, was baptised.
In 1961 the Dutch government installed a Commission on Modernising Mathematics Education (CML W), which as late as the nineteen seventies was followed
by similar commissions for other subject areas. The way CML W interpreted its
task foreshadowed later developments [24]. Soon the need was felt for a centre
that would combine and co-ordinate their numerous activities. Early in 1971 the
CML W was finally given an Institute on Developing Mathematics Education
(IOWO), which was loosely connected to Utrecht University. The mere name of
the institute implied a programme: its focus would be on educational rather than
on curriculum development From the start onwards, the task of IOWO was in-
--e"'''- 7e•e,-;-1 '1.'S -=-1P13)f i.n,l--'i:_--tce.r} -----'-'fhi -;:}"'.:---<?"';----_UT "3-----r::;'"-"'" -'-- ,./_-:'.-
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look into the future [37, 44, 46].
General educationalists found fault with this heterodox unprofessional approach.
Indeed, IOWO people could not answer questions about the curriculum or learning theories they adhered to, nor could they produce catalogues of learning objectives, simply because they did not have any. The only things they had to show
were a philosophy on mathematics and mathematics education -- this had never
been heard of -- and a grand experimental design, which, to make matters worse,
failed somehow to mention control experiments. To be hones- I admit that
IOWO was taken seriously in the long run -- or even dreaded -- by general educationalists, even by those who had other ideas on educational development and
research.
Before IOWO's arrival the educational "Landscape" had neatly been divided into
sectors of Training, Counselling, Retraining, Development, Innovation, Test Production, and Research, each represented by a single institute or a group of institutions, or this was the theory. At IOWO all this was integrated. So IOWO was a
notorious case of unfair competition. As such it was particularly feared in the politically most powerful among these sectors, which up to the present day has unsuccessfully struggled for a genuine educational identity of its own.
Anyway, one day the government decided to put an end to this chaos and, as behooves bureaucracy, tb'neatly divide the landscape (as far as subsidised) into sectors that were to respect each other's borders -- a ruling that afterwards was even
written into law. Although this law strictly forbids any integration, it allows cooperation between workers or working units of two or more institutions of different sectors, which in any particular case requires a tedious procedure of approval
by managing directors, boards and councils of the respective institutions. Yet bureaucracy is not almighty. Meaningful work is still being done and subsidy obtained on behalf of this work by people who know the loop-holes, though it is not
an easy way of getting work done.
For the IOWO the clearing of the "Landscape" meant its death sentence. Only a
short "research" leg should be spared, which was implanted (together with theeducational computer centre) as a department into the Faculty of Mathematics of
Utrecht University; since that time this leg has grown quite a bit stronger. Formally stripped of educational development, it is now known as Institute for Research on Mathematics Education and Computer Centre (OW tiOC).
IOWO, of course, had fought back against its own death sentence, and this struggle gave birth to the tenn "Developmental Research", which is the reason why I
told the story. Under pressure of public opinion and parliament the death sentence was commuted from decapitation into mutilation; as mentioned before, a
research leg should be spared so a broad definition of research was in the best interest of IOWO. Our integrated approach had been successful, and none of our
people could imagine or trust nor feel able to try another approach as promising
as the proven one. What's in a name ifonly it covers a good thing? If it was a lie,
it had short wings. Though it was not, it did not help. Even though the name
looked new, Developmental Research was not an ad hoc invention, or so I hope 164 CHAPTER3
to have convinced the reader. Il simply was the approach that had proven successful in the past and promised success in the future. Through its struggle for life
IOWO, understandably became conscious of its true identity, as well as of the
need for a name to distinguish it from others.
10. Developmental research - a conclusion
Did I emphasise developmental research unbecomingly and to the detriment of
other views in this section on Research in Mathematics Education? My proper
criterion for judging educational research was formulated as early as (3.2.2, 1):
What is the use of it?, interpreted in a broad sense. My answer, in general, was:
Change. How much can research contribute to this? Or rather, how far should research participate in change? My answer was: As Developmental Research. (Notice the switch from "can" to "should", and from "contribute" to "participate"!)
"Developmental research" is meant as an ideal yardstick and a critical touchstone, and it will be used as such in the next section, where developmental research implicit IO practice will be made explicit
3.3 PRACTICE OF MATHEMATICS EDUCATION
3.3.1 Practice
In (3.2.3, B) I described in detail the Educational Landscape, which along with
first line education includes what in my country is called Service for Education.
In the Netherlands this is quite an extensive, and for the most part govemmentsubsidised, bureaucratic structure, strictly compartmentalised by law (WOV),
comparable to the guild system of crafts of former centuries (3.2.3, 9).
In the preceding section I dealt with the Research compartment of the Landscape
(that is, with the emphasis on Mathematics Education). As a name for the remainder! could find nothing better than that of Practice, which on the one hand is or
should be the customer of Research, and on the other hand looks like the counterpart of Theory, which was dealt with in the first section of the present chapter.
"Practice" has both a descriptive and a normative meaning. If it is understood in
the descriptive sense, its qualitative evaluation covers such a broad range that it
defies any attempt at description, even when restricted to one single country and
one single type of institution or part of it. So the normative element cannot be dispensed with in organising a description. Moreover, individual practitioners or
groups of them act according to explicit or implicit norms, which compete with
or are derived from more general ones.These norms are subjected to change either pursued by the practitioners, or demanded by the public, the authorities and
the theoreticians; and, after all, if there is any use for descriptions, it is in general
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tion.
3.3.2A background of competence
How competent may I feel to deal with practice in its descriptive or normative
aspects or in a synthesis of both? Not until 1970 did I shift my interest fully to
education, and even though I participated for a decade in the everyday activities
of a number of schools and training institutions of various kinds, I should confess
that I have never taught school nor even teacher training classes. Although I tried
to look inside as deeply as I could and as is needed for being a participant, I remained an outsider, one who casts glances into the educational system and acts
within it, without factually being a part of it.
To make things worse, I know almost nothing about other educational systems
when compared with my knowledge of the Dutch one. Of course, I have read
quite a bit about some of them, but what does this mean? If I were to judge our
Dutch educational system on the strength of written and printed documents, I
would get a severely distorted picture. Not because people who write about education are liars, but for the simple reason that the bulk of what is written is wishful
thinking rather than description. As far as it is descriptive, it is most often written
from a distance where details, among which perhaps the most important, are
blurred, clouded and obscured. So I must warn you with regard to all that I am
going to say -- about what in practice is right or wrong, what can be improved
and how it should be done, what should be changed and what is better kept as it
is -- that I am my own spokesman and the interpreter of my own experiences,
which were acquired in my own country at the places I just happened to visit.
This, then, may involve ideas that apply nowhere else or, the other way round, it
may mean labouring the obvious.
To be honest, I am not as pessimistic as all that. ! fi were, ! would not dare to tackle practice. Even if various educational systems appear more different than in fact
they are, I firmly believe that these are deceptive appearances. In particular, if our
gaze is focused on nations that share similar social conditions, it is quite improbable that their educational systems diverge as much from each other as they make
it appear. Even in one and the same country, with a narrow or broad scale of diversity, attempts at producing change are defined to a higher degree by social determinants than by individual or group efforts, which of course, cannot at all be
dispensed with. To repeat a joke I am proud of: I believe that in my life-time nothing has had as much impact on education as the ball-point, the overhead projector, and xerox -- inventions that have not primarily been made on behalf of education. Or am I exaggerating? In a sense I am. Besides the ball-point, overhead
projector and xerox, one needs people who use them, and preferably, people who
use them in a creative way.
This restores my confidence: we are living in one world, most of us under similar
social conditions, and even though learning one's own language and learning to 166 CHAPTER3
teach it may mean different things in countries where people speak other languages, mathematics is the same all over the world, and so learning mathematics
and learning to teach mathematics and to develop mathematics education are the
same as well.
3.3.3 Taught and learned-- the subject matter
Let me reconsider my last sentence! Is that what I just claimed to be the same,
really the same all over the world? Or is, if anything, the teaching subject the
same? If one compares mathematics as taught in various countries, certainly at
the secondary level, one would be inclined to say it is not The Bourbaki caricature taught in France strongly differs from the Kolmogorov version in the Soviet
Union (or are they really taught as such?), and both of them differ from what is
taught at British O and A levels, and still more from what pupils are expected to
learn in highschool in the USA.
Well, there are broad chasms between what is being taught in various countries.
But does it matter? I don't think so. As I see i-- if things are to be compared, the
proper question to be asked is not what is taught here and there but what is
learned, what is really learned, what lastingly affects the minds of the learners. I
cannot believe that it differs that much in different countries. I don't believe it because I think it cannot possibly be so. How can people in various countries differ
intellectually so much that whatever they are taught they can learn such different

!!tings as those different kinds of mathematics? I would prefer to say that people
on the whole, and in particular young people at a certain age, are all the same under similar social conditions -- even beyond comparing averages. We need not be
haunted by feelings of inferiority when confronted with the awfully abslract
mathematics that, say, sixteen years olds are supposed to understand and to master according to, say, French, Gennan, or Soviet textbooks. Tests have made it
clear beyond any doubt that the vast majority of pupils have not the slightest idea
of what is meant by this highly sophisticated mathematics that they are supposed
to have learned. Every year at the national examinations of 16 years olds in my
own country I am struck by the wide gap (extending over many other subjects areas, indeed) between the boastful pretensions of the tests and the poor perfonnances -- a gap possible thanks to an evaluation of the results that belies the expectations of the test producers.
I like to call it the big lie of our educational system, and I am afraid that this characterisation applies to quite a few others: there is a wide gap between the inflated
demands and the allowable passing levels. Exams serve selection: they single out
those who can satisfy the most exacting demands. Formal tests are constructed in
order to discriminate, as test producers call it, which means a 50% failure rate.
But why should we ask the great majority to aim infinitely higher than they are
able to reach? Not only infinitely hi-her, I should say, but also in 'the wrong di-

ing simple mathematics at a reasonable level is a more dignified pursuit than
learning complex mathematics at no level of understanding at all.
It sounds almost trivial; this focusing on what learners did learn, rather than on
what they did not. It is the most natural thing during the learning process, so why
should it not be natural at its end? Why not adapt the demands qualitatively and
quantitatively, and with regard to substance and depth, to the learners' ability, as
proven by their accomplishments?
I used the plural "learners" intentionally, and if this is taken into account, my
statement is less trivial than it sounds. We teach classes, so why shouldn't we expect classes to learn? Life is cooperation. Why do examinations focus on individual performances and entirely disregard collective ones? But let us delay answering this question!
The wide gap between what is claimed to be taught and accepted as being learned
. . the big lie . . is inherent to any system that strictly relies on exterior control.
The gap is reduced or absent at institutions that are trusted to set standards of their
own. This is true in any case for my own country, where uniformity of teaching,
learning, and examining stops at the doors of institutions of higher learning and
the prospect of the examination is the only measure, that puts restrictions on the
freedom of teaching. I don't know enough about other countries, but I still remember tests at FRG universities a few decades ago, where the majority of freshmen who had learned a vast amount of Calculus at school, didn't master the most
elementary algebra. Have things changed in the mean time? I am still puzzled by
the conlrast when I look at the kind and level of mathematics that future primary
school teachers are supposed to master in my own country and in the FRG. Are
their students so much more mathematically minded than ours and is teaching 6-
10 year-olds elementary arithmetic so much more demanding in Germany than it
is in Holland to justify requirements in mathematics as high as theirs? Nowadays,
students who are able to fulfil such requirements are not very likely to aspire to
primary school jobs . . this is the case in my country, and! doubt whether they are
more likely to elsewhere.
3.3.4 Taught and learned -- the agents
Leaming and teaching -- aren't these not outmoded tenns? In Education the fashion is now to call it change and transformation from novice into expert, although
this is more than a merely linguistic fashion. Two well-defined states are assumed with respect to a certain knowledge or ability: the clean slate, and mastery.
The agent, responsible for this transformation, is supposed to be an expert himself, which means a product expert. The intended process that transforms novices
into experts is likened to that which takes place on an assembly line. It has been
well-designed on the strength of cognitive or instrumental analysis •• a kind of
thought-experiment, where the subject to be transformed from novice into expert
is supposed to be acted on, rather than to be an agent 168 CHAPTER3
To be sure, one can imagine more refined versions of the expert-novice model,
which may grant the learner more freedom: choosing among several assembly
lines, for instance, or even within a predesigned network -- the more complex the
more costly, and nobody can say where the break-even point is. Can the computer
be of great help? I doubt whether enthusiasm alone justifies optimistic expecta-
tions".
The expert-novice model is a way to organise learning processes. Learning is
time-consuming. To be efficient it must be organised, but organising is also
something that has to be learned. Machines in which a lot of learning is firmly
and flawlessly invested are different; they need not learn themselves, or if they
are of the kind that do need to learn, then the required learning process is invested
just as firmly and flawlessly. The technological progress is accompanied by an
ever increasing estrangement Everybody can make some tool like a hammer as
soon as he needs it. A lot of people can repair electrical switches, radio sets and
cars, but few people know how computers work, and still fewer dare repair them.
Yet, even in this context, one can try to account forlearning as a human activity.
Von Neumann once claimed that, in principle, any human activity can be taken
over by computers, provided it has been analysed in detail. No doubt this applies
to physical activities, taken over by robots. But brain and mind are different. Although the first computers were copies of human reckoners, albeit adapted to the
understanding of the computer, our contemporaries in this field apply strategies
of their own in order to more efficiently use those facilities by which computers
distinguish themselves from humans. Yet chess computers, when first designed,
owed nothing to any analysis of human chess; as a matter off act, such an analysis
had not even been attempted, and even now there is a difference between the
computers' strategies and the more complex ones of humans, which are less accessible to analysis.
Computers are human-made and some humans know how they work. There is no
fundamental knowledge about the working of the brain, and hardly any phenomenological knowledge about that of the mind. Nevertheless, we are able to predict
and influence our own and other people's mental activities, without being able to
say how we acquired this ability. The computer metaphor of brain and mind is
therefore mistaken in both directions. All in all, computers are much too streamlined, much too prestructured and too reliable to be useful as paradigms for learning to mathematise little worlds.
3.3.5 Taught and learned-- interlaced
Let us recapitulate: At the assembly line of the expert-novice model, the learner
is supposed to be acted on, rather than to be an agent On the other hand, the computer model lacks descriptive validity as required for its being normative; and as
a tool in the expert-novice relation, it still fails to even approach the measure of

back between teaching and learning is still a most promising way to improve instruction.
Indeed, through personal experience we know about individual learning processes, through generalisation about general ones, and through history about those of
mankind. It depends on one's viewpoint whether one qualifies this statement by
adding "a little" or "a lot". Anyway, the greater part of this knowledge cannot be
verbalised, let alone formalised, and this is the very reason why programmed in-
struction -- whether by computer or not -- has made little, if any, progress. Fortunately, just as modern travellers need not follow to the trails of old time voyagers, so learners ueed not repeal all the learning processes that were once required.
Most of these processes can be shortened or even reduced to merely accepting the
bare product, and teaching is a way of organising this reorganisation The big
problem is to distinguish the products which admit the strategy of transfer from
those for which the strategy of what I called guided reinvention is indispensable.
To what degree may estrangement be either useful or harmful?
All one needs to know in order to use a computer is its exterior reactions to one's
acts. This is astonishingly simple, although the style -- either formal or childish -
- of the manuals conjures up an image of complexity, which only betrays bad didactics. Even oral explanations may be counterproductive if supplied by experts
who lack the ability to place themselves in the shoes of the novice. Until further
notice the fasfest manner to learn to use computers is to use them and to ask for
help only if some problem seems to demand too much of one's solving ability.
Although this is an unsatisfactory and certainly not paradigmatic example of
guided reinvention, it should not be surprising that teaching how to use computers has not kept pace with their rapid evolution.
Other teaching subjects can boast a longer history. But history is not yet done; it
continues as times goes on. The learning paths leading from here to there form an
intricate and never completed labyrinth and each fresh learning process, be it individual or collective, asks for new teaching processes. Even assembly lines, as
useful as they may be, require modernisation, which can be a costly venture.
Teaching/learning processes, observed and reinforced, entangled and isolated,
necessarily require and fortunately allow more flexibility. To be sure, as a whole
they must be organised carefully, but the care needed does include caring for
flexibility. Interactivity means that both teachers and learners are agents as well
as being acted on. Taught and learned should coincide, or, even better, compete
with each other on equal terms. These are high demands, and even higher if learning is to mean long-term learning.
3.3.6 Change
"Taught and learned" is just one facet of the familiar antonym 'Theory versus
Practice" as I recalled in (3.1.l, 2). Let me now turn to Research as a source of
Practice! My answer in (3.2.3, 1) to the question "what is the use of it?" was:

Change.
Change, in particular under the name of "innovation", is tacitly understood to involve improvement Rapid changes are known as revolutions, yet revolution need
not imply innovation. Anyway the innovatory speed depends on the circumstances. This is even true in industry. There it lakes time to sell innovation to their directors, to develop new tools, to buy new machinery, to insiruct workers how to
use them, and to advertise the new products. Even if the time needed is considerable it is nothing when compared to that needed in education. (An estimate I just
heard was I to 10.) The reason why, is the greater amount and complexity offeedback required in education, and the lack of past experience.
Innovation, when thoroughly planned, affects each level in the Educational Landscape, a big hierarchy extending from first line instruction to research, where the
taught-learned relation is the glue that connects subsequent levels to one another
-- a large number of feedback spots here, each of which is of a high complexity.
Curriculum theory as developed in the sixties tried to account for this structure,
but in formal educational innovation projects the amount and complexity of required feedback are easily underestimated, which has happened in the past and
which still happens. This, then, is the reason why most of them failed in the past.
The few that somehow succeeded, did so because all levels of what I called the
Landscape were involved in the process of change; they underestimated, however, the time required for teaching/learning at each particular stage of dissemination. In fact, innovation itself is a learning process for the Landscape as a whole
and for its particular agents, some of whom may be subject matter experts who
nevertheless lack the expertise to recall their own learning processes and to analyse them in order to know what has to be transmitted to learners and how it
should be done. I f innovation is a learning process, it is quite a singular one, rather than one of guided reinvention. As far as it may be guided, it is so internally,
which requires a high measure of flexibility.
3.3.7 The agents of change
3.3.7.1 The educational developer as an organiser
Leaming is time-consuming, indeed. It depends on the learner and the learning
matter how much time is required, that is, how much estrangement can be allowed if learning processes are to be shortened. Strangely enough, as history has
shown us, this is a bigger problem in the service regions of the educational landscape than in the first line. Disregard for this fact is, in my opinion, the actual reason why time estimates at the various stages of the process of change have been
so far off the mark.
Though under another aspect, it was already discussed in (3.2.3, 6) how the "ex-

the researcher-- to "prove" the validity of proposed instruction to the satisfaction
of those who are expected to get acquainted with the proposal. In projects of developmental research the researcher need not rely on mere thought-experiments.
As such a project proceeds, thought-experiments are gradually superseded by series of ever broader factual ones, only to cede their indispensable role to farther
reaching and more profound new ones. At any stage, the results of the preceding
stage, rather than being imposed as products on the participants, need to be transmitted as inventions, in order to enable participants to reinvent them.
After (3.2.3), where gradual dissemination was to serve the progress of developmenial research, the tables were turned: research now serves dissemination. Even
so, dissemination is not understood here as something that happens once, after research and development have been completed. It is a gradual and, in a way, openended process. From the start onwards, dissemination is part of any developmental research: at firrst, this lakes place within the restricted company of the initial
actualisers who perform thought-experiments behind the desk. In the developmental process they compete with the professional developers in the activity of
adapting and modifying preconceived ideas. As dissemination begins to affect
ever extending circles of participants, developmental research includes keeping
a grip on wh-tjs occurring at an ever more distant periphery. Yet it not a one-way
influence. Signals from the broadening circle are fed back into the developmental
process -- this aspect of dissemination has already been brought to the fore in the
preceding subsection.
With regard to practice I made a distinction between descriptive and normative
aspects. Since educational development as such is not a widespread conception,
it is no wonder that norm by far outstriped description in the preceding exposition. At closer glance and regarding details quite a few realistic aspects may be
recovered as a description of what at present is subconsciously and unintentionally being carried out as educational development Yet let us now turn to other
agents in the process of change!
3 .3 .7.2 The teacher
In the Educational Landscape the teacher is the least specialised actor (not counting, of course, the frrst line learner), and the less specialised the teacher the lower
the age bracket of the instruction in question. Less specialised is likely to imply
less self-conscious of one's role, and for that matter, maybe less conceited. Was
the apparent lack of specialisation the reason why curriculum developers, in particular those of New Math, trusted the teacher to be able to implement brand new
subject matter in the classroom? On a tabula rasa one can write all one wants,
and this is what they did in the retraining courses with the subject matter to be
taught in the classroom -- somewhat broadened or seen from a higher viewpoint
When this strategy failed, some developers conceived the idea of"teacher-proof
curriculum". My dictionaries are not up-to-date enough to include items like 172 CHAPTER3
"teacher-proof' but, looking for related items, I feel inclined to believe that it
means stuff that even a teacher cannot spoil. Other developers realised that a
teacher was no more a tabula rasa than anybody else and that the deeper the imprints, the more likely they were to withstand attempts to change or erase them.
In the cognitive domain, background philosophy (as I called it in (3.1.3)) is the
totality of the deepest imprints. As far as mathematics is concerned, there can be
little doubt that such aspects as mathematising are lacking in pictures of mathematics which traditional instruction created in most people's and even in most
teachers' minds. The mechanistic style (3.1.5-3.1.6) of the mathematics they
were taught pervades the mathematics they learned. If they chose to become
teachers, they will continue teaching the mathematics they were taught, in spite
of new subjects unless emphasis in retraining is shifted from subjects to attitudes.
Most attitudes are acquired unconsciously. The easiest and most efficient way to
change them is actively participating in change this need not remain a privilege
of that group participating in projects of educational development which I characterised as the initial actualisers of thought-experiments. Participation can be
transmitted. At every stage of dissemination teachers are guided to reinvent primordial changes, aJbeit modified by the experiences at intermediate stages. AJthough unconsciously acquired attitudes are reinforced by reflection and knowledge of them may be helpful for transmission, the true thing to be transmitted in
the course of dissemination -- if it is to be efficient -- is the behaviour itself rather
than knowledge of it
Guided disse-inatlon aims as far as the ultimate frontiers. At every stage it is endangered by its own success. I f stopped too early, the campaign can be lost and
the instructional system be worse off than before. What does "too early" mean?
In medical terms: if the mechanistic view on mathematics is a disease then it
matters to judge how many and how large areas of reinfection may be I -ft with
impunity. But what about prevention?
3.3.7.3 The teacher student
Malhematics teacher training is a newcomer in the Landscape of Mathematics
Education. At any rate, this is true in my country. If there are other countries
which can boast a longer tradition of mathematics teacher training, I am not sure
what its impact has been on first line mathematics education. In most countries,
teachers were educated either as educational generalists in order to function in
primary education, or as subject matter specialists with a professional future in
secondary education. The first were educated at teacher training institutions, the
second at universities, although in my country extra-universitary examinations
kept a continuous stream flowing from the first to the second pool of teachers. As
a consequence, in secondary level mathematics, the majority, rather than being

However, the number of those who, prior lo the extra-universitary examinations,
had not had any training as a teacher was steadily increasing. Not until a few
years after World War II was some teacher training made compulsory for future
secondary school teachers, but, even so, for years this was thwarted and kept

down by universitary people in charge of the various disciplines, who jealously
guarded the scientific status of their students -- indeed, in this respect didacticians
had still to qualify themselves. At the present state of teacher training for primary
education, mathematics is integrated with its didactics; teacher training for lower
secondary education has mathematics and its didactics taught simultaneously and
by the same persons, while, for the upper secondary level, students are trained by
didacticians after the accomplishment of their study of mathematics, which is
still an unsatisfactory state of affairs.
I have re-counted the above in order was to draw attention to two basic issues regarding the relation between the two components of teacher training -- the scientific content and the didactical form. These issues are
equality of status,
the measure of integration that respects the level of both components.
Only where the mutual respect of scientific and didactic trainers and their cooperation can be taken for granted, can students be expected to take both components equally seriously and lo integrate them by themselves.
In [115], when dealing with "Major Problems of Mathematics Education", I
made the parallelism between mathematising and didactising explicit by drawing
a parallel between each of the problems in first line mathematics instruction and
its counterpart in teacher training, between each learning problem and a teaching
problem, which in turn shows up as a learning problem for the teacher student
So I started with drawing a parallel between
Why can Jennifer not do arithmetic? and
• Why can Jennifer not teach arithmetic?, only to continue via the pair
reflecting on one's own learning with its counterpart
reflecting on one's own teaching and, among many others,
developing a mathematical attitude with
• developing a didactical attitude to
• How can calculators and computers be used to arouse and increase mathematical understanding? paired with
How can calculators and computers be used to arouse and increase didactical
understanding?
To explain why I changed the paradigmatical "Johnny" into Jennifer, let me
quote myself from [115]:
Why can Johnny not do arithmetic?
Does this sound sexist? I will not change it into
Why can Mary not do arithmetic?,
lest it may sound even more sexist, suggesting that girls are less able than boys. As a
matter of fact both are wrong. My problem is not John Roe and Mary Doe. The prob-174 CHAPTER3
lem is, indeed, why many children do not learn arithmetic as they should, and it is a
major one because, more than anything else, failure in arithmetic may mean failure at
school and in life. My concern, however, is not, or not primarily, what is wrong in
classrooms and textbooks today that creates a host of underachievers.
Let me change the question! I now ask:
Why can Jennifer not do arithmetic?
Rather than an abstraction like John and Mary, Jeonifer is a living child (though I have
changed her name) whom I can describe in every detail. The two details that matter
here are that she was eight years old and could not do arithmetic. Meanwhile the question
Why can Jennifer not do arithmetic?
is not a question any more, because today Jennifer is eleven and excels in arithmetic.
Yet when she was eight, somebody who was observing her stumble with numbers succeeded in answering the question and after ten minutes of remedial teaching, Jennifer's problem had ceased to exist. Was it a miracle? Not at all. It was just an easy case.
But there are so many such cases. Noticed and unnoticed, cared for and uncared for.
But what about the less easy cases? They have grown out of those easy ones that remained unnoticed and uncared for.
Indeed, the parallelism between learning mathematics and learning lo teach
mathematics is the leading principle of what I have to say about mathematics
teacher training, which in practice is acted out by the parallel between mathematising and didactising. Still, in accordance with the tabula rasa metaphor, the second kind of learning is more difficult than the first Teacher students, in general,
belong to the large group of adults where the sources of what they once learned
by insight -- be it much or little -- are likely to have been clogged by the knowledge and skills they acquired in the meantime. To say it more concretely: they
neither care why multiplication by 100 is carried out by "adding two zeroes" nor
about the fact that you can argue about such a piece of knowledge, or why you
should do so. So they have to undergo remediation: thy must relearn such facts
while teaching children and observing their learning processes. The higher the
level of learning, the more paradoxical this conclusion may sound. I have often
argued that knowing a piece of mathematics too well may seriously impede one's
ability to teach it decently; as a teacher one should be so conscious of the learning
process that produced one's own excellence that one can reinvent it So the teacher student needs to relearn by observing the learning processes of less skilled people -- in fact, of children. But if this is so important, we are faced with one of the
big problems in teacher training. Although one can easily arrange for observing
short-term learning processes in the school environment, it is impractable and
hence impossible to do the like for long-term learning processes. Thought-experiments, as undertaken by textbook authors, cannot fill the gap if undertaken by
unskilled people. Lack of experience in long-term learning processes is the actual
cause for the depending of young and even older teachers on textbooks as their
dilly sources of Innowledge on 1011+-tenn learning processes. Inservke guidmce

and getting them to re-create.
Anyway, there is good reason to shift the centre of gravity of teacher training
from theory to practice, from the training institution to the classroom. From personal experience I know about such efforts in my own country. All I know about
them elsewhere stems from written sources. As far as such sources are theory, I
am not satisfied. I am afraid that the so-called classroom-based teacher training
is too theory-laden to be helpful. Yet, is it still a topic?
I believe that teaching should be learned in the classroom. This is more than a belief; I think it is a fact that almost all of what practising teachers have learned in
their lives about teaching and about how to teach was learned in everyday classroom practice.Teaching is learned by teaching -- one could claim -- like walking
is learned by walking. But this simple truth is an outright falsehood. It cannot be
denied that walking is learned by walking, but as soon as "walking" is replaced
by "swimming", it is clear that things are not that simple. To learn to swim one
needs aids, material and personal. Learning lo speak presupposes listening, and
learning to speak better requires personal and material interventions lo improve
speech; for that matter, even "walking learned by walking" does not apply to
handicapped persons, who, without help, would never learn to walk as we would
like them to do. But when I claimed teaching to be learned by teaching I meant-
- and people who if!ree with me likewise mean -- that actual teaching is an indispensable component in the process of learning to teach.
Not so long ago teaching, certainly at the secondary level, was learned by imita-
tion, and by trial and error. Imitation was here a crude form of observation, that
is, one repeats one's own learning processes, which were recorded from one's
own school-life, and supported by the textbooks used in the classroom. The trial
and error provided as much feedback as the guinea pigs in the classroom could
offer and the teacher was able to process.
No doubt passive observation and active trial and error are still the essential features of learning lo teach, just as they are of any kind of learning not sufficiently
supported by general theory. The one thing that has changed and should change
even more in order to further improvement is the organisation of both observation
and trial and error.
A forceful means of doing this is to observe learning processes, which I dealt with
in a larger context (including teacher training) in such detail in (2.3.i) that I need
not repeat it here.
I think that all would agree that a trainee should get in touch with the practice that
is to become his future practice, at the earliest opportunity. Training institutions
diverge in answering the question: how early is "early". After three years, four
years? Well, this may be correct if it is required by the quantity of the new subject
matter the student has to learn in order to teach. Primary teacher training is certainly not such a case.
On the other hand, being confronted with the practice (someone else's or one's
own) without any preparation may be counter!.Iroductive. Even trial and error pre-
suppose some knowledge about what is to be tried. But if the aim is to observe 176

learning processes, and mathematical ones, in particular, one has the opportunity
to start within the walls of the training institution, or even at home -- I mean by
observing one's own learning processes: Take a little mathematical problem,
solve it, tell me how you did it, and think about what you have learned while doing it! Isn't this a marvellous idea?
As early as (2.3.1) I answered this question. When applied to teacher training, we
may conclude that, even before being introduced to practice, the trainee has and
should have opportunities in the classroom of the training institution to observe
and to analyse learning situations. By this, however, I do not mean what in my
country is called institution practice: one of the students playing the teacher and
the others simulating being the pupils who are being taught a certain subject in a
similar way to what would happen in the genuine classroom. This is too artificial
an approach to be successful, and anyway it is not what I mean.
To make clear what I do mean, let me tell a little anecdote, a story from a lesson
in a teacher training institution I was observing. While solving a mathematical
problem at the blackboard, a student had committed a grave error but neither the
trainer nor the other students had intervened to correct it. The student went on and
on, and the trainer let her proceed as she did up to the unhappy end when she ran
into a flat contradiction, which proved that all she had done was wrong and that
the entire work would have to be done over again. Finally, when all this was finished, I intervened to pose the question of whether somebody had noticed any re-
markable feature. Nobody had. My guess is that even the trainer hadn't noticed.
Anyway, he hadn't asked this question -- I had. Even after I had specified my
question a bit and added hints, nobody could answer it. Finally, I myself had to
disclose the answer, that is, to draw their attention, and maybe even the trainer's,
to the fact that the trainer had not intervened to correct the student's mistake.
Hadn't they noticed that mistake as soon as it was made? Did they believe that
the trainer hadn't noticed it? For what reason could he have passed over that mis-
take? Was it a good reason? How should one act in such a situation? Nobody had
asked themselves such questions. So I simply had to explain the trainer's didactical trick and to reveal his intention of letting the student proceed as she did until
she ran into trouble.
What does this story mean? Observing learning processes is indispensable to
learning to teach, and it should start at the earliest opportunity: observing one's
own learning processes and those of one's learning group. But in order to beef-
ficient, observing as horizontal didactising has to be complemented vertically.
3.3.7.4 The textbook author
When I discussed change and its agents, I did so from the high viewpoint of Developmental Research as presented in (3.2.3). How far can it be approached by
Practice? Formal projects is the closest approach; the question of how close, is

possible. Projects that are designed too narrowly, in particular with respect to the
size of the instructional landscape involved (as most of them were in the past)
may be useless if not harmful.
So far, in practice, the most efficient agents of change were textbooks, and by the
look of it, they will maintain this status for the time being. So I may narrow my
former question by asking: "How far can textbooks approach the view on change
taken by developmental research!"
Mathematical textbooks -- including similar teaching aids -- have changed in the
last few decades. To conclude from reviews and the ways of reviewing, however,
they have changed less fundamentally than one would have liked in the wake of
their remarkable predecessors dating from at least half a century ago. I admit that
my first-hand knowledge of the present textbook literature is quite restricted, but
among the few textbooks I do know, I could signal some that have done the utmost to take the desirable view on change, or so I would guess. This is indeed
guess work. I never reviewed textbooks and with all respect to the host of reviewers, I wonder how much teaching experience is required to undertake the task of
reviewing textbooks before or without having used them. My reluctance stems
from confrontations at a semi-teaching level with a few of them: when asked for
help by children who had to work with these books, I suddenly saw, as it were,
through the lellfl!ers' eyes, how bad these, at frrst sight good-looking textbooks
were: they were "shaky from questions onwards I myself was not able to understand, to didactical tricks that backfired, up to an ill-conceived build-up of the
teaching matter as a whole.
I agree that what I called my "semi-teaching" experience is hardly representative.
Whether one likes it or not, textbooks are merchandise, and in the marketplace
good quality is what appeals to the needs and tastes of prospective customers. In
the present case, said customers are teachers who need these books as teaching
aids and who prefer to use what they like. This would seem to be a gloomy perspective for change and for the textbook author as an agent of change, were it not
that needs can be stimulated and tastes can be educated.
The traditional way to try this is teacher manuals. As far as they supply additional
information and disclose richer sources, I do not object to them, but as didactical
aids I do not trust them much, because of the serious doubts I have about their
factual use. In [87, p.119-121] I cited an investigation into the use ofteacher manuals, which included four mathematical textbooks. Only one quarter of the teachers questioned claimed to have used them. This does not look like much, but even
it is an over-estimate, since none of the four textbooks could boast a teacher's
manual. Not that this prevented the 25% self-styled users from answering the
most detailed questions about their ways of using them.
I prefer a textbook style where both learning and teaching at all levels of the
teaching/learning process, as well as local and global reflection on it, are implicit
in the text itself. For instance, the task of posing meta-questions should be shouldered by the textbook author, rather than leaving it to the teacher. This is at the
same time the most effective way to implicitly influence the teacher's attitude, 178 CHAPTER3
which incl-des asking ."ore of these questions wherever they are needed. By
meta-quesuons I mean, m the present didactical context, questions that start like
why do you think
and continue like
did I claim that . . .
did I call this . . .
did I supply you with this . . .
did I ask you . . .
did I include the condition . . .
did I not add this conclusion . . .
did I define this by . . . rather than by . . .
did I deal with this in this order rather than . . . ,
did I prescribe this task for a group of four
or start like
what do you think
and continue like
did you learn in . . .
did you unlearn in . . .
is the use of this . . .
should be added
could be missed in !his context?
For short, I mean didactising questions as the textbook author has asked himself
as an educational developer and implicitly answered by writing the textbook in
the w-y he eventually did. Framing these questions to fit the layout of the textbook is the way to have users reinvent !he educational development !hat produced
the textbook as though they had participated themselves in its production process; and by users I mean both teachers and learners, each on the level that is suitable to that group.
I started discussing the textbook author as an agent of change with a reference to
(3.2.3). From the above it has become clear that I would like him to bear witness
to his -ctiv.it7 as developm_en- researcher (whether he did so explicitly or not)
m !he !mphc!t way of !mpnnung the process into its product, thereby leaving the

users _as much freedom as they deserve, among other things by means of metaquesuons that suggest alternatives.
Would this strategy be understood and its tactics put to good use in the class-
-oom? I_t all depends on the existing infra-structure of teacher training and retrainmg, which should be accounted for as boundary conditions in any educational development whatever, and certainly in textbook production -- this is all part of the
Landscape I have been trying to deal with.
3.3.8 Mathematics for all
mE LANDSCAPE OF MATHEMATICS EDUCATION
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has been the !theme (or one of the themes) of innumerable meetings and conferences -- national and international -- and it has not ceased to be the subject of research and development. In discussions on general education, numeracy has become the companion of literacy. Indeed, as badly as literacy is needed in a world
full of written and printed matter, so indispensable does numeracy seem to be in
a world where number emerges at every turn to be used and -- maybe even more
often -- to be misused; a world which looks like it is ruled by number. But let me
add that, just as literacy is no more than one aspect of language, so is numeracy
one aspect of Mathematics for All -- which certainly does not cover all of it. Yet
as much as I have refrained from discussing subject matter in the present books
so little will I deal with the contents of Mathematics for All. Anyway, it will not
be the same thing for each particular learner; there is a great deal of diversity not
only of contents, but also or even more so, of breadth and depth of understanding.
As a matter of fact, this may already have been asserted with respect to numeracy,
and if you wish to add it, for literacy, as well.
Literacy and numeracy are spoken of as companions, but is this really a company
of equals? There is a habit of distinguishing what is called tee hnical reading from
reading comprehension. I f we do so with technical numeracy and number comprehension, and consider how closely reading comprehension is knitted to life
and reality, shouldn't we ask for numeracy to have an equally strong grip on
number in the real world? Or are literacy and numeracy different in principle?
Literacy is a relation between two expressions of language -- spoken and written
(or printed) -- or so it would seem, whereas numeracy is having the competence
to relate a manifold of manifestations of number to each other, which is harder to
learn; and this might be an argument to be satisfied with less numeracy than what
is required as literacy.
In fact, neither literacy nor numeracy are uniquely defined. Being illiterate can
mean many things, depending on the environment and circumstances; so what is
meant to be understood by literacy heavily depends on individuals and groups,
communicating with each other in all kind of situations. Numeracy is hardly any
different.
But what about mathematics for all? The answer to this question, even if it depends on the group in question, is not given by the mere intersection of all the
particular mathematics that concern !he individual members of the group, whether as subject matter or as competence; and this again is a property it shares with
literacy.
A most striking feature of language is the abyss between passive and active mastery. Each of us, whether illiterate or literately gifted, understands tremendously
more language than he is speaking or writing, and than he would ever speak or
write, or be able to speak or write, and this difference extends from mere vocabulary to grammar and style, and certainly to the mastery of foreign languages.
Nowhere in our mental and instrumental structure does passive mastery exceed
its active counterpart to such a degree as in language, and mathematics is no exception to this rule. No wonder -- one would say -- for language is the vehicle of 180 CHAPTER3
communication, and communicating vessels, whether wide or narrow, are in
some respect each other's equals.
Under the aspect of communication I shall look at Mathematics for All. Mathematics can boast a language, so specific that it looks like mathematics itself is a
language (which is often, though wrongly, identified with mathematics as such).
Language is mainly learned in the course of strongly interactive communication-
even reading, thanks to contents, is more interactive than it looks at first sight '
This is the reason (or one of the reasons) why I have pleaded for learning in small
groups of say, four learners and, for that matter, in groups of mixed ability [B7
p.60-63]. Relieve me of the task of repeating my arguments! In fact, these argu'.
ments are the direct consequences of my view on learning processes: they are
steered by reflection on one's own mental activities, which is stimulated by ob-
serving oneself by means of observing others, and reinforced by the distant levels
of the participants in the learning process.
Cooperative learning foreshadows cooperation in adult life and professions. In
this perspective Mathematics for All means: as much and as good active mathe-
matics as is needed to participate in even more and even better passive mathematics. I asked:
We teach classes, so why shouldn't we expect classes to learn? Life is cooperation.
-y do examinations focus on individual performances and entirely disregard collec-
uve ones? But let us delay answering this question!
The answer is, as you may expect: They should not This is a verbal answer, in-
deed. It will take much time and trouble before these words become deeds. A first
step on this long path is to formulate, besides individual objectives, cooperative
ones of Mathematics for All, and to describe specific tasks to be performed in cooperation with others.
EPILOGUE
Did I do justice to what I promised and explained in "Apology and Explication"?
I myself was astonished how little revisitation could add to the yield of former
visits except reorganisation of the contents.
Publishing "Lectures", which are no lectures at all is an old and venerable habit,
but I did not realise early enough that nowadays "Lectures" are even less suited
to reflect lectures. Indeed, lectures on subjects like those dealt with in this book,
are centred around a rich variety of overhead sheets, and elaborated on by informal talk that provides for the indispensable concreteness. Sometimes I have tried
surrogates but I could not continue this way indefinitely. In particular, I had to
refrain from dealing extensively with teaching matter. -
The reader might be disappointed that I did not tackle computer education, in-
eluding informatics afit is called on the continent. In the perspective of the
present book this does not distinguish itself from other mathematical subject mat·
ter. And the computer as a teacher is still a far cry away, or so it seems to me.
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1'
' I APPENDIX
Solution of the problem 1.3.3 , 15 (fig. 3)
1. The first step to deal with such problems is distinguishing states, and operations moving states into each other.
2. Under the conditions of the problem some states may be indistinguishable.
This is the case for states that arise from each other by rotations of the barrel
and by interchanging white and black. This remark also holds for the operations.
3. As a consequence one considers classes of states and classes of operations.
Two states (operations) belong to the same class if they arise from each other
by rotations or colour change.
4. This leaves us with four classes of states and three classes of operations, each
defined by a representative:
States Members in the class
v
• • O=o o 2
• 0
S=oo
8
• • P=oo
4
D=•o 2
0.
Operations Members in the class - means inverting the .di,sk,
+ means leavng 1t as it u
s=
- -
++
p=
++
d=
- -
- -
5. Although it is not strictly needed, one can draw up a table recording the possible results of operations applied to states:
183 184
D p
s
d 0 p
s
p p D,O s
s s s P,D,O
6. The system shall be moved into state 0. If it is state D, the operation d does
the job. So we first apply d. If it happens to be in state P, it then moves into
state D or 0. So if is not yet in state 0, we can move it into state Dor 0, and
if need be, by another d into 0. Anyway if the system was in state Dor P, it
moves into 0 by means of dpd (or part of it).
7. If, however, it is in state S, thens moves it into state P, D, or 0. So if we now
apply s, and if need be, another dpd (or part of ii), we are finished.
8. The solution is
dpdsdpd (or part of it).
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:
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1
=dl= l,
n= n n n n n
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16. There is no authentic formulation. See J.L.Kaput & l.Clement, 'Letter to the
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24. To my surprise I noticed that in France teaching arithmetic in the first grade
is restricted to addition, albeit up to 80.
25. An idea emphasised and a term coined by L.Streefland. Cp.'.
26. The idea of learning negative numbers in this intertwined way seems to be so
exotic that, after Van Hiele and my own experiments [146,154), it has not
found any followers. It was even overlooked by otherwise careful reviewers
of [146).
27. Parts of this lecture were edited in [87), which can also be consulted for details that are here omitted.
28. See also [166).
29. Unfortunately I lost the evidence.
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PREFACE

In Dutch "WISKOBAS" stands for a particular kind of mathematics in the elementary school (ages 6-12). In turn Wiskobas was one of the departments in the IOWO, the Institute for the Development of Mathematics Education. This institute was concerned with the development of material for mathematics education as well as the related research on the possibility of change from the then existing arithmetic instruction to the future mathematics education. The present publication Three Dimensions has three aims: to give a picture of the goals Wiskobas set for future mathematics education, at the same time to show how such goals can be described, and to show the theoretical framework of the Wiskobas curriculum.

The problem at hand is not at all simple. What is more, Wiskobas' ideas about mathematics education cannot literally be translated into strings of words. So how can we face the accusation that our objectives are unattainable and the goal itself irrational? In order to avoid this vagueness as much as possible and for-the sake of clarity, this book makes continuous use of illustrations of mathematics education. In these examples both the subject-matter and the methods of description of the goals are illustrated as explicitly as possible, while at the same time creating the opportunity to read between the lines. The reader is urged to follow carefully the mathematical material at the start of each chapter. This advice applies both to the more general education oriented, and to the more mathematical/ didactical reader.

In the completion of this task I received assistance from Jelle Sixma, Hans Freudenthal and the former Wiskobas team of the IOWO (now the OW & OC at the State University of Utrecht). A special word of thanks is due to Rob de Jong, Sylvia Pieters, Betty Dekker and Els Feijs, and to Hetty and Guus Vonk, Hans Freudenthal, Alan Bishop and the late Arthur Morley for their help in translating the Dutch version into English.

Chapters 1-V are translated from the original version. Chapter VI is new and was written in 1983. It contains an example of the goal description of a course, as opposed to the examples of goal descriptions in the other chapters which have a bearing on themes. Chapter VII was appended in 1986 with the idea that the three-dimensional goal description propagated here only becomes transparent when a clear view is provided for the framework of instruction theory. In our case, this concerns mathematics education as realised by the Wiskobas project. This theoretical framework is also described in three dimensions, which

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CHAPTER VII

Regarding or disregarding certain context factors, which is the essence of modeling the problem, is in fact a non-negligible part of the process of mathematising. Of course, this does not mean that each problem should be heavily contextually loaded or fitted into a larger thematic context. Sometimes it would even be undesirable, in particular if an elementary problem situation were to be assigned a model function.

A context problem is not merely a dressed-up word problem, with its application subordinated to mathematics. Alongside the logical connection a phenomenological one by which mathematics becomes ancillary to the

real problem is possible. In this case reality counts first, and mathematics
is a mere derivative. Any context problem, whatever it may be, includes a
(subjective) variable aspect which can elicit a realistic reconsideration or
motivation as soon as the answer is translated back to reality.
Context problems fulfil several functions, to wit:
- concept formation: at the onset of the course they offer the pupils
natural and motivating access to mathematics;
- model formation: they support the learning of formal operations,
procedures, notations, rules, together with providing other material
and visual models which fulfil an important support function for
thinking;
- applicability: they uncover reality as source and domain of application;
- training specific arithmetic abilities in applied situations.
Take, for instance, division: an elementary division problem can be the
start for learning long division. Conversely, it can be chosen as a concrete
base for a bare division problem or for understanding a specific property
of division. In both cases the context conveys significance in such a way
that the procedure and the operation will in the long run become
meaningful within the formal system. Moreover, such a division problem
can function as an application of tables learned previously or of long
division procedures, as well as an exercise in the applied situation of
money arithmetic. In brief, context problems function both horizontally
and vertically. On the one hand they serve to make mathematical knowledge and abilities applicable, and on the other hand they impart meaning
to formal operations. The urgent need for the latter function is witnessed
by the test scores of two division problems taken from a National Assessment research (NAEP; 1980):
- 6 /608 . . . was correctly answered by 69% and 65% of the 13 and
17 years olds, respectively;
- the problem 'Zé children, four per car; how many cars are needed?'
was correctly answered by no more than 3% of the nine years olds

(handheld calculators were allowed); 12% chose for the answer 6.5
and 7% for 65 cars.
20
Whitney's comment on the result:
These examples furnish further evidence that, right through the elementary schoolyears, the
attitude that 'schoolmath is something for itself, not for life outside school' is becoming
entrenched.
21
Children evidently do not always imagine a situation in connection with
the bare arithmetic problem (608 7 6) as, for instance, that of distributing
6 dollars and 8 cents among six persons. Conversely, m a realistic
transport situation they do not know what to do with a 'bare' answer (6.5).
In fact, a vast phenomenological exploration is not a luxury one_ can
dispense with but, on the contrary, is a sheer necessity. This necessrt.r. is
shunned by the formal, seemingly logical approach, devoid of any meamng
and leading to a kind of mathematics detached from and isolated m
reality. The phenomenological approach, on the contrary, does lustice to
all phenomena in which the mathematical concepts and structures under
consideration reveal themselves.
It would take us too far afield to sum up how-many-sided embedding in
the spirit of Freudenthal's didactical phenomenology of mathematical
structures takes pface in each particular theme and course of those we
displayed above and what, among others, Sawyer, Vergnaud, Hilton,
Winter have contributed to it. We only mention the broad embeddmg of:
- ratio with a reference to 'Gulliver' (Chapter I) and to the description
in the Mathematical Material for the present chapter: geometric
aspects, drawings, puzzles, stick-shadow, mixtures, densities, scale,
percentages, ...
- combinatoric problems (Chapter II): roads, tree diagrams, clothmg
combinations colouring, lattice, probability, . . . ;
- division (Ch-pter VI); extensively discussed in the preceding and
present chapter; .
- fractions described extensively in the present chapter: breakmg,
dividing,' distributing fairly, measuring, ratios, probability;
- many more examples of a many-sided approach in the other courses
sketched in the Mathematical Material for this chapter.
However rather than restricting our attention to the form, contents and
functions' of the particular context problems belonging to a certain teaching subject, we shall also consider them in their totality. _An important
criterion is completeness, in other words, whether the vanety of mathematical phenomena in the real world is sufficiently expressed .. In particular, problem situations should be viewed which, thanks to then mherent 50 CHAPTER II
The number of routes from a to bis 15, b to c is 3. The total number of
routes from a to c is 15 X 3 = 45. The difficulty with this problem lies in
the modeling, which means, in this case, taking goals as being "moves".
The problem is then replaced by an isomorphic problem, for which the
solution is partially known, in so far as it concerns the separate consideration of the goals in each half of the game.
Then the product rule from section 1.2 must be 'seized' with the
network. All in all this is quite a job. The mathematicians among the
readers have a different model at their disposal. This is a model from
probability, which gives: "Before half-time six goals were scored, two for
Heereveen, the possibilities for the score progression are (I) = 15: after
half-time GJ = 3: the total number (I). (I) = 45.
But even this reasoning is not as automatic as it looks. And why not?
Because, in any case, It must be discovered that the progress of the score
before half-time (4-2) - as far as the possibilities for the sequence are
concerned - is similar to the manner in which two marbles of the same
size can be placed in a row of six holes. And . . . that there is a formula to
fit this: (j), to be pronounced as "two out of six", meaning: (6 X 5)/
(1 x 2) = 15.
Another question is how to find such a formula. It is a fact that
knowing a formula does not imply its use in situations in which it can be
applied.
1.10 Families
Acting on the assumption that the chance per family of a boy being born is
as great as the chance for a girl, indicate the boy-girl situations that can be
expected in a sample of 160,000 families with four children. For example:
20, 000 families with four boys, 30, 000 families with three boys and one girl,
etc. Explain the answer!
The road highway network can also be used here. The boy/girl ratio
can be seen as the end of one route on the network:

Each of the 160,000 families is asked to follow one of the routes. The
route cannot be read from the destination - with the exceptI0n of the
extremes. i i 'th th e boys
For example, there are four possible routes for a fam y Wl. re .
and one girl (the first, second, third or fourth child can be a gu:. therefore.
four possbilities). .
The total distribution IS:
- one possible route for four boys; . .
- four possible routes for three boys and on.e glfl,
- six possible routes for two boys and two gu!s; .
- four possible routes for one boy and three girls,
_ one possible route for four girls.
_ Total: 16 possible routes.
From this it follows that:
- There is a chance of 1/16 for four boys; .
- there is chance of 4/16 for three boys and one glfl;
- there is a chance of 6/16 for two boys and two g1rls;
- there is a chance of 4/16 for one boy and three girls;
_ there is a chance of 1/16 for four girls;
To inte ret this calculation we have used the law .of large numbers.
- s that for this large number, the relative deV)allon will
;::::b;-- b/small. The term "approximately" takes care of the matter of
b blilt .
pr; :t w: have been somewhat too hasty and abstract in the explanallon
we have given. First of all we should have asked ourselves whether th-
havin four boys is the same or smaller than the chance o
ch' : e t-o bo s gand two girls. In general the first reaction is: "the chance
hav g. f Yb . maller but I don't know why, I just have the feeling
of having our oys s s ' . fy the chances. To
it is " That "feeling" is a good opportumty to quanll
1
'f
- b · -th coins and ask ourse ves i
simplify matters somewhat we can egm Wl f .
the chance of getting head-head is smaller than the chance o g-ttmg
head-tails. Analogously, is the chance of boy-boy :-alk::i:-:-!-o-o':--
of boy-girl in a family of two children? Then - · - , s is
in a course of study. . f link d -ences
The foundation should be laid in a varied senes o . . e expen 7 '
wbich will lead to familiarity with the concepts of probability and chance.
1.11 Mathematising
Mathematising is an organising activity. It refers to the essence ?f the mathematical activity, to the thread that runs through all mathemal!cs educal!o-
directed towards the acquisition of factual knowledge, the learrung o
i

concepts, the_ attainment of skills and the use of language and other
orgamsmg skills m solving problems that are, or are not, laced in a
mathematical context. In the ten counting problems organisingpoccur db
means of: ' e Y
- noticing similarities and differences (see 1
- l);
- bemg systematic (see l. l and following)'
- symbolising (see 1.2 and following); '
- generalhsng certain solutions (see 1.2 and follo .)-
form l f h wmg '
- u a mg t ese generalisations (see 1.2 and following)-
- formulatmg . . . and Vl - s u al" rnmg - one pro b !em in various ways ' (see 1..3)-
- precisely definmg concepts (see 1.4); '
- usmg different models (see 1.4);
= applymg_ a discovered rule in non-obvious cases (see 1.5);
schema!smg mcreasmgly a solution model (in this case: the network), which produces an algorithm-
= recognising the isomorphism of two -roblems (see 1.
7
);
- r-asonmg on the ba.s1s of considerations of symmetry (see 1.7);
discovermg. regularities through induction (see 1.8)-
-u thnderstandmg and explaining the underlying stn:cture of a pattern-
e proof (see 1.8); '
- proving the solution by means of the principle of mathematical
mduclion (see 1.8);
- explaining a relational link that is paradigmatic for all othe r nk
that the whole pattern is explained (see 1.8); r 1 s, so
- transfm:mmg and adapting a problem into a familiar model-
- combmmg a number of solutions, strategies and rules; '
- usmg formulae (substitution) (see 1.9);
- findmg a relat10nship and expressing it in a formula (see 1.9);
- 1(denn
1
fying a specifically mathematical element in a general context
see .10).
This. list of mathematical operations is certainly not com lete If
Gulliver theme had been included in the analysis, we could hav: add-d:
the
- estimating results;
- empirically experimen"-g •rn as a s
t a.
ri;_ · f
rn•g pomt or a mathematical
exploration;
- analogical reasoning.
;;i::::c-b-ms e:c!unt:i:;ed in mathematising can, depending on the level
. . e ac v1ty t es place, be very different. Sometimes it will be
difficult to detect the mathematical aspect of a problem as in th f
the family problem, where it is essential to fit the -roblem --tset-
network of the route problem. On another occasion it would be h-d t-

generalise using a specific case and to apply the discovered rules to
another problem, as occurred in the complicated "to and fro" problem.
It is also possible that the difficulty will l!e in detecting essential
similarities between a series of problems or of fundamental elements
within one problem, thus finding underlying structures at a greater depth.
It is a case of raising the level, which allows the original problem to be
seen differently.
This ever-progressing process of expansion and raising of the level is
essential for the mathematical activity; new problem areas are explored
and old areas are considered from a different point of view. Thus it can
happen that lower levels are used as an algorithmic basis for higher levels,
as has happened with the intersection-grid. It is also possible that initial
problem.s become the model for the solution of new problems.
We have given a short characterisation of mathematising. It will
however be apparent that rather than starting autonomously, this mathematisation process must be prompted and supported through didactical

action.
2. ACTING DIDACTICALLY
This sub-section gives a description of a mathematics lesson and a
reflection on it from the didactical point of view.
What was stated previously about mathematising is once more illustrated here by using a concrete piece of mathematics instruction. Where
previously the emphasis lay entirely on the learning aspect, now the
teaching aspect is also taken into consideration.
It will become clear how the teacher initiates and stimulates the
learning process, thus generating a learning process that has certain
specific traits.
2.1 A Mathematics Lesson
The problem treated in the mathematics lesson reads: A person travels
from a to c via b daily, taking what he considers to be the shortest route.
One day he decides that he will take a different "shortest" route abc. How
many times will he be able to change his route?
We will describe the most important moments of the lesson as it was
given to two sixth graders - Jolanda and Tineke. Neither girl had
encountered a combinatorial problem before.
8 238 MATHEMATICAL MATERIAL VII
metry, because of the motivating force of the problems arising in perceived
reality - this, at any rate, is the experience of Wiskobas.
For the sake of completeness and as a finishing touch to this short
description of the Wiskobas programme we repeat the question asked at
the end of the introduction in an adapted form:
I> Please check how the didactical characteristics of Wiskobas as outlined
in the introduction are globally expressed in the preceding description of
the six subject matter areas.
In the following basic text the specific question, as well as the general
one we have just asked, is placed in a framework for instruction theory.
CHAPTER VII
FRAMEWORK FOR INSTRUCTION THEORY
Wiskobas Viewed Three-dimensionally and Seen from the Outside
1. PREAMBLE
In the first chapter we distinguished four trends in mathematics education,
to wit the mechanistic (arithmetic), empiricist, structuralist and the realistic, represented by Wiskobas.
The fundamental ideas of these trends are to be c9nsidered as frameworks for instruction theory, embracing the basic views on mathematics
instruction, determining the architecture of the various courses, and
directing the didactical activity.
In what follows the framework for instruction theory, in particular
of the realistic trend, is described. We start with the global starting
points. Then we sketch the main lines of this framework in general terms,
that is paying no attention to the six subject matter areas described
above. Following our former terminology . ThiS .. is appropriately called a
one-dimensional description. The two-dimensional description shows the
impact of the general theory on the character of various courses as
concretised globally in textbook series. With the three-dimensional description we place ourselves on the level of the concrete instructional practice.
Throughout we use the examples of instruction and course descriptions
from preceding chapters of the book.
Finally, we discuss the place of the prevailing cognitive-psychological
theories with respect to the instruction theory of progressive mathematisation outlined here.
As emphasised before, the following description of such a theory also
serves to clarify the third dimension of the goal description, that is, in a
broader sense, the teaching/learning component. Indeed since we earlier
asserted that the instruction goals are, or can be, comprised in the
teaching/learning process itself, it would seem useful also to explore the
content-related characteristics of the teaching/learning process in the
Wiskobas curriculum, after the sketch of the formal characteristics in
Chapter II. Tiris turn to a more content-oriented and systematic description is now possible thanks to the solid foundation of concrete examples
and considerations which has in the meanwhile been laid down in this
study and on which the following description can be based. Tiris a
posteriori description of a theoretic framework reflects the historic process; indeed, only after the curriculum development had for the greater
part been completed were we able to discover the theoretical- patterns of
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the development and actualisation of instruction behind the three-dimensional goal descriptions.
1.1 Starting Points of a Realistic Instruction Theory
The starting points of a theory of realistic mathematics education can be
seen most clearly when placed against the background of the structuralist
viewpoint.
In 1962 a group of 7 5 prominent mathematicians from the USA
and Canada published a memorandum in 'The Mathematics Teacher',
directed against the oncoming movement of the so-called 'Modern mathematics education' of structuralist character. A few fragments from this
memorandum:
To know mathematics means to be able to do mathematics: to use mathematical
language with some fluency, to do problems, to criticize arguments, to find proofs, and
what may be the most important activity, to recognize a mathematical concept in, or to
extract it from, a given concrete situation.
Therefore, to introduce-new concepts without a sufficient background of concrete facts,
to introduce unifying concepts where there is no experience to unify, or to harp on the
introduced concepts without concrete applications which would challenge the students, is
worse than useless: premature formalization may lead to sterility; premature introduction
of abstractions meets resistance especially from critical minds who before accepting wish to
know why it is relevant and how it could be used . . .
. . . There are several levels of rigor. The student himself should learn to appreciate, to
find and to criticize proofs on the level corresponding to his experience and background. If
pushed prematurely to a too formal level he may be discouraged and disgusted . . .
. . . the best way to guide the mental development of the individual is to let him retrace
the mental development of the race - retrace its great lines, of course, and not the
thousand errors of detail . . .
. . . Of course, not all mathematicians have the same taste. Mathematics has many
aspects. It can be regarded as an instrument to understand the world around us: mathematics presumably possessed this value for Archimedes and Newton. Mathematics can also
be regarded as a game with arbitrary rules where the principal consideration is to stick to
the rules of the game: some such views may be considered suitable for certain problems of
foundations. There are several other aspects of mathematics and a professional mathematician may favor anyone. Yet when it comes to teaching, the choice is not a mere matter
of taste. We may expect that the intelligent youngste-r wants to explore the world around
him, but we cannot expect him to learn arbitrary rules: why just these and not others?
1
Here are Polya, Sawyer, Pollak, Courant, and many others.
Was there at that time unanimity among prominent mathematicians?
Within one year after this appeal for realism the first volume of Papy's
spectacular 'Mathematique moderne' appeared - the structuralist wave
proved irresistible.
2
But the opposition of prominent mathematicians continued.
Thom turned against formalism, premature introduction and, in particular, pleaded with much passion for geometry instruction.
3
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Whitney stressed the need for meaningful instruction, full of motivating
tasks bound to reality, as opposed to the stiff formalistic problems in
actual education. Moreover, he asked for selfrestraint from the teacher
when explaining things: the children should selfreliantly move towards
concept attainment.
4
Hilton equally stressed the crucial significance of long lasting learning
processes and the patience needed to abstain from pushing too hastily to
too high a formal level.
5
The learning process can only be continued as
long as, again and again, it is started from a concrete basis: there must be
some underlying matter to be abstracted from, to be generalised, to be
unified. At the same time he stipulated the importance for teachers,
counsellors, developers and researchers of doing their own mathematics in
order to avoid losing contact with mathematical activity.
Lakatos turned vehemently against a dogmatic mathematics of eternal
and unimpeachable truth, where quest, failure and adventure are suffocated.6 In his view textbooks should be rewritten in order to exhibit the
growth of discovery. Heuristically organised instruction need not necessarily follow the precise course of the historical emergence of the concepts
under consideration; it comes about by a rational reconstruction of the
process of their coming into being, and then keeping the learner in mind
- that is globally Lakatos' course of thought.
In the same sense Freude"nthal judged that 'one should recognise that
the learner is entitled to recapitulate in a fashion the learning process of
mankind.
7
This entails instruction not starting with the formal system,
which in fact is a final product, nor with embodiments (materialisations of
structures) nor with structural games. On the contrary, the phenomena by
which the concepts appear in reality should be the source of concept
formation. These actual phenomena elicit, as it were, the mathematical
structuring - this was the way it happened in history and so it should
happen in education too, according to Freudenthal. This so-called principle of reinvention in his course of thought is strongly associated with the
levels distinguished by Van Hiele in the learning process of the individual
and of mankind, to be dealt with later on.
Overall the global starting points of realistically oriented mathematics
education can be summarised as follows, in accordance \Vith the earlier
mentioned formal starting points in Chapter IT:
- paying much attention to 're-invention', that is recreating mathematical concepts and structures on the basis of intuitive notions in
the making or made (active),
- carrying on at various levels of concreteness and abstraction ('differentiated'),
- the programming of the instruction is guided by the historical-genetic
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rather than the subject matter systematic method ('vertically
planned'),
- reality bound, meaningful ('and mathematically rich') instruction.
Stating these principles localises them globally within the framework of
instruction theory without, however, imposing a sharper outline. This will
be done later on by one-, two- and three-dimensional descriptions.
2. ONE-DIMENSIONAL DESCRIPTION OF THE FRAMEWORK
FOR INSTRUCTION THEORY
We shall start by describing the levels distinguished by Van Hiele in
the learning process. Then we shall sketch Freudenthal's didactical phenomenology. Subsequently we will show how the theoretical construct of
the gradual progressive mathematisation can be fitted into that broader
connection of the level theory. And, finally, we shall view the onedimensional theoretic framework of realistic mathematics instruction
against the background of the three other trends of mathematics instruction mentioned above.
Strictly speaking we will be concerned with realistic mathematics
instruction as actualised by Wiskobas, viewed a posteriori. In fact when
that project of educational development was started, its theoretical conception could only be made visible by paradigms of teaching/learning
units. All the time the theory was still implicit in the action, that is, in the
creation of instruction. Gradually the produced instruction became a
matter of discussion. Communicating within and reflecting upon the
concrete activity of educational development required developing an
appropriate didactical language. Only as the work went on could a
theoretic framework be constructed, into which the well-known Van Hiele
levels could be fitted in and reinterpreted. One may even assert that,
until recently, the level theory in our interpretation had not played an
explicit role for the group. By this feature the progress of the Wiskobas
work in three phases reflects exactly the learning process on three levels
such as described by Van Hie le.
2.1 Van Hiele's Levels
As an analogy to the learning process of the Wiskobas group alluded to
above, here is a quote concerning that of the Van Hieles:
'When the Van Hieles started teaching they were just as unprepared as many other young
teachers; nobody had told them how to do it. Of course they had passively undergone
teaching, may be even observed their teachers' performing, but this was not enough. As
time went on, they got the opportunity of discussing their teaching with each other and
with others. They subjected their own actions to reflection. They observed themselves
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-hen teaching, recalled what they had done, and analysed it. Thinking is continued acting,
mdeed, but there are relative levels. At the higher level the acting of the lower becomes an
object of analysis. This is what the Van Hiele s recognized as a remarkable feature of
the learning process, namely of that in which they learned teaching. They transfered this
feature to the learning process of pupils who were learning mathematics. There they
discovered similar levels.'
8
In the Dutch literature levels of didactical activity are sometimes described
as follows:
- didaxis, the level where thinking is still embedded in didactical
activity;
- didactics, where acting is subjected to discussion and objectivation
though in a still incoherent and inconsistent language;
- didaxology, where acting didactically is viewed in the framework of a
coherent theory based on research results.
Van Hiele distinguishes similar levels in the process of learning mathematics, which, to point out the analogy, could be called mathemataxis,
mathematics, mathemaxology.9
According to this view learning proceeds
through these levels, whose order is respected in instruction - the level
theory is therefore both descriptive and prescriptive. The order interpreted by Piaget as levels of development according to Van Hiele rather
_represents phases in a learning pn;tcess that can manifest itself at every age
and every level.
What do the levels look like, and what are their characteristic features?
Each level has its specific organisation of thinking matter. At each level

the objects of mathematical thought represent different things. At the first
level, in the primary grades for instance, they are embedded in operations
with material, observable objects such as concrete quantities of numbers
of things, measuring numbers, visible models and figures. At the second
level, relations between numbers and figures are the object of investigation. The result of this exploration is that numbers and figures symbolise
properties. In other words, a certain organisation applies in the network of
relations. At the third level, the relations themselves are objects of thinking: the character of relations and the connection between properties are
settled, which makes it possible to derive them from each other. The
organisation now fits into a logical and connected system.
The bonds between the various levels across different courses are fairly
complex. So the third level of the arithmetic system represents the concrete basis for the first level in algebra instruction, and the third level of
'fractions' is the basis for 'probability', according to Van Hiele.
In general he asserts that true learning on a higher level is not feasible
as long as the learning process at the lower level has not, or not sufficiently, been completed. More specifically: traditional arithmetic instruction is insufficient and ineffective because it almost always starts at the
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third, the formal level. The symbols used refer insufficiently to first level
notions, for the simple reason that these notions have not been developed
to a degree that the connection can be made. The signal character of the
symbols is too weak, they do not refer to concrete (mental) objects, they
represent nothing, they are properly said to be no symbols at all, and at
that moment they don't allow the pupil to profit from the system of formal
rules according to which they are operated on. As Hofstadter would say,
there is a lack of active as well as of passive 'meaning'.
10
One cannot even
raise the question of real formal significance - it is merely formalism and
verbalism.
Compared with the traditional manner, instruction according to Van
Hiele goes in just the opposite direction. Instead of starting with formal
operations at the second, or even third level, and trying to apply the
matter learned in appropriate first level situations, one starts at the first
level with the phenomenological exploration of the manifestations of
mathematical concepts and structures, in order to continue with formal
operations on the second level, and subsequently on the third. Essentially
it is the same teaching/learning process by reflection and recursion, as
typified by Kilpatrick.11 The lower level is the basis of the higher; the
(mental) actions on the lower level become thinking matter at the higher
one.
In order to rise from one level to the next, the pupils pass through five
phases, to wit: information, constrained exploration, explicitation, free
orientation, integration. Still in brief, this means:
- getting acquainted with the subject;
- getting in touch with the relevant structured elements by means of
tasks;
- developing appropriate linguistic means for the field under consideration in order to describe and verbalise relations and structures;
- free e;ploration of the total field in question by using the now
available means of organisation;
- and, finally, condensing the learned matter and fitting it into the total
inventory of knowledge, abilities and insight.
Ths:se are briefly, and in rather general terms, the levels and phases of the
learning process distinguished by Van Hiele.
Let us now, even more briefly, indicate what we appreciate as valuable
and stimulating in Van Hiele's conception as well as point out instances of
incompleteness and imperfectness - in particular from the viewpoint of
educational development as carried out in the Wiskobas project.
To start with, let us say that, in our view, the levels distinguished by
Van Hiele in the macro learning process, that is, viewed in the long term,
are still of great value for learning and teaching mathematics as well as
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other subjects. If we restrict ourselves to mathematics, we may state that
Van Hiele's triad is well founded in the developmental and learning
process with respect to various subjects. This, then, is the reason why we
have decided to integrate it into our theoretical framework. And, anticipating the last part of this chapter, we venture to state that, bearing in
mind the most recent theoretical developments and research data, the
level theory is now even more topical than it has ever been since its first
publication in 1957.
12
It should, however, be pointed out that there are no strict criteria for
delimiting macro-levels independent of the contents (concepts, structures,
and so on). In short, more often than not there is no way to exactly
diagnose whether an actual teaching/learning process takes place on the
third level - that is, of the systematics of subject matter - or on the"
descriptive second level or the intuitive first one: there are no sharp
delimitations, and we will soon show why not. As a heuristic tool,
however, the principle oflevel structure applies fairly well.
On the other hand, one should say that the vulnerable spots in the
materialisation of the three levels in the teaching/learning process as
envisaged by Van Hiele can now also be more sharply identified than in
the past. At the time when Van Hiele launched his level theory, the
contents of mathematics education were constantly under discussion. At
any rate, there was no questipn at that time of mathematics-for-all, or
mathematics in primary school, mathematics for slow secondary school
students or - to stay close to the work of the Van Hieles in those days -
of geometry instruction in primary school. The question, in fact, of what
mathematics-for-all should mean and comprise was not under discussion
in those days. Therefore, when at the end of the sixties the question
became topical in the so-called modernisation of mathematics education,
this obstacle was bypassed in Dutch secondary education (and elsewhere)
by focussing primarily on new subject matter. In primary education,
however, Wiskobas could not avoid the central question. In other words,
with regard to primary education the development was not forced into an
- even somewhat loose - structuralist straitjacket. It gradually became
clear that we could not find in Van Hiele's level theory and its concretisation in his textbooks a ready answer to the following two questions:
1. How to shape concretely the phenomenological exploration at the
first level?
2. Which didactical acts should be performed to raise the pupils as
efficiently as possible from one level to the next?
Or, condensed into one question: What really is mathematics-for-all and
how should mathematics education essentially be structured?
These questions will be discussed in the following subsection. First via
Freudenthal's didactical phenomenology and then in the light of the
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concrete work of Wiskobas in which, for that matter, Freudenthal has
been strongly involved.
2.2 Freudenthal's Didactical Phenomenology
Freudenthal set out his didactical phenomenology with regard to Dienes'
structuralist procedure as follows:
In opposition to concept attainment by concrete embodiments I have placed the constitu-
ti on of mental objects based on phenomenology . . . .
What a didactical phenomenology can do is to prepare the converse approach: starting
from those phenomena that beg to be organised and from that starting point teaching the
learner to manipulate these means of organising. Didactical phenomenology is to be called
in to develop plans to realise such an approach. In the didactical phenomenology of length,
number, and so on, the phenomena organised by length, number and so on, are displayed
as broadly as possible. In order to teach groups, rather than starting from the group
concept and looking around for material that concretise this concept, one shall look first
for phenomena that might compel the learner to constitute the mental object that is being
mathematised by the group concept. If at a given age such phenomena are not available,
one gives up the - useless - attempts to instill the group concept.
For this converse approach I have avoided the term concept attainment intentionally.
Instead I speak of the constitution of mental objects, which in my view precedes concept
attainment and which can be highly effective even if it is not followed by concept
attainment.
13
Such a didactical phenomenology is of course not new. On the contrary it
is as old as instruction itself. Even the most mechanistic arithmetic instruction uses it when looking for appropriate applications. Even there subtraction, for instance, is not restricted to merely removing a part of a
quantity from a whole, nor does applied division appear only as fair
sharing of a quantity among a number of persons.
The novelty in the realistic conception is that reality does not only
function in applications but also serves as a source of concept formation,
that is, in order to first develop intuitive notions, or in Freudenthal's
terminology, to constitute mental objects. The emphasis is on laying a
solid basis for learning rather than only on a posteriori applications at the
end of the learning process. By this peculiarity more weight is placed on
this didactical phenomenology in the realistic approach than in the nonrealistic ones. For the same reason, the demands made on phenomenological analyses are higher. One is obliged to pay attention to the subtleties
required for the functioning of learning processes in which mental objects
or intuitive notions are formed - subtleties which we are not usually
conscious of, due to the simple fact that we ourselves have far outgrown
the first level of the mathematical concept under consideration or, conversely, because we have never discovered the wealth of the related
mathematical structures. In brief, we need knowledge of mathematics as a
phenomenon of phenomena, where mathematics is understood as a human
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activity, tied to reality: mathematics-for-all. This phenomenological orientation answers the first question posed with regard to Van Hiele's level
system. The didactical credo of Freudenthal's realistic approach is: put the
pupils in touch with the phenomena for which the mathematical structure
is the organising tool in order to let them shape these tools themselves in a
process of re-invention, and learn to handle and use these mathematical
organising tools in concept formation.
Freudenthal does not, to be sure, distinguish levels in the learning
process which are attained by reflection and recursion in the same way as
Van Hiele. In his view, however, there is no rigid tripartition but rather an,
in principle, unrestricted progression according to micro-levels which are
only relatively delimited against each other. This is a view to which we
subscribe too in the 'progressive mathematising' to be discussed later on.
Besides, we also support the macro-structuring according to Van Hiele's
levels which is in fact an adequate tool for describing the rough structure
of instructional courses. In particular, it is of fundamental importance to
distinguish between the level of subject matter systematics and, on the
lower levels, between an intuitive phenomenological level and a locallydescriptive one.
The desire to have reality function as a source of mathematising
together with the structuring acc;ording to micro-levels by reflection and
recursion processes, as well aS according to Van Hiele's macro levels,
gives the first outline of our framework for instruction theory.
In order to give an overall answer to the second central question 'How
does the teaching/learning process proceed from one level to the next?',
we will first look at the concept of mathematising and will then describe
the kind of instruction devised by Wiskobas in this respect.
2.3 Progressive Mathematising Guided by the Five Instruction Principles
Chapter I I described mathematising as the organising and structuring
activity in which acquired knowledge and abilities are called upon in order
to discover still unknown regularities, connections, structures. We distinguish horizantal and vertical mathematisation in order to account for
the difference between transforming a problem field into a mathematical
problem on the one hand, and processing within the mathematical system
on the other hand. In the horizontal component the way towards mathematics is paved via model formation, schematising, symbolising. The
vertical sketch is concerned with mathematical processing and level raising
in the structuring of the problem field under consideration. We admit that
this distinction between horizontal and vertical components is a bit
artificial given the fact that they may be strongly interrelated.
Progressive mathematising, both horizontally and vertically, as under-
stood by Wiskobas, is inspired by five educational tenets which are 248
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logically related to the realistic starting point of the level theory and the
didactical phenomenology. .
Jn brief, and leaving aside the subject matter, they are the followmg:
1. phenomenological exploration . .
In the first instructional phase of a relatively new sub3ect an extensive phenomenological exploration is undertaken. The mathematical
activities take place within a concrete context. The real phenomena
in which the concepts and structures under consideration manifest
themselves are explored as multifariously as possible. The aim is _to
acquire a rich collection of intuitive notions in which the essential
aspects of concepts and structures are preformed. This, then, is
laying the basis for concept formation.
2. bridging by vertical instruments .
From the very start, in elementary problem situations, a vanety of
'vertical' instruments such as models, schemas, diagrams and symbols
are offered, explored and developed. This is done with the i_ntention
of helping to bridge the level difference between the mtm!ive,
informal context bound operating at the first level and the reflective,
formal s-stematic one on the third. In conformity with the theoretical
construct of gradually progressive mathematisation this is more a
step-by-step progress rather than a single jump from one level to
another. This means that from the start more formal elements may
be involved in the learning process, although they take place 'locally',
that is are bound to specific cases and within a concrete reference
frame.' In other words, horizontal exploration is not free of vertical

features in the teaching/learning process, as illustrated by the following picture:
level 3
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level 2
I
.
I level 1
This sketch intends to show that, besides the macro-levels in the
learning process within a certain course, one can distinguish finer
FRAMEWORK FOR INSTRUCTION THEORY
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meshes and a stepwise structure of mathematising at the micro-level.
Paradigmatic context problems, models, schemas, diagrams and
symbols are the vehicles of progressive mathematisation. On the one
hand the sum of horizontal components in the above sketch represents the variety of real situations in which the concept under
consideration may manifest itself, and the abundance of notions
required for concept formation. On the other hand, the sum of the
vertical components represents the more formal, subject-areaoriented systematic knowledge and abilities acquired in the learning
process. Admittedly, this schematising of the process of mathematising is a bit artificial: the passage from one mental level to the next is
not a neat lattice path but rather a patch-work of paths, if not a
labyrinth, although one where the teacher, knowing the right path,
can help the pupils to find their way up to the top.
3. selfreliance: pupils' own constructions and productions
Nonetheless, the trip should be made by the pupil on his own two
legs, which means that a decisive influences comes from the pupils'
own constructions and productions in the learning process. The
individual contributions can in general always be seen in the differentiated production procedures used: solving problems on various
micro-levels as well i)S producing their own ideas with respect to free
tasks and problems within the subject matter area under consideration. In both cases the pupil indicates by bis activity at each moment
his location in the learning field and his progress in the process of
mathematising. This provides the teacher as well as the learner
himself with knowledge about the course of the learning process. In
brief, in this view, the constructive element, although often stressed,
is more than a credo, a nebulous thought, an elusive principle. It is
clearly distinguishable in the pupils' own constructions and productions; it is visible in the concrete work presented by the pupils.
4. interactivity
This means that the pupils are also confronted with the constructions
and productions of their fellows, which can stimulate them to
shorten their learning path, to help themselves up on the procedures
of others, to become aware of the drawbacks or advantages of their
own productions, and that copying others' work slavishly will not
aid their own progress. In brief, the learning process is part of
interactive instruction where individual work is combined with consulting fellow students, group discussion, collective work reviews,
presentation of one's own productions, evaluation of various contructions on various levels and explanation by the teacher.
5. intertwining
However, when outlining a course one should be fully aware of the
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global connection of the various domains, the intertwining of learning strands. This involves preconditions for the possibilities of
progressive mathematisation which lie to a certain extent outside the
course under consideration. Such conditions can include skills and
algorithms, as well as aspects of insight.
These then are the five general principles to be generally observed with
the intention of making the transition from one level to the next in the
progress of mathematising according to Wiskobas. The progress is realised
by reflection and recursion, by a cyclic process of phenomenological
exploration and mathematical structuring, by a stepwise articulation of the
teaching/learning process.
At the first approach, context problems are elementary but nevertheless
paradigmatic, which means that they function as forceful mental models.
Or, in the terminology of mathematisation: they possess a strong vertical
potential, they lend actual significance to more abstract models and to
symbols, while as context problems their horizontal range is broad
because they represent a broad domain of real phenomena, which may
emerge as applications. The pupil's own contribution to the learning
process is an essential factor. The cognitive intertwining of learning
strands, which may be relatively separated from the viewpoint of subject
oriented systematics, is accounted for. And finally, this view on and theory
of the process of mathematising implies interactive instruction.
This ends the abstract one-dimensional description of the tenets of
instruction theory connected to Van Hiele's level theory and Freudenthal's
phenomenology. We were able to risk making the exposition in general
terms because the concrete basis of orientation had been laid in advance
- a precaution taken in order not to sin conspicuously against the
principles of phenomenological theory adhered to. In fact, this theoretical
framework will be given a more concrete filling as soon as we proceed to
the two- and three-dimensional descriptions. But before tackling these, let
us cast a glance at the three other instruction theory trends, indicated
above.
2.4 Schematic Comparison of the Four Trends in Arithmetic/Mathematics
Instruction
At the close of this section we venture to pose the question: How is the
instruction theory construct of progressive mathematisation related to the
instruction theory conceptions of the three other trends of arithmetic/
mathematics instruction?
Well, mathematising with its horizontal and vertical aspects is shaped
according to the four trends as follows:
FRAMEWORK FOR JNSTRUCTION THEORY
Frameworks for instruction theory
Mathematising
Realistic
Structuralist
Empiricist
Mechanistic
horiz vert
+ +
+
+
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In realistic mathematics instruction careful attention is paid to both the
horizontal and the vertical component of mathematising. This means that
the phenomena from which the mathematical concepts and structures arise
are implicitly used both as source and domain of application. This,
according to the tenet of the theory, creates for the learner the possibility
of concept attainment by orienting himself to a variety of phenomena,
which benefits the building of formal mathematical concepts and structures and their application. Mathematising proceeds step by step. A
number of specific micro-levels can be distinguished whi.ch, considered
over a larger domain and in the long run, are gathered into the three Van
Hiele levels.
In structuralist instruction, organised by subject-oriented systematics,
the vertical component is dominant: operating within the mathematical
system is the principal part of the mathematical activity. The horizontal
component manifests itself only in an a posteriori application of what was
learned before within the formal system, and then only in a restricted way.
As a consequence, the real phenomena do not and cannot function as
models to support operating within the mathematical system. Instead of
the real phenomena, one uses embodiments and materialisations of mathematical concepts and structures (see Chapter I) or structural games in
order to create a concrete basis of orientation. There is, indeed, a structuring according to levels, although mainly by means of vertical connections in the sense of Bruner's 'modes of representation', 'enactive, iconic,
symbolic', which we will reconsider later on.
Following the empiricist trend, the horizontal component predominantly determines the course of instruction. This is already clear from
the brief characterisation in Chapter I . There is relatively little pressure to
pass from the first to the second level. Or, as Morley says: the emphasis is
on environmental rather than on mental operations.
14
Empiricist instruction cares little about formal mathematical goals. Many ideas are taken
from Piaget's psychology. Maturing of cognitive opportunities in a stimulating environment is preferred to consciously aiming at higher levels of
thought by offering paradigms, schemas, models and symbols as 'cultural
amplifiers'.
To end, the mechanistic approach is characterised by weakness of both
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the horizontal and the vertical component: no real phenomenon as a
source, little attention paid to applications, much emphasis on blind
memorising and automating of numerical facts and acts, no use for insight
in the operations of the formal system. A moderate form of this approach
is based on Gagn6's cumulative learning theory, but stricter forms most
often lean on behaviouristic principles. In instructional practice the
mechanistic approach promotes 'individually prescribed instruction', that
is the cultivation of solitary formal calculations. The subject matter is
offered in such an atomised and fragmented way that the pupils master the
petty instructional goals step by step under their own steam. Sometimes
this method is called 'mastery learning' but this is a very narrow interpretation of mastery, compared with that described in the preceding
chapters.
In what follows we will have the opportunity to refine the somewhat
schematic picture of frameworks for instruction theory sketched above by
adding subject areas (themes, learning strands) to the picture.
3. TWO-DIMENSIONAL DESCRIPTION OF A FRAMEWORK
FOR INSTRUCTION THEORY
In this section we explain the framework for instruction theory by means
of examples of themes and partial courses taken from the Mathematical
Material of the preceding chapters and the present chapter.
We start with a description of the various levels in the teaching/learning
process of the most important courses. We continue with a description of
the five tenets which determine instruction according to progressive
mathematisation. Again, the present section is closed by a brief comparison of the four principal trends which can be distinguished in the
textbooks of arithmetic/mathematics instruction. The comparison, then,
will be guided by the five tenets.
3.1. Progressive Mathematisation in the Wiskobas Programme
In all six areas of the Wiskobas programme as described in the introduction to the present chapter, we can discover progressive mathematising
according to Van Hiele's macro-levels, as well as the phenomenological
foundation required. by Freudenthal.
We shall review the corresponding six courses while restricting ourselves to the phasing on macro-level.
Remarking generally on the six subject matter areas, it should be
observed that, in most cases, the pupils do not aim at the third Van Hiele
level of subject-matter oriented systematics. Numbers and figures, relations between them and structures around them do not, for the pupils,
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reach the degree of concreteness needed for systematic operating and
reasoning. This is certainly true with regard to ratio and percentage,
common and decimal fractions, the metric system and geometry. For the
majority of the pupils concerned, only local organisation in the areas is
feasible and the application of the operations will not be developed in its
full breadth. The symbols have the character of signals of context-bound
situations, schemas and models, but they are not yet sufficiently detached
to be able to operate on them within the system with no reference at all to
the underlying concrete infrastructure. In short, in these cases the pupils
do not reach the instruction level that is chosen as the start of the
instruction in the mechanistic and structuralistic approaches, mostly after
a short 'concrete' introductory phase.
Thus far our characterisation has been 'negative', that is, in terms of
what pupils cannot do in the presystematic phases. We could however
also discuss in a positive sense, that is, as regards the applications they are
able to make and the essentially mathematical abilities they develop,
interpret meaningfully and apply, which are not attained if the starting
level is too high. This is so, at any rate, from the viewpoint of mathematics-for-all. After these general remarks on the macro-levels we will
now consider each particular subject area separately.
Basic abilities starts with a or~ad phenomenological exploration, followed
by schematising and structuring of calculating methods.
15
Even memorising is embedded in a process of progressive mathematising and the same
holds for the various forms of skillful reckoning. Many pupils, in the long
run, can operate at the 'pure' number level, discover properties, apply
them within the arithmetic system and recognise the basic operations in
various context situations, thus reaching the third level.
Column arithmetic offers the same picture that has been painted in detail
in Chapter VI. A number of pupils develop the standard algorithms, can
apply them broadly within the system of whole numbers, can use them in
context problems, and are able to make the algorithms themselves an
object of investigation.
Ratio activities start at the first level with qualitative and informal comparisons in mainly visually set problems and gradually pass into the stage
of more numerical and calculation strategies. The gap between is bridged
by all kinds of models. Finally, arithmetic processing becomes more and
more schematised, shortened and endowed with numerical precision, in
particular when dealing with percentages. It should be added, as said
before, that in primary school only a certain number of the pupils reach
the third Van Hiele level of more profound investigation of ratio, propor-
tion and percentage.
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Fractions shows the same macro-feature of progressive mathematising: a
broad phenomenological exploration, a well-knit coupling with ratio and,
from the outset, a great deal of attention lo the qualitative comparison of
fractions as the result of fair sharing.17 All ways of symbolising ratio
situations and fractions results, as well as their descriptions, are extensively stimulated. Models and model situations are offered, the pupils' own
production on the symbolic level is stimulated while, on the other hand,
the focus is not directed to ready rules. It is a long way from the first to
the second level, which means that some pupils can hardly transcend the
stage of context-bound calculations with fractions, while others are able to
perform and even interpret and apply them meaningfully. Formal fractions
are not used as a subject of extended investigation into the rational
number system at the primary school.
Measuring passes from !lie context-bound, qualitative work to more
formal numerical precision. Areas, for instance, are compared in the
following sequence: on qualitative grounds, with the naked eye, by estimation, and by covering with and transforming of figures. This leads to
measuring, first with natural, then with standard units, which in the long
run are to be refined.18
Finally, formulas are used for the areas of
rectangles and then of half rectangles, that is right-angled triangles, other
triangles, and quadrilaterals. In the latter cases structures and patterns -
which on the second level lead to certain rules and formulas - become a
subject of investigation and are extended to other instances. LIkewise, the
relation between perimeter and area can be investigated at various levels.
The phenomenological exploration of area is started by connecting area to
other magnitudes, such as weight, price and so on.
Geometry is not begun in the traditional way, at the second level, by
introducing mathematical objects such as point, line, plane, plane figures,
the angles, angle measures, translations, rotations, reflections, but by
offering the natural phenomena of spatial perceptions as the starting point
of instruction.19 Geometrical notions are developed by looking, building,
experimenting and trying out, which are later supplemented by reasoning
and reckoning. It is not assumed that elementary geometrical entities such
as point, line and plane are mentally constituted, nor are they defined.
They arise from activities concerning light and shadow.
That describes progressive mathematisation as a general characteristic of
certain Wiskobas courses. Of course the principle extends to secondary
education, where it has been concretised in the Wiskivon work of IOWO
and then in the Hewet project of OW & OC and of SLO-projects
(Foundation for Curriculum Development). In the next subsection the five
instruction tenets that are to shape the overall process of mathematising
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and mathematical instruction are made more precise with the intention of
relating them to the themes and courses sketched above.
3.2 The Five Tenets of the Framework for Instruction Theory Revisited
(1) Phenomenological exploration by means of contexts
With regard to form and function context problems is a broader conception than traditional word or text problems. A context problem looks like
a word problem, or may be embedded in a play, a game, described in a
story, offered as a press-cutting, represented by models and graphs, or as
a combmatton of such information resources. They can be part of themes
or proj-cts, _of ;"-ich the present study contains a variety of examples,
such as Gulliver', Roads problems', 'Grains on the Chessboard'.
As far as content is concerned, 'context' means:
a. that, in the case of an isolated, self-contained problem, one looks for
the 'surroundings' of the text or the presentation of the problem, in
other words, for that which is not explicitly formul<-)(ed or presented,
but does belong to the background assumptions;
b. that, in the case of a non-self-contained problem, one looks for the
surroundings explicitly evoked by the story, the theme, the location,
which, as a matter of fact, also carry their own background assumptions with them.
Thus with the context a (subjective!) variable element is introduced into
the problem, and this variable can be stronger or weaker depending on the
kmd of problem. For instance, in the problem of the fair sharing of 230
items among 10 persons, the variable is relatively weak, although the idea
of_ fairness is,. not necessarily unequivocal (among the 10 persons there
might be babies and adults). On the other hand, in the case of 234 items
to be shared among 10 persons the context may be more essential and it
will certainly be so as soon as the items are more precisely defined. The
context counts even more in a problem like 'our car does 1 O to the liter
(that is 1 liter of petrol for 10 kilometers); how much does it use during a
trip of 234 kmi'
In re.alistic mathematics instruction the pupils will certainly not have to
put then relevant knowledge and experience aside; on the contrary, they
will be. able to use them by noticing that the 10 is an average (driving on
the mam road 1s more economical than in the city unless one disregards
the 100 km/h limit) with the intention of arriving at a well-considered
answer. In _brief, we should be aware, at least at the first level of phenomenological exploratton, of the part played by the context in which the
problems are proposed - a part which may be subordinate as well as
principal. 256 CHAPTER VII
Regarding or disregarding certain context factors, which is the essence
of modeling the problem, 'is in fact a non-negligible part of the process of
mathematising. Of course, this does not mean that each problem should be
heavily contextually loaded or fitted into a larger thematic context. Sometimes it would even be undesirable, in particular if an elementary problem
situation were to be assigned a model function.
A context problem is not merely a dressed-up word problem, with its
application subordinated to mathematics. Alongside the logical connection
a phenomenological one by which mathematics becomes ancillary to the
real problem is possible. In this case reality counts first, and mathematics
is a mere derivative. Any context problem, whatever it may be, includes a
(subjective) variable aspect which can elicit a realistic reconsideration or
motivation as soon as the answer is translated back to reality.
Context problems fulfil several functions, to wit:
- concept formation: at the onset of the course they offer the pupils
natural and motivating access to mathematics;
- model formation: they support the learning of formal operations,
procedures, notations, rules, together with providing other material
and visual models which fulfil an important support function for
thinking;
- applicability: they uncover reality as source and domain of application;
- training specific arithmetic abilities in applied situations.
Take, for instance, division: an elementary division problem can be the
start for learning long division. Conversely, it can be chosen as a concrete
base for a bare division problem or for understanding a specific property
of division. In both cases the context conveys significance in such a way
that the procedure and the operation will in the long run become
meaningful within the formal system. Moreover, such a division problem
can function as an application of tables learned previously or of long
division procedures, as well as an exercise in the applied situation of
money arithmetic. In brief, context problems function both horizontally
and vertically. On the one hand they serve to make mathematical knowledge and abilities applicable, and on the other hand they impart meaning
to formal operations. The urgent need for the latter function is witnessed
by the test scores of two division problems taken from a National Assessment research (NAEF; 1980):
- 6 /608 . . . was correctly answered by 69% and 65% of the 13 and
1 7 years olds, respectively;
- the problem '26 children, four per car; how many cars are needed?
was correctly answered by no more than 3% of the nine years olds
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(handheld calculators were allowed); 12% chose for the answer 6 .5
and 7% for 65 cars.
20
Whitney's comment on the result:
These examples furnish further evidence that, right through_the ele-entary sc--olye-rs, -he
attitude that 'schoolmath is something for itself, not for hfe outside school is beco011ng
entrenched.
21
Children evidently do not always imagine a situation in connectio.n with
the bare arithmetic problem (608 -7 6) as, for instance, that of distnbutmg
6 dollars and 8 cents among six persons. Conversely, .m a realistic
transport situation they do not know what to do with a 'bare' answer (6.5).
In fact, a vast phenomenological exploratt0n is not a luxury one. ca.n
d
. th but on the contrary is a sheer necessity. This necessity is
ispense w1 , , . .
shunned by the formal, seemingly logical approach, devmd of any meamng
d 1 d
. to a kind of mathematics detached from and isolated m
an ea ing . .
reality. The phenomenological approach, on the contrary, does 1us!ice to
all phenomena in which the mathematical concepts and structures under
consideration reveal themselves. . . .
It would take us too far afield to sum up how .many-sided embeddmg m
the spirit of Freudenthal's didactical phenomenology of mathematical
structures takes place in each particular theme and course of those we
displayed above and what, among others, Sawyer, Vergnaud, Hilton,
Winter have contributed to it. We only ment10n the broad embeddmg of.
_ ratio with a reference to 'Gulliver' (Chapter I) and to the descriptio.n
in the Mathematical Material for the present chapter:. geometnc
aspects, drawings, puzzles, stick-shadow, mixtures, densities, scale,
percentages, ... ; .
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th'
_ combinatoric problems (Chapter II): rn.ads, tree diagrams, c o mg
combinations, colouring, lattice, probabilhty, . . . ;
_ division (Chapter VI); extensively discussed m the preceding and
present chapter; .
_ fractions, described extensively in the . present chapter: breakmg,
dividing, distributing fairly, measuring, ra!los, probability;
_ many more examples of a many-sided approach m the other courses
sketched in the Mathematical Material for this chapter.
However, rather than restricting our attention to -e form, cont-nts and
functions of the particular context problems belongmg to a certam teaching subject, we shall also consider them m thelf totality . . An nnportant
criterion is completeness, in other words, whether the vanety of mathematical phenomena in the real world is sufficiently expressed'. In parlicular, problem situations should be viewed which, thanks to thelf mherent 258 CHAPTER VII
manysideness, fulfil a number of different functions at the same time:
concept formation, modeling and applicability.
Such model problems - in the literal as well as in the technical sense -
are the core of realistic mathematics instruction because of the multitude
of horizontal and vertical impulses which they contribute to the process of
mathematising. They help to bridge the level difference in the learning
process. Let us give some examples:
- the handkerchief problem in 'Gulliver' (Chapter I);
- the roads problem in organised counting (Chapter U);
- the chess board for exponential growth (Chapter HI);
- the Walt Disney drawings for number systems (Chapter IV);
- the lattice in Freckleham (Chapter V);
- the fair sharing situations in long division (Chapter VI);
- the distribution situations in table arrangements (Chapter VII).
We will return to this from another viewpoint in what follows now.
(2) Bridging by vertical instruments
Among the vertical instruments, models primarily fulfil a bridging function
between the phenomenal appearances of mathematics in the physical,
social, imagined and 'real' reality on the one side and the formal system on
the other side, as handy, visual, verbal or symbolic representations, and
real world situations (paradigms), which reflect essential aspects of mathematical concepts and structures.
Besides serving as thought models in the service of concept formation
and applications, models can also serve to facilitate calculations to learn
algorithms, to explore regularities, and so on. In the present -tudy we
encountered a large number of instances of this last category, such as:
- the roads model, the tree diagrams, the rectangle model (Chapter H);
- the abacus and the number line (Chapter IV);
- the arrow diagram and the lattice (Chapter V);
- the distribution pattern (Chapter VI);
- and many more in the Mathematical Material for the present chapter.
As a consequence, the character and function of models have already been
discussed incidentally, especially in Chapter II. For this reason we now
restrict ourselves to the kind of models we would like to call situation
models. Situation models are context problems with paradigmatic value
which, thanks to this property, are able to function as models in the sense
of prototype rather than of mould. To this end, the problem must be
investigated and analysed sufficiently in order to be able to recall the
prototype to be copied. Because of this peculiarity they are particularly
important for the level theory of progressive mathematisation. In fact, in
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this view on instruction, they are typically first level - at least in pnnciple
since, as the pupils advance, many parts of mathematics can acqmre the
same concreteness, but let us disregard this for the moment.
Instead, we give one example of this kind of model from each chapt_er.
The handkerchief problem from 'Gulliver' is first of all an obtect of
exploration and functions later on as a model for the behaviour of area
and perimeter under linear magnification. . . .
The road problem functions similarly, together with the tree diagram m
organised counting as a model for combinatoric problems. .

The chessboard problems with grains of corn is an important basis for
(expo-imtial) growth problems (growth by. a certain percentage per year).
As a JTirst step an (approximate) doublmg time is figured out m order then
to use the chessboard model further. . .
The context of the Walt Disney drawings (characters with four fingers
on each hand) helps the pupils learn to count and reckon in base eight. The
fingers-hand model is then retained in order to do the same calculations m
base six and other number systems, and eventually, for instance, also in
base twelve, where two new symbols are needed - a unique opportunity
to demonstrate the force and function of digital symbols. Let us add that
the fingers-hand model, which initially provided strong support, might
later on block the process of generalisation - a feature inherent m the use
ofmodels. . .
In Freckleham if 'was seen how linear ordering can function m problems when reasoning rests on awareness of transitivity. In not_e 27 of
Chapter V we dislayed four isomorphic order problems which notwithstanding their isomorphism, differ considerably m difficulty. This is
because in certain cases, the linear order is not obtrusive enough to
impose the global thought model on the learner who, therefore, proceeds
by chains of local deductions, that is, on a higher level than neede_d.
Nmnerous psychological experiments have revealed the enormous difference arising from the way isomorphic problems are proposed, that
is whether they can be solved on a concrete and meaningful level or
whether, thanks to their specific adornment, it must b_e done on a more
formal level. The availability and possibility of using thmking models is of
crucial importance if a difference of level in such reasoning problems is to
be bridged - not only in reasoning problems but m mathemansmg as a
whole. - ff -
If the tong division procedure is to be developed, the situation o mr
sharing and distributing plays a principal part. This is true of the first
phase of the course, but just as true in later phases if shortemng ' s
stimulated and procedural mistakes are to be overcon:ie. More precisely: it
is the combination of fair sharing with its symbohsat10n m a schema that
opens the road to self-reliant development of the algorithm. . .
Table arrangement is another opportumty to make similar observa-260
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tions. Table arrangement (ratio) is symbolised in such a way (- means
three pizzas on a round table for four persons) that the pupils can easily
imagine the portions and the arrangement behind the symbol. Moreover
the ratio is clearly distinguished from the fraction (- in the present case) as
the result of the distribution per person.
In many of the preceding cases the main thing is the construction of
a meaningful situation (say at level one) with a fitting materialisation,
visualisation, and symbolisation (in some cases on a higher level); thus the
integration of an elementary situation and a model. This makes it possible
for similar models to fulfil both a horizontal and a vertical function. In
other words: abstracting, generalising, unifying, schematising, shortening,
reflecting, anticipating - all these can be made possible by modeling.
Not all models can function in so many ways. Artificial materialisations
and visualisations of mathematical concepts and structures, which primarily or exclusively function within the mathematical system, may lack
this breadth of power: MAB material, 'machines', sticks and similar
structural models, schemas and diagrams can be useful thanks to their
strong vertical power, yet phenomenologically viewed their power is
restricted. They lack the bridging quality required for the transition from
the first to the second Van Hiele level, on which we have insisted. (Not to
mention the traditional pancake models, which fulfil only a transitory and
onesided function.) As a matter of fact, the bridging function of situation
models consists mainly in conveying meaning to symbols, procedures,
formulas and so on: they stand as models for something real, which is
signaled by them. The degree to which this is indispensable was shown
earlier by the division problems from a NAEP investigation.
(3) Pupils' own constructions and productions
In the instruction theory of progressive mathematising the pupils' own
constructions and productions occupy a key position. Horizontal and
vertical mathematising comes about by the pupils' actions and their reflections on the actions. When speaking about their own constructions, we
stress their actions, while in speaking about production the stress is on
reflection. Of course, this is only a matter of more or less emphasis, as
they are inseparable. In any case, the examples show clearly that the
pupils' contribution to the design of the courses is quite essential. On
closer inspection, it is essentially the pupils themselves who construct the
courses, albeit guided by the teacher. Take for instance number systems,
long division, fractions - examples that have been discussed broadly. At
certain stages in the instructional process the pupils are stimulated to
produce their own problems. Producing simple, moderate, complex problems means that the pupil reflects on the path he himself has taken in his
learning process, and at the same time anticipates its continuation. These
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productions derive part of their enormous significance from. the fact that
the instructional course is essentially the result of the pupils' own contributions. The mere possibility of their own production protects pupils
from being forced into a subject matter oriented straitjacket. They are
relatively free, that is, as far as the space in which the pupils can construct
the instructional course permits.
Within the frame of their own constructions, their own production is, as
it were the condensation point of the teaching/learning process. For both
the pu-il himself and for the instructor the pupils' own contributions are
of eminent importance, both for mathematising and d1dact1Csmg . . The
reflective moment for the teacher as well as for the pupil 1s the s1gmf1cant
factor in the production. The pupil's production, as the result of instruction functions as the mirror image of the teacher's didactical activity. Its
prognostic
'value is unmistakable. .
In brief, the pupil's own production is both a learning and a teachmg
tool - that is to say, in instruction where pupils' own construct10ns
determine the progress of mathematising. This, in fact, is one of the most
characteristic features of realistic instruction theory.
Other trends in mathematical instruction, in particular the structuralist
one, pay no attention to pupils' own constructions, let. alone productions.
On the contrary, after a sometimes playful introductory penod little space
is left for the development of informal approaches and the!T gradual
transformation into formal methods, and consequently as little for own
productions.
(4) Interactive instruction
The pupils' own constructions and productions, as well as phenomenological exploration and modeling, can be efficient only 1f they are reahsed
in interactive instruction, that is, instruction where there 1s the opportunty
to consult - to participate, to negotiate, to cooperate, with review afterwards and where the teacher holds back from providing explanations. All
the examples of instruction we have presented. in this study involve this
interactive element, which clashes with the solitary rote calculat10n and
focus on pencil-and-paper learning of so-called individualised instruction.
In realistic programmes and textbooks mstruc\Jon 1s d1dac\Jcally organised in such a way that interaction and cooperat10n between the pupils and
with the teacher are coexistent with individual work. Various differentiation models are proposed, among which that of internal differentiation
within the classroom is considered as the most appropriate in The Netherlands. The subject matter is divided into blocks of more or less selfcontained units, each covering a few weeks of instruction. The members of
a year group always start together with such a block and for the most part
stay together during their way through the block. Each block mcludes 262 CHAPTER Vll
opportunities for differentiation according to subject matter, level and
speed, which are strongly determined by the specific content. It is true that
vertical classes wher- th- teacher is responsible for two or even more year
groups, are. a comphcatmg factor. In this case, quite drastic organisation
measures w1H need to be taken at class and school level in order to adapt
the system to interactive instruction.
. Realistic courses have to pay attention to this organisation aspect of
mstruct10n, and this obligation consequently extends to the theory of
instruction.
We -estrict ?urselves to these general points, as a more detailed
elabornhon of .differentiation would take us too far afield, in particular if
we WJSh to. do 1ustice to the great variety of possible practical elaborations
based on different textbook series.
. For a. more concrete example of unity in supply of instruction combmed w!th differentiation in demand we refer to column arithmetic,
realised accordmg to internal differentiation in interactive instruction
(Chapter Vl). The_ pupils start and stay together, they can help one
another up, there is opportumty for consultation and cooperation and,
conseque.ntly, for reflect10n and recursion. Each pupil gets the opportunity
to work md1V1dually and, within certain limits, to construct his own path
without isolating himself from the group.
(5) Intertwining of learning strands; the broader connection
The intertwining of learning strands has its deepest origin in the phenomenological start, that JS, m the real phenomena in which mathematical
structures and concepts manifest themselves. Rarely are the phenomena so
pure as to comprise one structure or one concept Only.
On the thematic level, intertwining of a number of mathematical
elements is clearly visible in 'Gulliver', 'Grains on the Chessboard' and
'Freckleham'-_ On the course level intertwining is visible in column arithme.1'c and_ skilful reckoning, in calculating and measuring, in fractions and
ratio, and 1n measurement and geometry.
Earlier on we remarked that the prerequisites for promoting progresswe m_athematisation are often partly found outside the course under
con.s1derahon. These prerequisites may concern arithmetic abilities as well
as those of a conceptual and insightful nature. The basic abilities in
particular pervade almost all partial courses - which is why they are
called basic. But, the conceptual connection is equally strong in the
subjects ment10ned above - measuring and arithmetic, fractions and
decimals.
The_ p_rinciple of many-sided embedding of concepts and structures,
which is inherent to the phenomenological level theory and to progressive
mathemahsmg unavmdably leads to intertwining.
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Yet in realistic instruction the ancillary function of mathematics in
reality does not lead, to subordinating mathematics to the contents, say, of
a project as it does according to the empiricist view. This instruction
hardly leaves any room for relatively self contained courses.
Giving in on this point means that the vertical component of mathematising is not sufficiently cared for. Mathematically and phenomenologically natural intertwining aims equally at promoting vertical mathematising. These general considerations with regard to this subject together
with the reference to the concrete examples of themes, courses and the
sketch of the Wiskobas progranune may suffice, since -laboration in detail
would certainly be as long as a complete textbook series manual.
3.3 Comparison of the Four Trends
In the foregoing sections we have shown with parts of a realistic
programme (Wiskobas) how to discern the features of progressive
mathematising, and how the five tenets that direct instruction are to be
recognised in such a prog'amme. The themes and courses described in
this study have continually been used as illustrations. If however, we
wished to contrast sharply and in detail the three oth-r trends with the
realistic one, we would have to describe how themes and courses are
shaped in textbooks wh!h represent those trends. We cannot possibly go
that far. All we can do is to check how the five tenets are dealt with in
mechanistic, structuralist and empiricist textbooks, one after the other. To
this end we will refine the schematic description by means of the
horizontal and vertical structure of mathematising.
In textbooks of the mechanistic trend one is struck by the scantiness
of its supply of problems: stereotyped word and text problems, as a
posteriori application of the subject matter learned within the formal
system, scholastic and preconceived, dressed-up arithmetic, where reality
is subordinated to applications. For this reason they are scarcely motivating in general and terribly insignificant. Little use, if any, is made of
models, let alone pupils' own constructions and productions. No interactivity is pursued and instruction is mainly individual. Speed is the main
criterion of differentiation. Rather than being intertwined, the partial
courses are isolated. Summarising, we may state that on each of the tenets
which direct progressive mathematising, the mechanistic trend diverges
from the realistic one.
Textbooks and programmes representing the structuralist trend are in
general not interested in context problems, except as a posteriori applications, where reality is a derivative of the mathematical problem. Material,
visual and symbolic models are used but only in a vertical function. That is
to say, they fulfil the bridging function between the formal system and its
natural and materialised 'embodiments' by means of structural material 264 CHAPTER VII
and games around it. In the initial phase, pupils are allowed to make their
own constructions; this is even stimulated, only to disappear later. Pupils'
own productions are not systematically used in the intermediate stages of
the learn1n- process. Instruction is, or at least can be, interactive. Partial
courses which are conn_ected might be intertwined if this is suggested by
some aspect of the subject matter, but phenomenological aspects do not
count in this respect.
In the empiricist methods all five tenets are active, although a strong
vertical component of mathematising is lacking. This implies that relatively
little use is made of situation models in modeling, schematising and
symbohsmg, that the pupils' own productions are not related to problems
of a more formal symbohc nature with situations in the background, that
mteraction takes place on too low a level since the pupils have no
su-1c1 -nt formal linguistic means of description and communication at
then disposal and_ finally, that the mathematical activities are integrated in
problems set withm themes and projects which do not allow any resolution of the intertwining.
While mechanistic and structuralist instruction methods are too formal
empiricist instruction is bound to remain too material. And while isolatio-
prevails in the first and second, integration dominates i- the empiricist
presentation. If the first two start in at too high a level, the third stays too
long on one that is too low.
This, _sharply outlined, is how_ things look from the perspective of a
realistic mstruction theory, which has itself been ontlined by the Van
Hiele s macro-levels and Freudenthal's instructional principle of phenomenology. In the present two-dimensional description of the framework
for mstruct10n theory some attention has also been paid to the microphases m . the teaching/learning process with regard to progressive
mathematisng and the tenets which direct the instruction. In the description, however, we have stayed at the textbook and teacher-manual level
that _is, rather far away from classroom practice. Or rather: we hav-

restncted ourselves to referrmg to descriptions of courses and of the
organisation of subject matter. But in some cases we were concerned with
more con-rete descriptions of themes. In this way three-dimensionality
was mphcitly present in the description of the framework for instruction
which will now be made more explicit. '
4. THREE-DIMENSIONAL DESCRIPTION OF A FRAMEWORK
FOR INSTRUCTION THEORY
In our experience and according to research the four trends which we
distingmshed in the textbook literature are also reflected in the minds of
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teachers. This is not surprising as textbook authors often are, or have
been, teachers.
This does not mean that teachers' ideas always harmonise with the
intentions of the textbook they use, firstly because, for various reasons, the
teacher might have no say in the choice and, secondly, because the basic
ideas of textbooks are not always clearly enough outlined to prevent
preposterous, and as often happens, obnoxious interpretations.
For our purposes the following questions are momentous:
- What connection does there exist between views on instruction and
instructional practice?
- How can the teachers' views on instruction be refined, and if need
be, adjusted?
- How could the main features of the theory of realistic mathematics
instruction be more efficiently described to teachers, counsellors,
authors and so on?
A few remarks on the first question. It appears from the research of
Thompson and others, that there is a direct relation between teachers'
views on mathematics and mathematics instruction on the one hand and
their school practice on the other.
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This may lead one, as expressed by
Cooney, to consider whether it would not be advisable to pay more attention in training and retraining to instruction theoretic forming of future
and present teachers-" This could be done by having them work on their
own level and then reflect" on their work, acting didactically and again
reflecting thereon, discussing alternatives to certain courses, presenting
research findings and so on, but also by the exposition of instruction
theory.
We are now concerned with the last of these options, and especially
with the three-dimensional theory description by means of one or more
examples of instructional themes or courses.
Let us start with long division (since this links directly with the threedimensional goal description in Chapter VI) and continue by summing up
the most salient points of the theoretic framework.
4.1 Progressive Mathematising of Long Division
First the preparation for long division: .In the first phase of initial instruction, division is anticipated, namely by way of fair sharing. Indeed, fair
sharing leads to the relations of more, less and equal. It can be evoked on
by counting and grouping. Fair sharing between more people leads to the
questions 'How many does each of them get?' 'How many are left?' What
to do with the remainder? At this stage dividing is not yet a formal
operation. Not until the basic operations of adding and subtracting, and to 266 CHAPTER Vii
a lesser degree multiplying are widely explored, can formal division
become an explicit objective.
Even then division is deeply embedded in context problems which
involve fair sharing, as well as partitioning and inverting the multiplication
represented in the rectangle pattern.
As this continues the field of bare division tables is explored, structured
and gradually memorised, using all kind of reckoning strategies. From the
very start much attention is paid to context dependency of the result in the
case of divisions with a remainder.
Why do we insist on the link between solving context problems (side by
side with bare sums) and the learning of multiplication and division
tables?
Often pupils do not discern the division in the context problem.
Instead, they try more primitive additive and subtractive methods or
multiplicative mixed with additive ones. By using the context problems as
a source, division formally viewed is thereby associated with repeated
addition and subtraction as well as multiplication. In other words, the
association with the other basic operations and their integration makes it
possible to form the concept of division by progressive schematising and
shortening from the other, already known, basic operations. Division
arises as it were via these 'primitive' operations, informal strategies, and
context dependent procedures. On the other hand, pupils can orient
themselves, when structuring the field of division tables, to the real
phenomena of dividing, which is essential in discovering properties.
In this way the horizontal mathematisation of learning to discern the
division in appropriate problem situations and the vertical mathematisation of more and more skilful and shortened reckoning go hand in hand.
Research has proved that, as a consequence:
1. context problems can function as models for operating within the
arithmetic system;
2. formal operations are anchored in the informal procedures used by
the pupils when solving context problems. Both together support
formal arithmetic as well as applications thereof.
Long division is approached according to the same policy of progressive
mathematising. Context problems still cover the first phase of orientation.
The long division scheme is developed by the pupils themselves as an
extreme way of shortening the distribution pattern (see the more detailed
description in Chapter VI). Bare arithmetical problems are a useful
exercise but they also motivate reducing the division 'tail' more and more.
Given bare arithmetical problems, pupils contrive elementary context
situations for distributing and partitioning, which conversely give them
support in bare arithmetic and help them to shorten procedures.
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Finally the long division schema itself explicitly becomes an object of
research. Schemas with empty places or dots in various places are helpful.
Equally important is calculating the remainder If, together with the problem itself, the answer is supplied as a decimal fraction. The remainder
itself is an object of investigation if criteria of divisibility by 10, 5, 2, 4, 9,
3, 6, and finally 11 and 7 are to be established. These, therefore, are
investigations in the formal system on Van Hiele's third level, which,
however, will not be attainable in the primary school or at least, not in its
full breadth and depth. Model situations of fair sharing and partitioning
can even form the background for the divisibility criteria. Attention is then
focused on the remainders in the various positions (10, l 00, 1000, . . .
divided by 2, 5, 10, 9 and so on).
More lines can be drawn connecting division with other subjects:
fractions, decimal fractions, ratios, percentages, averages. Altogether we
notice a diversity of horizontal and vertical mathematising moments and
their mutual links. Context problems are helpful in the gradual development and shortening of the long division scheme. Moreover, they contribute to linking division more solidly with adding, subtracting and
multiplying by means of the informal solutions offered by the pupils. The
phenomenological diversity also penetrates the pro-lem series: division as
distributing, partitioning, as the nverse of multiplymg, particularly 1f
related to magnitudes pr remainders. .
In the first phase, division is preponderantly context-bound, m the
second it is also performed as bare arithmetic within the system, and in
the third the schema of long division becomes a subject of investigation.
These are again the three Van Hiele levels, the phenomenological
foundation according to Freudenthal, and the progressive mathematisation
such as realised by Wiskobas, that is in the present case: mathematising
the procedures and shortening them under the influence of context
problems.
The fact that the standard algorithm is taken as the final product of a
process of algorithmisation rather than as starting point is, in a way,
paralleled by the historical development with its succession of divisio
ferrea, divisio aurea and galley division, each of which underwent processes of schematising and shortening.
24
Here long division is adduced to illustrate the framework for instruction
theory, sketched above. In particular, it reflects the five tenets of instruction directed towards progressive mathematising. The pupils themselves
construct the algorithm while starting from elementary context problems
and a notational schema of distributing, developed by themselves, and at
certain stages in the course they produce bare and dressed-up problems of
their own; the intertwining of division with the operations and other
activities is unmistakable, and interactivity is an indispensable prerequisite
for schematising and shortening. 268 CHAPTER VII
This three-dimensional description of the framework for instruction
theory could be completed by:
- a sketch of the results;
- a comparison with other approaches to long division instruction.
In order to avoid needless repetitions we refer instead to the Mathematical
Material of Chapter VI. As a matter of fact, the brevity of our exemplary
three-dimensional description is a consequence of the fact that we are able
to rely heavily on previous descriptions.
4.2 Two More Examples: Number Systems and Fractions
These two examples are again chosen because they offer the opportunity
of finding support from previous descriptions (to wit, number systems in
Chapter V and fractions in the present chapter). We can, however, give
one more reason: in both cases the pupils' own productions are clearly
seen. This then is the point to which we draw special attention. It appears
that the pupils' own productions in particular are especially suited as a
means for describing three-dimensionally the framework for instruction
theory. Those lessons enable us to look over the shoulders of the pupils at
a long stretch of the course.
With regard to the Land of Eight (see the Mathematical Material as
well as the description of the evaluation lesson in Chapter V) we draw
attention to the following point of theoretic relevance.
- The three Van Hiele levels can be seen in:
1. unreflected imitation of the counting sequence and in its notation,
particularly the carrying over algorithm;
2. consciously using the peculiarities of the decimal position system in
column arithmetic;
3. reflecting on the decimal system via non-positional and other positional systems.
- the phenomenological exploration takes place within mathematics;
- progressive schematising is made possible by the paradigm of Walt
Disney drawings functioning as situation model, and the pupil's own
production is stimulated by the desire to obtain passports to as many
number system countries as possible;
- all kind of micro-aspects of mathematising and its inherent difficulties can be demonstrated by using the pupils' own productions:
the asymmetry of the fingers-hand model in base three, the specific
difficulties of the system of base two, breaking the blockade of the
decimal system in base twelve - an inexhaustible wealth of examples
of progressive mathematising, progressing from level 2 to level 3.
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Moreover, the meaning of reflection on the micro-level (how did I
do it in the Land of Eight?) and on the macro-level (what did I do in
the past when I added columns?) is elucidated.
- by the interactivity attention can be drawn to didactical action, as
well as to levels of didactical action that reveal a certain analogy with
the levels of mathematical action and thought with regard to the
positional system. Mathematisings are, as it were, reduced to the
same denominator, that of the phenomenological level theory.
'Denominator' evokes 'Fractions', our second example in this subsection,
which was dealt with circumstantially in the Mathematical Material of the
present chapter. Again we stress the importance of the pupils' own
productions.
Well then, to start the three-dimensional description one could conveniently present one of the production lessons as an example, though this
might also be a drawback, considering the complexity of the subject of
fractions. On the other hand, there would seem to be no better method of
doing justice to the subtleties of mathematising than just to display pupils'
own productions. As examples we choose more advanced stages of the
course, where larger parts can be scanned to give aft overview of the
pupils' work. For this reason we will restrict ourselves now to the pupils'
own productions at the 'symbolic level, that is problems on decomposing
(l - . . .) and composing (H - - - - -) and will try to ascertain what the
pupils are actually thinking about with these operations.
The viewpoint of instruction theory suggests the following:
- within a group of 11 years olds, say, a considerable difference of
level is manifested in symbolic expression: some children relate the
symbols closely to situations of table arrangement, concrete partitioning or money problems (within the cluster of activities described
in the Wiskobas programme) whereas, for others, the symbols have
become meaningful in themselves. The last group are able to operate
adequately with them in the mathematical system, that is, as they do
with whole numbers, although even in this group there are gradations of abstracting, generalising and unifying. The first group,
however, have not yet (or not yet sufficiently) freed themselves from
the concrete embedding. But in this group also there exist (local)
gradations. A pupil may be able to operate entirely symbolically with
halves and fourths without any concrete representation, while not yet
succeeding with other denominators. Thus the progress of progressive mathematising can in a way be gathered from the pupils' own
productions;
- the various mental representations as concrete background for the
pupils' own productions on the symbolic level can be made the
subject of a discussion on the various aspects of the fraction concept
|
|

and its appearance in reality. Moreover, they can be points of
motivation for making clear the objectives of the phenomenological
exploration within the four clusters of activities;
- the momentous power of 'natural' symbolisations, such as that of
table arrangements, as well as both the kinship and the distinction
between fractions and ratios are elucidated. The last point again
touches on intertwining as a means of mutual support;
- the opportunity for making their own constructions is used in the
pupils' own productions, for instance for adding fractions by using,
among other things, the 'payment model': the 'abstract' search for a
common denominator or the 'concrete' one for an equal table
arrangement, are superseded by the search for an appropriate pizza
price which permits paying for the portions (of pizza parts) in order,
after this settlement, to then convert the price into a fraction. The
pupils can proceed in various ways; the price can vary, as can the
calculation of the price and the relevant table arrangements; prices
can be replaced by areas representing chocolate pieces; some problems are already being solved at the symbolic level.
- in interactive instruction pupils can advance various possibilities and
thereby benefit themselves by reflection;
- by means of a similar outline of levels, phenomenological explora:
tion, moments of progressive mathematising, verification of the five
tenets, a background description of alternative approaches, as well as
exposition of research results, is made possible.
Altogether, in an advanced stage of the course the pupils' own productions supply good opportunities for a three-dimensional description of the
framework for instruction theory in question. We have already given some
indications as to how to perform such a description based on previous
sketches. Under these circumstances a complete three-dimensional description would have given too many redundancies here. Moreover, it would
have been partly superfluous because, as we believe, the foregoing instances
show clearly enough what we mean by a three-dimensional description and
how it can be realised.
4.3 The Most Conspicuous Elements of the Framework for Instruction
Theory Seen from the Viewpoint of Implementation of Instructional
Ideas
The five tenets probably provide the most significant contribution to a
first acquaintance with the framework for instruction theory in question: context-models-constructions and productions-interactive instructionintertwining of learning strands. They can be used to throw light on
mathematisation at micro-level, Van Hiele's level theory, and Freudenthal's

phenomenological starting point. Such an introduction should be illustrated by examples of lessons and courses. Moreover, one can appeal to
the teachers' own experiences when doing the mathematical work on their
own level, on probability, say, where the disadvantages of too high, too
formal, too subject matter oriented approaches can be experienced personally. Difficult schoolwork and extensions thereof can also be useful in
this respect. Another tool might be the parallels between mathematising
and didactising, at micro- as well as macro-level. As a counterpart to
pupils' own production one can propose the design of lessons and formative and summative tests by the teacher. The impact of reflection and
recursion can be stressed for mathematising as well as f-r didactising.
Finally alternatives allied to the other trends could be used to show the
greatest contrast with the realistic trend.
These, then, are the main lines we have followed over the past years
when describing our framework for instruction theory in statu nascendi.
This description can be extended rather arbitrarily of course: the importance of doing mathematics self-reliantly, the reserve the teacher should
observe when explaining things, the accord between teacher's and textbook author's ideas, the importance of organisational measures with
regard to differentiated instiuction, the instrumental or constructive use of
textbooks.
7
An additional point might be the description of general teaching/learning theories and of specific theories which are represented by the four
trends (for text book a-thors this might even be the main point). The
relation of such theories to that of the phenomenological level theory of
progressive mathematisation should then be discussed. This is the subject
of the following and last content-oriented section of the present chapter -
an opportunity to view the framework for instruction theory from another
perspective than the realistic viewpoint.
5. THE BROADER FRAMEWORK FOR INSTRUCTION THEORY
Although both components of the teaching/learning process determine
each other, it would be an exaggeration to consider instruction as a selfreferential system. Indeed, learning exists independently of teaching. And
at school, at any rate, there is a strong mutual dependence. For this reason
we often speak of teaching/learning theories where theory building is
concerned. There are, however, shades of emphasis with regard to teaching and to learning in various theories. We shall discuss them, in particular, with an eye on their theoretical value for didactical action and
educational development. What kind of principles and rules for developing and implementing instruction can be obtained from various subject
matter-oriented didactical theories or more general information processing and activity theories, and how do they agree with or diverge from 272

the framework for instruction theory of progressive mathematising as
presented above?
This, then, is the central question we will try to answer in what follows.
We will start with ideas of Gagne, Dienes, Piaget and Bruner which have
had a considerable influence on curriculum and textbook development
and consequently on instruction of the particular trends we have distinguished. Then we will discuss approaches towards theory building,
emerging among others in the recently established TME group 'Theory of
Mathematics Education'. Finally, we will analyse the more general cognitive psychological research with reference to our central question.
5.1 Gagne, Dienes, Piaget, and Bruner
(l) Gagne and mechanism
Gagne's early work fits into the behaviourist tradition. His cumulative
learning theory consists of eight types of learning. The ordered set of
lower types is the prerequisite for learning the higher ones:
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Problem solving (type 8)
requires as prerequisites;
Principles (type 7),
which require as prerequisites:
Concepts (type 6),
which require as prerequisites:
Multiple d'iscriminations (type 5),
which –equire as prerequistes:
Verbal associations (type 4) or other chains (type 3),
which' require as prerequisites:
Stimulus-response connections (type 2).
It is probable, though not yet proved - Gagne says - that the basis of this
learning structure is formed by the type 1 'signal learning', that is, Pavlov's
classical conditioned response.
Overall the foundation of Gagne's learning theory, that is, the sequence
of the first three to five types of learning, is about the same as that of
Thorndike, Tolman, Hull and Skinner. And these were primarily the types
of learning which appear in many New Math projects of the sixties and
seventies, partly guided by Gagne for designing curricula. Use was made
of task analysis as sketched by Gagne, which starts with the question:
What knowledge and ability is the learner required to possess at the start
of instruction?
This analytic question led Gagne to decompose a complex task into
a hierarchy of simple tasks, which can be interpret;ed as parts of the
complex one. Mastery of the elementary tasks simplifies learning to
perform the complex ones, according to Gagne. The cumulative learning

theory is reflected in this hierarchy: the lower it is in the hierarchy, the
more elementary is the learning type.
The influence of this neo-behaviouristic learning theory on instruction
and, in particular, on mathematics textbooks, is momentous. Not only
numerous well-known projects of 'he sixties such as UMMaP and IPI are
modeled upon this theory, but alirlost all the more recent individualising
programmes are deeply rooted in it. This holds even for more traditionally
oriented textbooks: organisation of the subject matter according to the
principle of progressive complication as we called it above. Both mechanistic and structuralist textbooks display this feature, although the second
kind aims at higher learning types than the first.
It should, however, be acknowledged that the third edition of 'Conditions of Learning' (1977) reveals a drastic change in Gagne's views: the
eight learning types are not even mentioned.
With respect to mathematics Gagne's most recent views are clearly
expressed in an 'Invited Address', published in 1983. Here he presents a
three phase model for instruction in the frame of information processing
theory(!), which should also serve as a guide-line for research.
26
The first phase consists of converting a concretely stated problem mto
mathematical shape, the second of carrying out the mathematical operations, and the third of validating the result.
An example: 19 workers earn 142 dollars a week each. How much
must be paid per weel.< . in total?
Phase 1 consists of translating the problem into 19 X 142, phase 2 of
the algorithmic performance of the multiplication and phase 3 of checking
the result by estimate, say 20 X 140.
According to Gagne, these three aspects should be present in every
task analysis of mathematics instruction. He arrives at the following
recommendations for instruction with respect to these essential parts.
Firstly, special attention should be paid to 'translating' word problems,
which means a variety of different problems are offered which cover one
particular 'formula'.
Secondly, emphasis must be laid on the ability to carry out automatically certain arithmetic activities, that is, techniques and procedures
which may not require insight. In the case of 19 X 142, Gagne rejects
the use of concrete material, such as sticks and blocks, which neither
simplifies the calculation nor helps automatising, and for the same reason
he rejects exercises in other number systems.
Thirdly, there should be emphasis on checking of the answer through
estimation.
At present, Gagne asserts, too much emphasis is put 'on problem
solving as a more ambitious enterprise', on basing arithmetical procedures
and formulas on insight, and too little on automatising and validating the
result. Altogether, these recommendations clearly reveal the most impor-274

tant instructional principles on which this moderately mechanistic mathematics education is based. But, at the same time, the example chosen by
Gagne himself shows the vulnerability of his argument, since this elementary word problem can be solved excellently by informal methods. There
is no need for 'straight translation' into the formula' 19 X 142. On the
contrary: the informal procedure of adding up (19) notes of one hundred,
(76) notes of ten, and (38) notes of one dollar, taking it together or, if
need be, changing it, can be an excellent start for an msightful course on
the multiplication algorithm, which at the same time links up with informal
estimating procedures. In brief, translating, automatic calculating and
estimating are interrelated to a much higher degree than Gagne suggests
with this example.
Both Gagnt\'s early and more recent views give a fair picture of a
possiBe theoretic infrastructure of mechanistic mathematics education.
Task analysis accounts for the subject matter aspect of this view on
education: the structure of the teaching matter is determined by the
increasing complexity of the task according to subject matter systematics,
that is, of the teaching matter products in the subject matter domam.
According to this course of thought, the teaching/learning process moves
on one level, that of the systematics of the subject matter, and all that
counts are quantitative rather than qualitative distinctions. This, then,
distinguishes this theory from that of progressive mathematising in a way
that is marvellously illustrated by Gagnt\'s multiplication exampJe. Besides,
this example can also be used to compare the results of other instructional
approaches, which at this moment may be left aside.
(2) Dienes and structuralism
Dienes' teaching/learning theory shows a great many similarities to Van
Hiele's, at least viewed formally, since there is a considerable distinction
between them as regards to content.
Dienes' theory rests on four pillars: the dynamical principle, the construction principle, the principle of mathematical variability and that of
perceptual variability. . .
The dynamical principle is certainly the most conspicuous feature, smce
it structures the teaching/learning process according to phases, which m a
sense are similar to Van Hi el e's micro-levels. The first phase is that of free
play in an artificially constructed environment. The second is play controlled by rules using the same material. In the third phase, an isomorphism game is played: the identical structure of seemingly different games is
explored, and the children abstract the common rules which fit the vanous
embodiments of the game. In the fourth phase, the abstract10n is visually
and audibly expressed, in order to be verbalised in the fifth phase. The

sixth phase finally is the formal system with its derivation rules (rules of
the game):
Finally, there will be the theorems of the system, which are the parts reached from the
initial description by using the 'rules of play'.
27
The construction principle is included in this phasing of instruction: the
children themselves construct the rules of the game, carry out abstractions,
and design the various representations of the structures.
Mathematical variability is, as it were, built into the material by the
designer. He can change the number system without affecting the positional idea: by this change of basis the system's essence is put in bold
relief.
Perceptual variability is accounted for by the multiple embodiments of

the abstract mathematical structures. This prevents abstraction from being
tied too firmly to one of its embodiments, and thereby becoming immobilised. Thus far we have presented Dienes' instruction theory from a
bird's-eye view.28 As mentioned above these teaching/learning principles
apply to Van Hiele's level theory as well. Formally viewed, Yan Hiele also
knows this kind of phasing where acting on one level is subjected to
reflecting and exploration in the next phase. Level raising is in Van Hiele's
view also the result of the pupil's own activity. Mathematical variability, if
interpreted (as we did \ere) in the phenomenological sense, is accounted
for by the diversity of phenomenological manifestations of a certain
concept or structure in reality. Perceptual variability can be guaranteed
by multifarious realisations in the context of dramatising, games, word
problems, themes or mixtures of them all as described earlier. However,
where the contents are concerned, there are essential distinctions to be
made between the two theories.
Whereas in the first phases of Dienes' learning cycles the children come
in touch with concretised mathematical structures, with embodiments of
games in artificially created environments, the Van Hiele basic levels entail
an exploration of those aspects of reality for which the mathematical
concepts and structures (can) serve as means of organisation: 'embeddedness' versus 'embodiment'; the naturally organisable versus the artificially
organised matter; eliciting a structuring activity in an everyday life or
physical or imagined reality versus debasing mathematical structures and
forcing them into an artificially created environment.
Because of this essential distinction at the basis of the learning process,
constructive activity, and mathematical and perceptual variability mean
completely different things. As a matter of fact, in Dienes' view, the real
world of and for children is an obstacle to mathematics instruction. A
quotation:
Let us consider for a moment the example of -glc; we need to recognize that the natural 276
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environment in which the child lives does not embrace all the attributes which we consider
as logical. It is necessary, then, to invent an artificial environment.
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Not only for logic is this supposed to hold, but for mathematics as a
whole: the everyday reality is not pure enough, there is too much noise in
it for clean mathematics education according to Dienes. On the contrary,
in the framework of Van Hiele's level theory, reality is or can be the
source of mathematical activity and the starting point for both constructive
activity and what Dienes calls the principles of mathematical and perceptual variability.
Dienes' influence in some domains is certainly not negligible, although
he has apparently been interpreted in the wrong way. It is true that
the most important impulses in structuralism come from subject matter
systematics. But, whereas Dienes cares about the meaningfulness of
symbols, technical language, logical patterns and strategies, this is no
longer true of structuralists in general. On the contrary, they beheve that
after a short concrete introduction to the mathematical entities in question, the formal approach enables the pupils to build the mathematical
structures. The learning cycle as used by them is nominally 'exploration -
concept introduction - concept application'.
30
In the usually short introductory phase they use the materials, schemas, diagrams that have been
publicised by Dienes' work; but this way of working has in fact little in
common with that of Dienes, as little for that matter, as the sequence of
'actions first, then pictures, finally symbols' has to do with Bruner's view,
although it is ascribed to him. Let us, however, disregard this for a
moment.
(3) Piaget, Bruner, empiricism and mathematics education in general
To a considerable extent Dienes has borrowed his ideas on concept
formation from Piaget. But the system of stages perceived by Piaget in the
development of thought in general (thus admitting all possible contents)
during certain general age ranges, was applied by Dienes to concept
formation and, for that matter, both in the descriptive sense (i.e. learning)
and in the prescriptive one (i.e. teaching) - a transplantation of patterns,
which one also encounters in Bruner, Van Hiele and others.
Piaget's reserve with regard to influencing cognitive development by
instruction is well-known - we won't go into details, no more than we do
with the general characteristics of his own theory of development. Nor is
there any need for arguments stating that for Piaget himself it is sci-n
tifically absolutely illegitimate to transpose macro-levels of cogmnve
development into micro-stages of concept development - mdeed it flatly
contradicts any age-bound phasing. We would prefer to point out some
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consequences of Piaget's ideas for mathematics education, as stated by
Piaget himself.
According to Piaget, the first and central problem of mathematics
education is how to bridge the gap between natural, unreflective thinking
in a certain domain on the one hand, and the conscions reflective thinking
on structures in the same domain and its mathematical symbolisation on
the other. In general he remarks that the acquisition of knowledge, in the
mathematical domain as well, is operational. This means that children
don't learn primarily by transfer, but by construction of knowledge via
concrete action and operations. For this reason, instruction should be an
opportunity for children to form ideas themselves. Here one should,
however, account for the fact that children pass through certain phases in
their cognitive development, starting with that of concrete actions and
finishing with that of formal operations. Therefore, according to Piaget,
there is no point in bringing children into contact with the formal system
and the underlying course of thought until they have sufficiently experienced the underlying concrete operations and ideas. In brief, one should
take care in instruction when introducing technical symbols and laying
down the logical carpet of reasoning conditioned by the systematic subject
matter. Not until the solid experimental basis of concrete operational
thought has been laid, can formal operations thrive, as-is required in the
mathematical-logical appr,oach, by defining, reasoning and proving. For
the ability of reflecting: in particular, the learner needs a rich fertile
soil of concrete-operational structures in order to pull himself up to the
higher formal-operational level of the systematic subject matter dependent
method - this is Piaget's 'conditional' recommendation for (mathematics)
education.
In his later work Piaget also indicated the importance of social interaction between fellow-pupils. Diverging opinions of members of the group
can induce the pupil in the long run to reflect on his own ways of thought
- this is Piaget's version of what now is known as the socio-cognitive
conflict.
Piaget warns against tying 'concrete', in the context of thinking, too
closely to objects, that is, perceivable or visible outward patterns, and so
on. Operations, even concrete ones, are mental actions with certain
characteristics, which are not tied to handling material objects. In
(concrete or formal) operational thinking - rather than being at the mercy
of acting on materials - is allowed to mentally keep at a distance from it,
to raise oneself above it. Identifying 'concrete' with 'palpable', concrete
operational thinking with acting on material is not allowed - an identification that seems to occur in (initial) empiricist mathematics instruction.
For an even more profound reason we may mention Piaget here in
relation to the empiricist persuasion because of this reserve with regard to
instructional influence on cognitive development, which is also found in 278
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the empiricist practice. It is true that practicians of this investigationoriented instruction offer pupils a mathematical, copiously structured and
stimulating environment, but their anxiety with regard to cautious maturing of (mathematical) structures ends up being at the expense of deliberate
vertical mathematisation, as earlier mentioned. This empiricist practice is
(most often implicitly) motivated by Piaget's ideas: there is no point in
trying to accelerate cognitive development towards (formal) operat10nal
thinking on mathematical concepts and structures, since instruction cannot
come to grips with it. Not until the age of about twelve are children ready
for a more abstract, formal, subject matter orientation on mathematics.
It should, however, be remarked that Piaget's influence extends beyond
the empiricist persuasion to the other ones as well, albeit indirectly and
not always in agreement with Piaget's own intentions.
We have already mentioned Piaget's influence on Dienes and Van
Hiele, and this is not so very different in the case of Bruner either: his
well-known gradation enactive-iconic-symbolic (e-i-s) is connected to the
phases distinguished by Piaget as pre-operational thinking, concrete and
formal operations of thought.31
The triad e-i-s regards the ways of mental
representation (coding, processing, storing) used by children consecutively
in the process of fanning the concepts they learn; the lower form of
representation lays the basis for the higher one in the process of concept
attainment.
For use in instruction, the e-i-s sequence is translated into the didactical
requirement that instruction has to be organised in a way that the concepts
to be taught can be represented first enactively, then iconically and finally
symbolically - a sequence that widely agrees with Dienes'. Bruner's
didactical triad has (had) a gigantic influence on mathematics education,
and in research, too, the so-called 'action-image-symbol sequence of
mental activity' (has) enjoyed great popularity. Actually, research has
shown that the didactical guide-line of action first, then picture and finally
symbol, allegedly drawn from Bruner's work, is too simplistic a representation of instructional matter.32 As an example consider column arithmetic:
the connections between the actions on various levels, the completeness
and extension of arithmetical procedures, their verbalisation and representation, the relations between the procedures and the positional system,
the ability to refer to elementary context situations (such as adding up
monetary quantities), partition of objects, shortening of procedures, reflecting (and learning to reflect) are only a few examples among many that play
an important part and are not accounted for by what they consider as
Bruner's requirement. Actually, the same holds true for the cognitive
representation form as such.
This ends a sketch of Piaget's and Bruner's ideas which have influenced
(and still continue to influence) the development and implementation of
instruction, in particular via textbooks where the instructional implications
of theoretical views are elaborated, albeit often in a simplistic way.
FRAMEWORK FOR INSTRUCTION THEORY 279
(4) Conclusion
We have briefly sketched a few 'classical' theoretical frameworks used as
infrastructures of mechanistic, structuralist and empiricist instruction. That
is to say, instruction as documented by textbooks and, since textbooks are
the most powerful determinant in actual mathematics education, one may
conclude: everyday instructional practice.
Only incidentally have we touched on the question of agreement and
contradiction with the teaching theory of progressive mathematising, and
when we actually have done so, it was from the viewpoint of Van Hiele's
theory, which forms a part of it. Notwithstanding disagreements with
regard to contents there were important points of agreement in the formal
sense, that is, with respect to phasing, instructional principles, the central
role of reflecting and that of building structures of knowledge.
5.2 Recent Investigations of Subject Matter in Instruction Theory
The great variety of recent theoretic views is to an extent reflected by the
large number of terms used in various theories in the teaching/learning
process - terms that ind}cate ideas or concepts, such- as schema, script,
frame, nncro-world, domain of subjective experiences, knowledge structure, conceptual model, -<:bnceptual field etc. From this variety of theoretic
frameworks we will choose a few which are significant for developing and
realising mathematics instruction on the level of textbooks and didactic
action. This issue is inspired by subject matter didactics or by mathematics
instruction theory - in the next section we will view this from the general
psychological or instruction theoretic perspective.
As. in the preceding section we will touch upon the points which agree
and disagree with our own theory only incidentally and incompletely; most
often they are self-evident. Where this is not the case we take Dienes' four
instructior principles as a basis, because they can also serve as a comprehensive means of description for realistic instruction theory.
As an example of a cognitive science approach we choose Davis' work.
Bauersfeld's will illustrate interactional theory. The conceptual modeling
view will be exemplified by Lesh's ideas.
In no case can we speak of the approach because several approaches are
feasible within anyone viewpoint. Even in specific project bound research
such as that in which Lesh participates, various theoretic frameworks can
be distinguished. (Compare in this respect Behr, Lesh, Post and Silver,
1983, with Lesh, Landau and Hamilton, 1983, that is the Rational Number
Project with the Applied Mathematical Problem-Solving Project.)'
We choose these three examples, firstly because they are considered in
general as rather important studies which have attracted wide attention
secondly because their aim is a general theory within mathematics instruc'.
tion rather than restricted to certain aspects of the teaching/learning 280 CHAPTER VII
process or to certain subjects, as are many practical theories, and thirdly
because of the diversity of frameworks for instruction theory they represent. The last argument requires some comment: they represent the
theoretic starting points of information-processing and related views as
well as the more reserved scientific attitude towards the use of the
computer as metaphor for the teaching/learning process.
Once more we stress that these theories are primarily dealt with from
the viewpoint of their implications for developing and realising instruction.
This means that we disregard the sometimes overemphasised aspects of
learning theory; these tend to impose themselves, in particular, when
models of mind are viewed microscopically, as may be the case where
information processing and concept modeling are concerned.
(1) Davis' cognitive science approach
Davis distinguishes two views on teaching and learning mathematics, the
formal-mathematical view and the cognitive science approach.
They confront each other as follows.
i. The 'Notational View':
Mathematics deals with symbols

written on paper
with the manipulation of these symbols
according to rules
that are told to the students beforehand
by using 'na-ral language' (ordinary English, ordinary German, etc.).
2. A newer 'Cognitive Science' View:
Mathematics is a matter of ideas
which a student builds up in -s/her mind
(and this is something only the student himself/herself can do).
These ideas come from experiences,
and are not primarily coded in natural language statements.
New ideas are built up from ideas the student already
has in mind,
by combining them, revising them. etc. often in a
metaphoric way.
Effective learning requires not merely 'doing something',
but also reflection on what you have done after y'ou
have done it.
Therefore an important responsibility is to help students develop appropriate metalanguage and 'meta'-notatiorn.
34
In the above description Davis uses the term 'idea' where in the theoretic
framework (our terminology) of his cognitive science approach he would
speak of 'frame'.
35
A frame is a stable concept or, as Pavis would say, an
information representation structure. In the terminology of computer
programs it is a subroutine of a higher level programme. Examples of
FRAMEWORK FOR INSTRUCTION THEORY 281
frames, in Davis' sense, are the function concept, the division algorithm,
the negative number - that is, such as understood by the pupils, accordmg to their ideas and representations.
What matters in instruction is building mental representation structures
that are stable, meaningful and useful, which is often not quite that easy.
How adequate frame-building should be realised can be made clear by
means of two layers of mathematical thinking that he distinguishes: the
calculator layer and the layer of experience. In the first layer - that of the
calculation world - symbols, operations and routines need not necessarily
be meaningful. In the second layer, on the other hand, they are taken from
concrete experience and therefore must be meaningful. The so-called
paradigm teaching strategy, which has a central place in Davis' theory,
runs at founding the activity of operating mathematically within the
abstract-symbolic layer upon that of the experience-bound knowledge.
As an example of a teaching strategy Davis uses the introduction of
negative numbers via the pebbles-in-the-bag paradigm. There is a bag with
pebbles and a pile of single ones. A problem like 4 - 10 = . . . is assimilated to 'first 4 into and then 10 out of the bag, thus 6 less than before',
which yields the answer 4 - 10 = -6. The assimilation paradigm of the
pebbles-in-the-bag is:
- a forceful representation
- which procures 'an accurate isomorphic image of additions and
subtractions of positive integers
- tells how pupils should solve the given problems
- and is simple.
In general, Davis remarks on assimilation paradigms:
Given the studies of recent years, however, it has become clearer that learning is primarily
17etaphoric - we build representations for new ideas by taking representations of familiar
tdeas and modifying them as necessary, (. . .)
36
Examples of forceful representations according to Davis, are among
others: fair sharing, splitting, larger, further, straight, rectangular. Therefore, activities and situations formed around these concepts can function
excellently in the paradigm teaching strategy, according to Davis.
If we consider the implications of Davis' view on mathematics education_ -om the perspective of Dienes' four instructional principles, we can
put 1t m the following way.
The mathematical variability is structuralist rather than realistic: the
pebbles-in-the-bag paradigm is envisaged mainly because of its value for
calculating bare sums. Davis does not notice that such a paradigm could
possibly block the applicability of the envisaged structure. In other words,
not much attention is paid to the various manifestations of mathematical
structures in reality and their relation to the use of assimilation paradigms. 282 CHAPTER VII
With regard to perceptual variability Davis refers to Bruner's modes of
representation. However, when emphasising the use of experience he
refers to situations such as fair sharing, which can, indeed, be represented
mentally quite clearly, although it is not quite clear to which mode of
representation these real scripts might belong.
The meaning of 'meaningful' is sometimes structuralist (being able to
insightfully manipulate Dienes blocks in relations to arithmetic) and sometimes realistic (situations in which column arithmetic is applied), but more
often it is the former.
The dynamical and the construction principle are also cornerstones of
Davis' instructional framework, as appears from our first quotation. With
regard to contents these principles are determined by the moderate structuralist basic view on mathematical instruction with realistic influences
- at least according to our interpretation of Davis' cognitive science
approach to mathematics education.
Non-cognitive and interactive aspects are hardly mentioned by Davis.
They will be dealt with in (2).
Another information processing view on a theory of mathematics
education inspired by subject matter didactics, but with the 'expert system'
as central concept, is Nason and Cooper's, and with 'microworld' in the
centre Lawler's, about which we will say more in a moment.
37
In general the computer metaphor is not widely used in theorising
about mathematics education. And as far as it is done, one most often
subscribes to Kilpatrick's statement:
To conclude that the computer is the last metaphor for the mind requires the assumption,
first, that the computer will not change in its nature and, second, that cognitive theory m.ust
be computational. To conclude that the computer offers a complete metaphor for the rmnd
requires the assumption that all knowledge can be reduced to information and all wisdom
to knowledge.
38
(2) Bauersfeld's interactional theoretical framework
Bauersfeld, Krummheuer, Voigt et al. are busy developing a theory of
learning and teaching mathematics where emphasis is laid on the social
interactive component of the teaching/learning process - an interactional
theory.29 Within \his framework Bauersfeld's construct of 'domains of
subjective experiences' is central.
40
The meaning of this concept can be described very well by means
of Lawler's psychological interpretation of microworld (which differs from
Papert's didactical one).
41
A microworld is a cognitive structure in the
shape of a microcosm, but then a domain-specific structure, which is- quite
different from Piaget-style general cognitive schemas where the thinking
operations are considered to be independent of the contents. A microFRAMEWORK FOR INSTRUCTION THEORY 283
world organises the knowledge of a certain domain. For instance, young
children may possess various numerical worlds - a counting world, a
money world, a paper-and-pencil calculations world - which initially may
be clearly separated from each other. As a consequence, they might be
able to solve problems set in the money context (such as 25 + 35) and
unable to solve bare calculation sums (2 + 7), and conversely.
It is a general characteristic of these microworlds that they are not
governed from above by a central authority, a homunculus, in order to
find problems to work on but that they work autonomously and competitively in a kind of self-governing system. Or, as Hatfield formulated: 'a
child's microworld may be less like a parliament where some executive
officer attempts to keep order, and more like the floor of a stock
exchange, where the loudest voice gets heard.'
42
It depends on the context
in which the problem is set and on the relevant knowledge structure of the
microworld in question.
Different microworlds can be connected with each other in order to
cooperate rather than to compete. This requires shaping a new microworld, while however, allowing the old ones to subsist: the components
are, as it were, coexistent with the composed world. In creating microworlds the learning process is cumulative. -,
Within the connections made between the microworlds a kind of
hierarchy develops: mi'c-oworlds are constituted, connected and integrated
(to become new microworlds), while the relations of one layer become, as
it were, the thinking object of the next thanks to the active and reflective
engagement of the learner - the same genesis as that of the Van Hiele
levels, as Bauersfeld rightly remarks.
How are Lawler's microworlds and Bauerfeld's domains related to each
other?
One could state that the Bauersfeld's domains may be considered as an
extension of Lawler's. A domain of subjective experiences includes the
totality of subjectively relevant experiences in a specific class of situations,
that is not only the cognitive dimension of knowledge but also emotional
and motor experiences.
On the other hand, in a number of respects they can be described as
the systems of microworlds as understood above: they are also determined
by the domain specificity of the subject matter, cumulatively ordered and
in principle competitive. They do not allow themselves to be subdivided
into separate domains of lower and higher sorts of learning (knowledge
and metacognitive abilities), or into different experiential fields of knowing, feeling, willing - in other words, they are holistic. They can intertwine
with each other in a new domain. The active and reflective engagement of
the learner determines his learning, in this case the creating of new
domains. It is highly stimulated by social interactions. To quote Bauersfeld:
The subject does not form concepts as internal images or through copying other people's 184 CHAPTER VII
activities. The subject actively constructs meaning and relations through social situations of
interaction and negotiation.
43
Altogether, it appears that this model of domains of subjective experiences, which aims at understanding the coming into being, the nature and
the course of learning processes, sheds specific light on learning concepts
such as transfer, abstracting, generalising, visualising etc. In what follows
we will in particular consider a few consequences for instruction.
Firstly, the importance of social interaction has already been discussed.
It highlights the crux of the role of the teacher as mediator where new
domains are constituted and as a stimulator where meanings are conveyed,
engagement is activated and connections are made in interactive instruction.
Secondly, and partly as a consequence of the first point, the formal,
subject area oriented structure of the teaching/learning process is put
sharply into perspective and reduced. There is one more reason for this
within the domains model: only by the intertwining and integration of
domains (into new domains) are abstractions and generalisations made
possible. In terms of instruction theory this means first: constitute rich
context-bound domains, then organise them first locally and then globally
by means of the earlier mentioned active and interactive processes of
reflection (let me use this level theory formulation). Here is a concrete
example taken from initial instruction to illustrate this recommendation:
develop money arithmetic, elementary applied problems, mental arithmetic, column arithmetic etc. in an optimal way within given contexts and
do not restrict your activities by the girdle of the stepwise subject area
oriented structure of the world of numbers. Then, at an appropriate
moment, tie the strings between these arithmetic domains.
Thirdly, a conclusion drawn by Bauersfeld from the domain model,
which is of course connected to the previous ones, generally runs as
follows: allow informal subjective approaches also within the subject
matter system, and beware of aiming for standard methods, otherwise you
may ruin the fertile soil of learning. Or, formulated in the language of
domains: by repressing the informal element and imposing the formal one,
domains are isolated and cut off from fundamental experiential domains,
and too little opportunity is granted to holistic 'coincidental' learning of
knowledge, abilities, strategies and attitudes.
Fourthly and again closely connected to the preceding points, not only
does Bauersfeld on the grounds of his theory, show a certain reserve
towards the formal structuralist view but also structuralism of the Dienes
persuasion. This kind of instruction can, no doubt, fairly quickly claim
success, but this is, in fact, an illusion. It is, according to Bauersfeld, stenle
instruction, with no opportunity for interaction and communication, isolated from reality, and therefore from the fundamental domains of subjective experiences.
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Fifthly and last among the instructional implications of Bauersfeld's
domains model, which aim specifically at learning difficulties and remedial
instruction. Rather than founding diagnosis and remediation on subject
area logic and mathematics, Bauersfeld concludes:
Investigating relevant subjective experiential domains by means of long term case studies at
preschool age (and after) would lead to more trustworthy appreciations of, for instance,
what --nceptual distinctions are also subjectively important, what connections of meanings
are difficult or rather obvious or even self-evident, what kind of experiential domains are
appropriate as up to date foundations and which contextual contradictions could possibly
obstruct their tying up to the mathematical concept.
44
The kind of instruction Bauersfeld is aiming at, strongly emphasises:
subjective experiences and the development of comparing and connecting
domains by means of reflection, incited by interactive processes. In brief,
we may rightly conclude that the model of domains of subjective experiences generally leads to realistic mathematics instruction as described in
the preceding chapters.
So far, we have sketched Bauersfeld's attempt at forming an interactionalist theory. We should, however, mention that in particular with respect
to interactionism in the teaching/learning process, other theoretically

oriented research is being carried on in the mathematical area. First,
Brousseau's work ,on the crux of the so-called didactic contract for the
analysis and construction of teaching/learning situations.
45
Then, Balacheff
and Laborde's studies, which pay attention to the importance of the pupils'
own productions in a communicative context and to the resulting sociocognitive conflict.
46
Finally, Bishop's placing of interaction within the
broader context of the social dimension of mathematics education with its
cultural, societal, institutional, pedagogical and individual levels' and his
call for further development of this area of research in mathematics
instruction.
47
Bishop and Goffree also stress the importance of interactivity, and of communication and negotiation in the classroom.
The last paragraph was meant as additional information with regard to
an important aspect of Bauersfeld's and related work.
(3) Lesh 's conceptual model
In studies of Lesh et al. the theoretic construct of 'conceptual model' is
central. In this construct the following components are distinguished:
- a concept is decomposed into subconcepts; for instance, where the
number concept is concerned, this could be the numerosity aspect,
the measuring aspect; the ordinal aspect, the calculating aspect, the
label aspect; or where the concept of fraction is concerned, it could 286 CHAPTER VII
be the operative aspect, the part-whole aspect, the ratio aspect etc. -
or, in general the various structural manifestations of the concept in
the reality: the within concept networks;
- the relation between these within-concepts and those with related
concepts - such as fractions - are related to ratio, division,
geometry, decimals, magnitudes: the between-concept networks;
- the embedding in various systems of representation: concrete materials, pictures, symbols, real world situations; the system of modeling
processes for problem solving, that is, the way to translate problems
into mathematics, solving within mathematics and feeding back into
the original situations.
48
In this way a conceptual model filters the information couched in the
problem and organises and transforms its data. Lesh et al. are not only
concerned with the identification of the subjective conceptual models but
also with the search for the mechanisms that influence their origination
into being. This is, indeed, a typical learning theory concern: not only is
the status quo described but also how to model the students' modeling.
In the affirmative part of the research on conceptual models and on
the corresponding applied mathematical problem-solving processes, the
most striking results on rational numbers can be generally formulated as
follows:
- there is an 'instability of the subjective conceptual models': a minor
change in the problem can have a major impact on performance
with, as a consequence, a major regression in conceptualising in a
(short) period where instruction is not oriented towards the concept
in question;
- the external mode of representation reveals nothing about the degree
of difficulty of a problem: problems which are posed in terms of
concrete material are not necessarily easier than structurally equivalent ones which are formulated in the writing of symbols (possibly
surprising for Brunerians);
- representing a problem can be a quite active process in which pupils
alternate between modes of representation (manipulative models,
pictures, symbols, real scripts) and use several ones simultaneously;
- the internal representation influences the external representational
expression and, conversely, the externalisation allows the pupils to
refine the internal representation: a cyclic process of internal and
external structuring;
- pupils are inclined to work straightforwardly towards the precise
answer; there are few instances of estimates or approximations;
- pupils do not care whether different procedures applied to the same
problem lead to different answers (this recalls the 'microworlds' and
'domains of subjective experiences').
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All in all Lesh states that pupils can better be characterised as modelers
than as processors. And as far as processing is concerned, it is a processing of the conceptual models themselves rather than that of rough data.
According to Lesh, this is one of the reasons why problem solving and
concept formation should not be considered separately - 'thinking is
domain specific, says Lesh.
Passing from these theoretic facts to the constructive part of his
research, Lesh emphasises the role of the focus on real math real
situations and real pupils. In brief his choice is a realistic starting -oint.
Let us quote:
It is not necessarily the case that students first learn an idea, then add some general
p.roblem-solving processes, and finally (if ever) use the idea and processes of real
situations, that is, those in which some knowledge about the situation is needed to supplement the underlying mathematical ideas and processes.
We believe that applications and problem solving should not be reserved for consideration only after learning has occurred; they can and should be used as a context within
which the learning of mathematical ideas takes place.
49
Extensive research undertaken by Lesh, Behr, Post, Silver, Landau,
Hamilton, Wachsmuth and others has not yet made plausible that this
starting point yields better instructional results than other systems. One
may even state that a long term developmental process of realistic
mathematics instruction was not (yet) possible in this broad area within
the various projects." Nonetheless, the results of the affirmative research
as presented above and of the micro-constructive research are highly
valuable.
Lesh also mentions an experiment in computer-based instruction with a
realistic starting point as explained above:
. . . we have compared groups of students receiving a traditional 'teach first, apply later',
instructional approach to groups receiving a computer-utilities-based 'application first,
conceptualization refinement' approach. The results have shown that, for comparable
amounts of instructional time-on-task, the utilities groups consistently have outperformed
the control groups; and they have done so not only, on 'applications' or 'problem solving'
post-test items, and on items focussing on the meanings of the underlying concepts, but
also on 'computation' items associated with the topics - even though the utilities groups
had no practice at all on the pencil-and-paper computations.
50
If, summarising, we compare Lesli's conceptual model view with the theoretic framework of progressive mathematisation, we may ascertain a
close agreement with regard to the starting point of the global phenomenological basis, although more strongly than Lesh we emphases the pupils'
own productions, interactive learning (in which we agree with Bauersfeld,
Brousseau etc.) and the importance of teaching paradigms (as does Davis).
Nevertheless, agreement preponderates. Here we should also mention the
important part that Lesh plays in the discussion on subject-area-oriented
versus more general psychology-oriented research and with regard to the 288
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issue of various methods of analysis, such as task analysis, analysis of
children's cognitive characteristics, information processing analysis versus
conceptual analysis of mathematical ideas introduced by himself. However it would lead us too far away to discuss agreements and disagreemen;s between conceptual and didactic phenomenological analysis. It may
suffice to remark that they complement each other excellently as far as the
first is more research-oriented, whereas the second is based on broad and
profound contemplations from various perspectives and fundamental
thought experiments, and thereby stands at a distance from mstruct1Onal
practice.
The problem solving approach in concept formation, adhered to by Lesh
and others, is also shared by Vergnaud, who in this context speaks about
conceptual fields fitting related concepts and relations.
51
Vergnaud's
standpoint is that concepts develop gradually through problem solvmg
processes; he adheres to the re-invention principle which was mentlOned
earlier in connection with Freudenthal's didactical phenomenology. Vergnaud, however, at variance with Lesh, uses formal mathematical structures
as tools to analyse learning processes, which leads him to statements on
rational numbers such as:
It is true that most general properties of rational number cannot be expressed and
explained unless rational numbers are viewed as pure numbers. This is the case of crossmultiplying, for instance: a + b = c + d, ad= be, a + c = b + d.
What would it mean to multiply a and d if a were an M,-measure and d an
M--measure? But it is true and important that understanding multiplicative structures does
n-t rely upon rational numbers only, but upon linear and n-linear functions and vector
spaces too.
52
In spite of this statement, Vergnaud does not choose a formal point of
view. On the contrary, he turns sharply against structuralism. But by
means of such formally oriented statements he tries to briefly and sharply
analyse and typify the system of concepts and the structural variations in
theirapplicability in reality. . , .
Finally, again referring to rational numbers, we ment10n. Regme
Douady's work where the relation between conceptual acqu151t10n and
technical competence was studied.
53
She uses the instructional method of
'tool-object didactics' and, related to it, the 'interplay between settings'. In
the tool-object didactics she distinguishes six phases which are pas-ed
through after a set of meaningful problems have seen offered, which
initially cannot coll).pletely be solved by the pupils. It runs as follows:
1. available knowledge is applied explicitly
2. this newly acquired knowledge implicitly unfolds and
3. is made explicit and locally organised in order to be
4. globally inserted into the knowledge inventory, from which
5. its applicability is exercised, after which,
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6. the concept is brought in contact with other concepts in complex
situations.
The 'interplay of settings' means, according to Douady, that a certain
problem can externally be represented multifariously (by numbers, letters,
graphs, geometrically etc.) and that dealing with the problem in various
settings can yield unexpected results, new techniques and, finally, a richer
knowledge in the original setting. This is illustrated by multiplication of
fractions in a long term research project which precisely exemplified what
Lesh calls 'how to model the children's models'.
(4) Final remarks
Our presentation of the work of subject-area-oriented researchers has
shed new light on, and contributed more profound understanding of the
theoretic framework of progressive mathematisation. In a random order
we recall:
- the metaphoric character of learning;
- the importance of paradigms;
- the crux of interaction;
- the domain specific aspect of knowledge structures, frames, microworlds, conceptual models, domains of subjective experience or
however one wishes to indicate this 'idea';
- the embedding of problem solving into processes of conceptual
acquistion;
- the virtual independence of mutually related knowledge structures;
- the possibility of connecting and integrating them in reflective
processes;
- the interaction between internal and external representations in
problem solving;
- the instability and progressive structuring of conceptual models.
All these ideas and quite a few others can in principle serve to describe in
greater detail and to understand more profoundly the phenomenological
start, the level theory framework and the five tenets of progressive
mathematising.
It is worthwhile observing that subject-area inspired research globally
bears a non-mechanistic and non-structuralist signature, in contradiction
to the general psychological research to be dealt with in the next section.
5.3 The Almost Complete Absence of Instruction Theory Ideas in General
Cognitive Psychological Research
The implications of general psychological research for instruction theory
can be dealt with rather briefly, because most of it is affirmative learning 290 CHAPTER Vil
theory rather than constructive instruction theory, Resnick and Ford
express their cognitive psychological view as follows:
Instructional processes cannot be greatly clarified and improved unless we can both look
into the mind and chart changes in capability.
54
The primacy of the learning psychology component is above all conspicuous in the attitude towards developmental research via teaching
experiments,55 Macro-teaching experiments with groups of pupils where
the researcher determines the teaching matter sequence are in general not
considered as part of cognitive psychological research - this in contrast
with the activity-psychological work in the Davydov schooL
56
Microteaching experiments, however, that are thought-oriented rather than
curriculum-oriented, fit fairly well into cognitive psychological research,
They are usually undertaken with individual subjects,
Such a dichotomy of thought- and curriculum-oriented research, however, is hardly acceptable for subject area-oriented didactic research, as
we noticed earlier, With Davydov and his followers we are inclined to
believe that, in the domain of mathematics instruction, cognitive development is for a non-negligible part determined by instruction, which includes
the choice of the teaching matter sequence, For them, and also for us, this
means that instruction too should be an object of research, Therefore the

best and most appreciated instructional courses should be implicated in
developmental research. As long as, for instance, for learning the long
division procedures no new instructional environment is created in which
the pupils can themselves develop the standard algorithm, then according
to the realistic conception, no conclusions can be drawn of a teaching/
learning theory character about the part played by elementary contexts in
partition and distribution situations for the performance of mental actions,
the enchainment of informal and formal procedures, the meaning of
schematising and shortening in this chain, and so on - this would be our
subject area-oriented didactic reasoning.
(1) Learning-oriented versus teaching-oriented,'
background of a disagreement
Gal'perin distinguishes two kinds of analysis of the teaching/learning
process: the logical and the psychological analysis, The distinction is quite
useful if one tries to understand what is behind attributing various weights
to the learning and the teaching component in research and theory
building,
57
So, the cognitive psychological theories in general start from weak
mathematical-logical and powerful psychological structures or thinking
processes, whereas according to the subject area didactic persuasion it is
just the other way round - relatively speaking,
50
In other words: -eneral
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cognitive psychology views the mathematical structures primarily within
the subject area systematics, They are abstract in the sense of being
removed from the real manifestations and they are situated on the formal
symbolic level. Although general and comprehensive, as such they are
structurally poor. In particular, most information processing analysis
systems are satisfied with rather simple networks of knowledge structures,
where the structural element of mathematics is more conspicuous than the
phenomenological aspect There is hardly any analysis and research where
the real manifestations of structures and operations serve as source of
concept attainment. In brief, as far as teaching experiments are undertaken, they are mostly structuralistically organised, and thus grafted onto
rather weak contents structures and rather powerful thinking processes,
which in turn are intended to connect different knowledge structures,
As repeatedly emphasised, according to the didactic view, the mathematical content is not exclusively structured according to the subject area
systematics, The mathematical structures are primarily found in the context of real manifestations, There they are no less general and comprehensive than in the structuralist form but, in addition, their phenomenological
structure is richer, As we noticed above, structuralism is hardly espoused
by those who hold that view, More room is left for design in educational
development than is enjoyed in the cognitive psychological camp, since
one is not locked into in the organisational framework of the teaching
matter according t- subject area systematics, Learning long division,
mentioned above, is an example which illustrates the consequences.
(2) Learning-oriented versus teaching-oriented' causes of a disagreement
But why does cognitive psychology lean towards structuralism in contradistinction to subject area didactic approaches?
Well, the reasons have partly been pointed out in the discussion on
microworlds and domains of subjective experiences, Roughly stated, there
are two categories of 'model of mind', According to the first, mind is
unitary and mental operations are essentially domain-independent According to the second there is, as it were, a 'society of mind', Kilpatrick says:
The mind is a collection of partially autonomous smaller minds, each specialized to its own
purpose, that operate in parallel rather than sequential fashion.
59
In this kind of model, thinking is domain specific,
Cognitive psychological research prefers the first kind of model while
the second is the choice of subject area-oriented researchers and theoricians, The model of the all-purpose mind with its context-independent
mental operations fits smoothly, in particular, into the computer metaphoric picture of cognition: the information processing occurs step-by-step
and the information storing takes place at certain places and addresses, 292 CHAPTER VII
just as is the case with digital computers, where the programs are
expressed in high-level language. However, more recently 'It has been
pointed out from various sides that conventional computers are not such a
good model of cognitive processes (take arithmetic on the one hand and
geometry on the other), probably for the reason the brain operates not so
much sequentially as simultaneously, thus as a network of parallel computers which are more specialised in certain domains than a generalpurpose computer, perhaps something - _let's say - . like fifth generat10n
computers. This picture, then, would possibly be a bit more m agreement
with the model of mind prefered by subject area didactics.
60
(3) General cognitive psychological theories and
realistic mathematics instruction
It might seem too far-fetched to reduce the discussed disagreement in
orientation on either learning or teaching exclusively to disagreement on
models of mind. The researcher's or theoricians's unsophisticated view on
mathematics might be highly influential too. If this is correct we should not
be surprised that psychologists (as well as most mathematicians) choose a
structuralist approach. But starting from a general cognitive theory, which
might be either an activity theory or a cognitive psychological theory, _one
should in principle be able to arrive at structuralist as well as at realtstic
developments and actualisations of mathematics instruction. In other
words, in principle no impediments, not to mention obstacles, arise from
these theories - for instance in the case of column arithmetic - in the
way of choosing and investigating applicability more dosely_. .
Take for instance Gal'perin's theory of the stepwise buildmg of mental
actions.- 1 According' to this instructional procedure each mental activity
should be prepared by an activity on material or materialised level. that
already bears all features basic to the performance of the mental actlVity.
Thanks to this mirroring of the nuclear elements on the vanous levels of
performance (material-perceptive-mental) the pupils can orient themsel-es
from the very beginning completely and concretely towards the anthmetical activity. Then both in the course of column arithmetic according
to progressive complication and the course according to progressive
schematisation the performance can take place on different levels. The
difference is that, in the first case the phasing affects every partial instance
separately whereas, in the second, it concerns the totality of the course.
Exactly the same holds for Gal'perin's second parameter: m the first
case shortening takes place in each special instance, in the second
'vertically' across the whole course. .
The two other activity parameters, the degrees of generality and of
mastery, tend to be broadened by the realistic approach: not only column
arithmetic but also its applicablity is affected.
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In Gal'perin's theoretic framework this should not be a disadvantage at
all. Therefore starting from a general instruction theory of the activity
psychology kind, one can arrive at two essentially different teaching
courses. Gal'perin himself would hold logical analysis responsible for this
difference: obviously one can view mathematics in two radically different
ways.62
He would therefore recognise that a general framework for
instruction theory alone does not offer enough points of support for
developing acceptable mathematics instruction.
Indeed, in our example the essence of algorithmising can be found in
the straightforward conveying of insight into the standard procedures, as
well as in the process of schematising and shortening. Gal'perin - unlike
Davydov - leaves this choice to the subject area expert, thereby withdrawing a fundamental part of organisation of teaching matter from
psychological research. But in his own research he often implicitly takes
the structuralist stand although this is not inherent, as we have shown, to
his theory of instruction.
Mutatis mutandis this also holds for the instructional component of the
information-processing approach. Again, the structuralist design of the
micro-teaching experiments and the clinical investigation is not, or cannot
be, justified within the general theoretic framework, although there would
be every reason for not restricting oneself to the psychological component
of the learning. 'Thought-oriented versus teaching-oriented' is no valid
excuse; indeed, such a separation can hardly be maintained against the
earlier mentioned arguments on subject matter didactics.
From an instruction theory perspective one might regret the lack
of new thrusts supplied by general psychology. As mentioned above,
Davydov is an exception, although even his developmental research,
aiming at the developing of scientific concepts, is rather structuralistically
designed. However, we would be digressing too far here were we to
outline in which direction his orientation to both Bourbaki and Lakatos(!)
is leading.
63
(4) Nuances
Here our only intention is to show that general cogmve and activity
psychological theories cannot serve as instruction theories for mathematics. Hardly any rules for developing and actualising instruction can be
borrowed from them - none, at any rate, for realistically oriented mathematics instruction, unless one is satisfied with rules about what should not
be done. As a matter of fact, in this respect and with respect to early
instruction, affirmative cognitive psychological research can teach us
lessons. Striking examples can be found in the studies of Carpenter &
Moser, Riley, Greeno & Heller, and De Corte & Verschaffel.
64
From their
research one general positive conclusion for instruction can be drawn, 294 CHAPTER VII
namely that real situations can function excellently at the start of instruction (developing the number concept). This, then, is exactly the realistic
starting point, which in general is shunned in the micro-teaching experiments of general cognitive psychological research.
5 .4 Overview
Let us pass the foregoing as viewed through the framework for instruction
theory of progressive mathematising in review - short references to the
discussed theoriticians and researchers will suffice. By this sketchy review
we will demonstrate that the present chapter was not meant so much to be
a description of numerous frameworks for instruction theory but rather as
a means to deepen various nuclei of the realistic framework as described
here.
Starting points for mathematical instruction were expressis verbis discussed in the case of Davis' cognitive science approach. They appeared to
be broader than that of structuralistic mathematics instruction.
The framework for Van Hiele's level theory was put into relief against
the background of Dienes' learning cycles, as well as of Piaget's and
Bruner's theories.
In the micro-sphere of mathematising the levels in the learning process
appeared in our sketches of Lawler and Gal'perin.
The phenomenological approach can be deepened through Lesh's
conceptual models and Bauersfeld's domains of subjective experiences,
and by way of contrast, through the work of Gagne and Dienes.
The important part played by situation models (context models) was
recognised in Davis' paradigm teaching strategy, as was that of modes of
representations in Bruner's, Davis', Lesh's and Douady's theories, and the
relation between internal and external representations mainly in Lesh's
theory.
The constructive aspect is present in almost all the theories we discussed, although the pupils' own productions, such as defined by us as one
of the most powerful means of organisation and reflection, was not (or
almost not) explicitly mentioned by others.
Attention to the interactive aspect was paid in Piaget's (later) work, by
Bauersfeld, Brousseau, Balacheff, Laborde, Bishop and Goffree.
Intertwining of courses is found in the theories of Lesh, Bauersfeld,
Vergnaud and Douady and in a general sense, in cognitive theories.
The more recent didactically oriented theories show a trend towards a
kind of interactive realistic mathematics instruction. The more general
cognitive psychological theories, on the contrary, choose in their microteaching experiments a structuralist mathematics instruction. Although this
is not implied by their general psychological standpoint, their models of
mind, with the computer metaphor, emphasis more strongly psychological
FRAMEWORK FOR INSTRUCTION THEORY
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processes than rich content structures. This, then, is one of the reasons
why subject area didacticians react with reserve to the metaphorical use of
the computer for the learning process.
Finally, we point out that the theory of progressive mathematisation as
presented here should properly be viewed in close connectioh with its
operationalisation in the concrete Wiskobas programme covering a large
part of mathematics instruction from ages 4-12. The theoretic framework
is in a sense a composite of partial theories as developed and described
for a number of courses. Or.I more to the point: it is an attempt on the
third Van Hiele level to arrive at a global theoretical organisation via local
organisations on the second level which, in turn, came into being by reflection on developing and actualising mathematics instruction for the primary
school. Certainly more than is the case with many of the cited theoreticians, we have placed ourselves primarily on the level of educational
development, with the emphasis on textbooks. For this reason, our
analysis of the different persuasions was mainly concerned with the
textbooks and manuals available on the educational market, rather than
with material of certain projects or ideas of individual researchers of
theoreticians. In the present chapter, however, this textbook picture was
supplemented by that obtained from more theoretically oriented studies
related to instructional practice, from Gagne to Gal'p-erin.
5.5 Closure
We started the present study with a sketch of the cube of goals. With
Steiner we believe:
that value and goal orientation should be also important factors of every systems view of
mathematics education commencing with the fact that mathematics itself - like other
sciences - is not a value-free human activity.65
In this goal-space we made a theoretical Lissajous flight in which we
touched all teaching subject faces of the Wiskobas programme. The three
parameters by which this flight of thought was described are Van Hiele's
level theory, Freudenthal's clidactical phenomenology and Wiskobas' progressive mathematisation, operationalised in a concrete programme of

mathematics instruction at the primary school. And we considered this
flight from outside too.
The reason why the goal descriptions were followed by a description of
the framework for instruction theory is found in the great diversity of the
various concrete instructional situations which we described. Their paradigmatic force was not strong enough to evoke an adequate impression of
the curriculum in its totality. For this reason, we passed in review in this
last chapter the general instructional patterns of the themes and the
curriculum, that is, of Wiskobas' framework for instruction theory. (We
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I296 CHAPTER VII
say 'general', because we let the 'specific' instruction theories, placed
within the general framework, about for example, the basic facts, column
arithmetic fractions, area etc. out of view. A detailed study of the
theoretical framework as a whole would take a new volume.) Against this
background the goals themselves might be better understood. In the
chapters on goals as well as in those on theory we applied the so-called
one- two- and three-dimensional description, thus 'three dimensions as a
mod-' of goal and theory description in mathematics education'.
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NOTES
CHAPTER I
1
In this connection see:
Huhse K.: Theorie und Praxis der Curriculum-Entwicklung. Ein Bericht Uber Wege der
Curriculum-Reform in den USA mit Ausblicken auf Schweden und England, Berlin
1968.
For an overview of the situation around 1977 see:
Kapur, J. N.: 'New Mathematics movement, is it over?', International Journal of
Mathematics Education in Science and Technology8 (1977), 259-267.
1
This distinguishing of three directions in mathematics education alongside the existing
pluriform arithmetic instruction connects fairly well with the division made by Dienes and
Williams:
a. the basic-set approach;
b. the arithmetically-oriented approach;
c. the geometrically-oriented approach;
d. the science-oriented approach;
e. the symbol-game-oriented approach;
f. the object-game-oriented approach.
We consider a. and b. to belong tO the New Math and the innovative arithmetic instruction;
c. and d. agree in our view with the empirical trend;
e. and f. are in our division related to the structural direction.
These divisions are idealised, pure types: in reality blends will be encountered in which the
various approaches are found in varying proportions.
In this connection see:
Dienes, Z. P. (ed.): Mathematics in primary education, Hamburg 1966.
Williams, J. D. (ed.): Mathematics reform in the primary school, Hamburg 1967.
Achter, V. van: De modernisering van het rekenonderwijs op de basisschool, 's Hertogenbosch 1969.
Gelder, L. van, Wijdeveld, E. J., Goffree, F. and Krooshof, G.: Modeme wiskunde .en het
basisonderwijs, Groningen 1968.
3
The state of affairs in the "New M.ath" development in the USA is described in a report
by the "Conference Board of the Mathematical Sciences": Overview and analysis of school
mathematics-Grades K-12, Washington 1975.
The most impOrtant statement is: The overwhelming conclusion to be drawn from
these findings is that mathematics teachers and classrooms have changed far less in the
past 15 years than has been supposed." (p .. 77)
For the development in the Soviet Union see:
Maslov, G., Kuznetsova, L. and Leont'eva, M.: 'Improving the Teaching of Mathematics',
Soviet Education 19 (1977), 91-107.
4
Melis P.: 'Wiskunde en de basisschool', Persoon en Gemeenschap 28 (1975), p. 228.
Other sounds can be heard from this direction as well.
See, for example, the extensive report from a Belgian pedagogical work:
Vernieuwde wiskunde in de basisschool, Brussels 1976.
5
Biggs, E. E.: 'Forward and Back. Reflections after a decade of discovery mathematics',
Education 3 (1971), p. 84.
297

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77 quotations

Codes (35)#advantages of using RME, #Acontent where need help to relate to real life, #human activity, #real life, ##RME tasks, #RME, $$Sapproaches to teaching Mathematics, $$content related to real-life, $$learner understanding, $$mathematics education, $$meaningful learning of Maths, $$problem areas to teach/learn, $$teaching methods and strategies, $factors that influence teaching and learning, $learner aspects, $mathematics, $mathematics content, %context, %learner, %role players, %teacher, &&integration across strands, &&learning environment, &&student support, &motivation, &pedagogy, &social aspects, @@challenges when working with ICT, @@ICT systems: virtual learning environment (VLE), @@teacher-student ICT factors, @@technology as educational tool, @@value of using ICT, @Devices, @ICT, @Tools

Memos (0)

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Interviewer: Thank you very much sir, for coming and for being willing to take part in the research. I'm going to ask a few questions, you can answer freely, speak about the experience that you have from school, from any source that you have, as you feel comfortable to do. You said earlier that you are teaching grade eights currently?
Participant: yes
interviewer: Is it the only grade that you are teaching?
participant: Mathematics, it is only grade 8.
interviewer: what sections of the curriculum are most problematic for you?
Participant: At grade 8, it's .... for me, or for the learners?
Interviewer: both, let's hear what is a problem for you and what is a problem for the learners
participants: Uhm, it's algebra
interviewer: algebra, is that for you or for the learners?
Participants: for the learners, I don't have a problem. Learners do have a problem with algebra and integers in fact. If they do have a problem with integers, it is going to give them a problem when dealing with algebra
interviewer: right
participant: then the other one is geometry.
interviewer: okay, geometry
participant: yes. geometry gives them a hell of a problem.
Interviewer: why do you think that is the case?
participant: I think it's because they lack a proper foundation, the problem starts at the foundation phase.
interviewer: okay
Participant: so if they don't have a good foundation, they are going to have a problem at the higher grades
interviewer: so, it's a problem with foundations of geometry basically
Participant: yes. And then with integers, I'm not sure what's the problem.
Interviewer: Hmm. Perhaps it's because it's new to them, when do they learn about integers? do you start teaching them about integers in grade 8?
Participant: Ja, we start at grade 7, but it is just basic.
Interviewer: Hmm
Participant: actually they start at grade 8.
Interviewer: okay
Participant: Grade seven they are just introducing it, then they leave it
interviewer: all right. And for you personally, which sections of the curriculum are problematic for you to teach, or are you comfortable with the whole curriculum?
Participant: at grade 8 I'm comfortable.
interviewer: if you were to teach higher up, other than grade 8, what would be challenging for you then, should you teach higher grades?
participant: which grades?
Interviewer: anything higher than grade 8
Participants: great eight and nine I think I will be comfortable
interviewer: okay
Participant: because I have experience with those two grades, but I have never taught grade 10, 11, 12. So I think ..... I'm not sure which areas would give me a problem in grade 10, 11 and 12. I've never taught it before.
interviewer: okay. Perhaps I can mention a few then you can tell me from your years of studying and so on if there was a problem there or not. Something like probability? Do you know anything about probability?
participant: yes, they do it in grade 9, probability.
Interviewer: okay. Statistics also then?
participant: yes.
Interviewer: you are familiar with that?
participant: yes
Interviewer: okay, what about calculus?
participant: Ja, maybe calculus, I'm not sure about that.
Interviewer: okay, what about financial maths?
Participant: Ja, financial maths, it is also problematic to learners, financial maths
Interviewer: at grade 8 level?
participant: Grade 8 and nine, hmm.
Interviewer: Okay. The next question I want to ask is: how often are you able to relate the content that you are teaching to real life situations?
participant: how often?
Interviewer: Hmm.
participant: (laughs)
interviewer: you can be honest, you can say that you never do it, or you try but you're not sure how
participant: Hmm, there are topics that we can relate to real-life situations, the others they are a little bit more confusing or difficult to relate.
interviewer: True
Participant: So to tell the honest truth, it depends on the topic.
Interviewer: Right, So in line with what you are saying, which sections of work do you think will be easiest to relate to real-life?
Participant: financial maths is easy to relate, exponents will be a bit difficult, and then even integers is easier to relate.
interviewer: okay, any more topics?
Participant: Then, probability you can relate, what are the other topics? Data handling is easy to relate, even measurement is easy. I think exponents is problematic and algebra, functions.
interviewer: okay, so there are a few sections that you think can be related to real-life, like those that you have mentioned already.
Participant: yes
Interviewer: In which sections of the curriculum do you need help to link the content to real-life situations?
participant: Sections of the curriculum?
Interviewer: Yes.
participant: I think the ones that I have mentioned, even integers I think I need help, because learners are not doing well when it comes to integers. Then if they are not doing well, I used to think that maybe I need some sort of development.
interviewer: Good
Participant: And algebra.
Interviewer: Algebra, okay. Then you also mentioned functions?
Participant: Ja also functions.
Interviewer: Okay, good. So even though you do try to relate integers to real life, you still feel you need a little bit of support in that area?
Participant: Yes, because when you are teaching integers, it may look as if learners understand, but if you give them a test, they fail it.
interviewer: Right
Participant: But when you are teaching integers, you may think they understand.
interviewer: I see, do you think it's important to relate your teaching to real-life situations or is it not important?
participant: It is important
interviewer: Why do you say that?
Participant: Because if you relate it to real-life situations, it becomes easier for them to understand and they can also relate other things to the topic that you are teaching.
Interviewer: Good
participant: It is important.
Interviewer: It is important. So in that module of mine that you do at honours, it comes up there relating anything new to something familiar, is quite important. So in that respect, it helps the learners to understand better.
participant: And to remember.
Interviewer: And to remember. How do you feel about using technology in your teaching?
participant: Technology, such as?
Interviewer: Anything, think of what is available to you and even things that are not available to you. Perhaps the overhead projector is available to you, I'm not sure, think beyond that as well. Think in terms of electronic whiteboard, computers, handheld devices like tablets, cell phones, think of any kind of technology
Participant: At present we don't have technology devices at our school, but I think those devices if we can have those devices they are going to help us ... because it will ease the burden ... uhh ... especially of, how can I put it ... uhh ... let me just think about it for a while.
Interviewer: Okay, while you thinking about that, let me ask you this. Of the technology that I have mentioned now, which ones do you think would be the best to use? I realise at schools the technology is not available at the moment, but imagine it was a perfect world and you could get any technology to use in your classroom. Which ones do you think would work the best?
Participant: I think computers will work the best.
Interviewer: Okay, for each learner? Or for the teacher to use?
Participant: Both the teacher and the learner, computers will be the best, if they are used properly. Because they will have a negative effect at times
Interviewer: Of course, anything else? Any other form of technology?
Participant: Except for the? .... okay the tablet. About the cell phones, I don't think it's good
Interviewer: Okay, why do you say cell phones are not good?
Participant: (laugh) I think with cell phones, ... uhh ... you can say cell phones are not designed for teaching and learning, it's just for communication purposes, so if you use them for teaching and learning, I don't think it's a good idea, unless if you want to communicate with them, send them homework, maybe the results, something like that.
Interviewer: Okay, so again a form of communication?
Participant: Yes
Interviewer: This question links to what we have just spoken about: how do you feel about using mobile technology such as Whatsapp, SMS, for you to discuss with me and with other colleagues, possible ways of making the maths more realistic for your learners. So not for your learners, but more a communication between you and me and other maths teaching colleagues?
Participant: Ja, in that way it is good. Like I said, for communication, it is going to work.
Interviewer: Okay.
Participant: Even between the teacher and the learner. If the learner is phoning the teacher to discuss something related to school work, at least in that way it is good.
Interviewer: Okay. Do you have any experience with Whatsapp? Do you use Whatsapp?
Participant: Yes I do use Whatsapp.
Interviewer: How would you feel about using Whatsapp also to be part of a group, to communicate with me and other colleagues about issues about Mathematics. Would you be willing to join a group like that?
Participant: Yes.
Interviewer: Or would you prefer to use it just for your personal use?
Participant: No, I'd be willing to use it with other colleagues to discuss tasks related to Maths.
Interviewer: What experience do you have with Facebook?
Participant: No, I don't like Facebook
Interviewer: Why not?
Participant: (laugh)
Interviewer: Have you used it before?
Participant: No, not at all
Interviewer: Do you prefer not use it?
Participant: I prefer not use it, because even those people who do not know me will invite me even if I do not know them, to join them the Facebook and then everyone will get the information. So I prefer those ways which are a little bit personal.
Interviewer: Perhaps you are concerned about the security issues of Facebook?
Participant: Yes.
Interviewer: Okay, also this is a follow-up question: How do you feel about using Facebook to discuss Maths, issues related to Maths, for your learners, again it will most likely be with me and with other colleagues. Similar to the Facebook group that I've started for the BEd honours group, Would you be willing to do that? Or would you prefer not to be involved with Facebook?
Participant: I would prefer not to join.
Interviewer: Okay, rather something like Whatsapp would suit you better?
Participant: Yes

Interviewer: Okay that's fine. What experience do you have regarding interactive whiteboards?
Participant: Interactive whiteboards? Like the ones which are used at the University? I don't have any experience.
Interviewer: Other than attending contact sessions?
Participant: No
Interviewer: Haven't you attended contact sessions with the presentation has been done on the whiteboard?
Participant: Okay, okay (laugh) it's because the lecturer is there. So I think that one is good. For me it's good.
Interviewer: Okay, you are fortunate enough to attend the contact session at Potch, where the lecturer is. If you were to be a student elsewhere in South Africa and it wasn't possible for you to come to Potchefstroom, tell me how you would feel then about the interactive whiteboard. What you think they would experience it like?
Participant: Hey .... (laugh). I think for them it's good. It's good because they are getting what we are getting, but to tell the honest truth, I won't be happy. It's just like listening to the radio, but it's good. Sometimes we listen to the radio, and we get the information. And even the television, we watch the television and we get that information that we need from the television.
Interviewer: Without being there with the person that is presenting?
Participant: Yes it's good, it's good. We watch a match there, a rugby match, and then you just feel as if you are at the stadium.
Interviewer: Okay, how would you feel about using the interactive whiteboard also to discuss issues of making Maths more realistic for your learners? Again, imagine a group between you and me and other colleagues. If we were to for example meet over the whiteboard so that there might be somebody in KwaZulu Natal, they might be somebody in Polokwane and you at one place and me at another place, how would you feel about communicating through the interactive whiteboard?
Participant: I would feel good.
Interviewer: You would be happy to do that?
Participant: Yes
Interviewer: Any particular reason why?
Participant: It's just like the reasons I mentioned earlier.
Interviewer: Okay
Participant: That you are going to get that information looking at that person, even if it is in the screen, at least you can communicate with that person. And especially sharing those ideas of Mathematics, it will help a lot.
Interviewer: How do you think is the interactive whiteboard different to for example just a telephone conversation? Because that is also a way of communicating? Do you think there is a difference? Or do you think it is just a matter of talking?
Participant: There is a difference, because with a whiteboard, they will be those writings, and with a cell phone it will just be communication.
Interviewer: Good, so you can share documents and many other things?
Participant: Yes.
Interviewer: What type of technology will you personally be most comfortable with in this project?
Participant: In the project ....
Interviewer: To communicate with me and other colleagues, and should you be able to use it for your learners, which type of technology would you feel comfortable with?
Participant: To communicate with you and other colleagues, cell phone is okay, tablet because those are the ones which are available. The others I don't have, they are not available. But if the whiteboard was there I would also prefer the whiteboard.
Interviewer: Okay
Participant: Uh, ja, I think those are the ones.
Interviewer: And in your classroom for your learners? Imagine that there is no, you know, say for example, if it could be provided to you, what would you be willing to use in your classroom for your learners?
Participant: I like that with the whiteboard
Interviewer: The whiteboard? Okay.
Participant: Because if you are prepared, it could save time. You don't have to write again. As long as you prepare in time, then you just put your preparation there and you go on, so a lot of time will be saved
Interviewer: Good, so that'll work for you?
Participant: Yes. Even there the computer also, more or less that is almost the same as the whiteboard
Interviewer: So, you would like to have a computer in your classroom?
Participant: Yes
Interviewer: Any particular reason, other than for the presentation, would you use it for anything else other than presenting to your learners?
Participant: With a computer, I think there are other programmes which might help. When I was doing ACE in Potch, there are those programmes where you will be tested on a certain topic. Then the computer will immediately correct you and show you how to do this problem if you are doing it wrongly.
Interviewer: Good, so there is software that can be used on a computer?
Participant: Yes. It will help the learners, even with speed. Uh, marking, even that immediate feedback, because if you are given feedback immediately, you can try it again.
Interviewer: True. Mr B, is there anything else that you want to add or say? Anything you can think of relating to real-life teaching Maths, and relating to using technology in the classroom? Anything else you feel strongly about?
Participant: Uhm, I think we have said a lot. What about these calculators? Because with the grade eight, let me say grade six, seven and eight, I think that using calculators is not good. Because they tend to ....., how can I put it - to lose those skills, calculating skills or basic skills of Mathematics. Even a simple problem, they want to use a calculator instead of doing it themselves.
Interviewer: True. What about ... I know that you teach grade 8 to 9, but what about the higher standards? Do you feel the same about them and calculators? Or do you feel differently?
Participant: No, no, calculators at least with them, it saves time. Because there, they are doing more complicated things. For example with financial Maths, they have to use a calculator.
Interviewer: They don't really have a choice for certain sections?
Participant: There are so many formulas that they have to use.
Interviewer: Ja, trigonometry also?
Participant: Trigonometry also. You cannot do trig .... okay you can do some basic things, but trig needs a calculator.
Interviewer: Thank you, I think that wraps it up. Thank you very much for your time.
Participant: You are welcome.

P60: Mrs S 15 July 2016.docx

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91 quotations

Codes (28)##advantages of using RME, ##characteristics of RME, ##content where need help to relate to real life, ##real life, #aspects of RME, $$approaches to teaching Mathematics, $$content related to real-life, $$meaningful learning of Maths, $$problem areas to teach/learn, $$teaching methods and strategies, $factors that influence teaching and learning, $learner aspects, $mathematics content, %community, %context, %learner, %role players, %teacher, &&learning environment, &motivation, &social aspects, &TPCK, @@challenges when working with ICT, @@ICT systems: virtual learning environment (VLE), @@teacher-student ICT factors, @@technology as educational tool, @@value of using ICT, @Devices

Memos (0)

Document body:

Mrs S 15 July 2016

Interviewer: Welcome, ma'am and thank you for taking part in the research. Thank you for coming to Potchefstroom to talk to me. I am going to be asking you a few questions, and like I said you can answer freely, talk about your experience at school and even if it comes to something that perhaps you haven't used in your classroom before, feel free to think about possibilities. Perhaps if you were to use certain things in your class, what would you like to do? How would you like to change things? So we are looking at what you are doing currently, but also possibilities for future development. Let me ask you, what grade level do you teach at the moment?
Participant: I'm teaching Grade 12 Maths Grade 12 Maths literacy and Grade 11 Maths literacy and Grade 9 technology
Interviewer: So you have a full load
Participant: Yes, a full load
Interviewer: What sections of the curriculum are most problematic for you to teach?
Participant: Hmm ... Permutations, those factorials. Not data handling, but part of it ...
Interviewer: Counting techniques?
Participant: Yes, counting principles, yes.
Interviewer: That's also new to the syllabus?
Participant: Ja, it's new. I once did it at university, but I couldn't even remember it, but .... We even went to the workshop, but you know those workshops are only for three days. And we were doing so many things - we were doing geometry because they also introduced geometry. But we did it in our times - those theorems.
Interviewer: So they did everything in one workshop?
Participant: Yes, for three days.
Interviewer: Do you feel that it was not enough training?
Participant: Yes, all of us feel that it was not enough training, really it was not enough.
Interviewer: Any other sections that you find problematic? Perhaps you are comfortable to teach it, but you find it problematic, perhaps it's a section that the learners don't cope well with.
Participant: Okay, financial maths. In financial maths - future value and present value, where there is maybe a loan and then there is a gap for three years .... The learners sometimes don't understand that - what should be done there. It means that it is going to affect the value of n. So they struggle with financial Maths, and before and immediately. There is just confusion with them.
Interviewer: How long has that been in the curriculum, financial maths? That has been in the curriculum for quite some time but yet the children battle with it?
Participant: Ja, they battle. And also the children ... Uhm .. mostly the graphs. And we teach them graphs using the calculator, because they have got the tablet, but they can't use it.
Interviewer: Do they have tablets?
Participant: Yes, they have the tablet. We have the interactive whiteboard, but we the teachers are still struggling to use the interactive whiteboard. Most especially for Maths. They say that somebody from the department will come and help us, because if I want to write x squared, then I can't write x squared - I must use that rubber (eraser). There is a rubber that I have to write with. I can't use the text pen to write with.
Interviewer: Really? Are your interactive whiteboards the same as ours or is it a different kind?
Participant: I don't understand whether they are the same, but there are more or less the same. We are writing or maybe I can select a topic, and the children can listen. Then I can stop and explain.
Interviewer: So, are there pre-recorded lessons on the interactive whiteboard?
Participant: Ja, ja. For each and every topic
Interviewer: That's interesting. Tell me, is it part of the department's project?
Participant: Yes, it's part of the department's project.
Interviewer: Tell me more about the tablets that the learners have.
Participant: Every learner has a tablet in grade 12. So, this year it was supposed to be the grade 11's, next year grade 10's, the other year grade 9's. And also in the primary, there are. I don't know which grade they have started, but they have the tablet.
Interviewer: So, each learner has a tablet?
Participant: Ja.
Interviewer: And on the tablet, is there material loaded as well?
Participant: It is loaded. Some of the questions, previous question papers are there with memos, and they can research, they can do whatever they want there.
Interviewer: Do they have Internet access, or is it off-line?
Participant: It's off-line at night, but during the day it's online.
Interviewer: Okay, that's very interesting because for the project that I'm busy with, to know that you've got the whiteboard and the learners have tablets. Perhaps I can assist you with that. Assist you to assist them, so it's very good that you have those facilities.
Participant: Thank you, thank you, thank you. I'm still struggling using the interactive whiteboard.
Interviewer: Yes. Maybe I can help you with some of that. Back to the problem areas, just to summarise: you say that it is financial maths, counting techniques and graphs.
Participant: Ja, and functions.
Interviewer: How often are you able to relate the content that you are teaching to real-life situations?
Participant: Oh .... and also, in calculus. Not drawing .... The last part of calculus .....
Interviewer: Applications?
Participant: Ja, application of calculus. You know these children tend to forget, you know the area, the formulas, the surface area ...
Interviewer: So, the application of calculus is another problem area.
Participant: Ja, maximise ....
Interviewer: Good, how often are you able to relate their work to real life situations, or not at all?
Participant: I sometimes use real-life situations.
Interviewer: Sometimes? Where you can?
Participant: Ja.
Interviewer: Which sections of the work do you think will be the easiest to link to real-life situations? Forget about time for a moment, time can be an issue. What sections do you think will work best if you think of real-life examples?
Participant: Data handling, and also probability, and financial maths.
Interviewer: Calculus?
Participant: Yes, calculus, ja, and also calculus. Equations, no?
Interviewer: It will be difficult. In what sections of the curriculum do you need help to link the content to real-life situations? So, maybe those that you've mentioned now, which of those do you think you need help with to link it to real life?
Participant: Counting principles.
Interviewer: And also those others that you mentioned? Calculus, financial maths, data handling?
Participant: Yes.
Interviewer: What was the other one that you mentioned? I think it was just those.
Participant: Yes, financial maths
Interviewer: Data handling
Participant: Calculus
Interviewer: Probability also?
Participant: Ja, ja.
Interviewer: So, those areas you feel you need help to link it to real-life
Participant: Yes
Interviewer: Do you think it's important to relate your teaching to real life situations?
Participant: Yes, because learners can grasp quickly. They will remember that oh ... I can link maybe this glass, maybe with ... you know .... they can grasp easily.
Interviewer: Good. You know, in the module that I teach we always talk about linking prior knowledge to existing knowledge.
Participant: Ja, ja.
Interviewer: Do you think it has something to do with that?
Participant: Yes.
Interviewer: Because then we are linking the new work to something that they are familiar with.
Participant: Ja.
Interviewer: Good. How do you feel about using technology in your teaching?
Participant: Because I didn't even study or do technology from high school, even from teaching, I'm still struggling with it, I'm really ..... Technology as a subject or you said ...?
Interviewer: More technology in any teaching. So, think of technology in the Maths classroom. How do you feel about it? Are you positive, negative, excited? How do you feel?
Participant: I'm excited, but mostly in Maths there must be a lot of explanation, I can use maybe ..... They can listen, but I feel that if I write then the learner can, maybe I can write some examples the learners can .... ja
Interviewer: Your whiteboard at the moment, you can't actually write?
Participant: I can write, I can't use the text pen, because in our interactive whiteboard we can select the pens that you write with. So, the text pen writes like this and the other pen is my writing.
Interviewer: Okay?
Participant: It's the same like I write on the board.
Interviewer: So that's the problem now because you can't do it properly, the way you want to?
Participant: Yes, I can't use the text pen.
Interviewer: So, obviously because you want to write the sums for your learners ...
Participant: I wanted to write x cubed minus two x squared, I can't write it.
Interviewer: So, you can't write because it's typing it.
Participant: Yes, I must write it. And also ... Um ... like in Calculus maybe I want to draw a cylinder, a nice cylinder, not mine. I want a cylinder that is already drawn, you know ... I'm still struggling with that.
Interviewer: Back to how do you feel ....um .. What emotion do you think comes up when I say use technology in the Maths class? What do you feel?
Participant: I feel that maybe if I can maybe have more knowledge or used to do that, then maybe I can accept it.
Interviewer: But are you willing to use it?
Participant: I am really willing to.
Interviewer: Good, but you think that if you have a bit more training and a bit more help ...
Participant: There was training, and then the department sent someone from them to come. But, it was not only for Maths, you understand?
Interviewer: Oh, yes.
Participant: We were doing so many things, linking those things .... And it was just a short space of time.
Interviewer: Okay.
Participant: But they say in August again they are going to send someone again.
Interviewer: Okay, good. How do you feel about using mobile technology like things like Whatsapp, SMS to talk to me and to talk to other colleagues about how we can make the Maths more realistic for your learners?
Participant: Yes, like with my group of learners, I've got 33 learners for pure Maths
Interviewer: Grade 12?
Participant: Yes. We've got a group Whatsapp. If there is maybe something that they don't understand, we chat.
Interviewer: Excellent
Participant: Ja, or maybe I want to see them maybe next week I prepare them. Or if maybe it is something that I forgot to tell them in class, I Whatsapp them see page this on .... Because in their tablet there are also textbooks, and also on the interactive whiteboard, there are textbooks.
Interviewer: Right.
Participant: Yes, the textbooks that we are using.
Interviewer: Wonderful. That Whatsapp group works well for you to communicate with your learners?
Participant: Ja, but you know ... Our Maths teachers don't use only one textbook. We also use the old, old textbooks, it's very good, I also use it
Interviewer: There was almost more structure in those old textbooks.
Participant: Ja, ja. Those higher grade old textbooks, it was excellent. So, I usually take the answers, then I sometimes take a camera, and then I send it to them.
Interviewer: Do you take a picture?
Participant: Ja, I take a picture, and send it to them.
Interviewer: Okay excellent, that's also a nice idea. So, you actually use technology more than you realise.
Participant: Yes, I use it but ... you know what .... at school there are laptops for Maths. We were four maths teachers, the other one has passed on, so now we are three. So, we have to borrow it and then take it back.
Interviewer: Okay
Participant: Since the department said that every grade 12 teacher must have a laptop, but we are still waiting for them to give that to us, the laptop
Interviewer: For each one?
Participant: Ja, for each one. For each and every one.
Interviewer: So you have to share at the moment?
Participant: Yes
Interviewer: Okay, excellent
Participant: But I'm using, it's not a laptop, I'm using a computer at my home.
Interviewer: Okay, good. And the tablets, do you get to use the tablet? You don't have access like the learners have got?
Participant: No, no. The tablet is only for learners.
Interviewer: Okay
Participant: It's only for learners. But I can borrow one from the office. If I'm using it, then I use it now, then return it again.
Interviewer: Right. And for communication, say in this project, if we start working on things together ... would you be willing to join a Whatsapp group with me and say one or two other colleagues? Would that suit you?
Participant: Yes.
Interviewer: Good. That's the next question, how do you feel about being part ....
Participant: Yes, because it will be proper because there are some other challenging questions, maybe I can Whatsapp the Whatsapp group and they ....
Interviewer: Good, ja and then share ideas also and say I'm having a problem ...
Participant: Yes, or how do I tackle this topic? ..... you know, you understand?
Interviewer: Hmm, nice. What experience do you have with Facebook?
Participant: I'm not, I'm not, truly speaking, I haven't used Facebook. How is it?
Interviewer: I think it can be used in different ways. Obviously the main purpose is for social things, you know, to have your friends and communicate with friends, tell them what you're doing, tell them where you are et cetera. But there is an educational use to Facebook as well.
Participant: Is it? I'm not aware of that.
Interviewer: Yes

Participant: Then I'm going to download it (laugh)
Interviewer: What some people do ... For example I've started a Facebook page for the BEd Honours group, where students can join the page and ask each other questions and sometimes ... I always keep an eye on the page, so I can see if somebody asked a question but nobody was able to help, then I will step in and help. Try and answer the question, or give advice. I might say to the student send me and email and I will forward you some material or whatever the case may be.
Participant: Okay
Interviewer: So that is one way that I have started to use Facebook with my lecturing. One tends to think that it is just social, but there are people that use Facebook for educational purposes. So would you be willing in other words?
Participant: Yes
Interviewer: What experience do you have regarding interactive whiteboards? You've actually answer this question already, because you've got one in your classroom
Participant: Yes, yes, I can write x squared. Sometimes if I'm using the pen, the text pen sometimes it's quick.
Interviewer: Do you think it's better than a normal, old-fashioned whiteboard?
Participant: Yes, it's better because sometimes they advise us if we can have the sticks and put our things on the sticks, then put it there, then maybe everything that I'm going to do, step by step like you are using here.
Interviewer: Okay, good. So, you can prepare in advance and then bring your stick and present the work on the whiteboard?
Participant: Yes
Interviewer: So, there are advantages over a normal old-fashioned whiteboard?
Participant: Yes, very very much. Because the chalk is just a waste of time.
Interviewer: Hmm, because you have to erase, and then it's gone, if you want it again ....
Participant: Ja, that's because with that one if it is gone and you can just take it back and see it again.
Interviewer: Good, so in other words it can save time.
Participant: Yes and I can also save my lessons. I can even use it next year for the other groups.
Participant: Good, so there are advantages.
Interviewer: Yes. You know, I liked the overhead projectors also. I liked it, but unfortunately in our schools we don't have it
Participant: Really?
Interviewer: Ja, we don't have it.
Participant: Why did you like them, how did you use them?
Interviewer: Maybe explaining, writing, then they can do you know, yes ...
Participant: And that could also be more permanent because you could write on transparencies and keep them.
Interviewer: Yes.
Participant: That was also an advantage.
Interviewer: Back to the interactive whiteboards, how would you feel about discussing issues with me and with colleagues on the interactive whiteboards. Say for example, like our classes where we connect three or four different centres, different whiteboards together and you stand at your whiteboard and I stand at my whiteboard and some other colleagues stand at their whiteboards and we could communicate about how to make maths teaching easier for you. How do you feel about that?
Participant: You know, our whiteboard is not like yours, because I've seen you have something there in the ears. I don't know what is that.
Interviewer: It's a microphone, so that other people can hear my voice from other whiteboards all over the country.
Participant: Okay.
Interviewer: You see, ours works like this, and this is what I'm talking about over here. I stand in Potchefstroom, with a microphone and I present a lesson. You sit in the classroom here, but then they are 30 other centres around South Africa that also have whiteboards. They don't see me physically, but they see everything that's on the screen, they hear my voice and they can even stop and ask questions. So that is the interactive part of the whiteboards.
Participant: Okay.
Interviewer: So that is why we have a microphone, because then they can hear my voice clearly.
Participant: Yes okay. It's not like those may be Calculus is there and someone is talking.
Interviewer: No, it's not pre-recorded. It is live.
Participant: Ja, I understand, because in our interactive whiteboards there are pre-recorded lesson.
Interviewer: Oh, so you are used to using pre-recorded stuff.
Participant: Yes.
Interviewer: How would you feel about live, connecting live with other people?
Participant: Yes, yes. I think it is easy, it is not time-consuming.
Interviewer: Okay, what type of technology would you personally be most comfortable with should you get involved in this project with me? What technology would you say, yes, I'm willing to use this technology, to communicate, to receive information, to send information? What are you most comfortable with?
Participant: Like maybe Whatsapp or Facebook. Because ja, I've got Whatsapp, I am comfortable with Whatsapp. Facebook I'm not into that, but as you've just explained, I can do it. Twitter, I don't know how to tweet, I don't know. But Whatsapp, yes.
Interviewer: Okay, so more mobile technology would suit you?
Participant: Yes
Interviewer: Okay, good. Thank you Mrs S, that is all I want to ask you. You have helped me a lot already. You have given me lots of good information. I'm glad that you're involved with the project with the whiteboards and the tablets, because it's going to be really interesting working with you.
Participant: Ja because, the learners, they are not supposed to carry books but we've got many types of books like in Maths. We don't use only the textbook which is on the tablet, we normally have extra textbooks that we are using.
Interviewer: Okay.
Participant: But for the tablets, it was supposed to, they must just come with the tablet not with anything else
Interviewer: Okay
Participant: According to them, even the tests they are going to write on the tablet, then it marks everything. But we are still going to learn. They have just delivered those things, I don't know what is that ....
Interviewer: Clickers?
Participant: Yes, yes (laugh). We are still going to learn more. So it's going to be easy, we are no more going to mark.
Interviewer: Okay.
Participant: So they mark, I give them questions, they mark even the tests, everything. I'm going to just put the learners and then the marks will automatically .....
Interviewer: Who designs those tests? Do you design them or are they also sent by the Department?
Participant: Uhm, even the class work, they are also going to do it there. And the homework, they are also going to do it, then they do it at home.
Interviewer: Right.
Participant: So they say, no more paper for the children.
Interviewer: Okay.
Participant: Yes, as they said, but we are still waiting. Maybe after we reopen, because they came with the clickers.
Interviewer: Let me tell you about the clickers ...
Participant: Ja, the clickers. They say clickers. You click, then "ah ... yes, I've got it correct"
Interviewer: Yes, each learner is somehow connected to the whiteboard. Say for example you put a question on the board, then they can click the correct answer. Then you can ...
Participant: If you are wrong and if you're right, it shows.
Interviewer: That also helps you because say for example you have just taught a section and you put up a question relating to that section
Participant: Then I can see that most learners didn't understand that question, then I can go back ...
Interviewer: Exactly. That is excellent, it is exciting for you.
Participant: Yes, I am eager I am eager you know, to let them come and use that.
Interviewer: Nice. Thank you very much. I appreciate it very much that I can talk to you and I will definitely continue to talk to you. I am going to be designing things to help you, talking, finding out where can I help, what I can help with. I'm looking forward to working with you because I see that you are enthusiastic about your teaching and about the technology and about Maths.
Participant: Yo, I like Maths. My child is doing Maths, but, she is not so .... She's doing Maths, Physics, Life Science and Accounting. She is in Grade 12 now.
Interviewer: Hmm.
Participant: But she is so lazy. She is so lazy. Sometimes, there is this learning channel on the DSTV. I asked her to watch it, maybe she can learn more. I even watch the subject that I don't do, like accounting. Like the day before yesterday night I was watching Accounting. They were explaining and asking questions, and I got it correct. I was so excited.
Interviewer: Nice. That's another form of technology.
Participant: Yes, I always ask my learners to watch it. But, maybe only, let me say 10% of our children have got a good background, they have got a satellite dish. So mostly they are from shacks, where they do not have electricity, you know ...
Interviewer: So, that makes it difficult to encourage them to do that?
Participant: Yes, because I once asked them to watch that learning channel. Then I realised that most of our learners, 90% of them don't have that. Some they have the TV, but they don't have the dish, they can't afford to pay the dish every month.
Interviewer: Those channels, are they just on satellite?
Participant: Yes.
Interviewer: Are there any of those channels on a SABC?
Participant: There are on SABC, ... because the day before yesterday .... On SABC 1 or 2 or 3, I don't remember, they were also doing that topic, explaining the topic of probability.
Interviewer: But, obviously the learning channel is a better one.
Participant: Yes.
Interviewer: Good, you've given me some nice ideas today. Thank you very much. Is anything else you want to add? Anything else you want to talk about?
Participant: Uhm, what is the best way of teaching Maths? Are you teaching Maths?
Interviewer: I used to teach Maths at school, now I'm at the University I am lecturing, which is a bit different to teaching, but there are many things in common. At school level, the best way that I found is you must be well prepared.
Participant: Yes, because they are clever learners. They are clever learners, because I just tell them tomorrow we are going to do this topic. You can even notice that some learners are far from you.
Interviewer: They are not with you?
Participant: They are not behind, they are for ahead of you in the syllabus
Interviewer: Okay. Is that because of the tablets?
Participant: No.
Interviewer: Would they have done that on the own anyway?
Participant: Sometimes I have got some clever learners in the class. I just tell them tomorrow you are going to do uh ... let me say uh ... this topic. Then I ask this one to do this thing, uh or maybe ... I want to recall maybe a topic? Any topic?
Interviewer: Calculus?
Participant: Yes, in Calculus. Because in Calculus you can't do many things in one day. You can't do the limits, the graphs and also the rules and applications in one day.
Interviewer: Hmm.
Participant: So I asked maybe to ... ja like the rules of differentiation. I can ask one learner to do the first rule, the other one to do the next rule. Then I am just watching and I will assess.
Interviewer: So the learners present?
Participant: Yes, the learners present and they present it so well, they present it so well. In the meantime they are also learning, you understand? They are also learning. So, I also tell some other learners they must also do it, because you gain more knowledge by just standing there and going home to prepare the topic to do it. They come prepared and then they do it so nicely. Sometimes, you know, if you are a teacher, if you are a Maths teacher, maybe someone comes in. Because our department, last year, they said, we must also exchange, the teachers must also exchange. I can go and teach at your school a certain topic then you can come to my school and teach. So the learners they are not used to you. They are going to listen to you. Because to me, they are used to me. So, to the other teacher they are not .... They are going to listen clearly and understand maybe.
Interviewer: That in a way a nice idea, peer teaching.
Participant: Yes, peer teaching.
Interviewer: But, I like your idea of using the learners to sometimes show or explain a section.
Participant: Yes, I usually do it, and I am just facilitating and guiding. Maybe if there is confusion between the learners and the one who is teaching, then I can intervene and explain. And maybe I can explain it better, but if I can see that he has left some of the things ...
Interviewer: I like that idea very much. I think we are done. Is anything else you would like to talk about?
Participant: No.
Interviewer: Thank you so much for your time.
Participant: I thought it was difficult, you are going to ask me such difficult questions (laugh)
Interviewer: No. I just wanted to chat and hear what you think, how do you do things, how do you feel. Thank you so much for your time, I appreciate it.
Participant: Thank you Dorothy.

P61: Mr K.docx

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40 quotations

Codes (23)###emergent modelling, #characteristics of RME, ##principles of RME, ##real life, #RME, $$$approaches to teaching Mathematics, $$content related to real-life, $$learner understanding, $$problem areas to teach/learn, $factors that influence teaching and learning, $mathematics content, %community, %context, %teacher, &&integration across strands, &pedagogy, @@challenges when working with ICT, @@ICT systems: virtual learning environment (VLE), @@teacher-student ICT factors, @@technology as educational tool, @@value of using ICT, @Devices, @ICT

Memos (0)

Document body:

Mr K

Interviewer: Thank you for agreeing to meet with me. The purpose of the meeting is to get your ideas about things like the curriculum and especially the use of technology in the Maths curriculum. So, I would like to get your ideas and feelings about some of these issues. So, if you don't mind we can have a look at the questions.
Participant: Okay, I read through all of them.
Interviewer: Good, so then you can speak from experience, what you have worked with, what you have not worked with, and your opinions about these things is actually what I am interested in. What sections of the curriculum are most problematic for you to teach?
Participant: Uhm, ja. In this case I will dwell on grade 11 Mathematics. The curriculum is broad, but here I'm going to talk about content in grade 11 for example. We have a problem in functions, but the problem emanates from the fact that they don't understand for example exponential function - if learners don't understand how to solve exponential equations then when you go to functions it becomes very difficult. Yeah, that's one area, secondly trigonometry. I don't know what is supposed to be done, but learners in general struggle with trigonometry. So mainly its function and trigonometry, but in particular trigonometry is the major problem. Then, recently we have Euclidean geometry which has been introduced. Uhm, it's also a major problem, but I think the problem is not the learners but maybe the majority of teachers who are teaching that, they did not even do it during their times at school. It's something that was brought in as a crash programme for teachers to be taught as well. So, those are the main three areas: Euclidean geometry, trigonometry and functions.
Interviewer: Good, How often are you able to relate the content that you are teaching to real-life situations?
Participant: Uhm, for example, when you are talking about analytical geometry as a topic. There are areas like if you are driving, there are areas like what people call black spots
Interviewer: Right?
Participant: In most cases that had to do with gradient. The gradient of that particular road is not evenly balanced. In mathematical terms that has to do with gradient.
Interviewer: Right.
Participant: Uhm, in Financial Mathematics as well. Through and through it every day life situation where you talk of borrowing, investments and other areas of activities.
Interviewer: Good. Which section the work do you think would be easier to relate to real-life situation?
Participant: It's also.... Everything depends on the cognitive level of the learners you are teaching. For example if you're talking about calculus, maybe up to grade 12, not so many will have pre-requisites of the cognitive level to understand that calculus also works in real-life situations. Only when someone goes beyond grade 12, then they will be able to relate to what they have been taught in grade 12. But calculus on its own, like I've talked about gradient, it also comes in as part of calculus. How it is introduced in grade 11 and 12. It appears as if there is no link but calculus it also talks about gradient and so forth. That is real-life situations.
Interviewer: Absolutely. Do you think important to relate are teaching to real-life situation?
Participant: Uhm, so many writers have written different meanings of what Mathematics is. Some call it a natural science, some have called it a philosophical science, because in a natural science you don't have x and you don't have y, that's a bit philosophical. So, I believe in the prior, who talks about Mathematics is a natural science. So, you cannot do Mathematics without modelling. And, once you talk about modelling, you are now talking about real-life situations.
Interviewer: Excellent. How do you feel about using technology in your teaching?
Participant: Maybe I must am speaking from the background of being a traditional teacher. But how I think is also influenced by the way I was taught myself although I have changed a bit but what I have discovered is that the moment technology takes over everything, the learners lose the naturality of Mathematics. Now, it is only supposed to be used in conjunction with .... For example with Euclidean geometry, the curriculum statement does not give the teacher the opportunity for learners to discover where these theorems are coming from. Even, I happened to use the HeyMaths programme which was in the Northern Cape province. Now, the HeyMaths programme, if you want to show a learner that the angle at the centre is double the angle at the circumference attended by the same chord for example, you can do it by technology. But, it's just like watching a TV. How many learners will ...., maybe some learners can learn by seeing but some learners need hands-on for them to understand. So, now so the word technology the way we are using it is deeper because yes, for example if a learner uses a protractor to draw a circle, draw a radius and so forth, that is using technology, which means they can still have the correct feel of what they are doing. Maybe, the focus now, we have shifted away from that type of technology which we thought was very traditional, to this one that is so electronic. Everything now is very electronic. So, I feel learners have lost a lot because they now don't have hands-on experience, for example what I went through. It is easier to tell a learner by using a computer. When he knows things practically, it is much easier. But what is happening now, technology, we are just telling learners things which they don't have a feel for. That's my take on it.
Interviewer: You think we are neglecting the more practical side of Maths by focusing on the electronic technology?
Participant: Yes, we have moved from this technology which we think is traditional, we have moved out right to electronic technology.
Interviewer: Do you see any benefits of technology in Mathematics?
Participant: Okay, maybe we need to go back to..... We in Third World countries we have borrowed much of technology, especially electronic technology. Looking back to where these things have started, looking at China, Japan. They have benefited a lot, but maybe there are so many things that have to do with their culture. Because, yes, it is them who started it. It fuses well within their cultural confines. Now, what happens now is ... Are we having the same culture of the people who started these things? Our understanding is different. For example you will discover that ..... I am teaching at a school that is almost in a peri-urban area. We have the smart boards, we have the overheads but the teachers moves from one class to another. Therefore, we teachers do not have one class where we can say "This is my Maths class." Though it's not easy for me to move on with my screen, overhead projector, laptop et cetera
Interviewer: That makes it very difficult.
Participant: Yes, and my experience as well is that learners might be fascinated with the use of the computer and the overhead for one or two days, then they will lose interest. That is my experience.
Interviewer: How do you feel about mobile technology, things like Whatsapp, SMS to communicate with colleagues and even with me about any research project in the line of what we are talking about? Are you open to using mobile technology?
Participant: There are so many things which have changed now. We no longer live the way we used to live in communities that used to come together. It is completely different. Yeah, communicating in different forms.... You can go back to learners, you will discover that even at school, they no longer sit down together and talk. Everybody is busy on Whatsapp. It's no longer interesting to be a storyteller. No one will listen to you. Because there are things now that are taking their time. So, I don't think I will be having any problem. I have tried it even with my learners, but I also discovered that yes, you might capture very few who will be interested in ..... Because Mathematics on its own is a subject like for learners - it stresses them physically ....
Interviewer: It's challenging?
Participant: Ja, it's challenging. But you will get one or two that are interested, yes.
Interviewer: What did you do with your learners? Did you have a Whatsapp group, did you make use of SMS? How did you use technology with your learners?
Participant: I had a Whatsapp group.
Interviewer: Okay, good and on the group, you could give them information, or answer their questions? Was that the purpose?
Participant: Yes, like I give them homework, somebody's stuck along the way they don't have any, you know they don't have many resources even to refer to. I will give them one or two tips on how to continue with the problems.
Interviewer: That excellent, I like that idea. Did you find that it worked? Or was the focus not on the right thing?
Participant: It goes back to what I said, there are so many factors within the community which are inherent on how this will work. I did not think that I had enough support from their parents. Because learners in their own, they have a lot of things they need to do. There are so many things that are fascinating, more than doing Mathematics on Whatsapp.
Interviewer: True
Participant: Other social issues to discuss with. But, maybe if I could have involved the parents, it could have worked. So even now, I cannot say that it worked or did not work.
Interviewer: Good, but I like the idea that you have tried. How would you feel about being part of a Whatsapp group to discuss issues about making maths more realistic perhaps with other colleagues, myself, with other researchers. Would you be willing to do it?
Participant: I don't have a problem doing it, it's welcome.
Interviewer: Good, thank you
Participant: It's welcome
Interviewer: Facebook, you have any experience of Facebook in an educational context?
Participant: In an educational context, no, but I have followed LinkedIn. I have not done anything, but I follow the articles which I return, especially .... It seems there is not much writing on the South African context. I rarely find anyone, I think I have one guy who returns comments, but the majority is on the UK curriculum.
Interviewer: Right.
Participant: Even those who are in South Africa normally comment about the UK curriculum, not the South African curriculum.
Interviewer: That's interesting.
Participant: That's what I have discovered.
Interviewer: Would you, how would you feel about using Facebook to discuss issues about making maths more realistic for your learners, as a medium of communication? Or would you feel more comfortable with other means?
Participant: What I have observed is, yes ....., there are some Mathematics problems that are normally flouted there. But what I have discovered those are more of puzzles. But I think that the basic point is to make sure that it will catch the eye of everybody who is at school at the moment. Maybe to adapt the minds, but not suddenly to say you can talk of algebra. But I think much of puzzles, I've tried to follow on Facebook.
Interviewer: Good, you said you have an interactive whiteboard in your classroom. What experience you have in relation to interactive whiteboards?
Participant: Unfortunately at the moment, because we have only one interactive whiteboard in one of the computer labs. So, any specific teacher who wants to use it will have to discuss it with other teachers when and so on. So, I have not agreed to having a class with them because of the congestion of the programme of the curriculum. Because for example in grade 10 we have nine classes, grade 11 around seven, grade 12 we have close to two hundred learners. It has not been easy. But we have used overhead projectors, the interactive whiteboard .... It was only me going there to see how I can use it . Effectively, with learners it's not easy, depending on our environment, requires a more civilised programme to make sure that everyone benefits. But at the moment, it's not so easy. One whiteboard for the whole school.
Interviewer: If the situation were different and you had a whiteboard in your own class and you did not need to move in this situation would be different.
Participant: It's the teachers who move from one class to another. The learners remain in one class.
Interviewer: That makes it almost impossible to use it effectively.
Participant: Yes.
Interviewer: how would you feel about using an interactive whiteboard to discuss issues of making maths more realistic for your learners? Again as communication means?
Participant: I'm not understanding the question.
Interviewer: If it were possible for you to connect via Internet, almost like a Skype discussion, would you be willing to interact with other colleagues and with myself via interactive whiteboard, to discuss issues relevant to Mathematics?
Participant: It would be a pleasure. Because at my school, we already have telematics, but it's been done through Stellenbosch University.
Interviewer: Okay.
Participant: For all grade 10's to12's.
Interviewer: Okay. The final question - what type of technology will you be most comfortable with in a project such as this one? The project which you have read a little bit about, it is about integrating technology in maths education. So, I know a little bit how you feel about technology. Should you use technology, what type of technology would you be most comfortable with? You can even refer to the more traditional technology.

Participant: Like I've said, it would be much beneficial for example if I use the interactive whiteboard, almost all learners like it at our school. The whole generation of today, they can all use computers. Now, there are so many things that come into interplay, like the community for example. You have a situation here where there is no electricity in the community, electricity is gone. But, for me I think through my experience, I have tried both electronic and the traditional. I think they should all be used concurrently in such a way that for example after using the traditional one, you can move onto technology so that the learners can understand it better.
Interviewer: Right.
Participant: It will be like watching a film, they will understand it better. It will promote the situation of real-life examples. Even those who developed the technology, I'm talking about HeyMaths. You cannot develop it if you don't understand it using the traditional way.
Interviewer: You are absolutely correct.
Participant: Yes.
Interviewer: You need to understand the basics, and you need to understand the content before using all kind of technology to present it.
Participant: That's what I'm thinking.
Interviewer: that's one of the core problems, wanting to think that technology is the answer, but you need the basics before you can actually get to the technology. That's all I need to discuss with you Mr K. Thank you so much for your time and for your opinions, I really value them. Is there anything else that you would like to talk about in relation to what we have discussed today?
Participant: Not at the moment, I don't think so, but all in all it's too beneficial to use technology. It also saves time, instead of you writing notes on the board for one hour, it's easier using the technology.
Interviewer: Thank you so much for your time, I will keep contact and we will keep in touch.
Participant: Okay have a good day.
Interviewer: Have a good evening, bye bye.
Participant: Bye bye.

P62: Focus Group Interview 1.docx

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53 quotations

Codes (21)##characteristics of RME, ##principles of RME, $$content related to real-life, $$meaningful learning of Maths, $$teaching methods and strategies, $factors that influence teaching and learning, $learner aspects, $mathematics content, %community, %context, %learner, %role players, %teacher, @@challenges when working with ICT, @@teacher-student ICT factors, @@technology as educational tool, @@value of using ICT, @Devices, Positive aspects of app, Suggestions to improve app, Usability of the app

Memos (0)

Document body:

Focus Group Interview 1

P1: Ah, it's so easy!
P2: In this way you can attend to individual learners.
P1: So, you can even ask for the formula. Because sometimes they struggle to know which formula they should use.
P2: They can play with this and still arrive at the right answer.
P2: They can teach themselves with this, while they are revising and doing this, they can check on themselves. It means that the teacher is there. This one will rectify them - it will lead them to the right answers and so on.
I: Why is mobile technology suitable to present Mathematics content in a realistic way?
P2: Because the learners will be directly involved. They won't sit back and just listen to you saying everything. They will just type it in, rectify themselves. They are actively involved, and they can interact with others.
I: Is it different giving the app on a cell phone compared to using a computer?
P1: Cell phones, there are many. Let's say more than 90% of them they have cell phones.
P2: They can afford to buy cell phones rather than computers. Even though that can't afford to buy computers they can buy cell phones.
P1: Because computers are very expensive
P2: It's more accessible to them and it's easy to carry, where they are and while they are relaxing they can learn while you are sitting outside.
P1: It's good!
I: How can this app address the principles and characteristics of RME?
P2: The topics are intertwined. They can bring topics together. They can see how this one is intertwined with the next topic. They can see for themselves. And they are actively involved.
I: What about modelling?
P2: They can use formulas for themselves. They can try to put figures and see what they get, whether this formula is the right one to use you or not.
I: How can this app assist teachers to teach Maths in a meaningful way? Do you think it can?
P2: It can. It can. One, as a teacher - because each and every teacher wants their learners to be involved. You can get them involved if they have got this. You can communicate with them through this. Maybe it's at home, while they are doing homework they can ask you- I've been trying this and this, what is the problem? We can answer them and say "try this one". Before they come to class they know what to do.
P1: The weaker learners can be able to do it. If they can't then they click on the icon for help. It helps the weaker learners.
P2: There are learners who are so shy when they are in class they can work comfortably when they are alone. I think this will be so helpful especially for the weaker learners.
I: What knowledge of technology would a teacher need to effectively use the app? How easy was it for you to navigate the app?
P2: It wasn't a problem to navigate. The fact is we own cell phones and we have been fiddling with the app. To work on this technology, is easy. But for a person who has got this for the first time it will be a challenge. Based on what we know about learners, most of them do have cell phones. I think they can use it.
I: What about the teachers? Do teachers have a good knowledge of technology?
P1: It's a challenge for some teachers.
P2: It's a challenge for teachers. Based on the needs for the content, we must acquaint ourselves first with the content before we can use it here. Then there's that challenge. The teachers must know how to download it. Without downloading it, I will be stuck. There are some features that I know that I can apply. But, a teacher is a leader - they must always be in front of the learners. Teachers must learn more.
P1: They must be able to know it first before they can take it to the learners. Because sometime we don't understand what this yes means and what the no means.
P2: Most of the icons are familiar, they used this almost daily. Once they are stuck you can ask them what about this icon? The teachers and learners will be motivated to use this app. They will love it!
P1: The demotivated learners will also be motivated.
I: Why do you say that?
P1: Because they like technology. They like to use cell phones. It's like playing games - they like playing games, whenever they are they play games. This is like playing a game.
I: Which role-players will influence the use of an app such as this in the school situation?
P2: The Mathematics teacher is directly involved with this app. They must convince the parents, they must convince the Department that they need this app.
P1: The Mathematics teacher will influence its use.
P2: But you cannot just convince the Department, saying that let's buy this so that I can come out clean
P1: You just show the learners first. Then we call a parents meeting. The learners would already have told their parents about the app. We will just show them how easy it will be.
P2: The Mathematics teacher must encourage all the stakeholders to be involved in using the app. By showing them how possible, how easy and how economical it is to use it. But if you have no facts to support your statement then you will fail, the Department will fail to deliver this. But if there is a need and it can be shown to the stakeholders that if we apply this app - we save time, all learners are involved, all learners are interested. Maybe we should have one class where we use the traditional method and one classroom use this technology and show the difference between them.
I: School principals, will all school principals be happy for teachers to be using something like this?
P1: Not at all. They like teachers to work. They think we are lazy.
P2: Not at all. It involves money. This involves money. If for example you go to the principal and say that you want to use this app, you we will work out the feasibility - how much will it cost - It will work out to be thousands. If the principal had never seen it being applied, it will not be easy to convince them. It was for the Maths teacher to convince. That's why I said if I bring the principal to class and show him how wonderful this app works, then maybe he can buy the story. Then we'll be fine.
P1: It should mostly be used by the Maths learners, because the other learners will download pornography and other things. Cell phones are not allowed in the class.
I: Learners get busy with other things in the class if they have cell phone. That's why I ask the question.
P1: For me as a teacher I must move around the class and make sure that they are doing the right thing.
I: The teacher would have to convince the principal why it is important.
P1: Because I am also going to be actively involved to check whether they are doing the subject that they are supposed to do, not outside things.
P2: To guard against the wrong information that can be used here, some schools - not all, they are getting there, the schools are having tablets - that is for the school. They can only access that thing while they are in class and do what is expected of them and then it gets taken back.
I: It can be feasible?
P2: When I get to class, everyone gets his tablets and we start working. When I am finished with the period then I collect the tablets again. In class it must be this. But once they are there and they are used to it- even at home then they will know that with a cell phone that they must use it fruitfully and effectively. I must do the school work on the cell phone at home. Then we won't have problems if they come to school with cell phones but for now they are problems that we will first have to sort out.
I: How did you experience using this app?
P1: Knowing the stuff that you're supposed to learn, the content - I think it was easy. Like now it seemed should I give you the formula, we said no, then we could punch it in. We know the information already. We learn fast in that way.
I: What is easy to navigate?
P1: It was for the first time, but it was not so difficult. But once used, I'm telling you it will come easy to them.
I: How would you use this app to improve your teaching practice?
P2: At any time when they feel like using it, while revising they can just use it. Because with the classroom situation, we they are to revise, they want to come to the classroom for help. They can't do it on their own. But once they have this, they will do it there. When they encounter a problem they will contact the teacher and get help
P1: It will help them
I: Which sections of the curriculum would benefit most from an app such as this one?
P1: Calculus, mostly the real-life, financial Maths
P2: Financial Maths. The only one that will suffer will be Algebra. But the rest, even number patterns will be fine.
P1: Number patterns, because learners use different formulas, the wrong formulas. So this will help them.
P2: Even probability, I think it can work. It only Algebra that will be left out, not all of it but part of it - the rest will work perfectly with this.
I: Last question - What can be done to improve this app?
P2: I think putting all the questions that could be available - so that you can attempt each and every question that is possible. Otherwise if the question is not there and they are applying this - there will be a challenge, even if I don't know how to do it now. They will be a challenge; I can contact whoever to help me. So to improve it, let's have as many questions as possible , especially those in real life. Having all the questions will give an opportunity for interaction with other learners and with the teacher. Once they are loaded it will put a challenge to the user each and every time they open the phone. I think it can be improved in that way. Secondly, I would say how can I show this to someone else - how can this be shown to a colleague, while I am working, so that other colleagues can help me.
I: Do you mean connectivity between users?
P2: They should be away to contact fellow users, like with Whatsapp. I can say how do you do this? I couldn't get to the right answer, how do I tackle it? They can look into your steps and say no, you are using the wrong formula. Let's try this formula. By a friend, the learners feel so proud when they can help a fellow learner.
P1: It is good, but it is also bad. Those lazy learners will not learn, but they will know which formula to use - it helps them a lot.
I: Perhaps if the user keeps asking for help they should be something built in that can block it so that the user can try it themselves.
P2: Block it and then they can't access it anymore. Then they know they must try themselves.
P1: They must be able to think
I: What about having a tutor that they can talk to?
P2: It works nicely. A tutor can work on different learners with the same problem. While he helped this one, this one can help another one.
P1: Even for the teachers. In our district we have got a peer- learning group. We get together and discuss how we tackle different topics - how we tackle this and that topic.
P2: We have a WhatsApp group where we can communicate with fellow teachers. I can get information from her. Or I am doing this and my learner struggle to understand, how do I tackle it?
P1: We get advice from other teachers on what is easier. It's good to get a new idea from someone else. They find it very helpful.
P2: My school has tablets and in my grade 12 Maths classes there are only 21 learners, I am going to download the app to help them. What I like most is that the users can work on the Maths at home.
P2: This app can help with very large classes.
P1: It will be easier for the new teachers because they know technology. The older teachers they might struggle a bit. The app helps to make the lesson more interesting.
P2: Those who are lazy to write in those whose handwriting is horrible - the app can help - it cuts all of that out. If the users get used to working with technology, it will be a challenge for them to write.
P1: To manage time can be a problem. If the users are used to doing maths on the technology and they now have to write it out they will not know how much time they should spend on each question.
P2: They will be those that are so motivated by the app that they will just improve.

P65: Focus Group Interview 2.docx

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50 quotations

Codes (20)##characteristics of RME, ##real life, $$content related to real-life, $$meaningful learning of Maths, $$teaching methods and strategies, $factors that influence teaching and learning, $learner aspects, $mathematics content, %community, %context, %learner, %role players, %teacher, @@challenges when working with ICT, @@teacher-student ICT factors, @@technology as educational tool, @@value of using ICT, Positive aspects of app, Suggestions to improve app, Usability of the app

Memos (0)

Document body:

Focus Group Interview 2

P4: The terminology helps them to differentiate between the two concepts.
P4: I think it is good. I also think that it is going to challenge the users. Especially the part where they can choose if they want help will not. Most of them will choose the no - they do not ask for help, but if they get it wrong, they will get feedback immediately from the app. Those slower ones will go for the yes. As time goes on, they will improve and will opt for the no. And then they will keep on improving.
P3: They must try to answer it first, maybe three times - and then after not answering correctly the first three times it can then give them the right answer. So definitely they will go to yes (I need help) instead of no (I don't need help). At least if you can give them three chances of doing it, it gives them the opportunity of doing it rather than going to yes, they can do it themselves. It helps by giving them three chances to do some. It will help us with lazy learners.
P4: Those brighter ones will go for the no, they will want to do it for themselves. The slower learners will go for the yes, for the help. As time goes on they will also want to go for the no. So you are catering for both.
P3: In fact there will be a competition.
P4: They will be a competition.
P3: The first ones who have done a solution and find the solution to be correct, they will boast to the others who did not find a solution.
P4: It is also going to motivate the ones who didn't get the correct solution
P3: And even improve the pace at which they work
I: Why is mobile technology suitable to present Mathematics content in a realistic way?
P3: It is suitable.
P4: Because it gives the user a chance to do Mathematics anywhere, anytime. They will get immediate feedback.
P3: Anywhere, in a taxi, in a bus - where ever they are, they can do Maths. Rather than going home, having to take out books and then learning - it can be done anywhere at any time.
P4: They will refrain from doing other things on their phones - instead they will opt for doing Mathematics, especially those who like Mathematics. Those who think they have problems with maths, there will always carry this around, always try to do some mathematical problem, at any time.
P3: Every learner has a cell phone, and they like phones
P4: They all have cell phones
P3: It is simple - when their friends are together they will be able to do some Maths together. They do not need a suitable place to do it, anywhere they can do Maths - whether they are sitting or standing, they can do Maths.
P4: Even their parents will be motivated to buy them cell phones and then even improve to buy more smart phones and install mathematical programs on the cell phones.
I: How can this app address the principles and characteristics of RME?
P3: It was very interesting. There will be collaboration, when they are together. They will ask each other: Who has done the sum first?
P4: I think all the characteristics of RME are there. There is collaboration; users will be actively involved for sure - they are doing it themselves. Intertwining of strains is there: like with simple and compound interest - showing that you can use simple interest to get to the compound interest formula.
P3: At the end the answer is the same - using the other method.
I: How can this app assist teachers to teach Maths in a meaningful way? How would it help you?
P4: First it will save time for the teacher. Secondly, it will capture the minds of the learners and maintain that for the whole period, because learners enjoy doing things for themselves. If they are using cell phones, also like this - they will be actively involved, which will help the educator. And also the feedback - they will get immediate feedback. In the end it will help them to correct their sums. If they are getting it wrong, it will tell them - this is wrong.
P3: This is going to help the teacher in the class - getting feedback immediately. Even time is not going to be wasted.
P4: You can do many problems with the learners, and they can do even more - they can try to do more. A more advanced learner can do even more work. A slow learner you can give them more time.
P3: Maybe you consider specific topics for the slower learners, for those who are willing to learn you can give them more advanced content.
I: What knowledge of technology would a teacher need to effectively use the app?
P4: I don't think that will be a problem.
I: Think of all the teachers at your school - are all the teachers at your school okay with dealing with technology?
P4: I don't think the technology will hinder them. Let's take for example, all the maths educators can use a calculator - so I don't think it will give them a problem. They will get used to smart phones. As long as they get used to using a calculator, they will get used to using the smart phone. Those educators that are not Maths educators, they struggle to work with a calculator.
P3: If they can be showed how that is working, it won't be a problem. Even the calculators - how to use the calculator - we were never taught how to use a calculator, but in the end we know how to use the calculator. There are topics that we have to use a calculator, so we were never taught how to use a calculator.
P4: With the cell phone, nowadays we all have cell phones, we use cell phones and we are also encouraged to align ourselves with the current technology. So it shouldn't be a problem.
I: Which role-players will influence the use of an app such as this in the school situation?
P3: I think this is a question of whether the school has money to buy the technology. If the Department can provide the technology, then we can always use this technology. Just like textbooks - they provide textbooks, so they can provide mobile technology, then it will be simple.
I: What about the other role players?
P4: I think all of them play a role. Because let's say for example parents - in terms of the cell phones - if we can tell the parents to buy cell phones for the learners, then parents can buy it. As long as they realise that it is effectively used and can benefit the learners, they are going to buy it. On the part of the educator - the educator, the Mathematic educators should take the initiative to show that this really works. In terms of the HODs and principals, they are the ones that should come up with the implementation and encourage the educators to implement this. So, if the Head of Department can encourage educators to implement and if the educator is prepared to implement - then from the Department to the parents - everything will run smoothly. Because the Department I think they should come up with programs where they design programs for the schools then the HOD's and the principals should check that those programs are implemented.
P3: I feel that the other stakeholders should see to it that the technology is kept safe. So there it can be used by another group of learners the next year.
P4: If they use it, will it be for the school or for an individual learner?
I: I think you can do it either way. The learners can download the app on their personal phones as well.
I: How did you experience using this app?
P4: It will work. It is interesting.
P3: This is excellent, excellent!
I: Why do you think it will work?
P4: This is a new method of doing those things which are giving learners a problem. Nowadays people like this sort of things - they like using technology.
P3: They won't find it difficult to use this technology - they are hands on.
P4: This generation like using these devices. They are interested in using cell phone, tablets. It will capture their minds. So they are focused, they will be focused.
I: Which sections of the curriculum would benefit most from an app such as this one?
P4: I think Calculus, probability, Geometry in Grade 8
P3: Geometry, data handling, the Cartesian plane which falls under graphs.
P4: Even the theorem of Pythagoras is very abstract - here they can master it. I think they can master it with this app.
P3: If I think thoroughly, all of them will be suitable for the app. Any section would be suitable.
I: What can be done to improve this app?
P3: Like we already said, just to change where the learners have to choose between yes or no. They should be given more opportunity to do it on their own first. They should be forced to try the sums on their own first, and then the help can come. Twice or thrice is enough otherwise they might become demotivated.
P4: I think if you can make it to be audio-visual - that would be great.
P3: There was also no sound. The sound can tell them what to do.
P4: Maybe make it an option for learners - those who want sound maybe have that option, those who don't want sound can switch it all.
P3: They also like sound.
P4: It will also help those who battle to read. It will also improve their reading skills if they can hear and see at the same time.
I: How would you make it more visual?
P3: I have seen that you have put some colourful pictures in there - even in the formula - maybe add some pictures but not too much.
P4: It should also depend on the grades - maybe in the senior and FET phases you can limit the images. In the lower grade you can include more pictures.
P4: What about animation? It will be good especially for the grade fours, fives and sixes - because they like cartoons - you can add it there. If you can include animations, that will be good for the lower grades.
P3: It would be good if you could develop these apps for the whole curriculum - from foundation phase right through. Then they can get used to using it.
P4: You can do Maths with something that they know - make it real life.
P3: I think this will help in other subject too - if teachers that teach other subjects can see this, they would also want to use such apps. This will help them too.
P4: Disadvantages might be: I think it would limit their basic skills - those learners will not have the basic Mathematics skill like calculating. When it comes to exam time, they will not be able to use this. They will use paper and pencil. If they are used to the technology- it can be a problem for them to write exams with paper-and-pencil.
P3: I don't think they should only use this - in the end they should also write something. So, one should create opportunities for them to answer questions on paper-and-pencil as well - after learning the content, maybe use worksheets where they get to do the sums on paper.
P3: I am very impressed. In fact we are tired of chalkboards - it will be nice to use something new.

Codes Summary

(Commented codes are clickable)

All codes used: ####HLT {62-1} , ###design of activities {45-1}, ###didactical phenomenology {14-1}, ###emergent modelling {42-1}, ###guided reinvention {93-1}, ##advantages of using RME {26-1}, ##characteristics of RME {53-1}, ##content where need help to relate to real life {11-0}, ##disadvantages of using RME {6-1}, ##human activity {29-1}, ##mathematization {49-1}, ##principles of RME {22-4}, ##real life {252-1}, ##recommendations for RME based lessons {42-1}, ##RME tasks {32-1}, #advantages, disadvantages and recommendations for RME {0-4}, #aspects of RME {1-7}, #RME {361-8}, #RME theory {16-1}, $$approaches to teaching Mathematics {118-1}, $$modelling {72-1}, $$content related to real-life {27-1}, $$learner reasoning {4-1}, $$learner understanding {8-1}, $$mathematics education {44-2}, $$maths strategies {4-1}, $$meaningful learning of Maths {66-1}, $$problem areas to teach/learn {17-1}, $$teaching methods and strategies {105-6}, $factors that influence teaching and learning {249-4}, $learner aspects {36-3}, $mathematics content {125-3}, $mathematics {130-10}, $mathematics content {125-3}, %community {11-1}, %context {35-1}, %learner {19-1}, %role players {205-7}, %teacher {99-1}, &&integration across strands {6-1}, &&learning environment {7-1}, &&student support {5-1}, &distance education {13-1}, &motivation {14-1}, &pedagogy {71-6}, &social aspects {13-2}, &TPCK {6-1}, @@challenges when working with ICT {56-1}, @@ICT and RME {0-1}, @@ICT systems: virtual learning environment (VLE) {85-1}, @@teacher-student ICT factors {79-1}, @@technology as educational tool {97-1}, @@value of using ICT {241-1}, @Devices {327-1}, @ICT {38-11}, @Tools {64-1}, =methodology {97-2}, Needs Analysis {0-4}, Observations and experiences of App {0-5}, Positive aspects of app {26-1}, SLR {0-5}, Suggestions relating to App {0-4}, Suggestions to improve app {17-1}, Usability of the app {5-1}

Codes sorted by Alphabet:

####HLT {62-1}
###design of activities {45-1}
###didactical phenomenology {14-1}
###emergent modelling {42-1}
###guided reinvention {93-1}
##advantages of using RME {26-1}
##characteristics of RME {53-1}
##content where need help to relate to real life {11-0}

Codes sorted by Groundedness:

#RME {361-8}
@Devices {327-1}
##real life {252-1}
$factors that influence teaching and learning {249-4}
@@value of using ICT {241-1}
%role players {205-7}
$mathematics {130-10}
$mathematics content {125-3}

Codes sorted by Density:

@ICT {38-11}
$mathematics {130-10}
#RME {361-8}
#aspects of RME {1-7}
%role players {205-7}
&pedagogy {71-6}
$$teaching methods and strategies {105-6}
Observations and experiences of App {0-5}

Commented Codes only:

Primary Document Families

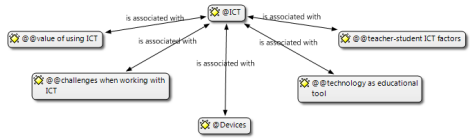
Code Families

Memo Families

Network Views

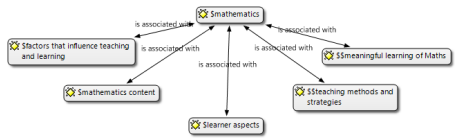
Nodes are prefixed with a single letter denoting its type: C= Code, M = Memo, Q = Quotation, P = Primary Document

Ch 6 ICT



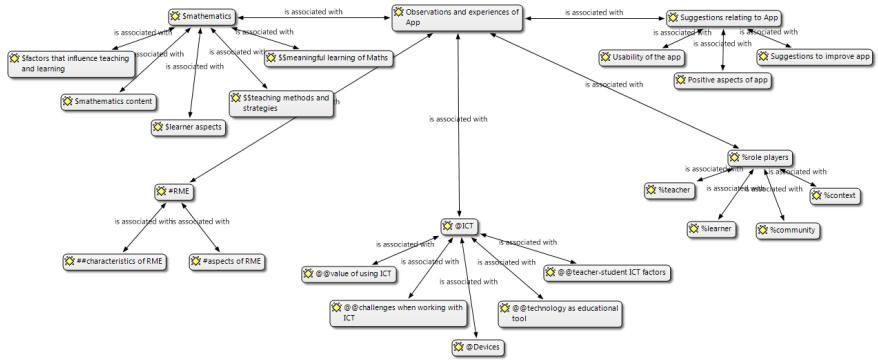
Nodes (6): CO:@@technology as educational tool {97-1}, CO:@@teacher-student ICT factors {79-1}, CO:@@challenges when working with ICT {56-1}, CO:@ICT {38-11}, CO:@Devices {327-1}, CO:@@value of using ICT {241-1}

Ch 6 Maths



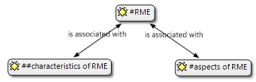
Nodes (6): CO:$factors that influence teaching and learning {249-4}, CO:$$meaningful learning of Maths {66-1}, CO:$$teaching methods and strategies {105-6}, CO:$mathematics content {125-3}, CO:$mathematics {130-10}, CO:$learner aspects {36-3}

Ch 6 overview



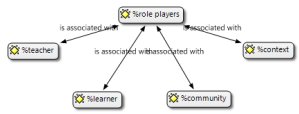
Nodes (25): CO:$mathematics {130-10}, CO:$learner aspects {36-3}, CO:$factors that influence teaching and learning {249-4}, CO:%context {35-1}, CO:%community {11-1}, CO:$mathematics content {125-3}, CO:#aspects of RME {1-7}, CO:##characteristics of RME {53-1}, CO:Usability of the app {5-1}, CO:$$teaching methods and strategies {105-6}, CO:$$meaningful learning of Maths {66-1}, CO:#RME {361-8}, CO:%learner {19-1}, CO:Positive aspects of app {26-1}, CO:@ICT {38-11}, CO:@Devices {327-1}, CO:Observations and experiences of App {0-5}, CO:Suggestions to improve app {17-1}, CO:Suggestions relating to App {0-4}, CO:@@challenges when working with ICT {56-1}, CO:%teacher {99-1}, CO:%role players {205-7}, CO:@@value of using ICT {241-1}, CO:@@technology as educational tool {97-1}, CO:@@teacher-student ICT factors {79-1}

Ch 6 RME



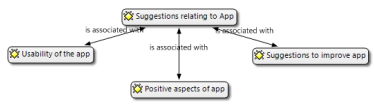
Nodes (3): CO:##characteristics of RME {53-1}, CO:#aspects of RME {1-7}, CO:#RME {361-8}

Ch 6 role players



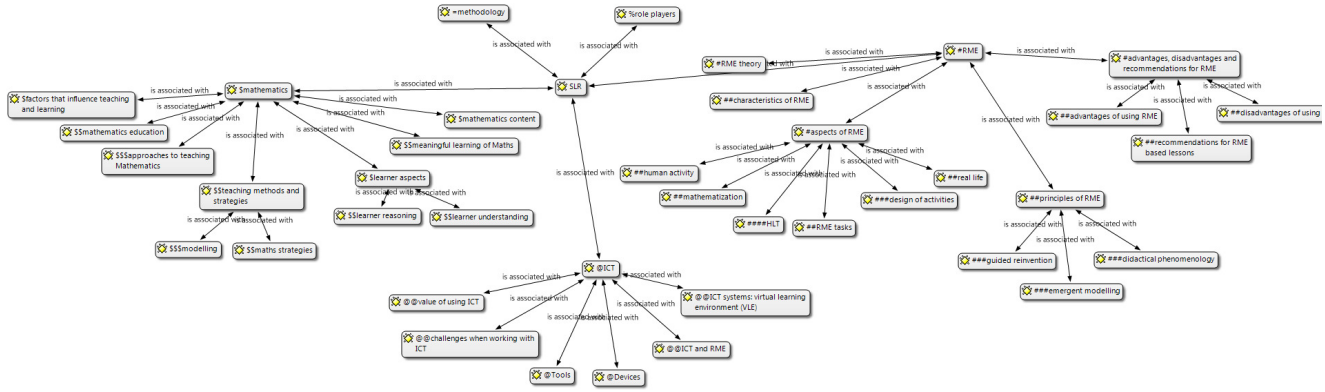
Nodes (5): CO:%context {35-1}, CO:%community {11-1}, CO:%learner {19-1}, CO:%teacher {99-1}, CO:%role players {205-7}

Ch 6 Suggestions for app



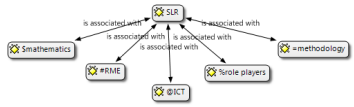
Nodes (4): CO:Suggestions relating to App {0-4}, CO:Usability of the app {5-1}, CO:Suggestions to improve app {17-1}, CO:Positive aspects of app {26-1}

Chapter 3 Detail



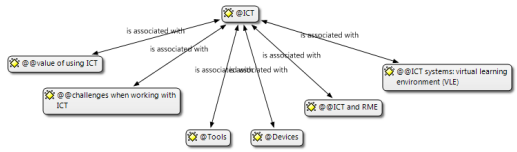
Nodes (40): CO:##principles of RME [22-4], CO:##mathematization [49-1], CO:##recommendations for RME based lessons [42-1], CO:#aspects of RME [1-7], CO:#advantages, disadvantages and recommendations for RME [0-4], CO:###emergent modelling [42-1], CO:###didactical phenomenology [14-1], CO:###guided reinvention [93-1], CO:##human activity [29-1], CO:##disadvantages of using RME [6-1], CO:##characteristics of RME [53-1], CO:$mathematics content [125-3], CO:##advantages of using RME [26-1], CO:=methodology [97-2], CO:$$maths strategies [4-1], CO:##real life [252-1], CO:#RME theory [16-1], CO:###design of activities [45-1], CO:####HLT [62-1], CO:##RME tasks [32-1], CO:@value of using ICT [241-1], CO:@Tools [64-1], CO:@@ICT and RME [0-1], CO:@Devices [327-1], CO:@@challenges when working with ICT [56-1], CO:$mathematics [130-10], CO:#RME [361-8], CO:%role players [205-7], CO:SLR [0-5], CO:@ICT [38-11], CO:$$meaningful learning of Maths [66-1], CO:$$learner reasoning [4-1], CO:$$teaching methods and strategies [105-6], CO:$factors that influence teaching and learning [249-4], CO:$$$modelling [72-1], CO:$$$approaches to teaching Mathematics [118-1], CO:@@ICT systems: virtual learning environment (VLE) [85-1], CO:$$mathematics education [44-2], CO:$$learner understanding [8-1], CO:$learner aspects [36-3]

Chapter 3 General



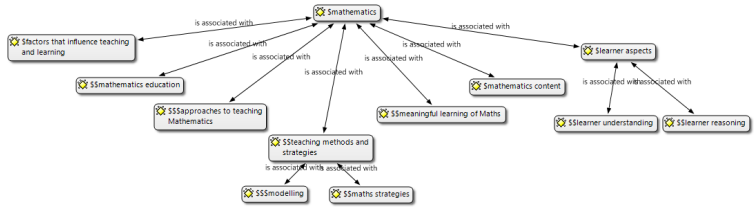
Nodes (6): CO:@ICT [38-11], CO:=methodology [97-2], CO:SLR [0-5], CO:#RME [361-8], CO:$mathematics [130-10], CO:%role players [205-7]

Chapter 3 ICT



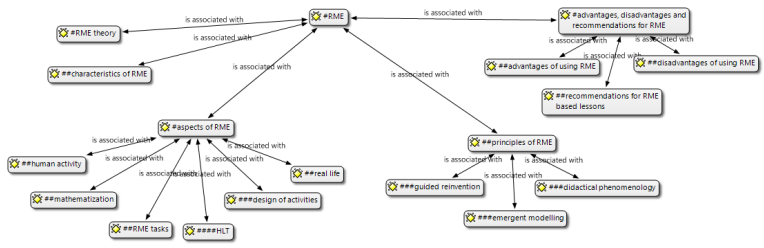
Nodes (7): CO:@ICT [38-11], CO:@Tools [64-1], CO:@@ICT systems: virtual learning environment (VLE) [85-1], CO:@Devices [327-1], CO:@@challenges when working with ICT [56-1], CO:@@ICT and RME [0-1], CO:@@value of using ICT [241-1]

Chapter 3 Maths



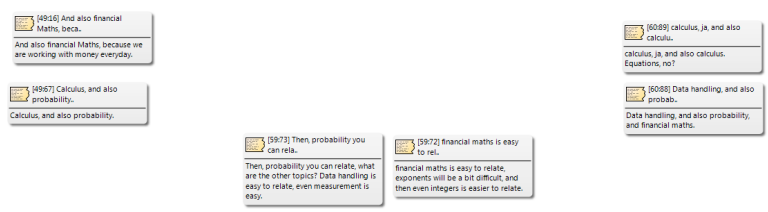
Nodes (12): CO:$$learner understanding [8-1], CO:$$mathematics education [44-2], CO:$$meaningful learning of Maths [66-1], CO:$maths strategies [4-1], CO:$factors that influence teaching and learning [249-4], CO:$$teaching methods and strategies [105-6], CO:$$learner reasoning [4-1], CO:$$$modelling [72-1], CO:$$$approaches to teaching Mathematics [118-1], CO:$mathematics [130-10], CO:$mathematics content [125-3], CO:$learner aspects [36-3]

Chapter 3 RME



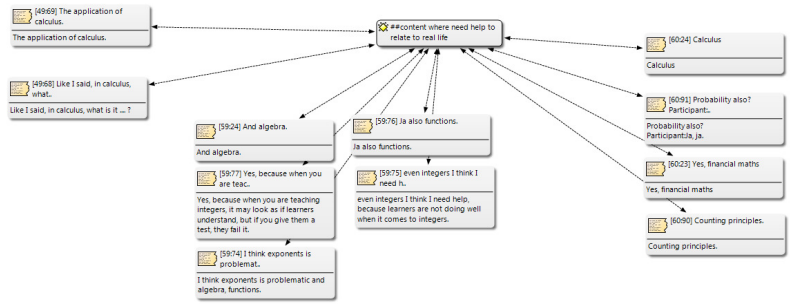
Nodes (18): CO:##RME tasks [32-1], CO:#aspects of RME [1-7], CO:##mathematization [49-1], CO:##recommendations for RME based lessons [42-1], CO:##real life [252-1], CO:#principles of RME [22-4], CO:#advantages, disadvantages and recommendations for RME [0-4], CO:#RME [361-8], CO:#RME theory [16-1], CO:###didactical phenomenology [14-1], CO:###emergent modelling [42-1], CO:####HLT [62-1], CO:###design of activities [45-1], CO:###guided reinvention [93-1], CO:##disadvantages of using RME [6-1], CO:##human activity [29-1], CO:##advantages of using RME [26-1], CO:##characteristics of RME [53-1]

Content easy to relate to real life

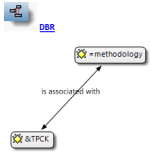


Nodes (6): QU:49:16 And also financial Maths, beca.. (36:36), QU:59:72 financial maths is easy to rel.. (58:58), QU:60:89 calculus, ja, and also calculu.. (54:54), QU:59:73 Then, probability you can rela.. (60:60), QU:49:67 Calculus, and also probability.. (34:34), QU:60:88 Data handling, and also probab.. (52:52)

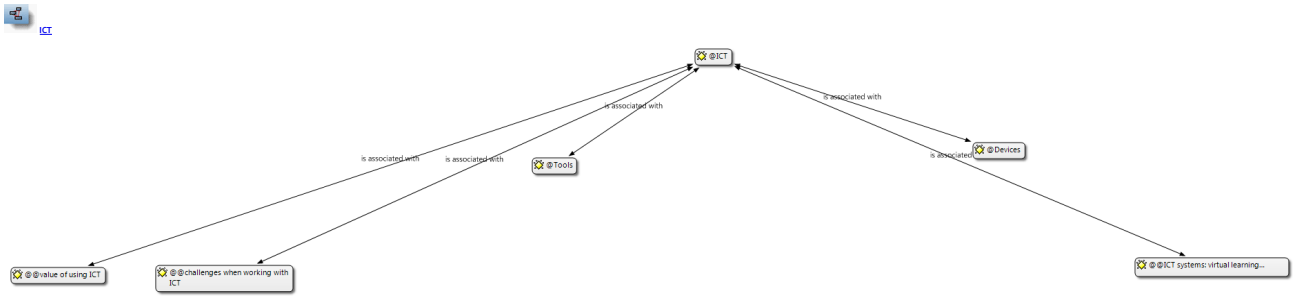
Content where help is needed to relate to real life



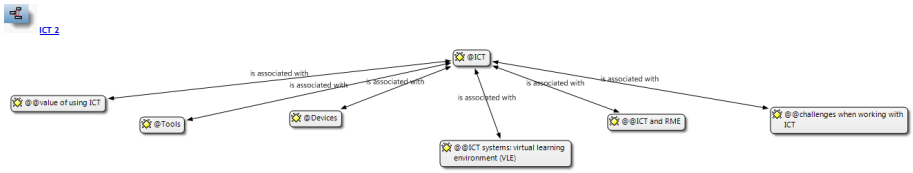
Nodes (12): CO:##content where need help to relate to real life [11-0], QU:59:76 Ja also functions. (70:70), QU:60:91 Probability also? Participant:.. (63:64), QU:49:68 Like I said, in calculus, what.. (52:52), QU:59:77 Yes, because when you are teac.. (72:72), QU:60:24 Calculus (62:62), QU:49:69 The application of calculus. (54:54), QU:<>59:24 And algebra. (68:68), QU:59:74 I think exponents is problemat.. (60:60), QU:60:90 Counting principles. (56:56), QU:59:75 even integers I think I need h.. (66:66), QU:60:23 Yes, financial maths (60:60)



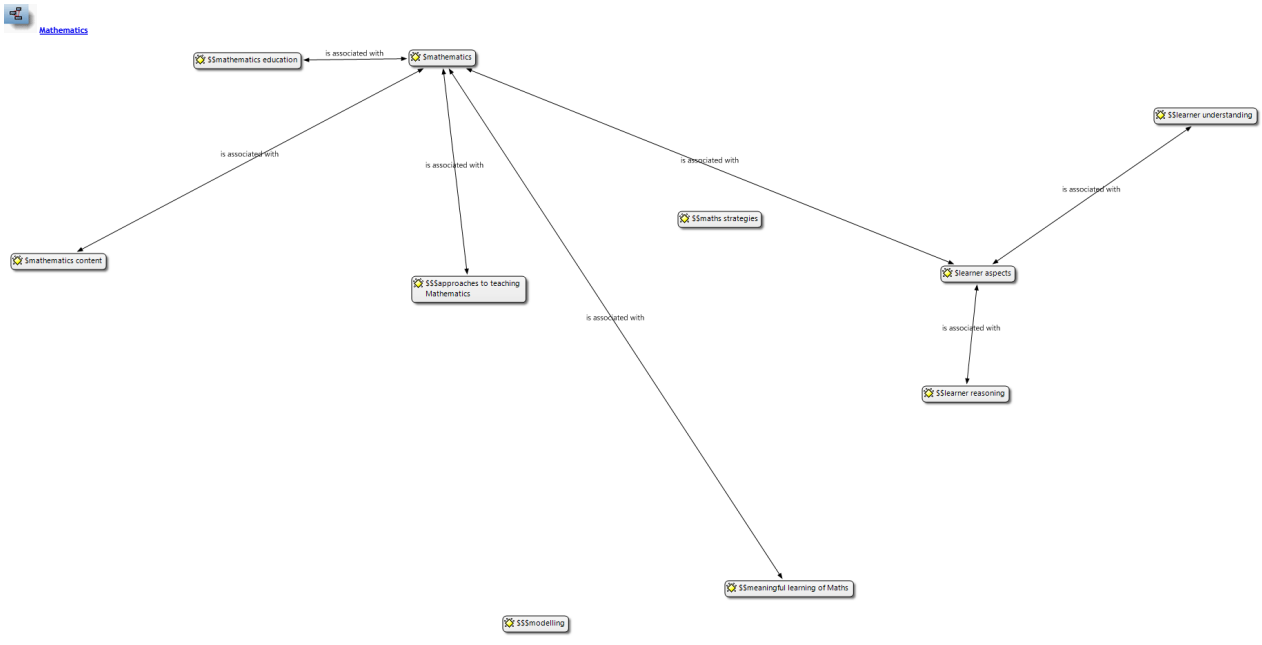
Nodes (2): CO:&TPCK [6-1], CO:=methodology [97-2]



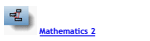
Nodes (6): CO:@Devices [327-1], CO:@Tools [64-1], CO:@ICT [38-11], CO:@@challenges when working with ICT [56-1], CO:@@value of using ICT [241-1], CO:@@ICT systems: virtual learning environment (VLE) [85-1]

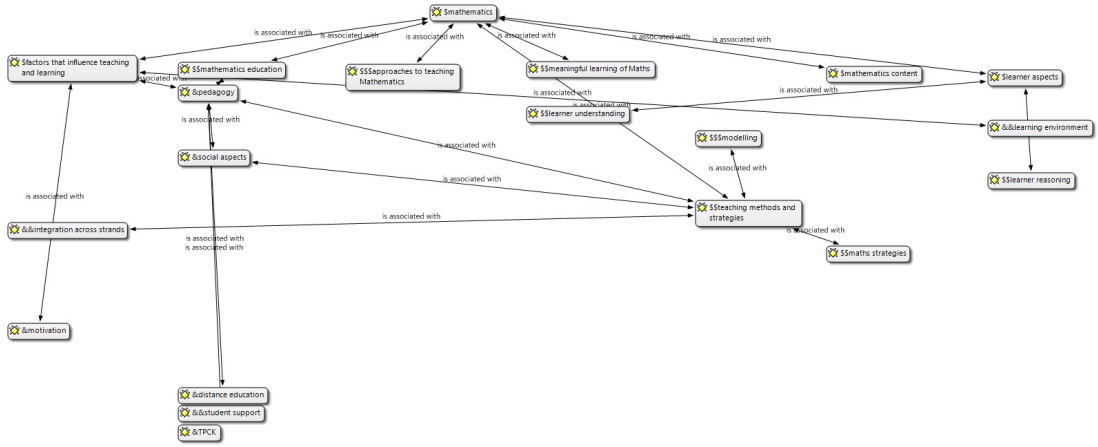


Nodes (7): CO:@ICT [38-11], CO:@Tools [64-1], CO:@@ICT and RME [0-1], CO:@Devices [327-1], CO:@@challenges when working with ICT [56-1], CO:@@value of using ICT [241-1], CO:@@ICT systems: virtual learning environment (VLE) [85-1]



Nodes (10): CO:$$$modelling [72-1], CO:$$maths strategies [4-1], CO:$$learner understanding [8-1], CO:$mathematics content [125-3], CO:$learner aspects [36-3], CO:$$meaningful learning of Maths [66-1], CO:$$$approaches to teaching Mathematics [118-1], CO:$$mathematics education [44-2], CO:$mathematics [130-10], CO:$$learner reasoning [4-1]





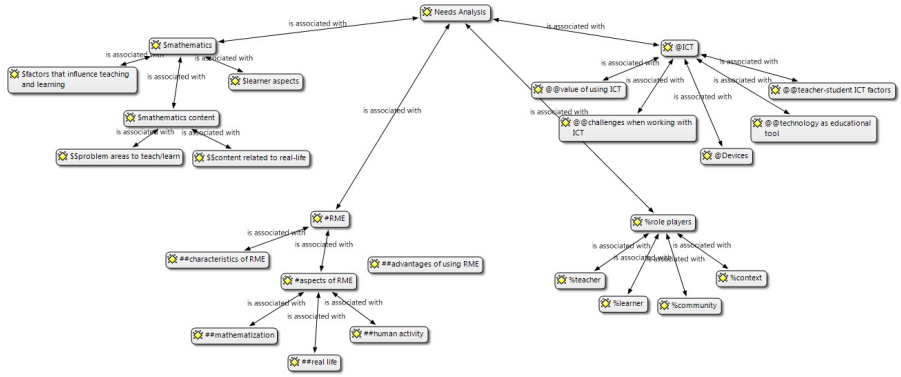
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Methodology 2



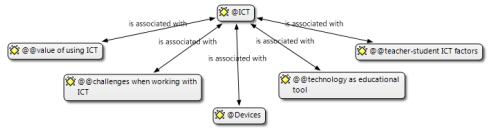
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Needs Analysis



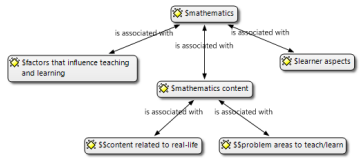
Nodes (25): CO:@@value of using ICT [241-1], CO:@Devices [327-1], CO:@@challenges when working with ICT [56-1], CO:$factors that influence teaching and learning [249-4], CO:@ICT [38-11], CO:@@technology as educational tool [97-1], CO:%learner [19-1], CO:%community [11-1], CO:%context [35-1], CO:@@teacher-student ICT factors [79-1], CO:%role players [205-7], CO:%teacher [99-1], CO:$$content related to real-life [27-1], CO:##real life [252-1], CO:##mathematization [49-1], CO:##human activity [29-1], CO:Needs Analysis [0-4], CO:#RME [361-8], CO:#aspects of RME [1-7], CO:$learner aspects [36-3], CO:$mathematics content [125-3], CO:$$problem areas to teach/learn [17-1], CO:##characteristics of RME [53-1], CO:##advantages of using RME [26-1], CO:$mathematics [130-10]

Needs Analysis ICT



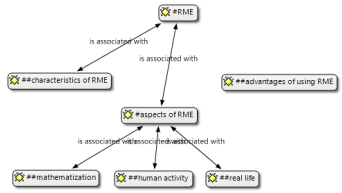
Nodes (6): CO:@ICT [38-11], CO:@@technology as educational tool [97-1], CO:@@teacher-student ICT factors [79-1], CO:@@value of using ICT [241-1], CO:@@challenges when working with ICT [56-1], CO:@Devices [327-1]

Needs Analysis Mathematics



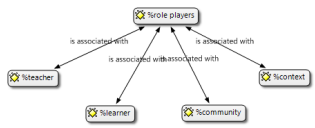
Nodes (6): CO:$$problem areas to teach/learn [17-1], CO:$mathematics content [125-3], CO:$$content related to real-life [27-1], CO:$learner aspects [36-3], CO:$factors that influence teaching and learning [249-4], CO:$mathematics [130-10]

Needs Analysis RME



Nodes (7): CO:##mathematization [49-1], CO:#aspects of RME [1-7], CO:##real life [252-1], CO:##advantages of using RME [26-1], CO:##human activity [29-1], CO:##characteristics of RME [53-1], CO:#RME [361-8]

Needs Analysis Role-players

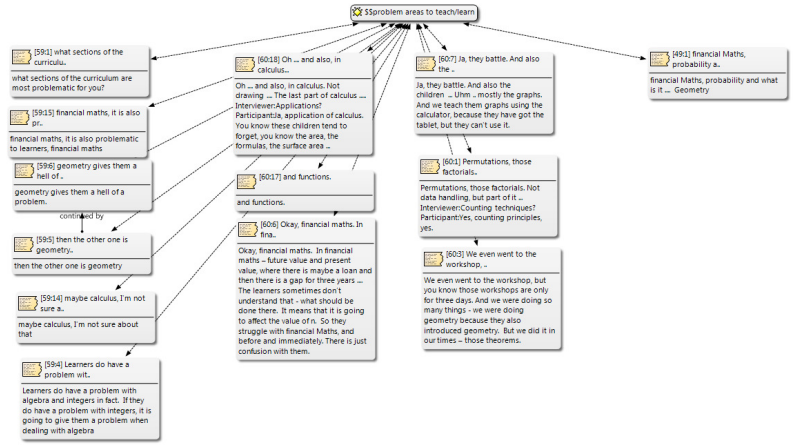


Nodes (0):

Pedagogy

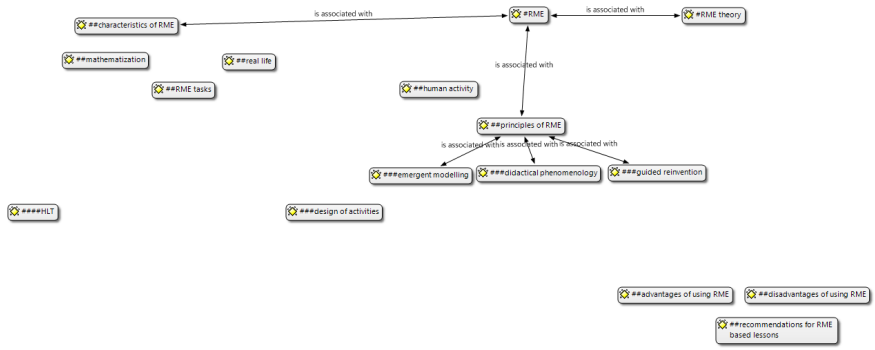
Nodes (9): CO:&&learning environment [7-1], CO:&pedagogy [71-6], CO:&&integration across strands [6-1], CO:$factors that influence teaching and learning [249-4], CO:$$teaching methods and strategies [105-6], CO:&motivation [14-1], CO:&social aspects [13-2], CO:&distance education [13-1], CO:&&student support [5-1]

Problem areas to treach/learn



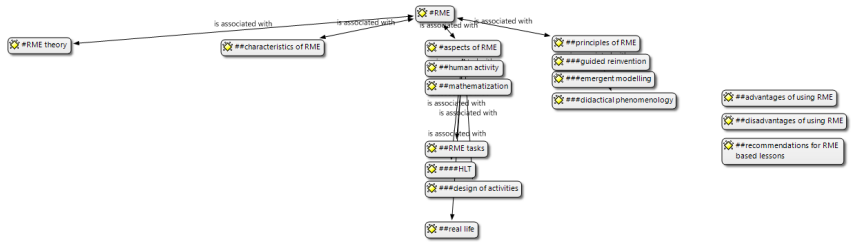
Nodes (14): CO:$$problem areas to teach/learn {17-1}, QU:60:3 We even went to the workshop, .. (12:12), QU:60:7 Ja, they battle. And also the .. (20:20), QU:<59:5 then the other one is geometry.. (14:14), QU:60:6 Okay, financial maths. In fina.. (18:18), QU:49:1 financial Maths, probability a.. (2:2), QU:60:1 Permutations, those factorials.. (8:10), QU:>59:6 geometry gives them a hell of .. (16:16), QU:60:18 Oh .... and also, in calculus... (42:44), QU:59:14 maybe calculus, I'm not sure a.. (44:44), QU:59:15 financial maths, it is also pr.. (46:46), QU:59:4 Learners do have a problem wit.. (12:12), QU:60:17 and functions. (40:40), QU:<59:1 what sections of the curriculu.. (7:7)

RME



Nodes (16): CO:##real life {252-1}, CO:##principles of RME {22-4}, CO:##human activity {29-1}, CO:###didactical phenomenology {14-1}, CO:##recommendations for RME based lessons {42-1}, CO:#RME theory {16-1}, CO:##characteristics of RME {53-1}, CO:#RME {361-8}, CO:####HLT {62-1}, CO:##disadvantages of using RME {6-1}, CO:##RME tasks {32-1}, CO:###design of activities {45-1}, CO:###guided reinvention {93-1}, CO:##mathematization {49-1}, CO:###emergent modelling {42-1}, CO:##advantages of using RME {26-1}

RME 2



Nodes (17): CO:##real life {252-1}, CO:##recommendations for RME based lessons {42-1}, CO:##mathematization {49-1}, CO:##principles of RME {22-4}, CO:#RME theory {16-1}, CO:#aspects of RME {1-7}, CO:##RME tasks {32-1}, CO:#RME {361-8}, CO:##human activity {29-1}, CO:###didactical phenomenology {14-1}, CO:###emergent modelling {42-1}, CO:####HLT {62-1}, CO:###design of activities {45-1}, CO:##characteristics of RME {53-1}, CO:##disadvantages of using RME {6-1}, CO:###guided reinvention {93-1}, CO:##advantages of using RME {26-1}

Role-players

%role players

Nodes (1): CO:%role players {205-7}

Code Neighbors List (Thesaurus)

The following is a thesaurus-style alphabetic list of all codes with their relations to other codes.

Each code-code relations is displayed in text form as a simple two argument proposition:

CODE_A < relation > CODE_B.

####HLT
<is associated with> **#aspects of RME**
###design of activities
#aspects of RME *<is associated with>* **###design of activities**
###didactical phenomenology
##principles of RME *<is associated with>* **###didactical phenomenology**
###emergent modelling
##principles of RME *<is associated with>* **###emergent modelling**
###guided reinvention
##principles of RME *<is associated with>* **###guided reinvention**
##advantages of using RME
#advantages, disadvantages and recommendations for RME *<is associated with>* **##advantages of using RME**
##characteristics of RME
#RME *<is associated with>* **##characteristics of RME**
##content where need help to relate to real life
##disadvantages of using RME
#advantages, disadvantages and recommendations for RME *<is associated with>* **##disadvantages of using RME**
##human activity
#aspects of RME *<is associated with>* **##human activity**
##mathematization
#aspects of RME *<is associated with>* **##mathematization**
##principles of RME
<is associated with> **###didactical phenomenology**
<is associated with> **###emergent modelling**
<is associated with> **###guided reinvention**
#RME *<is associated with>* **##principles of RME**
##real life
<is associated with> **#aspects of RME**
##recommendations for RME based lessons
#advantages, disadvantages and recommendations for RME *<is associated with>* **##recommendations for RME based lessons**
##RME tasks
<is associated with> **#aspects of RME**
#advantages, disadvantages and recommendations for RME
<is associated with> **##advantages of using RME**
<is associated with> **##disadvantages of using RME**
<is associated with> **##recommendations for RME based lessons**
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<is associated with> **##human activity**
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<is associated with> **#RME**
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##real life *<is associated with>* **#aspects of RME**
##RME tasks *<is associated with>* **#aspects of RME**
#RME
<is associated with> **##characteristics of RME**
<is associated with> **##principles of RME**
<is associated with> **#RME theory**
<is associated with> **Needs Analysis**
<is associated with> **SLR**
#advantages, disadvantages and recommendations for RME *<is associated with>* **#RME**
#aspects of RME *<is associated with>* **#RME**
Observations and experiences of App *<is associated with>* **#RME**
#RME theory
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$learner aspects *<is associated with>* **$$learner reasoning**
$$learner understanding

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$$maths strategies
<is associated with> $$teaching methods and strategies
$$meaningful learning of Maths
$mathematics *<is associated with>* $$meaningful learning of Maths
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@ICT *<is associated with>* @@challenges when working with ICT
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@ICT *<is associated with>* @@ICT and RME
@@ICT systems: virtual learning environment (VLE)
@ICT *<is associated with>* @@ICT systems: virtual learning environment (VLE)
@@teacher-student ICT factors
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@Devices
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Positive aspects of app
Suggestions relating to App *<is associated with>* Positive aspects of app
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Suggestions to improve app
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Usability of the app
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Code Hierarchy

####HLT *<is>* Root
###design of activities *<is>* Root
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####HLT *<is associated with>* #aspects of RME
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#advantages, disadvantages and recommendations for RME *<is associated with>* ##advantages of using RME
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@@teacher-student ICT factors <is associated with> @ICT
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@@technology as educational tool <is> Root
@@value of using ICT <is> Root
@Devices <is> Root
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Needs Analysis <is> Root
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#advantages, disadvantages and recommendations for RME <is associated with> #RME
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Positive aspects of app <is> Root
Suggestions relating to App <is associated with> Positive aspects of app
Observations and experiences of App <is associated with> Suggestions relating to App
SLR <is> Root
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Suggestions to improve app <is> Root
Suggestions relating to App <is associated with> Suggestions to improve app
Observations and experiences of App <is associated with> Suggestions relating to App
Usability of the app <is> Root