

**SYMMETRY ANALYSIS,  
CONSERVATION LAWS AND EXACT  
SOLUTIONS OF CERTAIN  
NONLINEAR PARTIAL  
DIFFERENTIAL EQUATIONS**

**by**

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# Declaration

I declare that the thesis for the degree of Doctor of Philosophy at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other university, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

Signed: .....

MR TANKI MOTSEPA

Date: .....

This thesis has been submitted with my approval as a University supervisor and would certify that the requirements for the applicable Doctor of Philosophy degree rules and regulations have been fulfilled.

Signed:.....

PROF CM KHALIQUE

Date: .....

# Declaration of Publications

Details of contribution to publications that form part of this thesis.

## Chapter 2

T Motsepa, CM Khalique, Lie group classification of a variable coefficients Gardner equation, submitted to *Communications in Nonlinear Science and Numerical Simulation*

## Chapter 3

T Motsepa, CM Khalique, Cnoidal and snoidal waves solutions and conservation laws of a generalized (2+1)-dimensional Kortweg-de Vries equation, accepted and to appear in *AIP Conference Proceedings* (Proceedings of the 14th Regional Conference on Mathematical Physics)

## Chapter 4

T Motsepa, CM Khalique, Travelling wave solutions of a coupled Kortweg-de Vries-Burgers system, *AIP Conference Proceedings* (Proceedings of the Progress in Applied Mathematics in Science and Engineering) 1705, 020027 (2016); doi: 10.1063/1.4940275

## Chapter 5

T Motsepa, CM Khalique, A study of a (2+1)-dimensional Kortweg-de Vries modified Kortweg-de Vries equation of mathematical physics, submitted to *Applied Mathematics Letters*

## Chapter 6

T Motsepa, CM Khalique, Exact solutions and conservation laws for a generalized improved Boussinesq equation, *AIP Conference Proceedings* (Proceedings of the 2nd International Conference on Mathematical sciences & Statistics), 1739, 020029 (2016); doi: 10.1063/1.4952509

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## **Chapter 8**

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## **Chapter 9**

T Motsepa, CM Khalique, Conservation laws and solutions of a generalized coupled (2+1)-dimensional Burgers system, submitted to *Computers & Mathematics with Applications*

## **Chapter 10**

T Motsepa, CM Khalique, T Aziz, Algebraic aspects of evolution partial differential equation arising in the study of constant elasticity of variance model from financial mathematics, submitted to *International Journal of Geometric Methods in Modern Physics*

## **Chapter 11**

T Motsepa, CM Khalique, ML Gandarias, Symmetry analysis and conservation laws of the Zoomeron equation, submitted to *Symmetry*

# Dedication

I dedicate this work to my family.

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# Abstract

In this research work we study some nonlinear partial differential equations which model many physical phenomena in science, engineering and finance. Closed-form solutions and conservation laws are obtained for such equations using various methods. The nonlinear partial differential equations that are investigated in this thesis are; a variable coefficients Gardner equation, a generalized (2+1)-dimensional Kortweg-de Vries equation, a coupled Korteweg-de Vries-Burgers system, a Kortweg-de Vries–modified Kortweg-de Vries equation, a generalized improved Boussinesq equation, a Kaup-Boussinesq system, a classical model of Prandtl’s boundary layer theory for radial viscous flow, a generalized coupled (2+1)-dimensional Burgers system, an optimal investment-consumption problem under the constant elasticity of variance model and the Zoomeron equation.

We perform Lie group classification of a variable coefficients Gardner equation, which describes various interesting physics phenomena, such as the internal waves in a stratified ocean, the long wave propagation in an inhomogeneous two-layer shallow liquid and ion acoustic waves in plasma with a negative ion. The Lie group classification of the equation provides us with four-dimensional equivalence Lie algebra and has several possible extensions. It is further shown that several cases arise in classifying the arbitrary parameters. Conservation laws are obtained for certain cases.

A generalized (2+1)-dimensional Korteweg-de Vries equation is investigated. This equation was recently constructed using Lax pair generating technique. The extended Jacobi elliptic method is employed to construct new exact solutions for this equation and obtain cnoidal and snoidal wave solutions. Moreover, conservation laws are derived using the multiplier method.

The coupled Korteweg-de Vries-Burgers system, which arises in mathematical

physics and has a wide range of scientific applications is studied and new travelling wave solutions are obtained by employing the  $(G'/G)$ -expansion method. The solutions obtained are expressed in two different forms, viz., hyperbolic functions and trigonometric functions. Also conservation laws are derived by employing the multiplier method.

The  $(2+1)$ -dimensional Kortweg-de Vries–modified Kortweg-de Vries equation, which arises in various problems in mathematical physics, is analysed. This equation has two integral terms in it. By an appropriate substitution, we transform this equation into a system of two partial differential equations, which does not have an integral term. We then work with the system of two equations and obtain its exact travelling wave solutions in terms of Jacobi elliptic functions. Furthermore, we derive conservation laws using the multiplier method. Finally, we revert the results obtained into the original variables of the  $(2+1)$ -dimensional Kortweg-de Vries–modified Kortweg-de Vries equation.

We analyse a nonlinear generalized improved Boussinesq equation, which describes nonlinear dispersive wave phenomena. Exact solutions are derived using the Lie symmetry analysis along with the simplest equation method. Moreover, conservation laws are constructed using the multiplier method.

We study a Kaup-Boussinesq system, which is used in the analysis of long waves in shallow water. Travelling wave solutions are obtained using direct integration and group-invariant solutions are constructed based on the optimal system of one-dimensional subalgebra. Moreover, conservation laws are derived using the multiplier method and the new conservation theorem.

Exact closed-form solutions of the Prandtl's boundary layer equation for radial flow models with uniform or vanishing mainstream velocity are derived using the  $(G'/G)$ -expansion method. Many new exact solutions are found for the bound-

ary layer equation, which are expressed in terms of hyperbolic, trigonometric and rational functions.

We study an integrable coupled (2+1)-dimensional Burgers system, which was introduced recently in the literature. The Lie symmetry analysis along with the Kudryashov approach are utilized to obtain new travelling wave solutions of the system. Furthermore, conservation laws of the system are derived using the multiplier method.

The optimal investment-consumption problem under the constant elasticity of variance (CEV) model is investigated from the perspective of Lie group analysis. The complete Lie symmetry group of the evolution partial differential equation describing the CEV model is derived. The Lie point symmetries are then used to obtain an exact solution of the governing model satisfying a standard terminal condition. Finally, we construct conservation laws of the underlying equation using a general theorem on conservation laws.

We study the (2+1)-dimensional Zoomeron equation which is an extension of the famous (1+1)-dimensional Zoomeron equation that has many applications in scientific fields. Firstly we derive the classical Lie point symmetries admitted by the equation and then obtain symmetry reductions and new group-invariant solutions based on the optimal system of one-dimensional subalgebras. Secondly we construct the conservation laws of the underlying equation using the multiplier method.

# Introduction

It is well-known that many physical phenomena in the real world are modelled by nonlinear partial differential equations (NLPDEs). Unlike linear differential equations, where the exact solution of any initial-value problem can be found, NLPDEs rarely enjoy this and other features. Moreover, basic properties, like existence and uniqueness of solutions, which are so obvious in the linear case, no longer hold for NLPDEs. As a matter of fact, some NLPDEs have no solutions with a given initial-value problem while others have infinitely many solutions. This means that the underlying theory behind systems of NLPDEs is more complicated than that for linear systems. Therefore it is imperative to study these NLPDEs from different points of view. One important aspect of studying NLPDEs is to find their exact explicit solutions. However, this is a very difficult task because there are no specific tools or techniques which can be used to find exact solutions of NLPDEs. Despite this fact, in recent years, many scientists have developed various methods of finding exact solutions of NLPDEs.

Some of these methods are the inverse scattering transform method [1], Bäcklund transformation [2], Darboux transformation [3], Hirota's bilinear method [4], the bilinear method and multilinear method [5], the nonclassical Lie group approach [6], the Clarkson-Kruskal's direct method [7], the deformation mapping method [8], the Weierstrass elliptic function expansion method [9], the transformed rational function method [10], the auxiliary equation method [11], the homogeneous balance

method [12], the simplest equation method [13], the extended tanh method [14], the Jacobian elliptic function expansion method [15], the sine-cosine method [16], the exp-function method [17], the  $(G'/G)$ -expansion method [18], multiple exp-function method [19], the  $F$ -expansion method [20], and the Lie symmetry method [21–26].

The Lie group analysis is one of the most powerful methods to determine solutions of NLPDEs. Sophus Lie (1842–1899), a Norwegian mathematician, with the inspiration from Galois' theory, discovered this method and showed that many of the known ad hoc methods of integration of ordinary differential equations could be derived in a systematic manner. In the past few decades Lie group method was revived by several researchers including Ovsianikov [21, 22].

A large number of differential equations that model real world problems involve parameters, arbitrary elements or functions. These parameters are usually determined experimentally. However, the Lie group classification method can be effectively used in obtaining the forms of these parameters systematically [26–30]. In 1881 Sophus Lie [31] was the first person to perform group classification on a linear second-order partial differential equation with two independent variables.

In the study of the solution process of differential equations, conservation laws play a central role. They also help in the numerical integration of partial differential equations [32] and theory of non-classical transformations [33, 34]. In recent years conservation laws have been used to construct exact solutions of differential equations [35, 36]. The Noether theorem [37] gives us a sophisticated and constructive way for obtaining conservation laws. It actually provides an explicit formula for finding a conservation law once a Noether symmetry corresponding to a Lagrangian is known for an Euler-Lagrange equation. However, there are differential equations, such as scalar evolution differential equations, which do not have a Lagrangian. In such cases, several methods have been developed by researchers about the con-

struction of conserved quantities. Comparison of several differential methods for computing conservation laws can be found in [38, 39].

This thesis is structured as follows:

In Chapter one we introduce the preliminaries that are needed in our study.

In Chapter two Lie group classification of a variable coefficients Gardner equation is performed. As a result the arbitrary functions and constants which appear in the system are specified. Conservation laws are obtained in certain cases.

Chapter three presents the cnoidal and snoidal wave solutions of a generalized (2+1)-dimensional Kortweg-de Vries equation using the extended Jacobi elliptic function method. Conservation laws are constructed using the multiplier approach.

In Chapter four travelling wave solutions of a coupled Kortweg-de Vries-Burgers system are obtained by employing the  $(G'/G)$ -expansion method. Moreover, conservation laws are derived using the multiplier method.

Chapter five studies exact travelling wave solutions in terms of Jacobi elliptic functions of a (2+1)-dimensional Kortweg-de Vries modified Kortweg-de Vries equation. Furthermore, conservation laws are derived using the multiplier method.

Chapter six analyses a nonlinear generalized improved Boussinesq equation. Exact solutions are derived using the Lie symmetry analysis and the simplest equation methods. Moreover, conservation laws are constructed using the multiplier method.

Chapter seven studies exact solutions and conservation laws of a Kaup-Boussinesq system. Travelling wave solutions are obtained using direct integration and employing the Lie symmetry analysis. Moreover, conservation laws are derived using the multiplier method and the conservation theorem due to Ibragimov.

In Chapter eight exact closed-form solutions of a Prandtl's boundary layer equation for radial flow models with uniform or vanishing mainstream velocity are derived

by employing the  $(G'/G)$ -expansion method.

Chapter nine deals with an integrable coupled (2+1)-dimensional Burgers system. Lie symmetry analysis along with Kudryashov approach are utilized to obtain new travelling wave solutions. Furthermore, conservation laws of the system are derived using the multiplier method.

In Chapter ten a group-invariant solution of an optimal investment-consumption problem under the constant elasticity of variance model is obtained. Finally, conservation laws of the underlying equation are constructed using a general theorem on conservation laws.

Chapter eleven studies a (2+1)-dimensional Zoomeron equation which is an extension of the famous (1+1)-dimensional Zoomeron equation. We compute Lie point symmetries admitted by the equation and then obtain symmetry reductions and new group-invariant solutions based on the optimal system of one-dimensional subalgebras. Moreover, we derive conservation laws of the underlying equation using the multiplier method.

Finally, in Chapter twelve a summary of the results of the thesis is presented and future work is deliberated.

Bibliography is given at the end.

# Chapter 1

## Preliminaries

In this chapter we present some preliminaries on the theory of Lie group analysis, conservation laws of differential equations and some methods for obtaining exact solutions of differential equations, which are used in this thesis and are based on references [18, 22–26, 37, 40, 41].

### 1.1 One-parameter transformation groups

Let  $x = (x^1, \dots, x^n)$  and  $u = (u^1, \dots, u^m)$  be the independent and dependent variables with coordinates  $x^i$  and  $u^\alpha$  ( $n$  and  $m$  finite), respectively. Consider a change of the variables  $x$  and  $u$  involving a real parameter  $a$ :

$$T_a : \bar{x}^i = f^i(x, u, a), \quad \bar{u}^\alpha = \phi^\alpha(x, u, a) \quad (1.1)$$

where  $a$  continuously ranges in values from a neighbourhood  $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$  of  $a = 0$ , and  $f^i$  and  $\phi^\alpha$  are differentiable functions.

**Definition 1.1 (Lie group)** A set  $G$  of transformations (1.1) is called a *continuous one-parameter (local) Lie group of transformations* in the space of variables

$x$  and  $u$  if

(i) For  $T_a, T_b \in G$  where  $a, b \in \mathcal{D}' \subset \mathcal{D}$  then  $T_b T_a = T_c \in G$ ,  $c = \phi(a, b) \in \mathcal{D}$   
(Closure)

(ii)  $T_0 \in G$  if and only if  $a = 0$  such that  $T_0 T_a = T_a T_0 = T_a$  (Identity)

(iii) For  $T_a \in G$ ,  $a \in \mathcal{D}' \subset \mathcal{D}$ ,  $T_a^{-1} = T_{a^{-1}} \in G$ ,  $a^{-1} \in \mathcal{D}$  such that  
 $T_a T_{a^{-1}} = T_{a^{-1}} T_a = T_0$  (Inverse)

We note that the associativity property follows from (i). The group property (i) can be written as

$$\begin{aligned}\bar{x}^i &\equiv f^i(\bar{x}, \bar{u}, b) = f^i(x, u, \phi(a, b)), \\ \bar{u}^\alpha &\equiv \phi^\alpha(\bar{x}, \bar{u}, b) = \phi^\alpha(x, u, \phi(a, b))\end{aligned}\tag{1.2}$$

and the function  $\phi$  is called the *group composition law*. A group parameter  $a$  is called *canonical* if  $\phi(a, b) = a + b$ .

**Theorem 1.1** For any  $\phi(a, b)$ , there exists the canonical parameter  $\tilde{a}$  defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)}, \text{ where } w(s) = \left. \frac{\partial \phi(s, b)}{\partial b} \right|_{b=0}.$$

## 1.2 Prolongation formulas

The derivatives of  $u$  with respect to  $x$  are defined as

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j D_i(u_i), \dots,\tag{1.3}$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots, \quad i = 1, \dots, n\tag{1.4}$$

is the total differential operator. The collection of all first derivatives  $u_i^\alpha$  is denoted by  $u_{(1)}$ , i.e.,

$$u_{(1)} = \{u_i^\alpha\} \quad \alpha = 1, \dots, m, \quad i = 1, \dots, n.$$

Similarly

$$u_{(2)} = \{u_{ij}^\alpha\} \quad \alpha = 1, \dots, m, \quad i, j = 1, \dots, n$$

and  $u_{(3)} = \{u_{ijk}^\alpha\}$  and likewise  $u_{(4)}$  etc. Since  $u_{ij}^\alpha = u_{ji}^\alpha$ ,  $u_{(2)}$  contains only  $u_{ij}^\alpha$  for  $i \leq j$ . In the same manner  $u_{(3)}$  has only terms for  $i \leq j \leq k$ . There is natural ordering in  $u_{(4)}$ ,  $u_{(5)}$   $\dots$ .

In group analysis all variables  $x, u, u_{(1)} \dots$  are considered functionally independent variables connected only by the differential relations (1.3). Thus, the  $u_s^\alpha$  are called differential variables [26].

We now consider a  $p$ th-order partial differential equations, namely

$$E_\alpha(x, u, u_{(1)}, \dots, u_{(p)}) = 0. \tag{1.5}$$

### 1.2.1 Prolonged or extended groups

Consider a one-parameter group of transformations  $G$  given by

$$\begin{aligned} \bar{x}^i &= f^i(x, u, a), & f^i|_{a=0} &= x^i, \\ \bar{u}^\alpha &= \phi^\alpha(x, u, a), & \phi^\alpha|_{a=0} &= u^\alpha. \end{aligned} \tag{1.6}$$

According to the Lie's theory, the construction of the symmetry group  $G$  is equivalent to the determination of the corresponding *infinitesimal transformations*:

$$\bar{x}^i \approx x^i + a \xi^i(x, u), \quad \bar{u}^\alpha \approx u^\alpha + a \eta^\alpha(x, u) \tag{1.7}$$

obtained from (1.1) by expanding the functions  $f^i$  and  $\phi^\alpha$  into Taylor series in  $a$ , about  $a = 0$  and also taking into account the initial conditions

$$f^i|_{a=0} = x^i, \quad \phi^\alpha|_{a=0} = u^\alpha.$$

Thus, we have

$$\xi^i(x, u) = \left. \frac{\partial f^i}{\partial a} \right|_{a=0}, \quad \eta^\alpha(x, u) = \left. \frac{\partial \phi^\alpha}{\partial a} \right|_{a=0}. \quad (1.8)$$

One can now introduce the *symbol* of the infinitesimal transformations by writing (1.7) as

$$\bar{x}^i \approx (1 + a X)x, \quad \bar{u}^\alpha \approx (1 + a X)u,$$

where

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}. \quad (1.9)$$

This differential operator  $X$  is known as the infinitesimal operator or generator of the group  $G$ . If the group  $G$  is admitted by (1.5), we say that  $X$  is an *admitted operator* of (1.5) or  $X$  is an *infinitesimal symmetry* of equation (1.5).

We now see how the derivatives are transformed.

The  $D_i$  transforms as

$$D_i = D_i(f^j) \bar{D}_j, \quad (1.10)$$

where  $\bar{D}_j$  is the total differential operator in transformed variables  $\bar{x}^i$ . So

$$\bar{u}_i^\alpha = \bar{D}_j(u^\alpha), \quad \bar{u}_{ij}^\alpha = \bar{D}_j(\bar{u}_i^\alpha) = \bar{D}_i(\bar{u}_j^\alpha), \dots$$

Now let us apply (1.6) and (1.10)

$$D_i(\phi^\alpha) = D_i(f^j) \bar{D}_j(\bar{u}^\alpha)$$

$$= D_i(f^j)\bar{u}_j^\alpha. \quad (1.11)$$

Thus

$$\left(\frac{\partial f^j}{\partial x^i} + u_i^\beta \frac{\partial f^j}{\partial u^\beta}\right)\bar{u}_j^\alpha = \frac{\partial \phi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \phi^\alpha}{\partial u^\beta}. \quad (1.12)$$

The quantities  $\bar{u}_j^\alpha$  can be represented as functions of  $x, u, u_{(i)}, a$ , for small  $a$ , that is, (1.12) is locally invertible

$$\bar{u}_i^\alpha = \psi_i^\alpha(x, u, u_{(1)}, a), \quad \psi_i^\alpha|_{a=0} = u_i^\alpha. \quad (1.13)$$

The transformations in  $x, u, u_{(1)}$  space given by (1.13) and (1.6) form a one-parameter group called the extension of the group  $G$  and denoted by  $G^{[1]}$ .

We let

$$\bar{u}_i^\alpha \approx u_i^\alpha + a\zeta_i^\alpha \quad (1.14)$$

be the infinitesimal transformation of the first derivatives so that the infinitesimal transformation of the group  $G^{[1]}$  is (1.7) and (1.14).

Higher-order prolongations of  $G$ , namely,  $G^{[2]}$ ,  $G^{[3]}$  can be obtained by derivatives of (1.11).

### 1.2.2 Prolonged generators

Using (1.11) together with (1.7) and (1.14) we get

$$\begin{aligned} D_i(f^j)(\bar{u}_j^\alpha) &= D_i(\phi^\alpha) \\ D_i(x^j + a\xi^j)(u_j^\alpha + a\zeta_j^\alpha) &= D_i(u^\alpha + a\eta^\alpha) \\ (\delta_i^j + aD_i\xi^j)(u_j^\alpha + a\zeta_j^\alpha) &= u_i^\alpha + aD_i\eta^\alpha \\ u_i^\alpha + a\zeta_i^\alpha + au_j^\alpha D_i\xi^j &= u_i^\alpha + aD_i\eta^\alpha \\ \zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \quad (\text{sum on } j). \end{aligned} \quad (1.15)$$

This is called the first prolongation formula. Likewise, one can obtain the second prolongation, viz.,

$$\zeta_{ij}^\alpha = D_j(\eta_i^\alpha) - u_{ik}^\alpha D_j(\xi^k), \quad (\text{sum on } k). \quad (1.16)$$

The higher prolongations can be found by induction (recursively) by this formula

$$\zeta_{i_1, i_2, \dots, i_p}^\alpha = D_{i_p}(\zeta_{i_1, i_2, \dots, i_{p-1}}^\alpha) - u_{i_1, i_2, \dots, i_{p-1} j}^\alpha D_{i_p}(\xi^j), \quad (\text{sum on } j). \quad (1.17)$$

The first to  $p$  prolongations of the group  $G$  form a group denoted by  $G^{[1]}, \dots, G^{[p]}$ , respectively. The corresponding prolonged generators are

$$\begin{aligned} X^{[1]} &= X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} \quad (\text{sum on } i, \alpha), \\ &\vdots \\ X^{[p]} &= X^{[p-1]} + \zeta_{i_1, \dots, i_p}^\alpha \frac{\partial}{\partial u_{i_1, \dots, i_p}^\alpha} \quad p \geq 1, \end{aligned}$$

where

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}.$$

### 1.3 Groups admitted by differential equations

**Definition 1.2 (Point symmetry)** The vector field

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha} \quad (1.18)$$

is a *point symmetry* of the  $p$ th-order partial differential equations (1.5), if

$$X^{[p]}(E_\alpha) = 0 \quad (1.19)$$

whenever  $E_\alpha = 0$ . This can also be written as

$$X^{[p]} E_\alpha|_{E_\alpha=0} = 0, \quad (1.20)$$

where the symbol  $|_{E_\alpha=0}$  means evaluated on the equation  $E_\alpha = 0$ .

**Definition 1.3 (Determining equation)** Equation (1.19) is called the *determining equation* of (1.5) because it determines all the infinitesimal symmetries of (1.5).

**Definition 1.4 (Symmetry group)** A one-parameter group  $G$  of transformations (1.1) is called a symmetry group of equation (1.5) if (1.5) is form-invariant in the new variables  $\bar{x}$  and  $\bar{u}$ , i.e.,

$$E_\alpha(\bar{x}, \bar{u}, \bar{u}_{(1)}, \dots, \bar{u}_{(p)}) = 0, \quad (1.21)$$

where the function  $E_\alpha$  is the same as in equation (1.5).

## 1.4 Infinitesimal criterion of invariance

**Definition 1.5 (Invariant)** A function  $F(x, u)$  is called an *invariant* of the group of transformation (1.1) if

$$F(\bar{x}, \bar{u}) \equiv F(f^i(x, u, a), \phi^\alpha(x, u, a)) = F(x, u) \quad (1.22)$$

identically in  $x$ ,  $u$  and  $a$ .

**Theorem 1.2 (Infinitesimal criterion of invariance)** A necessary and sufficient condition for a function  $F(x, u)$  to be an invariant is that

$$X(F) \equiv \xi^i(x, u) \frac{\partial F}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial F}{\partial u^\alpha} = 0. \quad (1.23)$$

It follows from the above theorem that every one-parameter group of point transformations (1.1) has  $n - 1$  functionally independent invariants, which can be taken to be the left-hand side of any first integrals

$$J_1(x, u) = c_1, \dots, J_{n-1}(x, u) = c_{n-1}$$

of the characteristic equations

$$\frac{dx^1}{\xi^1(x, u)} = \cdots = \frac{dx^n}{\xi^n(x, u)} = \frac{du^1}{\eta^1(x, u)} = \cdots = \frac{du^n}{\eta^n(x, u)}.$$

**Theorem 1.3 (Lie equations)** If the infinitesimal transformations (1.7) or its symbol  $X$  is given, then the corresponding one-parameter group  $G$  is obtained by solving the Lie equations

$$\frac{d\bar{x}^i}{da} = \xi^i(\bar{x}, \bar{u}), \quad \frac{d\bar{u}^\alpha}{da} = \eta^\alpha(\bar{x}, \bar{u}) \quad (1.24)$$

subject to the initial conditions

$$\bar{x}^i|_{a=0} = x, \quad \bar{u}^\alpha|_{a=0} = u.$$

## 1.5 Conservation laws

### 1.5.1 Fundamental operators and their relationship

Consider a  $p$ th-order system of partial differential equations of  $n$  independent variables  $x = (x^1, x^2, \dots, x^n)$  and  $m$  dependent variables  $u = (u^1, u^2, \dots, u^m)$  given by equation (1.5).

**Definition 1.6 (Euler-Lagrange operator)** The *Euler-Lagrange operator*, for each  $\alpha$ , is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1 i_2 \cdots i_s}^\alpha}, \quad \alpha = 1, \dots, m. \quad (1.25)$$

**Definition 1.7 (Lagrangian)** If there exists a function

$\mathcal{L} = \mathcal{L}(x, u, u_{(1)}, u_{(2)}, \dots, u_{(s)})$ ,  $s \leq p$ , with  $p$  the order of equation (1.5), such that

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} = 0 \quad \alpha = 1, \dots, m \quad (1.26)$$

then  $\mathcal{L}$  is called a Lagrangian of equation (1.5). Equation (1.26) is known as the Euler-Lagrange equation.

**Definition 1.8 (Lie-Bäcklund operator)** The Lie-Bäcklund operator is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \quad \xi^i, \eta^\alpha \in \mathcal{A}, \quad (1.27)$$

where  $\mathcal{A}$  is the space of differential functions [26]. The operator (1.27) is an abbreviated form of infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} \zeta_{i_1 i_2 \dots i_s}^\alpha \frac{\partial}{\partial u_{i_1 i_2 \dots i_s}^\alpha}, \quad (1.28)$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} \zeta_i^\alpha &= D_i(W^\alpha) + \xi^j u_{ij}^\alpha, \\ \zeta_{i_1 \dots i_s}^\alpha &= D_{i_1} \dots D_{i_s}(W^\alpha) + \xi^j u_{j i_1 \dots i_s}^\alpha, \quad s > 1, \end{aligned} \quad (1.29)$$

in which  $W^\alpha$  is the *Lie characteristic function* given by

$$W^\alpha = \eta^\alpha - \xi^i u_j^\alpha. \quad (1.30)$$

One can write the Lie-Bäcklund operator (1.28) in characteristic form as

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} D_{i_1} \dots D_{i_s}(W^\alpha) \frac{\partial}{\partial u_{i_1 i_2 \dots i_s}^\alpha}. \quad (1.31)$$

**Definition 1.9 (Conservation law)** The  $n$ -tuple vector  $T = (T^1, T^2, \dots, T^n)$ ,  $T^j \in \mathcal{A}$ ,  $j = 1, \dots, n$ , is a *conserved vector* of (1.5) if  $T^i$  satisfies

$$D_i T^i|_{(1.5)} = 0. \quad (1.32)$$

Equation (1.32) defines a local conservation law of system (1.5).

## 1.5.2 Noether Theorem

**Definition 1.10 (Noether operator)** The Noether operators associated with a Lie-Bäcklund symmetry operator  $X$  are given by

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s \geq 1} D_{i_1} \cdots D_{i_s} (W^\alpha) \frac{\delta}{\delta u_{i_1 i_2 \cdots i_s}^\alpha}, \quad i = 1, \dots, n, \quad (1.33)$$

where the Euler-Lagrange operators with respect to derivatives of  $u^\alpha$  are obtained from (1.25) by replacing  $u^\alpha$  by the corresponding derivatives. For example,

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s \geq 1} (-1)^s D_{j_1} \cdots D_{j_s} \frac{\partial}{\partial u_{i_1 j_2 \cdots j_s}^\alpha}, \quad i = 1, \dots, n, \quad \alpha = 1, \dots, m, \quad (1.34)$$

and the Euler-Lagrange, Lie-Bäcklund and Noether operators are connected by the operator identity [41]

$$X + D_i(\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i N^i. \quad (1.35)$$

**Definition 1.11 (Noether symmetry)** A Lie-Bäcklund operator  $X$  of the form (1.27) is called a Noether symmetry corresponding to a Lagrangian  $\mathcal{L} \in \mathcal{A}$ , if there exists a vector  $B^i = (B^1, \dots, B^n)$ ,  $B^i \in \mathcal{A}$  such that

$$X(\mathcal{L}) + \mathcal{L} D_i(\xi^i) = D_i(B^i). \quad (1.36)$$

If  $B^i = 0$  ( $i = 1, \dots, n$ ), then  $X$  is called a strict Noether symmetry corresponding to a Lagrangian  $\mathcal{L} \in \mathcal{A}$ .

**Theorem 1.4 (Noether Theorem)** For any Noether symmetry generator  $X$  associated with a given Lagrangian  $\mathcal{L} \in \mathcal{A}$ , there corresponds a vector  $T = (T^1, \dots, T^n)$ ,  $T^i \in \mathcal{A}$ , given by

$$T^i = N^i(\mathcal{L}) - B^i, \quad i = 1, \dots, n, \quad (1.37)$$

which is a conserved vector of the Euler-Lagrange differential equations (1.26).

In the Noether approach, we find the Lagrangian  $\mathcal{L}$  and then equation (1.36) is used to determine the Noether symmetries. Then, equation (1.37) will yield the corresponding Noether conserved vectors.

### 1.5.3 The multiplier method

The multiplier approach is an effective algorithmic for finding the conservation laws for partial differential equations with any number of independent and dependent variables. The algorithm was given in [40,42] using the multipliers presented in [24]. A local conservation law of a given system of differential equations arises from a linear combination formed by local multipliers with each differential equation in the system, where the multipliers depend on the independent and dependent variables as well as on the derivatives of the dependent variables of the given system of differential equations.

This method does not require the existence of a Lagrangian and reduces the calculation of conservation laws to solving a system of linear determining equations similar to that for finding Lie point symmetries.

A multiplier  $\Lambda_\alpha(x, u, u_{(1)}, \dots)$  has the property that

$$\Lambda_\alpha E_\alpha = D_i T^i \tag{1.38}$$

holds identically. Here  $E_\alpha$ ,  $D_i$  and  $T^i$  are defined by equations (1.5), (1.4) and (1.32), respectively. The right hand side of (1.38) is a divergence expression. The determining equation for the multiplier  $\Lambda_\alpha$  is

$$\frac{\delta(\Lambda_\alpha E_\alpha)}{\delta u^\alpha} = 0 \tag{1.39}$$

and once the multipliers are obtained the conserved vectors are constructed by invoking the homotopy operator [40].

### 1.5.4 A Conservation theorem due to Ibragimov

A new conservation theorem due to Ibragimov [41] provides the procedure for computing the conserved vectors associated with all symmetries of the system of  $p$ th-order partial differential equations (1.5).

**Definition 1.12 (Adjoint equations)** Consider a system of  $p$ th-order partial differential equations given by (1.5). Let

$$E_{\alpha}^{*}(x, u, v, \dots, u_{(p)}, v_{(p)}) = \frac{\delta(v^{\beta} E_{\beta})}{\delta u^{\alpha}}, \quad \alpha = 1, \dots, m, \quad (1.40)$$

where  $v = (v^1, \dots, v^m)$  are new dependent variables,  $v = v(x)$ , and define the system of *adjoint equations* to system (1.5) by

$$E_{\alpha}^{*}(x, u, v, \dots, u_{(p)}, v_{(p)}) = 0, \quad \alpha = 1 \dots, m. \quad (1.41)$$

**Theorem 1.5** Any system of partial differential equations (1.5) considered together with its adjoint system (1.41) has a Lagrangian

$$\mathcal{L} = v^{\beta} E_{\beta}^{*}(x, u, v, \dots, u_{(p)}). \quad (1.42)$$

**Theorem 1.6** Consider a system of partial differential equations (1.5). The adjoint system given by (1.41) inherits the symmetries of system (1.5). If system (1.5) admits a point transformation group with a generator (1.27), then the adjoint system (1.41) admits the operator (1.27) extended to the variables  $v^{\alpha}$  by the formula

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta^{\alpha} \frac{\partial}{\partial u^{\alpha}} + \eta_{*}^{\alpha} \frac{\partial}{\partial v^{\alpha}} \quad (1.43)$$

with  $\eta_{*}^{\alpha} = \eta_{*}^{\alpha}(x, u, v, \dots)$ .

**Theorem 1.7 (Ibragimov theorem)** Any infinitesimal symmetry (Lie point, Lie-Bäcklund, nonlocal) given by (1.27) of system (1.5) leads to a conservation law

$D_i(C^i) = 0$  for the system (1.5) and (1.41). The components of the conserved vector are given by the formula

$$\begin{aligned}
C^i = & \xi^i \mathcal{L} + W^\alpha \left[ \frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j \left( \frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} \right) + D_j D_k \left( \frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) - \dots \right] \\
& + D_j(W^\alpha) \left[ \frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k \left( \frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) + \dots \right] + D_j D_k(W^\alpha) \left[ \frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - \dots \right],
\end{aligned} \tag{1.44}$$

where  $W^\alpha$  is the Lie characteristic function given by (1.30) and  $\mathcal{L}$  is the formal Lagrangian (1.42) [41].

## 1.6 Exact solutions

In this section we recall some methods which can be used to determine exact solutions of differential equations.

### 1.6.1 Description of $(G'/G)$ –expansion method

We present a brief summary of the  $(G'/G)$ –expansion method for solving nonlinear ordinary differential equations [43]. The algorithm for the  $(G'/G)$ –expansion method is given in the following steps:

**Step 1:** Consider a nonlinear ordinary differential equation given in general form by

$$P[U(z), U', U'', U''', \dots] = 0, \tag{1.45}$$

where  $U$  is an unknown function of  $z$  and  $P$  is a polynomial in  $U$  and its derivatives.

**Step 2:** The solution of ODE (1.45) is written as a polynomial in  $(G'/G)$  as

follows:

$$U(z) = \sum_{i=0}^M \beta_i \left( \frac{G'}{G} \right)^i, \quad (1.46)$$

where  $G = G(z)$  satisfies the second-order linear ODE with constant coefficients, namely,

$$\frac{d^2G}{dz^2} + \lambda \frac{dG}{dz} + \mu G = 0, \quad (1.47)$$

where  $\lambda$  and  $\mu$  are constants and  $\beta_i$  ( $i = 0, 1, 2, \dots, M$ ) are constants to be determined. The integer  $M$  is found by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE (1.45).

**Step 3:** The substitution of (1.46) into (1.45) and then making use of ODE (1.47) leads to the polynomial equation in  $(G'/G)$ . Now by equating the coefficients of the powers of  $(G'/G)$  to zero, one obtains a system of algebraic equations which is solved for  $\beta_i$ 's.

**Step 4:** The solutions of ODE (1.45) are given by (1.46) by using the solutions of the algebraic system obtained in Step 3 for the constants  $\beta_i$ 's and making use of the solutions of (1.47) which are given by

$$G(z) = -\frac{\lambda}{2} + \delta_1 \frac{C_1 \sinh(\delta_1 z) + C_2 \cosh(\delta_1 z)}{C_1 \cosh(\delta_1 z) + C_2 \sinh(\delta_1 z)}, \quad \lambda^2 - 4\mu > 0, \quad (1.48)$$

$$G(z) = -\frac{\lambda}{2} + \delta_2 \frac{-C_1 \sin(\delta_2 z) + C_2 \cos(\delta_2 z)}{C_1 \cos(\delta_2 z) + C_2 \sin(\delta_2 z)}, \quad \lambda^2 - 4\mu < 0, \quad (1.49)$$

$$G(z) = -\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}, \quad \lambda^2 - 4\mu = 0, \quad (1.50)$$

where  $\delta_1 = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}$ ,  $\delta_2 = \frac{1}{2}\sqrt{4\mu - \lambda^2}$ ,  $C_1$  and  $C_2$  are arbitrary constants.

## 1.6.2 The simplest equation method

The simplest equation method was introduced by Kudryashov [13, 44] for finding exact solutions of nonlinear partial differential equations. Many researchers have

applied this method to solve nonlinear partial differential equations. The basic steps of the method are as follows:

Consider the nonlinear partial differential equation of the form

$$E_1(u, u_t, u_x, u_y, u_{tt}, u_{xx}, u_{yy}, \dots) = 0. \quad (1.51)$$

Using the transformation

$$u(t, x, y) = F(z), \quad z = k_1 t + k_2 x + k_3 y + k_4, \quad (k_1, \dots, k_4 \text{ constants}) \quad (1.52)$$

equation (1.51) reduces to an ordinary differential equation

$$E_2[F(z), k_1 F'(z), k_2 F'(z), k_3 F'(z), k_1^2 F''(z), k_2^2 F''(z), k_3^2 F''(z), \dots] = 0. \quad (1.53)$$

The simplest equations that we use here are the Bernoulli equation

$$H'(z) = aH(z) + bH^2(z) \quad (1.54)$$

and the Riccati equation

$$G'(z) = aG^2(z) + bG(z) + c, \quad (1.55)$$

where  $a, b$  and  $c$  are constants. We look for solutions of the nonlinear ordinary differential equation (1.53) that are of the form

$$F(z) = \sum_{i=0}^M A_i (G(z))^i, \quad (1.56)$$

where  $G(z)$  satisfies the Bernoulli or Riccati equation,  $M$  is a positive integer that can be determined by the balancing procedure and  $A_0, \dots, A_M$  are parameters to be determined.

The solution of Bernoulli equation (1.54) is given by

$$H(z) = a \left\{ \frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b \cosh[a(z+C)] - b \sinh[a(z+C)]} \right\}$$

where  $C$  is a constant of integration. For the Riccati equation (1.55), the solutions are

$$G(z) = -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left[ \frac{1}{2} \theta (z + C) \right] \quad (1.57)$$

and

$$G(z) = -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left( \frac{1}{2} \theta z \right) + \frac{\theta \operatorname{sech} [(\theta z)/2]}{C \theta \cosh [(\theta z)/2] - 2a \sinh [(\theta z)/2]} \quad (1.58)$$

with  $\theta = \sqrt{b^2 - 4ac}$  and  $C$  is a constant of integration.

### 1.6.3 The Kudryashov method

Here we present the Kudryashov method for finding exact solutions of nonlinear differential equations, which has recently appeared in [44]. We now give its description.

Consider a NLPDE, say, in two independent variables  $t$  and  $x$ , given by

$$E_1(t, x, u, u_t, u_x, u_{tt}, u_{xx}, \dots) = 0. \quad (1.59)$$

The algorithm consists of the following five steps:

- **Step 1.** The transformation  $u(x, t) = U(z)$ ,  $z = kx + \omega t$ , where  $k$  and  $\omega$  are constants, transforms equation (1.59) into the ODE

$$E_2(U, \omega U', kU', \omega^2 U'', k^2 U'', \dots) = 0. \quad (1.60)$$

- **Step 2.** The solution of equation (1.60) is expressed by a polynomial in  $Q$  as follows:

$$U(z) = \sum_{n=0}^N a_n (Q(z))^n, \quad (1.61)$$

where the coefficients  $a_n$  ( $n = 0, 1, 2, \dots, N$ ) are constants to be determined and  $Q(z)$  satisfies

$$Q'(z) = Q^2(z) - Q(z) \quad (1.62)$$

whose solution is given by

$$Q(z) = \frac{1}{1 + e^z}. \quad (1.63)$$

The positive integer  $N$  is determined by the balancing procedure.

- **Step 3.** Using (1.61) in equation (1.60) and making use of (1.62), equation (1.60) transforms into an equation in powers of  $Q(z)$ . Equating coefficients of powers of  $Q(z)$  to zero, we obtain the system of algebraic equations in the form

$$P_n(a_N, a_{N-1}, \dots, a_0, k, \omega, \dots) = 0, \quad (n = 0, \dots, N). \quad (1.64)$$

- **Step 4.** Solving the system of algebraic equations, we obtain values of coefficients  $a_n$ 's and relations for parameters of equation (1.60). As a result, we obtain exact solutions of equation (1.60) in the form (1.61).
- **Step 5.** The solution of the NLPDE (1.59) is then given by  $u(x, t) = U(kx + \omega t)$ .

#### 1.6.4 The extended Jacobi elliptic function method

In this subsection we describe the extended Jacobi elliptic function method which was introduced in [45] for finding exact solutions of nonlinear partial differential equations. The basic steps of the method are as follows: Consider a nonlinear partial differential equation in two variables

$$E(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots) = 0. \quad (1.65)$$

Making use of the transformation

$$u(t, x) = U(z), \quad z = x - \nu t \quad (1.66)$$

equation (1.65) transforms into a nonlinear ordinary differential equation

$$F(U(z), U'(z), -\nu U'(z), U''(z), -\nu U''(z), \nu^2 U''(z), \dots) = 0. \quad (1.67)$$

We consider solutions of (1.67) of the form

$$U(z) = \sum_{i=-M}^M A_i G(z)^i, \quad (1.68)$$

where  $A_i$ 's are constants to be determined,  $M$  will be determined by the homogeneous balance method and  $G(z)$  satisfies the following first-order ordinary differential equations [46]:

$$G'(z) + \sqrt{(1 - G^2(z))(1 - \omega + \omega G^2(z))} = 0, \quad (1.69)$$

$$G'(z) - \sqrt{(1 - G^2(z))(1 - \omega G^2(z))} = 0. \quad (1.70)$$

The solutions of the above equations are

$$G(z) = \text{cn}(z; \omega),$$

$$G(z) = \text{sn}(z; \omega),$$

respectively. Substituting (1.68) into (1.67) and making use of one of (1.69) or (1.70) at a time, we get a system of algebraic equations in  $A_i$  by equating the coefficients of the powers of  $G(z)$ . The solution of the algebraic system when substituted into (1.68) will give the solution of (1.67). Hence, the solution of (1.65) is found by making use of (1.66).

## 1.7 Concluding remarks

In this chapter we recalled some results from the Lie group analysis and conservation laws of partial differential equations which will be used in this thesis. In

addition, we presented algorithms of various methods that are used to find exact solutions of partial differential equations.

# Chapter 2

## Lie group classification of a variable coefficients Gardner equation

In this chapter we carry out Lie group classification of the variable coefficients Gardner equation (also called the general KdV equation) [47]

$$u_t + G(t)u^n u_x + H(t)u^{2n}u_x + R(t)u_x + F(t)u_{xxx} = 0, \quad (2.1)$$

which describes many physical phenomena, such as the long wave propagation in an inhomogeneous two-layer shallow liquid [48], ion acoustic waves in plasma with a negative ion [49] and the internal waves in a stratified ocean [50]. We first find equivalence transformations of (2.1).

The results of this chapter have been submitted [51] for possible publication.

## 2.1 Equivalence transformations

We recall that an equivalence transformation of a partial differential equation is an invertible transformation of both the independent and dependent variables mapping the PDE into a PDE of the same form, where the form of the transformed functions can, in general, be different from the form of the original function.

In this section we look for equivalence transformations of (2.1). We consider the one-parameter group of equivalence transformations in  $(t, x, u, F, G, H, R)$  given by

$$\begin{aligned}\tilde{t} &= t + \epsilon \tau(t, x, u) + O(\epsilon^2), \\ \tilde{x} &= x + \epsilon \xi(t, x, u) + O(\epsilon^2), \\ \tilde{u} &= u + \epsilon \eta(t, x, u) + O(\epsilon^2), \\ \tilde{F} &= F + \epsilon \omega^1(t, x, u, F, G, H, R) + O(\epsilon^2), \\ \tilde{G} &= G + \epsilon \omega^2(t, x, u, F, G, H, R) + O(\epsilon^2), \\ \tilde{H} &= H + \epsilon \omega^3(t, x, u, F, G, H, R) + O(\epsilon^2), \\ \tilde{R} &= R + \epsilon \omega^4(t, x, u, F, G, H, R) + O(\epsilon^2),\end{aligned}$$

where  $\epsilon$  is the group parameter. Therefore, the operator

$$Y = \tau \partial_t + \xi \partial_x + \eta \partial_u + \omega^1 \partial_F + \omega^2 \partial_G + \omega^3 \partial_H + \omega^4 \partial_R \quad (2.2)$$

is the generator of the equivalence group for (2.1) provided it is admitted by the extended system

$$u_t + G(t)u^n u_x + H(t)u^{2n}u_x + R(t)u_x + F(t)u_{xxx} = 0, \quad (2.3a)$$

$$F_x = F_u = G_x = G_u = H_x = H_u = R_x = R_u = 0. \quad (2.3b)$$

The prolonged operator for the extended system (2.3) has the form

$$\tilde{Y} = Y + \zeta_t \partial_{u_t} + \zeta_x \partial_{u_x} + \zeta_{xxx} \partial_{u_{xxx}} + \mu_x^1 \partial_{F_x} + \mu_u^1 \partial_{F_u} + \mu_x^2 \partial_{G_x} + \mu_u^2 \partial_{G_u}$$

$$+ \mu_x^3 \partial_{H_x} + \mu_u^3 \partial_{H_u} + \mu_x^4 \partial_{R_x} + \mu_u^4 \partial_{R_u}. \quad (2.4)$$

The variables  $\zeta$ 's and  $\mu$ 's are defined by the prolongation formulae

$$\begin{aligned} \zeta_t &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi), \\ \zeta_x &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi), \\ \zeta_{xx} &= D_x(\zeta_x) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi), \\ \zeta_{xxx} &= D_x(\zeta_{xx}) - u_{txx} D_x(\tau) - u_{xxx} D_x(\xi) \end{aligned}$$

and

$$\begin{aligned} \mu_x^1 &= \tilde{D}_x(\omega^1) - F_t \tilde{D}_x(\tau) - F_x \tilde{D}_x(\xi) - F_u \tilde{D}_x(\eta), \\ \mu_u^1 &= \tilde{D}_u(\omega^1) - F_t \tilde{D}_u(\tau) - F_x \tilde{D}_u(\xi) - F_u \tilde{D}_u(\eta), \\ \mu_x^2 &= \tilde{D}_x(\omega^2) - G_t \tilde{D}_x(\tau) - G_x \tilde{D}_x(\xi) - G_u \tilde{D}_x(\eta), \\ \mu_u^2 &= \tilde{D}_u(\omega^2) - G_t \tilde{D}_u(\tau) - G_x \tilde{D}_u(\xi) - G_u \tilde{D}_u(\eta), \\ \mu_x^3 &= \tilde{D}_x(\omega^3) - H_t \tilde{D}_x(\tau) - H_x \tilde{D}_x(\xi) - H_u \tilde{D}_x(\eta), \\ \mu_u^3 &= \tilde{D}_u(\omega^3) - H_t \tilde{D}_u(\tau) - H_x \tilde{D}_u(\xi) - H_u \tilde{D}_u(\eta), \\ \mu_x^4 &= \tilde{D}_x(\omega^4) - R_t \tilde{D}_x(\tau) - R_x \tilde{D}_x(\xi) - R_u \tilde{D}_x(\eta), \\ \mu_u^4 &= \tilde{D}_u(\omega^4) - R_t \tilde{D}_u(\tau) - R_x \tilde{D}_u(\xi) - R_u \tilde{D}_u(\eta), \end{aligned}$$

respectively, where

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + \cdots, \quad D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + \cdots$$

are the total derivative operators and

$$\begin{aligned} \tilde{D}_x &= \frac{\partial}{\partial x} + F_x \frac{\partial}{\partial F} + G_x \frac{\partial}{\partial G} + H_x \frac{\partial}{\partial H} + R_x \frac{\partial}{\partial R} + \cdots, \\ \tilde{D}_u &= \frac{\partial}{\partial u} + F_u \frac{\partial}{\partial F} + G_u \frac{\partial}{\partial G} + H_u \frac{\partial}{\partial H} + R_u \frac{\partial}{\partial R} + \cdots \end{aligned}$$

are the total derivative operators of the extended system (2.3). Applying (2.4) to the extended system (2.3) and then splitting on the derivatives of  $u$  we obtain the

following overdetermined system of linear partial differential equations:

$$\begin{aligned}
\tau_x = \tau_u = 0, \quad \omega_x^1 = \omega_u^1 = \omega_x^2 = \omega_u^2 = \omega_x^3 = \omega_u^3 = \omega_x^4 = \omega_u^4 = 0, \quad \xi_u = 0, \quad \eta_{uu} = 0, \\
\eta_{xu} - \xi_{xx} = 0, \quad \omega^1 + (\tau_t - 3\xi_x)F = 0, \quad \eta_t + \eta_x (R + Gu^n + Hu^{2n}) + F\eta_{xxx} = 0, \\
\omega^4 + \omega^2 u^n + \omega^3 u^{2n} + nGu^{n-1}\eta + 2nHu^{2n-1}\eta - \xi_t - F\xi_{xxx} + 3F\eta_{xxu} \\
+ (\tau_t - \xi_x) (R + Gu^n + Hu^{2n}) = 0.
\end{aligned}$$

Solving the above system we get

$$\begin{aligned}
\tau &= a(t), \\
\xi &= k_2 x + b(t), \\
\eta &= k_1 u, \\
\omega^1 &= (3k_2 - a'(t)) F, \\
\omega^2 &= -nk_1 G + (k_2 - a'(t)) G, \\
\omega^3 &= -2nk_1 H + (k_2 - a'(t)) H, \\
\omega^4 &= (k_2 - a'(t)) R + b'(t),
\end{aligned}$$

where  $k_1$  and  $k_2$  are constants and  $a(t)$  and  $b(t)$  are arbitrary functions of  $t$ .

Thus, the equivalence generators of class (2.1) are

$$\begin{aligned}
Y_1 &= u\partial_u - nG\partial_G - 2nH\partial_H, \\
Y_2 &= x\partial_x + 3F\partial_F + G\partial_G + H\partial_H + R\partial_R, \\
Y_a &= a(t)\partial_t - a'(t)F\partial_F - a'(t)G\partial_G - a'(t)H\partial_H - a'(t)R\partial_R, \\
Y_b &= b(t)\partial_x + b'(t)\partial_R.
\end{aligned}$$

Thus, the equivalence group corresponding to each of the equivalence generators is given by

$$Y_1 : \tilde{t} = t, \tilde{x} = x, \tilde{u} = ue^{c_1}, \tilde{F} = F, \tilde{G} = Ge^{-nc_1}, \tilde{H} = He^{-2nc_1}, \tilde{R} = R,$$

$$\begin{aligned}
Y_2 : \quad & \tilde{t} = t, \tilde{x} = xe^{c_2}, \tilde{u} = u, \tilde{F} = Fe^{3c_2}, \tilde{G} = Ge^{c_2}, \tilde{H} = He^{c_2}, \tilde{R} = Re^{c_2}, \\
Y_a : \quad & \tilde{t} = a(t), \tilde{x} = x, \tilde{u} = u, \tilde{F} = \frac{F}{a'(t)}, \tilde{G} = \frac{G}{a'(t)}, \tilde{H} = \frac{H}{a'(t)}, \tilde{R} = \frac{R}{a'(t)}, \\
Y_b : \quad & \tilde{t} = t, \tilde{x} = x + b(t)c_4, \tilde{u} = u, \tilde{F} = F, \tilde{G} = G, \tilde{H} = H, \tilde{R} = R + b'(t)c_4
\end{aligned}$$

and their composition gives

$$\begin{aligned}
\tilde{t} &= a(t), \quad \tilde{x} = (x + b(t)c_4)e^{c_2}, \quad \tilde{u} = ue^{c_1}, \quad \tilde{F} = \frac{Fe^{3c_2}}{a'(t)}, \\
\tilde{G} &= \frac{Ge^{c_2-nc_1}}{a'(t)}, \quad \tilde{H} = \frac{He^{c_2-2nc_1}}{a'(t)}, \quad \tilde{R} = \frac{(R + b'(t)c_4)e^{c_2}}{a'(t)}. \tag{2.5}
\end{aligned}$$

Since there are two arbitrary functions  $a(t)$  and  $b(t)$  in (2.5), one can rescale two of the arbitrary functions of (2.1) [52, 53]. Thus, we set  $\tilde{F} = \tilde{R} = 1$  by the equivalence transformation

$$\tilde{t} = \int Fe^{3c_2} dt, \quad \tilde{x} = \left( x + \int (Fe^{2c_2} - R) dt \right) e^{c_2}, \quad \tilde{u} = ue^{c_1}, \tag{2.6}$$

which transforms equation (2.1) into an equivalent equation

$$\tilde{u}_{\tilde{t}} + \tilde{u}_{\tilde{x}} + \tilde{G}(\tilde{t})\tilde{u}^{\tilde{n}}\tilde{u}_{\tilde{x}} + \tilde{H}(\tilde{t})\tilde{u}^{2\tilde{n}}\tilde{u}_{\tilde{x}} + \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} = 0,$$

where

$$\tilde{G} = \frac{Ge^{-(nc_1+2c_2)}}{F}, \quad \tilde{H} = \frac{He^{-2(nc_1+c_2)}}{F}.$$

Therefore, without loss of generality, we can confine our study to the equation

$$u_t + u_x + G(t)u^n u_x + H(t)u^{2n} u_x + u_{xxx} = 0. \tag{2.7}$$

## 2.2 Principal Lie algebra and classifying relations of (2.7)

The symmetry group of equation (2.7) will be generated by the vector field of the form

$$X = \tau(t, x, u) \frac{\partial}{\partial t} + \xi(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u}.$$

Applying the third prolongation of  $X$  to (2.7) and splitting on the derivatives of  $u$  yields the following overdetermined system of linear PDEs:

$$\begin{aligned} \tau_u = 0, \quad \xi_u = 0, \quad \eta_{uu} = 0, \quad \tau_x = 0, \quad 3\eta_{xu} - 3\xi_{xx} = 0, \quad \tau_t - 3\xi_x = 0, \\ \eta_t + G(t)u^n\eta_x + H(t)u^{2n}\eta_x + \eta_x + \eta_{xxx} = 0, \\ u^n G_t \tau + u^{2n} H_t \tau + nG(t)u^{n-1}\eta + 2nH(t)u^{2n-1}\eta - G(t)u^n \xi_x + G(t)u^n \tau_t \\ - H(t)u^{2n} \xi_x + H(t)u^{2n} \tau_t - \xi_t + \tau_t + 3\eta_{xxu} - \xi_x - \xi_{xxx} = 0. \end{aligned}$$

Solving the above system, we obtain

$$\tau = a(t), \quad \xi = d(t) + \frac{1}{3}xa'(t), \quad \eta = b(t, x) + u \left( c(t) + \frac{a'(t)}{3} \right), \quad (2.8a)$$

$$G(t)u^n b_x + H(t)u^{2n} b_x + b_t + b_x + b_{xxx} + u \left( c' + \frac{1}{3}a'' \right) = 0, \quad (2.8b)$$

$$\begin{aligned} nG(t)u^{n-1}b(t, x) + 2nH(t)u^{2n-1}b(t, x) - d' + \frac{2}{3}a' - \frac{1}{3}xa'' \\ + \left( G'a(t) + \frac{1}{3}nG(t)a' + \frac{2}{3}G(t)a' + nc(t)G(t) \right) u^n \\ + \left( H'a(t) + \frac{2}{3}nH(t)a' + \frac{2}{3}H(t)a' + 2nc(t)H(t) \right) u^{2n} = 0, \quad (2.8c) \end{aligned}$$

where  $a(t), b(t, x), c(t)$  and  $d(t)$  are arbitrary functions of their variables. In order to find the principal Lie algebra admitted by any equation of class (2.7) we solve equations (2.8) for arbitrary functions  $G$  and  $H$ . This results in  $\tau = \eta = 0$  and  $\xi = \text{const}$ . Hence the principal Lie algebra consists of one space translation symmetry, namely,

$$X_1 = \frac{\partial}{\partial x}.$$

## 2.3 Lie group classification

The analysis of equations (2.8b) and (2.8c) leads to the following five cases:

**Case 1**  $G(t) = A(\beta + t)^{\frac{1}{3}(-3\alpha n - n - 2)}$ ,  $H(t) = B(\beta + t)^{-\frac{2}{3}(3\alpha n + n + 1)}$  with  $n \neq 0, 1$  and  $A, B, \alpha, \beta$  constants

In this case the principal Lie algebra is extended by one operator, viz.,

$$X_2 = 3(t + \beta) \frac{\partial}{\partial t} + (2t + x) \frac{\partial}{\partial x} + u(3\alpha + 1) \frac{\partial}{\partial u}.$$

**Case 2**  $G(t) = Ae^{-\lambda nt}$ ,  $H(t) = Be^{-2\lambda nt}$ ,  $A, B, \lambda$  constants

The principal Lie algebra is extended by the operator

$$X_2 = \frac{\partial}{\partial t} + \lambda u \frac{\partial}{\partial u}.$$

**Case 3**  $G(t) = H(t) = 0$

The principal Lie algebra extends by four Lie point symmetries

$$\begin{aligned} X_2 &= \frac{\partial}{\partial t}, \\ X_2 &= 3t \frac{\partial}{\partial t} + (2t + x) \frac{\partial}{\partial x}, \\ X_4 &= u \frac{\partial}{\partial u}, \\ X_b &= b(t, x) \frac{\partial}{\partial u}, \end{aligned}$$

where  $b(t, x)$  satisfies  $b_t + b_x + b_{xxx} = 0$ .

**Case 4**  $G(t) = A(\beta + t)^{-\alpha-1} + \frac{6B\lambda}{3\alpha+1}(\beta + t)^{-\frac{2}{3}(3\alpha+2)}$ ,  $H(t) = B(\beta + t)^{-\frac{2}{3}(3\alpha+2)}$  with  $n = 1$  and  $\alpha \neq -1/3, -1/6$ ,  $A, B, \alpha, \beta, \lambda$  constants

The extension of the principal Lie algebra in this case is given by

$$\begin{aligned} X_2 &= (t + \beta) \frac{\partial}{\partial t} + \frac{1}{3} \left( 2t + x - \frac{54B\lambda^2(\beta + t)^{-2\alpha - \frac{1}{3}}}{(3\alpha + 1)(6\alpha + 1)} - 2\beta - \frac{3A\lambda(\beta + t)^{-\alpha}}{\alpha} \right) \frac{\partial}{\partial x} \\ &\quad + \left\{ \lambda + \left( \frac{1}{3} + \alpha \right) u \right\} \frac{\partial}{\partial u} \end{aligned}$$

**4.1**  $G(t) = \{A - 2B\lambda \ln(\beta + t)\}(\beta + t)^{-2/3}$ ,  $H(t) = B(\beta + t)^{-2/3}$

The principal Lie algebra extends by

$$X_2 = (t + \beta) \frac{\partial}{\partial t} + \frac{1}{3} \left[ 9\lambda \sqrt[3]{\beta + t} \{A + 6B\lambda - 2B\lambda \ln(\beta + t)\} + 2(\beta + t) + x \right] \frac{\partial}{\partial x}$$

$$+ \lambda \frac{\partial}{\partial u}$$

$$\mathbf{4.2} \quad G(t) = A(\beta + t)^{-5/6} + 12B\lambda(\beta + t)^{-1}, \quad H(t) = B(\beta + t)^{-1}$$

The extension of the principal Lie algebra is given by

$$X_2 = (t + \beta) \frac{\partial}{\partial t} + \frac{1}{3} \left[ 2 \left\{ \beta + t + 9A\lambda^6 \sqrt{\beta + t} + 18B\lambda^2 \ln(\beta + t) \right\} + x \right] \frac{\partial}{\partial x} + \left( \lambda + \frac{u}{6} \right) \frac{\partial}{\partial u}.$$

$$\mathbf{Case 5} \quad G(t) = Ae^{-\mu t} + \frac{2B\nu}{\mu} e^{-2\mu t}, \quad H(t) = Be^{-2\mu t}, \quad A, B, \nu, \mu \neq 0 \text{ constants}$$

The principal Lie algebra in this case is extended by

$$X_2 = \frac{\partial}{\partial t} - \frac{\nu}{\mu^2} e^{-2\mu t} (A\mu e^{\mu t} + B\nu) \frac{\partial}{\partial x} + (\nu + \mu u) \frac{\partial}{\partial u}.$$

$$\mathbf{5.1} \quad G(t) = A - 2B\nu t, \quad H(t) = B$$

The principal Lie algebra extends by the following operator:

$$X_2 = \frac{\partial}{\partial t} + \nu t(A - B\nu t) \frac{\partial}{\partial x} + \nu \frac{\partial}{\partial u}.$$

## 2.4 Symmetry reductions and group-invariant solutions

In this section we find symmetry reductions and group-invariant solutions for two particular cases of equation (2.7). In order to find symmetry reductions and group-invariant solutions, one has to solve the associated Lagrange system

$$\frac{dt}{\tau(t, x, u)} = \frac{dx}{\xi(t, x, u)} = \frac{du}{\eta(t, x, u)}.$$

**Case (i)**  $G(t) = A(\beta + t)^{\frac{1}{3}(-3\alpha n - n - 2)}$ ,  $H(t) = B(\beta + t)^{-\frac{2}{3}(3\alpha n + n + 1)}$

In this case equation (2.7) becomes

$$u_t + A(\beta + t)^{\frac{1}{3}(-3\alpha n - n - 2)}u^n u_x + B(\beta + t)^{-\frac{2}{3}(3\alpha n + n + 1)}u^{2n}u_x + u_x + u_{xxx} = 0.$$

We now find group-invariant solution of this equation under the symmetry

$$X = 3(t + \beta)\frac{\partial}{\partial t} + (2t + x)\frac{\partial}{\partial x} + u(3\alpha + 1)\frac{\partial}{\partial u}.$$

The two invariants are found from the solutions of the associated Lagrange system and are given by

$$I_1 = \frac{x - t - 3\beta}{\sqrt[3]{\beta + t}}, \quad I_2 = u(t + \beta)^{-\alpha - 1/3}.$$

Hence, the group-invariant solution in this case is

$$u = (t + \beta)^{\alpha + 1/3}f(z), \quad z = (x - t - 3\beta)(t + \beta)^{-1/3}$$

where  $f(z)$  satisfies the following nonlinear ODE:

$$3f'''(z) + f'(z)(3Af(z)^n + 3Bf(z)^{2n} - z) + f(z)(3\alpha + 1) = 0.$$

**Case (ii)**  $G(t) = Ae^{-\lambda nt}$ ,  $H(t) = Be^{-2\lambda nt}$

In this case equation (2.7) is given by

$$u_t + Ae^{-\lambda nt}u^n u_x + Be^{-2\lambda nt}u^{2n}u_x + u_x + u_{xxx} = 0.$$

We find group-invariant solutions of this equation using the operator

$$X_2 = \frac{\partial}{\partial t} + \lambda u \frac{\partial}{\partial u}.$$

This operator  $X_2$  has two invariants  $I_1 = x$  and  $I_2 = ue^{-\lambda t}$  and hence the group-invariant solution is

$$u = e^{\lambda t}f(x),$$

where  $f(x)$  satisfies the nonlinear ODE

$$f'''(x) + f'(x)(Af(x)^n + Bf(x)^{2n} + 1) + \lambda f(x) = 0.$$

## 2.5 Conservation laws

We now construct conservation laws for the variable coefficients Gardner equation (2.7) using the multiplier approach for two cases.

**Case A**  $G(t) = A(\beta + t)^{\frac{1}{3}(-3\alpha n - n - 2)}$ ,  $H(t) = B(\beta + t)^{-\frac{2}{3}(3\alpha n + n + 1)}$

In this case equation (2.7) becomes

$$u_t + At^{-(n+2+3\alpha n)/3}u^n u_x + Bt^{-2(n+1+3\alpha n)/3}u^{2n}u_x + u_x + u_{xxx} = 0 \quad \text{with } \beta = 0.$$

We look for second-order multiplier of the form

$$\Lambda = \Lambda(t, x, u).$$

Now following the procedure given in Section 1.5.3, the zeroth-order multiplier is given by  $\Lambda(t, x, u) = C_1 u + C_2$ , where  $C_1$  and  $C_2$  are arbitrary constants. Corresponding to the above multiplier we obtain the following two conservation laws:

$$\begin{aligned} T_1^t &= \frac{1}{(k+1)(l+1)} (B(l+1)t^{k+1}u^{2n+1}u_x + A(k+1)t^{l+1}u^{n+1}u_x) + tuu_x + \frac{1}{2}u^2, \\ T_1^x &= \frac{-1}{(k+1)(l+1)} (B(l+1)t^{k+1}u^{2n+1}u_t + A(k+1)t^{l+1}u^{n+1}u_t) - tuu_t \\ &\quad - \frac{1}{2}u_x^2 + uu_{xx}; \end{aligned}$$

$$\begin{aligned} T_2^t &= \frac{1}{(k+1)(l+1)} (B(l+1)t^{k+1}u^{2n}u_x + A(k+1)t^{l+1}u^n u_x) + tu_x + u, \\ T_2^x &= \frac{-1}{(k+1)(l+1)} (B(l+1)t^{k+1}u^{2n}u_t + A(k+1)t^{l+1}u^n u_t) + tu_t - u_{xx}, \end{aligned}$$

where  $l = -(2 + n + 3\alpha n)/3$ ,  $k = -2(1 + n + 3\alpha n)/3$  with  $\alpha \neq (1 - n)/(3n)$  and  $\alpha \neq (1 - 2n)/(6n)$ .

**Case B**  $G(t) = Ae^{-\lambda nt}$ ,  $H(t) = Be^{-2\lambda nt}$

Here equation (2.7) becomes

$$u_t + Ae^{-\lambda nt}u^n u_x + Be^{-2\lambda nt}u^{2n}u_x + u_x + u_{xxx} = 0.$$

The zeroth-order multiplier is  $\Lambda(t, x, u) = C_1u + C_2$  and hence corresponding to this multiplier we have the following two conserved vectors:

$$T_1^t = -\frac{1}{2\lambda n} (Be^{-2\lambda nt}u^{2n+1}u_x + 2Ae^{-\lambda nt}u^{n+1}u_x - 2\lambda ntuu_x - \lambda nu^2),$$

$$T_1^x = \frac{1}{2\lambda n} (Be^{-2\lambda nt}u^{2n+1}u_t + 2Ae^{-\lambda nt}u^{n+1}u_t - 2\lambda ntuu_t - \lambda nu_x^2 + 2\lambda nuu_{xx});$$

$$T_2^t = -\frac{1}{2\lambda n} (Be^{-2\lambda nt}u^{2n}u_x + 2Ae^{-\lambda nt}u^n u_x - 2\lambda ntu_x - 2\lambda nu),$$

$$T_2^x = \frac{1}{2\lambda n} (Be^{-2\lambda nt}u^{2n}u_t + 2Ae^{-\lambda nt}u^n u_t - 2\lambda ntu_t + 2\lambda nu_{xx}).$$

## 2.6 Concluding remarks

In this chapter we carried out Lie group classification of the Gardner equation with variable coefficients. This was achieved by first determining the equivalence transformations for the variable coefficients Gardner equation (2.1). The transformations were then used to rescale some arbitrary functions in equation (2.1), which simplified the original equation to an equivalent equation (2.7). We then studied equation (2.7). It was found that the equivalent Gardner equation (2.7) had a translation symmetry in space variable  $x$  as its kernel algebra. The functions  $G(t)$  and  $H(t)$  that were able to extend the principal Lie algebra were found to be exponential, power, logarithmic and linear functions. Symmetry reductions were performed for two cases which extended the principal Lie algebra. Finally, for two cases we obtained conservation laws using the multiplier method.

## Chapter 3

# Cnoidal and snoidal waves solutions and conservation laws of a generalized (2+1)-dimensional Kortweg-de Vries equation

A nonlinear integrable (2+1)-dimensional Korteweg-de Vries equation

$$4u_t - \alpha(t, y) (4uu_y + 2u_x \partial^{-1} u_y + u_{xxy}) - \beta(t, y) (6uu_x + u_{xxx}) = 0 \quad (3.1)$$

was constructed in [54] using the Lax pair generating technique. For  $\alpha(t, y) = -4$  and  $\beta(t, y) = -4$  the two-solitary wave solution was obtained by employing the singular manifold method and the Bäcklund transformation in terms of the singular manifold was derived [54].

Recently, in [55] the integrability of (3.1) was investigated for  $\alpha(t, y) = a(t)$  and  $\beta(t, y) = b(t)$ . By employing the binary Bell polynomials, its bilinear formalism, bilinear Bäcklund transformation, Lax pair and Darboux covariant Lax pair were precisely constructed.

In this chapter we study the generalized  $(2 + 1)$ -dimensional Korteweg-de Vries equation

$$u_t + 2auu_y + au_x \partial^{-1} u_y + 3auu_x + u_{xxx} + u_{xxy} = 0, \quad (3.2)$$

where  $a$  is a non-zero real-valued constant. This equation is obtained from (3.1) by taking  $\alpha = \beta = -1$  and then generalizing it by replacing 2 by  $a$  in the three terms. In order to study this equation we first eliminate the integral appearing in (3.2) by letting  $v = \int u_y dx$ . This substitution then transforms equation (3.2) into a system of two partial differential equations in the dependent variables  $u$  and  $v$ , namely,

$$u_t + 2auu_y + avu_x + 3auu_x + u_{xxx} + u_{xxy} = 0, \quad (3.3a)$$

$$u_y - v_x = 0. \quad (3.3b)$$

The results of this chapter have been accepted for publication [56].

## 3.1 Exact solutions of (3.2) and (3.3)

In this section we construct some solutions of (3.2) and (3.3).

### 3.1.1 Exact solution of (3.3) using its Lie symmetries

In this subsection we obtain exact solutions of (3.3) using the Lie point symmetries of (3.3). The vector field

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \phi^1 \frac{\partial}{\partial u} + \phi^2 \frac{\partial}{\partial v},$$

where the coefficients  $\xi^1, \xi^2, \xi^3, \phi^1$  and  $\phi^2$  are functions of  $t, x, y, u$  and  $v$  is a Lie point symmetry of the system (3.3) provided

$$X^{[3]}(u_t + 2auu_y + avu_x + 3auu_x + u_{xxx} + u_{xxy})|_{(3.3)} = 0, \quad (3.4a)$$

$$X^{[3]}(u_y - v_x) |_{(3.3)} = 0. \quad (3.4b)$$

Here  $X^{[3]}$  denotes the third prolongation of the vector field  $X$ . Writing out the two equations of (3.4) and splitting on the derivatives of  $u$  and  $v$  yields an overdetermined system of twenty-one linear partial differential equations. These are

$$\begin{aligned} \phi_v^1 &= 0, \quad \xi_v^1 = 0, \quad \xi_v^2 = 0, \quad \xi_v^3 = 0, \quad \xi_u^1 = 0, \quad \xi_u^2 = 0, \quad \xi_u^3 = 0, \quad \phi_{uu}^1 = 0, \\ \xi_y^1 &= 0, \quad \xi_x^1 = 0, \quad \xi_x^3 = 0, \quad \phi_y^1 - \phi_x^2 = 0, \quad \xi_y^2 + \phi_u^2 = 0, \quad 2\phi_{xu}^1 - \xi_{xx}^2 = 0, \\ 2a\phi^1 + 2au\xi_t^1 - \xi_t^3 - 2au\xi_y^3 + \phi_{xxu}^1 &= 0, \\ 3a\phi^1 + a\phi^2 + 3au\xi_t^1 + av\xi_t^1 - \xi_t^2 - 3au\xi_x^2 - av\xi_x^2 - 2au\xi_y^2 - \xi_{xx}^2 \\ - \xi_{xy}^2 + 3\phi_{xxu}^1 + 2\phi_{xyu}^1 &= 0, \quad \xi_t^1 - 2\xi_x^2 - \xi_y^3 = 0, \quad \xi_t^1 - 3\xi_x^2 - \xi_y^2 = 0, \\ \phi_t^1 + 3au\phi_x^1 + av\phi_x^1 + 2au\phi_y^1 + \phi_{xxx}^1 + \phi_{xxy}^1 &= 0, \\ 3\xi_{xx}^2 + 2\xi_{xy}^2 - 3\phi_{xu}^1 - \phi_{yu}^1 = 0, \quad \xi_x^2 - \xi_y^3 + \phi_u^1 - \phi_v^2 &= 0. \end{aligned}$$

Solving the above system we obtain the symmetry algebra of the system (3.3) spanned by operators

$$\begin{aligned} X_1 &= \frac{\partial}{\partial y}, \\ X_2 &= \frac{\partial}{\partial t}, \\ X_3 &= \mathcal{F}(t) \frac{\partial}{\partial x} + \frac{1}{a} \mathcal{F}'(t) \frac{\partial}{\partial v}, \\ X_4 &= 2at \frac{\partial}{\partial y} + \frac{\partial}{\partial u} - 3 \frac{\partial}{\partial v}, \\ X_5 &= (x - 3y) \frac{\partial}{\partial x} - 2y \frac{\partial}{\partial y} - 2u \frac{\partial}{\partial u} + (3u + v) \frac{\partial}{\partial v}, \\ X_6 &= 4t \frac{\partial}{\partial t} + (x + y) \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y} - 2u \frac{\partial}{\partial u} + (-u - 3v) \frac{\partial}{\partial v}, \\ X_7 &= 2t^2 \frac{\partial}{\partial t} + t(x + y) \frac{\partial}{\partial x} + 2ty \frac{\partial}{\partial y} + \left( \frac{y}{a} - 2tu \right) \frac{\partial}{\partial u} \\ &\quad + \left( \frac{x}{a} - \frac{2y}{a} - tu - 3tv \right) \frac{\partial}{\partial v}. \end{aligned}$$

We now use the three translation symmetries  $X_1$ ,  $X_2$  and  $X_3$  with  $\mathcal{F}(t) = 1$ . The sum of these three symmetries provides us with four invariants, viz.,

$$p = t - y, \quad q = x - y, \quad r = u, \quad s = v.$$

Using the above invariants, system (3.3) transforms into a coupled system of non-linear PDEs

$$r_p + asr_q + ar(r_q - 2r_p) - r_{pqq} = 0, \quad (3.5a)$$

$$r_p + r_q + s_q = 0 \quad (3.5b)$$

in two independent variables  $p$  and  $q$ . This new system has three Lie point symmetries, namely,

$$\Gamma_1 = \frac{\partial}{\partial p}, \quad \Gamma_2 = \frac{\partial}{\partial q}, \quad \Gamma_3 = aq \frac{\partial}{\partial q} - 4ap \frac{\partial}{\partial p} + (1 - 2ar) \frac{\partial}{\partial r} + (5ar + 3as - 1) \frac{\partial}{\partial s}$$

and utilizing the symmetry  $\alpha\Gamma_1 + \Gamma_2$  yields the invariants

$$\zeta = p - \alpha q, \quad H = r, \quad J = s.$$

Now these invariants transform the system (3.5) into the coupled system of non-linear ordinary differential equations

$$\alpha^2 H'''(\zeta) + a(\alpha + 2)H(\zeta)H'(\zeta) + \alpha\alpha J(\zeta)H'(\zeta) - H'(\zeta) = 0, \quad (3.6a)$$

$$(\alpha - 1)H'(\zeta) + \alpha J'(\zeta) = 0. \quad (3.6b)$$

Integrating equation (3.6b) and solving for  $J$ , we obtain

$$J(\zeta) = C_1 - \frac{\alpha - 1}{\alpha} H(\zeta), \quad (3.7)$$

where  $C_1$  is an arbitrary constant of integration. Substituting the value of  $J(\zeta)$  from equation (3.7) into equation (3.6a) leads to

$$\alpha^2 H'''(\zeta) + 3aH(\zeta)H'(\zeta) + (a\alpha C_1 - 1)H'(\zeta) = 0. \quad (3.8)$$

Integration of equation (3.8) twice with respect to  $\zeta$  yields

$$H'^2(\zeta) + \frac{a}{\alpha^2}H(\zeta)^3 + \frac{a\alpha C_1 - 1}{\alpha^2}H(\zeta)^2 + 2C_2H(\zeta) + C_3 = 0, \quad (3.9)$$

where  $C_2$  and  $C_3$  are arbitrary constants of integration. To solve equation (3.9), we assume that  $\lambda_1, \lambda_2$  and  $\lambda_3$  (where  $\lambda_1 \geq \lambda_2 \geq \lambda_3$ ) are roots of the algebraic equation [57]

$$\frac{a}{\alpha^2}H(\zeta)^3 + \frac{a\alpha C_1 - 1}{\alpha^2}H(\zeta)^2 + 2C_2H(\zeta) + C_3 = 0.$$

Thus, equation (3.9) can be rewritten in the form

$$H'(\zeta)^2 + \frac{a}{\alpha^2}(H(\zeta) - \lambda_1)(H(\zeta) - \lambda_2)(H(\zeta) - \lambda_3) = 0$$

and consequently its general solution is given by the Jacobi elliptic function as [57]

$$H(\zeta) = \lambda_2 + (\lambda_1 - \lambda_2)\text{cn}^2\left(\sqrt{\frac{a}{4\alpha^2}(\lambda_1 - \lambda_3)}\zeta; S^2\right), \quad S^2 = \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_3},$$

where  $\text{cn}$  is the elliptic cosine function [46]. Now  $J(\zeta)$  can be found from equation (3.7) by substituting the value of  $H(\zeta)$  into (3.7). Thus

$$J(\zeta) = C_1 - \frac{\alpha - 1}{\alpha} \left\{ \lambda_2 + (\lambda_1 - \lambda_2)\text{cn}^2\left(\sqrt{\frac{a}{4\alpha^2}(\lambda_1 - \lambda_3)}\zeta; S^2\right) \right\}$$

and consequently

$$u(t, x, y) = \lambda_2 + (\lambda_1 - \lambda_2)\text{cn}^2\left(\sqrt{\frac{a}{4\alpha^2}(\lambda_1 - \lambda_3)}\zeta; S^2\right),$$

$$v(t, x, y) = C_1 - \frac{\alpha - 1}{\alpha} \left\{ \lambda_2 + (\lambda_1 - \lambda_2)\text{cn}^2\left(\sqrt{\frac{a}{4\alpha^2}(\lambda_1 - \lambda_3)}\zeta; S^2\right) \right\},$$

where  $\zeta = t - \alpha x - (1 - \alpha)y$ , is a solution of the system (3.3).

### 3.1.2 Exact solutions of (3.2) and (3.3) using the extended Jacobi elliptic function method

In this section we employ the extended Jacobi elliptic function method [45, 58] to obtain more exact solutions of (3.2) and (3.3). We assume that the solutions of

the ordinary differential equations system (3.6) can be written in the form

$$H(\zeta) = \sum_{i=-M}^M A_i G(\zeta)^i, \quad J(\zeta) = \sum_{i=-N}^N B_i G(\zeta)^i,$$

where  $M$  and  $N$  are positive integers,  $A_i$  and  $B_i$  are constants to be determined. The function  $G(\zeta)$  satisfies the first-order ordinary differential equation

$$G'(\zeta) + \sqrt{(1 - G^2(\zeta))(1 - \omega + \omega G^2(\zeta))} = 0, \quad (3.10)$$

whose solution is given by [46]

$$G(\zeta) = \text{cn}(\zeta; \omega).$$

The balancing procedure yields  $M = N = 2$  and hence the solutions of (3.6) are of the form

$$H(\zeta) = A_{-2}G^{-2} + A_{-1}G^{-1} + A_0 + A_1G + A_2G^2, \quad (3.11a)$$

$$J(\zeta) = B_{-2}G^{-2} + B_{-1}G^{-1} + B_0 + B_1G + B_2G^2. \quad (3.11b)$$

The substitution of (3.11) into (3.6) and using (3.10) and then equating the coefficients of  $G^i$  to zero, one obtains the following algebraic system of thirteen equations in  $A_i$  and  $B_i$  ( $i = -2, -1, 0, 1, 2$ ):

$$(\alpha - 1)A_{-2} + \alpha B_{-2} = 0,$$

$$(\alpha - 1)A_{-1} + \alpha B_{-1} = 0,$$

$$(\alpha - 1)A_1 + \alpha B_1 = 0,$$

$$(\alpha - 1)A_2 + \alpha B_2 = 0,$$

$$A_1(\alpha(6\alpha\omega - aB_2) - 3a(\alpha + 2)A_2) - 2a\alpha A_2 B_1 = 0,$$

$$a(\alpha + 2)A_{-2}^2 + \alpha A_{-2}(aB_{-2} - 12\alpha(\omega - 1)) = 0,$$

$$a(\alpha + 2)A_2^2 + \alpha(aB_2 - 12\alpha\omega)A_2 = 0,$$

$$\alpha A_{-1}(aB_{-2} - 6\alpha(\omega - 1)) + aA_{-2}(3(\alpha + 2)A_{-1} + 2\alpha B_{-1}) = 0,$$

$$\begin{aligned}
& a\alpha (2A_2B_{-2} + A_1B_{-1}) - a\alpha A_{-1}B_1 - 2a\alpha A_{-2}B_2 = 0, \\
& a(\alpha + 2)A_{-1}^2 + 2A_{-2} (8\alpha^2\omega - 4\alpha^2 + a(\alpha + 2)A_0 + a\alpha B_0 - 1) + a\alpha A_{-1}B_{-1} = 0, \\
& a(\alpha + 2)A_1^2 + 2A_2 (8\alpha^2\omega - 4\alpha^2 + a(\alpha + 2)A_0 + a\alpha B_0 - 1) + a\alpha A_1B_1 = 0, \\
& A_{-1} (2\alpha^2\omega - \alpha^2 + a(\alpha + 2)A_0 + a\alpha B_0 - 1) - a\alpha A_1B_{-2} \\
& + aA_{-2} ((\alpha + 2)A_1 + 2\alpha B_1) = 0, \\
& A_1 (2\alpha^2\omega - \alpha^2 + a(\alpha + 2)A_0 + a\alpha B_0 - 1) + 2a\alpha A_2B_{-1} \\
& + aA_{-1} ((-\alpha - 2)A_2 + \alpha B_2) = 0.
\end{aligned}$$

Using Mathematica one can solve the above system of algebraic equations. This gives

$$\begin{aligned}
A_{-2} &= \frac{4\alpha^2(\omega - 1)}{a}, \quad A_{-1} = 0, \quad A_1 = 0, \quad A_2 = \frac{4\alpha^2\omega}{a}, \\
B_{-2} &= \frac{4\alpha(1 - \alpha)(\omega - 1)}{a}, \quad B_{-1} = 0, \quad B_0 = \frac{4\alpha^2(1 - 2\omega) - a(\alpha + 2)A_0 + 1}{a\alpha}, \\
B_1 &= 0, \quad B_2 = \frac{4\alpha\omega(1 - \alpha)}{a}.
\end{aligned}$$

As a result, a solution of the system (3.3) is

$$\begin{aligned}
u(t, x, y) &= A_{-2}\text{nc}(\zeta; \omega)^2 + A_0 + A_2\text{cn}(\zeta; \omega)^2, \\
v(t, x, y) &= B_{-2}\text{nc}(\zeta; \omega)^2 + B_0 + B_2\text{cn}(\zeta; \omega)^2,
\end{aligned}$$

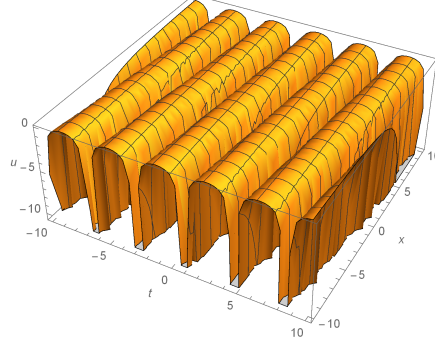
where  $\zeta = t - \alpha x - (1 - \alpha)y$ ,  $A_0$  is an arbitrary constant and  $\text{nc} = 1/\text{cn}$ . However, the solution of the (2+1)-dimensional KdV equation (3.2) is given by

$$u(t, x, y) = A_{-2}\text{nc}(\zeta; \omega)^2 + A_0 + A_2\text{cn}(\zeta; \omega)^2, \quad (3.12)$$

where

$$A_{-2} = \frac{4\alpha^2(\omega - 1)}{a}, \quad A_0 = \frac{1 + 4\alpha^2 - 8\alpha^2\omega}{a(\alpha + 2)}, \quad A_2 = \frac{4\alpha^2\omega}{a}.$$

A profile of the solution (3.12) is given in Figure 1.



**Figure 3.1:** Profile of the cnoidal wave solution (3.12)

Likewise, by using the auxiliary equation

$$G'(\zeta) - \sqrt{(1 - G^2(z))(1 - \omega G^2(\zeta))} = 0,$$

whose solution is given by [46]

$$G(\zeta) = \text{sn}(\zeta; \omega)$$

one can obtain solutions of (3.3) in terms of Jacobi elliptic sine functions. In fact, without giving details here, we can obtain the solution of (3.3) as

$$u(t, x, y) = A_{-2}\text{ns}(\zeta; \omega)^2 + A_0 + A_2\text{sn}(\zeta; \omega)^2,$$

$$v(t, x, y) = B_{-2}\text{ns}(\zeta; \omega)^2 + B_0 + B_2\text{sn}(\zeta; \omega)^2,$$

where  $\text{ns} = 1/\text{sn}$ ,

$$A_{-2} = -\frac{4\alpha^2}{a}, A_{-1} = 0, A_1 = 0, A_2 = -\frac{4\alpha^2\omega}{a}, B_1 = 0, B_2 = \frac{4\alpha\omega(\alpha - 1)}{a},$$

$$B_{-2} = \frac{4\alpha(\alpha - 1)}{a}, B_{-1} = 0, B_0 = \frac{1 - a(\alpha + 2)A_0 + 4\alpha^2(\omega + 1)}{a\alpha},$$

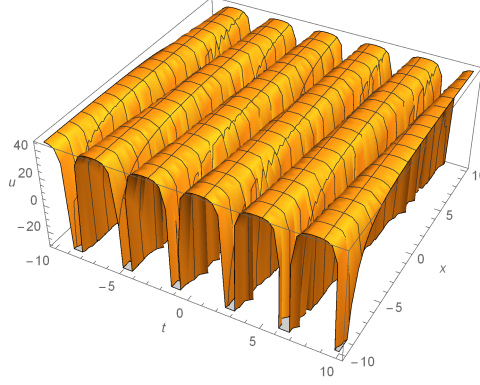
$\zeta = t - \alpha x - (1 - \alpha)y$  and  $A_0$  is an arbitrary constant. Nevertheless, the solution of the (2+1)-dimensional KdV equation (3.2) is given by

$$u(t, x, y) = A_{-2}\text{ns}(\zeta; \omega)^2 + A_0 + A_2\text{sn}(\zeta; \omega)^2, \quad (3.13)$$

where

$$A_{-2} = -\frac{4\alpha^2}{a}, \quad A_0 = \frac{1 + 4\alpha^2 + 4\alpha^2\omega}{a(\alpha + 2)}, \quad A_2 = -\frac{4\alpha^2\omega}{a}.$$

A profile of the solution (3.13) is given in Figure 2.



**Figure 3.2:** Profile of the snoidal wave solution (3.13)

A variety of further solutions of (3.2) can be constructed using Theorem 2.2 [59,60].

## 3.2 Conservation laws of (3.2)

We now construct conservation laws for the generalized (2+1)-dimensional Korteweg-de Vries equation (3.2) by employing the multiplier approach. For details of the multiplier method the reader is referred to Section 1.5.3.

For the coupled system (3.3), we obtain zeroth-order multipliers of the form,  $\Lambda_1 = \Lambda_1(t, x, y, u, v)$  and  $\Lambda_2 = \Lambda_2(t, x, y, u, v)$  and these are given by

$$\begin{aligned} \Lambda_1 &= -\frac{1}{2}a f_1(t)u^2 + f_1'(t)u_y + \frac{1}{2a}y^2 f_1''(t) - a f_2(t)u - y f_2'(t) + f_3(t), \\ \Lambda_2 &= f_1(t)u - \frac{1}{a}y f_1'(t) + f_2(t), \end{aligned}$$

where  $f_i$ ,  $i = 1, 2, 3$  are arbitrary functions of  $t$ . Corresponding to the above multipliers we obtain the following three nonlocal conserved vectors of (3.2):

$$\begin{aligned} T_1^t &= \frac{1}{2}f_1(t)u^2 - \frac{1}{a}yf_1'(t)u, \\ T_1^x &= -\frac{1}{2a}\left\{y^2f_1''(t)\int u_y dx + f_1'(t)\left(3ayu^2 + 2ayu\int u_y dx + 2yu_{xx} + 2yu_{xy}\right)\right. \\ &\quad \left.+ f_1(t)\left(au_xu_y - 2auu_{xx} - 2a^2u^3 - a^2u^2\int u_y dx - auu_{xy} + au_x^2\right)\right\}, \\ T_1^y &= \frac{1}{2a}\left(y^2f_1''(t)u - ayf_1'(t)u^2 + a^2f_1(t)u^3 + af_1(t)uu_{xx}\right); \end{aligned}$$

$$\begin{aligned} T_2^t &= f_2(t)u, \\ T_2^x &= yf_2'(t)\int u_y dx + f_2(t)\left(au\int u_y dx + \frac{3}{2}au^2 + u_{xx} + u_{xy}\right), \\ T_2^y &= \frac{1}{2}af_2(t)u^2 - yf_2'(t)u; \end{aligned}$$

$$\begin{aligned} T_3^t &= 0, \\ T_3^x &= -f_3(t)\int u_y dx, \\ T_3^y &= f_3(t)u. \end{aligned}$$

**Remark 2** We note that the conserved vector  $(T_3^t, T_3^x, T_3^y)$  gives us a trivial conservation law. We further note that since the functions  $f_i$ ,  $i = 1, 2$ , are arbitrary, one can construct infinitely many nonlocal conserved vectors of equation (3.2).

### 3.3 Concluding remarks

In this chapter, the generalized (2+1)-dimensional Korteweg-de Vries equation (3.2) was studied. The substitution  $v = \int u_y dx$  was made so as to remove the integral in equation (3.2) and then transformed (3.2) into a system of two partial

differential equations (3.3). New exact solutions were found and these were cnoidal and snoidal waves solutions. Furthermore, conserved vectors were constructed by employing the multiplier method.

# Chapter 4

## Travelling wave solutions of a coupled Korteweg-de Vries-Burgers system

In this chapter we study the coupled Korteweg-de Vries-Burgers (KdV-Burgers) system [61], which consists of two NLPDEs and is given by

$$u_t + uu_x + av_{xx} + bu_{xxx} = 0, \quad (4.1a)$$

$$v_t + vv_x + cv_{xx} + du_{xxx} = 0, \quad (4.1b)$$

where  $a, b, c$  and  $d$  are constants. In [61], the classical Lie group method was used to study (4.1). Symmetry reductions and some similarity solutions were obtained for (4.1).

In this chapter we employ the  $(G'/G)$ -expansion method to find new exact explicit solutions of the system (4.1). Wang et al. [18] introduced this method for finding solutions to NLPDEs. Using this method one can obtain travelling wave solutions. These solutions can be expressed by the hyperbolic, the trigonometric and the rational functions. This method is simple to use and can be employed to obtain

exact solutions for many NLPDEs. Furthermore, conservation laws are derived using the multiplier method.

The results of the chapter have been published in [62].

## 4.1 Exact explicit solutions of (4.1)

We first use the wave variable  $\xi = x - \nu t$ , where  $\nu$  is the wave speed and transform the system of nonlinear partial differential equations (4.1) into a nonlinear ordinary differential equation system. Thus by letting

$$u(x, t) = F(\xi), \quad v(x, t) = H(\xi), \quad \text{where } \xi = x - \nu t, \quad (4.2)$$

the system (4.1) transforms to the system of nonlinear ODEs

$$bF'''(\xi) + aH''(\xi) + F(\xi)F'(\xi) - \nu F'(\xi) = 0, \quad (4.3a)$$

$$dF'''(\xi) + cH''(\xi) + H(\xi)H'(\xi) - \nu H'(\xi) = 0, \quad (4.3b)$$

where  $'$  denotes the derivative with respect to  $\xi$ .

The  $(G'/G)$ -expansion method allows us to use the substitution

$$F(\xi) = \sum_{i=0}^N A_i (G'/G)^i \quad \text{and} \quad H(\xi) = \sum_{j=0}^M B_j (G'/G)^j, \quad (4.4)$$

where  $A_i$  ( $i = 0, 1, \dots, N$ ) and  $B_j$  ( $j = 0, 1, \dots, M$ ) are constants which need to be determined. The function  $G(\xi)$  in (4.4) satisfies

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \quad (4.5)$$

which is the second-order linear ODE with constant coefficients  $\lambda$  and  $\mu$ .

For our system (4.3), the balancing process gives  $M = N = 2$  and this leads to

$$F(\xi) = A_0 + A_1(G'/G) + A_2(G'/G)^2, \quad (4.6a)$$

$$H(\xi) = B_0 + B_1(G'/G) + B_2(G'/G)^2. \quad (4.6b)$$

Substituting (4.6) into (4.3), using (4.5), and then collecting the coefficients of each power of  $(G'/G)$  and setting each coefficient to zero, gives us the following system of twelve algebraic equations:

$$\begin{aligned}
2A_2^2 + 24bA_2 &= 0, \\
2B_2^2 + 24dA_2 &= 0, \\
6cB_2 - 2\lambda B_2^2 - 3B_1B_2 - 6dA_1 - 54d\lambda A_2 &= 0, \\
6aB_2 - 2\lambda A_2^2 - 54b\lambda A_2 - 3A_1A_2 - 6bA_1 &= 0, \\
a\mu B_1\lambda - b\mu A_1\lambda^2 - 6b\mu^2 A_2\lambda - 2b\mu^2 A_1 + \mu\nu A_1 - \mu A_0A_1 + 2a\mu^2 B_2 &= 0, \\
c\mu B_1\lambda - d\mu A_1\lambda^2 - 6d\mu^2 A_2\lambda - 2d\mu^2 A_1 + \mu\nu B_1 - \mu B_0B_1 + 2c\mu^2 B_2 &= 0, \\
aB_1\lambda^2 - bA_1\lambda^3 - 14b\mu A_2\lambda^2 - 8b\mu A_1\lambda + \nu A_1\lambda A_0A_1\lambda + 6a\mu B_2\lambda - \mu A_1^2 \\
- 16b\mu^2 A_2 + 2\mu\nu A_2 - 2\mu A_0A_2 + 2a\mu B_1 &= 0, \\
4aB_2\lambda^2 - 8bA_2\lambda^3 - 7bA_1\lambda^2 - A_1^2\lambda - 52b\mu A_2\lambda + 2\nu A_2\lambda - 2A_0A_2\lambda + 3aB_1\lambda \\
- 8b\mu A_1 + \nu A_1 - A_0A_1 - 3\mu A_1A_2 + 8a\mu B_2 &= 0, \\
2\nu A_2 - 38bA_2\lambda^2 - 12bA_1\lambda - 3A_1A_2\lambda + 10aB_2\lambda - A_1^2 - 2\mu A_2^2 - 40b\mu A_2 \\
- 2A_0A_2 + 2aB_1 &= 0, \\
cB_1\lambda^2 - dA_1\lambda^3 - 14d\mu A_2\lambda^2 - 8d\mu A_1\lambda + \nu B_1\lambda - B_0B_1\lambda + 6c\mu B_2\lambda - \mu B_1^2 \\
- 16d\mu^2 A_2 + 2c\mu B_1 + 2\mu\nu B_2 - 2\mu B_0B_2 &= 0, \\
2\nu B_2\lambda - 8dA_2\lambda^3 - 7dA_1\lambda^2 + 4cB_2\lambda^2 - B_1^2\lambda - 52d\mu A_2\lambda + 3cB_1\lambda - 2B_0B_2\lambda \\
- 8d\mu A_1 + \nu B_1 - B_0B_1 + 8c\mu B_2 - 3\mu B_1B_2 &= 0, \\
10cB_2\lambda - 38dA_2\lambda^2 - 12dA_1\lambda - 3B_1B_2\lambda - B_1^2 - 2\mu B_2^2 - 40d\mu A_2 + 2cB_1 \\
+ 2\nu B_2 - 2B_0B_2 &= 0,
\end{aligned}$$

Solving the above system of algebraic equations, with the aid of Mathematica, we

obtain the following values of the constants  $A_i, B_i, i = 0, 1, 2$ :

$$\begin{aligned}
A_0 &= 6b\lambda\sqrt{\lambda^2 - 4\mu} - 12b\mu + \nu, \quad A_1 = 12b\left(\sqrt{\lambda^2 - 4\mu} - \lambda\right), \quad A_2 = -12b, \\
B_0 &= \nu + \frac{6c\left(\lambda\sqrt{\lambda^2 - 4\mu} - 2\mu\right)}{5\sqrt{\lambda^2 - 4\mu}}, \quad B_1 = \frac{12c\left(\sqrt{\lambda^2 - 4\mu} - \lambda\right)}{5\sqrt{\lambda^2 - 4\mu}}, \\
B_2 &= -\frac{12c}{5\sqrt{\lambda^2 - 4\mu}}, \quad \mu = \frac{25bd\lambda^2 - c^2}{100bd}, \quad \text{when } ad = bc.
\end{aligned}$$

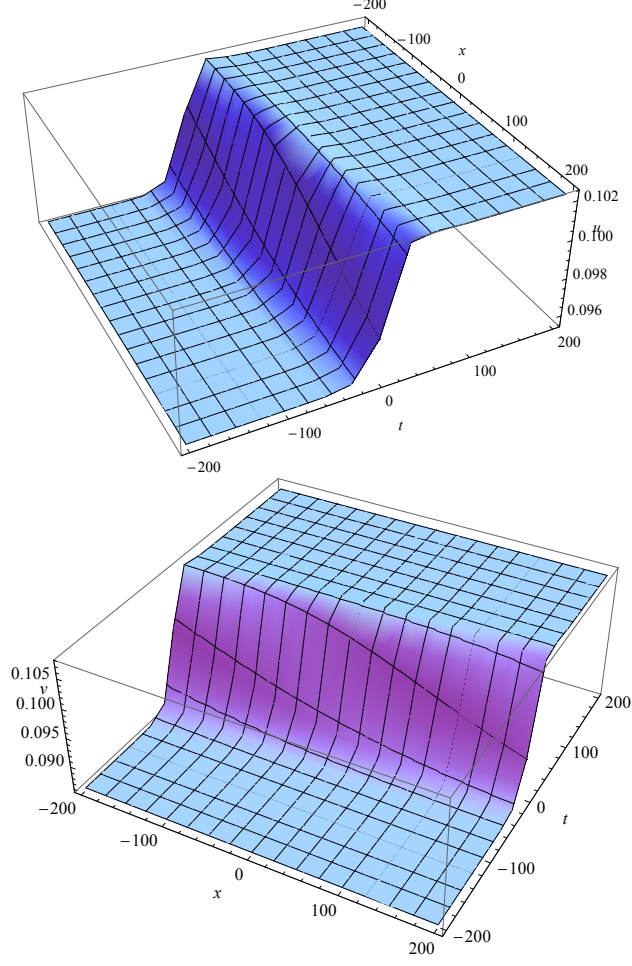
Thus, the following two types of travelling wave solutions of (4.1) are obtained:

When  $\lambda^2 - 4\mu > 0$ , we obtain the hyperbolic function solutions

$$\begin{aligned}
u_1(x, t) &= A_0 + A_1\left(\delta_1 \frac{C_1 \sinh(\delta_1 \xi) + C_2 \cosh(\delta_1 \xi)}{C_1 \cosh(\delta_1 \xi) + C_2 \sinh(\delta_1 \xi)} - \frac{\lambda}{2}\right) \\
&\quad + A_2\left(\delta_1 \frac{C_1 \sinh(\delta_1 \xi) + C_2 \cosh(\delta_1 \xi)}{C_1 \cosh(\delta_1 \xi) + C_2 \sinh(\delta_1 \xi)} - \frac{\lambda}{2}\right)^2, \quad (4.7)
\end{aligned}$$

$$\begin{aligned}
v_1(x, t) &= B_0 + B_1\left(\delta_1 \frac{C_1 \sinh(\delta_1 \xi) + C_2 \cosh(\delta_1 \xi)}{C_1 \cosh(\delta_1 \xi) + C_2 \sinh(\delta_1 \xi)} - \frac{\lambda}{2}\right) \\
&\quad + B_2\left(\delta_1 \frac{C_1 \sinh(\delta_1 \xi) + C_2 \cosh(\delta_1 \xi)}{C_1 \cosh(\delta_1 \xi) + C_2 \sinh(\delta_1 \xi)} - \frac{\lambda}{2}\right)^2, \quad (4.8)
\end{aligned}$$

where  $\xi = x - \nu t$ ,  $\delta_1 = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}$ ,  $C_1$  and  $C_2$  are arbitrary constants. We present the exact solution (4.7)-(4.8) graphically in Figure 1.



**Figure 4.1:** Exact solitary wave profiles of the solution (4.7)-(4.8)

When  $\lambda^2 - 4\mu < 0$ , we obtain the trigonometric function solutions

$$\begin{aligned}
 u_2(x, t) &= A_0 + A_1 \left( \delta_2 \frac{-C_1 \sin(\delta_2 \xi) + C_2 \cos(\delta_2 \xi)}{C_1 \cos(\delta_2 \xi) + C_2 \sin(\delta_2 \xi)} - \frac{\lambda}{2} \right) \\
 &\quad + A_2 \left( \delta_2 \frac{-C_1 \sin(\delta_2 \xi) + C_2 \cos(\delta_2 \xi)}{C_1 \cos(\delta_2 \xi) + C_2 \sin(\delta_2 \xi)} - \frac{\lambda}{2} \right)^2, \\
 v_2(x, t) &= B_0 + B_1 \left( \delta_2 \frac{-C_1 \sin(\delta_2 \xi) + C_2 \cos(\delta_2 \xi)}{C_1 \cos(\delta_2 \xi) + C_2 \sin(\delta_2 \xi)} - \frac{\lambda}{2} \right) \\
 &\quad + B_2 \left( \delta_2 \frac{-C_1 \sin(\delta_2 \xi) + C_2 \cos(\delta_2 \xi)}{C_1 \cos(\delta_2 \xi) + C_2 \sin(\delta_2 \xi)} - \frac{\lambda}{2} \right)^2,
 \end{aligned}$$

where  $\xi = x - \nu t$ ,  $\delta_2 = \frac{1}{2}\sqrt{4\mu - \lambda^2}$ ,  $C_1$  and  $C_2$  are arbitrary constants.

**Remark** We note that we are not able to obtain rational solution of equation (4.1) using the  $(G'/G)$ -expansion method.

## 4.2 Conservation laws of (4.1)

In this section we derive conservation laws of (4.1) using the multiplier method.

The associated multipliers are given by  $\Lambda_1 = C_1$ ,  $\Lambda_2 = C_2$  and corresponding to these multipliers we obtain the following conserved vectors of (4.1):

$$\begin{aligned} T_1^t &= u, \\ T_1^x &= \frac{1}{2}u^2 + av_x + bu_{xx}; \\ T_2^t &= v, \\ T_2^x &= \frac{1}{2}v^2 + cv_x + du_{xx}. \end{aligned}$$

**Remark** These conserved vectors can be derived directly from system (4.1). That is the system itself is in conserved form.

## 4.3 Concluding remarks

In this chapter, we obtained exact explicit travelling wave solutions of the coupled Korteweg-de Vries-Burgers system (4.1) by employing the  $(G'/G)$ -expansion method. The solutions obtained were expressed in the form of hyperbolic and trigonometric functions and were much more general than the solutions previously obtained in [61]. This means that the solutions obtained in [61] were special cases of our solutions. It should be noted that many researchers have recently employed this method to various nonlinear differential equations and have shown that this method provides a very effective and powerful mathematical tool for solving non-

linear differential equations in various fields of applied sciences. Conservation laws for the coupled Korteweg-de Vries-Burgers system (4.1) were constructed by the multiplier method.

# Chapter 5

## A study of a (2+1)-dimensional KdV-mKdV equation of mathematical physics

In this chapter we study a NLPDE known as (2+1)-dimensional KdV-mKdV equation which is given by [63]

$$u_t + u_{xxy} + 4uu_y - 4u^2u_y + 2u_x\partial_x^{-1}u_y - 2u_x\partial_x^{-1}(2uu_y) = 0 \quad (5.1)$$

and arises in various problems in many areas of theoretical physics. The (1+1)-dimensional KdV-mKdV equation describes the wave propagation of bounded particle with a harmonic force [64]. In particular, it describes the propagation of ion acoustic waves of small amplitude without Landau damping in plasma physics [65]. The propagation of thermal pulse through single crystal of sodium fluoride in solid physics can also be explained by this equation [66, 67].

To study the (2+1)-dimensional KdV-mKdV equation (5.1), we first introduce a new dependent variable  $v$  and set  $v = \partial_x^{-1}(u_y - 2uu_y) = \int(u_y - 2uu_y)dx$ . This allows us to remove the integral terms from the equation and replace equation (5.1)

by a system of two PDEs

$$\begin{aligned} u_t + u_{xxy} + 4uu_y - 4u^2u_y + 2u_xv &= 0, \\ v_x - u_y + 2uu_y &= 0. \end{aligned} \tag{5.2}$$

Firstly, we use the travelling wave variable to transform system (5.2) into a system of ordinary differential equations (ODEs) and then obtain its exact explicit solutions. Secondly, we construct conservation laws of equation (5.1) using the multiplier method.

This work has been submitted for publication [68] .

## 5.1 Exact solutions of (5.2)

We first use the travelling wave variable  $z = x + by + ct$  to reduce the system of PDEs (5.2) to a system of ODEs. Thus by letting

$$u(x, y, t) = U(z), \quad v(x, y, t) = V(z), \quad \text{where } z = x + by + ct, \tag{5.3}$$

the system (5.2) transforms to

$$bU'''(z) + 4b(1 - U(z))U(z)U'(z) + 2V(z)U'(z) + cU'(z) = 0, \tag{5.4a}$$

$$2bU(z)U'(z) - bU'(z) + V'(z) = 0, \tag{5.4b}$$

which is a system of nonlinear ODEs. We now decouple the above system by solving the second equation for  $V$ . Integrating (5.4b) with respect to  $z$ , we obtain

$$V(z) = -bU(z)^2 + bU(z) + c_1, \tag{5.5}$$

where  $c_1$  is an arbitrary constant of integration. The substitution of the value of  $V$  from (5.5) into (5.4a) gives the third-order nonlinear ODE

$$bU'''(z) - 6bU(z)^2U'(z) + 6bU(z)U'(z) + cU'(z) + 2c_1U'(z) = 0. \tag{5.6}$$

This equation can be solved in the following manner. Integrating (5.6) with respect to  $z$  yields

$$bU'' - 2bU^3 + 3bU^2 + (c + 2c_1)U + c_2 = 0, \quad (5.7)$$

where  $c_2$  is an arbitrary constant of integration. We multiply (5.7) by the integrating factor  $U'$  and obtain

$$bU'U'' - 2bU^3U' + 3bU^2U' + (c + 2c_1)UU' + c_2U' = 0. \quad (5.8)$$

Now integration of (5.8) with respect to  $z$  gives us

$$U'^2 - U^4 + 2U^3 + \frac{U^2}{b}(c + 2c_1) + \frac{2c_2U}{b} + \frac{2c_3}{b} = 0, \quad (5.9)$$

where  $c_3$  is an arbitrary constant of integration. The general solution of (5.9) is given via the Jacobi elliptic function [69] and is a bit cumbersome to write here. However, by imposing the asymptotic boundary conditions  $U, U', U'' \rightarrow 0$  for  $|z| \rightarrow \infty$ , we obtain  $c_2 = 0$  and  $c_3 = 0$  and one can write a special solution of (5.9) given by

$$U(z) = -\frac{4(c-k)^2 e^\theta}{4b^2 \sqrt{k-c} + 4b(c-k)(\sqrt{k-c} + e^\theta) + (k-c)^{3/2} e^{2\theta}},$$

where  $k = -2c_1$ ,  $\theta = \sqrt{(k-c)/b}(z+l)$  and  $l$  is an arbitrary constant of integration. Thus the exact solutions of (5.1) can be given in terms of the Jacobi elliptic functions and a special solution has the form

$$u(t, x, y) = -\frac{4(c-k)^2 e^\theta}{4b^2 \sqrt{k-c} + 4b(c-k)(\sqrt{k-c} + e^\theta) + (k-c)^{3/2} e^{2\theta}},$$

where

$$\theta = \sqrt{\frac{k-c}{b}}(x + by + ct + l).$$

## 5.2 Conservation laws of (5.1)

We now construct conservation laws for the (2+1)-dimensional KdV-mKdV equation (5.1) using the multiplier approach. For the coupled system (5.2), we obtain multipliers of the form,  $Q_1 = Q_1(t, x, y, u, v)$  and  $Q_2 = Q_2(t, x, y, u, v)$  that are given by

$$\begin{aligned} Q_1 &= u^2 f_1(t) - u f_1(t) - \frac{1}{2} y f_1'(t) + f_2(t) + 2u f_3(y-t) - f_3(y-t), \\ Q_2 &= u f_1(t) - \frac{1}{2} f_1(t) + f_3(y-t), \end{aligned}$$

where  $f_i$ ,  $i = 1, 2$  are arbitrary functions of  $t$  and  $f_3$  is an arbitrary function of  $y - t$ . Corresponding to the above multipliers we obtain the following nonlocal conserved vector of (5.1):

$$\begin{aligned} T^t &= \frac{1}{2} u^2 f_1(t) - \frac{1}{2} u f_1(t) + u f_3(y-t), \\ T^x &= \left\{ (u^2 - u) \int (u_y - 2uu_y) dx - \frac{1}{2} u_x u_y + \frac{1}{2} u u_{xy} - \frac{1}{2} u_{xy} \right\} f_1(t) \\ &\quad + \left( f_2(t) - \frac{1}{2} y f_1'(t) \right) \int (u_y - 2uu_y) dx + u_{xy} f_3(y-t) \\ &\quad + (2u - 1) f_3(y-t) \int (u_y - 2uu_y) dx, \\ T^y &= -\frac{1}{2} u (u^3 + u - u_{xx} - 2u^2) f_1(t) - \frac{1}{2} u (uy - y) f_1'(t) + u(u-1) f_2(t) \\ &\quad + u f_3(y-t). \end{aligned}$$

**Remark 2.** Due to the presence of the arbitrary functions,  $f_i$ ,  $i = 1, 2, 3$ , in the multipliers, one can obtain infinitely many nonlocal conservation laws.

## 5.3 Concluding remarks

In this chapter we studied the (2+1)-dimensional KdV-mKdV equation (5.1). We introduced a new variable  $v$  and wrote the (5.1) as a system of two PDEs, which did

not have integral terms. The travelling wave variable was then used to transform the system of PDEs into a nonlinear system of two ODEs. The resulting system of ODEs was decoupled and solved directly. As a result we obtained travelling wave solutions of (5.1) in terms of Jacobi elliptic functions. Furthermore, conservation laws of (5.1) were also computed using the multiplier approach. The conservation laws consisted of an infinite number of nonlocal conserved vectors.

# Chapter 6

## Exact solutions and conservation laws for a generalized improved Boussinesq equation

In this chapter we study the so called generalized improved Boussinesq equation (GIBE) [70, 71]

$$u_{tt} - (1 + \alpha u)u_{xx} - \alpha u_x^2 - u_{ttxx} = 0, \quad (6.1)$$

where  $\alpha$  is a real constant. This equation describes nonlinear dispersive wave phenomena, such as propagation of long waves on the surface of shallow water in both directions, propagation of ion-sound waves in a uniform isotropic plasma, and so on [72].

The aim of this chapter is two-fold. Firstly, we determine some exact solutions of (6.1) by using Lie symmetry method along with the simplest equation method. Secondly, conservation laws will be derived for (6.1) using the multiplier method.

The results of this chapter have been published. See [73].

## 6.1 Exact solutions of (6.1) using Lie symmetry and simplest equation methods

In this section we employ two methods namely Lie symmetry and simplest equation methods to obtain exact solutions of the generalized improved Boussinesq equation (6.1).

### 6.1.1 Lie point symmetries of (6.1)

We now calculate the Lie point symmetries of (6.1) and then use them to find exact solutions of (6.1).

The symmetry group of the generalized improved Boussinesq equation (6.1) will be generated by the vector field of the form

$$X = \tau(t, x, u) \frac{\partial}{\partial t} + \xi(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u}.$$

Applying the fourth prolongation  $\text{pr}^{(4)}X$  [24] to (6.1) one obtains an overdetermined system of linear partial differential equations

$$\begin{aligned} \tau_u = 0, \quad \xi_u = 0, \quad \eta_{uu} = 0, \quad \tau_x = 0, \quad \xi_t = 0, \quad \eta_{txu} = 0, \quad \xi_{xx} - 2\eta_{xu} = 0, \\ 2\xi_x - \eta_{xxu} = 0, \quad \tau_{tt} - 2\eta_{tu} = 0, \quad 2\alpha\xi_x - 2\alpha\tau_t - \alpha\eta_{xxu} - \alpha\eta_u = 0, \quad 2\eta_{tu} - \tau_{tt} = 0, \\ 2\xi_x - \alpha\eta - 2u\alpha\tau_t - \eta_{ttu} - 2\tau_t - u\alpha\eta_{xxu} + 2u\alpha\xi_x - \eta_{xxu} = 0, \\ \eta_{tt} - \eta_{ttxx} - u\alpha\eta_{xx} - \eta_{xx} = 0, \quad \xi_{xx} - 2u\alpha\eta_{xu} + u\alpha\xi_{xx} - 2\eta_{xu} - 2\alpha\eta_x = 0. \end{aligned}$$

Solving this system of PDEs we obtain the following three Lie point symmetries of (6.1):

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \alpha t \frac{\partial}{\partial t} - 2(1 + \alpha u) \frac{\partial}{\partial u}.$$

### 6.1.2 Symmetry reductions and group-invariant solutions of (6.1)

We first consider the operator  $X_3 = \alpha t \partial/\partial t - 2(1 + \alpha u) \partial/\partial u$  and obtain a group-invariant solution of (6.1) under  $X_3$ . Solving the associated Lagrange equations

$$\frac{dt}{\alpha t} = \frac{dx}{0} = \frac{du}{-2(1 + \alpha u)}$$

of  $X_3$ , we obtain two invariants  $J_1 = x$  and  $J_2 = t^2(1 + \alpha u)$ . Hence the group-invariant solution is given by

$$u(t, x) = \frac{1}{\alpha} \left( -1 + t^{-2} \phi(x) \right),$$

where  $\phi(x)$  satisfies

$$6\phi - \phi'^2 - 6\phi'' - \phi\phi'' = 0. \tag{6.2}$$

The solution of this second-order nonlinear ODE can be written in terms of elliptic integral of the first kind, which is quite complicated to write here. However, a particular solution of (6.2) is

$$\phi(x) = (C + x)^2 + 3,$$

where  $C$  is a constant of integration. Hence the group-invariant solution of (6.1) under the Lie point symmetry  $X_3$  is given by

$$u(t, x) = \frac{(C + x)^2 - t^2 + 3}{\alpha t^2}.$$

We now use the remaining two symmetries  $X_1$  and  $X_2$  to transform (6.1) to an ODE. The linear combination of the two symmetries,  $X_1 + \nu X_2$ , where  $\nu$  is a constant, yields the two invariants

$$z = x - \nu t, \quad F = u.$$

Now by considering  $F$  as the new dependent variable and  $z$  as new independent variable, the generalized improved Boussinesq equation (6.1) transforms to the fourth-order nonlinear ODE

$$\alpha F'^2 + (1 - \nu^2 + \alpha F) F'' + \nu^2 F'''' = 0. \quad (6.3)$$

We use the simplest equation method to solve equation (6.3).

### 6.1.3 Use of simplest equation method to obtain exact solutions

In this section we find solutions of the ODE (6.3) by using the simplest equation method. This will then give us the exact solutions of (6.1). The simplest equations that will be used are the Bernoulli and Riccati equations, whose solutions can be written in terms of elementary functions. See Section 1.6.2.

#### 6.1.3.1 Solutions of (6.1) using the Bernoulli equation as the simplest equation

The balancing procedure yields  $M = 2$  so the solutions of the ODE (6.3) can be written in the form

$$F(z) = A_0 + A_1 H(z) + A_2 H(z)^2. \quad (6.4)$$

Substituting this value of  $F(z)$  into (6.3) and making use of the Bernoulli equation, then equating all coefficients of the functions  $H^i$  to zero, we obtain the system of algebraic equations.

$$\begin{aligned} 120a^4 A_2 \nu^2 + 10\alpha a^2 A_2^2 &= 0, \\ A_1 b^2 \nu^2 - A_1 b^4 \nu^2 - \alpha A_0 A_1 b^2 - A_1 b^2 &= 0, \end{aligned}$$

$$\begin{aligned}
24a^4A_1\nu^2 + 336a^3A_2b\nu^2 + 12\alpha a^2A_1A_2 + 18\alpha aA_2^2b &= 0, \\
3aA_1b\nu^2 - 15aA_1b^3\nu^2 - 3a\alpha A_0A_1b - 3aA_1b - 16A_2b^4\nu^2 - 2\alpha A_1^2b^2 - 4\alpha A_0A_2b^2 \\
+ 4A_2b^2\nu^2 - 4A_2b^2 &= 0, \\
2a^2A_1\nu^2 - 2a^2\alpha A_0A_1 - 50a^2A_1b^2\nu^2 - 2a^2A_1 - 130aA_2b^3\nu^2 - 5a\alpha A_1^2b \\
- 10a\alpha A_0A_2b + 10aA_2b\nu^2 - 10aA_2b - 9\alpha A_1A_2b^2 &= 0, \\
6a^2A_2\nu^2 - 60a^3A_1b\nu^2 - 3\alpha a^2A_1^2 - 6\alpha a^2A_0A_2 - 330a^2A_2b^2\nu^2 - 6a^2A_2 \\
- 21\alpha aA_1A_2b - 8\alpha A_2^2b^2 &= 0.
\end{aligned}$$

This system, with the aid of Mathematica, has the solution

$$A_0 = \frac{\nu^2 - b^2\nu^2 - 1}{\alpha}, \quad A_1 = -\frac{12ab\nu^2}{\alpha}, \quad A_2 = -\frac{12a^2\nu^2}{\alpha}.$$

Therefore, a solution of (6.1) can be written as

$$\begin{aligned}
u(t, x) = A_0 + A_1a \left\{ \frac{\cosh[a(z + C)] + \sinh[a(z + C)]}{1 - b \cosh[a(z + C)] - b \sinh[a(z + C)]} \right\} \\
+ A_2a^2 \left\{ \frac{\cosh[a(z + C)] + \sinh[a(z + C)]}{1 - b \cosh[a(z + C)] - b \sinh[a(z + C)]} \right\}^2, \quad (6.5)
\end{aligned}$$

where  $z = x - \nu t$  and  $C$  is a constant of integration.

### 6.1.3.2 Solutions of (6.1) using the Riccati equation as the simplest equation

For this case, the balancing procedure gives  $M = 2$ . So the solutions of the ODE (6.3) are given by

$$F(z) = A_0 + A_1H + A_2H^2. \quad (6.6)$$

Following the above procedure, and using the Riccati equation, we obtain

$$A_0 = \frac{-8ac\nu^2 - b^2\nu^2 + \nu^2 - 1}{\alpha}, \quad A_1 = -\frac{12ab\nu^2}{\alpha}, \quad A_2 = -\frac{12a^2\nu^2}{\alpha}.$$

Thus the solutions of (6.1) are given by

$$u(t, x) = A_0 + A_1 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left[ \frac{1}{2} \theta (z + C) \right] \right\} + A_2 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left[ \frac{1}{2} \theta (z + C) \right] \right\}^2 \quad (6.7)$$

and

$$u(t, x) = A_0 + A_1 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left( \frac{1}{2} \theta z \right) + \frac{\theta \operatorname{sech} [(\theta z)/2]}{C \theta \cosh [(\theta z)/2] - 2a \sinh [(\theta z)/2]} \right\} + A_2 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left( \frac{1}{2} \theta z \right) + \frac{\theta \operatorname{sech} [(\theta z)/2]}{C \theta \cosh [(\theta z)/2] - 2a \sinh [(\theta z)/2]} \right\}^2,$$

respectively. Here  $z = x - \nu t$  and  $C$  is a constant of integration.

## 6.2 Conservation laws of (6.1)

We now construct conservation laws for the generalized improved Boussinesq equation (6.1). We employ the multiplier approach and look for multipliers of order zero. Following the procedure described in Section 1.5.3, we obtain multiplier of the form

$$Q = C_1 t x + C_2 t + C_3 x + C_4, \quad (6.8)$$

where  $C_i$ ,  $i = 1, 2, 3, 4$  are arbitrary constants and so corresponding to this multiplier we obtain the following four local conserved vectors of (6.1):

$$T_1^x = -xu + txu_t + xu_{xx} - txu_{txx},$$

$$T_1^t = tu + \frac{1}{2} \alpha t u^2 - txu_x - \alpha txuu_x;$$

$$T_2^t = -u + tu_t + u_{xx} - tu_{txx},$$

$$T_2^x = -tu_x - t\alpha uu_x;$$

$$T_3^t = xu_t - xu_{txx}$$

$$T_3^x = u + \frac{1}{2}\alpha u^2 - xu_x - \alpha xuu_x;$$

$$T_4^t = u_t - u_{txx},$$

$$T_4^x = -u_x - \alpha uu_x.$$

### 6.3 Concluding remarks

In this chapter we studied the generalized improved Boussinesq equation (6.1). Group-invariant solutions of (6.1) were found using the Lie symmetry methods. Secondly by employing Lie symmetry method along with the simplest equation method we obtained travelling wave solutions. Finally, the conservation laws for the generalized improved Boussinesq equation were computed by employing the multiplier method.

# Chapter 7

## Solutions and conservation laws for a Kaup-Boussinesq system

In this chapter we study the Kaup-Boussinesq (KB) system [74, 75]

$$u_t + v_x + uu_x = 0, \quad (7.1a)$$

$$v_t + \frac{1}{4}u_{xxx} + (uv)_x = 0, \quad (7.1b)$$

which describes the propagation of long waves with small amplitude in shallow water. Here  $v(x, t)$  denotes the height of the water surface above a horizontal bottom and  $u(x, t)$  denotes its velocity averaged over depth.

The KB system arises in several other physical applications including vibrations in a nonlinear string, nonlinear lattice waves, and ion sound waves [76]. The solution of the Kaup-Boussinesq system for internal waves was obtained and used as initial guess for solving the Hamiltonian system numerically [77]. He's variational method has been used to search for solitary solutions of KB system [78]. Travelling wave solutions for KB system were constructed using the  $(G'/G)$ -expansion method [79].

We first obtain some exact solutions of (7.1) using the direct integration and later employing the Lie symmetry method. In addition to this, conservation laws will be constructed for (7.1) using the multiplier method and the new conservation theorem due to Ibragimov.

The results of this chapter have been submitted for possible publication [80].

## 7.1 Exact solutions of (7.1)

In this section we employ two methods of solution and obtain exact solutions of the Kaup-Boussinesq system (7.1); direct integration and Lie group analysis.

### 7.1.1 Travelling wave solutions of (7.1) using wave variable

Let us assume that  $z = kx + lt + m$ , where  $k, l$  and  $m$  are constants and  $u(x, t) = F(z)$  and  $v(x, t) = G(z)$ .

Then (7.1) is transformed into a system of coupled nonlinear ordinary differential equations (ODEs)

$$kF(z)F'(z) + lF'(z) + kG'(z) = 0, \quad (7.2a)$$

$$\frac{1}{4}k^3F^{(3)}(z) + kG(z)F'(z) + kF(z)G'(z) + lG'(z) = 0. \quad (7.2b)$$

The integration of (7.2a) gives

$$G(z) = \frac{-kF(z)^2 - 2lF(z)}{2k} - C_1, \quad (7.3)$$

where  $C_1$  is a constant of integration. Inserting the value of  $G(z)$  from (7.3) into (7.2b), yields the third-order nonlinear ODE

$$k^4F'''(z) - 6k^2F(z)^2F'(z) - 12klF(z)F'(z) - 4(l^2 + C_1k^2)F'(z) = 0, \quad (7.4)$$

After integrating the above equation twice, we obtain

$$k^4 F'(z)^2 = k^2 F(z)^4 + 4klF(z)^3 + 4(l^2 + C_1 k^2) F(z)^2 + 2C_2 F(z) + 2C_3, \quad (7.5)$$

where  $C_2$  and  $C_3$  are arbitrary constants of integration. Let  $\alpha, \beta, \lambda, \gamma$  be the roots of the right hand side of (7.5) such that  $\gamma \leq \lambda \leq \beta \leq \alpha$ . Using a new variable  $q(z)$  and the parameter  $n$  [69] we have

$$F(z) = \frac{\beta(\lambda - \alpha)q(z)^2 + \alpha(\beta - \lambda)}{(\lambda - \alpha)q(z)^2 + \beta - \lambda}, \quad n^2 = \frac{(\beta - \lambda)(\alpha - \lambda)}{(\alpha - \gamma)(\beta - \lambda)}. \quad (7.6)$$

Equation (7.5) can be written as

$$\int \frac{dq}{\sqrt{(1 - q^2)(1 - n^2 q^2)}} = \frac{1}{2k} \sqrt{(\beta - \lambda)(\alpha - \gamma)}(z - z_0). \quad (7.7)$$

The expression on the left of equation (7.7) is an elliptic integral of the first kind, and therefore its solution is given by

$$q(z) = \text{sn}(\kappa, n), \quad (7.8)$$

where  $\kappa = \sqrt{(\beta - \lambda)(\alpha - \gamma)}(z - z_0)/(2k)$ . The solution of (7.5) via the Jacobi elliptic function [69] by substituting the expression for  $q(z)$  into (7.6) is given by

$$F(z) = \frac{\beta(\lambda - \alpha)\text{sn}^2(\kappa, n) + \alpha(\beta - \lambda)}{(\lambda - \alpha)\text{sn}^2(\kappa, n) + \beta - \lambda}.$$

We now find some particular solutions of (7.1) by taking  $C_1 = C_2 = C_3 = 0$ . Integrating (7.5) and reverting back to the original variables  $x$  and  $t$ , we obtain

$$u_1(x, t) = F(z) = \frac{2l \{ \cosh(2lz/k^2) + \sinh(2lz/k^2) \}}{1 - k \cosh(2lz/k^2) - k \sinh(2lz/k^2)} \quad (7.9)$$

and

$$u_2(x, t) = F(z) = \frac{2l \{ \cosh(2lz/k^2) - \sinh(2lz/k^2) \}}{1 - k \cosh(2lz/k^2) + k \sinh(2lz/k^2)}, \quad (7.10)$$

where  $z = kx + lt + m$ .

The corresponding solutions  $v_1(x, t)$  and  $v_2(x, t)$  can be obtained from the equation (7.3) and are given by

$$v_1(x, t) = G(z) = -\frac{2l^2 \{\sinh(2lz/k^2) + \cosh(2lz/k^2)\}}{k \{k \sinh(2lz/k^2) + k \cosh(2lz/k^2) - 1\}^2} \quad (7.11)$$

and

$$v_2(x, t) = G(z) = -\frac{2l^2 \{\sinh(2lz/k^2) + \cosh(2lz/k^2)\}}{k \{\sinh(2lz/k^2) + \cosh(2lz/k^2) - k\}^2}, \quad (7.12)$$

respectively.

## 7.1.2 Symmetry reductions and group-invariant solutions

### 7.1.2.1 Lie point symmetries of (7.1)

The symmetry group of the Kaup-Boussinesq system (7.1) will be generated by the vector field of the form

$$X = \xi^1(x, t, u, v) \frac{\partial}{\partial x} + \xi^2(x, t, u, v) \frac{\partial}{\partial t} + \eta^1(x, t, u, v) \frac{\partial}{\partial u} + \eta^2(x, t, u, v) \frac{\partial}{\partial v}.$$

Applying the third prolongation  $\text{pr}^{(3)}\Gamma$  [24] to (7.1) we obtain the following overdetermined system of fifteen linear partial differential equations:

$$\begin{aligned} \xi_u^1 &= 0, \quad \xi_v^1 = 0, \quad \xi_x^2 = 0, \quad \xi_u^2 = 0, \quad \xi_v^2 = 0, \quad \eta_{uu}^1 = 0, \quad \eta_v^1 = 0, \quad \eta_{xu}^1 - \xi_{xx}^1 = 0, \\ \xi_t^2 - 2\xi_x^1 &= 0, \quad 2\eta^2 + 4v\xi_x^1 + \xi_{xxx}^1 = 0, \quad \xi_x^1 - \eta_u^1 + \eta_v^2 = 0, \quad \eta_t^1 + u\eta_x^1 + \eta_x^2 = 0, \\ \eta^1 - \xi_t^1 - \eta_u^2 + u\xi_x^1 &= 0, \quad \xi_t^1 - \eta^1 - \eta_u^2 - u\xi_x^1 = 0, \quad 4\eta_t^2 + 4u\eta_x^2 + 4v\eta_x^1 + \eta_{xxx}^1 = 0. \end{aligned}$$

After some lengthy computations, the above system gives the following four Lie point symmetries:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad X_4 = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - u \frac{\partial}{\partial u} - 2v \frac{\partial}{\partial v}.$$

### 7.1.2.2 One-dimensional optimal system of subalgebras

In this subsection we present the optimal system of one-dimensional subalgebras for the equation (7.1) to obtain the optimal system of group-invariant solutions. The method which we use here for obtaining one-dimensional optimal system of subalgebras is that given in [24]. The adjoint transformations are given by

$$\text{Ad}(\exp(\epsilon X_i))X_j = X_j - \epsilon[X_i, X_j] + \frac{1}{2}\epsilon^2[X_i, [X_i, X_j]] - \dots .$$

The table of commutators of the Lie point symmetries of equation (7.1) and the adjoint representations of the symmetry group of (7.1) on its Lie algebra are given in Table 1 and Table 2, respectively. The Table 1 and Table 2 are used to construct the optimal system of one-dimensional subalgebras for equation (7.1).

**Table 1.** Commutator table of the Lie algebra of equation (7.1)

[ , ]	$X_1$	$X_2$	$X_3$	$X_4$
$X_1$	0	0	0	$X_1$
$X_2$	0	0	$X_1$	$2X_2$
$X_3$	0	$-X_1$	0	$-X_3$
$X_4$	$-X_1$	$-2X_2$	$X_3$	0

**Table 2.** Adjoint table of the Lie algebra of equation (7.1)

Ad	$X_1$	$X_2$	$X_3$	$X_4$
$X_1$	$X_1$	$X_2$	$X_3$	$X_4 - \epsilon X_1$
$X_2$	$X_1$	$X_2$	$X_3 - \epsilon X_1$	$X_4 - 2\epsilon X_2$
$X_3$	$X_1$	$X_2 + \epsilon X_1$	$X_3$	$X_4 + \epsilon X_3$
$X_4$	$e^\epsilon X_1$	$e^{2\epsilon} X_2$	$e^{-\epsilon} X_3$	$X_4$

From Tables 1 and 2 one can obtain an optimal system of one-dimensional subalgebras given by  $\{X_1, X_2, X_3, X_4, X_2 + X_3\}$ .

### 7.1.2.3 Group-invariant solutions and symmetry reductions of (7.1)

In this subsection we use the optimal system of one-dimensional subalgebras calculated above to obtain group-invariant solutions and symmetry reductions.

#### Case 1. $X_1$

The group-invariant solution corresponding to  $X_1$  is  $u = f(\xi)$ ,  $v = g(\xi)$  where  $\xi = t$  is the group-invariant of  $X_1$ . The substitution of this solution into the equation (7.1) and solving the ODE gives the solution  $u(x, t) = C_1$ ,  $v(x, t) = C_2$ , where  $C_1$  and  $C_2$  are arbitrary constants.

#### Case 2. $X_2$

The symmetry  $X_2$  gives rise to the group-invariant solution

$$u = f(\xi), \quad v = g(\xi), \quad (7.13)$$

where  $\xi = x$  is an invariant of the symmetry  $X_2$ . The insertion of (7.13) into (7.1) results in the system of ODEs, whose solution is given by

$$u(x, t) = 2\sqrt{C_1} \sec\left(2\sqrt{C_1}x + C_2\right), \quad (7.14)$$

$$v(x, t) = C_1 - 2C_1 \sec^2\left(2\sqrt{C_1}x + C_2\right), \quad (7.15)$$

where  $C_1$  and  $C_2$  are arbitrary constants of integration. One can also obtain the following rational function solutions under the symmetry  $X_2$ :

$$u(x, t) = \frac{1}{C_1 \pm x}, \quad v(x, t) = -\frac{1}{2(C_1 \pm x)^2}, \quad (7.16)$$

where  $C_1$  and  $C_2$  are arbitrary constants of integration.

#### Case 3. $X_3$

The symmetry  $X_3$  gives rise to the group-invariant solution

$$u = \frac{x}{t} + f(\xi), \quad v = g(\xi), \quad (7.17)$$

where  $\xi = t$  is an invariant of the symmetry  $X_3$ . The insertion of (7.17) into (7.1) results in the system of ODEs, whose solution is given by

$$u(x, t) = \frac{x + C_1}{t}, \quad v(x, t) = \frac{C_2}{t}, \quad (7.18)$$

where  $C_1$  and  $C_2$  are arbitrary constants of integration.

#### Case 4. $X_4$

The symmetry  $X_4$  gives rise to the group-invariant solution

$$u = \frac{1}{x}f(\xi), \quad v = \frac{1}{x^2}g(\xi), \quad (7.19)$$

where  $\xi = t/x^2$  is an invariant of the symmetry  $X_4$ . The insertion of (7.19) into (7.1) results in the system of ODEs

$$2\xi(ff' + g') - f' + f^2 + 2g = 0, \quad (7.20)$$

$$4\xi^3 f''' + 24\xi^2 f'' + \xi(27f' + 4fg' + 4gf') + 3f + 6fg - 2g' = 0. \quad (7.21)$$

Thus we have symmetry reduction.

#### Case 5. $X_2 + X_3$

The symmetry  $X_2 + X_3$  gives rise to the group-invariant solution

$$u = t + f(\xi), \quad v = g(\xi), \quad (7.22)$$

where  $\xi = x - t^2/2$  is an invariant of the symmetry  $X_2 + X_3$ . The insertion of (7.22) into (7.1) results in the system of ODEs

$$ff' + g' + 1 = 0, \quad (7.23)$$

$$f''' + 4(fg' + f'g) = 0. \quad (7.24)$$

## 7.2 Conservation laws of (7.1)

In this section we construct conservation laws for the Kaup-Boussinesq system (7.1) using the multiplier method.

### 7.2.1 Construction of conservation laws using the multiplier method

We look for second-order multipliers of the form

$$\Lambda_1 = \Lambda_1(x, t, u, v, u_x, v_x, u_{xx}, v_{xx}), \quad \Lambda_2 = \Lambda_2(x, t, u, v, u_x, v_x, u_{xx}, v_{xx}).$$

Now following the procedure given in Section 1.5.3, these multipliers are given by

$$\begin{aligned} \Lambda_1 &= \frac{1}{6} C_2 (6 u u_{xx} + 3 u_x^2 + 4 v_{xx} + 12 u^2 v + 12 v^2) + (C_1 t + C_4 + 4 C_3 u) v + C_5 \\ &\quad + C_3 u_{xx}, \\ \Lambda_2 &= \frac{1}{6} C_2 u_{xx} + \frac{1}{6} C_2 u^3 + \frac{1}{2} C_3 u^2 + \frac{1}{4} (C_1 t + 4 C_2 v + C_4) u + C_3 v - \frac{1}{4} C_1 x + C_6, \end{aligned}$$

where  $C_i, i = 1, 2, \dots, 6$  are arbitrary constants. Thus, corresponding to the above multipliers we obtain the following six local conserved vectors of (7.1):

$$\begin{aligned} T_1^t &= (tu - x)v, \\ T_1^x &= \frac{1}{4} (tu - x) u_{xx} - \frac{1}{8} tu_x^2 + \frac{1}{4} u_x + (tu - x) uv + \frac{1}{2} tv^2; \\ T_2^t &= \frac{1}{3} u^2 u_{xx} + \frac{1}{3} v u_{xx} + \frac{1}{3} u v_{xx} + \frac{1}{6} u u_x^2 + \frac{2}{3} u^3 v + 2 u v^2, \\ T_2^x &= \frac{1}{12} u_{xx}^2 + u v u_{xx} + \frac{1}{6} u^3 u_{xx} + \frac{1}{3} (u v_{tx} - u^2 u_{tx} - v u_{tx} + u u_x u_t + 2 u u_x v_x) \\ &\quad + \frac{1}{3} v_x u_t - \frac{1}{6} u_x^2 v + \frac{1}{3} v_x^2 + \frac{1}{3} u_x v_x + \frac{1}{4} u^2 u_x^2 + \frac{2}{3} u^4 v + \frac{2}{3} v^3 + 3 u^2 v^2; \end{aligned}$$

$$T_3^t = \frac{1}{2} u u_{xx} + 2u^3 + 2v^2,$$

$$T_3^x = \frac{1}{2} u^2 u_{xx} + v u_{xx} - \frac{1}{2} u u_{tx} + \frac{1}{2} u_x u_t + 2u^3 v + 4uv^2;$$

$$T_4^t = uv,$$

$$T_4^x = \frac{1}{4} u u_{xx} - \frac{1}{8} u_x^2 + u^2 v + \frac{1}{2} v^2;$$

$$T_5^t = u,$$

$$T_5^x = \frac{1}{2} u^2 + v;$$

$$T_6^t = 4v,$$

$$T_6^x = u_{xx} + 4uv.$$

## 7.2.2 Construction of conservation laws using the new conservation theorem

In this subsection we construct conservation laws for (7.1), by applying the new conservation theorem [41]. See also Section 1.5.4.

The coupled Kaup-Boussinesq system together with its adjoint equation are given by

$$E_{\alpha_1} \equiv u_t + u u_x + v_x = 0,$$

$$E_{\alpha_2} \equiv v_t + v u_x + u v_x + \frac{1}{4} u_{xxx} = 0, \quad (7.25)$$

$$E_{\alpha_1}^* \equiv p_t + u p_x + q_x = 0,$$

$$E_{\alpha_2}^* \equiv q_t + u q_x + v p_x + \frac{1}{4} p_{xxx} = 0. \quad (7.26)$$

One can easily verify that the third-order Lagrangian for the system of equations (7.25) and (7.26) is given by

$$L = p(v_t + vu_x + uv_x) + q(u_t + uu_x + v_x) - \frac{1}{4}p_x u_{xx}. \quad (7.27)$$

We recall that the coupled Kaup-Boussinesq system admits the following four Lie point symmetries:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad X_4 = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - u \frac{\partial}{\partial u} - 2v \frac{\partial}{\partial v}.$$

Thus we have the following four cases:

(i) For the Lie point symmetry  $X_1 = \partial_x$  the corresponding Lie characteristic functions are  $W^1 = -u_x$ ,  $W^2 = -v_x$ ,  $W^3 = -p_x$  and  $W^4 = -q_x$ . Thus, using the Ibragimov theorem [41], the components of the conserved vector are given by

$$\begin{aligned} T_1^t &= -qu_x - pv_x, \\ T_1^x &= qu_t + pv_t - \frac{1}{4}u_x p_{xx} + \frac{1}{4}p_x u_{xx}. \end{aligned}$$

(ii) The Lie point symmetry  $X_2 = \partial_t$  has the Lie characteristic functions that are given by  $W^1 = -u_t$ ,  $W^2 = -v_t$ ,  $W^3 = -p_t$  and  $W^4 = -q_t$ . Hence by the application of Ibragimov theorem [41], the conserved vector  $(T_2^t, T_2^x)$  is given by

$$\begin{aligned} T_2^t &= qv_x + quu_x + puv_x + pu_xv - \frac{1}{4}p_x u_{xx}, \\ T_2^x &= -qv_t - quu_t - puv_t - pu_tv - \frac{1}{4}u_t p_{xx} + \frac{1}{4}p_x u_{tx} + \frac{1}{4}p_t u_{xx}. \end{aligned}$$

(iii) The symmetry generator  $X_3 = t \partial_x + \partial_u$  has the Lie characteristic functions given by  $W^1 = 1 - tu_x$ ,  $W^2 = -tv_x$ ,  $W^3 = -tp_x$  and  $W^4 = -tq_x$  and hence in this case one can obtain the conserved vector whose components are

$$\begin{aligned} T_3^t &= q - tqu_x - tpv_x, \\ T_3^x &= qu + pv + tqu_t + tpv_t + \frac{1}{4}tp_x u_{xx} + \frac{1}{4}p_{xx} - \frac{1}{4}tu_x p_{xx}. \end{aligned}$$

(iv) Finally, we consider the symmetry generator  $X_4 = 2t\partial_t + x\partial_x - u\partial_u - 2v\partial_v$ , which has the Lie characteristic functions

$$\begin{aligned} W^1 &= -u - 2tu_t - xu_x, & W^2 &= -2v - 2tv_t - xv_x, \\ W^3 &= p - 2tp_t - xp_x, & W^4 &= -2tq_t - xq_x. \end{aligned}$$

By invoking Ibragimov theorem [41], the components of conserved vector are given by

$$\begin{aligned} T_4^t &= -qu - 2vp - xqu_x - xpv_x + 2tqv_x + 2tquu_x + 2tpuv_x + 2tpu_xv - \frac{1}{2}tp_xu_{xx}, \\ T_4^x &= -2qv - qu^2 - 3puv + pxv_t + qxu_t - 2tqv_t - 2tpuv_t - 2tpvu_t - 2tquu_t \\ &\quad + \frac{1}{2}p_xu_x - \frac{1}{4}pu_{xx} + \frac{1}{2}tp_tu_{xx} + \frac{1}{4}xp_xu_{xx} - \frac{1}{4}up_{xx} - \frac{1}{4}xu_xp_{xx} + \frac{1}{2}tp_xu_{tx} \\ &\quad - \frac{1}{2}tu_t p_{xx}. \end{aligned}$$

**Remark.** Due to the presence of arbitrary functions  $p(x, t)$  and  $q(x, t)$ , the conservation laws obtained by this method are infinitely many.

### 7.3 Concluding remarks

In this chapter we studied the Kaup-Boussinesq system (7.1). Exact solutions of this system were found using two distinct methods. Firstly, using the direct integration we obtained travelling wave solutions of (7.1). Secondly, we obtained an optimal system of one-dimensional subalgebras and found group-invariant solutions based on the optimal system of one-dimensional subalgebras. Also, the conservation laws for the Kaup-Boussinesq system (7.1) were derived by two approaches; using the multiplier method and the conservation theorem due to Ibragimov.

# Chapter 8

## Classical model of Prandtl's boundary layer theory for radial viscous flow: Application of $(G'/G)$ –expansion method

Prandtl [81] initiated the concept of a boundary layer in large Reynolds number flows in 1904 and showed how the Navier-Stokes equation could be simplified to yield approximate solutions. Prandtl introduced boundary layer theory to understand the flow behavior of a viscous Newtonian fluid near a solid boundary. Prandtl's boundary layer equations arise in various physical models of fluid mechanics. The equations of the boundary layer theory have been the subject of considerable interest, since they represent an important simplification of the original Navier-Stokes equations. These equations arise in the study of steady flows produced by wall jets, free jets and liquid jets, the flow past a stretching plate/surface, flow induced due to a shrinking sheet and so on. These boundary layer equations

are usually solved subject to specific boundary conditions depending upon the physical model investigation.

Blasius [82] solved the Prandtl's boundary layer equations for a flat moving plate problem and found a power series solution of the model. Falkner and Skan [83] generalized the Blasius problem by considering the boundary layer flow over wedge inclined at certain angle. Sakiadis [84] studied the boundary layer flow over a continuously moving rigid surface with a constant speed. Crane [85] was the first one who investigated the boundary layer flow due to a stretching surface and developed the exact solutions of boundary layer equations. Gupta and Gupta [86] extended Crane's work and for the first time introduced the concept of heat transfer with the stretching sheet boundary layer flow.

Schlichting [87] was the first to apply the boundary layer theory to the steady flow produced by a free two-dimensional jet emerging into a fluid at rest and solved the resulting ordinary differential equation numerically. Later, Bickley [88] solved the differential equation analytically. The concept of the boundary layer to laminar jets is discussed fully in standard texts on boundary layer theory such as by Schlichting [89] and Rosenhead [90]. More recently, the similarity solution of axisymmetric non-Newtonian wall jet with swirl effects was obtained by Kolar [91]. Naz et al. [92] and Mason [93] studied the general boundary layer equations for two-dimensional and radial flows by using the classical Lie group approach and recently Naz et al. [94] provided the similarity solutions of the Prandtl's boundary layer equations by implementing the non-classical symmetry method.

In this chapter we employ the  $(G'/G)$ -expansion method to find some new class of exact closed-form solutions of Prandtl's boundary layer equation for radial flow models with constant or uniform main stream velocity.

The results of this chapter have been published [95].

## 8.1 Mathematical model

The Prandtl's boundary layer equation, for the stream function  $\phi(r, \theta)$ , for radial flow with uniform or vanishing mainstream velocity is [90]

$$\frac{1}{r} \frac{\partial \phi}{\partial \theta} \frac{\partial^2 \phi}{\partial r \partial \theta} - \frac{1}{r^2} \left( \frac{\partial \phi}{\partial \theta} \right)^2 - \frac{1}{r} \frac{\partial \phi}{\partial r} \frac{\partial^2 \phi}{\partial \theta^2} - \nu \frac{\partial^3 \phi}{\partial \theta^3} = 0, \quad (8.1)$$

where  $(r, \theta)$  denote the cylindrical polar coordinates and  $\nu$  is the kinematic viscosity. The velocity components  $u(r, \theta)$  and  $v(r, \theta)$ , in the  $r$  and  $\theta$  directions, are related to stream function  $\phi(r, \theta)$  as

$$u(r, \theta) = \frac{1}{r} \frac{\partial \phi}{\partial \theta}, \quad v(r, \theta) = -\frac{1}{r} \frac{\partial \phi}{\partial r}.$$

By the use of Lie group theoretic method of infinitesimal transformations [24], the general form of similarity solution for equation (8.1) is

$$\phi(r, \theta) = r^{2-\beta} H(\xi), \quad \xi = \frac{\theta}{r^\beta}, \quad (8.2)$$

where  $\beta$  is the constant determined from further conditions and  $\xi = \theta/r^\beta$  is the similarity variable. By the substitution of equation (8.2) into (8.1), we obtain the third-order nonlinear ODE in  $H(\xi)$ , viz.,

$$\nu \frac{d^3 H}{d\xi^3} + (2 - \beta) H \frac{d^2 H}{d\xi^2} + (2\beta - 1) \left( \frac{dH}{d\xi} \right)^2 = 0. \quad (8.3)$$

Equation (8.3) is the general form of Prandtl's boundary layer equation for radial flow of a viscous incompressible fluid. The boundary layer equation is usually solved subject to certain boundary conditions depending upon the particular physical model under investigation. Here, we find the exact closed-form solutions of (8.3) using the  $(G'/G)$ -expansion method.

## 8.2 Application of the $(G'/G)$ –expansion method

In this section, we employ the  $(G'/G)$ –expansion method to obtain solutions of Prandtl's boundary layer equation (8.3).

We assume that the solutions of (8.3) are of the form

$$H(\xi) = \sum_{i=0}^M A_i \left( \frac{G'(\xi)}{G(\xi)} \right)^i,$$

where  $G(\xi)$  satisfies the second-order linear ODE with constant coefficients, viz.,

$$\frac{d^2G}{d\xi^2} + \lambda \frac{dG}{d\xi} + \mu G = 0 \quad (8.4)$$

with  $\lambda$  and  $\mu$  being constants.

The balancing procedure yields  $M = 1$ , so the solution of the ODE (8.3) is of the form

$$H(\xi) = A_0 + A_1 \left( \frac{G'(\xi)}{G(\xi)} \right). \quad (8.5)$$

Now substituting (8.5) into (8.3), making use of the ODE (8.4), collecting all terms with same powers of  $(G'/G)$  and equating each coefficient to zero, yields the following system of algebraic equations:

$$\begin{aligned} 2\beta A_1^2 \mu^2 - \beta A_0 A_1 \lambda \mu - A_1 \lambda^2 \mu \nu + 2A_0 A_1 \lambda \mu - 2A_1 \mu^2 \nu - A_1^2 \mu^2 &= 0, \\ 3\beta A_1^2 \lambda \mu - \beta A_0 A_1 \lambda^2 - 2\beta A_0 A_1 \mu - A_1 \lambda^3 \nu + 2A_0 A_1 \lambda^2 - 8A_1 \lambda \mu \nu + 4A_0 A_1 \mu &= 0, \\ \beta A_1^2 \lambda^2 - 3\beta A_0 A_1 \lambda + 2\beta A_1^2 \mu - 7A_1 \lambda^2 \nu + A_1^2 \lambda^2 + 6A_0 A_1 \lambda - 8A_1 \mu \nu + 2A_1^2 \mu &= 0, \\ \beta A_1^2 \lambda - 2\beta A_0 A_1 - 12A_1 \lambda \nu + 4A_1^2 \lambda + 4A_0 A_1 &= 0, \\ 3A_1^2 - 6A_1 \nu &= 0. \end{aligned}$$

Solving this system of algebraic equations, with the aid of Mathematica, we obtain

$$A_0 = \lambda \nu, \quad A_1 = 2\nu, \quad \lambda = 2\sqrt{\mu}.$$

Substituting these values of  $A_0$ ,  $A_1$  and the corresponding solution of ODE (8.3) into Eq. (8.5), we obtain the following three types of solutions of Eq. (8.1):

**Case 1:** When  $\lambda^2 - 4\mu > 0$

For this case we obtain the hyperbolic function solution given by

$$H(\xi) = \lambda\nu + 2\nu \left( -\frac{\lambda}{2} + \delta \frac{C_1 \sinh(\delta\xi) + C_2 \cosh(\delta\xi)}{C_1 \cosh(\delta\xi) + C_2 \sinh(\delta\xi)} \right),$$

where  $\delta = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}$ ,  $C_1$  and  $C_2$  are arbitrary constants.

Reverting back to the original variables  $(r, \theta)$ , the corresponding stream function is given by

$$\phi(r, \theta) = r^{2-\beta} \left[ \lambda\nu + 2\nu \left( -\frac{\lambda}{2} + \delta \frac{C_1 \sinh\left(\delta \frac{\theta}{r^\beta}\right) + C_2 \cosh\left(\delta \frac{\theta}{r^\beta}\right)}{C_1 \cosh\left(\delta \frac{\theta}{r^\beta}\right) + C_2 \sinh\left(\delta \frac{\theta}{r^\beta}\right)} \right) \right].$$

**Case 2:** When  $\lambda^2 - 4\mu < 0$

Here we obtain the trigonometric function solution

$$H(\xi) = \lambda\nu + 2\nu \left( -\frac{\lambda}{2} + \epsilon \frac{-C_1 \sin(\epsilon\xi) + C_2 \cos(\delta\xi)}{C_1 \cos(\epsilon\xi) + C_2 \sin(\epsilon\xi)} \right),$$

where  $\epsilon = \frac{1}{2}\sqrt{4\mu - \lambda^2}$ ,  $C_1$  and  $C_2$  are arbitrary constants. The corresponding stream function is given as

$$\phi(r, \theta) = r^{2-\beta} \left[ \lambda\nu + 2\nu \left( -\frac{\lambda}{2} + \epsilon \frac{-C_1 \sin\left(\epsilon \frac{\theta}{r^\beta}\right) + C_2 \cos\left(\epsilon \frac{\theta}{r^\beta}\right)}{C_1 \cos\left(\epsilon \frac{\theta}{r^\beta}\right) + C_2 \sin\left(\epsilon \frac{\theta}{r^\beta}\right)} \right) \right].$$

**Case 3:** When  $\lambda^2 - 4\mu = 0$

For this case we obtain the rational function solution

$$H(\xi) = \lambda\nu + 2\nu \left( -\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2\xi} \right).$$

In the form of stream function, the solution is expressed as

$$\phi(r, \theta) = r^{2-\beta} \left[ \lambda\nu + 2\nu \left( -\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 \frac{\theta}{r^\beta}} \right) \right],$$

where  $C_1$  and  $C_2$  are arbitrary constants.

### 8.3 Concluding remarks

We have employed the  $(G'/G)$ -expansion method for obtaining exact closed-form solutions of the well-known Prandtl's boundary layer equation for radial flow models with uniform main stream velocity. The advantage of this method is that there is no need to apply the initial and boundary conditions at the outset. This method yields a general solution with free parameters which can be identified by the specific conditions. Also the general solutions obtained by  $(G'/G)$ -expansion method are not approximate solutions. Prandtl's boundary layer equations arise in various physical models of fluid dynamics and thus the exact solutions obtained may be very useful and significant for the explanation of some practical physical models dealing with Prandtl's boundary layer theory.

# Chapter 9

## Conservation laws and solutions of a generalized coupled (2+1)-dimensional Burgers system

We consider the integrable coupled (2+1)-dimensional Burgers system

$$\begin{aligned}u_t - 2uu_x - v_{xx} &= 0, \\v_{yt} - 2uv_{xy} - 2u_xv_y - u_{xxy} &= 0,\end{aligned}\tag{9.1}$$

which was studied in [96] and multiple kink solutions were obtained. The system (9.1) was in fact introduced in [43] and the binary Bell polynomial approach was applied to it in [97]. It should be noted that when  $u = v$  and  $x = y$  in (9.1), the system reduces to the standard Burgers equation. Actually, this system is a nonlinear version of the bilinear system

$$(D_t + D_x^2) f \cdot g = 0,$$

$$D_y (D_t + D_x^2) f \cdot g = 0$$

under the dependent variable transformations

$$u = \left( \ln \frac{g}{f} \right)_x, \quad v = (\ln fg)_x.$$

In this chapter we study a generalized coupled (2+1)-dimensional Burgers system

$$u_t - a u u_x - v_{xx} = 0, \quad (9.2a)$$

$$v_{ty} - a u_x v_y - a u v_{xy} - u_{xxy} = 0, \quad (9.2b)$$

where  $a$  is a constant using Lie group analysis. We obtain new exact solutions of (9.2) using Lie group method along with the Kudryashov approach [44]. In addition, conservation laws are derived using the multiplier method [24, 40].

The results of this chapter have been submitted in [98].

## 9.1 Exact solutions of (9.2)

Firstly, we compute the Lie point symmetries of system (9.2). The symmetry group of (9.2) is generated by the vector field

$$\begin{aligned} X = & \tau(t, x, y, u, v) \frac{\partial}{\partial t} + \xi(t, x, y, u, v) \frac{\partial}{\partial x} + \phi(t, x, y, u, v) \frac{\partial}{\partial y} \\ & + \eta(t, x, y, u, v) \frac{\partial}{\partial u} + \varphi(t, x, y, u, v) \frac{\partial}{\partial v}. \end{aligned}$$

We apply the third prolongation  $\text{pr}^{(3)}X$  [24] on (9.2). This gives us an overdetermined system of twenty-eight linear partial differential equations given by

$$\begin{aligned} \eta_v = 0, \quad \tau_v = 0, \quad \xi_v = 0, \quad \phi_v = 0, \quad \varphi_{vv} = 0, \quad \varphi_u = 0, \quad \tau_u = 0, \quad \xi_u = 0, \quad \phi_u = 0, \\ \eta_{uu} = 0, \quad \varphi_y = 0, \quad \tau_y = 0, \quad \xi_y = 0, \quad \eta_{yu} = 0, \quad \tau_x = 0, \quad \phi_x = 0, \quad \eta_{xxu} = 0, \quad \eta_{xxy} = 0, \\ \phi_t = 0, \quad \xi_{xx} - 2\varphi_{xv} = 0, \quad \eta_t - a u \eta_x - \varphi_{xx} = 0, \quad \eta_u - \tau_t - \varphi_v + 2\xi_x = 0, \end{aligned}$$

$$\begin{aligned}
au\xi_x - a\eta - au\tau_t - \xi_t &= 0, \quad \xi_{xx} - 2\eta_{xu} = 0, \quad 2\xi_x - \tau_t - \eta_u + \varphi_v = 0, \\
a\tau_t + a\eta_u - a\xi_x &= 0, \quad au\xi_x - a\eta - au\tau_t - \xi_t = 0, \quad \varphi_{tv} - au\varphi_{xv} - a\eta_x = 0.
\end{aligned}$$

After some tedious workings, the solution of the above system gives the following five Lie point symmetries:

$$\begin{aligned}
X_1 &= \phi(y)\frac{\partial}{\partial y}, \quad X_2 = q(t)\frac{\partial}{\partial v}, \quad X_3 = xw(t)\frac{\partial}{\partial v}, \\
X_4 &= h(t)\frac{\partial}{\partial x} - \frac{1}{a}h'(t)\frac{\partial}{\partial u} - \frac{1}{2a}x^2h''(t)\frac{\partial}{\partial v}, \\
X_5 &= \tau(t)\frac{\partial}{\partial t} + \frac{1}{2}x\tau'(t)\frac{\partial}{\partial x} - \left[ \frac{1}{2}\tau'(t)u + \frac{1}{2a}x\tau''(t) \right] \frac{\partial}{\partial u} - \left[ \frac{1}{2}\tau'(t)v + \frac{1}{12a}x^3\tau'''(t) \right] \frac{\partial}{\partial v}.
\end{aligned}$$

We now use the three symmetries  $X_1$ ,  $X_4$  and  $X_5$  with  $\phi(y) = h(t) = \tau(t) = 1$ . The combination of these symmetries gives the four invariants, viz.,  $G = v$ ,  $F = u$ ,  $g = x - y$ ,  $f = t - y$ . Using these invariants results in a system of PDEs in two variables  $f$  and  $g$  given by

$$F_f - G_{gg} - aF_gF = 0, \quad (9.3a)$$

$$-aF(-G_{fg} - G_{gg}) - aF_g(-G_f - G_g) + F_{fgg} - G_{fg} - G_{ff} + F_{ggg} = 0. \quad (9.3b)$$

The Lie point symmetries of the above system (9.3) include  $\Gamma_1 = \partial/\partial f$  and  $\Gamma_2 = \partial/\partial g$ . The symmetry  $\Gamma_1 + \nu\Gamma_2$  yields three invariants  $P = F$ ,  $R = G$ ,  $z = g - \nu f$  and using these invariants, (9.3) reduces to the system of ODEs

$$(\nu + aP(z))P'(z) + R''(z) = 0, \quad (9.4a)$$

$$(\nu - 1)\{aP'(z)R'(z) + P'''(z) + R''(z)(\nu + aP(z))\} = 0. \quad (9.4b)$$

We now employ Kudryashov method [44] to obtain exact solutions of system (9.4). Consider the solution of (9.4) in the form

$$P(z) = \sum_{i=0}^M A_i H(z)^i, \quad R(z) = \sum_{j=0}^N B_j H(z)^j \quad (9.5)$$

where  $A_i$  ( $i = 0, \dots, M$ ),  $B_j$  ( $j = 0, \dots, N$ ),  $N$  and  $M$  are constants to be determined and  $H(z)$  satisfies

$$H'(z) = H(z)^2 - H(z). \quad (9.6)$$

The solution of (9.6) can be written in terms of the elementary function as

$$H(z) = \frac{1}{1 + e^z}. \quad (9.7)$$

In our case the balancing procedure yields  $M = N = 1$ . Substituting (9.5) into (9.4) and making use of (9.6), we obtain

$$\begin{aligned} (H(z) - 1)H(z) (aA_1^2H(z) + A_1(aA_0 + \nu) + B_1(2H(z) - 1)) &= 0, \\ (H(z) - 1)H(z) (B_1(2H(z) - 1)(aA_0 + \nu) + A_1(3(aB_1 + 2)H(z)^2 \\ - 2(aB_1 + 3)H(z) + 1)) &= 0. \end{aligned}$$

Splitting on the powers of  $H(z)$  we get an algebraic system given by

$$\begin{aligned} A_1(-aA_0 - \nu) + B_1 &= 0, \quad A_1(aA_0 + \nu) - aA_1^2 - 3B_1 = 0, \\ aA_1^2 + 2B_1 &= 0, \quad 3A_1(\nu - 1)(aB_1 + 2) = 0, \\ aA_0B_1\nu - aA_0B_1 - A_1\nu + A_1 + B_1\nu^2 - B_1\nu &= 0, \\ 2A_1\nu(aB_1 + 3) - aA_0B_1(\nu - 1) - 2B_1\nu(aA_0 + \nu) + 2B_1(aA_0 + \nu) \\ - 2A_1(aB_1 + 3) + A_1(\nu - 1) + B_1(1 - \nu)\nu &= 0, \\ 2B_1(\nu - 1)(aA_0 + \nu) - 3A_1\nu(aB_1 + 2) - 2A_1(\nu - 1)(aB_1 + 3) + 3A_1(aB_1 + 2) &= 0. \end{aligned}$$

The solution of this algebraic system is

$$A_0 = \frac{1 - \nu}{a}, \quad A_1 = -\frac{2}{a}, \quad B_0 = B_0, \quad B_1 = -\frac{2}{a}$$

and hence the solution of the generalized coupled (2+1)-dimensional Burgers system (9.2) is given by

$$u(t, x, y) = \frac{1 - \nu}{a} - \frac{2}{a(1 + e^{-\nu t + x - (1 - \nu)y})},$$

$$v(t, x, y) = B_0 - \frac{2}{a(1 + e^{-\nu t + x - (1-\nu)y})},$$

where  $B_0$  is an arbitrary constant.

## 9.2 Conservation laws of (9.2)

In this section we derive conservation laws of the generalized coupled (2+1)-dimensional Burgers system (9.2) using the multiplier method.

We look for second-order multipliers and obtain

$$\begin{aligned} \Lambda_1 &= -\frac{1}{2a}x^2f_1''(t)v_y + \left(u_xu_y - 2uu_{xy} - 3v_xv_y - \frac{4}{a}u_{ty} - 3au^2v_y\right)f_1(t) + \frac{1}{2a}xf_2'(t)v_y \\ &\quad - \left(\frac{1}{a}u_y + \frac{2}{a}xu_{xy} + 2uxv_y\right)f_1'(t) + \left(\frac{1}{a}u_{xy} + uv_y\right)f_2(t) + f_3(t)v_y + f_4(y); \\ \Lambda_2 &= -\frac{1}{6a^2}x^3f_1'''(t) + \frac{1}{4a^2}x^2f_2''(t) - au^3f_1(t) - \frac{1}{2a}x^2f_1''(t)u + \frac{1}{2a}xf_2'(t)u \\ &\quad - \frac{1}{a}f_1(t)u_{xx} - \frac{2}{a}xf_1'(t)v_x + \frac{1}{a}f_2(t)v_x - \frac{1}{a}f_1'(t)v - \frac{3}{a}f_1(t)v_t + \frac{1}{a}xf_3'(t) \\ &\quad - xf_1'(t)u^2 + \frac{1}{2}u^2f_2(t) - 3f_1(t)uv_x + f_3(t)u + f_5(y) + f_6(t), \end{aligned}$$

where  $f_i$ ,  $i = 1, 2, \dots, 6$  are arbitrary functions of  $t$  and so corresponding to these multipliers we obtain the following six local conserved vectors of (9.2):

$$\begin{aligned} T_1^t &= -\frac{1}{6a^2}x^3v_yf_1'''(t) - \frac{1}{2a}ux^2v_yf_1''(t) - \left(\frac{1}{a}xuu_{xy} + \frac{1}{2a}uu_y + \frac{1}{a}xv_xv_y + xu^2v_y \right. \\ &\quad \left. + \frac{1}{2a}vv_y\right)f_1'(t) + \left(\frac{2u_yv_{xx}}{a} - \frac{2u_tu_y}{a} - \frac{3v_tv_y}{2a} - \frac{2uu_{ty}}{a} - \frac{3vv_{ty}}{2a} - au^3v_y \right. \\ &\quad \left. - \frac{u_{xx}v_y}{2a} + \frac{3vu_{xxy}}{2a} - \frac{2u^2u_{xy}}{3} - 2uv_xv_y + vu_xv_y + uvv_{xy} + \frac{5}{3}uu_xu_y\right)f_1(t), \\ T_1^x &= \left(\frac{x^3u_{xy}}{6a^2} - \frac{x^2u_y}{2a^2} + \frac{ux^3v_y}{6a}\right)f_1'''(t) + \left(\frac{ux^2u_{xy}}{4a} - \frac{x^2u_xu_y}{4a} - \frac{uxu_y}{2a} - \frac{v xv_y}{2a} \right. \\ &\quad \left. - \frac{vx^2v_{xy}}{4a} + \frac{x^2v_xv_y}{4a} + \frac{1}{2}u^2x^2v_y\right)f_1''(t) + \left(au^3xv_y - \frac{v xv_{ty}}{a} + u^2xu_{xy} \right. \\ &\quad \left. + \frac{2xu_{xy}v_x}{a} - \frac{u_yv_x}{a} - \frac{vu_{xy}}{a} - \frac{xu_tu_y}{a} + 2uxv_xv_y\right)f_1'(t) + \left(a^2u^4v_y - \frac{3v_{tx}u_y}{2a}\right) \end{aligned}$$

$$\begin{aligned}
& + \frac{uv_{txy}}{2a} - \frac{2vu_{txy}}{a} - \frac{v_{ty}u_x}{2a} + \frac{2u_{ty}v_x}{a} + \frac{3}{4}au^3u_{xy} + 3au^2v_xv_y - \frac{3}{4}au^2u_xu_y \\
& + \frac{3v_tu_{xy}}{2a} + \frac{u_xu_{xxy}}{2a} + \frac{u_{xx}u_{xy}}{2a} - \frac{u_{xxx}u_y}{2a} - \frac{uu_{xxy}}{2a} + \frac{4}{3}u^2u_{ty} - uvv_{ty} - \frac{1}{3}u^2v_{xxy} \\
& - \frac{1}{3}uu_xv_{xy} - \frac{1}{3}vu_xu_{xy} - \frac{4}{3}u_xu_yv_x + \frac{1}{3}u_x^2v_y - \frac{2}{3}uu_yv_{xx} + \frac{1}{3}vu_{xx}u_y + \frac{1}{3}uvu_{xxy} \\
& + 2uv_tv_y - vu_tv_y + \frac{5}{3}uvxu_{xy} + \frac{1}{3}uvyu_{xx} - \frac{1}{3}uu_tv_y - vv_xv_{xy} + v_x^2v_y \Big) f_1(t), \\
T_1^y &= \frac{1}{6a^2}vx^3f_1^{(4)}(t) + \frac{1}{a^2}uxf_1^{(3)}(t) + \left( \frac{u^2}{2a} + \frac{x^2uu_{xx}}{4a} + \frac{xuu_x}{a} + \frac{v^2}{2a} + \frac{x^2vv_{xx}}{4a} + \frac{xvv_x}{a} \right) f_1''(t) \\
& + \left( \frac{vu_{xx}}{2a} + \frac{uxu_{tx}}{a} + \frac{5uu_t}{2a} + \frac{v_xv_{tx}}{a} + \frac{2vv_t}{a} - \frac{1}{3}u^2u_x \right) f_1'(t) \\
& + \left( \frac{2uu_{tt}}{a} + \frac{3vv_{tt}}{2a} + \frac{1}{4}au^3u_{x^2} + \frac{1}{4}au^2u_x^2 - \frac{uv_{txx}}{2a} + \frac{vu_{txx}}{2a} + \frac{uu_{xxx}}{2a} + \frac{1}{3}u^2v_{xxx} \right. \\
& \left. - \frac{2}{3}u^2u_{tx} - \frac{1}{3}uvu_{xxx} + uu_{xx}v_x + uu_xv_{xx} - \frac{1}{3}uu_tv_x + vv_xv_{xx} \right) f_1(t); \\
T_2^t &= \frac{1}{4a^2}x^2v_yf_2''(t) + \frac{1}{2a}f_2'(t)xuv_y + \left( \frac{1}{2a}uu_{xy} + \frac{1}{2a}v_xv_y + \frac{1}{2}u^2v_y \right) f_2(t), \\
T_2^x &= \left( \frac{1}{2a^2}xu_y - \frac{1}{4a}x^2uv_y - \frac{1}{4a^2}x^2u_{xy} \right) f_2''(t) \\
& + \left( \frac{v_xv_{xy}}{4a} - \frac{v_xv_y}{4a} + \frac{vv_y}{4a} - \frac{uxu_{xy}}{4a} + \frac{xu_xu_y}{4a} + \frac{uu_y}{4a} - \frac{1}{2}u^2xv_y \right) f_2'(t) \\
& + \left( \frac{vv_{ty}}{2a} - \frac{1}{2}au^3v_y - \frac{u_{xy}v_x}{a} + \frac{u_tv_y}{2a} - \frac{1}{2}u^2u_{xy} - uv_xv_y \right) f_2(t), \\
T_2^y &= -\frac{1}{4a^2}x^2vf_2'''(t) - \frac{1}{2a^2}uf_2''(t) - \left( \frac{1}{4a}uxu_{xx} + \frac{1}{2a}uu_x + \frac{1}{4a}v_xv_{xx} + \frac{1}{2a}vv_x \right) f_2'(t) \\
& - \frac{1}{2a}f_2(t)uu_{tx} - \frac{1}{2a}f_2(t)vv_{tx}; \\
T_3^t &= \frac{1}{a}xv_yf_3'(t) + f_3(t)uv_y, \\
T_3^x &= \left( \frac{1}{a}u_y - uxv_y - \frac{1}{a}xu_{xy} \right) f_3'(t) + \left( \frac{1}{2}u_xu_y - au^2v_y - \frac{1}{2}uu_{xy} - \frac{1}{2}v_xv_y + \frac{1}{2}vv_{xy} \right) f_3(t), \\
T_3^y &= -\frac{1}{a}vx f_3''(t) - \frac{1}{2}f_3(t)(uu_{xx} + vv_{xx});
\end{aligned}$$

$$\begin{aligned}
T_4^t &= f_4(y)u, \\
T_4^x &= -f_4(y)\left(v_x + \frac{1}{2}au^2\right), \\
T_4^y &= 0;
\end{aligned}$$

$$\begin{aligned}
T_5^t &= f_5(y)v_y, \\
T_5^x &= -f_5(y)(auv_y + u_{xy}), \\
T_5^y &= 0;
\end{aligned}$$

$$\begin{aligned}
T_6^t &= f_6(t)v_y, \\
T_6^x &= -f_6(t)(auv_y + u_{xy}), \\
T_6^y &= -f_6'(t)v.
\end{aligned}$$

**Remark:** Due to the presence of arbitrary functions in the multipliers, we obtain infinitely many conserved vectors for our system.

### 9.3 Concluding remarks

In this chapter we studied a generalized coupled (2+1)-dimensional Burgers system (9.2). The Lie group analysis together with the Kudryashov approach were employed to obtain new travelling wave solutions of the system. Moreover, conservation laws of the system were derived for the first time. The multiplier approach was used to construct the conservation laws.

# Chapter 10

## Algebraic aspects of evolution partial differential equation arising in the study of constant elasticity of variance model from financial mathematics

One of the classical financial problems is the modelling of optimal investment-consumption decisions under uncertainty. Since the pioneering work of Merton [99], the investment-consumption problems have been widely studied in the literature with different extensions and applications. Cox [100] and Cox and Ross [101] have derived the renowned constant elasticity of variance (CEV) option pricing model and Schroder [102] has subsequently extended the model by expressing the CEV option pricing formula in terms of the noncentral Chi-square distribution. The CEV model is usually applied to analyze the option and asset pricing formula,

as was investigated by Beckers [103], Davydov and Linetsky [104], Emanuel and Macbeth [105] and recently by Hsu et al. [106].

In this chapter, we study the optimal investment-consumption problem under the CEV model [107]. Literature survey witnesses that a few studies [108, 109] have recently reported that the solutions of the CEV models have been found. However, the exact solutions of the CEV model to an optimal investment-consumption problem have not been reported in the existing literature, which is the main focus of this particular work.

Many studies on the applications of Lie symmetry group methods to differential equations are found in the literature, since the earliest works of Lie on the subject. In recent years a number of studies have been reported in which Lie's theory has been employed to problems in finance. Regarding the relation between Lie groups and some financial mathematics problems, we would like to point out the existence of several recent works which consistently show the application of well-known properties of Lie symmetry method to several financial mathematics problems. One of the initial investigations is due to [110], who performed the reduction and integration of the Black-Scholes equation. Goard [111, 112] also studied particular cases of bond-pricing equation via Lie group approach. Sinkala et al. [27, 113] uncovered the invariant properties of some well-known financial mathematics models. In recent times, the Lie group approach has been widely applied to other partial differential equations from finance. A few important studies to mention are [28, 114–116].

In this chapter, we discuss the optimal investment-consumption problem under the CEV model from the viewpoint of Lie symmetry approach. The symmetry structure allows the reduction of the evolution partial differential equation to an ordinary differential equation and the group-invariant solution to the problem is then obtained. Some nontrivial conservation laws are also derived for the CEV model by employing a general theorem proved in [41].

The results of this chapter have been submitted [117].

## 10.1 Formulation of the model

In this section, we briefly discuss the optimal investment-consumption problem under CEV process formulated in [107]. Consider a financial market with two assets which are continuous over the interval  $[0, T]$ . One asset is the bond with price  $p_t$  at time  $t$  and satisfies the equation

$$dp_t = \alpha p_t dt, \quad \text{with } p_t = p_0 > 0,$$

where the constant  $\alpha > 0$  denotes the interest rate of the bond and is called the elasticity parameter. The other asset is the stock with price  $s_t$  at time  $t$ , with the price process  $s_t$  given by the following constant elasticity of variance model

$$ds_t = s_t(\beta dt + K s_t^\gamma dw_t), \quad \text{with } s_t = s_0 > 0,$$

where  $\beta$  ( $\beta > \alpha$ ) is the expected instantaneous rate of return of the stock,  $K$  and  $\gamma$  are the constant parameters,  $K s_t^\gamma$  is defined as the instantaneous volatility of the stock and  $w_t$  is the total wealth at any time  $t$  also called a one-dimensional adapted Brownian motion.

The underlying market is modelled by assuming that the investor invests the market value of his wealth  $\varphi_t$  into the stock at time  $t$ . Clearly, the amount invested in the bond is  $x_t - \varphi_t$ , in which  $x_t$  represents the wealth of the investor at time  $t$ . In addition, assume that short-selling of stocks and borrowing at the interest rate of the bond are allowed and transaction cost is not taken into consideration. Thus, the wealth process  $x_t$  corresponding to trading strategy is subject to equation

$$dx_t = [\alpha x_t + (\beta - \alpha) \varphi_t - c_t] dt + \varphi_t K s_t^\gamma dw_t, \quad \text{with } x_t = x_0 > 0,$$

where  $c_t$  is the consumption rate. The optimal investment-consumption problem under the CEV model is governed by the (1+1) evolution partial differential equation of the form [118]

$$\begin{aligned} \frac{\partial G}{\partial t} + (\lambda\alpha - \delta)G + 2\gamma^2 K^2 x \frac{\partial^2 G}{\partial x^2} - \frac{\lambda(\beta - \alpha)^2}{2(\lambda - 1)K^2} xG + 2 \left[ \frac{\lambda}{(\lambda - 1)} \gamma(\beta - \alpha) - \gamma\beta \right] x \frac{\partial G}{\partial x} \\ + \gamma(2\gamma + 1)K^2 \frac{\partial G}{\partial x} - \frac{2\lambda}{(\lambda - 1)} \gamma^2 K^2 x \frac{(\partial G/\partial x)^2}{G} + (1 - \lambda)\pi^{\frac{1}{1-\lambda}} G^{\frac{\lambda}{\lambda-1}} = 0, \end{aligned} \quad (10.1)$$

with  $\lambda, \alpha, \delta, K, \gamma, \beta$  and  $\pi$  being constants. The corresponding terminal condition for PDE (10.1) is given by [107]

$$G(T, x) = 1 - \pi.$$

In addition, using of the variable change [107],

$$G(t, x) = \bar{F}(t, x)^{1-\lambda} \quad (10.2)$$

and substituting (10.2) into (10.1), results in a linear second-order evolution partial differential equation in  $\bar{F}(t, x)$ , viz.,

$$\begin{aligned} \frac{\partial \bar{F}}{\partial t} + \left[ \frac{\lambda\alpha - \delta}{1 - \lambda} + \frac{\lambda}{2(\lambda - 1)^2} \left( \frac{\beta - \alpha}{K} \right)^2 x \right] \bar{F} + 2 \left[ \frac{\lambda}{(\lambda - 1)} \gamma(\beta - \alpha) - \gamma\beta \right] x \frac{\partial \bar{F}}{\partial x} \\ + \gamma(2\gamma + 1)K^2 \frac{\partial \bar{F}}{\partial x} + 2\gamma^2 K^2 x \frac{\partial^2 \bar{F}}{\partial x^2} + \pi^{\frac{1}{\lambda-1}} = 0. \end{aligned} \quad (10.3)$$

The terminal condition takes the form

$$\bar{F}(T, x) = (1 - \pi)^{\frac{1}{1-\lambda}}.$$

Finally, assuming

$$\bar{F}(T, x) = \pi^{\frac{1}{\lambda-1}} \int_t^T F(u, x) du + (1 - \pi)^{\frac{1}{1-\lambda}} F(t, x),$$

then PDE (10.3) transforms into

$$\frac{\partial F}{\partial t} + \left[ \frac{\lambda\alpha - \delta}{1 - \lambda} + \frac{\lambda}{2(\lambda - 1)^2} \left( \frac{\beta - \alpha}{K} \right)^2 x \right] F + \left[ \frac{2\lambda}{(\lambda - 1)} \gamma(\beta - \alpha) - 2\gamma\beta \right] x \frac{\partial F}{\partial x}$$

$$+ \gamma(2\gamma + 1)K^2 \frac{\partial F}{\partial x} + 2\gamma^2 K^2 x \frac{\partial^2 F}{\partial x^2} = 0. \quad (10.4)$$

with the terminal condition

$$F(T, x) = 1, \quad \text{with } t \in [0, T]. \quad (10.5)$$

## 10.2 Lie point symmetries

In this section, we calculate the Lie point symmetries admitted by (10.4). These symmetries will be used to obtain the closed-form group-invariant solution of the PDE (10.4).

In order to facilitate the calculations when employing the Lie group approach, we rewrite the PDE (10.4) in the form

$$\frac{\partial F}{\partial t} + \varepsilon x \frac{\partial^2 F}{\partial x^2} + (cx + d) \frac{\partial F}{\partial x} + (a + bx)F = 0, \quad (10.6)$$

where

$$a = \frac{\lambda\alpha - \delta}{1 - \lambda}, \quad b = \frac{\lambda(\beta - \alpha)^2}{2(\lambda - 1)^2 K^2}, \quad c = 2 \left( \frac{\lambda\gamma(\beta - \alpha)}{(\lambda - 1)} - \gamma\beta \right),$$

$$d = \gamma(2\gamma + 1)K^2, \quad \varepsilon = 2\gamma^2 K^2.$$

We now look for the transformations of the independent variables  $t, x$  and the dependent variable  $F$  of the form

$$\bar{t} = \bar{t}(t, x, F, \epsilon), \quad \bar{x} = \bar{x}(t, x, F, \epsilon), \quad \bar{F} = \bar{F}(t, x, F, \epsilon), \quad (10.7)$$

which constitute a group where  $\epsilon$  is the group parameter such that PDE (10.6) is left invariant. From Lie's theory, the transformations in equation (10.7) are obtained in terms of the infinitesimal transformations

$$\bar{t} \simeq t + \epsilon\tau(t, x, F), \quad \bar{x} \simeq x + \epsilon\xi(t, x, F), \quad \bar{F} \simeq F + \epsilon\eta(t, x, F)$$

or the operator

$$X = \tau(t, x, F) \frac{\partial}{\partial t} + \xi(t, x, F) \frac{\partial}{\partial x} + \eta(t, x, F) \frac{\partial}{\partial F},$$

which is a generator of the Lie point symmetry of PDE (10.6) if the following determining equation holds, viz.,

$$X^{[2]}[F_t + \varepsilon x F_{xx} + (cx + d)F_x + (a + bx)F] |_{(10.6)} = 0. \quad (10.8)$$

Expanding the symmetry condition (10.8) and separating by powers of the derivatives of  $F$ , as  $\tau$ ,  $\xi$ , and  $\eta$  are independent of the derivatives of  $F$ , leads to the overdetermined system of linear homogeneous partial differential equations

$$\begin{aligned} \tau_F &= 0, \\ \xi_F &= 0, \\ \eta_{FF} &= 0, \\ \tau_x &= 0, \\ \varepsilon \xi + \varepsilon x \tau_t - 2\varepsilon x \xi_x &= 0, \\ (a + bx)\eta + bF\xi + \tau_t F(a + bx) - \eta_F F(a + bx) + \eta_x(cx + d) + \varepsilon x \eta_{xx} + \eta_t &= 0, \\ c\xi + \tau_t(cx + d) - \xi_x(cx + d) + 2\varepsilon x \eta_{xF} - \varepsilon x \xi_{xx} - \xi_t &= 0. \end{aligned}$$

The solution of the above system yields

$$\begin{aligned} \tau &= C_1 + \frac{2}{\theta} C_3 e^{\theta t} - \frac{2}{\theta} C_4 e^{-\theta t}, \\ \xi &= 2C_3 x e^{\theta t} + 2C_4 x e^{-\theta t}, \\ \eta &= C_2 F - \frac{1}{\varepsilon \theta} C_3 F e^{\theta t} (2a\varepsilon + d\theta + c\theta x - \theta^2 x - cd) \\ &\quad - \frac{1}{\varepsilon \theta} C_4 F e^{-\theta t} (-2a\varepsilon + cx\theta + cd + d\theta + \theta^2 x), \end{aligned}$$

where  $\theta = \sqrt{c^2 - 4b\varepsilon}$ . Thus, the Lie point symmetries of the PDE (10.6) are

$$X_1 = \frac{\partial}{\partial t},$$

$$\begin{aligned}
X_2 &= \frac{2}{\theta} e^{\theta t} \frac{\partial}{\partial t} + 2x e^{\theta t} \frac{\partial}{\partial x} - \frac{1}{\varepsilon \theta} F e^{\theta t} (2a\varepsilon + d\theta + c\theta x - \theta^2 x - cd) \frac{\partial}{\partial F}, \\
X_3 &= -\frac{2}{\theta} e^{-\theta t} \frac{\partial}{\partial t} + 2x e^{-\theta t} \frac{\partial}{\partial x} - \frac{1}{\varepsilon \theta} F e^{-\theta t} (-2a\varepsilon + cx\theta + cd + d\theta + \theta^2 x) \frac{\partial}{\partial F}, \\
X_4 &= F \frac{\partial}{\partial F}.
\end{aligned}$$

### 10.3 Group-invariant solution

We now obtain the closed-form group-invariant solution for the PDE (10.6) by making use of the Lie symmetry algebra calculated in the previous section.

We consider the linear combination of the Lie point symmetries, namely,

$$X = a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4 \quad (10.9)$$

and use the terminal conditions

$$X(t - T)|_{t=T, F=1} = 0 \quad (10.10)$$

and

$$X(F - 1)|_{t=T, F=1} = 0. \quad (10.11)$$

The condition (10.10) yields

$$a_1 + \frac{2a_2}{\theta} e^{\theta T} - \frac{2a_3}{\theta} e^{-\theta T} = 0, \quad (10.12)$$

while condition (10.11) gives

$$\begin{aligned}
&a_4 + a_2 \left( \frac{cd}{\varepsilon \theta} - \frac{2a}{\theta} - \frac{d}{\varepsilon} + \frac{c^2 x}{\varepsilon \theta} - \frac{4bx}{\theta} - \frac{cx}{\varepsilon} \right) e^{\theta T} \\
&+ a_3 \left( \frac{2a}{\theta} - \frac{cd}{\varepsilon \theta} - \frac{d}{\varepsilon} - \frac{c^2 x}{\varepsilon \theta} + \frac{4bx}{\theta} - \frac{cx}{\varepsilon} \right) e^{-\theta T} = 0.
\end{aligned} \quad (10.13)$$

Splitting equation (10.13) on powers of  $x$  yields

$$a_4 + a_2 \left( \frac{cd}{\varepsilon \theta} - \frac{2a}{\theta} - \frac{d}{\varepsilon} \right) e^{\theta T} + a_3 \left( \frac{2a}{\theta} - \frac{cd}{\varepsilon \theta} - \frac{d}{\varepsilon} \right) e^{-\theta T} = 0, \quad (10.14)$$

$$a_2 \left( \frac{c^2}{\varepsilon\theta} - \frac{4b}{\theta} - \frac{c}{\varepsilon} \right) e^{\theta T} + a_3 \left( -\frac{c^2}{\varepsilon\theta} + \frac{4b}{\theta} - \frac{c}{\varepsilon} \right) e^{-\theta T} = 0. \quad (10.15)$$

Solving equations (10.12), (10.14) and (10.15), we obtain

$$a_1 = \frac{a_2 c(\theta - c)}{b\varepsilon\theta} e^{\theta T}, \quad a_3 = \frac{a_2(2b\varepsilon + c(\theta - c))}{2b\varepsilon} e^{2\theta T}, \quad a_4 = -\frac{a_2(\theta - c)(ac - 2bd)}{b\varepsilon\theta} e^{\theta T}. \quad (10.16)$$

Substituting the values of  $a_1$ ,  $a_3$  and  $a_4$  from (10.16) into (10.9) and solving  $X(F) = 0$ , we obtain the two invariants

$$J_1 = \frac{x e^{\theta t}}{(c + \theta)e^{2\theta t} - 2c e^{\theta(t+T)} + (c - \theta)e^{2\theta T}},$$

$$J_2 = F \left( (c + \theta)e^{\theta t} + (\theta - c)e^{\theta T} \right)^{d/\varepsilon}$$

$$\times \exp \left( at + \frac{4b\varepsilon x e^{2\theta t} + x(c - \theta)(c + \theta)e^{2\theta T}}{2\varepsilon \left( (c + \theta)e^{2\theta t} - 2c e^{\theta(t+T)} + (c - \theta)e^{2\theta T} \right)} - \frac{cdt}{2\varepsilon} - \frac{d\theta t}{2\varepsilon} \right).$$

Thus, the group-invariant solution of the PDE (10.6) is given by  $J_2 = G(J_1)$ , which yields

$$F(t, x) = G(z) \left\{ (c + \theta)e^{\theta t} + (\theta - c)e^{\theta T} \right\}^{-\frac{d}{\varepsilon}}$$

$$\times \exp \left( -at + \frac{4b\varepsilon x e^{2\theta t} + x(c - \theta)(c + \theta)e^{2\theta T}}{2\varepsilon \left( -(c + \theta)e^{2\theta t} + 2c e^{\theta(t+T)} + (\theta - c)e^{2\theta T} \right)} + \frac{cdt}{2\varepsilon} + \frac{d\theta t}{2\varepsilon} \right), \quad (10.17)$$

where  $z = J_1$ . Now substituting the value of  $F$  from (10.17) into PDE (10.6), we obtain a second-order ODE for  $G(z)$ , viz.,

$$2e^{\theta T} (2bdG(z) + c^2 z G'(z)) + 8bzG(z)e^{2\theta T} (2b\varepsilon - c^2) - dG'(z) - \varepsilon z G''(z) = 0. \quad (10.18)$$

Solving this reduced ODE (10.18) for  $G(z)$ , we obtain

$$G(z) = C_1 \exp(4bz e^{\theta T}) - C_2 2^{\frac{d}{\varepsilon}-1} z^{1-\frac{d}{\varepsilon}} \exp(4bz e^{\theta T}) \left( -\frac{\theta^2 z e^{\theta T}}{\varepsilon} \right)^{\frac{d}{\varepsilon}-1}$$

$$\times \Gamma \left[ 1 - \frac{d}{\varepsilon}, -\frac{2e^{\theta T} z \theta^2}{\varepsilon} \right],$$

where  $C_1$  and  $C_2$  are constants of integration and  $\Gamma[.,.]$  is the incomplete Gamma function [119]. The terminal condition dictates that we take  $C_2 = 0$ , hence the above solution for  $G(z)$  takes the form

$$G(z) = C_1 \exp(4bze^{\theta T}).$$

Substituting the value of  $G$  into (10.17), the solution for  $F(x, t)$  is written as

$$\begin{aligned} F(t, x) = & C_1 \exp(4bze^{\theta T}) \left\{ (c + \theta) \exp(\theta t) + (\theta - c) e^{\theta T} \right\}^{-\frac{d}{\varepsilon}} \\ & \times \exp \left[ -at + \frac{4b\varepsilon x e^{2\theta t} + x(c - \theta)(c + \theta) \exp(2\theta T)}{2\varepsilon (-(c + \theta)e^{2\theta t} + 2ce^{\theta(t+T)} + (\theta - c)e^{2\theta T})} + \frac{cdt}{2\varepsilon} + \frac{d\theta t}{2\varepsilon} \right]. \end{aligned} \quad (10.19)$$

Finally, making use of the terminal condition  $F(x, T) = 1$  from (10.19), we obtain

$$C_1 = \left( (\theta - c)e^{\theta T} + (c + \theta)e^{2\theta T} \right)^{d/\varepsilon} \exp \left( aT - \frac{cdT}{2\varepsilon} - \frac{d\theta T}{2\varepsilon} \right).$$

Therefore, the solution of (1+1) evolution PDE (10.6) satisfying the terminal condition is given by

$$\begin{aligned} F(t, x) = & \left( \frac{2\theta e^{\theta T}}{(c + \theta)e^{\theta t} + (\theta - c)e^{\theta T}} \right)^{\frac{d}{\varepsilon}} \\ & \times \exp \left[ \frac{(t - T)(d(c + \theta) - 2a\varepsilon)}{2\varepsilon} - \frac{2bx(e^{\theta t} - e^{\theta T})}{c(e^{\theta t} - e^{\theta T}) + \theta(e^{\theta t} + e^{\theta T})} \right]. \end{aligned}$$

## 10.4 Conservation laws

In this section we derive conservation laws for the (1+1) evolution partial differential equation (10.6) using the conservation theorem due to Ibragimov.

The adjoint equation to the equation (10.6) is

$$(a + bx - c)w - (cx + d - 2\varepsilon)w_x + \varepsilon x w_{xx} - w_t = 0 \quad (10.20)$$

and hence the second-order Lagrangian of the equation (10.6) and its adjoint equation (10.20) is written as

$$\mathcal{L} = w[F_t + \varepsilon x F_{xx} + (cx + d)F_x + (a + bx)F].$$

We now apply conservation theorem to each Lie point symmetry of equation (10.6) by following the procedure given in Section 1.5.4. We start with  $X_1 = \partial/\partial t$ . Corresponding to symmetry  $X_1$  we obtain the conserved vector with components

$$\begin{aligned} T_1^t &= w((a + bx)F + F_x(cx + d) + \varepsilon x F_{xx} + F_t) - F_t w, \\ T_1^x &= -F_t(w(cx + d - \varepsilon) + \varepsilon x w_x) - \varepsilon x w F_{tx}. \end{aligned}$$

Likewise, the symmetry  $X_2$  provides us with the conserved vector whose components are

$$\begin{aligned} T_2^t &= \frac{w e^{\theta t}}{\varepsilon \theta} [2\varepsilon \{F_x(x(c - \theta) + d) + \varepsilon x F_{xx}\} - F(d\theta + cx\theta + 2b\varepsilon x - c^2x - cd)], \\ T_2^x &= \frac{e^{\theta t}}{\varepsilon \theta} [\varepsilon \{2\varepsilon x w_x(xF_x\theta + F_t) - w x F_x(2a\varepsilon + d\theta + cx\theta + 4b\varepsilon x - c^2x - cd) \\ &\quad + 2F_t(d - \varepsilon - x\theta + cx) + 2\varepsilon x F_{tx} + F w c(d^2 - 2\varepsilon x(a + 2bx) - d(2x\theta + \varepsilon)) \\ &\quad + 2a\varepsilon(x\theta - d + \varepsilon) - d^2\theta + c^2x(2d - x\theta) + d\varepsilon\theta + 2b\varepsilon x^2\theta - 4bd\varepsilon x + c^3x^2 \\ &\quad + \varepsilon x w_x(2a\varepsilon + d\theta + cx\theta + 4b\varepsilon x - c^2x - cd)], \end{aligned}$$

where  $\theta = \sqrt{c^2 - 4b\varepsilon}$ .

The conserved vectors associated with the symmetries  $X_3$  and  $X_4$  are given by

$$\begin{aligned} T_3^t &= -\frac{w e^{-\theta t}}{\varepsilon \theta} [F\{c(x\theta + d) + d\theta - 2b\varepsilon x + c^2x\} + 2\varepsilon \{F_x(x(\theta + c) + d) + \varepsilon x F_{xx}\}], \\ T_3^x &= \frac{e^{-\theta t}}{\varepsilon \theta} [2\varepsilon w(F_t(x(\theta + c) + d - \varepsilon) + \varepsilon F F_{tx}) - x F_x - 2\varepsilon(a + 2bx) \\ &\quad + c(x\theta + d) + d\theta + c^2x + 2\varepsilon x w_x(xF_x\theta - F_t) + 2F w c\varepsilon x(a + 2bx) \\ &\quad + d(\varepsilon - 2x\theta) - d^2 + 2a\varepsilon(x\theta + d - \varepsilon) - d^2\theta - c^2x(x\theta + 2d) + d\varepsilon\theta + 2b\varepsilon x^2\theta \\ &\quad + 4bd\varepsilon x - c^3x^2 + \varepsilon x w_x(-2\varepsilon(a + 2bx) + c(x\theta + d) + d\theta + c^2x)], \end{aligned}$$

$$T_4^t = wF,$$

$$T_4^x = F(w(cx + d - \varepsilon) - \varepsilon xw_x) + \varepsilon xwF_x,$$

respectively.

## 10.5 Concluding remarks

The evolution (1+1) PDE (10.4) describing the optimal investment-consumption problem under the CEV model [107] satisfied the classical Black–Scholes–Merton equation with boundary conditions which differ from those often used in the most common cases. It is well-known that the evolution (1+1) PDE (10.4) can be related to the standard heat equation by means of equivalence transformations. However, in view of the direct connection of PDE (10.4) to the classical heat equation the general solution can be obtained, but that was not the focus of this chapter. We here solved the PDE (10.4) subject to the terminal condition (10.5) by utilizing the Lie group method. This is not to suggest that the model cannot be solved by other methods, but the usefulness of the Lie symmetry approach is that it is algorithmic and not subject to the possession of special insight into the way a relevant solution to an evolution PDE can be obtained. The methodology used is simple. We found a four-dimensional Lie symmetry algebra for evolution PDE (10.4). Using the nontrivial Lie point symmetry operator, we have shown that the governing PDE can be transformed into a second-order variable coefficient ODE. The reduced ODE is solved to obtain a new exact closed-form solution of the CEV model which also satisfies the terminal condition. The Lie approach gives for the first time the closed-form solution of the optimal investment-consumption problem under discussion. Finally, we constructed conservation laws corresponding to the four Lie point symmetries by employing a general theorem on conservation laws. This is the first time that the evolution PDE (10.4) for optimal investment-

consumption problem has been considered from the view point of group theoretical approach and the conservation laws have been derived in the literature.

# Chapter 11

## Symmetry analysis and conservation laws of the Zoomeron equation

In this chapter we study the (2+1)-dimensional Zoomeron equation [120]

$$\left(\frac{u_{xy}}{u}\right)_{tt} - \left(\frac{u_{xy}}{u}\right)_{xx} + 2(u^2)_{tx} = 0, \quad (11.1)$$

which is an extension of the famous (1+1)-dimensional Zoomeron equation that has many applications in scientific fields. Equation (11.1) has attracted some attention in recent years. Many authors have found closed-form solutions of this equation. For example, the  $(G'/G)$ -expansion method [120, 121], the extended-tanh method [122], the tanh-coth method [123], the sine-cosine function method [124, 125] and the modified simple equation method [126] have been used to find closed-form solutions of (11.1). The (2+1)-dimensional Zoomeron equation with power-law nonlinearity was studied in [127] from Lie point symmetries point of view and symmetry reductions and some solutions were obtained. Also in [127] the authors have given a brief history of the (1+1)-dimensional Zoomeron equation.

In this chapter we first derive the classical Lie point symmetries admitted by equation (11.1) and consequently use them to find an optimal system of one-dimensional subalgebras. These are then used to determine exact group-invariant solutions and symmetry reductions of (11.1). Furthermore, we derive the conservation laws of (11.1) using the multiplier method.

The results of this chapter have been submitted for publication [128].

## 11.1 Exact solutions and symmetry reductions of (11.1)

In this section, firstly we compute Lie point symmetries admitted by (11.1) and then use them to construct the optimal system of one-dimensional subalgebras. Thereafter, we obtain group-invariant solutions based on the optimal system of one-dimensional subalgebras.

### 11.1.1 Lie point symmetries of (11.1)

A vector field of the form

$$X = \xi^1(t, x, y, u) \frac{\partial}{\partial t} + \xi^2(t, x, y, u) \frac{\partial}{\partial x} + \xi^3(t, x, y, u) \frac{\partial}{\partial y} + \eta(t, x, y, u) \frac{\partial}{\partial u}$$

is a Lie point symmetry of (11.1) if

$$X^{[4]} \left[ \left( \frac{u_{xy}}{u} \right)_{tt} - \left( \frac{u_{xy}}{u} \right)_{xx} + 2(u^2)_{tx} \right] \Big|_{(11.1)} = 0, \quad (11.2)$$

where  $X^{[4]}$  is the fourth prolongation of  $X$  given by [24]

$$\begin{aligned} X^{[4]} = & \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial u} + \eta^t \frac{\partial}{\partial u_t} + \eta^x \frac{\partial}{\partial u_x} + \eta^{tt} \frac{\partial}{\partial u_{tt}} + \eta^{tx} \frac{\partial}{\partial u_{tx}} + \eta^{xx} \frac{\partial}{\partial u_{xx}} \\ & + \eta^{xy} \frac{\partial}{\partial u_{xy}} + \eta^{txy} \frac{\partial}{\partial u_{txy}} + \eta^{xxy} \frac{\partial}{\partial u_{xxy}} + \eta^{ttxy} \frac{\partial}{\partial u_{ttxy}} + \eta^{xxxy} \frac{\partial}{\partial u_{xxxy}}. \end{aligned}$$

Expanding the determining equation (11.2) and splitting on the derivatives of  $u$  yields an overdetermined system of linear partial differential equations

$$\begin{aligned}\xi_{tt}^1 &= 0, & \xi_{tx}^1 &= 0, & \xi_t^3 &= 0, & \xi_t^2 &= 0, & \eta_t &= 0, & \xi_x^3 &= 0, & \xi_x^1 &= 0, \\ \eta_x &= 0, & \xi_y^2 &= 0, & \xi_y^1 &= 0, & \eta_y &= 0, & \xi_u^3 &= 0, & \xi_u^2 &= 0, & \xi_u^1 &= 0, \\ u\eta_u - \eta &= 0, & \xi_x^2 - \xi_t^1 &= 0, & 2\eta + u(\xi_t^1 + \xi_y^3) &= 0.\end{aligned}$$

Solving the above system we obtain

$$\xi^1 = c_1 + c_2t, \quad \xi^2 = c_5 + c_2x, \quad \xi^3 = c_3 - (c_2 + 2c_4)y, \quad \eta = c_4u,$$

where  $c_i$ ,  $i = 1, 2, 3, 4$  are constants. Thus, the Lie point symmetries of the Zoomeron equation (11.1) are given by

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y}, \quad X_4 = 2y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u}, \quad X_5 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y},$$

which generate a five-dimensional Lie algebra  $L_5$ .

### 11.1.2 Optimal system of one-dimensional subalgebras

In this subsection we use the Lie point symmetries of (11.1) obtained in the previous subsection to compute an optimal system of one-dimensional subalgebras. We employ the method given in [24], which takes a general element from the Lie algebra and reduces it to its simplest equivalent form by using the chosen adjoint transformations

$$\text{Ad}(\exp(\varepsilon X_i))X_j = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} (\text{ad}X_i)^n(X_j) = X_j - \varepsilon[X_i, X_j] + \frac{\varepsilon^2}{2!}[X_i, [X_i, X_j]] - \dots,$$

where  $\varepsilon$  is a real number and  $[X_i, X_j]$  denotes the commutator defined by

$$[X_i, X_j] = X_iX_j - X_jX_i.$$

The table of commutators of the Lie point symmetries of equation (11.1) and the adjoint representations of the symmetry group of (11.1) on its Lie algebra are given in Table 1 and Table 2, respectively. Then Table 1 and Table 2 are used to construct the optimal system of one-dimensional subalgebras for equation (11.1).

**Table 1.** Lie brackets for equation (11.1)

$[ , ]$	$X_1$	$X_2$	$X_3$	$X_4$	$X_5$
$X_1$	0	0	0	0	$X_1$
$X_2$	0	0	0	0	$X_2$
$X_3$	0	0	0	$2X_3$	$-X_3$
$X_4$	0	0	$-2X_3$	0	0
$X_5$	$-X_1$	$-X_2$	$X_3$	0	0

**Table 2.** Adjoint representation of subalgebras

Ad	$X_1$	$X_2$	$X_3$	$X_4$	$X_5$
$X_1$	$X_1$	$X_2$	$X_3$	$X_4$	$X_5 - \varepsilon X_1$
$X_2$	$X_1$	$X_2$	$X_3$	$X_4$	$X_5 - \varepsilon X_2$
$X_3$	$X_1$	$X_2$	$X_3$	$X_4 - 2\varepsilon X_3$	$X_5 + \varepsilon X_3$
$X_4$	$X_1$	$X_2$	$e^{2\varepsilon} X_3$	$X_4$	$X_5$
$X_5$	$e^\varepsilon X_1$	$e^\varepsilon X_2$	$e^{-\varepsilon} X_3$	$X_4$	$X_5$

Using Tables 1 and 2 we can construct an optimal system of one-dimensional subalgebras, which is given by  $\{X_3, X_4, X_5, X_1 + X_3, X_2 + X_3, X_1 + X_4, X_2 + X_4, X_4 + X_5, X_1 + X_2 + X_3, X_1 + X_2 + X_4\}$ .

### 11.1.3 Symmetry reductions and group-invariant solutions

We now obtain group-invariant solutions based on the optimal system of one-dimensional subalgebras computed in the previous subsection. We start with the symmetry operator  $X_3$ .

**Case 1.**  $X_3 = \partial/\partial y$

This symmetry provides us with three invariants

$$s = t, \quad r = x, \quad U = u,$$

which give group-invariant solution  $u = U(s, r)$  and reduces (11.1) to

$$(UU_s)_r = 0. \tag{11.3}$$

The Lie point symmetries of the above equation are

$$\begin{aligned} \Gamma_1 &= f_1(s) \frac{\partial}{\partial s}, \quad \Gamma_2 = f_2(r) \frac{\partial}{\partial r}, \quad \Gamma_3 = \frac{f_3(s)}{U} \frac{\partial}{\partial U}, \quad \Gamma_4 = \frac{f_4(r)}{U} \frac{\partial}{\partial U}, \\ \Gamma_5 &= (U + U^{-1}) \frac{\partial}{\partial U}, \quad \Gamma_6 = (U - U^{-1}) \frac{\partial}{\partial U}. \end{aligned}$$

Taking  $\Gamma_1 - \nu\Gamma_2$  with  $f_1(s) = f_2(r) = 1$ , we obtain two invariants  $z = r + \nu s$  and  $F = U$ . Making use of these invariants, (11.3) reduces to

$$(FF')' = 0, \tag{11.4}$$

whose solution is given by

$$F(z) = \pm \sqrt{a_1 z + a_2},$$

where  $a_1$  and  $a_2$  are constants of integration. Hence the solution of (11.1) is

$$u(t, x, y) = \pm \sqrt{a_1(x + \nu t) + a_2}.$$

**Case 2.**  $X_4 = 2y\partial/\partial y - u\partial/\partial u$

The associated Lagrange system to the operator  $X_4$  yields three invariants

$$s = t, \quad r = x, \quad U = uy^{1/2},$$

which give group-invariant solution  $u = y^{-1/2}U(s, r)$  and transforms (11.1) to

$$\left(\frac{U_r}{U}\right)_{ss} - \left(\frac{U_r}{U}\right)_{rr} - 4(U^2)_{rs} = 0. \quad (11.5)$$

The Lie point symmetries of the above equation are

$$\Gamma_1 = \frac{\partial}{\partial s}, \quad \Gamma_2 = \frac{\partial}{\partial r}, \quad \Gamma_3 = 2s\frac{\partial}{\partial s} + 2r\frac{\partial}{\partial r} - U\frac{\partial}{\partial U}.$$

The symmetry  $\Gamma_1 + \nu\Gamma_2$  gives the two invariants  $z = r - \nu s$  and  $F = U$ . Using these invariants (11.5) transforms to the nonlinear third-order ordinary differential equation

$$\left(\frac{F'}{F}\right)'' - \frac{4\nu}{1-\nu^2}(F^2)'' = 0. \quad (11.6)$$

Integrating (11.6) twice with respect to  $z$ , we obtain

$$F'(z) - \frac{4\nu}{1-\nu^2}(F(z))^3 - k_1zF(z) - k_2F(z) = 0, \quad (11.7)$$

where  $k_1$  and  $k_2$  are constants of integration. The solution of this equation is given by

$$F(z) = \pm \sqrt{\frac{\sqrt{k_1}(1-\nu^2) \exp\{(k_1z + k_2)^2/k_1\}}{k_3\sqrt{k_1}(1-\nu^2) \exp\{k_2^2/k_1\} - 4\nu\sqrt{\pi} \operatorname{erfi}((k_1z + k_2)/\sqrt{k_1})}},$$

where  $k_3$  is a constant of integration and  $\operatorname{erfi}(z)$  is the imaginary error function [129]. Thus, a solution of (11.1) is

$$u(t, x, y) = \pm y^{-1/2} \sqrt{\frac{\sqrt{k_1}(1-\nu^2) \exp\{(k_1(x - \nu t) + k_2)^2/k_1\}}{k_3\sqrt{k_1}(1-\nu^2) \exp\{k_2^2/k_1\} - 4\nu\sqrt{\pi} \operatorname{erfi}((k_1(x - \nu t) + k_2)/\sqrt{k_1})}}.$$

**Case 3.**  $X_1 + X_4 = \partial/\partial t + 2y\partial/\partial y - u\partial/\partial u$

The associated Lagrange system to  $X_1 + X_4$  yields three invariants

$$s = x, \quad r = ye^{-2t}, \quad U = e^t u,$$

which give group-invariant solution  $u = e^{-t}U(s, r)$  and transforms (11.1) to

$$\begin{aligned} U \left( U_{sr} (4r^2 U_{rr} - U_{ss}) + 4r U_r (3U_{sr} + 2r U_{srr}) - 2U_s U_{ssr} \right) + 8U^4 (r U_{sr} + U_s) + 8r U_r U_s U^3 \\ + U^2 (U_{sssr} - 4(U_{sr} + r(3U_{srr} + r U_{srrr}))) + 2(U_s^2 - 4r^2 U_r^2) U_{sr} = 0. \end{aligned} \quad (11.8)$$

The Lie point symmetries of the above equation are

$$\Gamma_1 = \frac{\partial}{\partial s}, \quad \Gamma_2 = 2r \frac{\partial}{\partial r} - U \frac{\partial}{\partial U}.$$

The symmetry  $\Gamma_2$  gives the two invariants  $z = s$  and  $F = r^{1/2}U$  and using these invariants, (11.5) transforms to the nonlinear third-order ordinary differential equation

$$\left( \frac{F'}{F} \right)'' = 0. \quad (11.9)$$

Integrating (11.9) twice with respect to  $z$ , we obtain

$$F'(z) = k_1 z F(z) + k_2 F(z), \quad (11.10)$$

where  $k_1$  and  $k_2$  are constants of integration. The solution of this equation is given by

$$F(z) = k_3 \exp \left( \frac{k_1}{2} z^2 + k_2 z \right),$$

where  $k_3$  is a constant of integration. Thus, a solution of (11.1) is

$$u(t, x, y) = k_3 y^{-1/2} \exp \left( \frac{k_1}{2} x^2 + k_2 x \right),$$

which is a steady-state solution.

**Case 4.**  $X_1 + X_2 + X_3$

The associated Lagrange system to this symmetry operator gives three invariants, viz.,

$$s = x - t, \quad r = y - t, \quad U = u,$$

which give group-invariant solution  $u = U(s, r)$  and reduces (11.1) to

$$\begin{aligned} & U^2 (U_{srrr} + 2U_{ssrr}) - 4U^4 (U_{sr} + U_{ss}) - 4U_s U^3 (U_r + U_s) + 2U_r (U_r + 2U_s) U_{sr} \\ & - U (U_{rr} U_{sr} + 2 (U_{sr}^2 + U_s U_{srr} + U_r (U_{srr} + U_{ssr}))) = 0. \end{aligned} \quad (11.11)$$

The Lie point symmetries of the above equation are

$$\Gamma_1 = \frac{\partial}{\partial s}, \quad \Gamma_2 = \frac{\partial}{\partial r}, \quad \Gamma_3 = s \frac{\partial}{\partial s} + r \frac{\partial}{\partial r} - U \frac{\partial}{\partial U}.$$

The symmetry  $\Gamma_1 - \nu \Gamma_2$  gives the two invariants  $z = r + \nu s$  and  $F = U$ . Using these invariants (11.11) transforms to the nonlinear fourth-order ordinary differential equation

$$\left( \frac{F''}{F} \right)'' - \frac{2(\nu + 1)}{2\nu + 1} (F^2)'' = 0. \quad (11.12)$$

Integrating (11.12) twice with respect to  $z$ , we obtain

$$F'' - \frac{2(\nu + 1)}{2\nu + 1} F^3 - k_1 z F - k_2 F = 0, \quad (11.13)$$

where  $k_1$  and  $k_2$  are constants of integration. This equation can not be integrated in the closed form. However, by taking  $k_1 = 0$  one can obtain its solution in the closed form in the following manner. Multiplying (11.13) with  $k_1 = 0$  by  $F'$  and integrating, we obtain

$$F'^2 = \frac{2(\nu + 1)}{2\nu + 1} F^4 + k_2 F^2 + k_3, \quad (11.14)$$

where  $k_3$  is a constant of integration. The solution of this equation is given by

$$F(z) = \sqrt{\frac{2k_3(2\nu + 1)}{C}} \operatorname{sn} \left( \sqrt{\frac{C}{2(2\nu + 1)}} z + k_4, 2 \sqrt{\frac{-k_3(\nu + 1)}{Ck_2 + 4k_3 + 4k_3\nu}} \right),$$

where  $k_4$  is a constant of integration,

$$C = \sqrt{4k_2^2\nu^2 + 4k_2^2\nu + k_2^2 - 16k_3\nu^2 - 24k_3\nu - 8k_3 - 2k_2\nu - k_2}$$

and  $\text{sn}$  is the Jacobi elliptic sine function [119]. Thus, a solution of (11.1) is

$$u(t, x, y) = \sqrt{\frac{2k_3(2\nu + 1)}{C}} \text{sn} \left( \sqrt{\frac{C}{2(2\nu + 1)}} (y + \nu x - (\nu + 1)t) + k_4, 2n^2 \right),$$

where  $n^2 = \sqrt{-k_3(\nu + 1)/(Ck_2 + 4k_3 + 4k_3\nu)}$ .

Likewise, one may obtain more group-invariant solutions using the other symmetry operators of the optimal system of one-dimensional subalgebras. For example, the symmetry operator  $X_2 + X_3$  of the optimal system gives us the group-invariant solution (2.9) obtained by Morris and Leach [127] in terms of the Airy functions.

## 11.2 Conservation laws of (11.1)

In this section we employ Anco and Bluman method [40, 42] to derive conservation laws for the Zoomeron equation (11.1).

Applying this method, we obtain the following equations for the multipliers:

$$\begin{aligned} Q_u(t, x, y, u) &= 0, \\ Q_{ty}(t, x, y, u) &= 0, \\ Q_{yyy}(t, x, y, u) &= 0, \\ Q_{tt}(t, x, y, u) - Q_{yy}(t, x, y, u) &= 0. \end{aligned}$$

Solving the above equations we get the four multipliers given by

$$\begin{aligned} Q_1 &= \frac{1}{2} (t^2 + y^2) f_1(x), \\ Q_2 &= f_2(x)y, \end{aligned}$$

$$Q_3 = f_3(x)t,$$

$$Q_4 = f_4(x).$$

Corresponding to these multipliers, we obtain four conservation laws. Thus, the multiplier

$$Q_1 = \frac{1}{2} (t^2 + y^2) f_1(x)$$

gives the conservation law with the following conserved vector:

$$\begin{aligned} T_1^t &= f_1(x) \left\{ \frac{1}{2} (t^2 + y^2) \left( \frac{u_t^2 u_x}{u^3} - \frac{u_x u_{tt}}{u^2} \right) + \frac{t u_x u_t}{u^2} - 2y u^2 \right\} \\ &\quad + f_1'(x) \left\{ \frac{1}{2} (t^2 + y^2) \left( \frac{u_{tt}}{u} - \frac{1}{2} \frac{u_t^2}{u^2} \right) - \frac{t u_t}{u} \right\}, \\ T_1^x &= f_1(x) \left\{ \frac{1}{2} (t^2 + y^2) \left( \frac{2u_t u_{tt}}{u^2} - \frac{u_{ttt}}{u} - \frac{u_t^3}{u^3} \right) - \frac{1}{2} \frac{u_t^2 t}{u^2} + \frac{u_t}{u} \right\}, \\ T_1^y &= f_1(x) \left\{ \frac{1}{2} (t^2 + y^2) \left( 4u u_t + \frac{u_{txy}}{u} - \frac{u_y u_{tx}}{u^2} \right) - \frac{y u_{tx}}{u} \right\}. \end{aligned}$$

Likewise, the multiplier

$$Q_2 = f_2(x)y$$

yields

$$\begin{aligned} T_2^t &= f_2(x)y \left( 4u u_y - \frac{u_x u_{tt}}{u^2} + \frac{u_t^2 u_x}{u^3} \right) + f_2'(x)y \left( \frac{u_{tt}}{u} - \frac{1}{2} \frac{u_t^2}{u^2} \right), \\ T_2^x &= f_2(x)y \left( \frac{2u_t u_{tt}}{u^2} - \frac{u_{ttt}}{u} - \frac{u_t^3}{u^3} \right), \\ T_2^y &= f_2(x) \left( \frac{y u_{txy}}{u} - \frac{y u_y u_{t2}}{u^2} - \frac{u_{tx}}{u} \right) \end{aligned}$$

as conserved vector.

Similarly, the multiplier

$$Q_3 = f_3(x)t$$

results in the following conserved vector

$$T_3^t = f_3(x) \left( 4t u u_y - \frac{t u_x u_{tt}}{u^2} + \frac{t u_t^2 u_x}{u^3} + \frac{u_x u_t}{u^2} \right) + f_3'(x) \left( \frac{t u_{tt}}{u} - \frac{1}{2} \frac{t u_t^2}{u^2} - \frac{u_t}{u} \right),$$

$$T_3^x = f_3(x) \left( \frac{2tu_t u_{tt}}{u^2} - \frac{tu_t^3}{u^3} - \frac{1}{2} \frac{u_t^2}{u^2} - \frac{tu_{ttt}}{u} \right),$$

$$T_3^y = f_3(x) \frac{(tuu_{txy} - 2u^4 - tu_y u_{tx})}{u^2}.$$

Lastly, the multiplier

$$Q_4 = f_4(x)$$

gives the conserved vector whose components are

$$T_4^t = f_4(x) \left( 4uu_y - \frac{u_x u_{tt}}{u^2} + \frac{u_t^2 u_x}{u^3} \right) + f_4'(x) \left( \frac{u_{tt}}{u} - \frac{1}{2} \frac{u_t^2}{u^2} \right),$$

$$T_4^x = f_4(x) \left( \frac{2u_t u_{tt}}{u^2} - \frac{u_{ttt}}{u} - \frac{u_t^3}{u^3} \right),$$

$$T_4^y = f_4(x) \left( \frac{u_{txy}}{u} - \frac{u_y u_{tx}}{u^2} \right).$$

### 11.3 Concluding remarks

In this chapter we studied the (2+1)-dimensional Zoomeron equation (11.1). The classical Lie point symmetries were obtained and used to construct an optimal system of one-dimensional subalgebras. This system was then used to obtain symmetry reductions and new group-invariant solutions of (11.1). Furthermore, we derived the conservation laws for (11.1) by employing the Anco and Bluman method. It should be noted that since we had arbitrary functions in the multipliers we obtained infinitely many conservation laws for equation (11.1).

# Chapter 12

## Conclusions

The exact solutions and conservation laws of NLPDEs are important for the explanation of some practical physical problems. The aim of this thesis was to obtain exact solutions and derive conservation laws of some NLPDEs using different methods.

Chapter one provided relevant literature, definitions and theorems of the important concepts that were used in this work.

In Chapter two we carried out Lie group classification of the Gardner equation with variable coefficients. This was achieved by first determining the equivalence transformations for the variable coefficients Gardner equation (2.1). The transformations were then used to rescale some arbitrary functions in equation (2.1), which simplified the original equation to an equivalent equation (2.7). We then studied equation (2.7). It was found that the equivalent Gardner equation (2.7) had a translation symmetry in space variable  $x$  as its kernel algebra. The functions  $G(t)$  and  $H(t)$  that were able to extend the principal Lie algebra were found to be exponential, power, logarithmic and linear functions. Symmetry reductions were performed for two cases which extended the principal Lie algebra. Finally, for two

cases we obtained conservation laws using the multiplier method.

The generalized (2+1)-dimensional Korteweg-de Vries equation (3.2) was studied in Chapter three. New exact solutions were found and these were cnoidal and snoidal waves solutions. Furthermore, conserved vectors were constructed by employing the multiplier method.

In Chapter four the travelling wave solutions of the coupled Korteweg-de Vries-Burgers system were obtained by employing the  $(G'/G)$ -expansion method. The solutions obtained were expressed in the form of hyperbolic and trigonometric functions. Conservation laws for the system were constructed using the multiplier method.

Chapter five studied the (2+1)-dimensional KdV-mKdV equation. The travelling wave solutions in terms of Jacobi elliptic functions were obtained for this equation. Furthermore, conservation laws of the equation were derived using the multiplier approach.

Chapter six dealt with the generalized improved Boussinesq equation. Group-invariant solutions of the equation were found by employing the Lie symmetry method along with the simplest equation method. Moreover, the conservation laws were constructed using the multiplier method.

In Chapter seven the Kaup-Boussinesq system was studied. Exact solutions of this system were found using two distinct methods. Firstly, the travelling wave solutions were obtained using the direct integration. Secondly, an optimal system of one-dimensional subalgebras was found and group-invariant solutions based on the optimal system of one-dimensional subalgebras were obtained. Also, the conservation laws were derived using two approaches; the multiplier method and the conservation law theorem due to Ibragimov.

In Chapter eight we employed the  $(G'/G)$ -expansion method to obtain exact solu-

tions of the Prandtl's boundary layer equation for radial flow models with uniform main stream velocity.

The generalized coupled (2+1)-dimensional Burgers system was investigated in Chapter nine. The Lie group analysis together with the Kudryashov approach were employed to obtain new travelling wave solutions of the system. Moreover, conservation laws of the system were derived by employing the multiplier approach.

In Chapter ten we obtained the closed-form group-invariant solution which also satisfied the terminal condition of the evolution (1+1) PDE describing the optimal investment-consumption problem under the CEV model. Finally, we constructed conservation laws by employing a general theorem on conservation laws. We note that this is the first time that the evolution PDE for optimal investment-consumption problem has been considered from the view point of group theoretical approach and the conservation laws have been derived in the literature.

In Chapter eleven we studied the (2+1)-dimensional Zoomeron equation. The classical Lie point symmetries were obtained and used to construct an optimal system of one-dimensional subalgebras. Symmetry reductions and new group-invariant solutions were obtained based on the optimal system. Furthermore, we derived infinitely many conservation laws for the (2+1)-dimensional Zoomeron equation by employing the Anco and Bluman method.

In future work, we plan to use conservation laws derived in this work to obtain group-invariant solutions.

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