GROUP INVARIANT SOLUTIONS AND CONSERVATION LAWS OF CERTAIN NONLINEAR EVOLUTION EQUATIONS IN MATHEMATICAL PHYSICS

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Thesis accepted in fulfilment of the requirements for the degree Doctor of Philosophy in Applied Mathematics at the North-West University

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by

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Thesis submitted for the degree of Doctor of Philosophy in Applied Mathematics at the Mafikeng Campus of the North-West University

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Declaration

I declare that the thesis for the degree of Doctor of Philosophy at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other University, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

Signed: ....

MR TS MORETLO

Date: ....

This thesis has been submitted with my approval as a University supervisor and would certify that the requirements applicable for the Doctor of Philosophy degree rules and regulations have been fulfilled.

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Declaration of Publications

Details of contribution to publications that form part of this thesis.

Chapter 2

T. S. Moretlo, B. Muatjetjeja, A. R. Adem, On the solutions of a (3+1)-dimensional novel KP-like equation, Iranian Journal of Science and Technology Transaction A: Sience 45 (2021) 1037–1041.

Chapter 3

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Chapter 4

T. S. Moretlo, A. R. Adem, B. Muatjetjeja, A generalized $(2 + 1)$-dimensional Bogoyavlenskii-Kadomtsev-Petviashvili (BKP) equation: Multiple exp-function algorithm; conservation laws; similarity solutions, Communications in Nonlinear Science and Numerical Simulation 106 (2022) 106072.

Chapter 5

T. S. Moretlo, B. Muatjetjeja, A. R. Adem, A generalized dispersive water waves system: Conservation laws; Symmetry reduction; Travelling wave solutions; Symbolic Computation, Partial Differential Equations in Applied Mathematics 7 (2023) 100465.

Chapter 6

T. S. Moretlo, A. R. Adem, B. Muatjetjeja, An extended $(2 + 1)$-dimensional coupled Burgers system in fluid mechanics: Symmetry reductions; Kudryashov method; Conservation laws, International Journal of Theoretical Physics 62 (2023) 1–12.

Dedication

I dedicate this work to all those who supported me throughout this venture.

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Abstract

This research project aims to study some nonlinear partial differential equations that arise in many branches of physics such as particle physics, fluid dynamics, plasma astrophysics, ocean dynamics, atmospheric science, computational fluid mechanics, cosmology, condensed matter physics, statistical physics, nonlinear acoustics, vehicular traffic, electronic transport, etc. Exact solutions, conservation laws and soliton solutions are derived for such equations using various methods. The nonlinear partial differential equations that are studied in this research work are two $(3+1)$-dimensional Kadomtsev-Petviashvili (KP) like equations, a generalized $(2+1)$-dimensional Bogoyavlenskii-Kadomtsev-Petviashvili (BKP) equation, a generalized dispersive water waves system and an extended $(2+1)$-dimensional coupled Burgers system in fluid mechanics.

The classical symmetry approach will be employed to search for exact solutions of a first $(3+1)$-dimensional KP like equation. Thereafter, we will derive the admitted conserved vectors of the aforementioned equation.

We employ some ansatz methods to derive topological soliton solutions of a second $(3+1)$-dimensional KP like equation. Furthermore, mixed solutions consisting of singular and periodic solutions and others are derived. Moreover, other analytical solutions based on modern group analysis are obtained. In addition, low-order conservation laws are constructed.

We further, determine novel exact solutions of a generalized $(2+1)$-dimensional Bogoyavlenskii-Kadomtsev-Petviashvili (BKP) equation by utilizing the multiple exp-function algorithm and the modern group analysis method. Then, we compute conserved currents using the invariance and multiplier technique.

Symmetry analysis is performed for a generalized dispersive water waves system. This symmetry analysis will lead to similarity reductions and new exact solutions

with the aid of the simplest equation method. The solutions obtained include the solitary waves and the travelling wave solutions. In addition, conservation laws are derived using the multiplier approach.

Finally, we determine novel exact solutions of an extended $(2 + 1)$-dimensional coupled Burgers system in fluid mechanics by the Lie symmetry method in conjunction with the Kurdyshov method. Conservation laws of the above-mentioned system are generated.

List of Acronyms

DE:	Differential equations
PDE:	Partial differential equations
NLPDE:	Nonlinear partial differential equations
NLEE:	Nonlinear evolution equations
KdV:	Kortweg-de Vries
KP:	Kadomtsev-Petviashvili
BKP:	Bogoyavlenskii-Kadomtsev-Petviashvili

Introduction

Differential equations arise in almost every field in life whether in the natural sciences, applied sciences, social sciences, economics or business problems. There are countless methods to solve differential equations both analytically and numerically. The Lie point symmetries and conservation laws play a significant role in the solution process of differential equations. Nonlinear partial differential equations (NLPDEs) are some of the most suitable mathematical representations of regular occurrences and studying them cannot be overemphasized. Thus, it is very important to investigate the exact solutions of NLPDEs.

However, a massive amount of work has been done in the last few decades and wonderful progress has been made in finding exact solutions of NLPDEs. In order to obtain the exact solutions, a number of methods have been proposed in literature. Some of the well-known methods include the solitary wave ansatz method [1, 2], Hirotas bilinear method [3], homogeneous balance method [4] and Lie group analysis [5–12].

There is no doubt that conservation laws play a remarkable role in the study of differential equations (DE). The mathematical idea of conservation laws comes from the formulation of well-known physical conserved quantities such as mass, momentum and energy. The high number of conservation laws for a partial differential equation (PDE) provides the insight that the PDE is strongly integrable.

Finding the conservation laws of differential equations is often the primary step towards finding the exact solutions. Thus, it is essential to study conservation laws of PDEs.

Recently, many research activities on solitary waves theory, predominantly on integrable systems, have attracted a lot of researchers. This is due to the fact that solitary waves theory has found a lot applications in many areas of nonlinear sciences, such as engineering, plasma physics, biology and other fields of mathematical physics. In the past decade, researchers have confined their application of solitary waves theory to $(1 + 1)$ and $(2 + 1)$-dimensional equations [13]. However, it was later found that solitary waves theory plays a significant role in the study of higher dimensional integrable equations.

This research project is structured as follows:

In Chapter one, we present the preliminaries that are going to be needed in our study.

In Chapter two, we employ classical symmetry approach to search for exact solutions, then find the admitted conserved vectors of a first $(3 + 1)$-dimensional KP like equation.

In Chapter three, we show that a second $(3 + 1)$-dimensional KP like equation admits topological soliton solutions. We also derive other analytical solutions based on modern group analysis and construct low-order conservation laws.

In Chapter four, we determine the exact solutions of a generalized (2+1)-dimensional Bogoyavlenskii-Kadomtsev-Petviashvili (BKP) equation and compute the assosiated conserved currents using the invariance and multiplier technique.

In Chapter five, similarity reductions and new exact solutions are constructed for a generalized dispersive water waves system with the aid of the simplest equation method. Conservation laws are also derived using the variational method for the aforesaid system.

Chapter six, is concerned with the Lie symmetry method in conjunction with the Kurdyshov method of an extended $(2+1)$-dimensional coupled Burgers system in fluid mechanics. Conservation laws of the system at hand are shown.

Finally in Chapter seven, a summary of the results of the research project are presented and future work is suggested.

Bibliography is given at the end.

Chapter 1

Preliminaries

In this chapter, we present some preliminaries on Lie symmetry analysis, conservation laws and some methods for obtaining exact solutions of differential equations, which will be used throughout this work and are based on references [5–12].

1.1 One-parameter group of continuous transformations

Let $x = (x^1, ..., x^n)$ be the independent variables with coordinates x^i and $u = (u^1, ..., u^m)$ be the dependent variables with coordinates u^α (n and m finite). Consider a change of the variables x and u involving a real parameter a:

$$T_a : \bar{x^i} = f^i(x, u, a), \ \bar{u^\alpha} = \phi^\alpha(x, u, a), \tag{1.1}$$

where a continuously ranges in values from a neighborhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$, and f^i and ϕ^α are differentiable functions.

Definition 1.1 (Lie group) A set G of transformations (1.1) is called a continuous one-parameter (local) Lie group of transformations in the space of variables

x and u if

(i) For T_a, $T_b \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then $T_b T_a = T_c \in G$, $c = \phi(a, b) \in \mathcal{D}$ (Closure),

(ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0 T_a = T_a T_0 = T_a$ (Identity),

(iii) For $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that $T_a T_{a^{-1}} = T_{a^{-1}} T_a = T_0$ (Inverse).

We note that the associativity property follows from (i). The group property (i) can be written as

$$\begin{aligned} \bar{\bar{x}}^i &\equiv f^i(\bar{x}, \bar{u}, b) = f^i(x, u, \phi(a, b)), \\ \bar{\bar{u}}^\alpha &\equiv \phi^\alpha(\bar{x}, \bar{u}, b) = \phi^\alpha(x, u, \phi(a, b)) \end{aligned} \tag{1.2}$$

and the function ϕ is called the group composition law. A group parameter a is called canonical if $\phi(a, b) = a + b$.

Theorem 1.1 For any $\phi(a, b)$, there exists the canonical parameter $\tilde{a}$ defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)}, \text{ where } w(s) = \left.\frac{\partial \phi(s, b)}{\partial b}\right|_{b=0}.$$

1.2 Prolongations

The derivatives of u with respect to x are defined as

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j D_i(u_i), \cdots, \tag{1.3}$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, ..., n \tag{1.4}$$

is the operator of total differentiation. The collection of all first derivatives u_i^α is denoted by $u_{(1)}$, i.e.,

$$u_{(1)} = \{u_i^\alpha\} \quad \alpha = 1, ..., m, \quad i = 1, ..., n.$$

Similarly

$$u_{(2)} = \{u_{ij}^\alpha\} \quad \alpha = 1, ..., m, \quad i, j = 1, ..., n$$

and $u_{(3)} = \{u_{ijk}^\alpha\}$ and likewise $u_{(4)}$ etc. Since $u_{ij}^\alpha = u_{ji}^\alpha$, $u_{(2)}$ contains only u_{ij}^α for $i \leq j$. In the same manner $u_{(3)}$ has only terms for $i \leq j \leq k$. There is natural ordering in $u_{(4)}$, $u_{(5)}$ $\cdots$.

In group analysis, all variables $x, u, u_{(1)} \cdots$ are considered functionally independent variables connected only by the differential relations (1.3). Thus the u_s^α are called differential variables [9].

We now consider a pth-order system of partial differential equations, namely

$$E_\alpha(x, u, u_{(1)}, ..., u_{(p)}) = 0. \tag{1.5}$$

Prolonged or extended groups

If $z = (x, u)$, one-parameter group of transformations G is

$$\bar{x^i} = f^i(x, u, a), \quad f^i|_{a=0} = x^i,$$

$$\bar{u^\alpha} = \phi^\alpha(x, u, a), \quad \phi^\alpha|_{a=0} = u^\alpha. \tag{1.6}$$

According to Lie's theory, the construction of the symmetry group G is equivalent to the determination of the corresponding infinitesimal transformations:

$$\bar{x^i} \approx x^i + a\,\xi^i(x, u), \quad \bar{u^\alpha} \approx u^\alpha + a\,\eta^\alpha(x, u) \tag{1.7}$$

obtained from (1.1) by expanding the functions f^i and ϕ^α into Taylor series in a, about $a = 0$ and also taking into account the initial conditions

$$f^i\big|_{a=0} = x^i, \quad \phi^\alpha|_{a=0} = u^\alpha.$$

Thus, we have

$$\xi^i(x,u) = \frac{\partial f^i}{\partial a}\bigg|_{a=0}, \quad \eta^\alpha(x,u) = \frac{\partial \phi^\alpha}{\partial a}\bigg|_{a=0}. \tag{1.8}$$

One can now introduce the *symbol* of the infinitesimal transformations by writing (1.7) as

$$\bar{x^i} \approx (1 + a\,X)x, \quad \bar{u^\alpha} \approx (1 + a\,X)u,$$

where

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}. \tag{1.9}$$

This differential operator X is known as the infinitesimal operator or generator of the group G. If the group G is admitted by (1.5), we say that X is an admitted operator of (1.5) or X is an infinitesimal symmetry of equation (1.5).

We now see how the derivatives are transformed.

The D_i transforms as

$$D_i = D_i(f^j)\bar{D}_j, \tag{1.10}$$

where $\bar{D}_j$ is the total differentiations in transformed variables $\bar{x}^i$. So

$$\bar{u}_i^\alpha = \bar{D}_j(u^\alpha), \qquad \bar{u}_{ij}^\alpha = \bar{D}_j(\bar{u}_i^\alpha) = \bar{D}_i(\bar{u}_j^\alpha), \cdots.$$

Applying (1.6) and (1.10), we obtain

$$\begin{aligned} D_i(\phi^\alpha) &= D_i(f^j)\bar{D}_j(\bar{u}^\alpha) \\ &= D_i(f^j)\bar{u}_j^\alpha, \end{aligned} \tag{1.11}$$

and so

$$\left(\frac{\partial f^j}{\partial x^i} + u_i^\beta \frac{\partial f^j}{\partial u^\beta}\right)\bar{u}_j^\alpha = \frac{\partial \phi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \phi^\alpha}{\partial u^\beta}. \tag{1.12}$$

The quantities $\bar{u}_j^\alpha$ can be represented as functions of $x, u, u_{(i)}$, i.e., (1.12) is locally invertible:

$$\bar{u}_i^\alpha = \phi_i^\alpha(x, u, u_{(1)}, a), \quad \phi^\alpha|_{a=0} = u_i^\alpha. \tag{1.13}$$

The transformations in $x, u, u_{(1)}$ space given by (1.6) and (1.13) form a one-parameter group (one can prove this but we do not consider the proof) called the first prolongation or just extension of the group G and denoted by $G^{[1]}$.

Letting

$$\bar{u}_i^\alpha \approx u_i^\alpha + a\zeta_i^\alpha \tag{1.14}$$

to be the infinitesimal transformation of the first derivatives so that the infinitesimal transformation of the group $G^{[1]}$ is (1.7) and (1.14).

Higher-order prolongations of G, viz., $G^{[2]}$, $G^{[3]}$ can be obtained by derivatives of (1.11).

Prolonged generators

Using (1.11) together with (1.7) and (1.14) we get

$$\begin{aligned}
D_i(f^j)(\bar{u}_j^\alpha) &= D_i(\phi^\alpha), \\
D_i(x^j + a\xi^j)(u_j^\alpha + a\zeta_j^\alpha) &= D_i(u^\alpha + a\eta^\alpha), \\
(\delta_i^j + aD_i\xi^j)(u_j^\alpha + a\zeta_j^\alpha) &= u_i^\alpha + aD_i\eta^\alpha, \\
u_i^\alpha + a\zeta_i^\alpha + au_j^\alpha D_i\xi^j &= u_i^\alpha + aD_i\eta^\alpha, \\
\zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \quad \text{(sum on } j).
\end{aligned} \tag{1.15}$$

This is called the first prolongation formula. Likewise, one can obtain the second prolongation, viz.,

$$\zeta^{\alpha}_{ij} \;=\; D_j(\eta^{\alpha}_i) - u^{\alpha}_{ik} D_j(\xi^k), \quad \text{(sum on } k). \tag{1.16}$$

By induction (recursively)

$$\zeta^{\alpha}_{i_1,i_2,\ldots,i_p} \;=\; D_{i_p}(\zeta^{\alpha}_{i_1,i_2,\ldots,i_{p-1}}) - u^{\alpha}_{i_1,i_2,\ldots,i_{p-1}\,j} D_{i_p}(\xi^j), \quad \text{(sum on } j). \tag{1.17}$$

The first and higher prolongations of the group G form a group denoted by $G^{[1]}, \cdots, G^{[p]}$. The corresponding prolonged generators are

$$\begin{aligned} X^{[1]} &= X + \zeta^{\alpha}_i \frac{\partial}{\partial u^{\alpha}_i} \quad \text{(sum on } i, \alpha), \\ &\vdots \\ X^{[p]} &= X^{[p-1]} + \zeta^{\alpha}_{i_1,\ldots,i_p} \frac{\partial}{\partial u^{\alpha}_{i_1,\ldots,i_p}} \quad p \geq 1, \end{aligned}$$

where

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^{\alpha}(x,u)\frac{\partial}{\partial u^{\alpha}}.$$

1.3 Group admitted by a partial differential equation

Definition 1.2 (Point symmetry) The vector field

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^{\alpha}(x,u)\frac{\partial}{\partial u^{\alpha}} \tag{1.18}$$

is a point symmetry of the pth-order partial differential equation (1.5) if

$$X^{[p]}(E_{\alpha}) = 0, \tag{1.19}$$

whenever $E_{\alpha} = 0$. This can also be written as

$$X^{[p]} E_{\alpha}\big|_{E_{\alpha}=0} = 0, \tag{1.20}$$

where the symbol $|_{E_\alpha=0}$ means evaluated on the equation $E_\alpha = 0$.

Definition 1.3 (Determining equation) Equation (1.19) is called the determining equation of (1.5) because it determines all the infinitesimal symmetries of (1.5).

Definition 1.4 (Symmetry group) A one-parameter group G of transformations (1.1) is called a symmetry group of equation (1.5) if (1.5) is form-invariant (has the same form) in the new variables $\bar{x}$ and $\bar{u}$, i.e,

$$E_\alpha(\bar{x}, \bar{u}, \bar{u_{(1)}}, \cdots, \bar{u_{(p)}}) = 0, \tag{1.21}$$

where the function E_α is the same as in equation (1.5).

1.4 Infinitesimal criterion of invariance

Definition 1.5 (Invariant) A function $F(x, u)$ is called an invariant of the group of transformation (1.1) if

$$F(\bar{x}, \bar{u}) \equiv F(f^i(x, u, a),\, \phi^\alpha(x, u, a)) = F(x, u) \tag{1.22}$$

holds identically in x, u and a.

Theorem 1.2 (Infinitesimal criterion of invariance) A necessary and sufficient condition for a function $F(x, u)$ to be an invariant is that

$$X\,F \equiv \xi^i(x, u)\frac{\partial F}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial F}{\partial u^\alpha} = 0\,. \tag{1.23}$$

It follows from the above theorem that every one-parameter group of point transformations (1.1) has $n-1$ functionally independent invariants, which can be taken to be the left-hand side of any first integrals

$$J_1(x,u) = c_1, \cdots, J_{n-1}(x,u) = c_n$$

of the characteristic equations

$$\frac{dx^1}{\xi^1(x,u)} = \cdots = \frac{dx^n}{\xi^n(x,u)} = \frac{du^1}{\eta^1(x,u)} = \cdots = \frac{du^n}{\eta^n(x,u)}.$$

Theorem 1.3 (Lie equations) If the infinitesimal transformation (1.7) or its symbol X is given, then the corresponding one-parameter group G is obtained by solving the Lie equations

$$\frac{d\bar{x^i}}{da} = \xi^i(\bar{x},\bar{u}), \quad \frac{d\bar{u^\alpha}}{da} = \eta^\alpha(\bar{x},\bar{u}) \tag{1.24}$$

subject to the initial conditions

$$\bar{x^i}\big|_{a=0} = x, \quad \bar{u^\alpha}\big|_{a=0} = u\,.$$

1.5 Conservation laws

1.5.1 Fundamental operators and their relationship

Consider a pth-order system of partial differential equations of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$, given by equation (1.5).

Definition 1.6 (Euler-Lagrange operator) The Euler-Lagrange operator, for each α, is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\ldots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}, \qquad \alpha = 1,\ldots,m. \tag{1.25}$$

Definition 1.7 (Lie-Bäcklund operator) The Lie-Bäcklund operator is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \quad \xi^i, \eta^\alpha \in \mathcal{A}, \tag{1.26}$$

where $\mathcal{A}$ is the space of differential functions [9]. The operator (1.26) is an abbreviated form of infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} \zeta^\alpha_{i_1 i_2 \ldots i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}, \tag{1.27}$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} \zeta^\alpha_i &= D_i(W^\alpha) + \xi^j u^\alpha_{ij} \\ \zeta^\alpha_{i_1 \ldots i_s} &= D_{i_1} \ldots D_{i_s}(W^\alpha) + \xi^j u^\alpha_{j i_1 \ldots i_s}, \quad s > 1, \end{aligned} \tag{1.28}$$

in which W^α is the *Lie characteristic function* given by

$$W^\alpha = \eta^\alpha - \xi^i u^\alpha_j. \tag{1.29}$$

One can write the Lie-Bäcklund operator (1.27) in characteristic form as

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} D_{i_1} \ldots D_{i_s}(W^\alpha) \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}. \tag{1.30}$$

Definition 1.8 (Conservation law) The n-tuple vector $T = (T^1, T^2, \ldots, T^n)$, $T^j \in \mathcal{A}, \quad j = 1, \ldots, n$, is a *conserved vector* of (1.5) if T^i satisfies

$$D_i T^i|_{(1.5)} = 0. \tag{1.31}$$

The equation (1.31) defines a local conservation law of system (1.5).

1.5.2 Multiplier method

The multiplier approach is an effective algorithmic method for finding the conservation laws for partial differential equations with any number of independent and dependent variables. Authors in [17] gave this algorithm by using the multipliers presented in [10]. A local conservation law of a given differential system arises from a linear combination formed by local multipliers (characteristics) with each differential equation in the system, where the multipliers depend on the independent and dependent variables as well as at most a finite number of derivatives of the dependent variables of the given differential equation system.

The advantage of this approach is that it does not require the use or existence of a variational principle and reduces the calculation of conservation laws to solving a system of linear determining equations similar to that for finding symmetries.

A multiplier $\Lambda_\alpha(x, u, u_{(1)}, \ldots)$ has the property that

$$\Lambda_\alpha E_\alpha = D_i T^i \tag{1.32}$$

holds identically, where E_α, D_i are defined by equations (1.5), (1.4) and T^i is defined in definition (1.31).

The right hand side of (1.32) is a divergence expression. The determining equation for the multiplier Λ_α is

$$\frac{\delta(\Lambda_\alpha E_\alpha)}{\delta u^\alpha} = 0. \tag{1.33}$$

Once the multipliers are obtained, the conserved vectors are constructed by invoking the homotopy operator [17].

1.6 Exact solutions

In this section we recall a method which can be used to determine exact solutions of differential equations.

1.6.1 The simplest equation method

In this subsection we recall the simplest equation method developed by Kudryashov [24,25] for finding exact solutions of nonlinear partial differential equations. Several researchers have recently applied this method to various nonlinear partial differential equations and shown that this method provides a very effective and powerful mathematical tool for solving nonlinear differential equations in various fields of applied sciences (see, for example, papers [26, 27]). The basic steps of the method are as follows:

Consider the nonlinear partial differential equation of the form

$$E_1(u, u_t, u_x, u_y, u_{tt}, u_{xt}, u_{xx}, u_{yy} \cdots) = 0. \tag{1.34}$$

Using the following transformation

$$u(t, x, y) = F(z), \quad z = k_1 t + k_2 x + k_3 y + k_4 \tag{1.35}$$

reduces equation (1.34) to an ordinary differential equation

$$E_2[F(z), k_1 F'(z), k_2 F'(z), k_3 F'(z), k_1^2 F''(z), k_2^2 F''(z), k_3^2 F''(z), \cdots] = 0. \tag{1.36}$$

The simplest equations that we use here are the Riccati and Bernoulli equations,

$$H'(z) = aH^2(z) + bH(z) + c, \tag{1.37}$$

$$H'(z) = aH(z) + bH^2(z), \tag{1.38}$$

respectively. It should be noted that when $a = 1$, $b = -1$ and $c = 0$, equation (1.37) leads to the Kudryashov method [24, 28–31]. We look for solutions of the nonlinear ordinary differential equation (1.36) that are of the form

$$F(z) = \sum_{i=0}^{M} A_i (H(z))^i, \tag{1.39}$$

where $H(z)$ satisfies the Bernoulli or Riccati equation, M is a positive integer that can be determined by balancing procedure and $A_0, \cdots, A_M$ are parameters to be determined.

The solution of Bernoulli Equation (1.38) that we will use here is given by

$$H(z) = a\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}, \tag{1.40}$$

where C is a constant of integration. For the Riccati equation (1.37), the solutions to be used are

$$H(z) = -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right] \tag{1.41}$$

and

$$H(z) = -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)} \tag{1.42}$$

with $\theta = \sqrt{b^2 - 4ac}$ and C is a constant of integration.

1.7 Concluding remarks

In this chapter we presented a brief introduction to Lie group analysis and conservation laws of partial differential equations and gave some results which will be used throughout this research project. In addition, we presented algorithms of various methods that are used to find exact solutions of partial differential equations.

Chapter 2

On the solutions of a $(3+1)$-dimensional KP-like equation

In the last few decades, nonlinear evolution equations (NLEEs) have been widely used as models to describe physical phenomena in various fields of sciences, particularly in plasma physics, plasma, dynamics, and other areas of nonlinear science. Due to the great role played by these equations, it is therefore crucial to search for exact solutions to these NLEEs. Finding closed-form solutions to these equations is a very difficult task. It is for this reason, that leads to the utilization of a variety of powerful and effective methods to obtain exact solutions of NLEEs. Some of the methods found in the literature include the Hirota's bilinear method [39], Jacobi elliptic function expansion method [40], Darboux transformation method [41], the Tanh and Sine-Cosine method [42] and the classical symmetry method [8, 10, 43].

In the study of nonlinear evolution equations, the Kadomtsev-Petviashvili (KP) is one of the equations that attracted several researchers. This equation accounts for various interesting waves in nonlinear science, see [37] and references therein. It is worth mentioning that the extension of the Kadomtsev-Petviashvili(KP) leads to the well known B-type Kadomtsev-Petviashvili (BKP), where the u_{xxxx} term is replaced by u_{xxxy} [37]. Since NLEEs provide a lot of information in nonlinear fields, this resulted in the derivation of a new $(3+1)$-dimensional KP-like equation [37] namely,

$$\delta\, u_{tx} + \mu\, u_{ty} + \alpha\, u_{tz} - \beta\, u_{xxxy} - 3\,(u_{xx}\, u_y + u_x u_{xy}) + \gamma\, u_{xx} = 0. \tag{2.1}$$

Here $\delta, \mu, \alpha, \beta$ and γ are arbitrary parameters.

The results of this chapter have been accepted for publication [38].

Motivated by the results in [44–46], the authors in [37] studied equation (2.1). They searched for versions of resonant phenomena as much as they could. Since the equation under study is new, a lot of information about the equation is not yet known in literature.

Thus the authors in [37] are the first ones to study this $(3+1)$-dimensional KP-like equation. They obtained new and generalized resonant multi-soliton solutions. Although authors in [37] have given an effort to solve this nonlinear evolution equation, there is no unified method. See for example [47–49] and references therein. To the best of our knowledge, this is the first time that the Lie symmetry method is being applied to search for exact solutions to the equation under study. It is also worth mentioning that, this is the first time that the conservation laws for the equation under study are being derived. It should also be noted that the methods applied in this thesis give unique solution sets from the newly reported solutions and it is worth asserting that the method employed in [37], cannot be used to construct conservation laws.

This chapter is structured in two fold. Firstly, we search for the admitted point symmetries of a (3 + 1)-dimensional KP-like equation (2.1). Then we perform the similarity reductions and construct solutions of the underlying equation. In addition, we will derive the low-order conserved vectors of equation (2.1).

2.1 Lie point symmetries

This section searches for the admitted symmetry generators for equation (2.1). Consider the vector field of the form

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \xi^4 \frac{\partial}{\partial z} + \eta \frac{\partial}{\partial u},$$

where the coefficients of the partial derivatives are functions of t, x, y, z and u. The extension of the vector field X to $\mathrm{pr}^{(4)} X$ and acting on equation (2.1), leads to an overdetermined system of partial differential equations. On solving the resulting system of equations with the aid of Maple package, one obtains the following nine Lie point symmetries:

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial x}, X_3 = \frac{\partial}{\partial z}, X_4 = F_1(z)\,\frac{\partial}{\partial u}, X_5 = F_2(t)\,\frac{\partial}{\partial u}, \\
X_6 &= 3\,\frac{\partial}{\partial y} + \gamma\,\frac{\partial}{\partial u}, X_7 = 3\delta z\,\frac{\partial}{\partial x} + 3\alpha y\,\frac{\partial}{\partial y} + 3\alpha z\,\frac{\partial}{\partial z} + \alpha\gamma y\,\frac{\partial}{\partial u}, \\
X_8 &= 9\alpha t \frac{\partial}{\partial t} + (3\alpha x - 3\delta z)\,\frac{\partial}{\partial x} + (\alpha\gamma y - 3\,\alpha u)\,\frac{\partial}{\partial u}, \\
X_9 &= -3\alpha t\,\frac{\partial}{\partial t} + 3\delta z\,\frac{\partial}{\partial x} + 3\mu z\,\frac{\partial}{\partial y} + 3\alpha z\,\frac{\partial}{\partial z} + \gamma\mu z\,\frac{\partial}{\partial u}.
\end{aligned}$$

Case 1. The addition of the scaling symmetries, namely $X_1, \tau X_2$ and X_3 and solving the resulting Lagrange equations, gives four invariants, viz.,

$$f = y, \quad g = t - z, \quad h = x - \tau z, \quad \phi = u, \tag{2.2}$$

where τ is a travelling wave constant.

The substitution of the above similarity transformation into equation (2.1) gives

$$-\alpha\,\tau\,\phi_{gh}+\gamma\,\phi_{hh}-3\,\phi_{h}\,\phi_{fh}-3\,\phi_{f}\,\phi_{hh}+\mu\,\phi_{fg}-\beta\,\phi_{fhhh}+\delta\,\phi_{gh}-\alpha\,\phi_{gg}=0, \quad (2.3)$$

where ϕ is a function of (f,g,h). Solving equation (2.3) and making use of the invariants (2.2), we conclude that the group-invariant solution of equation (2.1) is

$$\begin{aligned} u(t,x,y,z) &= 2\beta c_4 \tanh\left(yc_2+\frac{(\delta c_4-\alpha\tau c_4+\mu c_2+\Theta)\,(t-z)}{2\alpha}+(x-\tau z)\,c_4+c_1\right) \\ &\quad +c_5, \end{aligned} \quad (2.4)$$

where c_1,c_2,c_3,c_4,c_5 are constants and

$$\Theta=\sqrt{\alpha^2\tau^2c_4^2-16\alpha\beta c_2c_4^3-2\alpha\mu\tau c_2c_4-2\alpha\delta\tau c_4^2+\mu^2c_2^2+2\delta\mu c_2c_4+4\alpha\gamma c_4^2+\delta^2c_4^2}.$$

The associated profile solution of equation (2.4) is given by figure 2.1.

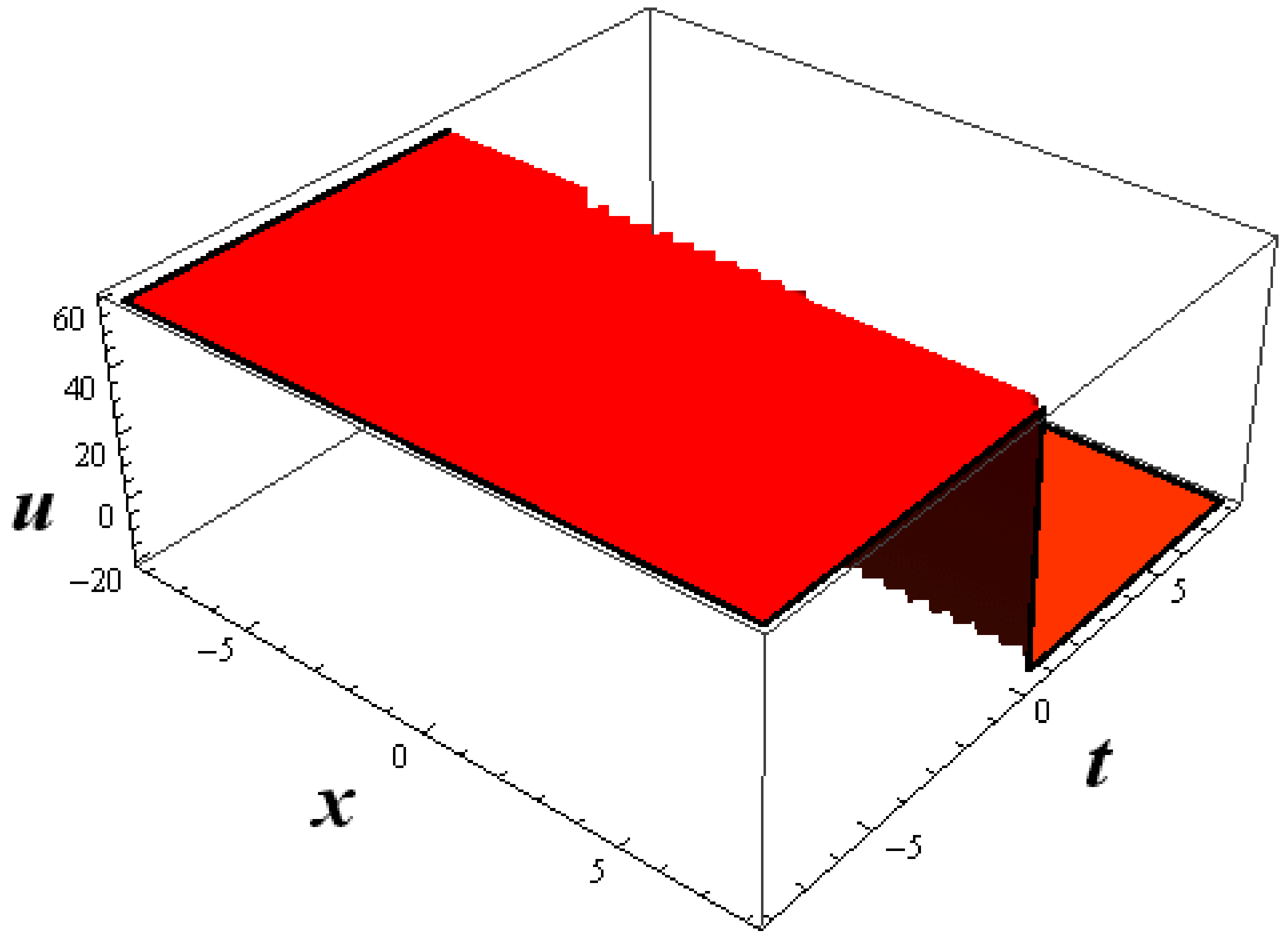

Figure 2.1: Evolution of travelling wave solution (2.4) for $y = 1, z = 1, \alpha = 1, \beta = -1, \mu = 1, \delta = 0.0001, \gamma = 0.001, \tau = 0.0001, c_1 = 10, c_2 = 5, c_4 = 20, c_5 = 20$.

Case 2.

We now choose symmetry X_6 and solve the characteristics equation to get the following four invariants

$$f = x, \quad g = z, \quad h = t, \quad \phi = u - \frac{1}{3}\gamma y.$$

The insertion of these invariants into equation (2.1) and solving the resulting partial deferential equation, yields

$$u(t, x, y, z) = \varphi\,(z, x) + \psi\,(t, -\alpha x + \delta z) + \frac{1}{3}\,\gamma y \tag{2.5}$$

as the group-invariant solution of equation (2.1).

Case 3.

We now take the symmetry combination of ϵX_2 and X_4, and get

$$f = y, \quad g = z, \quad h = t, \quad \phi = u - \frac{\Phi(z)}{\epsilon}$$

as the invariants. Making use of these invariants, equation (2.1) reduces to

$$\mu\,\phi_{fh} + \alpha\,\phi_{gh} = 0. \tag{2.6}$$

Solving equation (2.6) and reverting back into the original variables, we conclude that the group-invariant solution of equation (2.1) is

$$u(t,x,y,z) = \frac{\epsilon\varphi\,(z,y) + \epsilon\psi\,(t, -\alpha\,y + \mu\,z) + x\Phi\,(z)}{\epsilon}. \tag{2.7}$$

Case 4.

Invoking the combination of X_2 and X_6, leads to the following invariants

$$f = z, \quad g = t, \quad h = x - y - \frac{1}{3}\gamma y, \quad \phi = u - \frac{1}{3}\gamma y.$$

Employing these similarity transformations into equation (2.1), the group-invariant solution of equation (2.1) is

$$\begin{aligned} u(t,x,y,z) \quad = \quad & 2\beta c_4 \tanh\left(zc_2 + \frac{4\beta t\,(3+\gamma)\,c_4^4}{\gamma\mu c_4 - 3\alpha c_2 - 3\delta c_4 + 3\mu c_4} + \left(x - y - \frac{1}{3}\,\gamma\,y\right)c_4 + c_1\right) \\ & + c_5 + \frac{1}{3}\,\gamma y, \end{aligned} \tag{2.8}$$

and the associated profile solution is given in figure 2.2.

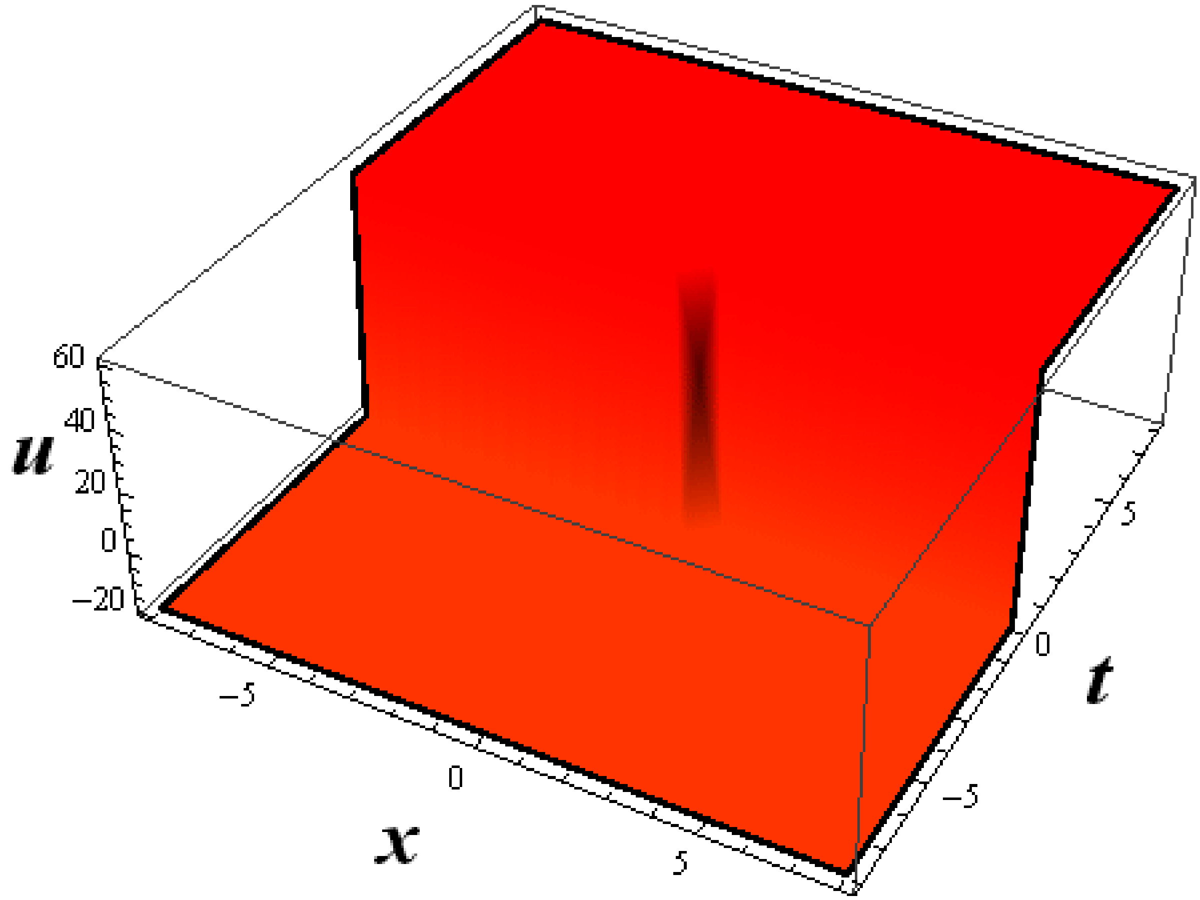

Figure 2.2: Evolution of travelling wave solution (2.10) for $y = 1, z = 1, \alpha = 1, \beta = -1, \mu = 1, \delta = 0.01, \gamma = 1, c_1 = 10, c_2 = 5, c_4 = 20, c_5 = 20$.

2.2 Conservation laws

This section is devoted to the derivation of low-order conservation laws of equation (2.1) via the multiplier approach [43]. Conservation laws are of great significance for NLEEs because they provide an insight into integrability and also provide physical meaning of the problem. The multiplier approach searches for local conservation laws. A local conservation law is a continuity equation

$$D_t T^1 + D_x T^2 D_y T^3 + D_z T^4 = 0 \tag{2.9}$$

that holds for all solutions of equation (2.1). T^1 is a conserved density while T^2, T^3 and T^4 denote spatial fluxes. $T^i (i = 1, 2, 3, 4)$ are functions of t, x, y, z, u and derivatives of u. The second-order multipliers Λ of equation (2.1) are determined by invoking the Euler Lagrange operator [8, 10] on equation (2.1) and getting

$$\frac{\delta}{\delta u}(\Lambda)\bigg(\delta\, u_{tx} + \mu\, u_{ty} + \alpha\, u_{tz} - \beta\, u_{xxxy} - 3\ (u_{xx}\, u_y + \ u_x u_{xy}) + \gamma\, u_{xx}\bigg) = 0. \quad (2.10)$$

Solving the above equation prompts this proposition.

Proposition. *The* $(3+1)$*-dimensional KP-like equation (2.1) admits the second order multiplier of the form*

$$\Lambda \ = \ u_x\, C_1 + F_2\,(z, y) + F_3\,(t, -\alpha\, y + \mu\, z)\,. \quad (2.11)$$

Remark 1. It is overwhelming that although we searched for multipliers of second order, our computations depicted multipliers of first order (2.11) as if we constructed first order multipliers. This is something very astonishing.

Thus corresponding to the above multiplier we derive the following low-order conserved vectors:

$$\begin{aligned}
T_1^1 &= -\frac{1}{4}\,\delta u u_{xx} - \frac{1}{4}\,\mu u u_{xy} - \frac{1}{4}\,\alpha u u_{xz} + \frac{1}{4}\,\alpha u_x u_z + \frac{1}{4}\,\mu u_x u_y + \frac{1}{4}\,\delta u_x^2,\\
T_1^2 &= \frac{1}{2}\,\gamma u_x^2 - \frac{3}{2}\,u_x^2 u_y - \frac{1}{8}\,\beta\, u_{xxx} u_y + \frac{3}{8}\,\beta u_{xx} u_{xy} - \frac{5}{8}\,\beta u_x u_{xxy} + \frac{1}{4}\,\delta u u_{tx} + \frac{1}{2}\,\mu u u_{ty}\\
&\quad + \frac{1}{2}\,\alpha u u_{tz} + \frac{1}{4}\,\delta u_t u_x - \frac{1}{8}\,\beta u u_{xxxy},\\
T_1^3 &= -\frac{1}{4}\,\mu u u_{tx} + \frac{1}{8}\,\beta u u_{xxxx} - \frac{1}{2}\,u_x^3 - \frac{1}{4}\,\beta u_x u_{xxx} + \frac{1}{8}\,\beta u_{xx}^2 + \frac{1}{4}\,\mu u_t u_x,\\
T_1^4 &= -\frac{1}{4}\,\alpha u u_{tx} + \frac{1}{4}\,\alpha u_t u_x;
\end{aligned}$$

$$\begin{aligned}
T_2^1 &= -\frac{1}{2}\,\alpha u D_1 F\,(z, y) - \frac{1}{2}\,\mu u D_2 F\,(z, y) + \frac{1}{2}\,\alpha u_z F\,(z, y) + \frac{1}{2}\,\mu u_y F\,(z, y)\\
&\quad + \frac{1}{2}\,\delta u_x F\,(z, y)\,,
\end{aligned}$$

$$
\begin{aligned}
T_2^2 &= \frac{3}{4}\,uu_x D_2F\,(z,y) + \frac{3}{4}\,uu_{xy}F\,(z,y) - \frac{9}{4}\,u_xu_yF\,(z,y) + \frac{1}{4}\,\beta u_{xx}D_2F\,(z,y) \\
&\quad + \gamma u_xF\,(z,y) + \frac{1}{2}\,\delta u_tF\,(z,y) - \frac{3}{4}\,\beta u_{xxy}F\,(z,y)\,, \\
T_2^3 &= -\frac{3}{4}\,u_x^2F\,(z,y) - \frac{3}{4}\,uu_{xx}F\,(z,y) - \frac{1}{4}\,\beta u_{xxx}F\,(z,y) + \frac{1}{2}\,\mu u_tF\,(z,y)\,, \\
T_2^4 &= \frac{1}{2}\,\alpha u_tF\,(z,y)\,;
\end{aligned}
$$

$$
\begin{aligned}
T_3^1 &= \frac{1}{2}\,\alpha u_z\,F\,(t,-\alpha\,y+\mu\,z) + \frac{1}{2}\,\mu u_y\,F\,(t,-\alpha\,y+\mu\,z) + \frac{1}{2}\,\delta u_x\,F\,(t,-\alpha\,y+\mu\,z)\,, \\
T_3^2 &= \frac{3}{4}\,uu_{xy}\,F\,(t,-\alpha\,y+\mu\,z) - \frac{9}{4}\,u_xu_y\,F\,(t,-\alpha\,y+\mu\,z) + \gamma u_x\,F\,(t,-\alpha\,y+\mu\,z) \\
&\quad + \frac{1}{2}\,\delta u_t\,F\,(t,-\alpha\,y+\mu\,z) - \frac{3}{4}\,\beta u_{xxy}\,F\,(t,-\alpha\,y+\mu\,z) - \frac{3}{4}\,\alpha uu_x\,D_2F\,(t,-\alpha\,y+\mu\,z) \\
&\quad - \frac{1}{4}\,\alpha\beta u_{xx}\,D_2F\,(t,-\alpha\,y+\mu\,z) - \frac{1}{2}\,\delta u\,D_1F\,(t,-\alpha\,y+\mu\,z)\,, \\
T_3^3 &= -\frac{1}{2}\,\mu uD_1F\,(t,-\alpha\,y+\mu\,z) - \frac{3}{4}\,u_x^2\,F\,(t,-\alpha\,y+\mu\,z) - \frac{3}{4}\,uu_{xx}\,F\,(t,-\alpha\,y+\mu\,z) \\
&\quad - \frac{1}{4}\,\beta u_{xxx}\,F\,(t,-\alpha\,y+\mu\,z) + \frac{1}{2}\,\mu u_t\,F\,(t,-\alpha\,y+\mu\,z)\,, \\
T_3^4 &= -\frac{1}{2}\,\alpha u\,D_1F\,(t,-\alpha\,y+\mu\,z) + \frac{1}{2}\,\alpha u_t\,F\,(t,-\alpha\,y+\mu\,z)\,.
\end{aligned}
$$

Remark 2. Based on the confounding results from the aforementioned proposition, we conclude that both first and second order multipliers of a (3 + 1)-dimensional KP-like equation (2.1) are identical and hence yield identical conservation laws, where (D_1, D_2) represent the differentiation of the arbitrary functions with respect to their first and second arguments respectively.

2.3 Conclusions

In this chapter, we constructed solutions of a (3+1)-dimensional KP-like equation, which arises in the analysis of versions of resonant phenomena. In addition, we derived the conserved vectors of the underlying equation. It is shown that it is possible that one equation can have identical multipliers, although their orders are distinct and hence resulting in identical conservation laws. We have excluded the zeroth order multipliers, as it is distinct from the first and the second-order multipliers. It remains to be thoroughly investigated whether all the multipliers of a $(3+1)$-dimensional KP-like equation are identical or not. However, this will be reported elsewhere. Although authors in [37] have given an effort to solve the nonlinear evolution equation, there is no unified method. To the author's best ability, this is the first time that the symmetry method is being applied to search for exact solutions of the equation under study. It is also worth mentioning that, this is the first time that the conservation laws for the equation are derived. It should also be noted that the methods applied in this thesis give unique solution sets from the new reported solutions [37]. Furthermore, future work will be devoted to studying the ansatz methods mentioned in [50, 51, 53] and applying them to the underlying equation.

Chapter 3

Soliton solutions and other analytical solutions of a $(3+1)$-dimensional KP like equation

Recently, many research activities on solitary waves theory, predominantly on integrable systems, have attracted a lot of researchers. This is due to the fact that solitary waves theory has found a lot of applications in many areas of nonlinear sciences, such as engineering, plasma physics, biology and other fields of mathematical physics. In the past decade, researchers have confined their application of solitary waves theory to $(1+1)$ and $(2+1)$-dimensional equations [13]. However, it was later found that solitary waves theory plays a significant role in the study of higher dimensional integrable equations. It is this reason, that motivated authors in [37] to establish a $(3+1)$-dimensional KP like equation given by

$$\alpha\, u_{tz} - \beta\, u_{xxxy} - 3\,(u_{xx}\, u_y + u_x\, u_{xy}) + \gamma\,(u_{xx} + u_{xy} + u_{xz}) = 0, \qquad (3.1)$$

where (α, β, γ) are arbitrary parameters.

The results of this chapter have been accepted for publication [52].

Authors in [37] used the simplified linear superposition principle to derive resonant multi-soliton solution of equation (3.1). It can easily be noticed that equation (3.1) is a natural extension of the famous B-type Kadomtsev-Petviashvili (BKP) equation [37] and references therein. To the best of our knowledge, topological soliton solutions, singular and periodic solutions and point symmetries of the aforesaid equation have not be reported in the literature.

The objectives of this work is two fold. Firstly, we will implement the ansatz methods so as to derive topological soliton solutions. Furthermore, we will employ the Tan-Cot method to attain singular and periodic solutions. In addition, the symmetry method will be invoked to obtain some other analytical solutions and lastly construct some low-order conservation laws via the multiplier approach.

3.1 Soliton solutions

This section aims to compute topological 1-soliton solution of equation (3.1). This will be attained via a hypothesis method. In order to search for dark soliton solutions or shock waves or kinks, we begin our hypothesis [13, 54–56] in the form of

$$u(t, x, y, z) = \lambda \tanh^p \tau, \tag{3.2}$$

where the wave variable τ is defined by $\tau = \eta x + \delta y + \varphi z - \nu t$ while η, δ and φ are unknown free parameters representing the inverse width of the wave. ν is the velocity of the soliton and p is a positive exponent that will be determined.

The hypothesis (3.2) yields

$$
\begin{aligned}
u_x &= \lambda\,\eta\,p(\tanh^{p-1}\tau - \tanh^{p+1}\tau), && (3.3)\\
u_{xx} &= \lambda\,\eta^2\,p(p+1)\tanh^{p+2}\tau + \lambda\,\eta^2\,p(p-1)\tanh^{p-2}\tau - 2\lambda\,\eta^2\,p^2\tanh^p\tau, && (3.4)\\
u_{xy} &= \lambda\,\eta\,\delta\,p(p+1)\tanh^{p+2}\tau + \lambda\,\eta\,\delta\,p(p-1)\tanh^{p-2}\tau - 2\lambda\,\eta\,\delta\,p^2\tanh^p\tau, && (3.5)\\
u_{xz} &= \lambda\,\eta\,\varphi\,p(p+1)\tanh^{p+2}\tau + \lambda\,\eta\,\varphi\,p(p-1)\tanh^{p-2}\tau - 2\lambda\,\eta\,\varphi\,p^2\tanh^p\tau, && (3.6)\\
u_{tz} &= -\lambda\,\varphi\,\nu\,p(p+1)\tanh^{p+2}\tau - \lambda\,\varphi\,\nu\,p(p-1)\tanh^{p-2}\tau + 2\lambda\,\varphi\,\nu\,p^2\tanh^p\tau, && (3.7)\\
u_y &= \lambda\,\delta\,p\tanh^{p-1}\tau - \lambda\,\delta\,p\tanh^{p+1}\tau, && (3.8)\\
u_y u_{xx} &= -\lambda^2\,\eta^2\,\delta\,p^2(p+1)\tanh^{2p+3}\tau + \lambda^2\,\eta^2\,\delta\,p^2(p-1)\tanh^{2p-3}\tau \\
&\quad +\lambda^2\,\eta^2\,\delta\,p^2(3p+1)\tanh^{2p+1}\tau - \lambda^2\,\eta^2\,\delta\,p^2(3p-1)\tanh^{2p-1}\tau, && (3.9)\\
u_x u_{xy} &= -\lambda^2\,\eta^2\,\delta\,p^2(p+1)\tanh^{2p+3}\tau + \lambda^2\,\eta^2\,\delta\,p^2(p-1)\tanh^{2p-3}\tau \\
&\quad +\lambda^2\,\eta^2\,\delta\,p^2(3p+1)\tanh^{2p+1}\tau - \lambda^2\,\eta^2\,\delta\,p^2(3p-1)\tanh^{2p-1}\tau, && (3.10)\\
u_{xxxy} &= -\lambda\,\eta^3\,\delta\,p(p+3)(p+2)(p+1)\tanh^{p+4}\tau + \lambda\,\eta^3\,\delta\,p(p-1)(p-2)(p-3)\tanh^{p-4}\tau \\
&\quad -4\,\lambda\,\eta^3\,\delta\,p(p+1)(p^2+2p+2)\tanh^{p+2}\tau - 4\,\lambda\,\eta^3\,\delta\,p(p-1)(p^2-2p+2)\tanh^{p-2}\tau \\
&\quad +2\,\lambda\,\eta^3\,\delta\,p^2(3p^2+5)\tanh^p\tau. && (3.11)
\end{aligned}
$$

The subustition of equations (3.3)–(3.11) into equation (3.1), gives

$$
\begin{aligned}
&-\lambda\,\eta^3\,\delta\,\beta p(p+3)(p+2)(p+1)\tanh^{p+4}\tau - \lambda\,\eta^3\,\delta\,\beta p(p-1)(p-2)(p-3)\tanh^{p-4}\tau \\
&+(4\,\eta^3\,\delta\,\beta(p^2+2p+2) - \alpha\,\varphi\,\nu + \gamma\,\eta(\eta+\delta+\varphi))\lambda\,p(p+1)\tanh^{p+2}\tau \\
&+(4\,\eta^3\,\delta\,\beta(p^2-2p+2) - \alpha\,\varphi\,\nu + \gamma\,\eta(\eta+\delta+\varphi))\lambda\,p(p-1)\tanh^{p-2}\tau \\
&+6\,\lambda^2\,\eta^2\,\delta p^2(p+1)\tanh^{2p+3}\tau - 6\,\lambda^2\,\eta^2\,\delta p^2(p-1)\tanh^{2p-3}\tau \\
&-6\,\lambda^2\,\eta^2\,\delta p^2(3p+1)\tanh^{2p+1}\tau + 6\,\lambda^2\,\eta^2\,\delta p^2(3p-1)\tanh^{2p-1}\tau \\
&+2(\alpha\,\varphi\,\nu - \eta^3\,\delta\,\beta(3p^2+5) - \gamma\,\eta(\eta+\delta+\varphi))\tanh^p\tau = 0. \qquad (3.12)
\end{aligned}
$$

To seek for the smallest positive integer p, we equate the highest linear term in tanh with the least nonlinear term in $\tanh\tau$.

This can be achieved by equating powers $\tanh^{p+4}\tau$ and $\tanh^{2p+3}\tau$, to get

$$p + 4 = 2p + 3,$$

therefore $p = 1$. Substituting $p = 1$ into the powers of $\tanh\tau$ only, equation (3.12) reduces to

$$\begin{aligned}
&-\lambda\,\eta^3\,\delta\,\beta p(p+3)(p+2)(p+1)\tanh^5\tau - \lambda\,\eta^3\,\delta\,\beta p(p-1)(p-2)(p-3)\tanh^{-3}\tau \\
&+(4\,\eta^3\,\delta\,\beta(p^2+2p+2) - \alpha\,\varphi\,\nu + \gamma\,\eta(\eta+\delta+\varphi))\lambda\,p(p+1)\tanh^3\tau \\
&+(4\,\eta^3\,\delta\,\beta(p^2-2p+2) - \alpha\,\varphi\,\nu + \gamma\,\eta(\eta+\delta+\varphi))\lambda\,p(p-1)\tanh^{-1}\tau \\
&+6\,\lambda^2\,\eta^2\,\delta p^2(p+1)\tanh^5\tau - 6\,\lambda^2\,\eta^2\,\delta p^2(p-1)\tanh^{-1}\tau \\
&-6\,\lambda^2\,\eta^2\,\delta p^2(3p+1)\tanh^3\tau + 6\,\lambda^2\,\eta^2\,\delta p^2(3p-1)\tanh^1\tau \\
&+2(\alpha\,\varphi\,\nu - \eta^3\,\delta\,\beta(3p^2+5) - \gamma\,\eta(\eta+\delta+\varphi))\tanh\tau = 0. \qquad (3.13)
\end{aligned}$$

Splitting equation (3.13) with respect to the powers of $\tanh\tau$ and simplifying, yields

$$\lambda = \frac{(p+2)(p+3)\beta\eta}{6p} \qquad (3.14)$$

and

$$\nu = \frac{-3\beta\delta\eta^3((p+3)(p+2)(3p-1) - 2p(3p^2+5))}{6p\alpha\varphi} + \frac{\gamma\eta(\eta+\delta+\varphi)}{\alpha\varphi}. \qquad (3.15)$$

Setting $p = 1$ into (3.14) and (3.15), we obtain

$$\lambda = 2\beta\eta \qquad (3.16)$$

and

$$\nu = \frac{\gamma\eta(\eta+\delta+\varphi) - 4\beta\delta\eta^3}{\alpha\varphi}. \qquad (3.17)$$

Consequently, the dark (optical) soliton solution for equation (3.1) is

$$u = 2\,\beta\,\eta\,\tanh\left(\eta\,x + \delta\,y + \varphi\,z - \frac{\gamma\eta\,(\eta+\delta+\varphi)\,t - 4\,\beta\,\delta\,\eta^3 t}{\alpha\,\varphi}\right). \qquad (3.18)$$

Remark 1. We observe that equation (3.1) admits an optical or shock wave soliton solution if and only if $\alpha\,\varphi \neq 0$. To the best of our knowledge, this crucial observation is reported here for the first time. This observation cannot be found anywhere in the literature. We now present the profile solution of equation (3.18) subject to some choice of the arbitrary parameters.

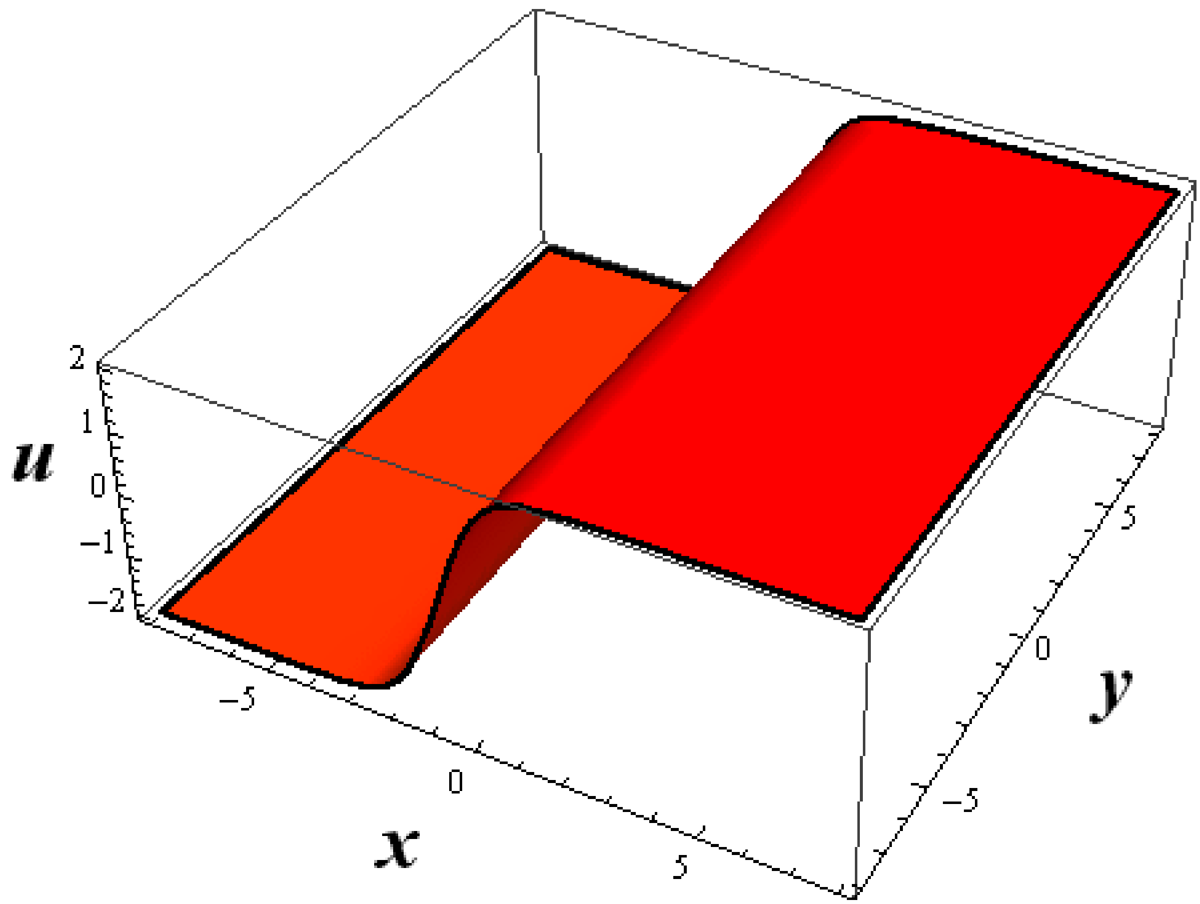

Figure 3.1: Kink shape soliton profile of (3.18) corresponding to $\eta = 1, \varphi = 1, \delta = 0.01, \gamma = 0.1, \alpha = 0.1, \beta = 1, t = 1, z = 1$.

In a like manner, we can derive the singular kink solution in the form

$$u = 2\,\beta\,\eta\,\coth\left(\eta\,x + \delta\,y + \varphi\,z - \frac{\gamma\eta\,(\eta + \delta + \varphi)\,t - 4\,\beta\,\delta\,\eta^3 t}{\alpha\,\varphi}\right). \qquad (3.19)$$

Remark 2. It is worth noting that a singular kink soliton solution does exists for equation (3.1). However, this singular kink soliton solution can only exist provided that the product of α and φ is not zero. This is a very remarkable observation that is being mentioned here for the first time. Figure 3.2 below, gives a graphical presentation of solution (3.19) with respect to some choice of arbitrary constants.

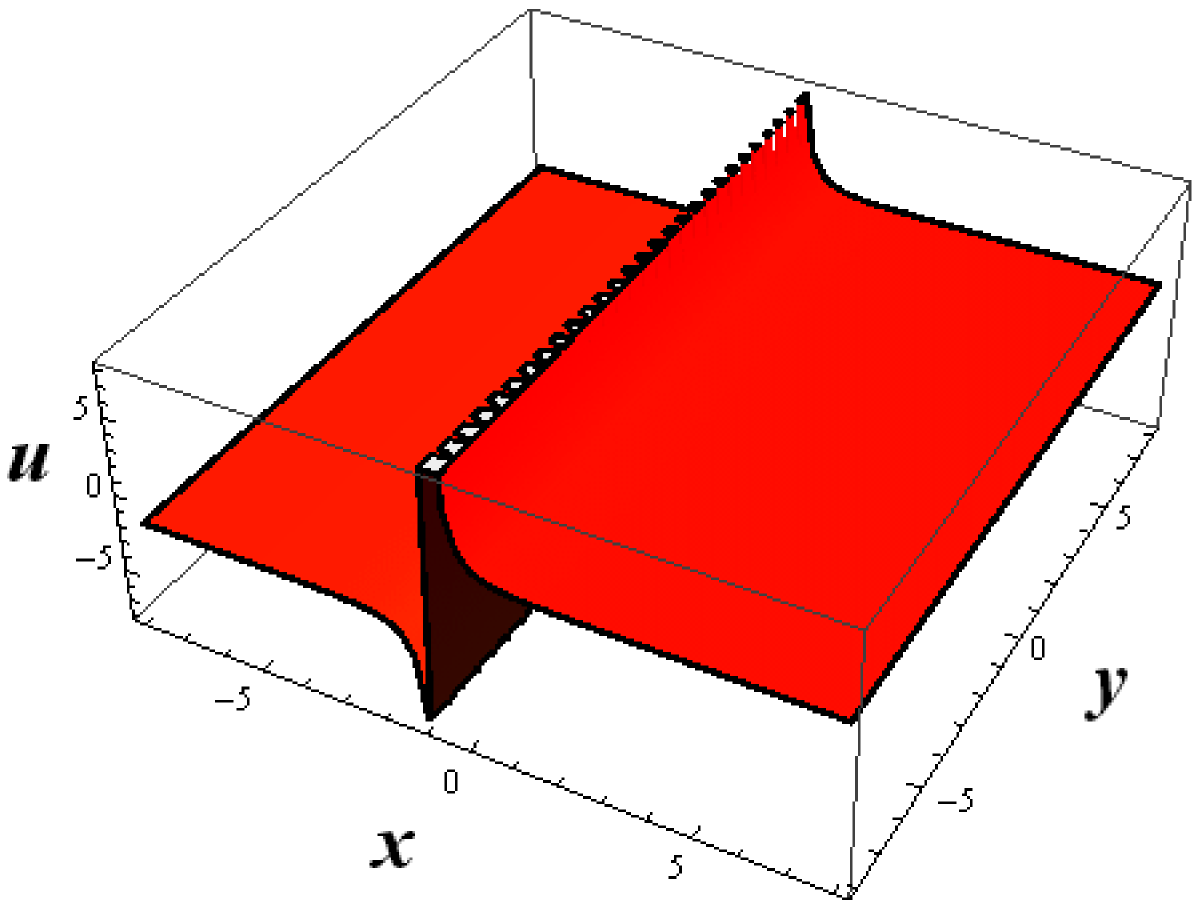

Figure 3.2: Singular kink shape soliton profile of (3.19) with $\eta = 1, \varphi = 1, \delta = 0.01, \gamma = 0.1, \alpha = 0.1, \beta = 1, t = 1, z = 1$.

3.2 Periodic solutions

In this section, we will implement the Tan-Cot ansatz method to derive periodic solutions of equation (3.1). The starting hypothesis is of the form

$$u(x, y, z, t) = \lambda \tan \tau, \tag{3.20}$$

where the wave variable τ is defined as $\tau = \eta x + \delta y + \varphi z - \nu t$. Proceeding as before and simplify, we obtain the amplitude and velocity of the wave as

$$\lambda = -2\beta\eta, \tag{3.21}$$

and

$$\nu = \frac{\gamma\eta(\eta + \delta + \varphi) + 4\beta\delta\eta^3}{\alpha\varphi}, \tag{3.22}$$

respectively.

This in turn gives the periodic solution of equation (3.1) as

$$u = -2\,\beta\,\eta\,\tan\left(\eta\,x + \delta\,y + \varphi\,z - \frac{\gamma\eta\,(\eta + \delta + \varphi)\,t + 4\,\beta\,\delta\,\eta^3 t}{\alpha\,\varphi}\right). \tag{3.23}$$

The graphical representation of solution (3.23) is presented in figure 3.3 below.

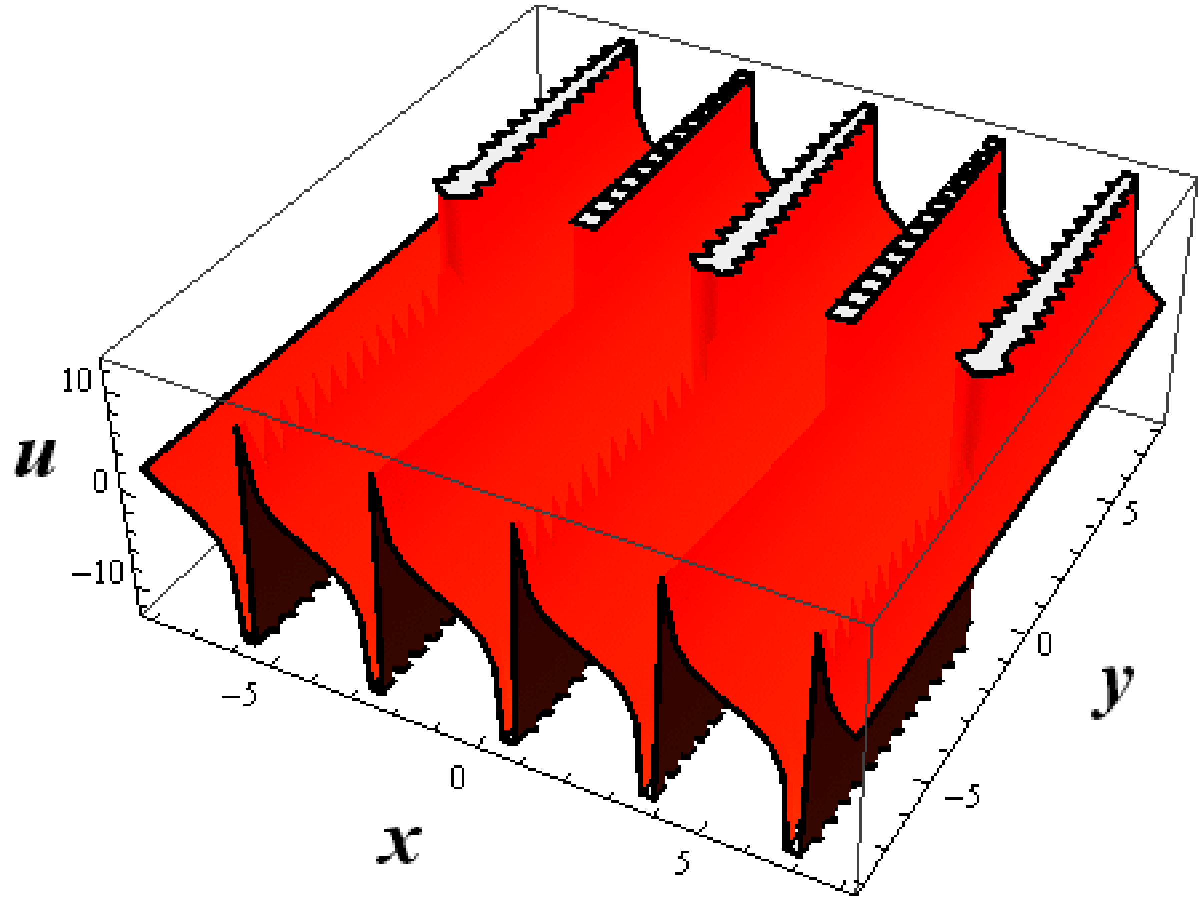

Figure 3.3: Periodic profile of solution (3.23) for $\eta = 1, \varphi = 1, \delta = 0.01, \gamma = 0.1, \alpha = 0.1, \beta = 1, t = 1, z = 1$.

Without loss of generality, the singular solution of equation (3.1) is

$$u = 2\,\beta\,\eta\,\cot\left(\eta\,x + \delta\,y + \varphi\,z - \frac{\gamma\eta\,(\eta + \delta + \varphi)\,t + 4\,\beta\,\delta\,\eta^3 t}{\alpha\,\varphi}\right). \qquad (3.24)$$

The profile of solution (3.24) is presented in figure 3.4 subject to the choice of arbitrary parameters.

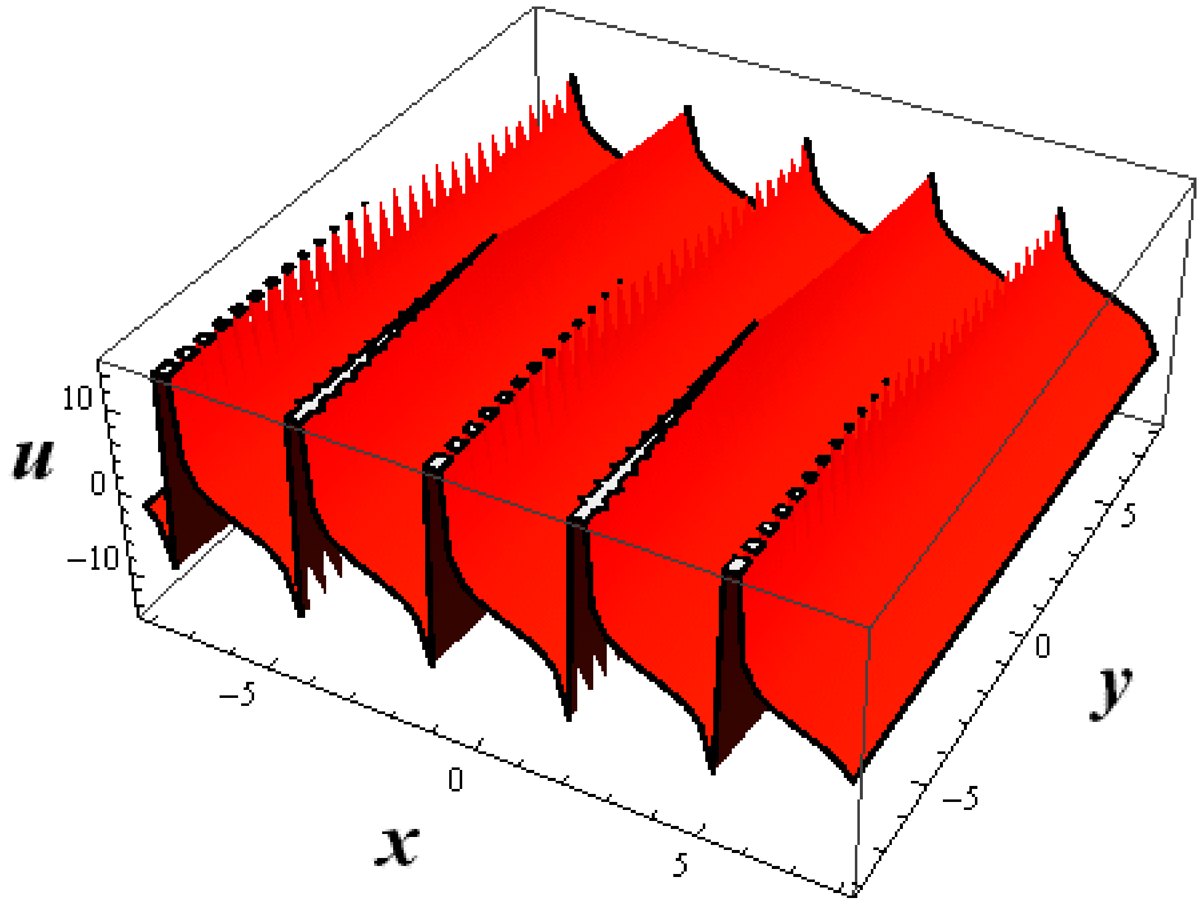

Figure 3.4: Periodic profile of solution (3.24) for $\eta = 1, \varphi = 1, \delta = 0.01, \gamma = 0.1, \alpha = 0.1, \beta = 1, t = 1, z = 1$.

Remark 3. A commendable observation indicates that periodic solutions do exist for equation (3.1). However, the existence of these solutions implies that the product of the parameters α and φ ($\alpha\varphi \neq 0$). To the authors knowledge, this outstanding observation is testified for the first time here.

3.3 Group invariant solutions

In order to derive group-invariant solutions of equation (3.1), one needs to obtain the admitted generators of equation (3.1). This is attained by considering the vector field of the form

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial u}, \tag{3.25}$$

where $\xi^i (i = 1, 2, 3)$ and η are functions of (t, x, y, z). Applying the fourth extension of equation (3.25) to equation (3.1) and solving the resulting system of linear partial differential equations, we conclude that equation (3.1) admits infinitely many point symmetries spanned by

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial x}, X_3 = \frac{\partial}{\partial z}, X_4 = F_1(z)\,\frac{\partial}{\partial u}, X_5 = F_2(t)\,\frac{\partial}{\partial u}, X_6 = 3\,\frac{\partial}{\partial y} + \gamma\,\frac{\partial}{\partial u}, \\
X_7 &= \alpha t \frac{\partial}{\partial t} + \gamma t \frac{\partial}{\partial x} - \alpha z \frac{\partial}{\partial z}, X_8 = 3\alpha t\,\frac{\partial}{\partial t} + 3\gamma t\,\frac{\partial}{\partial x} + 3\alpha y\,\frac{\partial}{\partial y} + \alpha\gamma y\,\frac{\partial}{\partial u}, \\
X_9 &= 9\alpha t\,\frac{\partial}{\partial t} + (3\alpha x + 6\gamma t)\,\frac{\partial}{\partial x} + (2\alpha\gamma x + \alpha\gamma y - 3\alpha u)\,\frac{\partial}{\partial u}.
\end{aligned}$$

In the theory of Lie symmetry analysis [8–10], it is well-known that a combination of symmetries will always remain symmetries of the problem at hand. Thus, invoking the linear combination of X_1, X_2 and X_3, and solving the associated Lagrange system, one obtains the four invariants, viz.,

$$f = y, \quad g = t - z, \quad h = x - z \quad \phi = u.$$

Using these invariants together with equation (3.1) and treating ϕ as function of (f, g, h), we get

$$3\,\phi_h\,\phi_{fh} + 3\,\phi_f\,\phi_{hh} - \gamma\,\phi_{fh} + \beta\,\phi_{fhhh} + \alpha\,\phi_{gg} + \gamma\,\phi_{gh} + \alpha\,\phi_{gh} = 0.$$

This in turn gives the group-invariant solution of equation (3.1) as

$$u = 2\beta c_4 \tanh\left(yc_2 + \frac{(\gamma c_4 + \alpha c_4 - \Omega)(t-z)}{2\alpha} + (x-z)c_4 + c_1\right) + c_5, \quad (3.26)$$

where

$$\Omega = \sqrt{\alpha^2 c_4^2 - 16\alpha\beta c_2 c_4^3 + 4\alpha\gamma c_2 c_4 + \alpha\gamma c_4^2 + \gamma^2 c_4^2}$$

and c_i are constants. The graphically representation of this solution is given in figure 3.5.

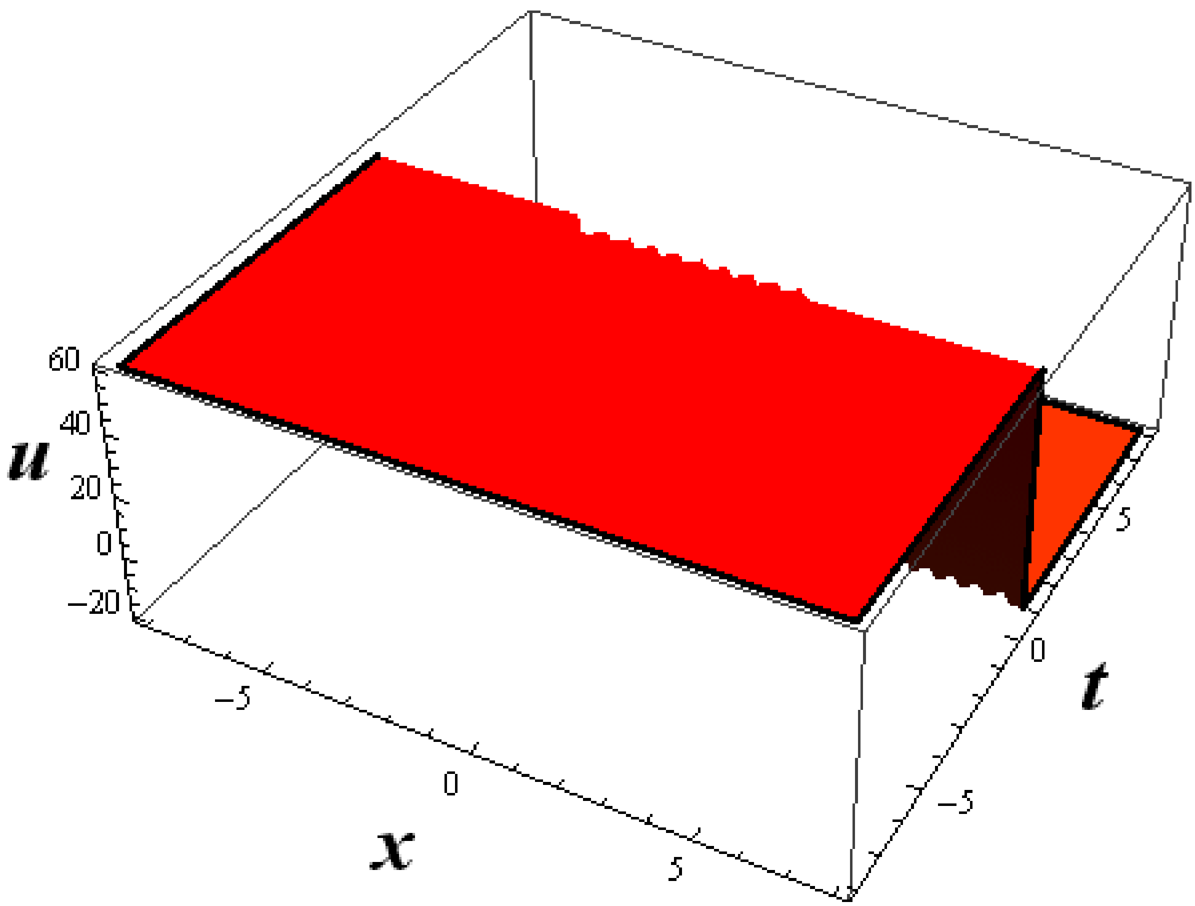

Figure 3.5: Anti-kink shape soliton profile of (3.26) for the values $y = 1, z = 1, \alpha = 1, \beta = -1, \mu = 1, \delta = 0.001, \gamma = 0.0001, c_1 = 10, c_2 = 5, c_4 = 20, c_5 = 20$.

We now make use of the generator X_6, which yields the following similarity variables

$$f = x, \quad g = z, \quad h = t \quad \phi = u - \frac{1}{3}\gamma y.$$

Consequently, equation (3.1) reduces to

$$\gamma\,\phi_{fg} + \alpha\,\phi_{gh} = 0.$$

Thus, we get

$$u = F_1\left(t, x\right) + F_2\left(z, -\alpha x + \gamma t\right) + \frac{1}{3}\,\gamma y, \tag{3.27}$$

as the group-invariant solution of equation (3.1) where F_i are arbitrary elements with respect to their arguments.

Thirdly, we employ linear combination of X_2 and X_4, which leads to four invariants, namely

$$f = y, \quad g = z, \quad h = t \quad \phi = u - x,\, F_3(t),$$

which transform equation(3.1) into

$$\phi_{gh} = 0,$$

and the resulting group-invariant solution of equation (3.1) is

$$u = xF_3\left(t\right) + F_4\left(y, t\right) + F_5\left(y, z\right), \tag{3.28}$$

with F_i being arbitrary functions with respect to their arguments.

Lastly, we consider the symmetry combination of X_2 and X_6 and get the following four invariants:

$$f = z, \quad g = t, \quad h = x - y - \frac{1}{3}\gamma y \quad \phi = u - \frac{1}{3}\gamma y,$$

which in turn reduce equation (3.1) into

$$6\,\gamma\,\phi_h\,\phi_{hh} - \gamma^2\,\phi_{hh} + \beta\,\gamma\,\phi_{hhhh} + 3\,\alpha\,\phi_{fg} + 3\,\gamma\,\phi_{fh} + 18\,\phi_h\,\phi_{hh} - 3\,\gamma\,\phi_{hh} + 3\,\beta\,\phi_{hhhh} = 0,$$

and the associated solution of equation (3.1) is

$$\begin{aligned} u &= 2\beta c_4 \tanh\left(z c_2 + \frac{(\gamma^2 c_4 - 4\beta\gamma c_4^3 - 12\beta c_4^3 - 3\gamma c_2 + 3\gamma c_4)\, t c_4}{3\alpha c_2} + \left(x - y - \frac{1}{3}\,\gamma\, y\right) c_4 + c_1 \right) \\ &\quad + c_5 + \frac{1}{3}\,\gamma y. \end{aligned} \tag{3.29}$$

The corresponding profile representation of solution (3.29) is given in figure 3.6.

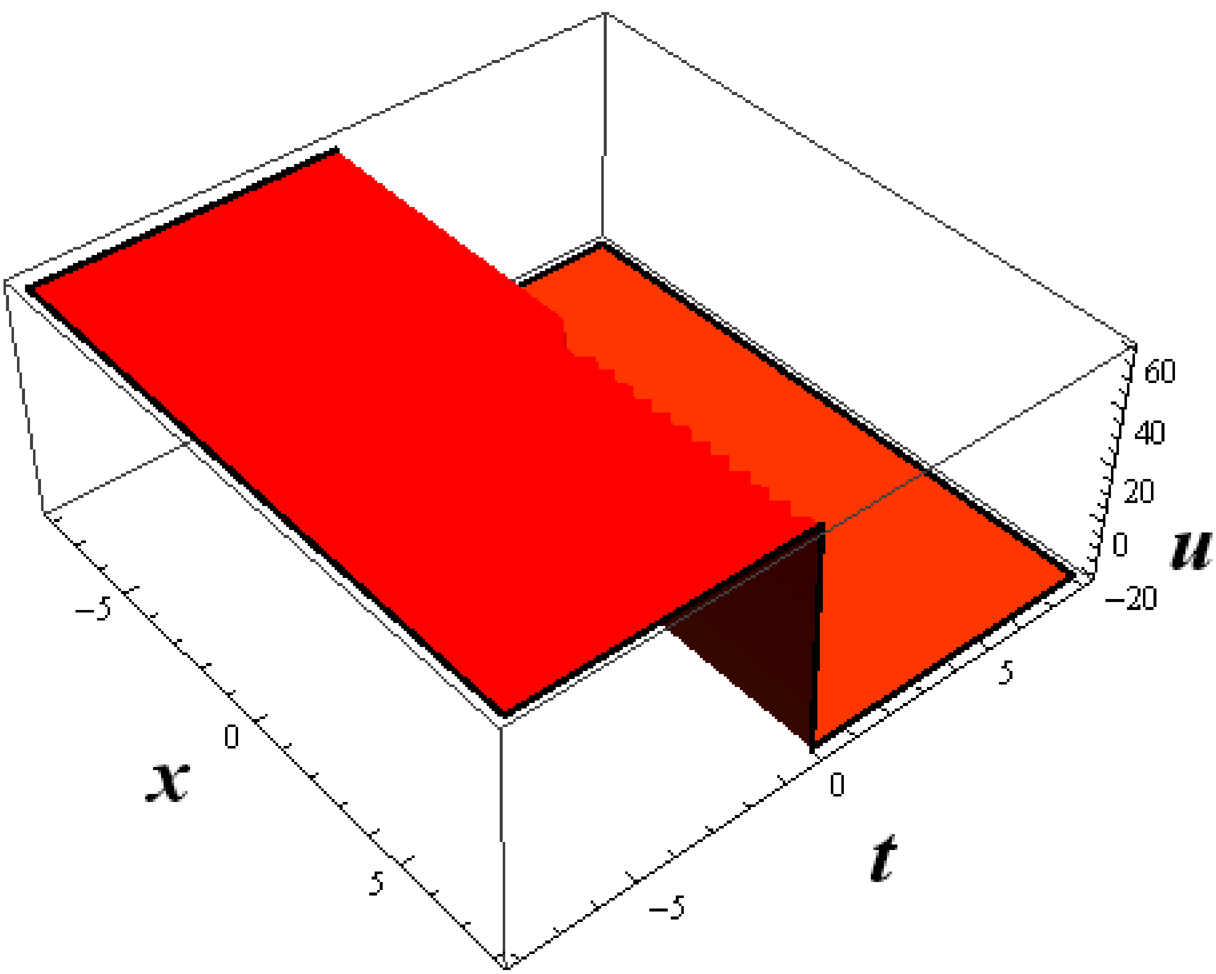

Figure 3.6: Anti-kink shape soliton profile of (3.29) for the values $y = 1, z = 1, \alpha = 1, \beta = -1, \gamma = 1, c_1 = 10, c_2 = 5, c_4 = 20, c_5 = 20.$

3.4 Local conservation laws

This section is devoted to the construction of low-order conservation laws of equation (3.1). This will be achieved via the multiplier method [16]. In order to derive the multiplier for equation (3.1), one needs to solve the resulting system of linear differential equations that arise from the expansion of

$$\frac{\delta(\Lambda E)}{\delta u} = 0, \tag{3.30}$$

where $\frac{\delta}{\delta u}$ is the Euler Lagrange operator, Λ denotes the multiplier function which in this context is assumed to be of order zero and E represents equation (3.1) [10,22]. The analysis of equation (3.30) prompts the following lemma.

Lemma 1. *Let Λ be a zeroth order conservation law multiplier, then a $(3+1)$-dimensional novel KP like equation admits infinitely many zeroth-order multipliers of the form*

$$\Lambda \;=\; C_1 u_x + F\,(y,z) + G\,(t,y)\,,$$

where F and G are arbitrary functions with respect to their arguments.

Proof. A straightforward but lengthy computation from $\frac{\delta}{\delta u}(\Lambda E) = 0$. □

Remark 4. It is worth pointing out that a $(3+1)$-dimensional KP like equation admits identical zeroth-order, first-order and second-order multipliers. This is a commendable observation which is mentioned here for the first time. In general, if one increases the order of the multiplier, then one aims to get higher-order multipliers which in turn leads to higher order conservation laws, but this is not the case with a $(3+1)$-dimensional KP like equation.

Without loss of generality, we now give the corresponding conserved vectors associated with the above multiplier, namely

$$
\begin{aligned}
T_1^t &= \frac{1}{4}\,\alpha u u_{xz} - \frac{1}{4}\,\alpha u_x u_z, \\
T_1^x &= \frac{1}{2}\,\gamma u_x^2 - \frac{3}{2}\,u_x^2 u_y - \frac{1}{8}\,\beta\, u_{xxx} u_y + \frac{3}{8}\,\beta u_{xx} u_{xy} - \frac{5}{8}\,\beta u_x u_{xxy} + \frac{1}{4}\,\gamma u u_{xy} + \frac{1}{4}\,\gamma u_x u_y \\
&\quad + \frac{1}{4}\,\gamma u_x u_z + \frac{1}{4}\,\gamma u u_{xz} + \frac{1}{2}\,\alpha u u_{tz} + \frac{1}{4}\,\delta u_t u_x - \frac{1}{8}\,\beta u u_{xxxy}, \\
T_1^y &= -\frac{1}{4}\,\gamma u u_{xx} + \frac{1}{8}\,\beta u u_{xxxx} - \frac{1}{2}\,u_x^3 - \frac{1}{4}\,\beta u_x u_{xxx} + \frac{1}{8}\,\beta u_{xx}^2 + \frac{1}{4}\,\gamma u_x^2, \\
T_1^z &= \frac{1}{4}\,\gamma u_x^2 - \frac{1}{4}\,\gamma u u_{xx} - \frac{1}{4}\,\alpha u u_{tx} + \frac{1}{4}\,\alpha u_t u_x;
\end{aligned}
$$

$$
\begin{aligned}
T_F^t &= -\frac{1}{2}\,\alpha\left(u\,D_z\,F\left(z,y\right) - F\left(z,y\right)u_z\right), \\
T_F^x &= \frac{3}{4}\,u u_x D_y\,F\left(z,y\right) + \frac{3}{4}\,u u_{xy} F\left(z,y\right) - \frac{9}{4}\,u_x u_y F\left(z,y\right) + \frac{1}{4}\,\beta u_{xx} D_y\,F\left(z,y\right) \\
&\quad + \gamma u_x F\left(z,y\right) - \frac{3}{4}\,\beta u_{xxy} F\left(z,y\right) - \frac{1}{2}\,\gamma u\,D_z\,F\left(z,y\right) - \frac{1}{2}\,\gamma u\,D_y\,F\left(z,y\right) \\
&\quad + \frac{1}{2}\,\gamma u_y F(y,z) + \frac{1}{2}\,\gamma u_z F(y,z), \\
T_F^y &= -\frac{3}{4}\,u_x^2 F\left(z,y\right) - \frac{3}{4}\,u u_{xx} F\left(z,y\right) - \frac{1}{4}\,\beta u_{xxx} F\left(z,y\right) + \frac{1}{2}\,\gamma u_x F\left(z,y\right), \\
T_F^z &= \frac{1}{2}\,\alpha u_t F\left(z,y\right) + \frac{1}{2}\,\gamma u_x F\left(z,y\right);
\end{aligned}
$$

$$
\begin{aligned}
T_G^t &= \frac{1}{2}\,\alpha u_z\,G\left(t,y\right), \\
T_G^x &= \frac{3}{4}\,u u_{xy}\,G\left(t,y\right) - \frac{9}{4}\,u_x u_y\,G\left(t,y\right) + \gamma u_x\,G\left(t,y\right) - \frac{1}{2}\,\gamma u\,D_y\,G\left(t,y\right) \\
&\quad + \frac{1}{2}\,\gamma u_y\,G\left(t,y\right) + \frac{1}{2}\,\gamma u_z\,G\left(t,y\right) - \frac{3}{4}\,\beta u_{xxy}\,G\left(t,y\right) + \frac{3}{4}\,u u_x\,D_y\,G\left(t,y\right) \\
&\quad - \frac{1}{4}\,\alpha\beta u_{xx}\,D_y\,G\left(t,y\right), \\
T_G^y &= \frac{1}{2}\,\gamma u_x\,G\left(t,y\right) - \frac{3}{4}\,u_x^2\,G\left(t,y\right) - \frac{3}{4}\,u u_{xx}\,G\left(t,y\right) - \frac{1}{4}\,\beta u_{xxx}\,G\left(t,y\right), \\
T_G^z &= -\frac{1}{2}\,\alpha u\,D_t\,G\left(t,y\right) + \frac{1}{2}\,\gamma u_x\,G\left(t,y\right) + \frac{1}{2}\,\alpha u_t\,G\left(t,y\right).
\end{aligned}
$$

3.5 Conclusions

In this chapter we obtained topological soliton solutions and periodic solutions of a $(3+1)$-dimensional KP like equation. In addition, other analytical solutions based on Lie symmetry method have been attained. Furthermore, conservation laws of the aforesaid equation were derived by using the multiplier method. The correctness of the obtained solutions have been verified with Maple software package by back substitution. It is anticipated that the solutions obtained here can be used as benchmarks against the numerical simulations of the underlying equation.

Chapter 4

A generalized $(2+1)$-dimensional Bogoyavlenskii-Kadomtsev-Petviashvili (BKP) equation: Multiple exp-function algorithm; Conservation Laws; Similarity Solutions

In the recent decades with the precipitous developments in nonlinear science, exact solutions of nonlinear partial differential equations have stimulated huge activity among many scientists and engineers [43, 57–83].

Nonlinear partial differential equations [43, 57–73, 76–83] model a plethora of nonlinear phenomena. Such phenomena arise in oceanography, aerospace industry, meteorology, nonlinear mechanics, population ecology, biology, fluid mechanics, plasma physics to mention but a few.

In the reflection of understanding these physical phenomena, it is of gigantic significance that one has to solve these mathematical models that regulate them. However, it is well-known that there is no orderly method that exists for assembling closed-form solutions to nonlinear partial differential equations. Notwithstanding this fact, scientists have devised numerous efficient procedures in finding practical special solutions to these equations.

A $(2+1)$-dimensional Bogoyavlenskii–Kadomtsev–Petviashvili (BKP) equation

$$u_{xxt} + u_{xxxxy} + 12u_{xx}u_{xy} + 8u_x u_{xxy} + 4u_{xxx}u_y = u_{yyy} \tag{4.1}$$

is a leeway of the Bogoyavlenskii-Schiff equation and Kadomtsev-Petviashvili equation [85–87]. This equation can be employed as a model for evolutionary shallow water waves [85]. The BKP equation is representative of the higher dimensional KP hierarchy and has been assessed in the preceding literature [85–87]. In [86], this equation was obtained by a reduction for the well-known three-dimensional Kadomtsev-Petviashvili equation [87] which describes the dissemination of nonlinear waves in plasmas and fluid dynamics. In this chapter, we will investigate a generalized BKP equation

$$u_{xxt} + u_{xxxxy} + \beta u_{xx}u_{xy} + \alpha u_x u_{xxy} + \delta u_{xxx}u_y = u_{yyy}. \tag{4.2}$$

The results of this chapter have been published in [84].

4.1 Multiple exp-function method

The multiple exp-function algorithm is autonomous in erecting bilinear forms. It commences that the multisoliton solutions can be conveyed as polynomials of exponential functions. The multiple exp-function algorithm, is basically a generalization of Hirota's perturbation scheme. Furthermore, the ensuing solutions contain generic phase shifts and wave frequencies.

4.1.1 Application of the multiple exp-function method to (4.2)

This subsection aims to apply the multiple exp-function method to solve equation (4.2) and hence obtain one-wave, two-wave and three-wave solutions of the aforesaid equation.

In order to obtain the one-wave solution, the starting hypothesis is of the form

$$u(x,y,t)=\frac{p}{q}, \tag{4.3}$$

where

$$p=A_0+A_1e^{k_1x+l_1y-\omega_1t}, \tag{4.4a}$$

$$q=B_0+B_1e^{k_1x+l_1y-\omega_1t}. \tag{4.4b}$$

Substituting equations (4.4) into equation (4.3) and thereafter inserting the resulting equation into equation (4.2) and solving the resulting systems of equations, we get

$$\alpha=-\frac{\delta A_1-12k_1}{A_1}, \tag{4.5}$$

$$\beta=\frac{12k_1}{A_1}, \tag{4.6}$$

$$\omega_1=\frac{l_1\left(k_1^4-l_1^2\right)}{k_1^2}. \tag{4.7}$$

Thus, the one-wave solution of equation (4.2) is given by equation (4.3) together with equations (4.5)–(4.7).

For the two-wave solutions the hypothesis is

$$u(x,y,t)=\frac{p}{q}, \tag{4.8}$$

where

$$p = 2\,k_1 e^{k_1x+l_1y-\omega_1t} + 2\,k_2 e^{k_2x+l_2y-\omega_2t} + 2\,A_{12}\,(k_1+k_2)\,e^{k_1x+l_1y-\omega_1t}e^{k_2x+l_2y-\omega_2t},$$

$$q = 1 + e^{k_1x+l_1y-\omega_1t} + e^{k_2x+l_2y-\omega_2t} + A_{12}e^{k_1x+l_1y-\omega_1t}e^{k_2x+l_2y-\omega_2t}.$$

By following the same procedure as above, we obtain the following two cases for two-wave solutions of equation (4.2)

Case I:

$$u(x,y,t) = \frac{p}{q},$$
$$\alpha = 4,$$
$$\beta = 6,$$
$$\delta = 2,$$
$$A_{12} = \frac{k_1^4k_2^2 - 2k_1^3k_2^3 + k_1^2k_2^4 + k_1^2l_2^2 - 2k_1k_2l_1l_2 + k_2^2l_1^2}{k_1^4k_2^2 + 2k_1^3k_2^3 + k_1^2k_2^4 + k_1^2l_2^2 - 2k_1k_2l_1l_2 + k_2^2l_1^2},$$
$$\omega_1 = \frac{l_1\,(k_1^4 - l_1^2)}{k_1^2},$$
$$\omega_2 = \frac{l_2\,(k_2^4 - l_2^2)}{k_2^2};$$

Case II:

$$u(x,y,t) = \frac{p}{q},$$
$$\alpha = \frac{3k_1^4k_2^2 - 3k_1^2k_2^4 - 3k_1^2l_2^2 + 3k_2^2l_1^2}{k_2^2k_1^2\,(k_1-k_2)\,(k_1+k_2)},$$
$$\beta = 6,$$
$$\delta = \frac{3k_1^4k_2^2 - 3k_1^2k_2^4 + 3k_1^2l_2^2 - 3k_2^2l_1^2}{k_2^2k_1^2\,(k_1-k_2)\,(k_1+k_2)},$$
$$A_{12} = \frac{k_1^3l_2 + 2k_1^2k_2l_1 - 3k_1^2k_2l_2 - 3k_1k_2^2l_1 + 2k_1k_2^2l_2 + k_2^3l_1}{k_1^3l_2 + 2k_1^2k_2l_1 + 3k_1^2k_2l_2 + 3k_1k_2^2l_1 + 2k_1k_2^2l_2 + k_2^3l_1},$$
$$\omega_1 = \frac{l_1\,(k_1^4 - l_1^2)}{k_1^2},$$
$$\omega_2 = \frac{l_2\,(k_2^4 - l_2^2)}{k_2^2}.$$

In the case of three-wave solutions the hypothesis is

$$u(x,y,t)=\frac{p}{q}, \tag{4.9}$$

where

$$\begin{aligned} p &= 2\,k_1e^{-\omega_1t+k_1x+l_1y}+2\,k_2e^{-\omega_2t+k_2x+l_2y}+2\,k_3e^{-\omega_3t+k_3x+l_3y}\\ &\quad +2\,A_{12}\,(k_1+k_2)\,e^{-\omega_1t+k_1x+l_1y}e^{-\omega_2t+k_2x+l_2y}+2\,A_{13}\,(k_1+k_3)\,e^{-\omega_1t+k_1x+l_1y}e^{-\omega_3t+k_3x+l_3y}\\ &\quad +2\,A_{23}\,(k_2+k_3)\,e^{-\omega_2t+k_2x+l_2y}e^{-\omega_3t+k_3x+l_3y}\\ &\quad +2\,A_{12}A_{13}A_{23}\,(k_1+k_2+k_3)\,e^{-\omega_1t+k_1x+l_1y}e^{-\omega_2t+k_2x+l_2y}e^{-\omega_3t+k_3x+l_3y},\\ q &= 1+e^{-\omega_1t+k_1x+l_1y}+e^{-\omega_2t+k_2x+l_2y}+e^{-\omega_3t+k_3x+l_3y}+A_{12}e^{-\omega_1t+k_1x+l_1y}e^{-\omega_2t+k_2x+l_2y}\\ &\quad +A_{13}e^{-\omega_1t+k_1x+yl_1}e^{-t\omega_3+xk_3+yl_3}+A_{23}e^{-t\omega_2+xk_2+yl_2}e^{-t\omega_3+xk_3+yl_3}\\ &\quad +A_{12}A_{13}A_{23}e^{-\omega_1t+k_1x+yl_1}e^{-t\omega_2+xk_2+yl_2}e^{-t\omega_3+xk_3+yl_3}, \end{aligned}$$

Following the same procedure as above, we obtain the following three-wave solutions of equation (4.2) as

$$\begin{aligned} & u(x,y,t)=\frac{p}{q},\\ & \alpha=4,\\ & \beta=6,\\ & \delta=2,\\ & A_{12}=\frac{k_1^4k_2^2-2k_1^3k_2^3+k_1^2k_2^4+k_1^2l_2^2-2k_1k_2l_1l_2+k_2^2l_1^2}{k_1^4k_2^2+2k_1^3k_2^3+k_1^2k_2^4+k_1^2l_2^2-2k_1k_2l_1l_2+k_2^2l_1^2},\\ & A_{13}=\frac{k_1^4k_3^2-2k_1^3k_3^3+k_1^2k_3^4+k_1^2l_3^2-2k_1k_3l_1l_3+k_3^2l_1^2}{k_1^4k_3^2+2k_1^3k_3^3+k_1^2k_3^4+k_1^2l_3^2-2k_1k_3l_1l_3+k_3^2l_1^2},\\ & A_{23}=\frac{k_2^4k_3^2-2k_2^3k_3^3+k_2^2k_3^4+k_2^2l_3^2-2k_2k_3l_2l_3+k_3^2l_2^2}{k_2^4k_3^2+2k_2^3k_3^3+k_2^2k_3^4+k_2^2l_3^2-2k_2k_3l_2l_3+k_3^2l_2^2},\\ & \omega_1=\frac{l_1\,(k_1^4-l_1^2)}{k_1^2},\\ & \omega_2=\frac{l_2\,(k_2^4-l_2^2)}{k_2^2},\\ & \omega_3=\frac{l_3\,(k_3^4-l_3^2)}{k_3^2}. \end{aligned}$$

4.2 Conserved currents

This segment explores local conserved currents of a generalized $(2+1)$-dimensional Bogoyavlenskii-Kadomtsev-Petviashvili (BKP) equation (4.2). Consider a partial differential equation $E = 0$, with three independent variables (t, x, y) and one dependent field variable u. Given $T = (T^t, T^x, T^y)$ such that the divergence $\partial_t T^t + \partial_x T^x + \partial_y T^y = 0$, then (T^t, T^x, T^y) are referred to as conserved currents and the divergence equation $\partial_t T^t + \partial_x T^x + \partial_y T^y = 0$ is referred to as local conservation laws. It should be distinguished that (T^t, T^x, T^y) are functions of the temporal variable, spatial variables, field variable and the derivatives of the field variable.

We now recall that if the divergence equation $\partial_t T^t + \partial_x T^x + \partial_y T^y$, holds for all solutions of $E = 0$, then there exists a function Λ called the multiplier such that $\Lambda E = \partial_t T^t + \partial_x T^x + \partial_y T^y$. The function Λ depends on (t, x, y, u) and the derivatives of the field variable. The exact value of the function Λ is derived from $\varepsilon_u(\Lambda E) = 0$, where ε_u is the Euler-Lagrange operator. Without loss of generality, we can now state the theorem below.

Theorem 4.1. *A generalized* $(2+1)$*-dimesional Bogoyavlenskii-Kadomtsev-Petviashvili equation (4.2) establishes the second order multiplier of the form*

$$\Lambda = uc_1 + \frac{x\left(y\left(yF\left(t\right) + 2\,G\left(t\right) + 2\,K\left(t\right)\right)\right)}{2} + \frac{y^2 L\left(t\right)}{2} + yM\left(t\right) + N\left(t\right)$$

for $\alpha = \beta - \delta$.

Proof. A straight forward but lengthy computation from $\varepsilon_u(\Lambda E) = 0$. □

Theorem 4.2. A generalized (2+1)-dimesional Bogoyavlenskii-Kadomtsev-Petviashvili equation (4.2) admits a set of conserved currents associated with the first-order multiplier, namely

$$T_1^t = -\frac{u_x^2}{6} + \frac{uuxx}{3},$$

$$T_1^x = -\frac{u^2 u_{xxy}\,\delta}{9} + \frac{\left(\left(40 u_x u_{xy} + 35 u_{xx} u_y\right)\delta + 36 u_{xxxy} + 30 u_{tx}\right)u}{45} - \frac{5 u_x^2 u_y\,\delta}{9}$$

$$-\frac{(15u_t+27u_{xxy})\,u_x}{45}-\frac{u_{xxx}u_y}{5}+\frac{2u_{xx}u_{xy}}{5},$$

$$T_1^y = \frac{u^2u_{xxx}\delta}{9}+\frac{(30u_xu_{xx}\delta+18u_{xxxx}-90u_{yy})\,u}{90}-\frac{u_x^3\delta}{9}-\frac{u_xu_{xxx}}{5}+\frac{u_y^2}{2}+\frac{u_{xx}^2}{10};$$

$$T_F^t = -\frac{(u_x-xu_{xx})\,y^2F(t)}{6},$$

$$T_F^x = \left(\frac{u-xu_x}{6}\right)yF'(t)+\frac{1}{6}\left(\left(-\frac{uu_{xxy}}{2}+2u_xu_{xy}+\frac{5u_{xx}u_y}{2}\right)\delta+\frac{12u_{xxxy}}{5}\right)xF(t)$$
$$+\frac{xu_{tx}F(t)}{3}+\frac{1}{6}\left((uu_{xy}-2u_xu_y)\,y\delta-\frac{9yu_{xxy}}{5}-yu_t-\left(uu_{xx}+u_x^2\right)x\delta\right)$$
$$-\frac{6xu_{xxx}}{5}+\frac{uu_x\delta}{3}+\frac{2u_{xx}}{5},$$

$$T_F^y = \frac{\left((uu_{xxx}+3u_xu_{xx})\,x\delta+\frac{6xu_{xxxx}}{5}-6xu_{yy}-(uu_{xx}+u_x^2)\,y^2\delta-\frac{6y^2u_{xxx}}{5}\right)F(t)}{12}$$
$$+(xyu_y-u_x)\,F(t);$$

$$T_G^t = -\frac{(u_x-xu_{xx})\,yG(t)}{3},$$

$$T_G^x = \frac{2y\,\left(\frac{u-xu_x}{2}\right)G'(t)}{3}+\frac{1}{3}\left(\left(-\frac{uu_{xxy}}{2}+2u_xu_{xy}+\frac{5u_{xx}u_y}{2}\right)x\delta+\frac{12xu_{xxxy}}{5}\right)G(t)$$
$$+\frac{2xu_{tx}G(t)}{3}+\frac{1}{3}\left((uu_{xy}-2u_xu_y)\,y\delta-\frac{9yu_{xxy}}{5}-yu_t-\left(uu_{xx}+u_x^2\right)x\delta-\frac{3xu_{xxx}}{5}\right)$$
$$+\frac{uu_x\delta}{3}+\frac{2u_{xx}}{5},$$

$$T_G^y = \frac{\left((uu_{xxx}+3u_xu_{xx})\,x\delta+\frac{6xu_{xxxx}}{5}-6xu_{yy}-(uu_{xx}+u_x^2)\,y\delta-\frac{6yu_{xxx}}{5}+6xu_y\right)G(t)}{6};$$

$$T_K^t = -\frac{y\,(u_x-xu_{xx})\,K(t)}{3},$$

$$T_K^x = \frac{(20u-10xu_x)\,K'(t)}{30}-\frac{1}{6}\left((uu_{xxy}-4u_xu_{xy}-5u_{xx}u_y)\,x\delta-\frac{24xu_{xxxy}}{5}\right)K(t)$$
$$-\frac{2xu_{tx}K(t)}{3}+\frac{1}{6}\left((-2uu_{xy}+4u_xu_y)\,\delta+\frac{18u_{xxy}}{5}+2u_t\right),$$

$$T_K^y = \frac{\left((uu_{xxx}+3u_xu_{xx})\,x\delta-uu_{xx}\delta-u_x^2\delta-\left(6u_{yy}-\frac{6u_{xxxx}}{5}\right)x-\frac{6u_{xxx}}{5}\right)K(t)}{6};$$

$$
\begin{aligned}
T_L^t &= \frac{y^2 u_{xx}\, L(t)}{6}, \\
T_L^x &= -\frac{1}{12}\left(2yu_x L'(t) - (uu_{xxy} + 4u_x u_{xy} + 5u_{xx}u_y)\, y\delta - \frac{24yu_{xxxy}}{5} - 4yu_{tx}\right) y \\
&\quad -\frac{1}{12}\left(\left(2uu_{xx} + 2u_x^2\right)\delta + \frac{12u_{xxx}}{5}\right) L(t),
\end{aligned}
$$

$$
T_L^y = \frac{\left(\left((uu_{xxx} + 3u_x u_{xx})\,\delta + \frac{6u_{xxxx}}{5} - 6u_{yy}\right) y^2 + 12u_{yy} - 12u\right) L(t)}{12};
$$

$$
\begin{aligned}
T_M^t &= -\frac{yu_{xx}\, M(t)}{3}, \\
T_M^x &= -\frac{yu_x\, M'(t)}{3} - \frac{1}{6}\left((uu_{xxy} - 4u_x u_{xy} - 5u_{xx}u_y)\, y\delta - \frac{24yu_{xxxy}}{5} - 4yu_{tx}\right) \\
&\quad +\frac{1}{6}\left(\left(uu_{xx} + u_x^2\right)\delta + \frac{6u_{xxx}}{5}\right) M(t),
\end{aligned}
$$

$$
T_M^y = \frac{\left((uu_{xxx} + 3u_x u_{xx})\, y\delta + \frac{6yu_{xxxx}}{5} - 6yu_{yy} + 6u_y\right) M(t)}{6};
$$

$$
\begin{aligned}
T_N^t &= \frac{u_{xx}\, N(t)}{3}, \\
T_N^x &= -\frac{u_x\, N'(t)}{3} - \frac{\left((uu_{xxy} - 4u_x u_{xy} - 5u_{xx}u_y)\,\delta - \frac{24u_{xxxy}}{5} - 4u_{tx}\right) N(t)}{6}, \\
T_N^y &= \frac{((5uu_{xxx} + 15u_x u_{xx})\,\delta + 6u_{xxxx} - 30u_{yy})\, N(t)}{30}.
\end{aligned}
$$

Proof. It can easily be shown that after some dreary but straight forward manipulation, the divergence equation $\partial_t T^t + \partial_x T^x + \partial_y T^y = 0$, holds for all solutions of a generalized $(2+1)$-dimesional Bogoyavlenskii-Kadomtsev-Petviashvili equation (4.2).

Remark 1. We succinctly discuss the significance and physical illumination that ascends from the computed conservation laws. Conservation laws reside in enormously crucial areas both at the foundations of nonlinear science and in its applications. Mathematical expressions of physical laws, such as conservation of energy, momentum and mass are fundamentally conservation laws. Imperative physical information about the complex behaviour in non-linear systems is confined in conservation laws. The arbitrary functions in the multipliers lead to an infinitely many conservation laws.

4.3 Symmetry analysis of (4.2)

This section utilizes the Lie symmetry method to develop the generators of equation (4.2) and thereafter, utilize them to derive exact solutions of equation (4.2). Consider the vector field of the form

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial u},$$

where (ξ^i, η) for $(i = 1, 2, 3)$ are functions of (t, x, y, u). Applying the fifth prolongation $\mathrm{pr}^{(5)}X$ to equation (4.2), expanding and thereafter splitting the monomials leads to a linear overdetermined system partial differential equations. The integration of this system leads to the following symmetry generators:

$$X_1 = \frac{\partial}{\partial t}, X_2 = -x\frac{\partial}{\partial x} - 2y\,\frac{\partial}{\partial y} - 4t\,\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial u}, X_3 = \alpha\, K_1\,(t)\,\frac{\partial}{\partial y} + x\,K_1'\,(t)\,\frac{\partial}{\partial u},$$
$$X_4 = \delta\, K_2\,(t)\,\frac{\partial}{\partial x} + y\,K_2'\,(t)\,\frac{\partial}{\partial u}, X_5 = K_3\,(t)\,\frac{\partial}{\partial u}.$$

4.3.1 Symmetry reduction and exact solutions of (4.2)

We shall first begin with X_2 and get the following three invariants

$$f = \frac{t}{y^2},\ g = \frac{x}{\sqrt{y}},\ \theta = u\,\sqrt{y}. \tag{4.1}$$

By considering θ as the new dependent variable and (g, f) as new independent variables, equation (4.2) transforms to a nonlinear PDE in two independent variables, viz.,

$$64f^3\,\theta_{fff}+48f^2\,g\,\theta_{ffg}+12fg^2\,\theta_{fgg}-16f\delta\,\theta_f\theta_{ggg}-16f\alpha\,\theta_g\theta_{fgg}-16f\beta\,\theta_{fg}\theta_{gg}$$
$$+g^3\,\theta_{ggg}-4g\alpha\,\theta_g\theta_{ggg}-4g\delta\,\theta_g\theta_{ggg}-4g\beta\,\theta_{gg}^2+336f^2\,\theta_{ff}+132fg\,\theta_{fg}+12g^2\theta_{gg}$$
$$-4\delta\,\theta\theta_{ggg}-12\alpha\,\theta\theta_{gg}-8\beta\,\theta_g\theta_{gg}+300f\theta_f-16f\,\theta_{fgggg}+33g\,\theta_g-4g\,\theta_{ggggg}+15\,\theta$$
$$+8\,\theta_{fgg}-20\,\theta_{gggg}=0. \tag{4.2}$$

The Lie point symmetries of equation (4.2) are as follows:

$$\Gamma_1=4\alpha f^{\frac{3}{2}}\,\frac{\partial}{\partial f}+g\sqrt{f}\alpha\,\frac{\partial}{\partial g}-\left(\sqrt{f}\alpha\,\theta+\frac{g}{\sqrt{f}}\right)\frac{\partial}{\partial\theta},\ \Gamma_2=\frac{1}{\sqrt[4]{f}}\frac{\partial}{\partial\theta},$$
$$\Gamma_3=4\delta\,\sqrt[4]{f}\,\frac{\partial}{\partial g}+\frac{1}{f^{\frac{3}{4}}}\frac{\partial}{\partial\theta}. \tag{4.3}$$

Now using Γ_3, we get the following two invariants:

$$r=f,\ \phi=\frac{4f\delta\theta-g}{4f\delta} \tag{4.4}$$

and this leads to the following nonlinear ordinary differential equation:

$$64\,r^3\phi'''\,(r)+336\,r^2\phi''\,(r)+300\,r\phi'\,(r)+15\,\phi\,(r)=0, \tag{4.5}$$

with

$$\phi\,(r)=\frac{rc_2+\sqrt{r}\,c_3+c_1}{r^{\frac{5}{4}}}, \tag{4.6}$$

where c_1, c_2 and c_3 are arbitrary constants. Now using the invariants (4.1) and (4.4) together with equation (4.6), we conclude that the group-invariant solution of equation (4.2) is

$$u(t,x,y)=\frac{4\,\nu^{\frac{3}{2}}\,y^{\frac{9}{2}}\delta c_3+xy^4\,\nu^{\frac{5}{4}}+4ty^{\frac{5}{2}}\delta c_1+4t^2\sqrt{y}\,\delta c_2}{4y^5\,\nu^{\frac{9}{4}}\delta}, \tag{4.7}$$

where $\nu=\frac{t}{y^2}$.

Now considering the combinations of X_1 and X_2, we obtain the following three invariants

$$m = \frac{4t-1}{4y^2},\ n = \frac{x}{\sqrt{y}},\ \vartheta = u\sqrt{y}. \tag{4.8}$$

Taking ϑ as the new dependent variable and (m, n) as new independent variables, equation (4.2) reduces to a nonlinear partial differential equation, namely

$$\begin{aligned}
&64m^3\,\vartheta_{mmm} + 48m^2\,n\,\vartheta_{mmn} + 12mn^2\,\vartheta_{mnn} - 16m\delta\,\vartheta_m\vartheta_{nnn} - 16m\,\beta\vartheta_{mn}\vartheta_{nn}\\
&-16m\alpha\,\vartheta_n\vartheta_{mnn} + n^3\,\vartheta_{nnn} - 4n\alpha\,\vartheta_n\vartheta_{nnn} - 4n\delta\,\vartheta_n\vartheta_{nnn} - 4n\beta\,\vartheta_{nn}^2 + 336m^2\,\vartheta_{mm}\\
&+132mn\,\vartheta_{mn} + 12n^2\vartheta_{nn} - 4\delta\,\vartheta\vartheta_{nnn} - 12\alpha\,\vartheta_n\vartheta_{nn} - 8\beta\,\vartheta_n\vartheta_{nn} + 300m\vartheta_m\\
&-16m\,\vartheta_{mnnnn} + 33n\,\vartheta_n - 4n\,\vartheta_{nnnnn} + 15\,\vartheta + 8\,\vartheta_{mnn} - 20\,\vartheta_{nnnn} = 0.
\end{aligned} \tag{4.9}$$

The Lie point symmetries of equation (4.9) are as follows:

$$\begin{aligned}
&\Omega_1 = 4\alpha m^{\frac{3}{2}}\frac{\partial}{\partial m} + n\sqrt{m}\alpha\frac{\partial}{\partial n} - \left(\sqrt{m}\alpha\,\vartheta + \frac{n}{\sqrt{m}}\right)\frac{\partial}{\partial\vartheta},\ \Omega_2 = \frac{1}{\sqrt[4]{m}}\frac{\partial}{\partial\vartheta},\\
&\Omega_3 = 4\sqrt[4]{m}\,\delta\frac{\partial}{\partial n} + \frac{1}{m^{\frac{3}{4}}}\frac{\partial}{\partial\vartheta}.
\end{aligned} \tag{4.10}$$

Again, using Ω_2 and Ω_3, we get the following two invariants:

$$l = m,\ \varphi = \frac{4m^{\frac{3}{2}}\delta\vartheta - \sqrt{m}n - mn}{4m^{\frac{3}{2}}\delta} \tag{4.11}$$

and this leads to the following nonlinear ordinary differential equation:

$$64\,l^3\varphi'''(l) + 336\,l^2\varphi''(l) + 300\,l\varphi'(l) + 15\,\varphi(l) = 0, \tag{4.12}$$

whose solution is

$$\varphi(l) = \frac{lc_1 + \sqrt{l}\,c_2 + c_3}{l^{\frac{5}{4}}}. \tag{4.13}$$

Now using the invariants (4.8) and (4.11) together with equation (4.13), we conclude that the group-invariant solution of equation (4.2) is

$$u(t,x,y) \;=\; \frac{1}{4y^5\,\kappa^{\frac{11}{4}}\,\delta}\left(4\sqrt[4]{4}\,y^{\frac{9}{2}}\,\kappa^{\frac{5}{2}}\delta c_1 + 16\sqrt[4]{4}\,y^{\frac{9}{2}}\,\kappa^{\frac{3}{2}}\delta c_3 + 2xy^4\,\kappa^{\frac{9}{4}} + 4xy^4\,\kappa^{\frac{7}{4}}\right)$$

$$-\frac{1}{4y^5\,\kappa^{\frac{11}{4}}\,\delta}\left(128\sqrt{2}\,t^2\,\sqrt{y}\,\delta+64\,\sqrt{2}\,.t\,\sqrt{y}\,\delta-8\sqrt{2}\,\sqrt{y}\,\delta\right)c_2, \quad (4.14)$$

where $\kappa=\frac{4\,t-1}{y^2}$.

Also, considering the combinations of X_3 and X_5, we obtain the following three invariants

$$\nu=x,\ \omega=t,\ \psi=u-\frac{yL_1\left(t\right)}{\alpha}. \quad (4.15)$$

By considering ψ as the new dependent variable and (ν,ω) as new independent variables, equation (4.2) transforms to a nonlinear PDE in two independent variables, viz.,

$$L_1\left(\omega\right)\delta\,\psi_{\nu\nu\nu}+\alpha\psi_{\nu\nu\omega}=0. \quad (4.16)$$

The Lie point symmetries of equation (4.16) are as follows:

$$\Psi_1=\left(2\nu\delta\int L_1\left(\omega\right)\,d\omega-\nu^2\alpha\right)\frac{\partial}{\partial\nu}+\left(\psi\delta\int L_1\left(\omega\right)\,d\omega-\nu\psi\alpha\right)\frac{\partial}{\partial\psi},\ \Psi_2=\nu\frac{\partial}{\partial\nu},$$
$$\Psi_3=\psi\frac{\partial}{\partial\psi},\ \Psi_4=L_2(\omega)\frac{\partial}{\partial\omega},\ \Psi_5=L_3(\nu,\omega)\frac{\partial}{\partial\psi},\ \Psi_6=L_4(\omega)\frac{\partial}{\partial\nu}. \quad (4.17)$$

Employing Ψ_4 and Ψ_6, we get the following two invariants

$$s=-\nu+\int^{\omega}\frac{L_4\left(a\right)}{L_2\left(a\right)}da,\ \varrho=\rho, \quad (4.18)$$

and this leads to the following linear ordinary differential equation

$$\varrho'''\left(s\right)=0. \quad (4.19)$$

The solution of the above equation is

$$\varrho\left(s\right)=\frac{1}{2}s^2c_1+sc_2+c_3, \quad (4.20)$$

where c_1,c_2 and c_3 are arbitrary constants. Now using the invariants (4.15) and (4.18) together with equation (4.20), we conclude that the group-invariant solution of equation (4.2) is

$$u(t,x,y)\ =\ \frac{1}{2\alpha}\left(\alpha\left(\int^{t}\frac{L_4\left(a\right)}{L_2\left(a\right)}da\right)^2c_1-2x\alpha\int^{t}\frac{L_4\left(a\right)}{L_2\left(a\right)}\,da\,c_1+x^2\alpha c_1\right)$$

$$+2\alpha \int^{t} \frac{L_4(a)}{L_2(a)} da\, c_2 - 2x\alpha c_2 + \frac{1}{2\alpha}\left(2yL_1(t) + 2\alpha c_3\right). \tag{4.21}$$

Remark 2. In many applications, group invariant solutions capture the limiting behaviour of problems that are far away from their initial or boundary conditions.

4.4 Concluding remarks

A generalized $(2+1)$-dimensional Bogoyavlenskii-Kadomtsev-Petviashvili (BKP) equation which is an augmentation of the Bogoyavlenskii-Schiff equation and Kadomtsev –Petviashvili equation was probed. We computed novel exact solutions by utilizing the multiple exp-function algorithm and the modern group analysis method. Finally, we computed conserved currents using the invariance and multiplier technique. These research findings can well mimic complex waves and their dynamics in fluids.

Chapter 5

A generalized dispersive water waves system: Conservation laws; Symmetry reduction; Travelling wave solutions; Symbolic Computation

An integrable generalized dispersive water waves system

$$u_t - \alpha u_{xx} - u u_x - g v_x = 0, \tag{5.1a}$$

$$v_t - v u_x - u v_x - w w_x + \alpha v_{xx} + 2\beta w_{xx} = 0, \tag{5.1b}$$

$$w_t - \frac{1}{2} w u_x - \frac{1}{2} u w_x - \beta u_{xx} + \alpha v_{xx} + 2\beta w_{xx} = 0, \tag{5.1c}$$

was formulated by extending the Hamiltonian form of the Benney system via the addition of the free-surface velocity.

The results of this chapter have been published in [88].

The nonlinear evolution equations for example such as the above play an essential role in studying nonlinear wave propagation [43, 57–73, 76–81]. Investigating the exact solutions of NLEEs is a difficult task, and only in certain exceptional cases can one write down the solutions explicitly. Despite this fact, various methods of solving these types of NLEEs have been proposed in recent years.

Among the methods mentioned earlier, the Lie group analysis method, also called the symmetry method, is one of the most effective methods to determine nonlinear partial differential equations [8–10, 39, 89–96]. In the second half of the 19th century and about 200 years after Leibniz and Newton introduced the concept of the derivative, solving ordinary differential equations (ODEs) had become one of the most critical problems in applied mathematics. Sophus Lie (1842-1899) got interested in this problem and, with inspiration from Galois' theory for solving algebraic equations, he discovered what is known today as Lie group analysis. He showed that the most known methods of integration of ordinary differential equations, which had seemed artificial, could be derived in a unified manner using his theory of continuous transformation groups. Recently there have been considerable developments in symmetry methods for differential equations, as evidenced by the number of research papers, books and much new symbolic software devoted to the subject.

The aim of this chapter is to compute conservation laws and travelling wave solutions of

$$u_t - \alpha u_{xx} - uu_x - gv_x = 0, \tag{5.2a}$$

$$v_t - vu_x - uv_x - ww_x + \alpha v_{xx} + 2\beta w_{xx} = 0, \tag{5.2b}$$

$$w_t - \frac{1}{2}wu_x - \frac{1}{2}uw_x - \beta u_{xx} + \alpha v_{xx} + 2\beta w_{xx} = 0. \tag{5.2c}$$

Note that in the system (5.1), $u = u(x,t), v = v(x,t), w = w(x,t)$, g is gravitational constant and α, β are arbitrary constants.

It should be noted that by setting $\alpha = 0$, $\beta = 0$ system (5.1) is reduced to an integrable extension of the classical long wave equation [97]. When $\alpha = 0$, $\beta = 0$, $w = 0$, system (5.1) collapses to the dispersive long wave equation [98], which includes several special cases such as the Broer-Kaup equation [99,100], the classical Boussinesq equation [101], the Jaulent-Miodek equation [102] and the two-boson equation [103].

5.1 Conservation laws

Conservation laws play an essential role in the solution process of DEs. Finding the conservation laws of a system of DEs is often the first step towards finding the solution. The existence of a large number of conservation laws of a system of partial differential equations (PDEs) is a strong indication of its integrability.

A conservation law of system (5.1) is a total space-time divergence expression that vanishes on the solution space ε of system (5.1),

$$D_t T^t + D_x T^x|_{\varepsilon} = 0, \tag{5.3}$$

where D_t and D_x are the total derivative operators while T^t is a conserved density and T^x is a spatial flux. To determine the conservation law for system (5.1), we will implement the multiplier method. Since the joint Euler operator annihilates the total divergence, we get

$$\begin{aligned}
&\frac{\delta}{\delta u}\Bigg((u_t - \alpha u_{xx} - u u_x - g v_x)\Lambda_1 + (v_t - v u_x - u v_x - w w_x + \alpha v_{xx} + 2\beta w_{xx})\Lambda_2 \\
&+(w_t - \frac{1}{2} w u_x - \frac{1}{2} u w_x - \beta u_{xx} + \alpha v_{xx} + 2\beta w_{xx})\Lambda_3\Bigg) = 0, \qquad (5.4) \\
&\frac{\delta}{\delta v}\Bigg((u_t - \alpha u_{xx} - u u_x - g v_x)\Lambda_1 + (v_t - v u_x - u v_x - w w_x + \alpha v_{xx} + 2\beta w_{xx})\Lambda_2 \\
&+(w_t - \frac{1}{2} w u_x - \frac{1}{2} u w_x - \beta u_{xx} + \alpha v_{xx} + 2\beta w_{xx})\Lambda_3\Bigg) = 0, \qquad (5.5)
\end{aligned}$$

$$\frac{\delta}{\delta w}\Bigg((u_t-\alpha u_{xx}-uu_x-gv_x)\Lambda_1+(v_t-vu_x-uv_x-ww_x+\alpha v_{xx}+2\beta w_{xx})\Lambda_2$$
$$+(w_t-\frac{1}{2}wu_x-\frac{1}{2}uw_x-\beta u_{xx}+\alpha v_{xx}+2\beta w_{xx})\Lambda_3\Bigg)=0, \qquad (5.6)$$

where Λ_1, Λ_2 and Λ_3 are the second-order multipliers to be determined. The expansion of equations (5.4)-(5.6), splitting and simplifying yields the following multipliers:

$$\Lambda_1 = \frac{1}{2}\left(2\,uv-2\,\alpha v_x+w^2-4\,\beta w_x\right)C_1+vC_2+C_3, \qquad (5.7)$$
$$\Lambda_2 = \frac{1}{2}\left(u^2+2\,\alpha u_x+2\,gv\right)C_1+uC_2+C_5, \qquad (5.8)$$
$$\Lambda_3 = (uC_1+2\,C_2)\,w+2\,\beta u_xC_1+C_4. \qquad (5.9)$$

Corresponding to the above multipliers, we have the following conserved vectors for system (5.1):

$$T_1^t = \frac{1}{2}\,u^2v-\frac{1}{2}\,\alpha uv_x+\frac{1}{2}\,uw^2-\beta\,uw_x+\frac{1}{2}\,gv^2+\frac{1}{2}\,\alpha\,u_xv+\beta u_xw,$$
$$T_1^x = 2\,\alpha\,\beta u_x\,w_x-\beta uu_xw+2\,\beta gvw_x+g\alpha vv_x-\frac{1}{2}\,u^3v-\frac{1}{2}\,u^2w^2-\beta^2u_x^2+\beta uw_t$$
$$+\frac{1}{2}\,\alpha uv_t-guv^2+\frac{1}{2}\,\alpha u^2v_x+\beta u^2w_x+\alpha^2u_xv_x-\frac{1}{2}\,\alpha u_xw^2-\frac{1}{2}\,\alpha u_tv-\frac{1}{2}\,gvw^2$$
$$-\beta u_tw-\alpha uu_xv;$$
$$T_2^t = uv+w^2,$$
$$T_2^x = -\frac{1}{2}\,gv^2-uw^2-u^2v+\alpha uv_x+2\beta uw_x-\alpha u_xv-2\beta u_xw;$$
$$T_3^t = u,$$
$$T_3^x = -\alpha u_x-gv-\frac{1}{2}\,u^2;$$
$$T_4^t = w,$$
$$T_4^x = -\beta u_x-\frac{1}{2}\,uw;$$
$$T_5^t = v,$$
$$T_5^x = \alpha v_x+2\beta w_x-uv-\frac{1}{2}\,w^2.$$

5.2 Symmetry reductions and exact solutions of (5.1)

The symmetry generator of the generalized dispersive water waves system (5.1) will be generated by the vector field

$$\begin{aligned} X &= \xi^1(t,x,u,v,w)\frac{\partial}{\partial t} + \xi^2(t,x,u,v,w)\frac{\partial}{\partial x} + \eta^1(t,x,u,v,w)\frac{\partial}{\partial u} + \eta^2(t,x,u,v,w)\frac{\partial}{\partial v} \\ &\quad + \eta^3(t,x,u,v,w)\frac{\partial}{\partial w}. \end{aligned} \tag{5.10}$$

The invocation of the second prolongation $\mathrm{pr}^{(2)}X$ [43, 67–73] to system (5.1) and splitting the monomials leads to linear overdetermined system of partial differential equations. Solving the resulting system of partial differential equations, one obtains two-dimensional Lie algebra spanned by the following linearly independent generators:

$$X_1 = \frac{\partial}{\partial t},$$
$$X_2 = \frac{\partial}{\partial x}.$$

5.2.1 Symmetry reduction of (5.1)

In order to construct a symmetry reduction, we need to solve the associated Lagrange equations

$$\begin{aligned} &\frac{dt}{\xi^1(t,x,u,v,w)} = \frac{dx}{\xi^2(t,x,u,v,w)} = \frac{du}{\eta^1(t,x,u,v,w)} \\ &= \frac{dv}{\eta^2(t,x,u,v,w)} = \frac{dw}{\eta^3(t,x,u,v,w)}. \end{aligned} \tag{5.11}$$

Employing the characteristics equations (5.11) with respect to $X_1 + \tau X_2$ (τ denotes the speed of wave), yields the following invariants $z = x - \tau t$, $E = u$, $F = v$ and $G = w$. Implementing these invariants, system (5.1) transforms to

$$E\left(z\right)E'\left(z\right) + gF'\left(z\right) + \tau E' + \alpha\, E''\left(z\right) = 0, \tag{5.12a}$$

$$\frac{1}{2}\,G\left(z\right)E'\left(z\right)+\frac{1}{2}\,E\left(z\right)G'\left(z\right)+\tau G'+\beta\,E''\left(z\right)=0, \tag{5.12b}$$

$$F\left(z\right)E'\left(z\right)+E\left(z\right)F'\left(z\right)+G\left(z\right)G'\left(z\right)+\tau F'-\alpha\,F''\left(z\right)$$

$$-2\,\beta\,G''\left(z\right)=0. \tag{5.12c}$$

5.2.2 Exact solutions using ansatz method

In this subsection we use an ansatz method to conduct the integration of the system of nonlinear ordinary differential equations computed via symmetry analysis in the previous subsection. The basic idea in ansatz method is as follows. Let us consider the solutions of system (5.12) in the form

$$E(z)=\sum_{i=-M}^{M}A_i(H(z))^i,\;\; F(z)=\sum_{i=-N}^{N}B_i(H(z))^i,\;\; G(z)=\sum_{i=-P}^{P}C_i(H(z))^i, \tag{5.13}$$

where $H(z)$ is a solution of Bernoulli or Riccati equation, M is a positive integer and $A_{-M},\cdots,A_M,B_{-N},\cdots,B_N,C_{-P},\cdots,C_P,$ are parameters to be computed. The Bernoulli equation is

$$H'(z)=aH(z)+bH^2(z), \tag{5.14}$$

where a and b are constants whose solution is given by

$$H(z)=a\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\}. \tag{5.15}$$

The Riccati equation is

$$H'(z)=aH^2(z)+bH(z)+c, \tag{5.16}$$

where a, b and c are constants. We shall employ the solutions

$$H(z)=-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right], \tag{5.17}$$

and

$$H(z)=-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}, \tag{5.18}$$

where $\theta^2=b^2-4ac$ and C is a constant of integration.

5.2.3 Solutions of (5.1) using the ansatz of the Bernoulli equation

The application of the balancing procedure to system (5.12) yields $M = 2, N = 4$ and $P = 1$. Thus, the solutions of equation (5.12) are of the form

$$
\begin{aligned}
E(z) &= \frac{A_{-2}}{H^2} + \frac{A_{-1}}{H} + A_0 + A_1 H + A_2 H^2, && (5.19a)\\
F(z) &= \frac{B_{-4}}{H^4} + \frac{B_{-3}}{H^3} + \frac{B_{-2}}{H^2} + \frac{B_{-1}}{H} + B_0 + B_1 H + B_2 H^2 + B_3 H^3 \\
&\quad + B_4 H^4, && (5.19b)\\
G(z) &= \frac{C_{-1}}{H} + C_0 + C_1 H. && (5.19c)
\end{aligned}
$$

Substituting equation (5.19) into equation (5.12) and making use of equation (5.14) and then equating the coefficients of the function H^i to zero, we obtain an algebraic system of equations in terms of A_i with $i = -2...2$, B_i with $i = -4...4$ and C_i with $i = -1...1$. Solving the resultant system of algebraic equations leads to the following cases, summarized in tables (5.1)–(5.3).

Table 5.1: Solution of equation (5.1) via Bernoulli equation

Case	**Solutions**
1	$u(t,x) = A_0 + A_1 a\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\},$
	$v(t,x) = B_1 a\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\},$
	$w(t,x) = C_0 + C_1 a\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\},$
	$\tau = -\frac{\alpha^3 a+2\,\beta^2 g}{\alpha^2},$
	$A_0 = \frac{3\beta^2 g}{\alpha^2},$
	$A_1 = -2\,\alpha b,$
	$B_1 = \frac{2\,\beta^2 b}{\alpha},$
	$C_0 = \frac{\beta^3 g}{\alpha^3},$
	$C_1 = -2\beta b,$
	where $z = x - \tau t.$

Table 5.2: Solution of equation (5.1) via Bernoulli equation (continued)

Case	Solutions
2	$u(t,x) = A_1 a\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\},$
	$v(t,x) = B_0 + B_1 a\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\}\mathrm{c},$
	$w(t,x) = C_0 + C_1 a\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\}\mathrm{z},$
	$\tau = -\frac{\alpha^3 a+\alpha^2 A_0-\beta^2 g}{\alpha^2},$
	$A_1 = -2\,\alpha b,$
	$B_0 = -\frac{\beta^2\left(\alpha^2 A_0-3\,\beta^2 g\right)}{\alpha^4},$
	$B_1 = \frac{2\,\beta^2 b}{\alpha},$
	$C_0 = \frac{\beta\left(\alpha^2 A_0-2\,\beta^2 g\right)}{\alpha^3},$
	$C_1 = -2\beta b,$
	where $z = x - \tau t$.

Table 5.3: Solution of equation (5.1) via Bernoulli equation (continued)

Case	Solutions
3	$u(t,x) = A_0 + A_1 a\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\},$
	$v(t,x) = B_0 + B_1 a\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\}$
	$+B_2 a^2\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\}^2,$
	$w(t,x) = C_0 + C_1 a\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\},$
	$A_0 = \frac{\alpha^3 a+\alpha^2\tau+\beta^2 g}{\alpha^2},$
	$A_1 = 2\alpha b,$
	$B_0 = -\frac{\beta^2\left(\alpha^3 a-\alpha^2\tau-2\beta^2 g\right)}{\alpha^4},$
	$B_1 = \frac{2b\left(2\alpha^3 a+\beta^2 g\right)}{\alpha g},$
	$B_2 = -\frac{4\alpha^2 b^2}{g},$
	$C_0 = -\frac{\beta\left(\alpha^3 a-\alpha^2\tau-\beta^2 g\right)}{\alpha^3},$
	$C_1 = -2\beta b,$
	where $z = x - \tau t$.

5.2.4 Solutions of (5.1) using the ansatz of the Riccati equation

In this subsection we use the ansatz of the Riccati equation. Substituting system (5.19) into system (5.12) and making use of equation (5.16), we obtain an algebraic system of equations in terms of A_i, B_i and C_i where $i = -2...2, i = -4...4$ and $i = -1...1$ respectively. By equating all coefficients of the functions H^i to zero and solving the resulting system of equations leads to the following cases summarized in tables (5.4)–(5.12).

Table 5.4: Solution of equation (5.1) via Riccati equation

Case	Solution 1	Solution 2
1	$u(t,x) = A_0 + A_1\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}$,	$u(t,x) = A_0 + A_1\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}$,
	$v(t,x) = B_1\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}$,	$v(t,x) = B_1\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}$,
	$w(t,x) = C_0 + C_1\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}$,	$w(t,x) = C_0 + C_1\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}$,
	$A_0 = \frac{3\beta^2 g}{\alpha^2}$, $A_1 = 2\alpha a\chi$, $B_1 = -\frac{2\beta^2 a\chi}{\alpha}$, $C_0 = \frac{\beta^3 g}{\alpha^3}$,	$C_1 = 2\beta a\chi$, where χ is any root of $\alpha^3\chi^2 b - \beta^2\chi g - \alpha^2 = 0$.
	This quadratic equation will appear in the proceeding	cases and will be ommitted.
2	$u(t,x) = A_0 + A_1\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}$,	$u(t,x) = A_0 + A_1\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}$,
	$v(t,x) = B_0 + B_1\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}$,	$v(t,x) = B_0 + B_1\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}$,
	$w(t,x) = C_0 + C_1\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}$,	$w(t,x) = C_0 + C_1\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}$,
	$A_0 = -\frac{\alpha^2 - \alpha^3\tau^2 b - \beta^2\tau g}{\alpha^2\tau}$, $A_1 = 2\alpha\tau a$, $B_1 = -\frac{2\beta^2\tau}{\alpha}$,	$C_0 = -\frac{1}{\alpha\tau}\left(-2\alpha\beta\tau^2 b - \frac{(\alpha^2 - \alpha^3\tau^2 b - \beta^2\tau g)\beta}{\alpha^2} + 2\beta\right)$, $C_1 = 2\beta\tau a$,
	$B_0 = -\frac{1}{2\alpha\tau^2 a}\left(-\frac{2\beta\tau a}{\alpha}\left(-2\alpha\beta\tau^2 b - \frac{(\alpha^2 - \alpha^3\tau^2 b - \tau\beta^2 g)\beta}{\alpha^2}\right.\right.$	$\left.\left.+2\beta\right)\right) + \frac{1}{2\alpha\tau^2 a}\left(\frac{2\beta^2\tau a}{\alpha}\left(\alpha\tau^2 b - \frac{\alpha^2 - \alpha^3\tau^2 b - \tau\beta^2 g}{\alpha^2} + 1\right) + 4\beta^2\tau^3 ab\right)$.

Table 5.5: Solution of equation (5.1) via Riccati equation (Continued)

Case	Solution 1	Solution 2
3	$u(t,x)=A_{-1}\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-1}+A_0,$	$u(t,x)=A_{-1}\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\mathrm{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-1}+A_0,$
	$v(t,x)=B_{-1}\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-1}+B_0,$	$v(t,x)=B_{-1}\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\mathrm{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-1}+B_0,$
	$w(t,x)=C_{-1}\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-1}+C_0,$	$w(t,x)=C_{-1}\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\mathrm{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-1}+C_0,$
	$A_{-1}=-2\,\alpha\tau c,\ A_0=-\frac{\alpha^3\tau^2b-\beta^2\tau g+\alpha^2}{\alpha^2\tau},\ B_{-1}=\frac{2\,\beta^2\tau c}{\alpha},$	$C_{-1}=-2\beta\tau c,\ C_0=-\frac{1}{\alpha\tau}\left(2\,\alpha\beta\tau^2b-\frac{\left(\alpha^3\tau^2b-\beta^2\tau g+\alpha^2\right)\beta}{\alpha^2}+2\,\beta\right).$
	$B_0=\frac{\beta}{\alpha^2\tau}\left(2\,\alpha\beta\tau^2b-\frac{\left(\alpha^3\tau^2b-\beta^2\tau g+\alpha^2\right)\beta}{\alpha^2}+2\,\beta\right)$	$-\frac{\beta^2}{\alpha^2\tau}\left(7\,\alpha\tau^2b+\frac{\alpha^3\tau^2b-\beta^2\tau g+\alpha^2}{\alpha^2}-1-16\,\beta^2\tau^3bc\right).$
4	$u(t,x)=A_0+A_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\},$	$u(t,x)=A_0+A_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\mathrm{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\},$
	$v(t,x)=B_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}$	$v(t,x)=B_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\mathrm{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right.$
	$+B_2\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2,$	$+B_2\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\mathrm{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^2,$
	$w(t,x)=C_0+C_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\},$	$w(t,x)=C_0+C_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\mathrm{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\},$
	$\tau=\sigma,\ A_0=-\frac{\alpha^3b\sigma^2-\beta^2g\sigma+\alpha^2}{\alpha^2\sigma},\ A_1=-2\,\alpha a\sigma,$	$B_1=-\frac{2\,a\sigma\left(2\,\alpha^3b\sigma-\beta^2g\right)}{\alpha g},\ B_2=-\frac{4\,\alpha^2a^2\sigma^2}{g},\ C_0=\frac{\beta\left(\alpha^3b\sigma^2+\beta^2g\sigma+\alpha^2\right)}{\alpha^3\sigma},$
	$C_1=2\beta a\sigma$, where σ is any root of $4\,\alpha^6ac\sigma^3-\alpha^3\beta^2bg\sigma^2$	$-2\,\beta^4g^2\sigma-\alpha^2\beta^2g=0$. This quadratic equation will appear in the
	ensuing cases and will be ommitted.	

Table 5.6: Solution of equation (5.1) via Riccati equation (Continued)

Case	Solution 1	Solution 2
5	$u(t,x)=A_0+A_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\},$ $v(t,x)=B_0+B_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\},$ $+B_2\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2,$ $w(t,x)=C_0+C_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\},$ $A_0=-\frac{\alpha^3\tau^2b-\beta^2\tau g+\alpha^2}{\alpha^2\tau},\ A_1=-2\,\alpha\tau a,$ $C_0=-\frac{1}{\alpha\tau}\left(-2\,\alpha\beta\tau^2b+\frac{(\alpha^3\tau^2b-\beta^2\tau g+\alpha^2)\beta}{\alpha^2}-2\,\beta\right),$ $B_0=-\frac{\beta}{\alpha^2\tau}\left(-2\,\alpha\beta\tau^2b+\frac{(\alpha^3\tau^2b-\beta^2\tau g+\alpha^2)\beta}{\alpha^2}-2\,\beta\right)$ $-4\,\beta^2\tau^3ab+\frac{8\,\alpha^3\tau^4a^2c}{g}.$	$u(t,x)=A_0+A_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\},$ $v(t,x)=B_0+B_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\},$ $+B_2\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^2,$ $w(t,x)=C_0+C_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\},$ $B_1=-\frac{1}{g}\left(2\,\alpha^2\tau^2ab+\frac{2\,a\left(\alpha^3\tau^2b-\beta^2\tau g+\alpha^2\right)}{\alpha}-2\,\alpha a\right),\ B_2=-\frac{4\,\alpha^2\tau^2a^2}{g},$ $C_1=2\beta\tau a,$ $-\frac{1}{g}\left(\alpha\tau^2b+\frac{\alpha^3\tau^2b-\beta^2\tau g+\alpha^2}{\alpha^2}-1\right)\left(2\,\alpha^2\tau^2ab+\frac{2\,a\left(\alpha^3\tau^2b-\beta^2\tau g+\alpha^2\right)}{\alpha}-2\,\alpha a\right)$
6	$u(t,x)=A_0+A_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\},$ $v(t,x)=B_0+B_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\},$ $w(t,x)=C_0+C_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\},$ $\tau=\chi, A_0=-\frac{2}{\chi},\ A_1=2\,\alpha a\chi,$	$u(t,x)=A_0+A_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\},$ $v(t,x)=B_0+B_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\},$ $w(t,x)=C_0+C_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\},$ $B_0=\frac{\beta^2\left(\beta^2g\chi+2\,\alpha^2\right)}{\alpha^4\chi}, B_1=-\frac{2\,\beta^2a\chi}{\alpha},\ C_0=2\,\beta b\chi,\ C_1=2\beta a\chi.$

Table 5.7: Solution of equation (5.1) via Riccati equation (Continued)

Case	Solution 1	Solution 2
7	$u(t,x) = A_{-1}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-1} + A_0,$	$u(t,x) = A_{-1}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-1} + A_0,$
	$v(t,x) = B_{-2}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-2},$	$v(t,x) = B_{-2}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-2}$
	$+B_{-1}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-1}$	$+B_{-1}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-1},$
	$w(t,x) = C_{-1}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-1} + C_0,$	$w(t,x) = C_{-1}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-1} + C_0,$
	$\tau = \sigma,\ A_{-1} = 2\,\alpha c\sigma,\ A_0 = \frac{\alpha^3 b\sigma^2 + \beta^2 g\sigma - \alpha^2}{\alpha^2\sigma},$	$B_{-2} = -\frac{4\,\alpha^2 c^2\sigma^2}{g},\ B_{-1} = -\frac{2\,c\sigma\left(2\,\alpha^3 b\sigma + \beta^2 g\right)}{\alpha g},\ C_{-1} = -2\beta c\sigma,$
	$C_0 = -\frac{\beta\left(\alpha^3 b\sigma^2 - \beta^2 g\sigma - \alpha^2\right)}{\alpha^3\sigma}.$	
8	$u(t,x) = A_{-1}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-1} + A_0,$	$u(t,x) = A_{-1}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-1} + A_0,$
	$v(t,x) = B_{-2}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-2}$	$v(t,x) = B_{-2}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-2}$
	$+B_{-1}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-1} + B_0$	$+B_{-1}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-1} + B_0,$
	$w(t,x) = C_{-1}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-1} + C_0,$	$w(t,x) = C_{-1}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-1} + C_0,$
	$A_{-1} = 2\alpha\tau c,\ A_0 = -\frac{\alpha^2 - \alpha^3\tau^2 b - \beta^2\tau g}{\alpha^2\tau},\ B_{-2} = -\frac{4\,\alpha^2\tau^2 c^2}{g},$	$B_{-1} = -\frac{1}{g}\left(2\,\alpha^2\tau^2 bc - \frac{2\,c\left(\alpha^2 - \alpha^3\tau^2 b - \beta^2\tau g\right)}{\alpha} + 2\,\alpha c\right),\ C_{-1} = -2\beta\tau c,$
	$C_0 = -\frac{1}{\alpha\tau}\left(2\,\alpha\beta\tau^2 b + \frac{\left(\alpha^2 - \alpha^3\tau^2 b - \beta^2\tau g\right)\beta}{\alpha^2} - 2\,\beta\right).$	
	$B_0 = -\frac{\beta}{\alpha^2\tau}\left(-2\,\alpha\beta\tau^2 b + \frac{\left(\alpha^3\tau^2 b - \beta^2\tau g + \alpha^2\right)\beta}{\alpha^2} - 2\,\beta\right)$	$-\frac{1}{g}\left(\alpha\tau^2 b + \frac{\alpha^3\tau^2 b - \beta^2\tau g + \alpha^2}{\alpha^2} - 1\right)\left(2\,\alpha^2\tau^2 ab + \frac{2\,a\left(\alpha^3\tau^2 b - \beta^2\tau g + \alpha^2\right)}{\alpha} - 2\,\alpha a\right)$
	$-4\,\beta^2\tau^3 ab + \frac{8\,\alpha^3\tau^4 a^2 c}{g}.$	

Table 5.8: Solution of equation (5.1) via Riccati equation (Continued)

Case	Solution 1	Solution 2
9	$u(t,x) = A_0 + A_1\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\},$	$u(t,x) = A_0 + A_1\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\},$
	$v(t,x) = B_1\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\},$	$v(t,x) = B_1\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\},$
	$w(t,x) = C_0 + C_1\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\},$	$w(t,x) = C_0 + C_1\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\},$
	$A_0 = \frac{3\beta^2 g}{\alpha^2},\ A_1 = 2\,\alpha a\chi,\ B_1 = -\frac{2\beta^2 a\chi}{\alpha},$	$C_0 = \frac{\beta^3 g}{\alpha^3},\ C_1 = 2\beta a\chi.$
10	$u(t,x) = A_{-1}\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-1} + A_0,$	$u(t,x) = A_{-1}\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-1} + A_0,$
	$v(t,x) = B_{-2}\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-2},$	$v(t,x) = B_{-2}\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-2}$
	$+B_{-1}\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-1} + B_0$	$+B_{-1}\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-1} + B_0,$
	$w(t,x) = C_{-1}\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-1} + C_0,$	$w(t,x) = C_{-1}\left\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-1} + C_0,$
	$\tau = \chi,\ A_{-1} = 2\alpha c\chi,\ A_0 = -\frac{2}{\chi},\ B_{-2} = \frac{4c^2\left(\beta^2 g\chi+\alpha^2\right)}{\alpha bg},$	$B_{-1} = \frac{2c\left(\beta^2 g\chi+2\alpha^2\right)}{\alpha g},\ C_{-1} = -2\beta c\chi,\ C_0 = -2\beta b\chi,$
	$B_0 = \frac{8\alpha^5\beta^2 abcg\chi - 4\beta^6 acg^3\chi - 2\alpha^5\beta^2 b^3 g\chi + 3\beta^6 b^2 g^3\chi + 4\alpha^7 abc}{\alpha^4 b^2 g(\beta^2 g\chi+\alpha^2)}$	$-\frac{4\alpha^2\beta^4 acg^2 + 3\alpha^2\beta^4 b^2 g^2}{\alpha^4 b^2 g(\beta^2 g\chi+\alpha^2)}.$

Table 5.9: Solution of equation (5.1) via Riccati equation (Continued)

Case	Solution 1	Solution 2
11	$u(t,x)=A_{-1}\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-1}+A_0,$ $v(t,x)=B_{-1}\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-1}+B_0,$ $w(t,x)=C_{-1}\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-1}+C_0$ $+C_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\},$ $\tau=\chi,\ A_{-1}=-2\,\alpha c\chi,\ A_0=-\frac{2}{\chi},\ B_{-1}=\frac{2\,\beta^2 c\chi}{\alpha},$	$u(t,x)=A_{-1}\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-1}+A_0,$ $v(t,x)=B_{-1}\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-1}+B_0,$ $w(t,x)=C_{-1}\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-1}+C_0$ $+C_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\},$ $B_0=\frac{\beta^2\left(3\,\beta^2 g\chi+2\,\alpha^2\right)}{\alpha^4\chi},\ C_{-1}=-2\beta c\chi,\ C_0=-2\beta b\chi,\ C_1=-4\beta a\chi$
12	$u(t,x)=A_0+A_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\},$ $v(t,x)=B_0+B_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\},$ $w(t,x)=C_{-1}\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-1}+C_0$ $+C_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\},$ $\tau=\chi,\ A_0=\frac{2}{\chi},\ A_1=2\,\alpha a\chi,\ B_0=\frac{\beta^2\left(3\,\beta^2 g\chi+2\,\alpha^2\right)}{\alpha^4\chi},$	$u(t,x)=A_0+A_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\},$ $v(t,x)=B_0+B_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\},$ $w(t,x)=C_{-1}\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-1}+C_0$ $+C_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\},$ $B_1=-\frac{2\,\beta^2 a\chi}{\alpha},\ C_{-1}=4\beta c\chi,\ C_0=2\beta a\sigma,\ C_1=2\beta a\chi.$

Table 5.10: Solution of equation (5.1) via Riccati equation (Continued)

Case	Solution 1	Solution 2
13	$u(t,x)=A_{-1}\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-1}+A_0,$ $v(t,x)=B_{-2}\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-2}$ $+B_{-1}\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-1}+B_0,$ $w(t,x)=C_{-1}\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-1}+C_0$ $+C_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\},$ $\tau=\chi,\ A_{-1}=-2\,\alpha c\chi,\ A_0=-\frac{2}{\chi},\ B_{-2}=\frac{4(\beta^2 g\chi+\alpha^2)c^2}{\alpha bg},$ $B_0=\frac{8\,\alpha^5\beta^2 abcg\chi-4\,\beta^6 acg^3\chi-2\,\alpha^5\beta^2 b^3 g\chi+3\,\beta^6 b^2 g^3\chi+4\,\alpha^7 abc}{\alpha^4 b^2 g(\beta^2 g\chi+\alpha^2)}$	$u(t,x)=A_{-1}\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\mathrm{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-1}+A_0,$ $v(t,x)=B_{-2}\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\mathrm{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-2}$ $+B_{-1}\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\mathrm{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-1}+B_0,$ $w(t,x)=C_{-1}\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\mathrm{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-1}+C_0$ $+C_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\mathrm{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\},$ $B_{-1}=-\frac{2\,c\left(\beta^2 g\chi+2\alpha^2\right)}{\alpha g},\ C_{-1}=-2\beta c\chi,\ C_0=-2\beta b\chi,\ C_1=-4\beta a\chi,$ $-\frac{4\,\alpha^2\beta^4 acg^2+3\,\alpha^2\beta^4 b^2 g^2}{\alpha^4 b^2 g(\beta^2 g\chi+\alpha^2)}.$
14	$u(t,x)=A_0+A_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\},$ $v(t,x)=B_0+B_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}$ $+B_2\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{2},$ $w(t,x)=C_{-1}\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-1}+C_0$ $+C_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\},$ $\tau=\chi,\ A_0=\frac{2}{\chi},\ A_1=-2\,\alpha a\chi,\ B_1=-\frac{2\,a\left(\beta^2 g\chi+2\,\alpha^2\right)}{\alpha g},$ $B_0=\frac{8\,\alpha^5\beta^2 abcg\chi-4\,\beta^6 acg^3\chi-2\,\alpha^5\beta^2 b^3 g\chi+3\,\beta^6 b^2 g^3\chi+4\,\alpha^7 abc}{\alpha^4 b^2 g(\beta^2 g\chi+\alpha^2)}$	$u(t,x)=A_0+A_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\mathrm{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\},$ $v(t,x)=B_0+B_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\mathrm{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}$ $+B_2\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\mathrm{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{2},$ $w(t,x)=C_{-1}\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\mathrm{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-1}+C_0$ $+C_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\mathrm{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\},$ $B_2=-\frac{4\,a^2\left(\beta^2 g\chi+\alpha^2\right)}{\alpha bg},\ C_{-1}=4\beta c\chi,\ C_0=2\beta b\chi,\ C_1=2\beta a\chi,$ $-\frac{4\,\alpha^2\beta^4 acg^2+3\,\alpha^2\beta^4 b^2 g^2}{\alpha^4 b^2 g(\beta^2 g\chi+\alpha^2)}.$

Table 5.11: Solution of equation (5.1) via Riccati equation (Continued)

Case	Solution 1	Solution 2
15	$u(t,x) = A_{-1}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-1} + A_0$	$u(t,x) = A_{-1}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-1} + A_0$
	$+A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}$,	$+A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}$,
	$v(t,x) = B_{-1}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-1} + B_0$	$v(t,x) = B_{-1}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-1} + B_0$
	$+B_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}$	$+B_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}$
	$+B_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2$,	$+B_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^2$,
	$w(t,x) = C_{-1}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-1} + C_0$	$w(t,x) = C_{-1}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-1} + C_0$
	$+C_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}$,	$+C_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}$,
	$\tau = \chi$, $A_{-1} = 2\,\alpha c\chi$, $A_0 = \frac{2\beta^2 g}{\alpha^2}$, $A_1 = -2\alpha\chi$,	$B_{-1} = \frac{2\,\beta^2 c\chi}{\alpha}$, $B_0 = \frac{\beta^4 g}{\alpha^4}$, $B_1 = \frac{2\,a\left(3\,\beta^2 g\chi + 2\,\alpha^2\right)}{\alpha g}$, $B_2 = \frac{4\,a^2\left(\beta^2 g\chi + \alpha^2\right)}{\alpha b g}$,
	$C_{-1} = -2\beta c\chi$, $C_1 = 2\beta a\chi$.	

Table 5.12: Solution of equation (5.1) via Riccati equation (Continued)

Case	Solution 1	Solution 2
16	$u(t,x) = A_{-1}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-1} + A_0$	$u(t,x) = A_{-1}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-1} + A_0$
	$+A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\},$	$+A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\},$
	$v(t,x) = B_{-2}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-2}$	$v(t,x) = B_{-2}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-2}$
	$+B_{-1}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-1} + B_0$	$+B_{-1}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-1} + B_0$
	$+B_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\},$	$+B_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\},$
	$w(t,x) = C_{-1}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^{-1}$	$w(t,x) = C_{-1}\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^{-1}$
	$+C_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\},$	$+C_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\},$
	$\tau = \chi,\ A_{-1} = 2\alpha a\chi,\ A_0 = \frac{2\beta^2 g}{\alpha^2},\ 2\,\alpha a\chi,$	$B_{-2} = -\frac{4c^2\left(\beta^2 g\chi + \alpha^2\right)}{\alpha b g},\ B_{-1} = -\frac{6c\left(\beta^2 g\chi + 2\alpha^2\right)}{\alpha g},\ B_0 = \frac{\beta^4 g}{\alpha^4},\ B_1 = -\frac{2\beta^2 a\chi}{\alpha},$
	$C_1 = -2\beta c\chi,\ C_{-1} = 2\beta a\chi.$	

The familiarity of closed-form solutions of nonlinear ordinary and partial differential equations enables numerical solvers and supports stability analysis.

Although many efforts have been dedicated to solving nonlinear evolution equations, there is no unified method. To the best of our knowledge, this is the first time that Lie point symmetry analysis in conjunction with an ansatz method have been applied to this underlying equation.

5.3 Concluding remarks

Symmetry analysis was performed for the generalized dispersive water waves system, which arises in many branches of physics such as particle physics and fluid dynamics. The similarity reductions and new exact solutions are constructed. Subsequently, conservation laws are derived using the multiplier approach also simplest equation method was used to obtain solutions of a generalized dispersive water waves system. In addition, we derive the conservation laws of the underlying system. It is also worth mentioning that this is the first time that the conservation laws for the equation under study are derived.

Chapter 6

An extended $(2+1)$-dimensional coupled Burgers system in fluid mechanics: Symmetry reductions; Kudryashov method; Conservation laws

The Burgers-type equation [72, 104–112] arises in different fields of science such as in plasma astrophysics, ocean dynamics, atmospheric science, computational fluid mechanics, cosmology, condensed matter physics, statistical physics. In this research project, we investigate the following extended $(2+1)$-dimensional coupled Burgers system in fluid mechanics

$$u_t + \alpha(u_{xx} + u_{yy}) + \beta(uu_x + u_y v) + \gamma(uu_x + uv_y) = 0, \tag{6.1a}$$

$$v_t + \alpha(v_{xx} + v_{yy}) + \beta(uv_x + vv_y) + \gamma(u_x v + vv_y) = 0, \tag{6.1b}$$

where $u(x, y, t)$ and $v(x, y, t)$ are the velocity components in fluid related problems and α, β, γ are real constants.

The results of this chapter have been published in [113].

An exceptional case of system (6.1) is given by

$$u_t - \frac{1}{R_e}(u_{xx} + u_{yy}) + uu_x + u_y v = 0, \tag{6.2a}$$

$$v_t - \frac{1}{R_e}(v_{xx} + v_{yy}) + uv_x + vv_y = 0, \tag{6.2b}$$

which is similar to the incompressible Navier-Stokes equations without the pressure and continuity considerations [111]. This system is said to constitute an appropriate model for developing computational algorithms for solving the incompressible Navier-Stokes equations [111], which is a suitable test case, because the equation structure is similar to that of the incompressible fluid flow momentum equations [112]. This system is also used in models for the study of hydrodynamical turbulence and wave processes in nonlinear media [112]. Note that $u(x, y, t)$ and $v(x, y, t)$ are the velocity components in fluid-related problems [110, 111], R_e is the Reynolds number. In contrast, $\frac{1}{R_e}$ represents the viscosity [110], the total or material derivative including the convective term is used while the diffusive and convective terms are linked with R_e [111].

The nonlinear evolution equations, for instance such as the above system, perform a crucial part in studying nonlinear wave propagation [43, 57–73, 76–81]. Deriving the exact solutions of NLEEs is a complex goal, and only in certain remarkable cases can one extract the solutions. However tools of integrating these types of NLEEs have been investigated in recent years [8–10, 39, 89–96].

6.1 Symmetry reductions (6.1)

The symmetry generator of an extended (2+1)-dimensional coupled Burgers system in fluid mechanics (6.1) will be generated by the vector field

$$\begin{aligned} X \;=\; & \xi^1(t,x,y,u,v)\frac{\partial}{\partial t}+\xi^2(t,x,y,u,v)\frac{\partial}{\partial x}+\xi^3(t,x,y,u,v)\frac{\partial}{\partial y}+\eta^1(t,x,y,u,v)\frac{\partial}{\partial u} \\ & +\eta^2(t,x,y,u,v)\frac{\partial}{\partial v}, \end{aligned} \tag{6.3}$$

where

$$\mathbf{X}^{[\mathbf{2}]}\left\{\begin{array}{l} u_t+\alpha(u_{xx}+u_{yy})+\beta(uu_x+u_yv)+\gamma(uu_x+uv_y), \\ \\ v_t+\alpha(v_{xx}+v_{yy})+\beta(uv_x+vv_y)+\gamma(u_xv+vv_y) \end{array}\right\}\Bigg|_{(6.1)}=0. \tag{6.4}$$

Expanding the above equation and splitting on the derivatives of u and v, where $\mathbf{X}^{[\mathbf{2}]}$ is the second prolongation, leads to linear overdetermined system of partial differential equations. Solving the resulting system of equations, one obtains

$$\begin{aligned} \xi^1_t\,(t,x,y,u,v) &= -2tC_4+C_3, \\ \xi^2_x\,(t,x,y,u,v) &= -yC_1-xC_4+C_2, \\ \xi^3_y\,(t,x,y,u,v) &= xC_1-yC_4+C_5, \\ \eta^1_u\,(t,x,y,u,v) &= -vC_1+uC_4, \\ \eta^2_v\,(t,x,y,u,v) &= uC_1+vC_4. \end{aligned} \tag{6.5}$$

Thus, the infinitesimal symmetries of system (6.1) form the five-dimensional Lie algebra spanned by the following linearly independent generators:

$$X_1 = \frac{\partial}{\partial x},$$
$$X_2 = \frac{\partial}{\partial y},$$
$$X_3 = \frac{\partial}{\partial t},$$
$$X_4 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v},$$
$$X_5 = -2t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}.$$

6.1.1 Symmetry reductions of (6.1)

In order to construct symmetry reductions and exact solutions, we need to implement the associated Lagrange equations

$$\frac{dt}{\xi^1(t,x,y,u,v)} = \frac{dx}{\xi^2(t,x,y,u,v)} = \frac{dy}{\xi^3(t,x,y,u,v)} = \frac{du}{\eta^1(t,x,y,u,v)} = \frac{dv}{\eta^2(t,x,y,u,v)} \quad (6.6)$$

The linear combination of the translation symmetries $\Gamma = a_1X_1 + a_2X_2 + a_3X_3$, where a_1, a_2, a_3 are constants, reduces system (6.1) to a partial differential equation (PDE) in two independent variables. The symmetry Γ yields the following four invariants

$$f = a_2x - a_1y, \quad g = a_3x - a_1t, \quad h = u, \quad \phi = v.$$

Hence system (6.1) reduces to a PDE in two independent variables given below

$$\begin{cases} -a_1\phi_g + \alpha\left(a_2\left(a_2\phi_{ff} + a_3\phi_{fg}\right) + a_3\left(a_2\phi_{fg} + a_3\phi_{gg}\right) + a_1^2\phi_{ff}\right) + \beta a_2\phi\left(\phi_f + a_3\phi_g\right) \\ -\beta a_1\psi\phi_f + \gamma\left(\phi\left(a_2\phi_f + a_3\phi_g\right) - a_1\phi\psi_f\right) = 0, \\ -a_1\psi_g + \alpha\left(a_2\left(a_2\psi_{ff} + a_3\psi_{fg}\right) + a_3\left(a_2\psi_{fg} + a_3\psi_{gg}\right) + a_1^2\psi_{ff}\right) + \beta a_2\phi\left(\psi_f + a_3\psi_g\right) \\ -\beta a_1\psi\phi_f + \gamma\left(\psi\left(a_2\phi_f + a_3\phi_g\right) - a_1\psi\psi_f\right) = 0. \end{cases} \quad (6.7)$$

We now further reduce equation (6.7) using its symmetries. The above equation has the two translation symmetries, namely

$$\Upsilon_1 = \frac{\partial}{\partial f}, \quad \Upsilon_2 = \frac{\partial}{\partial g}.$$

Taking a linear combination $b_1\Upsilon_1 + b_2\Upsilon_2$ of the above symmetries, yields the invariants

$$z = b_2 f - b_1 g, \quad \phi = E, \quad \psi = F.$$

Now treating E and F as new dependent variables, z as the new independent variable, system (6.1) reduces to a system of nonlinear ordinary differential equations

$$\begin{cases} a_1b_1F' + \left(\left(a_2b_2^2 - a_3b_1b_2\right)a_2 - \left(a_2b_1b_2 - a_3b_1^2\right)a_3 + a_1^2b_2^2\right)\alpha F'' + \left(a_2b_2F - a_3b_1\right)\beta F' \\ -\beta a_1b_2GF' + \left(a_2b_2F - a_3b_1\right)\gamma F' - a_1b_2FG' = 0, \\ a_1b_1G' + \left(\left(a_2b_2^2 - a_3b_1b_2\right)a_2 - \left(a_2b_1b_2 - a_3b_1^2\right)a_3 + a_1^2b_2^2\right)\alpha G'' + \left(a_2b_2F - a_3b_1\right)\beta G' \\ -\beta a_1b_2GG' + \left(a_2b_2G - a_3b_1\right)\gamma F' - a_1b_2GG' = 0. \end{cases} \tag{6.8}$$

6.2 Exact solutions using Kudryashov method

The intention of this segment is to introduce the algorithm of the Kudryashov method for finding exact solutions of nonlinear evolution equations. The Kudryashov method was one of the original procedures for acquiring exact solutions of nonlinear partial differential equations. Let us briefly recall the basic steps of the Kudryashov method. Consider for instance a scalar nonlinear partial differential equation in the form

$$E_1[u_t, u_x, u_y, \cdots] = 0. \tag{6.9}$$

We use the following ansatz

$$u(x, y, t) = F(z), \quad z = k_1x + k_2y + k_3t + k_4. \tag{6.10}$$

From equation (6.9), we obtain the nonlinear ordinary differential equation

$$E_2[k_1F'(z), k_2F'(z), k_3F'(z), k_1^2F''(z), k_2^2F''(z), k_3^2F''(z), \cdots] = 0, \tag{6.11}$$

which has a solution of the form

$$F(z) = \sum_{i=0}^{M} A_i(H(z))^i, \tag{6.12}$$

where

$$H(z) = a\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\},$$

satisfies the equation

$$H'(z) = aH(z) + bH^2(z), \tag{6.13}$$

M is a positive integer and $A_0, \cdots, A_M$ are parameters to be determined.

6.2.1 Application of the Kudryashov method

This subsection aims to apply Kudryashov method to derive solutions of equation (6.8) in the form

$$F(z) = \sum_{i=0}^{M} A_i H(z)^i, \tag{6.14a}$$

$$G(z) = \sum_{i=0}^{M} B_i H(z)^i. \tag{6.14b}$$

Reverting back to our underlying system (6.1), the solution structure takes the form

$$u(x,y,t) = \sum_{i=0}^{M} A_i\left(a\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}\right)^i, \tag{6.15}$$

$$v(x,y,t) = \sum_{i=0}^{M} B_i\left(a\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}\right)^i, \tag{6.16}$$

$$z = b_2 (a_2 x - a_1 y) - b_1 (-a_1 t + a_3 x).$$

The above solution structures are attained by replacing equation (6.15) into equation (6.8) and making use of equation (6.13), and then equating all coefficients of the functions H^i to zero. Thus, by solving the resulting system of algebraic equations leads to the following possible solutions with their associated cases, summarized in tables (6.1)–(6.4).

Table 6.1: Solution of equation (6.1) via Kudryashov method

Case	Solutions
1: M = 1	$u(x,y,t) = \sum_{i=0}^{1} A_i \left(a \left\{ \frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]} \right\} \right)^i,$
	$v(x,y,t) = \sum_{i=0}^{1} B_i \left(a \left\{ \frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]} \right\} \right)^i,$
	where $z = b_2(a_2x - a_1y) - b_1(-a_1t + a_3x)$, $\beta = -\frac{2\alpha b a_1^2 b_2^2 + 2\alpha b a_2^2 b_2^2 - 4\alpha b a_2 a_3 b_1 b_2 + 2\alpha b a_3^2 b_1^2 + \gamma A_1 a_2 b_2 - \gamma A_1 a_3 b_1}{A_1(a_2b_2 - a_3b_1)},$
	$A_0 = \frac{A_1\left(a\alpha a_1^2 b_2^2 + a\alpha a_2^2 b_2^2 - 2a\alpha a_2 a_3 b_1 b_2 + a\alpha a_3^2 b_1^2 + a_1 b_1\right)}{2\alpha b\left(a_1^2 b_2^2 + a_2^2 b_2^2 - 2a_2 a_3 b_1 b_2 + a_3^2 b_1^2\right)},\ B_0 = 0,\ B_1 = 0\ ,\ b = \frac{A_1\left(a\alpha a_1^2 b_2^2 + a\alpha a_2^2 b_2^2 - 2a\alpha a_2 a_3 b_1 b_2 + a\alpha a_3^2 b_1^2 + a_1 b_1\right)}{2\alpha A_0\left(a_1^2 b_2^2 + a_2^2 b_2^2 - 2a_2 a_3 b_1 b_2 + a_3^2 b_1^2\right)},$
	$B_0 = \frac{a\alpha a_1^2 b_2^2 + a\alpha a_2^2 b_2^2 - 2a\alpha a_2 a_3 b_1 b_2 + a\alpha a_3^2 b_1^2 + \beta A_0 a_2 b_2 - \beta A_0 a_3 b_1 + \gamma A_0 a_2 b_2}{(\gamma+\beta) a_1 b_2} - \frac{\gamma A_0 a_3 b_1 + a_1 b_1}{(\gamma+\beta) a_1 b_2},$
	$B_1 = \frac{A_1\left(a\alpha a_1^2 b_2^2 + a\alpha a_2^2 b_2^2 - 2a\alpha a_2 a_3 b_1 b_2 + a\alpha a_3^2 b_1^2 + \beta A_0 a_2 b_2 - \beta A_0 a_3 b_1\right)}{A_0 a_1 b_2(\gamma+\beta)} + \frac{A_1(\gamma A_0 a_2 b_2 - \gamma A_0 a_3 b_1 + a_1 b_1)}{A_0 a_1 b_2(\gamma+\beta)},$
and	
	$\alpha = -\frac{a_1 b_1}{a\left(a_1^2 b_2^2 + a_2^2 b_2^2 - 2a_2 a_3 b_1 b_2 + a_3^2 b_1^2\right)},\ A_0 = 0,\ B_0 = 0,\ B_1 = \frac{a\beta A_1 a_2 b_2 - a\beta A_1 a_3 b_1 + a\gamma A_1 a_2 b_2 - a\gamma A_1 a_3 b_1 - 2b a_1 b_1}{a a_1 b_2(\gamma+\beta)}$, respectively.

Table 6.2: Solution of equation (6.1) via Kudryashov method (Continued)

Case	Solutions
2: M = 3	$u(x,y,t)=\sum_{i=0}^{3} A_i\left(a\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\}\right)^i,$ $v(x,y,t)=\sum_{i=0}^{3} B_i\left(a\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\}\right)^i,$ $\beta=-\frac{5\gamma}{3},\ A_0=\frac{aA_1}{3b},\ A_2=\frac{3aA_3}{b},\ B_0=\frac{aB_1}{3b},\ B_2=0,\ B_3=0,\ a_1=-\frac{B_1\gamma}{3\alpha bb_2},\ a_2=\frac{a_3a\gamma B_1}{9b},\ b_1=\frac{a\gamma B_1b_2}{9b},$
and	$\alpha=-\frac{3a_1b_1}{a\left(a_1^2b_2^2+a_2^2b_2^2-2a_2a_3b_1b_2+a_3^2b_1^2\right)},\ \beta=-\frac{5\gamma}{3},\ A_0=\frac{aA_1}{3b},\ A_2=\frac{3aA_3}{b},\ B_0=\frac{a\gamma A_1a_2b_2-a\gamma A_1a_3b_1+9ba_1b_1}{3b_2a_1\gamma b},$ $B_1=\frac{a\gamma A_1a_2b_2-a\gamma A_1a_3b_1+9ba_1b_1}{\gamma a_1ab_2},\ B_2=\frac{3aA_3(a_2b_2-a_3b_1)}{ba_1b_2},\ B_3=\frac{A_3(a_2b_2-a_3b_1)}{a_1b_2},$ respectively.
3: M = 4	$u(x,y,t)=\sum_{i=0}^{4} A_i\left(a\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\}\right)^i,$ $v(x,y,t)=\sum_{i=0}^{4} B_i\left(a\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\}\right)^i,$ $\beta=-\frac{3\gamma}{2},\ A_0=\frac{aA_1}{4b},\ A_2=\frac{6a^2A_4}{b^2},\ A_3=\frac{4aA_4}{b},\ B_0=\frac{aB_1}{4b},\ B_2=0,\ B_3=0,\ B_4=0,\ a_1=-\frac{B_1\gamma}{4\alpha bb_2},\ a_2=\frac{a_3a\gamma B_1}{8b},$ $b_1=\frac{a\gamma B_1b_2}{8b},$
and	$\alpha=-\frac{2a_1b_1}{a\left(a_1^2b_2^2+a_2^2b_2^2-2a_2a_3b_1b_2+a_3^2b_1^2\right)},\ \beta=-\frac{3\gamma}{2},\ A_0=\frac{aA_1}{4b},\ A_3=\frac{2A_2b}{3a},\ A_4=\frac{b^2A_2}{6a^2},\ B_0=\frac{a\gamma A_1a_2b_2-a\gamma A_1a_3b_1+8ba_1b_1}{4b_2a_1\gamma b},$ $B_1=\frac{a\gamma A_1a_2b_2-a\gamma A_1a_3b_1+8ba_1b_1}{\gamma b_2a_1a},\ B_2=\frac{A_2(a_2b_2-a_3b_1)}{a_1b_2},\ B_3=\frac{2A_2b(a_2b_2-a_3b_1)}{3aa_1b_2},\ B_4=\frac{b^2A_2(a_2b_2-a_3b_1)}{6a^2a_1b_2},$ respectively.

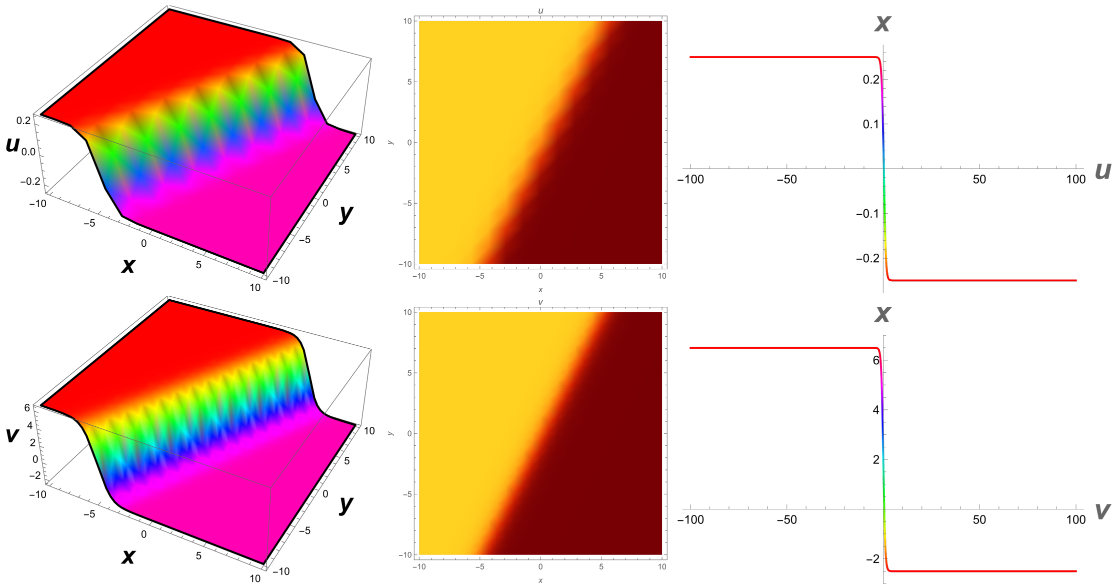

Figure 6.1: Evolution of the travelling wave solutions of table (6.2) case 3.

Table 6.3: Solution of equation (6.1) via Kudryashov method (Continued)

Case	Solutions
4: M = 5	$u(x,y,t)=\sum_{i=0}^{5} A_i\left(a\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\}\right)^i$,
	$v(x,y,t)=\sum_{i=0}^{5} B_i\left(a\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\}\right)^i$,
	$\beta=-\frac{7\gamma}{5},\ A_0=\frac{aA_1}{5b},\ A_1=A_1,\ A_2=\frac{10a^3A_5}{b^3},\ A_3=\frac{10a^2A_5}{b^2},\ A_4=\frac{5aA_5}{b},\ A_5=A_5,\ B_0=\frac{aB_1}{5b},\ B_2=0,\ B_3=0,$
	$B_4=0,\ B_5=0,\ a_1=-\frac{B_1\gamma}{5\alpha bb_2},\ a_2=\frac{3a_3a\gamma B_1}{25b},\ a_3=a_3,\ b_1=\frac{3a\gamma B_1b_2}{25b},\ b_2=b_2,$
and	
	$\alpha=-\frac{5a_1b_1}{3a\left(a_1^2b_2^2+a_2^2b_2^2-2a_2a_3b_1b_2+a_3^2b_1^2\right)},\ \beta=-\frac{7\gamma}{5},\ A_0=\frac{aA_1}{5b},\ A_3=\frac{bA_2}{a},\ A_4=\frac{b^2A_2}{2a^2},\ A_5=\frac{A_2b^3}{10a^3},$
	$B_0=\frac{3a\gamma A_1a_2b_2-3a\gamma A_1a_3b_1+25ba_1b_1}{15b_2a_1b\gamma},\ B_1=\frac{3a\gamma A_1a_2b_2-3a\gamma A_1a_3b_1+25ba_1b_1}{3\gamma b_2a_1a},\ B_2=\frac{A_2(a_2b_2-a_3b_1)}{a_1b_2},\ B_3=\frac{bA_2(a_2b_2-a_3b_1)}{aa_1b_2},$
	$B_4=\frac{b^2A_2(a_2b_2-a_3b_1)}{2a^2a_1b_2},\ B_5=\frac{A_2b^3(a_2b_2-a_3b_1)}{10a^3a_1b_2}$, respectively.

Table 6.4: Solution of equation (6.1) via Kudryashov method (Continued)

Case	Solutions
5: M = 6	$u(x,y,t)=\sum_{i=0}^{6} A_i\left(a\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\}\right)^i,$
	$v(x,y,t)=\sum_{i=0}^{6} B_i\left(a\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\}\right)^i,$
	$\beta=-\frac{4\gamma}{3},\ A_0=\frac{aA_1}{6b},\ A_2=\frac{15a^4A_6}{b^4},\ A_3=\frac{20a^3A_6}{b^3},\ A_4=\frac{15a^2A_6}{b^2},\ A_5=\frac{6aA_6}{b},\ B_0=\frac{aB_1}{6b},\ B_2=0,\ B_3=0,$
	$B_4=0,\ B_5=0,\ B_6=0,\ a_1=-\frac{B_1\gamma}{6\alpha bb_2},\ a_2=\frac{a_3\gamma B_1a}{9b},\ b_1=\frac{\gamma B_1b_2a}{9b},$
and	
	$\alpha=-\frac{3a_1b_1}{2a\left(a_1^2b_2^2+a_2^2b_2^2-2a_2a_3b_1b_2+a_3^2b_1^2\right)},\ \beta=-\frac{4\gamma}{3},\ A_0=\frac{aA_1}{6b},\ A_3=\frac{4bA_2}{3a},\ A_4=\frac{b^2A_2}{a^2},\ A_5=\frac{2A_2b^3}{5a^3},\ A_6=\frac{b^4A_2}{15a^4},$
	$B_0=\frac{a\gamma A_1a_2b_2-a\gamma A_1a_3b_1+9ba_1b_1}{6b_2a_1b\gamma},\ B_1=\frac{a\gamma A_1a_2b_2-a\gamma A_1a_3b_1+9ba_1b_1}{\gamma b_2a_1a},\ B_2=\frac{A_2(a_2b_2-a_3b_1)}{a_1b_2},\ B_3=\frac{4bA_2(a_2b_2-a_3b_1)}{3aa_1b_2},$
	$B_4=\frac{b^2A_2(a_2b_2-a_3b_1)}{a^2a_1b_2},\ B_5=\frac{2A_2b^3(a_2b_2-a_3b_1)}{5a^3a_1b_2},\ B_6=\frac{b^4A_2(a_2b_2-a_3b_1)}{15a^4a_1b_2}$, respectively.

6.3 Conservation laws

A conservation law of system (6.1) is a total space-time divergence expression that vanishes on the solution space ε of system (6.1),

$$D_tT^t + D_xT^x + D_yT^y|_{\varepsilon} = 0, \tag{6.17}$$

where D_t, D_x and D_y are the total derivative operators while T^t is a conserved density and T^x, T^y are the spatial fluxes. To determine the conservation law for system (6.1), we will implement the multiplier method. Since the joint Euler operator annihilates the total divergence, we get

$$\frac{\delta}{\delta u}\Big((u_t + \alpha(u_{xx} + u_{yy}) + \beta(uu_x + u_yv) + \gamma(uu_x + uv_y))\Lambda_1 + (v_t + \alpha(v_{xx} + v_{yy}) + \beta(uv_x + vv_y) + \gamma(u_xv + vv_y))\Lambda_2\Big) = 0, \tag{6.18}$$

$$\frac{\delta}{\delta v}\Big((u_t + \alpha(u_{xx} + u_{yy}) + \beta(uu_x + u_yv) + \gamma(uu_x + uv_y))\Lambda_1 + (v_t + \alpha(v_{xx} + v_{yy}) + \beta(uv_x + vv_y) + \gamma(u_xv + vv_y))\Lambda_2\Big) = 0, \tag{6.19}$$

where Λ_1 and Λ_2 are the second-order multipliers to be determined. The expansion of equations (6.18) – (6.19) and thereafter splitting and simplifying, yields the following multipliers

$$\Lambda_1 = yc_1 + c_2, \tag{6.20}$$

$$\Lambda_2 = -xc_1 + c_3, \tag{6.21}$$

where $\beta = \gamma$ and c_i, $(i = 1, 2, 3)$ are arbitrary constants. The multipliers Λ_1 and Λ_2 of system (6.1) have the property

$$D_tT^t + D_xT^x + D_yT^y = (u_t + \alpha(u_{xx} + u_{yy}) + \beta(uu_x + u_yv) + \gamma(uu_x + uv_y))\Lambda_1 + (v_t + \alpha(v_{xx} + v_{yy}) + \beta(uv_x + vv_y) + \gamma(u_xv + vv_y))\Lambda_2, \tag{6.22}$$

for the arbitrary functions $u(t,x,y)$ and $v(t,x,y)$, where the predetermined arguments of T^t , T^x and T^y are of some order in derivatives of the field variables u, v and w. The computations for T^t, T^x and T^y from equation (6.22) reveal that corresponding to the above multipliers we have the following conserved vectors for system (6.1):

$$
\begin{aligned}
T_1^t &= uy - vx,\\
T_1^x &= \gamma u^2 y - \gamma uvx + \alpha u_x y - \alpha v_x x + \alpha v,\\
T_1^y &= \gamma uvy - \gamma v^2 x + \alpha u_y y - \alpha v_y x - \alpha u;
\end{aligned}
$$

$$
\begin{aligned}
T_2^t &= u,\\
T_2^x &= \gamma u^2 + \alpha u_x,\\
T_2^y &= \gamma uv + \alpha u_y;
\end{aligned}
$$

$$
\begin{aligned}
T_3^t &= v,\\
T_3^x &= \gamma uv + \alpha v_x,\\
T_3^x &= \gamma v^2 + \alpha v_y.
\end{aligned}
$$

We succinctly discuss the significance and physical illumination that ascend from the computed conservation laws. Conservation laws reside in enormously crucial areas both at the foundations of nonlinear science and in its applications.

Mathematical expressions of physical laws, such as conservation of energy, momentum and mass are fundamentally conservation laws. Imperative physical information about the complex behaviour in non-linear systems is confined in conservation laws.

6.4 Concluding remarks

This work was concerned with Burgers-type equations that appeared in plasma astrophysics, ocean dynamics, atmospheric science, computational fluid mechanics, cosmology, condensed matter physics, statistical physics, nonlinear acoustics, vehicular traffic and electronic transport. We determined novel type exact solutions by the Lie symmetry method in conjunction with Kurdyshov method. Finally, conservation laws of the above mentioned system were generated. These new research findings can well mimic complex waves and their dynamics in fluids. Some diverse interaction phenomena which have great implication to the nonlinear waves in fluid mechanics have been shown graphically.

Chapter 7

Conclusions

In this research project the aim was to construct exact solutions and derive conservation laws of some nonlinear partial differential equations using various methods. In chapter one we gave some important definitions and results from Lie group theory and conservation laws, which were later used to carry out the calculations in this thesis.

In chapter two we constructed solutions of a first $(3+1)$-dimensional KP-like equation, which arises in the analysis of versions of resonant phenomena. In addition, we derived the conserved vectors of the underlying equation. It is shown that it is possible that one equation can have identical multipliers, although their orders are distinct and hence resulting in identical conservation laws. We have excluded the zeroth order multiplier, as it is distinct from the first and the second-order multipliers. It remains to be investigated whether all the multipliers of a $(3 + 1)$-dimensional KP-like equation are identical or not. Furthermore, future work, will be devoted to studying the ansatz methods mentioned in [50, 51, 53] and applying them to the underlying equation.

In chapter three we obtained topological soliton solutions and periodic solutions of a second (3+1)-dimensional KP like equation. In addition to this, other analytical solutions that were based on the Lie symmetry method were attained. Furthermore, conservation laws of the aforesaid equation were derived by using the multiplier method. The correctness of the obtained solutions have been verified with Maple software package by back substitution. It is anticipated that the solutions obtained here could be used as benchmarks against numerical simulations.

Chapter four dealt with exact solutions by utilizing the multiple exp-function algorithm and the modern group analysis method. Finally, we computed conserved currents using the invariance and multiplier technique. These research findings can well mimic complex waves and their dynamics in fluids.

Chapter five was devoted to similarity reductions and new exact solutions of a generalized dispersive water waves system. The integration of a generalized dispersive water waves system was performed using the Lie group analysis method. The solutions obtained include the solitary waves and the travelling wave solutions. Time and space symmetries resulted in novel similarity reduction and new exact solutions. The solutions obtained include solitary waves. The familiarity of closed-form solutions of nonlinear ordinary and partial differential equations enables numerical solvers and supports stability analysis. Although many efforts have been dedicated to solving nonlinear evolution equations, there is no unified method. To the best of our knowledge, this was the first time that Lie point symmetry analysis in conjunction with an ansatz method has been applied to this underlying equation. In addition, we derived the conservation laws of the underlying system. It is also worth mentioning that this was the first time that the conservation laws for the equation under study were derived.

In chapter six we determined novel type exact solutions by the Lie symmetry method in conjunction with Kurdyashov methodv for an extended (2+1)-dimensional coupled Burgers system in fluid mechanics. Finally, conservation laws of the above-mentioned system were generated. These new research findings can well mimic complex waves and their dynamics in fluids. Some diverse interaction phenomena were shown graphically. In this research project, the methods used to obtain new exact solutions can be extended to solve other nonlinear partial differential equations of physical interest.

In future, the computed conserved quantities obtained in this study will be utilized for the construction of closed form solutions for the corresponding nonlinear partial differential equations studied in this research project.

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