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A Bayesian generalized Eyring-Weibull accelerated life testing model

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Abstract
In this paper, a novel approach to a Bayesian accelerated life testing model is presented. The Weibull distribution is used as the life distribution and the generalized Eyring model as the time transformation function. This is a model that allows for the use of more than one stressor, whereas other commonly used acceleration models, such as the Arrhenius and power law models, incorporate one stressor. The use of the generalized Eyring-Weibull model developed in this paper is demonstrated in a case study, where Markov chain Monte Carlo methods are utilized to generate samples for posterior inference.

KEYWORDS
accelerated life testing, Bayes, generalized Eyring model, Markov chain Monte Carlo, Weibull distribution

1 | INTRODUCTION

Reliability and life testing are vitally important in the manufacturing of consumer and capital goods, particularly in the engineering fields. It is crucial to determine whether a product will perform its prescribed function, without failure, for a desired period of time. Reliability and life testing provide a theoretical as well as practical framework by which the life characteristics of a product, component or system can be quantified. An extensive discussion on the objectives of, and need for, reliability and life testing can be found in Kececioglu.[1]

In this day and age, manufacturers are pressured to provide durable products, timely and at competitive prices. Performing life tests on high-reliability materials, components and systems can be a tedious and unproductive task. Many modern products are made to endure for years or even decades. Obtaining sufficient failure data for these products at normal use conditions can be very costly and time consuming. This longevity obstacle has led to a rising interest in reliability engineering and the development of accelerated life testing (ALT). In ALT, products are tested in a more severe than normal use environment, by applying or fluctuating one or more stressors (stress variables) at accelerated levels, in order to induce early failures.[2] This failure data is then extrapolated to predict the reliability characteristics of the products at normal use conditions.[3]

Stressors can include temperature, pressure, voltage, wattage, humidity, loads, vibration amplitude, use rate, and so forth.[1,4] Appertaining to the understanding of the physics of failure, a functional relationship, known as a time transformation function (TTF), is assumed between the stressors and the parameters of the life distribution (see, e.g., refs. [5–8]).

The most commonly used models for acceleration include the Arrhenius, inverse power law, and Eyring models. Various other acceleration models are discussed in Escobar and Meeker,[4] Kececioglu,[1] and Thiraviam.[9]

In this paper, we develop a new Bayesian ALT model that can accommodate multiple stressors, where lifetimes are modeled with the Weibull distribution. Literature on multiple-stressor models is limited and largely restricted to the frequentist paradigm. The importance of multiple-stressor models is discussed in Section 1.2 and a motivation for using the Bayesian paradigm is provided in Section 1.3. The Bayesian approach often results in complicated and mathematically intractable models. However, recent developments in Markov chain Monte Carlo (MCMC) methods and Bayesian analysis software now allows one to draw inference from such models. Using advanced MCMC methods to draw posterior samples, we are able to estimate the model parameters as well as the predictive reliability and mean time to failure (MTTF) at use stress levels, as illustrated in the case study.

1.1 | Bayesian accelerated life testing

The development of Bayesian methods to draw inference from ALT models has experienced a surge in recent years. Ahmad[10] provides a general overview of early Bayesian methods in ALT. These methods include reducing high dimensional integration problems via semi-sufficient statistics, the use of semi-parametric inference, a Kalman filter approach, and failure rate strategies. Achcar and Louzada-Neto[11] consider an exponential distribution with the Eyring model. Type-II censored data are used and Jeffreys' priors are employed for the model parameters, where Laplace approximations are utilized for integrals that are difficult to solve analytically. Chaloner and Larntz[12] examine experimental Bayesian design for ALT models, where lifetimes follow either a Weibull or a log-normal distribution. A dynamic general linear model setup for ALT models, that uses linear Bayesian methods for inference, is presented in Mazzuchi and Soyer.[13] Lifetimes are assumed to be exponential with the power law as the TTF. Achcar[14] explores Laplace approximation methods for deriving posterior densities for various commonly used life distributions and TTFs.

Van Dorp et al.[15] develop a Bayesian model for step-stress ALT, where the lifetimes are exponentially distributed, and provide Bayesian point estimates and credibility intervals for parameters at normal use conditions. Dietrich and Mazzuchi[16] discuss the design of experiments in ALT, pointing out some problems that emerge from typical regression and ANOVA techniques, and propose an alternative analysis procedure based on Bayesian considerations. Mazzuchi et al.[17] present a Bayesian approach, based on general linear models, for inference from ALT models. The lifetimes are assumed to be Weibull distributed and the power law is used as TTF on the scale parameter. Erkanli and Soyer[18] provide simulation-based designs for ALT, using a Bayesian decision theory approach, where lifetimes follow an exponential distribution with the power law as TTF. Perdona and Louzada-Neto[19] propose an ALT model where lifetimes are exponential and a general log-non-linear TTF is used. A uniform prior is used and Laplace approximations are utilized to find marginal posterior distributions.

Van Dorp and Mazzuchi[20] develop a general Bayes inference model for ALT, which is compatible with various types of stress loadings, where the lifetimes are exponentially distributed and strict conformity to a specific TTF is not compulsory. MCMC methods are used for inference and to derive posterior quantities. This model is also used in Van Dorp et al.[21] to compare constant stress, step-stress and profile stress ALT models within a single Bayesian inferential framework, and is extended to the Weibull distribution in Van Dorp and Mazzuchi.[22] León et al.[23] present Bayesian inferences from ALT models, making use of MCMC methods, where items originate from different groups with random effects. Although the procedure is applied to an elementary example, it can also be used with multiple random effects and acceleration factors. Barriga et al.[24] consider Bayesian methods for ALT where the exponentiated Weibull is used as the life distribution and the Arrhenius model is the TTF. MCMC methods are used to sample from the posterior for inference, where a mixture of normal and gamma priors are imposed on the parameters.

Soyer[25] reviews Bayesian designs for ALT by comparing Bayesian decision theory, linear Bayesian, and Bayesian simulation-based designs. An exponential life distribution with the power law as TTF is used throughout the review. A basic parametric Bayesian ALT model, where lifetimes follow a Weibull distribution with the power law as TTF, is discussed in Soyer et al.[26] and then extended to allow for a more dynamic TTF. The two extensions are a hierarchical Bayesian model and a Markov model where inferences are made via MCMC methods. Pan[3] provides a Bayesian approach, and introduces a calibration factor, which allows for a combination of field data and ALT data to be used for reliability prediction. A log-linear TTF is used with the exponential and Weibull life distributions, and MCMC methods are employed for intricate posterior distributions. Upadhyay and Mukherjee[27] present a Bayesian comparison between accelerated Weibull and

Birnbaum-Saunders models, where the TTF is the inverse power law. Independent vague priors for the model parameters are chosen and MCMC techniques are used to generate posterior samples.

Yuan et al.[28] propose a semi-parametric Bayesian approach for ALT where the TTF is log-linear and no assumption is made on the form of the life distribution. The authors employ a Dirichlet process mixture model, using a Weibull kernel, to model failure times. Model fitting is performed via a simulation-based algorithm that incorporates Gibbs sampling. Mukhopadhyay and Roy[29] consider Bayesian analysis for ALT where log-lifetimes follow a distribution from the log-concave log-location-scale family and a linear TTF is used. After a general discussion, the article focuses on using the log-normal life distribution with the Arrhenius model as TTF. MCMC techniques are used to obtain posterior samples for inference. Polson and Soyer[30] propose a Bayesian decision theory approach to select an optimal stress-testing schedule. A simulation-based approach is used to simultaneously calculate the pre-posterior expected utility and find the optimal design. In Sha,[31] classical and Bayesian inference on a progressive-stress ALT model with a generalized Birnbaum-Saunders life distribution is presented. MCMC methods are used for Bayesian analysis and the author states that the proposed Bayesian method is not only efficient and accurate, but outperforms the classical likelihood-based approach to ALT.

1.2 | The importance of multiple-stressor models

From the Bayesian ALT literature, it is clear that there is a lack of multiple-stressor ALT models. The main reason for this is that the models that incorporate more than one stressor usually become very complicated and inference is not straightforward. However, multiple-stress models are often more applicable in industry, as pointed out by Nelson[2] and discussed next.

The normal use environment for many products can include various stressors and it may be necessary to investigate the interactions between these stressors. Some multiple-stressor ALT models, such as Black's model and Peck' s model, do not include an interaction term for the stressors (see, e.g., ref. [2]). Black' s model has been applied where microelectronic failures due to electromigration is explored and includes temperature and current density as the accelerated stressors. Peck' s model incorporates temperature and humidity as stressors and has been used to study electronic failures due to the failure of epoxy packaging. In many cases, the stressors may exhibit interactions, which require multiple-stressor ALT models such as the generalized Eyring model or Zhurkov's model (see, e.g., ref. [2]). An electronic device, for example, may be simultaneously affected by stressors such as temperature, voltage, and current while operating, which are known to exhibit interactions. The generalized Eyring model has been applied in a range of ALT experiments, since it incorporates temperature and a non-thermal stressor, such as humidity, voltage or current. Zhurkov' s model has been used for investigating the rupture of solids, where temperature and tensile stress are used as stressors. It should be noted that Zhurkov' s model, Black' s model and Peck' s model are special cases of the generalized Eyring model.

For some ALT experiments, a single stress can not be accelerated enough to induce sufficient failures within the time constraints of the study. This can occur, for example, when the testing equipment can not handle extreme levels of a stressor without failing. Accelerating one stressor too much could also result in the product failing via another failure mode which is not under investigation for the study. For such cases, additional stressors can be used to accelerate failures, without having to accelerate one stressor to extreme levels.

Many products have multiple failure modes while operating under their normal use conditions and a researcher may want to investigate these different failure modes. A single stressor might only accelerate some of these failure modes, while using various combinations of multiple stressors could explore a wider range of failure modes.

1.3 | A Bayesian dual-stress ALT model

There are also several motivations to explore multiple-stressor ALT models specifically from a Bayesian perspective. First, data sets from ALT experiments are usually small and Bayesian models tend to handle small sample sizes better than frequentist models. Second, the advances in MCMC techniques allow for Bayesian inference for many ALT models that may have mathematically intractable posterior distributions. Third, within the frequentist setup there are certain requirements for parameter estimation, which is not the case under the Bayesian paradigm. For example, when using the generalized Eyring TTF in an ALT model, the frequentist approach requires that there are at least four different accelerated levels of the stressors. For most life distributions, there are also requirements on the number of failures that occur. Fourth, the

Bayesian paradigm allows for the inclusion of the expert beliefs of engineers regarding the parameters of the ALT model. This can, in some cases, be achieved via the use of subjective priors that incorporate this prior knowledge.

This leads to the motivation for this paper, where a Bayesian approach to the generalized Eyring model with the Weibull distribution as life distribution is explored. The Weibull distribution is used as the life distribution due to its flexibility and applicability in the engineering field. The generalized Eyring model allows for an ALT model with multiple stressors. We will investigate the dual-stress model, while the methodology can be easily extended to include more stressors via an adjustment to the generalized Eyring TTF. The generalized Eyring TTF incorporates one thermal stressor, usually temperature, and one non-thermal stressor, such as humidity, voltage or current. The generalized Eyring-Weibull (GEW) model is an intricate model due to the number of unknown parameters, which complicates both classical and Bayesian inferences. However, the use of MCMC techniques makes it possible to work with complex models in the Bayesian setup.

The layout for the remainder of the paper is as follows. The GEW model and the general likelihood formulation for this model is given in Section 2. Due to the mathematically intractable posterior distribution, the log-concavity of the GEW model is assessed in the appendix to determine which MCMC methods can be utilized for posterior sampling. In Section 3, the GEW model is applied to a data set as part of a case study. In this case study, we also demonstrate how to incorporate expert opinions via the hyperparameter choices. The paper concludes in Section 4 with some final remarks on the GEW model.

2 | THE GENERALIZED EYRING-WEIBULL MODEL

Let X be a continuous random variable that follows a Weibull distribution with scale parameter α and shape parameter β ($\alpha > 0, \beta > 0$). The probability density function (PDF) of X is given by

$$f(x|\alpha,\beta) = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1} \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right], \quad x \geq 0, \tag{2.1}$$

and the reliability function by

$$R(x) = 1 - F(x) = \exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right]. \tag{2.2}$$

Consider two stressors, one thermal and one non-thermal. Let there be k distinct accelerated levels of the stressors, denoted by $\{T_i, S_i\}, i = 1, \dots, k$, where $T_i, i = 1, \dots, k$, are the accelerated levels of the thermal stressor and $S_i, i = 1, \dots, k$, are the accelerated levels of the non-thermal stressor. During the ALT experiment, an item is exposed to the constant application of one of these k distinct stress-level combinations. A common assumption in the literature is that the Weibull scale parameter α is dependent on the stress levels, whereas the shape parameter β is not (see, e.g., refs. [17, 26, 27]). The reparameterization of α given by the generalized Eyring model is

$$\alpha_i = \frac{1}{T_i} \exp\left(\theta_1 + \frac{\theta_2}{T_i} + \theta_3 V_i + \frac{\theta_4 V_i}{T_i}\right), \tag{2.3}$$

where θ_1, θ_2, θ_3, and θ_4 are unknown model parameters,[32] and V_i is a function of the non-thermal stressor S_i.[4]

For a lifetime subjected to the i^{th} level of the stressors, it follows from (2.1) and (2.3) that the Weibull PDF can be written as

$$\begin{aligned} f(x_i|\theta_1,\theta_2,\theta_3,\theta_4,\beta) = {} & \left[T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\right]^{\beta} \beta x_i^{\beta-1} \\ & \times \exp\left\{-\left[x_i T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\right]^{\beta}\right\}. \end{aligned} \tag{2.4}$$

From (2.2) and (2.3), the Weibull reliability function at some time τ can be written as

$$R(\tau) = \exp\left\{ -\left[\tau T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\right]^{\beta} \right\}. \tag{2.5}$$

Suppose that n_i items are tested at each of the k different stress levels and the test is terminated at time τ_i. Denote the failure times by $x_{ij}, j = 1, \dots, n_i, i = 1, \dots, k$. The likelihood function, in general, is then given by

$$L(\underline{x}|\theta_1, \theta_2, \theta_3, \theta_4, \beta) = \prod_{i=1}^{k} \left[\prod_{j=1}^{r_i} f(x_{ij}|\theta_1, \theta_2, \theta_3, \theta_4, \beta) \right] [R(\tau_i)]^{n_i - r_i}.$$

Note that for complete samples $r_i = n_i$. For type-I censoring, r_i is the number of failures that occur before censoring time τ_i, where $\tau_i < \infty, i = 1, \dots, k$ are predetermined censoring times for the k different stress levels. For type-II censoring, $\tau_i = x_{i(r_i)}$, where $x_{i(r_i)}$ is the r_i^{th} ordered failure time and $r_i, i = 1, \dots, k$ are the pre-chosen number of failures after which censoring occurs for the k different stress levels. From (2.4) and (2.5), the likelihood function for the GEW model is then given by

$$L(\underline{x}|\theta_1, \theta_2, \theta_3, \theta_4, \beta) = \beta^{\sum_{i=1}^{k} r_i} \exp\left(-\theta_1 \beta \sum_{i=1}^{k} r_i - \theta_2 \beta \sum_{i=1}^{k} \frac{r_i}{T_i} - \theta_3 \beta \sum_{i=1}^{k} r_i V_i - \theta_4 \beta \sum_{i=1}^{k} \frac{r_i V_i}{T_i} \right)$$

$$\begin{aligned} &\times \exp\left\{ -\sum_{i=1}^{k} (n_i - r_i) \left[\tau_i T_i \exp\left(-\theta_1 - \tfrac{\theta_2}{T_i} - \theta_3 V_i - \tfrac{\theta_4 V_i}{T_i}\right)\right]^{\beta} \right\} \\ &\times \exp\left\{ -\sum_{i=1}^{k} \sum_{j=1}^{r_i} \left[x_{ij} T_i \exp\left(-\theta_1 - \tfrac{\theta_2}{T_i} - \theta_3 V_i - \tfrac{\theta_4 V_i}{T_i}\right)\right]^{\beta} \right\} \left[\prod_{i=1}^{k} \prod_{j=1}^{r_i} T_i^{\beta} x_{ij}^{\beta - 1} \right]. \end{aligned} \tag{2.6}$$

Assume that the priors on the unknown parameters θ_1, θ_2,θ_3, θ_4 and β are independent (see, e.g., refs. [26, 27, 33]), and given by

$$\pi(\theta_1, \theta_2, \theta_3, \theta_4, \beta) = \pi(\theta_1)\pi(\theta_2)\pi(\theta_3)\pi(\theta_4)\pi(\beta).$$

The joint posterior distribution is then given by

$$\pi(\theta_1, \theta_2, \theta_3, \theta_4, \beta|\underline{x}) \propto L(\underline{x}|\theta_1, \theta_2, \theta_3, \theta_4, \beta)\pi(\theta_1, \theta_2, \theta_3, \theta_4, \beta).$$

Various prior distributions have been considered for the Weibull distribution. Soland[34] affirms that there is no conjugate family of continuous joint prior distributions for the two-parameter Weibull distribution, and proposes a gamma prior for the scale parameter and a discrete prior for the shape parameter. Tsokos[35] suggests using an inverse gamma prior for the scale parameter and a uniform prior for the shape parameter. A compound inverse gamma prior on the scale parameter, with either a discrete, inverse gamma, or uniform prior on the shape parameter is proposed in Papadopoulos and Tsokos.[36] Kundu[37] assumes a gamma prior for the scale parameter and a non-specific log-concave prior density with support $(0, \infty)$ for the shape parameter. Banerjee and Kundu[38] considers gamma priors on both the scale and shape parameters.

We take the above into consideration and let each parameter of the GEW model have a gamma prior distribution. The gamma prior is versatile enough to allow for a flat prior if no prior knowledge is available regarding the parameter values, or to incorporate expert knowledge by specifying subjective priors for the parameters. The priors for the GEW are thus given by

$$\theta_1 \sim \Gamma(c_0, c_1), c_0, c_1 > 0, \pi(\theta_1) \propto \theta_1^{c_0 - 1} \exp(-c_1 \theta_1)$$

$$\theta_2 \sim \Gamma(c_2, c_3), c_2, c_3 > 0, \pi(\theta_2) \propto \theta_2^{c_2 - 1} \exp(-c_3 \theta_2)$$

$$\theta_3 \sim \Gamma(c_4, c_5), c_4, c_5 > 0, \pi(\theta_3) \propto \theta_3^{c_4 - 1} \exp(-c_5 \theta_3)$$

$$\theta_4 \sim \Gamma(c_6, c_7)\ , c_6, c_7 > 0\ , \pi(\theta_4) \propto \theta_4^{c_6-1} \exp\left(-c_7\theta_4\right)$$

$$\beta \sim \Gamma(c_8, c_9)\ , c_8, c_9 > 0\ , \pi(\beta) \propto \beta^{c_8-1} \exp\left(-c_9\beta\right),$$

and the joint prior is then given by

$$\pi(\theta_1, \theta_2, \theta_3, \theta_4, \beta) \propto \theta_1^{c_0-1}\theta_2^{c_2-1}\theta_3^{c_4-1}\theta_4^{c_6-1}\beta^{c_8-1} \exp\left(-c_1\theta_1 - c_3\theta_2 - c_5\theta_3 - c_7\theta_4 - c_9\beta\right). \tag{2.7}$$

Using (2.6) and (2.7) the joint posterior is given by

$$\begin{aligned}
&\pi\left(\theta_1, \theta_2, \theta_3, \theta_4, \beta \middle| \underline{x}\right) \\
&\propto \theta_1^{c_0-1}\theta_2^{c_2-1}\theta_3^{c_4-1}\theta_4^{c_6-1}\beta^{c_8-1} \exp\left(-c_1\theta_1 - c_3\theta_2 - c_5\theta_3 - c_7\theta_4 - c_9\beta\right) \\
&\times \beta^{\sum_{i=1}^{k} r_i} \exp\left(-\theta_1\beta \sum_{i=1}^{k} r_i - \theta_2\beta \sum_{i=1}^{k} \frac{r_i}{T_i} - \theta_3\beta \sum_{i=1}^{k} r_i V_i - \theta_4\beta \sum_{i=1}^{k} \frac{r_i V_i}{T_i}\right) \\
&\times \exp\left\{-\sum_{i=1}^{k} (n_i - r_i)\left[\tau_i T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\right]^{\beta}\right\} \\
&\times \exp\left\{-\sum_{i=1}^{k}\sum_{j=1}^{r_i} \left[x_{ij} T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\right]^{\beta}\right\} \left[\prod_{i=1}^{k}\prod_{j=1}^{r_i} T_i^{\beta} x_{ij}^{\beta-1}\right].
\end{aligned}$$

Due to the complexity of the posterior, MCMC methods are used to draw posterior samples to base inference on. To determine which MCMC methods are suitable, the log-concavity of the full conditional posteriors of the GEW model must be assessed. We show that the full conditional posteriors for the GEW model are all log-concave, subject to the conditions $c_0, c_2, c_4, c_6 \geq 1$, and at least one failure occurring (see the appendix for details). Under these conditions, it is possible to use the adaptive rejection sampling (ARS) method of Gilks and Wild[39] to sample from the full conditional posteriors at each iteration of the Gibbs sampler. Alternatively, MCMC techniques that can handle non-log-concave functions, such as adaptive rejection Metropolis sampling (ARMS), proposed by Gilks et al.[40] or slice sampling, presented in Neal,[41] can be used.

3 | CASE STUDY

In this section, a case study is performed where the GEW model is applied to a real ALT data set. The accelerated data relates to failure times (in hours) obtained from a test on a high reliability device, where the accelerated stressors are temperature and humidity. The failures in these devices occur mainly due to two events. The first is decreased adhesion between layers of the material used in the devices, caused by high temperatures. The second is that moisture enters the devices via openings created by the decreased adhesion. It has furthermore been observed that the devices typically do not fail at high temperatures without moisture in the air, nor at high air moisture levels at low temperature. It is thus important to use a dual-stress ALT model, which takes into account temperature and humidity, to predict the reliability of the devices at use stress levels. It is also of particular importance to consider the interaction between the two stressors, which makes the generalized Eyring model an ideal choice for the TTF.

The data set consists of 21 devices tested at constant accelerated levels of the stressors, where the test was terminated after all devices had failed. The normal use conditions of the devices are a temperature of $T_u = 313$ K and a relative humidity of $S_u = 0.5$. There are three accelerated levels on which the devices were tested, as displayed in Table 1.

We demonstrate the use of the GEW model on this data set by considering vague gamma priors, for the case where there is no prior information available, and subjective gamma priors, for the case where expert judgement from an engineer wants to be incorporated.

For a vague gamma prior, the hyperparameters are chosen which result in a large prior variance.[42] Various combinations of $\{0.01, 0.05, 0.5, 1\}$ for the parameters of vague gamma priors are proposed by Lesaffre and Lawson.[43] We follow a similar

TABLE 1 Failure times for 21 devices under accelerated stressors.

#	Temperature (K)	Humidity	Failure time	#	Temperature (K)	Humidity	Failure time
1	333	0.9	521	12	353	0.8	504
2	333	0.9	561	13	353	0.9	115
3	333	0.9	575	14	353	0.9	119
4	333	0.9	599	15	353	0.9	150
5	333	0.9	609	16	353	0.9	152
6	333	0.9	684	17	353	0.9	153
7	333	0.9	709	18	353	0.9	155
8	333	0.9	713	19	353	0.9	156
9	353	0.8	345	20	353	0.9	164
10	353	0.8	357	21	353	0.9	199
11	353	0.8	439				

TABLE 2 Prior specifications for the generalized Eyring-Weibull models.

Model	θ_1	θ_2	θ_3	θ_4	β
GEW_1	$\Gamma(1, 0.01)$	$\Gamma(1, 0.01)$	$\Gamma(1, 0.01)$	$\Gamma(1, 0.01)$	$\Gamma(1, 0.01)$
GEW_2	$\Gamma(1, 0.0001)$	$\Gamma(1, 0.0001)$	$\Gamma(1, 0.0001)$	$\Gamma(1, 0.0001)$	$\Gamma(1, 0.0001)$
GEW_3	$\Gamma(1, 0.000001)$	$\Gamma(1, 0.000001)$	$\Gamma(1, 0.000001)$	$\Gamma(1, 0.000001)$	$\Gamma(1, 0.000001)$
GEW_4	$\Gamma(361, 190)$	$\Gamma(361, 190)$	$\Gamma(361, 190)$	$\Gamma(361, 190)$	$\Gamma(361, 190)$
GEW_5	$\Gamma(484, 220)$	$\Gamma(484, 220)$	$\Gamma(484, 220)$	$\Gamma(484, 220)$	$\Gamma(484, 220)$

Abbreviation: GEW, generalized Eyring-Weibull.

approach to choose hyperparameter values to investigate the robustness of the GEW model in terms of vague gamma priors. The hyperparameters which result in vague priors with increasing variance are given in Table 2, where these models are indicated by GEW_1, GEW_2 and GEW_3.

Hypothetically, suppose that two engineers have sound expert knowledge regarding the parameter values of the GEW model for this data set. Engineer A is of the opinion that all the model parameters are close to 1.9 and Engineer B is of the opinion that all the model parameters are close to 2.2. We can choose the hyperparameters to include this expert knowledge to varying degrees by having gamma priors centered around the values specified by the engineers, while adjusting the variance according to how strong the expert beliefs are. Both engineers are relatively certain of there opinions and suggest that a small variance is used for the prior distributions. We use GEW_4 and GEW_5 to indicate the models where the opinions of Engineer A and Engineer B are taken into account, respectively (see Table 2). We choose the prior for GEW_4 to have an expected value of 1.9 with a variance of 0.01, and similarly the prior for GEW_5 to have an expected value of 2.2 with a variance of 0.01.

The Bayesian data analysis software OpenBUGS is used in this case study to draw posterior samples to base inference on. The ARS and slice sampling techniques are available in OpenBUGS to sample from log-concave and non-log-concave densities, respectively. The models are implemented in OpenBUGS and convergence diagnostics are performed by running multiple Markov chains with different starting values. The modified Gelman-Rubin statistic, proposed by Brooks and Gelman,[44] trace plots, and autocorrelation plots are used to assess the convergence of the Markov chains. The Gelman-Rubin statistic and trace plots indicate satisfactory convergence before 5000 iterations of the Markov chain and very stable convergence when reaching 20,000 iterations. The autocorrelation is somewhat high for the parameters θ_1 and θ_2 in the cases where vague gamma priors are employed. This could indicate that it may take longer to explore the marginal posteriors for these parameters. However, since computational intensity is not an issue, we can generate enough samples to sufficiently explore the posteriors. We decide to initiate a single Markov chain for each model, with a burn-in of 50,000 iterations. Each chain is then run for another 100,000 iterations, with no thinning applied, to obtain posterior samples to base inference on.

The deviance information criterion (DIC), proposed by Spiegelhalter et al.[45] is the most popular measure used to compare various models in Bayesian ALT. Alternatively, Bayes factors can be used to perform model selection for complex ALT models (see, e.g., ref. [46]). The DIC not only takes into consideration the goodness-of-fit for the model, but also

TABLE 3 Deviance information criterion for the generalized Eyring-Weibull models.

Model	DIC
GEW_1	277.6
GEW_2	277.6
GEW_3	277.7
GEW_4	293.7
GEW_5	285.6

Abbreviations: DIC, deviance information criterion; GEW, generalized Eyring-Weibull.

penalizes the model for complexity in terms of overparameterization. The models with the smaller DIC values will typically be favored above models with larger values for the DIC, but there are other considerations that also need to be taken into account. For a parameter vector $\boldsymbol{\theta}$, with likelihood function $L(\underline{x}|\boldsymbol{\theta})$, the deviance can be defined as

$$D(\boldsymbol{\theta}) = -2\ln\left[L\left(\underline{x}|\boldsymbol{\theta}\right)\right].$$

Let $\overline{D}$ be the posterior mean of the deviance, and

$$\hat{D}\left(\overline{\boldsymbol{\theta}}\right) = -2\ln\left[L\left(\underline{x}\middle|\overline{\boldsymbol{\theta}}\right)\right],$$

be a point estimate for the deviance, with $\overline{\boldsymbol{\theta}}$ being the posterior mean of $\boldsymbol{\theta}$. The DIC can then be calculated as

$$\text{DIC} = \overline{D} + p_D,$$

where p_D is the effective number of parameters given by $p_D = \overline{D} - \hat{D}(\overline{\boldsymbol{\theta}})$.

The DIC values for the various GEW models are given in Table 3. It is clear that the GEW_1, GEW_2 and GEW_3 models have the lowest DIC values, that is, the models where vague priors are used. These three models fit the data equally well, which could be an indication that the GEW model is quite robust for these choices of vague priors. The models where the use of subjective priors are demonstrated indicate a worse fit, compared to those where vague priors are used, with the GEW_4 model having the worst fit amongst the models considered in this case study.

The summary statistics for the marginal posterior distributions of the GEW models are provided in Table 4. For the models with vague priors, very similar summary statistics are produced. This gives further evidence that the GEW model is reasonably robust for these vague priors. As expected, for the models using subjective priors, the central location measures of the marginal posteriors are pulled towards the central location of the priors. The extent to which this occurs can be adjusted by imposing gamma priors with the same means, but with smaller or larger variances, depending on how strong the prior beliefs on the parameters are. Plots of the marginal posteriors for the GEW_3 and GEW_4 models are displayed in Figures 1 and 2, to show the effect of using a subjective prior. It should be noted that the marginal posteriors for the GEW_1, GEW_2 and GEW_3 are almost identical, whereas using subjective priors results in marginal posteriors that are pulled towards the central locations of the prior distributions.

Finally, the aim of these Bayesian ALT models is to obtain the predictive reliability for the item being tested. The predictive reliability at normal use stress levels is given by

$$R\left(x_u|\underline{x}\right) = \int\int\int\int\int R(x_u|\theta_1,\theta_2,\theta_3,\theta_4,\beta)\pi\left(\theta_1,\theta_2,\theta_3,\theta_4,\beta|\underline{x}\right)d\theta_1 d\theta_2 d\theta_3 d\theta_4 d\beta, \tag{3.1}$$

where $R(x_u|\theta_1,\theta_2,\theta_3,\theta_4,\beta)$ is the Weibull reliability function at use stress levels T_u and S_u. $R(x_u|\underline{x})$ can be estimated from the MCMC output as follows:

1. Sample $\theta_1, \theta_2, \theta_3, \theta_4$ and β from the posterior M times, where M is a large number.
2. Calculate the integral in (3.1) by the Monte Carlo average

$$R\left(x_u|\underline{x}\right) \approx \frac{1}{M}\sum_{m=1}^{M} R\left(x_u\middle|\theta_1^{(m)},\theta_2^{(m)},\theta_3^{(m)},\theta_4^{(m)},\beta^{(m)}\right),$$

TABLE 4 Summary statistics for the generalized Eyring-Weibull models.

Model	Parameter	Mean	Standard deviation	Median	95% Credibility interval Lower bound	Upper bound
GEW_1	θ_1	0.9399	0.9202	0.6443	0.0229	3.4480
	θ_2	4.4402	0.9272	4.6870	1.9670	5.5600
	θ_3	1.2399	1.0787	0.9594	0.0408	3.9760
	θ_4	1.4977	1.2534	1.1920	0.0554	4.6469
	β	2.2446	0.4369	2.2170	1.4650	3.1720
GEW_2	θ_1	0.9166	0.9104	0.6262	0.0232	3.4250
	θ_2	4.4625	0.9193	4.7170	1.9860	5.5530
	θ_3	1.2479	1.0894	0.9708	0.0410	4.0000
	θ_4	1.4933	1.2421	1.1880	0.0533	4.5880
	β	2.2500	0.4354	2.2230	1.4750	3.1750
GEW_3	θ_1	0.9484	0.9314	0.6500	0.0244	3.5080
	θ_2	4.4277	0.9302	4.6740	1.9650	5.5530
	θ_3	1.2323	1.0565	0.9626	0.0405	3.9150
	θ_4	1.5298	1.2590	1.2320	0.0548	4.6990
	β	2.2425	0.4359	2.2160	1.4650	3.1700
GEW_4	θ_1	2.3811	0.1007	2.3800	2.1840	2.5810
	θ_2	2.4259	0.0986	2.4250	2.2340	2.6210
	θ_3	1.9593	0.1026	1.9580	1.7640	2.1660
	θ_4	1.9637	0.1028	1.9620	1.7670	2.1710
	β	1.7831	0.0929	1.7810	1.6060	1.9700
GEW_5	θ_1	2.4621	0.0900	2.4610	2.2880	2.6410
	θ_2	2.4930	0.0876	2.4920	2.3230	2.6670
	θ_3	2.2288	0.1008	2.2270	2.0350	2.4300
	θ_4	2.2323	0.1010	2.2300	2.0390	2.4360
	β	2.1286	0.0939	2.1270	1.9480	2.3170

Abbreviation: GEW, generalized Eyring-Weibull.

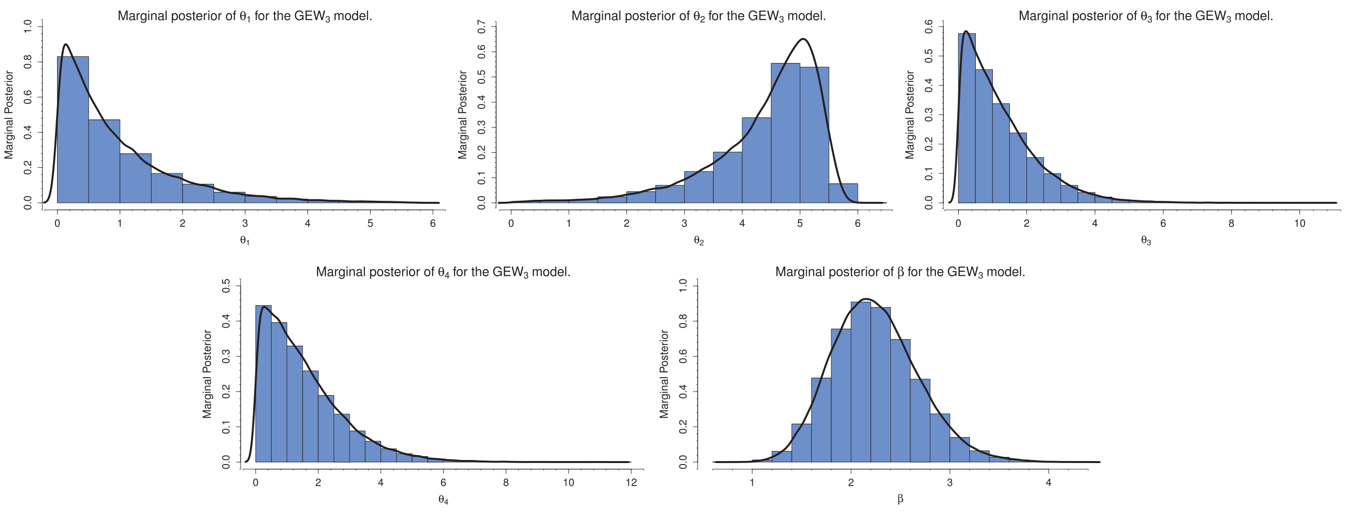

FIGURE 1 Marginal posterior distributions for the GEW_3 model. GEW, generalized Eyring-Weibull.

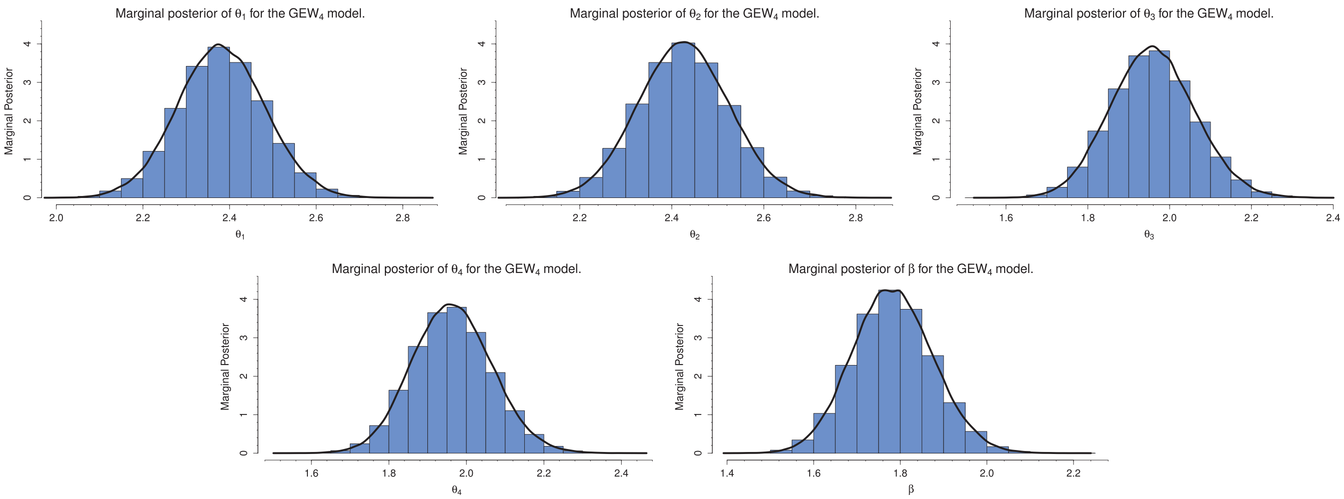

FIGURE 2 Marginal posterior distributions for the GEW_4 model. GEW, generalized Eyring-Weibull.

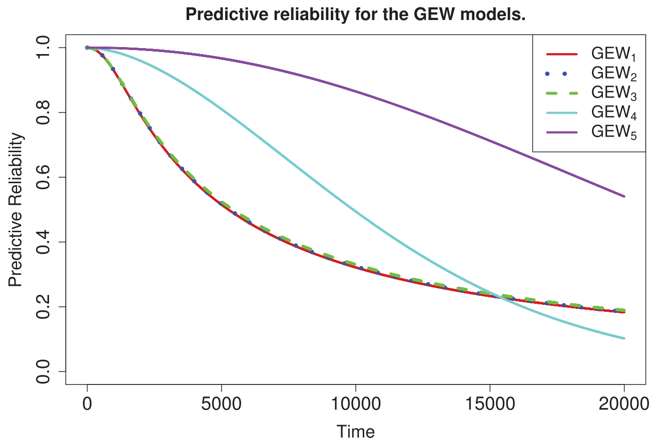

FIGURE 3 Predictive reliability of the GEW models at the use stress levels $T_u = 313$ and $S_u = 0.5$. GEW, generalized Eyring-Weibull.

which is the expected reliability at time x_u, using the posterior samples $\{\theta_1^{(m)}, \theta_2^{(m)}, \theta_3^{(m)}, \theta_4^{(m)}, \beta^{(m)}\}, m = 1, \dots, M$ from an MCMC procedure. Figure 3 shows the predictive reliability for the GEW models. It is clear that the GEW model is quite robust for the vague priors considered in this case study. A notable difference in the predictive reliability is observed for the GEW_4 and GEW_5 models. This difference would obviously depend on the restrictiveness of the subjective prior imposed to incorporate expert beliefs, as well as the amount of ALT data available. The use of subjective priors can result in either an underestimation or overestimation of the predictive reliability, compared to the data driven results obtained from employing vague priors, depending on the choice of hyperparameters for the GEW model.

Useful information regarding the items tested in this ALT experiment can also be deduced from the MTTF function at various stressor levels. The Weibull MTTF function under the i^{th} level of the stressors, is given by

$$\text{MTTF} = \frac{1}{T_i} \exp\left(\theta_1 + \frac{\theta_2}{T_i} + \theta_3 V_i + \frac{\theta_4 V_i}{T_i}\right) \Gamma\left(1 + \frac{1}{\beta}\right),$$

where $\Gamma(\cdot)$ denotes the gamma function. We consider the GEW_3 model and use the posterior means as parameter estimates, which are the Bayes estimates under a squared error loss. Figure 4 displays a surface plot of the Weibull MTTF for stressor levels ranging from the use stress levels up to the highest stress levels considered in this ALT experiment, that is, $T_i \in [313, 353]$ and $S_i \in [0.5, 0.9]$. At use stress levels, the MTTF is estimated to be 7320.55 h. Since the temperature and humidity would probably fluctuate somewhat under normal use, one can also look at the MTTF for some tolerance level

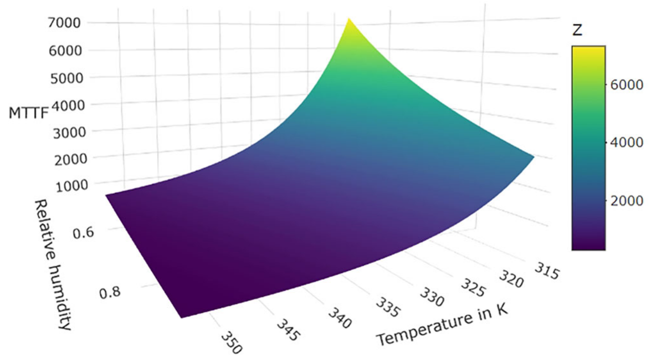

FIGURE 4 Surface plot of the Weibull MTTF under the GEW_3 model for various levels of the stressors. GEW, generalized Eyring-Weibull; MTTF, mean time to failure.

of the stressors to determine the effect of such fluctuations. For example, the MTTF for the stress levels $T_i \in [313, 318]$ and $S_i \in [0.5, 0.6]$ ranges between 7320.55 h at $T_i = 313$, $S_i = 0.5$ and 3864.71 h at $T_i = 318$, $S_i = 0.6$.

4 | CONCLUSIONS

The GEW model presented in this paper is a Bayesian ALT model that incorporates two stressors, one thermal and one non-thermal. This is a versatile model that can also be extended to include more stressors, by extending the generalized Eyring TTF to include more stressors. The formulation of the model also allows for complete samples, type-I censoring, and type-II censoring. The posterior of the GEW model, however, is mathematically intractable and MCMC techniques must be used to generate posterior samples for inference. The log-concavity of the model is also assessed to determine which MCMC techniques are appropriate.

The use of the GEW model is demonstrated in a case study, where it is applied to an ALT data set with temperature and humidity as the accelerated stressors. Different vague priors were considered in this application and the model was found to be robust for the choices of vague priors. Subjective priors are also investigated for this application in an attempt to illustrate how expert opinions from engineers can be incorporated into the model. Summary statistics, the predictive reliability and other inference were presented for the priors under consideration. It is recommended, unless sound expert knowledge regarding the model parameters are available, that vague priors are used for the model.

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DATA AVAILABILITY STATEMENT

The data that supports the findings of this study are provided in Table 1 of this article.

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APPENDIX

In this appendix, we determine which MCMC techniques are suitable to use for the GEW model. Many MCMC techniques, such as ARS, ARMS, and slice sampling, require that the full conditional posteriors of the parameters are specified up to at least proportionality. The full conditional posteriors of the GEW model are given by

$$
\begin{aligned}
&\pi\left(\theta_1 \middle| \underline{x}, \theta_2, \theta_3, \theta_4, \beta\right) \\
&\propto \theta_1^{c_0-1} \exp\left(-c_1\theta_1\right) \exp\left(-\theta_1\beta\sum_{i=1}^{k} r_i\right) \\
&\times \exp\left\{-\sum_{i=1}^{k}(n_i-r_i)\left[\tau_i T_i \exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3 V_i-\frac{\theta_4 V_i}{T_i}\right)\right]^{\beta}\right\} \\
&\times \exp\left\{-\sum_{i=1}^{k}\sum_{j=1}^{r_i}\left[x_{ij} T_i \exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3 V_i-\frac{\theta_4 V_i}{T_i}\right)\right]^{\beta}\right\}, \\
&\pi\left(\theta_2 \middle| \underline{x}, \theta_1, \theta_3, \theta_4, \beta\right) \\
&\propto \theta_2^{c_2-1} \exp\left(-c_3\theta_2\right) \exp\left(-\theta_2\beta\sum_{i=1}^{k} \frac{r_i}{T_i}\right) \\
&\times \exp\left\{-\sum_{i=1}^{k}(n_i-r_i)\left[\tau_i T_i \exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3 V_i-\frac{\theta_4 V_i}{T_i}\right)\right]^{\beta}\right\} \\
&\times \exp\left\{-\sum_{i=1}^{k}\sum_{j=1}^{r_i}\left[x_{ij} T_i \exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3 V_i-\frac{\theta_4 V_i}{T_i}\right)\right]^{\beta}\right\}, \\
&\pi\left(\theta_3 \middle| \underline{x}, \theta_1, \theta_2, \theta_4, \beta\right) \\
&\propto \theta_3^{c_4-1} \exp\left(-c_5\theta_3\right) \exp\left(-\theta_3\beta\sum_{i=1}^{k} r_i V_i\right) \\
&\times \exp\left\{-\sum_{i=1}^{k}(n_i-r_i)\left[\tau_i T_i \exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3 V_i-\frac{\theta_4 V_i}{T_i}\right)\right]^{\beta}\right\} \\
&\times \exp\left\{-\sum_{i=1}^{k}\sum_{j=1}^{r_i}\left[x_{ij} T_i \exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3 V_i-\frac{\theta_4 V_i}{T_i}\right)\right]^{\beta}\right\}, \\
&\pi\left(\theta_4 \middle| \underline{x}, \theta_1, \theta_2, \theta_3, \beta\right)
\end{aligned}
$$

$$
\begin{aligned}
&\propto \theta_4^{c_6-1}\exp(-c_7\theta_4)\exp\left(-\theta_4\beta\sum_{i=1}^{k}\frac{r_iV_i}{T_i}\right)\\
&\times\exp\left\{-\sum_{i=1}^{k}(n_i-r_i)\left[\tau_iT_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\right]^{\beta}\right\}\\
&\times\exp\left\{-\sum_{i=1}^{k}\sum_{j=1}^{r_i}\left[x_{ij}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\right]^{\beta}\right\},
\end{aligned}
$$

$$
\begin{aligned}
&\pi(\beta|\underline{x},\theta_1,\theta_2,\theta_3,\theta_4)\\
&\propto \beta^{c_8-1}\exp(-c_9\beta)\beta^{\sum_{i=1}^{k}r_i}\\
&\times\exp\left(-\theta_1\beta\sum_{i=1}^{k}r_i-\theta_2\beta\sum_{i=1}^{k}\frac{r_i}{T_i}-\theta_3\beta\sum_{i=1}^{k}r_iV_i-\theta_4\beta\sum_{i=1}^{k}\frac{r_iV_i}{T_i}\right)\\
&\times\exp\left\{-\sum_{i=1}^{k}(n_i-r_i)\left[\tau_iT_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\right]^{\beta}\right\}\\
&\times\exp\left\{-\sum_{i=1}^{k}\sum_{j=1}^{r_i}\left[x_{ij}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\right]^{\beta}\right\}\left[\prod_{i=1}^{k}\prod_{j=1}^{r_i}T_i^{\beta}x_{ij}^{\beta-1}\right].
\end{aligned}
$$

Next, the log-concavity of the full conditional posteriors must be assessed. A twice-differentiable function $f(x)$ is said to be log-concave if the second derivative of its natural log is non-positive on its domain (see, e.g., ref. [47]), thus if

$$
\frac{\partial^2\ln[f(x)]}{\partial x^2}\le 0\quad\forall x.
$$

Theorem A1. *The full conditional posterior distributions of the GEW model are all log-concave on their domains, provided that* $c_0, c_2, c_4, c_6, \sum_{i=1}^{k} r_i \ge 1$.

Proof. Let ℓ_{θ_1} denote the natural log of the full conditional posterior for θ_1, where

$$
\begin{aligned}
\ell_{\theta_1} &= \ln\left[\pi(\theta_1|\underline{x},\theta_2,\theta_3,\theta_4,\beta)\right]\\
&=(c_0-1)\ln(\theta_1)-c_1\theta_1-\theta_1\beta\sum_{i=1}^{k}r_i-\sum_{i=1}^{k}(n_i-r_i)\left[\tau_iT_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\right]^{\beta}\\
&\quad-\sum_{i=1}^{k}\sum_{j=1}^{r_i}\left[x_{ij}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\right]^{\beta}.
\end{aligned}
$$

The first and second derivatives of ℓ_{θ_1} are calculated to be

$$
\begin{aligned}
\frac{\partial\ell_{\theta_1}}{\partial\theta_1}&=\frac{c_0-1}{\theta_1}-c_1-\beta\sum_{i=1}^{k}r_i+\sum_{i=1}^{k}\beta(n_i-r_i)\left[\tau_iT_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\right]^{\beta}\\
&\quad+\sum_{i=1}^{k}\sum_{j=1}^{r_i}\beta\left[x_{ij}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\right]^{\beta},\\
\frac{\partial^2\ell_{\theta_1}}{\partial\theta_1^2}&=\frac{1-c_0}{\theta_1^2}-\sum_{i=1}^{k}\beta^2(n_i-r_i)\left[\tau_iT_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\right]^{\beta}
\end{aligned}
$$

$$-\sum_{i=1}^{k}\sum_{j=1}^{r_i}\beta^2\left[x_{ij}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\right]^{\beta}.$$

In this ALT context, temperature is measured in kelvin such that $T_i \geq 0$. Since the Weibull distribution is used, the failure times and the censoring times are positive, that is, $x_{ij} \geq 0$, $\tau_i \geq 0$, and the shape parameter of the distribution is positive, that is, $\beta > 0$. Furthermore, $r_i \leq n_i$ since there can not be more failures than items tested, and $\exp(\cdot) \geq 0$. It is thus straightforward to see that the second derivative of ℓ_{θ_1} is non-positive on the domain of θ_1, provided that $c_0 \geq 1$.

Similarly, the second derivatives of the other full conditional posteriors for the GEW model can be derived as

$$\frac{\partial^2\ell_{\theta_2}}{\partial\theta_2^2}=\frac{1-c_2}{\theta_2^2}-\sum_{i=1}^{k}\frac{\beta^2}{T_i^2}(n_i-r_i)\left[\tau_iT_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\right]^{\beta}$$
$$-\sum_{i=1}^{k}\sum_{j=1}^{r_i}\frac{\beta^2}{T_i^2}\left[x_{ij}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\right]^{\beta},$$

$$\frac{\partial^2\ell_{\theta_3}}{\partial\theta_3^2}=\frac{1-c_4}{\theta_3^2}-\sum_{i=1}^{k}\beta^2V_i^2(n_i-r_i)\left[\tau_iT_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\right]^{\beta}$$
$$-\sum_{i=1}^{k}\sum_{j=1}^{r_i}\beta^2V_i^2\left[x_{ij}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\right]^{\beta},$$

$$\frac{\partial^2\ell_{\theta_4}}{\partial\theta_4^2}=\frac{1-c_6}{\theta_4^2}-\sum_{i=1}^{k}\frac{\beta^2V_i^2}{T_i^2}(n_i-r_i)\left[\tau_iT_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\right]^{\beta}$$
$$-\sum_{i=1}^{k}\sum_{j=1}^{r_i}\frac{\beta^2V_i^2}{T_i^2}\left[x_{ij}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\right]^{\beta},$$

$$\frac{\partial^2\ell_{\beta}}{\partial\beta^2}=\frac{1-c_8}{\beta^2}-\frac{1}{\beta^2}\sum_{i=1}^{k}r_i-\sum_{i=1}^{k}\left\{(n_i-r_i)\left[\tau_iT_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\right]^{\beta}\right.$$
$$\left.\times\ln^2\left[\tau_iT_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\right]\right\}$$
$$-\sum_{i=1}^{k}\sum_{j=1}^{r_i}\left[x_{ij}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\right]^{\beta}\ln^2\left[x_{ij}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\right].$$

Following the same reasoning as above, and since $V_i^2 \geq 0$, $\ln^2(\cdot) \geq 0$, it is easy to see that all the full conditional posteriors of the GEW model are log-concave on their domains, provided that $c_0, c_2, c_4, c_6, \sum_{i=1}^{k} r_i \geq 1$. □

Comment: The conditions for log-concavity stated in Theorem A1 are not optimal and could possibly be relaxed in specific cases. Optimal conditions would depend on the magnitude of the negative terms in the second derivatives and the parameter values. ARS can be used to generate posterior samples where log-concavity holds, while ARMS or slice sampling can be used for non-log-concave densities.

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