

# Injective and surjective hulls of classical $p$ -compact operators with application to unconditionally $p$ -compact operators

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**Abstract.** The purpose of this paper is to present a brief discussion of some properties of the injective and surjective hulls of the Banach operator ideal of classical  $p$ -compact operators and to relate these ideals to the classes of unconditionally  $p$ -compact and quasi unconditionally  $p$ -nuclear operators that were introduced and studied by J. M. Kim [Studia Math. 224 (2014), 133–142].

**1. Introduction.** The theory of  $p$ -compact operators was initiated by Sinha and Karn [16]. Since then, there has been huge interest in this class of operators. The family  $(\mathcal{K}_p, k_p)$  of  $p$ -compact operators on Banach spaces is a Banach ideal of operators. Among many important properties, it is for instance shown in [4] that an operator is  $p$ -compact if and only if its adjoint is quasi  $p$ -nuclear.

Apparently, when the authors of [16] introduced the “new” class of  $p$ -compact operators, they were not aware of the fact that in the late seventies and early eighties of the previous century a different ideal of “ $p$ -compact operators” was independently introduced and studied in the book [14] and in the papers [8] and [9]. The first reference to this fact in the literature was in the paper [13] by Eve Oja, where the author highlighted the fact that there are two notions of  $p$ -compact operators in the literature, and also discussed the difference between them. Oja [13] proposed to call the “older” class of  $p$ -compact operators the “classical  $p$ -compact operators” and to denote the Banach ideal of classical  $p$ -compact operators by  $(K_p, \|\cdot\|_{K_p})$ . We will adopt this notation in the present paper.

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After almost ten years of discussion of the (new) class of  $p$ -compact operators, the authors of [1] were the first to apply an operator ideal approach, obtaining the main results of the theory in a much simpler way. Using the same departure point, Pietsch [15] also revisited the main results of the theory, adding a thorough study on the maximal hull of  $\mathcal{K}_p$ . The relation  $\mathcal{K}_p^{\text{dual}} \subseteq \mathcal{K}_p^{\text{inj}}$  is proved in [15]. This inclusion may be strict.

In 2014, Ju Myung Kim [11] introduced two new classes of operators, called “unconditionally  $p$ -compact operators” and “quasi unconditionally  $p$ -nuclear operators”. Our purpose is to find precise relationships between these two classes of operators and the ideal  $\mathcal{K}_p$ .

**2. Preliminaries.** Throughout the paper we work with Banach spaces  $X, Y, Z$ , etc. over the same scalar field  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$  and denote the space of bounded linear operators from  $X$  to  $Y$  by  $\mathcal{L}(X, Y)$ . The continuous dual space  $\mathcal{L}(X, \mathbb{K})$  of  $X$  is denoted by  $X^*$ , whereas  $B_X$  denotes the closed unit ball of  $X$ ; and  $\mathcal{K}(X, Y)$  is the space of compact linear operators. As usual, the space of absolutely  $p$ -summable (scalar) sequences (for  $1 \leq p < \infty$ ) is denoted by  $\ell_p$ , and the space of (scalar) null sequences by  $c_0$ .

We now recall some definitions and notation in the literature.

For  $1 \leq p < \infty$ , the space of all weakly  $p$ -summable sequences in a Banach space  $X$  is denoted by  $\ell_p^w(X)$ ; recall that it is a Banach space with norm

$$\|(x_i)\|_p^w := \sup \left\{ \left( \sum_{i=1}^{\infty} |\langle x_i, x^* \rangle|^p \right)^{1/p} : x^* \in X^*, \|x^*\| \leq 1 \right\}.$$

This space is isometrically identified with  $\mathcal{L}(\ell_{p'}, X)$  (where  $1/p + 1/p' = 1$ ) by the mapping  $(x_i) \mapsto E_{(x_i)}$ , where for  $(x_i) \in \ell_p^w(X)$  the linear operator

$$E_{(x_i)} : \ell_{p'} \rightarrow X : (\lambda_i) \mapsto \sum_{i=1}^{\infty} \lambda_i x_i$$

is bounded, with  $\|E_{(x_i)}\| = \|(x_i)\|_p^w$ . In the case of  $p = \infty$  we consider the space  $c_0^w(X)$  of weak null sequences in  $X$ .

The space of all weak\*  $p$ -summable sequences in the dual space  $X^*$  of a Banach space  $X$  is denoted by  $\ell_p^{w*}(X^*)$ . Recall that it is a Banach space with norm

$$\|(x_i^*)\|_p^{w*} := \sup \left\{ \left( \sum_{i=1}^{\infty} |\langle x, x_i^* \rangle|^p \right)^{1/p} : x \in X, \|x\| \leq 1 \right\}.$$

This space is isometrically identified with  $\mathcal{L}(X, \ell_p)$  by the mapping  $(x_i^*) \mapsto F_{(x_i^*)}$ , where for a fixed  $(x_i^*) \in \ell_p^{w*}(X^*)$  the linear operator

$$F_{(x_i^*)} : X \rightarrow \ell_p : x \mapsto (\langle x, x_i^* \rangle)_i$$

is bounded with  $\|F_{(x_i^*)}\| = \|(x_i^*)\|_p^{w*}$ . In the case of  $p = \infty$  we consider the space  $c_0^{w*}(X^*)$  of weak\* null sequences in  $X^*$ . Note that  $\ell_p^{w*}(X^*) = \ell_p^w(X^*)$ , but  $c_0^w(X^*) \subseteq c_0^{w*}(X^*)$ , where  $c_0^w(X^*)$  is isometrically isomorphic to the space  $\mathcal{W}(X, c_0)$  of weakly compact operators.

In general it is not true that

$$\lim_{n \rightarrow \infty} \|(x_i) - (x_1, \dots, x_n, 0, 0, \dots)\|_p^w = 0.$$

The closed subspace  $\ell_p^u(X)$  of  $\ell_p^w(X)$  consisting of those sequences for which this is true, is a Banach space with respect to the norm  $\|(\cdot)\|_p^w$ . If  $1 < p \leq \infty$ , then the identification of  $(x_i) \in \ell_p^u(X)$  with  $E_{(x_i)} : \ell_{p'} \rightarrow X : (\lambda_i) \mapsto \sum_{i=1}^{\infty} \lambda_i x_i$  defines an isometric isomorphism between  $\ell_p^u(X)$  and the space  $\mathcal{K}(\ell_{p'}, X)$  of compact linear operators. Since it is well known that  $(x_n) \in \ell_1^u(X)$  if and only if  $(x_n)$  is unconditionally summable in  $X$ , it is natural to call the elements of  $\ell_p^u(X)$  *unconditionally  $p$ -summable*. Similarly, the subspace  $\ell_p^u(X^*)$  of  $\ell_p^{w*}(X^*)$  consisting of all sequences  $(x_i^*) \in \ell_p^{w*}(X^*)$  such that

$$\lim_{n \rightarrow \infty} \|(x_i^*) - (x_1^*, \dots, x_n^*, 0, 0, \dots)\|_p^{w*} = 0$$

is isometrically isomorphic to the space  $\mathcal{K}(X, \ell_p)$  (for  $1 \leq p \leq \infty$ ) by the isometry  $(x_i^*) \mapsto F_{(x_i^*)}$ . Refer (for instance) to the book [3, p. 92] or the paper [8] for these facts.

Recall the definition of an operator ideal:

DEFINITION 2.1. An *ideal* of operators between Banach spaces is an assignment  $\mathcal{A}$  which associates with every pair  $(X, Y)$  of Banach spaces a subset  $\mathcal{A}(X, Y)$  of  $\mathcal{L}(X, Y)$  such that the following conditions are satisfied for arbitrary Banach spaces  $W, X, Y, Z$ :

- (I1)  $x^* \otimes y : X \rightarrow Y : x \mapsto x^*(x)y$  belongs to  $\mathcal{A}(X, Y)$ ,  $\forall x^* \in X^*, \forall y \in Y$ .
- (I2)  $S_1 + S_2 \in \mathcal{A}(X, Y)$ ,  $\forall S_1, S_2 \in \mathcal{A}(X, Y)$ .
- (I3)  $RST \in \mathcal{A}(X, Y)$ ,  $\forall T \in \mathcal{L}(X, W)$ ,  $\forall S \in \mathcal{A}(W, Z)$ ,  $\forall R \in \mathcal{L}(Z, Y)$ .

If  $\alpha$  is an assignment that associates with each  $S \in \mathcal{A}(X, Y)$  a real number  $\alpha(S)$ , then we call  $\alpha$  an *ideal norm* if each pair  $(\mathcal{A}(X, Y), \alpha(\cdot))$  is a normed space (i.e.  $\alpha$  defines a norm on the vector space  $\mathcal{A}(X, Y)$ ) and if moreover:

- (IN1)  $\alpha(x^* \otimes y) = \|x^*\| \|y\|$ ,  $\forall x^* \in X^*, \forall y \in Y$ .
- (IN2)  $\alpha(RST) \leq \|R\| \alpha(S) \|T\|$ ,  $\forall T \in \mathcal{L}(X, W)$ ,  $\forall S \in \mathcal{A}(W, Z)$ ,  $\forall R \in \mathcal{L}(Z, Y)$ .

If each  $\mathcal{A}(X, Y)$  is a Banach space with respect to the ideal norm  $\alpha$ , then we call  $(\mathcal{A}, \alpha)$  a *Banach operator ideal*.

It is clear that  $\mathcal{A}(X, Y) \subseteq \mathcal{L}(X, Y)$  for all pairs  $(X, Y)$ . The ideal  $(\mathcal{L}, \|\cdot\|)$  of bounded linear operators is thus the largest Banach ideal of operators. All finite rank bounded linear operators from  $X$  to  $Y$  belong to  $\mathcal{A}(X, Y)$

for all operator ideals  $\mathcal{A}$  and all Banach spaces  $X, Y$ . Thus the ideal  $\mathcal{F}$  of finite rank bounded linear operators is the smallest Banach operator ideal. The reader is referred to Pietsch's book [14] for information about operator ideals.

Given a Banach space  $X$ , recall from [10, p. 421] that

$$X^\infty := \ell_\infty(B_{X^*}) = \{f : B_{X^*} \rightarrow \mathbb{K} : f \text{ is bounded}\}, \quad X^1 := \ell_1(B_X).$$

There is a linear and isometric embedding  $J_X : X \rightarrow X^\infty$  and there is a canonical norm one surjection  $Q_X^1 \in \mathcal{L}(X^1, X)$ ; they are given by

$$J_X(x) = \langle x, \cdot \rangle \quad \text{and} \quad Q_X^1((\lambda_x)_{x \in B_X}) = \sum_{x \in B_X} \lambda_x x.$$

The space  $X^\infty$  has the *extension property* (or it is an *injective* Banach space), i.e. if  $Y, Z$  are Banach spaces such that  $Z$  is a subspace of  $Y$ , then for any  $T \in \mathcal{L}(Z, X^\infty)$  there exists  $\tilde{T} \in \mathcal{L}(Y, X^\infty)$  such that  $\|\tilde{T}\| = \|T\|$  and  $\tilde{T}x = Tx$  for all  $x \in Z$ . The space  $X^1$  has the *lifting property*, i.e. if  $Y$  and  $Z$  are Banach spaces such that there is a surjection  $S \in \mathcal{L}(Y, Z)$ , then for any given  $\epsilon > 0$  and each  $T \in \mathcal{L}(X^1, Z)$  there is a  $\tilde{T} \in \mathcal{L}(X^1, Y)$  such that  $S \circ \tilde{T} = T$  and  $\|\tilde{T}\| \leq (1 + \epsilon)\|S\|\|T\|$ .

Recall from [14, Theorem 4.7.9] that an operator ideal  $\mathcal{A}$  is *surjective* if for every surjection  $Q \in \mathcal{L}(Z, X)$  and every  $T \in \mathcal{L}(X, Y)$  it follows from  $TQ \in \mathcal{A}(Z, Y)$  that  $T \in \mathcal{A}(X, Y)$ . The smallest surjective ideal which contains  $\mathcal{A}$ , denoted by  $\mathcal{A}^{\text{sur}}$ , is called the *surjective hull* of  $\mathcal{A}$ . Also recall from [14, Theorem 4.6.9] that an operator ideal  $\mathcal{A}$  is *injective* if and only if for every (metric) injection  $J \in \mathcal{L}(Y, Y_0)$  and every  $T \in \mathcal{L}(X, Y)$  it follows from  $JT \in \mathcal{A}(X, Y_0)$  that  $T \in \mathcal{A}(X, Y)$ . The smallest injective operator ideal  $\mathcal{A}^{\text{inj}}$  that contains  $\mathcal{A}$  is called the *injective hull* of  $\mathcal{A}$ . Refer to [14, pp. 109 and 111] for the following concrete definitions of the injective and surjective hulls of an operator ideal  $\mathcal{A}$ : For any pair of Banach spaces  $X, Y$  we have

$$\begin{aligned} \mathcal{A}^{\text{inj}}(X, Y) &= \{S \in \mathcal{L}(X, Y) : J_Y \circ S \in \mathcal{A}(X, Y^\infty)\}, \\ \mathcal{A}^{\text{sur}}(X, Y) &= \{S \in \mathcal{L}(X, Y) : S \circ Q_X^1 \in \mathcal{A}(X^1, Y)\}. \end{aligned}$$

If  $(\mathcal{A}, \alpha)$  is a Banach operator ideal, then so are the ideals  $(\mathcal{A}^{\text{inj}}, \alpha^{\text{inj}})$  and  $(\mathcal{A}^{\text{sur}}, \alpha^{\text{sur}})$ , where for any Banach spaces  $X, Y$  we have

$$\begin{aligned} \alpha^{\text{inj}}(S) &= \alpha(J_Y \circ S) \quad \text{for all } S \in \mathcal{A}^{\text{inj}}(X, Y), \\ \alpha^{\text{sur}}(S) &= \alpha(S \circ Q_X^1) \quad \text{for all } S \in \mathcal{A}^{\text{sur}}(X, Y). \end{aligned}$$

Since  $\|J_Y\| = \|Q_X^1\| = 1$ , it is clear that  $\alpha^{\text{inj}}(S) \leq \alpha(S)$  and  $\alpha^{\text{sur}}(S) \leq \alpha(S)$  for all  $S \in \mathcal{A}(X, Y)$ .

Recall that the dual Banach operator ideal  $(\mathcal{A}^{\text{dual}}, \alpha^{\text{dual}})$  of the Banach ideal  $(\mathcal{A}, \alpha)$  is defined by

$$T \in \mathcal{A}^{\text{dual}}(X, Y) \Leftrightarrow T^* \in \mathcal{A}(Y^*, X^*) \quad \text{and} \quad \alpha^{\text{dual}}(T) = \alpha(T^*).$$

It is well known that the ideal  $(\mathcal{K}, \|\cdot\|)$  of compact operators on Banach spaces is both injective and surjective.

**3. The ideal of classical  $p$ -compact operators and its injective and surjective hulls.** In [8] and [9] the Banach operator ideal of classical  $p$ -compact operators is introduced and studied—although in those two papers and in the book [14], these operators were called  $p$ -compact. Refer to [8] for the following definition:

**DEFINITION 3.1.** Let  $1 \leq p < \infty$ . An operator  $T : X \rightarrow Y$  is called *classical  $p$ -compact* (or  *$c$ - $p$ -compact*) if there are compact operators  $P : X \rightarrow \ell_p$  and  $Q : \ell_p \rightarrow Y$  such that  $T = Q \circ P$ . Also,  $T$  is called  *$\infty$ -nuclear* if this condition holds with  $\ell_p$  replaced by  $c_0$ .

For a pair  $X, Y$  of Banach spaces the norm on the vector space  $K_p(X, Y)$  of all  $c$ - $p$ -compact operators from  $X$  to  $Y$  is defined by

$$\|T\|_{K_p} = \inf\{\|P\| \|Q\| : P \in \mathcal{K}(X, \ell_p), Q \in \mathcal{K}(\ell_p, Y), T = Q \circ P\}.$$

It is shown in [8] that  $(K_p, \|\cdot\|_{K_p})$  is a Banach operator ideal and that  $T \in K_p(X, Y)$  if and only if  $T$  has a representation

$$(\dagger) \quad Tx = \sum_i x_i^*(x) y_i, \quad \forall x \in X,$$

where  $(x_n^*) \in \ell_p^u(X^*)$  and  $(y_n) \in \ell_{p'}^u(Y)$ . Also,

$$\|T\|_{K_p} = \inf\left\{\|(x_n^*)\|_p^{w*} \|(y_n)\|_{p'}^w : T = \sum_{n=1}^{\infty} x_n^* \otimes y_n\right\}.$$

Here it is assumed that  $\ell_{\infty}$  is replaced by  $c_0$  in the case of  $p = \infty$ .

It is shown in [8] that the factorization condition for  $T \in K_p(X, Y)$  can be relaxed to  $T = Q \circ P$  where either  $P \in \mathcal{K}(X, \ell_p)$  and  $Q \in \mathcal{L}(\ell_p, Y)$  (for  $1 \leq p \leq \infty$ ) or  $P \in \mathcal{L}(X, \ell_p)$  and  $Q \in \mathcal{K}(\ell_p, Y)$  (for  $1 < p \leq \infty$ ), also in the definition of the norm  $\|\cdot\|_{K_p}$ . It is therefore clear that  $\mathcal{K}(X, \ell_p) = K_p(X, \ell_p)$ , with  $\|P\|_{K_p} = \|P\|$  for all  $P \in \mathcal{K}(X, \ell_p)$ . Similarly, by [8, Theorem 2.5], the representation of  $T \in K_p(X, Y)$  as in  $(\dagger)$  above can be relaxed to  $(x_n^*) \in \ell_p^u(X^*)$  and  $(y_n) \in \ell_{p'}^w(Y)$  (for  $1 \leq p \leq \infty$ ) or  $(x_n^*) \in \ell_p^{w*}(X^*)$  and  $(y_n) \in \ell_{p'}^u(Y)$  (for  $1 < p \leq \infty$ ).

The reader is referred to the literature on tensor products of normed spaces (see, for instance, the books [3] and [6]) for the definitions of the concepts of “reasonable cross norm” and “tensor norm”, as well as the concept of “ $\alpha$ -nuclear operator” with  $\alpha$ -nuclear norm  $N_{\alpha}(\cdot)$ .

Refer to [6], Proposition 1.5.8, for the following fact (due to Grothendieck):

**THEOREM 3.2.** *Let  $\alpha$  be a tensor norm and assume that  $T \in \mathcal{L}(X, Y)$  is  $\alpha$ -nuclear into  $Y^{**}$  (or more precisely suppose that  $C_Y T : X \rightarrow Y^{**}$  is  $\alpha$ -nuclear, where  $C_Y : Y \hookrightarrow Y^{**}$  denotes the canonical embedding) and  $X^*$  has the approximation property. Then  $T : X \rightarrow Y$  is  $\alpha$ -nuclear with*

$$N_\alpha(T) = N_\alpha(C_Y T).$$

In [9] it is shown that if  $X$  and  $Y$  are Banach spaces, then the formula

$$w_p(u) := \inf \|(x_i)\|_p^w \|(y_i)\|_{p'}^w,$$

where the infimum is taken over all representations  $u = \sum_i x_i \otimes y_i$ , defines a reasonable cross norm on  $X \otimes Y$ , and moreover  $w_p$  is a tensor norm. It is also shown in [9], using the representation  $(\dagger)$  of  $T \in K_p(X, Y)$  given above, that an operator belongs to  $K_p(X, Y)$  if and only if it is  $w_p$ -nuclear for all  $1 \leq p \leq \infty$ . This implies that if either  $X^*$  or  $Y$  has the approximation property, then  $X^* \tilde{\otimes}_{w_p} Y$  (the completion of  $(X^* \otimes Y, w_p)$ ) may be isometrically identified with  $K_p(X, Y)$ . By Theorem 3.2, it then follows that if  $T : X \rightarrow Y$  is  $c$ - $p$ -compact as an operator into  $Y^{**}$  and  $X^*$  has the approximation property, then  $T : X \rightarrow Y$  is also  $c$ - $p$ -compact and  $\|T\|_{K_p} = \|C_Y T\|_{K_p}$ .

In [7] some characterizations of the injective and surjective hulls of a generalized version of the ideal  $K_p$  (where the space  $\ell_p$  is replaced by any  $BK$ -space with  $AK$ ) are discussed. Based on the results in [7]–[9], we will now discuss some characterizations of the injective and surjective hulls of the operator ideal  $K_p$ . For the case  $p = \infty$  we recall the following well-known fact due to Terzioğlu [17], also discussed for instance in the book [5]:

**LEMMA 3.3.** *Let  $T : X \rightarrow Y$  be a bounded linear operator between Banach spaces. Then the following are equivalent:*

- (1)  *$T$  is compact.*
- (2) *There exists a norm-null sequence  $(x_n^*)$  in  $X^*$  such that*

$$\|Tx\| \leq \sup_n |\langle x, x_n^* \rangle|, \quad \forall x \in X.$$

- (3) *For some closed subspace  $Z$  of  $c_0$  there are compact operators  $P : X \rightarrow Z$  and  $Q : Z \rightarrow Y$  such that  $T = Q \circ P$ .*

Following the same arguments as for the more general case in [7], we have the following characterization of the injective hull of the ideal  $K_p$ :

**PROPOSITION 3.4.** *Let  $1 \leq p \leq \infty$ . Given any pair  $X, Y$  of Banach spaces and  $T \in \mathcal{L}(X, Y)$ , the following are equivalent:*

- (a)  *$T \in K_p^{\text{inj}}(X, Y)$ .*
- (b) *There exists a closed subspace  $\Sigma$  of  $\ell_p$  such that  $T = Q \circ P$  for some  $P \in \mathcal{K}(X, \Sigma)$  and  $Q \in \mathcal{K}(\Sigma, Y)$ .*
- (c) *There exists a closed subspace  $\Sigma$  of  $\ell_p$  such that  $T = Q \circ P$  for some  $P \in \mathcal{K}(X, \Sigma)$  and  $Q \in \mathcal{L}(\Sigma, Y)$ .*

If  $1 < p \leq \infty$ , then the above assertions are equivalent to

(d) There exists a closed subspace  $\Sigma$  of  $\ell_p$  such that  $T = Q \circ P$  for some  $P \in \mathcal{L}(X, \Sigma)$  and  $Q \in \mathcal{K}(\Sigma, Y)$ .

We have

$$\|T\|_{K_p}^{\text{inj}} = \inf\{\|Q\| \|P\| : T = Q \circ P\},$$

where the infimum is taken over all relevant factorizations.

**COROLLARY 3.5.** *Let  $X, Y$  be Banach spaces, where  $Y$  is injective (has the extension property). Then  $K_p(X, Y) = K_p^{\text{inj}}(X, Y)$  with  $\|T\|_{K_p} = \|T\|_{K_p}^{\text{inj}}$ .*

*Proof.* If  $T \in K_p^{\text{inj}}(X, Y)$ , then let  $T = Q \circ P$ , where  $P \in \mathcal{K}(X, \Sigma)$  and  $Q \in \mathcal{K}(\Sigma, Y)$  and  $\Sigma$  is a closed subspace of  $\ell_p$ . Since  $Y$  is injective, the operator  $Q$  extends to a bounded linear operator  $\tilde{Q} : \ell_p \rightarrow Y$  such that  $\|Q\| = \|\tilde{Q}\|$  and  $T = \tilde{Q} \circ P$ . By [8, Theorem 2.3],  $T \in K_p(X, Y)$  and

$$\|T\|_{K_p} \leq \|P\| \|\tilde{Q}\| = \|P\| \|Q\|.$$

Since the factorization  $T = Q \circ P$  was arbitrary, it follows that  $\|T\|_{K_p} \leq \|T\|_{K_p}^{\text{inj}}$ . ■

**THEOREM 3.6.** *Let  $1 \leq p \leq \infty$  and let  $X, Y$  be Banach spaces and  $T \in \mathcal{L}(X, Y)$ . Then  $T \in K_p^{\text{inj}}(X, Y)$  if and only if there is a sequence  $(x_n^*) \in \ell_p^u(X^*)$  such that*

$$\|Tx\| \leq \|(\langle x, x_n^* \rangle)\|_p \quad \text{for all } x \in X.$$

In this case,

$$\|T\|_{K_p}^{\text{inj}} = \inf\{(\|x_n^*\|_p^w)^* : \|Tx\| \leq \|(\langle x, x_n^* \rangle)\|_p, \forall x \in X\}.$$

*Proof.* Assume  $T \in K_p^{\text{inj}}(X, Y)$ . Then  $J_Y \circ T \in K_p(X, Y^\infty)$ . Therefore, there are  $(a_n) \in \ell_p^u(X^*)$  and  $(y_n) \in \ell_p^u(Y^\infty)$  such that

$$J_Y \circ T = \sum_{i=1}^{\infty} a_i \otimes y_i.$$

Set  $x_n^* = \|(y_i)\|_p^w a_n$  for all  $n$ . Then  $(x_n^*) \in \ell_p^u(X^*)$  and

$$\begin{aligned} \|Tx\| &\leq \|(J_Y \circ T)x\| = \sup_{\|y^*\|_{(Y^\infty)^*} \leq 1} |\langle (J_Y \circ T)x, y^* \rangle| \\ &= \sup_{\|y^*\|_{(Y^\infty)^*} \leq 1} \left| \sum_{i=1}^{\infty} \langle x, a_i \rangle \langle y_i, y^* \rangle \right| \leq \left( \sum_{i=1}^{\infty} |\langle x, x_i^* \rangle|^p \right)^{1/p}. \end{aligned}$$

Thus, with each representation  $J_Y \circ T = \sum_{i=1}^{\infty} a_i \otimes y_i$  we associate a sequence  $(x_n^*) \in \ell_p^u(X^*)$  for which the desired inequality holds and which also satisfies

$\|(x_i^*)\|_p^{w^*} = \|(y_i)\|_{p'}^w \|(a_i)\|_p^{w^*}$ . Therefore,

$$\|T\|_{K_p}^{\text{inj}} = \|J_Y \circ T\|_{K_p} \geq \inf\{\|(x_n^*)\|_p^{w^*} : \|Tx\| \leq \|(\langle x, x_n^* \rangle)\|_p, \forall x \in X\}.$$

Conversely, suppose  $\|Tx\| \leq \|(\langle x, x_n^* \rangle)\|_p$  for all  $x \in X$ , where  $(x_n^*) \in \ell_p^u(X^*)$ . Define  $P : X \rightarrow \ell_p$  by  $Px = (\langle x, x_n^* \rangle)$ . Then the operator  $P$  is compact and  $\|P\| = \|(x_i^*)\|_p^{w^*}$ . Clearly,  $\|Tx\| \leq \|Px\|_p$  for all  $x \in X$ . Therefore,  $S : P(X) \rightarrow Y : Px \mapsto Tx$  defines a bounded linear operator with  $\|S\| \leq 1$ . Let  $Q : \overline{P(X)} \rightarrow Y$  be the continuous linear extension of  $S$ , so  $\|Q\| \leq 1$ . By Proposition 3.4 we have  $T \in K_p^{\text{inj}}(X, Y)$  and  $\|T\|_{K_p}^{\text{inj}} \leq \|P\| = \|(x_i^*)\|_p^{w^*}$ . Since this is true for all  $(x_n^*) \in \ell_p^u(X^*)$  for which the inequality holds, we have

$$\|T\|_{K_p}^{\text{inj}} \leq \inf\{\|(x_n^*)\|_p^{w^*} : \|Tx\| \leq \|(\langle x, x_n^* \rangle)\|_p, \forall x \in X\}. \blacksquare$$

We now turn to a discussion of the surjective hull of the operator ideal  $(K_p, \|\cdot\|_{K_p})$ . The discussion again has its roots in [7]–[9]. Therefore, we will only give proofs of results that are important in the context of the present paper. The following characterization of the surjective hull of the ideal  $K_p$  follows from the general case in [7], where  $\ell_p$  is replaced by a  $BK$ -space with the  $AK$ -property:

**PROPOSITION 3.7.** *Let  $1 < p \leq \infty$  and  $X, Y$  Banach spaces. For  $T \in \mathcal{L}(X, Y)$  the following are equivalent:*

- (a)  $T \in K_p^{\text{sur}}(X, Y)$ .
- (b) *There exists a compact factorization  $T = Q \circ P$  through a quotient space  $Z$  of  $\ell_p$ , i.e.  $P \in \mathcal{K}(X, Z)$  and  $Q \in \mathcal{K}(Z, Y)$ .*
- (c) *There exists a factorization  $T = Q \circ P$  through a quotient space  $Z$  of  $\ell_p$ , where  $P \in \mathcal{L}(X, Z)$  and  $Q \in \mathcal{K}(Z, Y)$ .*

We have

$$\|T\|_{K_p}^{\text{sur}} = \inf\{\|Q\| \|P\| : T = Q \circ P\},$$

where the infimum is taken over all relevant factorizations.

**REMARK 3.8.** Although the case  $p = 1$  is excluded in the previous result, it is easily verified that for each  $T \in K_1^{\text{sur}}(X, Y)$  there exists a compact factorization  $T = Q \circ P$  through a quotient space  $Z$  of  $\ell_1$ .

**PROPOSITION 3.9.** *Let  $1 < p \leq \infty$  and  $T \in \mathcal{L}(X, Y)$ . Then  $T \in K_p^{\text{sur}}(X, Y)$  if and only if there exists an operator  $S \in \mathcal{K}(\ell_p, Y)$  such that  $T(B_X) \subseteq S(B_{\ell_p})$ . Moreover,*

$$\|T\|_{K_p}^{\text{sur}} = \inf\{\|S\| : S \in \mathcal{K}(\ell_p, Y) \text{ and } T(B_X) \subseteq S(B_{\ell_p})\}.$$

For each  $T \in K_1^{\text{sur}}(X, Y)$  there exists an operator  $S \in \mathcal{K}(\ell_1, Y)$  such that  $T(B_X) \subseteq S(B_{\ell_1})$ .

*Proof.* Let  $1 < p \leq \infty$ . If the condition holds, then since  $\mathcal{K}(\ell_p, Y) = K_p(\ell_p, Y)$  with  $\|S\|_{K_p} = \|S\|$ , it follows from a general result for operator ideals [14, p. 112] that  $T \in K_p^{\text{sur}}(X, Y)$  and  $\|T\|_{K_p}^{\text{sur}} \leq \|S\|_{K_p} = \|S\|$  for all  $S$  satisfying the condition.

The converse follows from Proposition 3.7 (or Remark 3.8 in case of  $p = 1$ ) and the lifting property of  $X^1$ : For  $\epsilon > 0$  apply Proposition 3.7 to obtain a compact factorization  $T = Q \circ P$  through a quotient space  $Z$  of  $\ell_p$  such that  $\|Q\| \|P\| < \|T\|_{K_p}^{\text{sur}} + \epsilon$ . Let  $\delta > 0$ . Using the lifting property, we obtain  $R \in \mathcal{L}(X^1, \ell_p)$  such that  $\Theta_p \circ R = P \circ Q_X^1$  and  $T \circ Q_X^1 = (Q \circ \Theta_p) \circ R$ , where  $\Theta_p : \ell_p \rightarrow Z$  is the quotient mapping and

$$\|R\| < \|\Theta_p\| \|P \circ Q_X^1\| (1 + \delta) \leq \|P\| (1 + \delta).$$

Then  $S := \|R\|(Q \circ \Theta_p) \in \mathcal{K}(\ell_p, Y)$  and for each  $x \in B_X$  there exists  $(\alpha_{i,x})_i \in B_{\ell_p}$  such that  $S((\alpha_{i,x})_i) = Tx$ . Finally,

$$\|S\| \leq \|R\| \|Q\| < (1 + \delta) \|P\| \|Q\| < (1 + \delta) (\|T\|_{K_p}^{\text{sur}} + \epsilon),$$

for all  $\delta > 0$ , i.e.  $\|S\| \leq \|T\|_{K_p}^{\text{sur}} + \epsilon$  for all  $\epsilon > 0$ . ■

REMARK 3.10. Let  $1 \leq p \leq \infty$  and let  $T \in K_p^{\text{sur}}(X, Y)$ . It follows from the compactness of  $TQ_X^1$  and the surjectivity of the ideal of compact operators that  $T \in \mathcal{K}(X, Y)$ . Thus, by the Grothendieck characterization of compact sets, it follows that

$$T(B_X) \subseteq c_0\text{-co}(\{y_n\}) := \left\{ \sum_n \alpha_n y_n : (\alpha_n) \in B_{\ell_1} \right\} \quad \text{for some } (y_n) \in c_0(Y).$$

Theorem 3.11 below follows from Proposition 3.9, by using the fact that  $\mathcal{K}(\ell_p, Y)$  is isometrically identified with  $\ell_{p'}^u(Y)$ . It was, however, also obtained in [12] (and its operator ideal version earlier in [2]), using the operator ideal method from [1] together with results from [8] and [9].

THEOREM 3.11. *Let  $1 < p \leq \infty$ . An operator  $T \in \mathcal{L}(X, Y)$  belongs to  $K_p^{\text{sur}}(X, Y)$  if and only if there exists  $(y_n) \in \ell_{p'}^u(Y)$  such that*

$$T(B_X) \subseteq p'\text{-co}(\{y_n\}) := \left\{ \sum_n \alpha_n y_n : (\alpha_n) \in B_{\ell_p} \right\}.$$

*In this case,*

$$\|T\|_{K_p}^{\text{sur}} = \inf \{ \|(y_i)\|_{p'}^w : T(B_X) \subseteq p'\text{-co}(\{y_n\}) \}.$$

REMARK 3.12. If  $T^* \in K_1^{\text{sur}}(Y^*, X^*)$ , then  $T^*$  is compact, and so  $T \in \mathcal{K}(X, Y) = K_\infty^{\text{inj}}(X, Y)$  by Lemma 3.3. For any compact factorization  $T = Q \circ P$  through some closed subspace  $Z$  of  $c_0$ , we have  $\|T^*\|_{K_1}^{\text{sur}} \leq \|Q\| \|P\|$ . Thus

$$\|T^*\|_{K_1}^{\text{sur}} \leq \|T\|_{K_\infty}^{\text{inj}}.$$

THEOREM 3.13. *Let  $1 \leq p < \infty$  and  $1/p + 1/p' = 1$ . Then:*

- (i)  $T \in K_p^{\text{inj}}(X, Y) \Leftrightarrow T^* \in K_{p'}^{\text{sur}}(Y^*, X^*)$ . In this case  $\|T\|_{K_p}^{\text{inj}} = \|T^*\|_{K_{p'}}^{\text{sur}}$ .  
Therefore,  $K_p^{\text{inj}} = (K_{p'}^{\text{sur}})^{\text{dual}}$ .
- (ii)  $T \in K_{p'}^{\text{sur}}(X, Y) \Leftrightarrow T^* \in K_p^{\text{inj}}(Y^*, X^*)$ . In this case  $\|T\|_{K_{p'}}^{\text{sur}} = \|T^*\|_{K_p}^{\text{inj}}$ .  
Therefore,  $K_{p'}^{\text{sur}} = (K_p^{\text{inj}})^{\text{dual}}$ .

*Proof.* The proof of (i) depends on the factorization results of Propositions 3.4 and 3.7 above, as well Schauder's Theorem for compact operators. Similarly, from the same results it follows that if  $T \in K_{p'}^{\text{sur}}(X, Y)$  then  $T^* \in K_p^{\text{inj}}(Y^*, X^*)$  and  $\|T^*\|_{K_p}^{\text{inj}} \leq \|T\|_{K_{p'}}^{\text{sur}}$ .

If, conversely,  $T^* \in K_p^{\text{inj}}(Y^*, X^*)$ , then by the first part of the theorem we have  $T^{**} \in K_{p'}^{\text{sur}}(X^{**}, Y^{**})$  with  $\|T^{**}\|_{K_{p'}}^{\text{sur}} = \|T^*\|_{K_p}^{\text{inj}}$ . Thus,

$$T^{**}Q_{X^{**}}^1 \in K_{p'}((X^{**})^1, Y^{**}).$$

Since  $X^1$  has the lifting property, for each  $\epsilon > 0$  there exists  $S \in \mathcal{L}(X^1, (X^{**})^1)$  such that  $\|S\| \leq 1 + \epsilon$  and  $Q_{X^{**}}^1 S = C_X Q_X^1$ . Moreover, since

$$T^{**}C_X Q_X^1 = T^{**}Q_{X^{**}}^1 S \in K_{p'}(X^1, Y^{**}),$$

we have  $C_Y T Q_X^1 \in K_{p'}(X^1, Y^{**})$ . Also,

$$\|C_Y T Q_X^1\|_{K_{p'}} = \|T^{**}Q_{X^{**}}^1 S\|_{K_{p'}} \leq (1 + \epsilon)\|T^{**}Q_{X^{**}}^1\|_{K_{p'}}.$$

Since  $(X^1)^*$  has the approximation property, it follows by the discussion following Theorem 3.2, but now for the class  $K_{p'}$  of  $w_{p'}$ -nuclear operators, that  $T Q_X^1 \in K_{p'}(X^1, Y)$  and  $\|T Q_X^1\|_{K_{p'}} = \|C_Y T Q_X^1\|_{K_{p'}}$ . Thus,  $T \in K_{p'}^{\text{sur}}(X, Y)$  and

$$\|T\|_{K_{p'}}^{\text{sur}} = \|T Q_X^1\|_{K_{p'}} \leq (1 + \epsilon)\|T^{**}\|_{K_{p'}}^{\text{sur}} = (1 + \epsilon)\|T^*\|_{K_p}^{\text{inj}}. \blacksquare$$

**4. Unconditionally  $p$ -compact and unconditionally quasi  $p$ -nuclear operators.** In [11] the concept of relatively *unconditionally  $p$ -compact* set is defined as follows:

DEFINITION 4.1. A subset  $A$  of  $X$  is called *relatively unconditionally  $p$ -compact* (or relatively  $u$ - $p$ -compact) if there exists  $(x_n) \in \ell_p^u(X)$  such that

$$A \subseteq p\text{-co}(\{x_n\}) := \left\{ \sum_n \alpha_n x_n : (\alpha_n) \in B_{\ell_{p'}} \right\}.$$

Using this definition, it is natural to introduce the concept of unconditionally  $p$ -compact operator as in [11, p. 135]:

DEFINITION 4.2. A linear operator  $T : X \rightarrow Y$  is said to be  *$u$ - $p$ -compact* if  $T(B_X)$  is a relatively  $u$ - $p$ -compact subset of  $Y$ . The collection of all

$u$ - $p$ -compact operators from  $X$  to  $Y$  is denoted by  $\mathcal{U}_p(X, Y)$ . A norm  $u_p$  is defined on  $\mathcal{U}_p(X, Y)$  by

$$u_p(T) = \inf\{\|(y_n)\|_p^u : (y_n) \in \ell_p^u(Y) \text{ and } T(B_X) \subseteq p\text{-co}(\{y_n\})\}.$$

For  $1 \leq p < \infty$ ,  $(\mathcal{U}_p, u_p)$  is a Banach operator ideal [11, Theorem 2.1]; this has also been observed in [12] through operator ideal methods.

In the same paper [11, p. 136], the author also introduces *quasi unconditionally  $p$ -nuclear operators* (quasi  $u$ - $p$ -nuclear operators) as follows:

DEFINITION 4.3. Let  $1 \leq p \leq \infty$ . A linear operator  $T : X \rightarrow Y$  is called *quasi  $u$ - $p$ -nuclear* if there exists  $(x_n^*) \in \ell_p^u(X^*)$  such that

$$\|Tx\| \leq \|(x_n^*(x))\|_p \quad \text{for every } x \in X.$$

The collection of all quasi  $u$ - $p$ -nuclear operators from  $X$  to  $Y$  is denoted by  $\mathcal{N}_{up}^Q(X, Y)$ . The norm

$$\nu_{up}^Q(T) = \inf\{\|(x_n^*)\|_p^w : \|Tx\| \leq \|(x_n^*(x))\|_p, \forall x \in X\}$$

turns the vector space  $\mathcal{N}_{up}^Q(X, Y)$  into a Banach space.

Comparing Definition 4.3 with Theorem 3.6 yields:

THEOREM 4.4. Let  $1 \leq p \leq \infty$ . Then  $\mathcal{N}_{up}^Q(X, Y) = K_p^{\text{inj}}(X, Y)$  and  $\nu_{up}^Q(T) = \|T\|_{K_p}^{\text{inj}}$  for all  $T \in \mathcal{N}_{up}^Q(X, Y)$ .

Theorem 4.4 allows us to immediately conclude that  $(\mathcal{N}_{up}^Q, \nu_{up}^Q)$  is an injective Banach operator ideal.

There is also a natural relationship between the “classical” operator ideal  $(K_p, \|\cdot\|_{K_p})$  and the ideal  $(\mathcal{U}_p, u_p)$ . Comparing Theorem 3.11 with Definition 4.2, we get:

THEOREM 4.5. Let  $1 \leq p < \infty$ . Then  $(\mathcal{U}_p, u_p) = (K_{p'}^{\text{sur}}, \|\cdot\|_{K_{p'}}^{\text{sur}})$ .

Theorem 4.5 was also obtained in [12] (and its operator ideal version in [2]). An immediate consequence of Theorem 4.5 is that  $(\mathcal{U}_p, u_p)$  is a surjective Banach operator ideal.

Theorems 4.4 and 4.5 and results in the previous section yield several results of [11], based upon the (operator ideal) theory developed in [7]–[9]. For instance, as a direct consequence of Corollary 3.5 and Theorem 4.4, we obtain [11, Lemma 2.6]:

PROPOSITION 4.6. Let  $1 \leq p \leq \infty$ . Suppose  $Y$  is injective. Then  $T \in \mathcal{N}_{up}^Q(X, Y)$  if and only if  $T \in K_p(X, Y)$  and  $\nu_{up}^Q(T) = \|T\|_{K_p}$ .

The results in Proposition 4.7 below follow from Theorems 4.5, 4.4 and 3.13. They were also obtained in [11], using results and techniques from the theory of  $p$ -compact operators (mostly from [4]); except that Proposition

4.7(b) is an improvement to [11, Theorem 2.4]: the inequality in [11] is now replaced by equality.

**PROPOSITION 4.7.** *Let  $1 \leq p < \infty$ . For Banach spaces  $X$  and  $Y$  we have:*

- (a)  $T \in \mathcal{N}_{up}^Q(X, Y) \Leftrightarrow T^* \in \mathcal{U}_p(Y^*, X^*)$ . In this case  $\nu_{up}^Q(T) = u_p(T^*)$ .
- (b)  $T \in \mathcal{U}_p(X, Y) \Leftrightarrow T^* \in \mathcal{N}_{up}^Q(Y^*, X^*)$ . In this case  $\nu_{up}^Q(T^*) = u_p(T)$ .

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