THE NON - LINEAR WAVE EQUATION WITH DISSIPATION

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A PROJECT IN SYMMETRIES OF DIFFERENTIAL EQUATIONS

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EVERYTHING IS MADE POSSIBLE BY GOD.

ABSTRACT

The aim of the project is to discuss the non-linear wave equations whose coefficients are dependent on first order spatial derivatives. We construct the principal Lie algebra, the equivalence Lie algebra, and the extensions by one of the principal Lie algebra. We further construct the optimal system of one-dimensional subalgebras for first three extended five-dimensional Lie algebras. These are finally used to determine invariant solutions of some examples.

0.1 INTRODUCTION

The Lie group analysis of differential equations is the area of mathematics pioneered by Sophus Lie in the 19th century (1849-1899). The first general solution of the problem of classification was given by Sophus Lie for an extensive class of partial differential equations. [1]. Since then many researchers have done work on various families of differential equations. The results of their work have been captured in several outstanding literary works [2,3].The preliminary group classification by Ibragimov, Torrisi and Valenti [2], gave us up to thirty three equivalence classes of submodels of the wave model of the form

$$u_{tt} = f(x, u_x)u_{xx} + g(x, u_x). \quad (0.1)$$

The present work examines a model which represents families of the nonlinear wave with dissipation, namely

$$u_{tt} + u_t = f(u_x)u_{xx} + g(u_x). \quad (0.2)$$

In this work we use the results of one-dimensional optimal systems (i) of

the equivalence Lie algebra to obtain X_5 and hence the classification of the family (0.2) above ,(ii) of the extended principal Lie algebra of equation (0.2) to calculate the invariant solutions of some examples. The method followed in the construction of the one-dimensional optimal systems is found in the paper by Ibragimov, Torrisi and Valenti [2].In section (0.2) while constructing the principal Lie algebra, we also show

how to determine the Lie point symmetries .We proceed to construct the equivalence Lie algebra, and give the extensions by one of the principal algebra of equation (0.2). In this section we also show the method of determining invariant solutions.In section (0.4) we construct the one-dimensional optimal systems of extended principal Lie algebras L_5.In section (0.5) we calculate the invariant solutions of some one-dimensional subalgebras of each extended algebra L_5.

0.2 PRELIMINARY GROUP CLASSIFICATION

In this section while constructing the principal Lie algebra, we also show how to determine the Lie point symmetries .We proceed to construct the equivalence Lie algebra, and give the extensions by one of the principal algebra of equation (0.2). In this section we also show the method of determining invariant solutions.

0.2.1 PRINCIPAL LIE ALGEBRA

The principal Lie algebra L_p of the non-linear wave equation with dissipation namely

$$u_{tt} + u_t = f(u_x)u_{xx} + g(u_x),$$

is determined as follows:

Let the generator of equation(0.2) be given by

$$X = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{0.3}$$

The second prolongation of (0.3) is given by

$$\widetilde{X}^2 = X + \zeta^t\frac{\partial}{\partial u_t} + \zeta^x\frac{\partial}{\partial u_x} + \zeta^{tt}\frac{\partial}{\partial u_{tt}} + \zeta^{xx}\frac{\partial}{\partial u_{xx}}, \tag{0.4}$$

where

$$\begin{aligned}
\zeta^t &= D_t(\eta) - u_t D_t(\xi^1) - u_x D_t(\xi^2), \\
\zeta^x &= D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2), \\
\zeta^{tt} &= D_t(\zeta^t) - u_{tt} D_t(\xi^1) - u_{tx} D_t(\xi^2), \\
\zeta^{xx} &= D_x(\zeta^x) - u_{tx} D_x(\xi^1) - u_{xx} D_x(\xi^2).
\end{aligned} \tag{0.5}$$

The operators D_t and D_x denote the total derivatives with respect to t and x respectively as follows:

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + \ldots.. \\
D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x} + \ldots..
\end{aligned} \tag{0.6}$$

The determining equation of (0.2) is given by

$$\widetilde{X}^2 \left(u_{tt} + u_t - f(u_x)u_{xx} - g(u_x)\right)|_{(0.2)} = (\zeta^{tt} + \zeta^t - f\zeta^{xx} - f^{u_x}\zeta^x u_{xx} - g\zeta^x)|_{(0.2)} = 0. \tag{0.7}$$

In cases of arbitrary f and g it follows that

$$\zeta^{xx} = \zeta^x = 0, \text{ and } \zeta^{tt} + \zeta^t = 0. \tag{0.8}$$

From the equation (0.8) we have that

$$\begin{aligned}
&\zeta^{tt} + \zeta^t = \eta_{tt} + u_t\left(2\eta_{tu} - \xi^1_{tt} - 2u_x\xi^2_{tu}\right) + u_t^2\left(\eta_{uu} - 2\xi^1_{tu} - u_x\xi^2_{uu}\right) \\
&-u_t^3\xi^1_{uu} - u_{tx}\left(2\xi^1_t + 2u_x\xi^2_u + u_t\xi^2_u\right) + \left(-u_t - f(u_x)u_{xx} - g(u_x)\right) \\
&\left(\eta_u - 2\xi^1_t - 3u_t\xi^1_u\right) + \eta_t + u_t\left(\eta_u - \xi^1_t\right) - u_t^2\xi^1_u - u_x\xi^2_t - u_tu_x\xi^2_u = 0.
\end{aligned} \tag{0.9}$$

From equation (0.9) we obtain

$$\begin{aligned} &\xi^2_u = \xi^1_t = 0. \\ &\xi^1_u = \eta_u = 0. \\ &\xi^2_t = 0. \\ &\eta_{tt} + \eta_t = 0 \quad \Rightarrow \quad \eta = c_1 + c_2 e^{-t}. \end{aligned} \tag{0.10}$$

Thus we have that

$$\xi^1 = c, \qquad \xi^2 = c, \qquad \eta = c_1 + c_2 e^{-t}.$$

Thus the principal Lie algebra L_p of the non-linear wave equation with dissipation namely

$$u_{tt} + u_t = f(u_x)u_{xx} + g(u_x),$$

is spanned by the following generators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = \frac{\partial}{\partial u}, \quad X_4 = e^{-t}\frac{\partial}{\partial u}. \tag{0.11}$$

EQUIVALENCE LIE ALGEBRA AND EXTENSIONS OF THE PRINCIPAL LIE ALGEBRA

We now state the details for determining the equivalence Lie algebra for the equation (0.2) .

The family of non-linear waves $u_{tt} + u_t = f(u_x)u_{xx} + g(u_x)$, can be written as a system of differential equations

$$\begin{aligned} &u_{tt} + u_t = f^1 u_{xx} + f^2 \\ &f^k_x = f^k_t = f^k_u = f^k_{u_t} = 0 \end{aligned} \tag{0.12}$$

k = 1, 2. The equivalence Lie algebra element for the system (4) is given by the generators

$$E = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} + \mu^k\frac{\partial}{\partial f^k}$$

where $\xi = \xi(x,t,u)$, $\tau = \tau(x,t,u)$, $\eta = \eta(x,t,u)$, $\mu^k = \mu^k(x,t,u,u_x,u_t,f^1,f^2)$.

We now introduce the following total derivatives

$$\widetilde{D_\alpha} = \frac{\partial}{\partial \alpha} + f^k_\alpha\frac{\partial}{\partial f^k} + f^k_{\alpha t}\frac{\partial}{\partial f^k_t} + f^k_{\alpha x}\frac{\partial}{\partial f^k_x} + f^k_{\alpha u}\frac{\partial}{\partial f^k_u} + f^k_{\alpha u_t}\frac{\partial}{\partial f^k_{u_t}} + \ldots$$

for $\alpha \in \{x,t,u,u_t\}$.

The extension of the equivalence algebra element E, takes the form

$$\widetilde{E} = E + \zeta^t\frac{\partial}{\partial u_t} + \zeta^x\frac{\partial}{\partial u_x} + \zeta^{xx}\frac{\partial}{\partial u_{xx}} + \varpi^k_t\frac{\partial}{\partial f^k_t} + \varpi^k_x\frac{\partial}{\partial f^k_x} + \varpi^k_u\frac{\partial}{\partial f^k_u} + \varpi^k_{u_t}\frac{\partial}{\partial f^k_{u_t}},$$

where

$$\zeta^i = D_i(\eta) - u_t D_i(\tau) - u_x D_i(\xi)$$

$$\zeta^{ij} = D_i(\zeta^i) - u_{jt} D_i(\tau) - u_{jx} D_i(\xi)$$

for $i, j \in \{x,t\}$ and

$$\varpi^k_\alpha = \widetilde{D_\alpha}(\mu^k) - f^k_t\widetilde{D_\alpha}(\tau) - f^k_x\widetilde{D_\alpha}(\xi) - f^k_u\widetilde{D_\alpha}(\eta) - f^k_{u_t}\widetilde{D_\alpha}(\zeta^t) - f^k_{u_x}\widetilde{D_\alpha}(\zeta^x)$$

where $\alpha \in \{x,t,u,u_t\}$, k = 1, 2.

The invariance condition for the system of equations (0.12)is given by

$$\widetilde{E}(u_{tt} + u_t - f^1 u_{xx} - f^2)|_{(0.12)} = 0 \qquad (0.13)$$

$$\widetilde{E}(f^k_\alpha) = 0 \text{ for } \alpha \in \{x,t,u,u_t\}. \qquad (0.14)$$

We thus obtain

$$\zeta^{tt} + \zeta^{t} - \mu^{1} u_{xx} - f^{'} \zeta^{xx} - \mu^{2} = 0$$

and

$$\varpi^{k}_{\alpha} = 0 \text{ for } \alpha \in \{x, t, u, u_t\}.$$

From the equations (0.13) we have

$$(\mu^{k})_{\alpha} = (\zeta^{x})_{\alpha} = 0, \alpha \in \{x, t, u, u_t\}$$

and $k = 1, 2$, which implies that the μ^k are independent of x, t, u, u_t and hence

$$\mu^{k} = \mu^{k}(u_x, f^{1}, f^{2}), \qquad k = 1, 2.$$

Furthermore $(\zeta^{x})_{\alpha} = 0$ yields

$$\begin{aligned} \xi &= a_1 x + a_2 u + p(t) \\ \tau &= \tau(t) \\ \eta &= b_1 u + b_2 x + q(t) \end{aligned} \tag{0.15}$$

where $a_1, a_2; b_1, b_2$ are constants. The equations (0.15), together with the invariance condition yield

$$\begin{aligned} \xi &= a_1 x + a_2 \\ \tau &= a_3 \\ \eta &= a_4 u + a_5 t + a_6 x + a_7 \\ \mu^{1} &= 2 a_1 f^{1} \\ \mu^{2} &= a_5 + a_4 f^{2}. \end{aligned} \tag{0.16}$$

For the model $u_{tt} + u_t = f(u_x)u_{xx} + g(u_x)$, we have

$$\mu^1 = 2a_1 f$$

$$\mu^2 = a_5 + a_4 g.$$

Therefore we obtain a 7-dimensional equivalence algebra for the non-linear wave equation (0.2), which is spanned by the following operators

$$\begin{aligned} &E_1 = \tfrac{\partial}{\partial x}, \quad E_2 = \tfrac{\partial}{\partial t}, \quad E_3 = \tfrac{\partial}{\partial u}, \quad E_4 = x\tfrac{\partial}{\partial u} \\ &E_5 = u\tfrac{\partial}{\partial u} + g\tfrac{\partial}{\partial g}, \quad E_6 = t\tfrac{\partial}{\partial u} + \tfrac{\partial}{\partial g}, \quad E_7 = x\tfrac{\partial}{\partial x} + 2f\tfrac{\partial}{\partial f}. \end{aligned} \tag{0.17}$$

The classification of the equation (0.2) is obtained by extending the principal Lie algebra $X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = \frac{\partial}{\partial u}, \quad X_4 = e^{-t}\frac{\partial}{\partial u}$ by X_5 as follows:

One-dimensional Optimal System

In order to determine X_5 and hence the classification of equation (0.2) we will give details of the determination of the one-dimensional optimal systems L_4 below. Since f and g depend on u_x, we prolong the equivalence operators E_i (0.17), to the following operators

$$\widetilde{E}_i = E_i + \zeta^x \frac{\partial}{\partial u_x}, \text{ for } i = 1, 2, ..., 7.$$

Therefore we have

$$\begin{aligned} &\widetilde{E}_i = E_i, \text{ for } i = 1, 2, 3 \\ &\widetilde{E}_4 = x\frac{\partial}{\partial u} + \frac{\partial}{\partial u_x}, \quad \widetilde{E}_5 = u\frac{\partial}{\partial u} + g\frac{\partial}{\partial g} + u_x\frac{\partial}{\partial u_x} \\ &\widetilde{E}_6 = E_6, \qquad E_7 = x\frac{\partial}{\partial x} + 2f\frac{\partial}{\partial f} - u_x\frac{\partial}{\partial u_x}, \end{aligned} \tag{0.18}$$

We form new operators Z_i by projecting each $\widetilde{E}_i$ (0.18), onto the (u_x, f, g)-subspace of the $(x, t, u, u_t, u_x, f, g)-$space. We have

$$pr(\widetilde{E}_i) = 0, \text{ for } i = 1, 2, 3$$

$$Z_i = pr(\widetilde{E}_{i+3}), \text{ for } i = 1, 2, 3, 4.$$

$$Z_1 = pr(\widetilde{E}_5) = \frac{\partial}{\partial u_x}$$

$$Z_2 = g\frac{\partial}{\partial g} + u_x\frac{\partial}{\partial u_x}, Z_3 = \frac{\partial}{\partial g},$$

$$Z_4 = 2f\frac{\partial}{\partial f} - u_x\frac{\partial}{\partial u_x},$$

We now consider the algebra L_4, which is spanned by Z_1, Z_2, Z_3, Z_4. We wish to determine the optimal system of one-dimensional subalgebras of the algebra L_4. The non-zero structure constants of L_4 are as follows:

$$[Z_1, Z_2] = Z_1 \quad, [Z_1, Z_4] = -Z_1, \quad [Z_2, Z_3] = -Z_3,$$

The generators of the adjoint algebra L_4^A are given by

$$A_1 = Z_1\frac{\partial}{\partial Z_2} - Z_1\frac{\partial}{\partial Z_4}$$

$$A_2 = -Z_1\frac{\partial}{\partial Z_1} - Z_3\frac{\partial}{\partial Z_3}$$

$$A_3 = Z_3\frac{\partial}{\partial Z_3} \tag{0.19}$$

$$A_4 = Z_1\frac{\partial}{\partial Z_1}$$

In order to obtain the elements of the adjoint group G^A or the group of inner automorphisms of the algebra L_4, we integrate the equations (0.19) to obtain a four parameter Lie group:

$$A_1 : \overline{Z}_2 = Z_2 + a_1 Z_1, \qquad \overline{Z}_4 = Z_4 - a_1 Z_1$$

$$A_2 : \overline{Z}_1 = a_2^{-1} Z_1, \qquad \overline{Z}_3 = a_2^{-1} Z_3$$

$$A_3 : \overline{Z}_2 = Z_2 + a_3 Z_3,$$

$$A_4 : \overline{Z}_1 = a_4 Z_1$$

A matrix representation of an arbitrary element of the adjoint group G^A is of the form

$$M = \begin{bmatrix} a_2^{-1} a_4 & a_1 & 0 & -a_1 \\ 0 & 1 & 0 & 0 \\ 0 & a_2^{-1} a_3 & a_2^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

If we let $Z \in L_4$ be given by

$$Z = e^1 Z_1 + e^2 Z_2 + e^3 Z_3 + e^4 Z_4$$

$$Z \equiv e = (e^1, e^2, e^3, e^4),$$

then $\overline{e} = Me$ defines an equivalence relation in L_4 and hence subdivides this algebra into equivalence classes. The components of Z map as follows under M :

$$\begin{aligned} \overline{e}^1 &= a_2^{-1} a_4 e^1 + a_1 (e^2 - e^4) \\ \overline{e}^2 &= e^2 \\ \overline{e}^3 &= a_2^{-1} a_3 e^2 + a_2^{-1} e^3 \\ \overline{e}^4 &= e^4 \end{aligned}$$

Therefore the optimal system of one-dimensional subspaces of L_4, obtained through the adjoint group G^A, are as follows:

Z	**Generator**	**Restrictions**
$Z^{(1)}$	$\alpha Z_2 + Z_4$	$\alpha \neq 1$
$Z^{(2)}$	$\alpha Z_2 + \beta Z_3 + Z_4$	$\alpha \neq \beta$
$Z^{(3)}$	$Z_1 + Z_2 + Z_4$	
$Z^{(4)}$	$Z_1 + Z_2 + \alpha Z_3 + Z_4$	
$Z^{(5)}$	Z_3	
$Z^{(6)}$	$Z_3 + Z_4$	
$Z^{(7)}$	$Z_1 + Z_3$	

Consider

$$Z^{(1)} = \alpha Z_2 + Z_4,$$

with $\alpha \neq 1$.

$$\begin{aligned} Z^{(1)} &= \alpha(g\frac{\partial}{\partial g} + u_x\frac{\partial}{\partial u_x}) + 2f\frac{\partial}{\partial f} - u_x\frac{\partial}{\partial u_x} \\ &= \alpha g\frac{\partial}{\partial g} + 2f\frac{\partial}{\partial f} + (\alpha - 1)u_x\frac{\partial}{\partial u_x}. \end{aligned}$$

From the characteristic equation

$$\frac{dg}{\alpha g} = \frac{df}{2f} = \frac{du_x}{(\alpha - 1)u_x},$$

we obtain

$$f = u_x^{\frac{2}{\alpha-1}} \qquad \text{and} \qquad g = u_x^{\frac{\alpha}{\alpha-1}}.$$

To obtain the extending vector X_5, we let

$$\widetilde{Z} = \alpha E_5 + E_7$$

$$= \alpha(u\frac{\partial}{\partial u} + g\frac{\partial}{\partial g}) + x\frac{\partial}{\partial x} + 2f\frac{\partial}{\partial f}.$$

Let X_5 be the projection of $\widetilde{Z}$ onto the $(x, t, u)-$ space, i.e

$$X_5 = x\frac{\partial}{\partial x} + \alpha u\frac{\partial}{\partial u}.$$

For the vectors $Z^{(i)}$, $i = 2, 3, \cdots, 7$, we proceed in a similar manner in order to determine the functions f, g and the extension vector X_5. The classification for equation (0.2) is given in the following table:

$Z^{(i)}$	$f\,(u_x)$	$g\,(u_x)$	X_5	Restrictions
$Z^{(1)}$	$u_x^{\frac{2}{\alpha-1}}$	$u_x^{\frac{2}{\alpha-1}}$	$x\frac{\partial}{\partial x} + \alpha u\frac{\partial}{\partial u}$	$\alpha \neq 1$
$Z^{(2)}$	$u_x^{\frac{2}{\alpha-1}}$	$\alpha^{-1}\left(u_x^{\frac{2}{\alpha-1}} - \beta\right)$	$x\frac{\partial}{\partial x} + (\alpha u + \beta t)\frac{\partial}{\partial u}$	$\beta \neq \alpha$
$Z^{(3)}$	e^{2u_x}	C	$x\frac{\partial}{\partial x} + (u + x)\frac{\partial}{\partial u}$	
$Z^{(4)}$	e^{2u_x}	αu_x	$x\frac{\partial}{\partial x} + (u + x + \alpha t)\frac{\partial}{\partial u}$	
$Z^{(5)}$	none	none		
$Z^{(6)}$	u_x^{-2}	$-\ln u_x$	$x\frac{\partial}{\partial x} + t\frac{\partial}{\partial u}$	
$Z^{(7)}$	C	u_x	$(t + x)\frac{\partial}{\partial u}$	

In what follows we will give the classification for equation (0.2) for the listed generators X_5.

1. If $X_5 = x\frac{\partial}{\partial x} + (x + u)\frac{\partial}{\partial u}$ then $f = e^{2u_x}$, and $g = c$
2. If $X_5 = x\frac{\partial}{\partial x} + (x + u + \alpha t)\frac{\partial}{\partial u}$ then $f = e^{2u_x}$, and $g = \alpha u_x$
3. If $X_5 = (x + t)\frac{\partial}{\partial u}$ then $f = c$, and $g = u_x$
4. If $X_5 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial u}$ then $f = u_x^{-2}$, and $g = -\ln u_x$
5. If $X_5 = x\frac{\partial}{\partial x} + \alpha u\frac{\partial}{\partial u}$ then $f = u_x^{\frac{2}{\alpha-1}}$, and $g = u_x^{\frac{2}{\alpha-1}}$ for $\alpha \neq 1$
6. If $X_5 = x\frac{\partial}{\partial x} + (\alpha u + \beta t)\,\frac{\partial}{\partial u}$ then $f = u_x^{\frac{2}{\alpha-1}}$, and $g = \alpha^{-1}(u_x^{\frac{2}{\alpha-1}} - \beta)$ for $\alpha \neq \beta$

Each extension will give us a five-dimensional Lie algebra L_5. From the above we will concentrate on the first four whose equations are given by the following

$$u_{tt} + u_t = e^{2u_x} u_{xx} + c. \tag{0.20}$$

$$u_{tt} + u_t = e^{2u_x} u_{xx} + \alpha u_x \tag{0.21}$$

$$u_{tt} + u_t = c u_{xx} + u_x. \tag{0.22}$$

$$u_{tt} + u_t = u_x^{-2} u_{xx} + \ln u_x. \tag{0.23}$$

From the latter we have five-dimensional Lie algebras for each of the equations (0.20) to (0.23). We will only construct optimal systems of one-dimensional Lie subalgebras for the first three equations. We will then calculate the invariant solutions using some of these one-dimensional subalgebras.

0.2.2 INVARIANT SOLUTIONS.

A useful feature of the symmetry group is that it conserves the set of solutions in the differential equations admitting this group. That is, the symmetry transformations merely permute the integral curves among themselves. Such integral curves are termed invariant solutions.

Invariant solutions of differential equations are determined from the symmetries as follows:

Suppose we have the differential equation of the form

$$F_\sigma\left(x, y, y', \ldots\ldots, y^{(n)}\right) = 0, \qquad \sigma = 1, \ldots\ldots s$$

admitting a group G, having a q-parameter subgroup H generated by

$$X = \xi_v^i\,(x, y)\,\frac{\partial}{\partial x^i} + \eta_v^\alpha\,(x, y)\,\frac{\partial}{\partial y^\alpha}, \qquad v = 1, \ldots\ldots q, \qquad \alpha = 1, \ldots\ldots m.$$

We determine the functionally independent invariants from the above generator.

We express one invariant in terms of the other, substitute this in the original equation and solve the resulting equation.

We determine the invariant solutions.(See section 0.4). In our case we obtained the regular invariants.[See [4] for the difference between regular and singular invariants.]

0.3 ONE DIMENSIONAL OPTIMAL SYSTEMS

0.3.1 INNER AUTOMORPHISMS

We construct the inner automorphisms for each one of the equations (0.20) to (0.23). We will show the details of the construction of the inner automorphisms using one equation only, and for the other three, we shall simply state the results , because the procedure is the same.

We consider the equation $u_{tt} + u_t = e^{2u_x} u_{xx} + c$, whose set of generators is given by the following point symmetries $X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial t}$, $X_3 = \frac{\partial}{\partial u}$, $X_4 = e^{-t}\frac{\partial}{\partial u}$, $X_5 = x\frac{\partial}{\partial x} + (u + x)\frac{\partial}{\partial u}$.

These generators form the basis for the Lie algebra L_5.

The non-zero commutators are given by

$$\begin{array}{lll} [X_1, X_5] = X_1 + X_2; & [X_2, X_4] = -X_4; & [X_3, X_5] = X_3; \\ [X_4, X_2] = X_4; & [X_4, X_5] = X_4; & [X_5, X_1] = -\left(X_1 + X_2\right); \\ [X_5, X_3] = -X_3; & [X_5, X_4] = -X_4. & \end{array} \tag{0.24}$$

ADJOINT GROUP FOR ALGEBRA L_5

In order to determine the inner automorphims, the procedure is found in the paper by Ibragimov, Torrisi, and Valenti [see[2]]. We denote by A the elements of the adjoint algebra adL_5. The operators

$$A_\alpha = [X_\alpha, X_\beta]\frac{\partial}{\partial X_\beta}$$

form the basis elements of the algebra adL_5.

Using the above commutators (0.24), we get that

$$\begin{aligned} A_1 &= \left(X_1 + X_3\right)\tfrac{\partial}{\partial X_5} \\ A_2 &= -X_4\tfrac{\partial}{\partial X_4} \\ A_3 &= X_3\tfrac{\partial}{\partial X_5} \\ A_4 &= X_4\tfrac{\partial}{\partial X_5} + X_4\tfrac{\partial}{\partial X_2} \\ A_5 &= -\left(X_1 + X_3\right)\tfrac{\partial}{\partial X_1} - X_3\tfrac{\partial}{\partial X_3} - X_4\tfrac{\partial}{\partial X_4}. \end{aligned} \tag{0.25}$$

To obtain the Lie group of transformations, we integrate the above equations (0.25) , and obtain the inner automorphisms.

For A_1 we have that

$$\frac{d\overline{x}^5}{da_1} = \overline{x}^1 + \overline{x}^3. \tag{0.26}$$

Integrating the above equation (0.26) with respect to a_1, we obtain that

$$\overline{x}^5 = a_1\overline{x}^1 + a_1\overline{x}^3 + c.$$

Since we have that $\overline{x}^i|_{a_1=0} = x^i$ for $i = 1, 2, 3, 4, 5$, then $\overline{x}^5|_{a_1=0} = x^5$.

Hence

$$\overline{x}^5 = a_1x^1 + a_1x^3 + x^5 \quad ; \quad \text{and} \quad \overline{x}^i = x^i \quad for \quad i = 1, 2, 3, 4.$$

Proceeding in the same way , we obtain a list of inner automorphisms given by

$$\begin{aligned}
A_1 &: \overline{x}^5 = a_1x^1 + a_1x^3 + x^5 \quad ; \quad \text{and} \quad \overline{x}^i = x^i \quad for \quad i = 1, 2, 3, 4. \\
A_2 &: \overline{x}^4 = e^{-a_2}x^4 \ ; \quad \overline{x}^i = x^i \quad for \quad i = 1, 2, 3, 5. \\
A_3 &: \overline{x}^5 = a_3x^3 + x^5 \quad ; \quad \text{and} \quad \overline{x}^i = x^i \quad for \quad i = 1, 2, 3, 4. \\
A_4 &: \overline{x}^2 = a_4x^4 + x^2; \ \overline{x}^5 = a_4x^4 + x^5 \quad \overline{x}^i = x^i \quad for \quad i = 1, 3, 4. \\
A_5 &: \overline{x}^1 = -e^{-a_5}x^1 - e^{-a_5}x^3; \overline{x}^3 = e^{-a_5}x^3; \ \overline{x}^4 = e^{-a_5}x^4 \quad \overline{x}^i = x^i \quad for \quad i = 1, 2, 3, 4.
\end{aligned} \tag{0.27}$$

The group of transformations (0.27) can be represented by matrices $M_1(a_1), \ldots M_5(a_5)$ corresponding to the inner automorphisms related to the algebra elements $A_1, \ldots A_5$ respectively as follows:-

$$M_1(a_1) = \begin{bmatrix} 1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & a_1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} ; \; M_2(a_2) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & e^{-a_2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_3(a_3) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & a_3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} ; \; M_4(a_4) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & a_4 & 0 & 1 & a_4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_5(a_5) = \begin{bmatrix} -e^{-a_5} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -e^{-a_5} & 0 & e^{-a_5} & 0 & 0 \\ 0 & a_4 & 0 & e^{-a_5} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The product of the matrices is given by

$$M = M_1(a_1) \times M_2(a_2) \times M_3(a_3) \times M_4(a_4) \times M_5(a_5)$$

$$= \begin{bmatrix} -e^{-a_5} & 0 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 & 0 \\ -e^{-a_5} & 0 & e^{-a_5} & 0 & a_1 + a_3 + a_4 \\ 0 & a_4 & 0 & e^{-a_2}e^{-a_5} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{0.28}$$

We will work with the coordinates of the decomposition $X = \sum_{i=1}^{5} e^i X_i$,that is the vectors $e = (e^1, e^2, e^3, e^4, e^5)$.

If we let $X \in L_5$ then $X = \sum_{i=1}^{5} e^i X_i$, that is $X \equiv e = (e^1, e^2, e^3, e^4, e^5)$.The equation $\overline{e} = Me$ defines an equivalence relation in L_5, and hence subdivides this algebra into equivalence classes. The components of X map as follows under M (0.28):

$$\begin{aligned} \overline{e}^1 &= -e^{-a_5}e^1 + a_1 e^5 \\ \overline{e}^2 &= e^2 \\ \overline{e}^3 &= -e^{-a_5}e^1 + -e^{-a_5}e^3 + (a_1 + a_3 + a_4)\, e^5 \\ \overline{e}^4 &= a_4 e^2 + e^{-a_2}e^{-a_5}e^4 \\ \overline{e}^5 &= e^5. \end{aligned} \tag{0.29}$$

If we let $e^{-a_5} = a_5^{'} > 0, e^{-a_2} = a_2^{'} > 0$, then the transformations (0.29) may be re-written as

$$\begin{aligned}
\bar{e}^1 &= -a_5' e^1 + a_1 e^5 & a_5' > 0 \\
\bar{e}^2 &= e^2 \\
\bar{e}^3 &= -a_5' \left(e^1 + e^3\right) + \left(a_1 + a_3 + a_4\right) e^5 \; ; \; a_5' > 0 \\
\bar{e}^4 &= a_4 e^2 + a_5' a_2' e^4 & ; \; a_5', a_2' > 0 \\
\bar{e}^5 &= e^5,
\end{aligned} \tag{0.30}$$

where a_1, a_3, a_4 are real parameters, while a_2' and a_5' are positive real parameters.

0.3.2 CONSTRUCTION OF THE OPTIMAL SYSTEM OF ONE DIMENSIONAL SUBALGEBRAS OF L_5.

In this section we will construct optimal systems of one dimensional subalgebras for only three equations (0.10 to 0.22). For the construction method see Ibragimov, Torrisi, and Valenti [2].

The transformation (0.30) leaves invariant the components e^2 and e^5. We have to look over all possibilities for these components and in every case simplify other components by means of transformation (0.30).

Case 1: $e^2 \neq 0, e^5 \neq 0$.

If $a_5' = 1, a_1 = \frac{e^1}{e^5}$, then $\bar{e}^1 = 0$.

If $a_2' = 1, a_4 = -\frac{e^4}{e^2}$, then $\bar{e}^4 = 0$

The vector $\bar{e}$ is then given by

$$\bar{e} = \left(0, e^2, -e^3 + \left(a_3 + a_4\right) e^5, 0, e^5\right).$$

Setting $\frac{e^3}{e^5} = a_3 + a_4$, then $\bar{e} = \left(0, e^2, 0, 0, e^5\right)$.

Dividing the above equation by e^5, the vector $\overline{e}$ is transformed to

$$\overline{e} = (0, \alpha, 0, 0, 1) \tag{0.31}$$

where $\alpha = \frac{e^2}{e^5} \neq 0$.

Case 2 : $e^2 = 0, e^5 \neq 0$.

The vector $\overline{e}$ is then given by

$$\overline{e} = \left(-a_5' e^1 + a_1 e^5, 0, -a_5' \left(e^1 + e^3\right) + \left(a_1 + a_3 + a_4\right) e^5, a_2' a_5' e^4, e^5\right).$$

If we let $a_5' = 1$, and $a_1 = \frac{e^1}{e^5}$ then $\overline{e}^1 = 0$.

If $a_2' = 1$,and $\frac{e^3}{e^5} = a_3 + a_4$, then $\overline{e} = (0, 0, 0, e^4, e^5)$.

Dividing the above equation by e^5, the vector $\overline{e}$ is transformed to

$$\overline{e} = (0, 0, 0, \beta, 1) \tag{0.32}$$

where $\beta = \frac{e^4}{e^5} \neq 0$.

Case 3 : $e^2 \neq 0, e^5 = 0$.

The vector $\overline{e}$ then is given by

$$\overline{e} = \left(-a_5' e^1, e^2, -a_5' \left(e^1 + e^3\right), a_4 e^2 + a_2' a_5' e^4, 0\right).$$

Subcase 1: If $-e^1 \neq e^3$, $a_2' = 1$, $a_5' = 1$ and $a_4 = -\frac{e^4}{e^2}$

then $\overline{e} = (-e^1, e^2, -e^1 - e^3, 0, 0)$.

Dividing the above equation by e^2, the vector $\overline{e}$ is transformed to

$$\overline{e} = (\theta, 1, \theta + \tau, 0, 0) \tag{0.33}$$

where $\theta = -\frac{e^1}{e^2} \neq 0, \tau = -\frac{e^3}{e^2} \neq 0$.

Subcase 2: If $e^1 = -e^3, a_2' = 1, \quad a_5' = 1$ and $a_4 = -\frac{e^4}{e^2}$

then the vector $\bar{e} = (-e^1, e^2, 0, 0, 0)$

Dividing the above equation by e^2, the vector $\bar{e}$ is transformed to

$$\bar{e} = (\theta, 1, 0, 0, 0) \tag{0.34}$$

where $\theta = -\frac{e^1}{e^2} \neq 0$.

Case 4 : $e^2 = 0, e^5 = 0$.

The vector $\bar{e}$ then is given by

$$\bar{e} = \left(-a_5' e^1, 0, -a_5' \left(e^1 + e^3\right), a_2' a_5' e^4, 0\right).$$

Let $\quad a_5' = 1$ and $a_4 = 1$.

The vector $\bar{e}$ is then given by

$$\bar{e} = \left(-e^1, 0, -e^1 - e^3, e^4, 0\right).$$

Subcase 1: $e^1 \neq 0, e^4 \neq 0, e^1 = -e^3$

The vector $\bar{e}$ is then given by

$$\bar{e} = \left(-e^1, 0, 0, e^4, 0\right).$$

Dividing the above equation by e^4, the vector $\bar{e}$ is transformed to

$$\bar{e} = (\beta, 0, 0, 1, 0) \tag{0.35}$$

where $\beta = -\frac{e^1}{e^4} \neq 0$.

Subcase 2: $e^1 \neq 0, e^4 \neq 0, e^1 \neq -e^3$.

The vector $\overline{e}$ is then given by

$$\overline{e} = \left(e^1, 0, -e^1 - e^3, e^4, 0\right).$$

Dividing the above equation by e^4, the vector $\overline{e}$ is transformed to

$$\overline{e} = (\beta, 0, \beta + \phi, 1, 0) \tag{0.36}$$

where $\beta = -\frac{e^1}{e^4} \neq 0, \phi = -\frac{e^3}{e^4} \neq 0$.

Subcase 3: $e^1 \neq 0, e^4 = 0, e^1 = -e^3$.

The vector $\overline{e}$ is then given by

$$\overline{e} = \left(-e^1, 0, 0, 0, 0\right).$$

Dividing the above equation by -e^1, the vector $\overline{e}$ is transformed to

$$\overline{e} = (1, 0, 0, 0, 0) \tag{0.37}$$

Subcase 4 :$e^1 \neq 0, e^4 = 0, e^1 = -e^3$.

The vector $\overline{e}$ is then given by

$$\overline{e} = \left(-e^1, 0, -e^1 - e^3, 0, 0\right).$$

Dividing the above equation by -e^1, the vector $\overline{e}$ is transformed to

$$\overline{e} = (1, 0, 1 + \rho, 0, 0) \tag{0.38}$$

where $\rho = \frac{e^3}{e^1} \neq 0$.

Subcase 5 : $e^1 = 0, e^4 = 0, e^3 \neq 0.$

The vector $\overline{e}$ is then given by

$$\overline{e} = \left(0, 0, e^3, 0, 0\right).$$

Dividing the above equation by e^3, the vector $\overline{e}$ is transformed to

$$\overline{e} = (0, 0, 1, 0, 0). \tag{0.39}$$

Subcase 6 : $e^1 = 0, e^4 \neq 0, e^3 = 0.$

The vector $\overline{e}$ is then given by

$$\overline{e} = \left(0, 0, 0, e^4, 0\right).$$

Dividing the above equation by e^4, the vector $\overline{e}$ is transformed to

$$\overline{e} = (0, 0, 0, 1, 0). \tag{0.40}$$

The list of possibilities is given by :

$(0, \alpha, 0, 0, 1)$, where $\alpha = \frac{e^2}{e^5} \neq 0.$

$(0, 0, 0, \beta, 1)$, where $\beta = -\frac{e^1}{e^5} \neq 0, \gamma = \frac{e^3}{e^5} \neq 0.$

$(\theta, 1, \theta + \tau, 0, 0)$, $\theta = -\frac{e^1}{e^2} \neq 0, \tau = -\frac{e^3}{e^2} \neq 0$

$(1, 0, 0, 0, 0)$.

$(\beta, 0, 0, 1, 0)$, where $\beta = -\frac{e^1}{e^4} \neq 0.$

$(\beta, 0, \beta + \gamma, 1, 0)$, where $\beta = -\frac{e^1}{e^4} \neq 0, \gamma = \frac{e^3}{e^4} \neq 0.$

$(1, 0, 1 + \rho, 0, 0)$ where $\rho = \frac{e^3}{e^1} \neq 0.$

$(0, 0, 1, 0, 0)$

$(0, 0, 0, 1, 0)$.

Summarizing the results (0.31) to (0.40), we obtain the following optimal system of one dimensional subalgebras of L_5.

$$X^{(1)} = \alpha X_2 + X_5$$

$$X^{(2)} = \beta X_4 + X_5$$

$$X^{(3)} = \theta X_1 + X_2 + (\theta + \tau) X_3$$

$$X^{(4)} = X_2$$

$$X^{(5)} = \beta X_1 + X_4$$

$$X^{(6)} = \beta X_1 + + (\beta + \gamma) X_3 + X_4$$

$$X^{(7)} = X_1$$

$$X^{(8)} = X_1 + (1 + \rho) X_3$$

$$X^{(9)} = X_3$$

$$X^{(10)} = X_4$$

where $\alpha, \beta, \gamma, \rho, \theta, \tau$ are arbitrary constants.

We now state the results of the other three equations.

Consider the equation $u_{tt}+u_t = e^{2u_x}u_{xx}+\alpha u_x$, whose the set of generators is given by the following point symmetries $X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial t}$, $X_3 = \frac{\partial}{\partial u}$, $X_4 = e^{-t}\frac{\partial}{\partial u}$, $X_5 = x\frac{\partial}{\partial x} + (u + x + \alpha t)\frac{\partial}{\partial u}$. The non-zero commutators are given by

$$\begin{array}{lll} [X_1, X_5] = -(X_1 + X_3); & [X_2, X_4] = -X_4; & [X_2, X_5] = \alpha X_3; \\ [X_3, X_5] = X_3; & [X_4, X_2] = X_4; & [X_4, X_5] = X_4; \\ [X_5, X_1] = X_1 + X_3; & [X_5, X_3] = -X_3; & [X_5, X_4] = -X_4. \end{array} \tag{0.41}$$

Using the above commutators (0.41), we obtain the following adjoint algebra generators.

$$\begin{aligned} A_1 &= (X_1 + X_3)\frac{\partial}{\partial X_5} \\ A_2 &= -X_4\frac{\partial}{\partial X_4} \\ A_3 &= X_3\frac{\partial}{\partial X_5} \\ A_4 &= X_4\frac{\partial}{\partial X_5} + X_4\frac{\partial}{\partial X_2} \\ A_5 &= -(X_1 + X_3)\frac{\partial}{\partial X_1} - X_3\frac{\partial}{\partial X_3} - X_4\frac{\partial}{\partial X_4} \end{aligned} \tag{0.42}$$

Their inner automorphisms will be given by the following list

$$\begin{array}{l} A_1 : \overline{x}^5 = a_1x^1 + a_1x^3 + x^5 \quad ; \quad \text{and} \quad \overline{x}^i = x^i \quad for \quad i = 1,2,3,4. \\ A_2 : \overline{x}^4 = e^{-a_2}x^4 \ ; \overline{x}^5 = \alpha a_2 x^3 + x^5; \quad \overline{x}^i = x^i \quad for \quad i = 1,2,3,. \\ A_3 : \overline{x}^5 = a_3x^3 + x^5 \quad ; \quad \text{and} \quad \overline{x}^i = x^i \quad for \quad i = 1,2,3,4. \\ A_4 : \overline{x}^2 = a_4x^4 + x^2; \ \overline{x}^5 = a_4x^4 + x^5; \quad \overline{x}^i = x^i \quad for \quad i = 1,3,4. \\ A_5 : \overline{x}^1 = e^{-a_5}(x^1 - x^3); \overline{x}^2 = \alpha a_5 e^{-a_5}x^3 + x^2; \overline{x}^3 = e^{-a_5}x^3; \ \overline{x}^4 = e^{-a_5}x^4 \quad \overline{x}^5 = x^5. \end{array} \tag{0.43}$$

The matrices $M_1(a_1), \ldots M_5(a_5)$, corresponding to the $A_1, \ldots A_5$ respectively for the above table (0.43) are given by :-

$$M_1(a_1) = \begin{bmatrix} 1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & a_1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} ; \; M_2(a_2) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \alpha a_2 \\ 0 & 0 & 0 & e^{-a_2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_3(a_3) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & a_3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} ; \; M_4(a_4) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & a_4 & 0 & 1 & a_4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_5(a_5) = \begin{bmatrix} e^{-a_5} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -e^{-a_5} & \alpha e^{-a_5} & e^{-a_5} & 0 & 0 \\ 0 & 0 & 0 & e^{-a_5} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The product of the matrices is given by

$$M = M_1(a_1) \times M_2(a_2) \times M_3(a_3) \times M_4(a_4) \times M_5(a_5)$$

$$= \begin{bmatrix} e^{-a_5} & 0 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 & 0 \\ -e^{-a_5} & \alpha a_5 e^{-a_5} & e^{-a_5} & 0 & a_1+\alpha a_2+a_4 \\ 0 & a_4 e^{-a_2} & 0 & e^{-a_2}e^{-a_5} & a_4 e^{-a_2} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (0.44)$$

The group of transformations of vectors $\overline{e}$ is given by the following table

$$\begin{aligned} \overline{e}^1 &= a_5' e^1 + a_1 e^5 && ;\ a_5' > 0 \\ \overline{e}^2 &= e^2 \\ \overline{e}^3 &= a_5' e^1 + \alpha \left(a_5'\right)^2 e^2 + a_5' e^3 + \left(a_1 + \alpha a_2 + a_3\right) e^5 && ;\ a_5' > 0 \\ \overline{e}^4 &= a_2' a_4 e^2 + a_2' a_5' e^4 + a_2' a_4 e^5 && ;\ a_2', a_5' > 0 \\ \overline{e}^5 &= e^5. \end{aligned} \qquad (0.45)$$

where $e^{-a_5} = a_5' > 0, e^{-a_2} = a_2' > 0.$

The optimal system of one dimensional subalgebras of L_5 is given by the following table (0.46)

$$\begin{aligned} X^{(1)} &= \gamma X_2 + X_5. \\ X^{(2)} &= X_1 + X_2. \\ X^{(3)} &= \alpha X_3 + X_5. \\ X^{(4)} &= X_1 + X_4. \end{aligned} \qquad (0.46)$$

where $\beta = \frac{e^3}{e^5} \neq 0, a_1 = \frac{e^1}{e^5} \neq 0, \rho = \frac{e^3}{e^2} \neq 0, \gamma = \frac{e^2}{e^5} \neq 0, \theta = \frac{e^3}{e^1} \neq 0, \phi = \frac{e^1}{e^2} \neq 0,$

are arbitrary constants.

Consider the equation $u_{tt} + u_t = c u_{xx} + u_x$ whose the set of generators are given by $X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial t}$, $X_3 = \frac{\partial}{\partial u}$, $X_4 = e^{-t}\frac{\partial}{\partial u}$, $X_5 = (x+t)\frac{\partial}{\partial u}$, the non-zero commutators are given by

$$\begin{aligned} &[X_1, X_5] = X_3; \quad [X_2, X_4] = -X_4; \quad [X_2, X_5] = X_3; \\ &[X_4, X_2] = X_4; \quad [X_5, X_1] = -X_3; \quad [X_5, X_2] = -X_3. \end{aligned} \tag{0.47}$$

Using the commutators (0.47), we get that

$$\begin{aligned} A_1 &= X_3\frac{\partial}{\partial X_5} \\ A_2 &= -X_4\frac{\partial}{\partial X_4} + X_3\frac{\partial}{\partial X_5} \\ A_3 &= \text{Identity} \\ A_4 &= X_4\frac{\partial}{\partial X_2} \\ A_5 &= -X_3\frac{\partial}{\partial X_1} - X_3\frac{\partial}{\partial X_2} \end{aligned} \tag{0.48}$$

for which the inner automorphisms will be given by the following table

$$\begin{aligned} &A_1 : \overline{x}^5 = a_1x^3 + x^5 \quad ; \quad \text{and} \quad \overline{x}^i = x^i \quad for \quad i = 1,2,3,4. \\ &A_2 : \overline{x}^4 = e^{-a_2}x^4 \ ; \overline{x}^5 = a_2x^3 + x^5; \quad \overline{x}^i = x^i \quad for \quad i = 1,2,3. \\ &A_3 : \text{Identity} \\ &A_4 : \overline{x}^2 = a_4x^4 + x^2; \quad \overline{x}^i = x^i \quad for \quad i = 1,3,4,5. \\ &A_5 : \overline{x}^1 = -a_5x^3 + x^1; \overline{x}^2 = -a_5x^3 + x^2; \quad \overline{x}^i = x^i; for \quad i = 3,4,5. \end{aligned} \tag{0.49}$$

The matrices $M_1(a_1), ...M_5(a_5)$ corresponding $A_1, ...A_5$ respectively for the above table (0.49) are given by :-

$$M_1(a_1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & a_1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} ; M_2(a_2) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & a_2 \\ 0 & 0 & 0 & e^{-a_2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_4(a_4) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & a_4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} ; M_5(a_5) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -a_5 & -a_5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The product of the matrices is given by

$$\begin{aligned} M &= M_1(a_1) \times M_2(a_2) \times M_4(a_4) \times M_5(a_5) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -a_5 & -a_5 & 1 & 0 & a_1 + a_2 \\ 0 & 0 & 0 & e^{-a_2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned} \tag{0.50}$$

The group of transformations of vectors $\bar{e}$ is given by the following table

$$
\begin{aligned}
\overline{e}^1 &= e^1 \\
\overline{e}^2 &= e^2 \\
\overline{e}^3 &= -a_5(e^1 + e^2) + e^3 + (a_1 + a_2)e^5 \\
\overline{e}^4 &= a_2' a_4 e^4 \\
\overline{e}^5 &= e^5
\end{aligned} \tag{0.51}
$$

where $e^{-a_2} = a_2' > 0$.

The optimal system of one dimensional subalgebras of L_5 is given by the following table (0.52)

$$
\begin{aligned}
X^{(1)} &= \alpha X_1 + \beta X_2 + \gamma X_3 + X_4 + X_5. \\
X^{(2)} &= \alpha X_1 + \beta X_2 + X_4 + X_5. \\
X^{(3)} &= \alpha X_1 + \beta X_2 + X_5. \\
X^{(4)} &= \beta X_2 + \gamma X_3 + X_4 + X_5. \\
X^{(5)} &= \beta X_2 + X_4 + X_5. \\
X^{(6)} &= \beta X_2 + X_5. \\
X^{(7)} &= \gamma X_3 + X_4 + X_5. \\
X^{(8)} &= X_4 + X_5. \\
X^{(9)} &= X_5.
\end{aligned} \tag{0.52}
$$

$$
\begin{aligned}
X^{(10)} &= X_3 + X_5. \\
X^{(11)} &= X_4. \\
X^{(12)} &= X_1 + \tau X_2 + \varkappa X_3 + X_4. \\
X^{(13)} &= X_1 + \tau X_2 + X_4. \\
X^{(14)} &= X_1 + \tau X_2. \\
X^{(15)} &= \alpha X_1 + \gamma X_3 + X_4 + X_5. \\
X^{(16)} &= \alpha X_1 + X_4 + X_5. \\
X^{(17)} &= \alpha X_1 + X_5.
\end{aligned}
$$

where $\alpha = \frac{e^1}{e^5} \neq 0, \beta = \frac{e^2}{e^5} \neq 0, \gamma = \frac{e^3}{e^5} \neq 0, \tau = \frac{e^2}{e^1} \neq 0, \varkappa = \frac{e^3}{e^1} \neq 0$, are arbitrary constants.

For the equation (0.23), namely $u_{tt}+u_t = u_x^{-2}u_{xx}+\ln u_x$ whose the set of generators is given by $X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = \frac{\partial}{\partial u}, \quad X_4 = e^{-t}\frac{\partial}{\partial u}, X_5 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial u}$, the non zero commutators are given by

$$
\begin{aligned}
&[X_1, X_5] = X_1; \quad [X_2, X_4] = -X_4; \quad [X_2, X_5] = X_3; \\
&[X_4, X_2] = X_4; \quad [X_5, X_1] = -X_1; \quad [X_5, X_2] = -X_3.
\end{aligned} \tag{0.53}
$$

Using the table of commutators (0.53), we get that

$$\begin{aligned} A_1 &= X_1 \tfrac{\partial}{\partial X_5} \\ A_2 &= -X_4 \tfrac{\partial}{\partial X_4} + X_3 \tfrac{\partial}{\partial X_5} \\ A_3 &= \text{Identity} \\ A_4 &= X_4 \tfrac{\partial}{\partial X_2} \\ A_5 &= -X_1 \tfrac{\partial}{\partial X_1} - X_3 \tfrac{\partial}{\partial X_2} \end{aligned} \tag{0.54}$$

for which the inner automorphisms will be given by the following table

$$\begin{aligned} A_1 &: \overline{x}^5 = a_1 x^1 + a_1 x^3 + x^5 \quad ; \quad \text{and} \quad \overline{x}^i = x^i \quad for \quad i = 1,2,3,4. \\ A_2 &: \overline{x}^4 = e^{-a_2} x^4 \ ; \overline{x}^5 = a_2 x^3 + x^5; \quad \overline{x}^i = x^i \quad for \quad i = 1,2,3. \\ A_3 &: \text{Identity} \\ A_4 &: \overline{x}^2 = a_4 x^4 + x^2; \quad \overline{x}^i = x^i \quad for \quad i = 1,3,4,5. \\ A_5 &: \overline{x}^1 = e^{-a_5} x^1; \overline{x}^2 = -a_5 x^3 + x^2; \quad \overline{x}^i = x^i; for \quad i = 3,4,5. \end{aligned} \tag{0.55}$$

The matrices $M_1(a_1), ...M_5(a_5)$ corresponding to $A_1, ...A_5$ respectively for the above table (0.55) are given by :-

$$M_1(a_1) = \begin{bmatrix} 1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} ; M_2(a_2) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & a_2 \\ 0 & 0 & 0 & e^{-a_2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_4\left(a_4\right)=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&a_4&0&1&0\\0&0&0&0&1\end{bmatrix};\ M_5\left(a_5\right)=\begin{bmatrix}e^{-a_5}&0&0&0&0\\0&1&0&0&0\\-e^{-a_5}&-a_5&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}.$$

The product of the matrices is given by

$$\begin{aligned}M &= M_1\left(a_1\right)\times M_2\left(a_2\right)\times M_4\left(a_4\right)\times M_5\left(a_5\right)\\ &= \begin{bmatrix}e^{-a_5}&0&0&0&a_1\\0&1&0&0&0\\0&-a_5&1&0&a_2\\0&a_4e^{-a_2}&0&e^{-a_2}&0\\0&0&0&0&1\end{bmatrix}.\end{aligned} \tag{0.56}$$

The group of transformations of vectors $\overline{e}$ is given by the following table

$$\begin{aligned}&\overline{e}^1=-a_5'e^1+a_1e^5 &&;\ a_5'>0\\ &\overline{e}^2=e^2\\ &\overline{e}^3=-a_5'e^2+e^3+a_2e^5\\ &\overline{e}^4=a_2'a_4e^2+a_2'e^4 && a_2>0\\ &\overline{e}^5=e^5\end{aligned} \tag{0.57}$$

where $e^{-a_5}=a_5'>0, e^{-a_2}=a_2'>0$.

0.4 INVARIANT SOLUTIONS

In this section we calculate the invariant solutions using some one dimensional subalgebras of L_5 for each of the four equations (0.20) to (0.23). We will use ordinary differential equations to find the solutions . If the solution is not immediate we will integrate by using two symmetries which form an algebra.

Consider the equation $u_{tt} + u_t = e^{2u_x} u_{xx} + c$, whose set of generators is given by $X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial t}$, $X_3 = \frac{\partial}{\partial u}$, $X_4 = e^{-t}\frac{\partial}{\partial u}$, $X_5 = x\frac{\partial}{\partial x} + (u+x)\frac{\partial}{\partial u}$.

We will use the one dimensional subalgebra $X = X_1 + (1+\rho)\, X_3$ i.e.

$$X = \frac{\partial}{\partial x} + (1+\rho)\frac{\partial}{\partial u}. \tag{0.58}$$

The characteristic equation of the above generator (0.58) is given by

$$\frac{dt}{0} = \frac{du}{k} = \frac{dx}{1} \qquad \text{where } k = 1 + \rho. \tag{0.59}$$

From equation (0.59) the invariants are given by

$$I_1 = u - kx \qquad ; \qquad I_2 = t. \tag{0.60}$$

If we define $I_1 = \phi\,(I_2)$ for some function ϕ, then

$$u\,(t, x) = kx + \phi\,(t)\,. \tag{0.61}$$

The substitution of (0.61) into equation (0.20) asserts that

$$\begin{aligned} u_t &= \phi^{'}(t) \\ u_{tt} &= \phi^{''}(t) \\ u_x &= k \\ u_{xx} &= 0 \end{aligned}$$

hence

$$u_{tt} + u_t - e^{2u_x}u_{xx} - c = \phi^{''}(t) + \phi^{'}(t) - c = 0. \tag{0.62}$$

The equation (0.62) simplifies to

$$\phi^{''}(t) + \phi^{'}(t) = c, \tag{0.63}$$

which is a second order ODE whose solution is given by

$$\phi(t) = c_1 + c_2e^{-t} + ct - c. \tag{0.64}$$

Thus the invariant solution of (0.20) is given by

$$u(t,x) = kx + c_1 + c_2e^{-t} + ct - c, \tag{0.65}$$

where $k = 1 + \rho$.

Consider the equation $u_{tt} + u_t = e^{2u_x}u_{xx} + \alpha u_x$ which has the following set of generators $X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial t}$, $X_3 = \frac{\partial}{\partial u}$, $X_4 = e^{-t}\frac{\partial}{\partial u}$, $X_5 = x\frac{\partial}{\partial x} + (u + x + \alpha t)\frac{\partial}{\partial u}$.

We will use the one dimensional subalgebra $X = X_1 + X_4$ i.e.

$$X = \frac{\partial}{\partial x} + e^{-t}\frac{\partial}{\partial u}. \tag{0.66}$$

The characteristic equation of the above generator (0.66) is given by

$$\frac{dt}{0} = \frac{du}{e^{-t}} = \frac{dx}{1} \tag{0.67}$$

From equation (0.67) the invariants are given by

$$I_1 = u - xe^{-t} \qquad ; \qquad I_2 = t. \tag{0.68}$$

If we define $I_1 = \phi(I_2)$ for some function ϕ, then

$$u(t,x) = xe^{-t} + \phi(t). \tag{0.69}$$

The substitution of (0.69) into equation (0.21) asserts that

$$\begin{aligned} u_t &= -xe^{-t} + \phi'(t) \\ u_{tt} &= xe^{-t} + \phi''(t) \\ u_x &= e^{-t} \\ u_{xx} &= 0, \end{aligned}$$

hence

$$u_{tt} + u_t - e^{2u_x}u_{xx} - \alpha u_x = \phi''(t) + \phi'(t) - \alpha e^{-t} = 0. \tag{0.70}$$

The equation (0.70) simplifies to

$$\phi''(t) + \phi'(t) = \alpha e^{-t}, \tag{0.71}$$

which is a non-linear second order ODE whose solution is given by

$$\phi(t) = c_1 + c_2 e^{-t} + \alpha e^{-t} - \alpha t e^{-t}.$$

The invariant solution of $u_{tt} + u_t = e^{2u_x} u_{xx} + \alpha u_x$ is given by

$$u(t, x) = x e^{-t} + c_1 + c_2 e^{-t} + \alpha e^{-t} - \alpha t e^{-t}. \tag{0.72}$$

Consider the equation $u_{tt} + u_t = c u_{xx} + u_x$ whose set of generators is given by $X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial t}$, $X_3 = \frac{\partial}{\partial u}$, $X_4 = e^{-t} \frac{\partial}{\partial u}$, $X_5 = (x+t)\frac{\partial}{\partial u}$.

We will use the one dimensional subalgebras $X = \alpha X_1 + X_5$ and $X = \beta X_2 + X_5$ i.e. $X = \alpha\frac{\partial}{\partial x} + (x+t)\frac{\partial}{\partial u}$, and $X = \beta\frac{\partial}{\partial t} + (x+t)\frac{\partial}{\partial u}$ respectively to calculate the invariant solutions of (0.22).

Consider the one dimensional subalgebra

$$X = \alpha\frac{\partial}{\partial x} + (x+t)\frac{\partial}{\partial u}. \tag{0.73}$$

The characteristic equation of (0.73) is given by

$$\frac{dx}{\alpha} = \frac{du}{x+t} = \frac{dt}{0}. \tag{0.74}$$

From equation (0.74) the invariants are given by $I_1 = \alpha u - \frac{1}{2}(x+t)^2$, $I_2 = t$.

If we let I_1 be a function of I_2,

$$u(t,x) = \frac{1}{\alpha}\left\{\frac{(x+t)^2}{2} + \phi(t)\right\} \text{ where } \phi(t) = I_1 \text{ i.e } I_1 = \phi(I_2)\,. \tag{0.75}$$

The substitution of (0.75) into (0.22) asserts that

$$\begin{aligned} u_t &= \tfrac{1}{\alpha}\left\{(x+t)-\phi'(t)\right\} \\ u_{tt} &= \tfrac{1}{\alpha}(1-\phi''(t)) \\ u_x &= \tfrac{1}{\alpha}(x+t) \\ u_{xx} &= \tfrac{1}{\alpha}. \end{aligned} \tag{0.76}$$

Hence $u_{tt}+u_t-cu_{xx}-u_x=\frac{1}{\alpha}\left\{1-\phi''(t)+(x+t)-(x+t)-c-\phi'(t)\right\}=0$, simplifies to

$$\phi''(t)+\phi'(t)=1-c. \tag{0.77}$$

Solving the equation (0.77) we obtain that

$$\phi(t)=c_1-c_2e^{-t}+(1-t)(1-c). \tag{0.78}$$

Therefore the invariant solution of (0.22) is given by

$$u(t,x)=\frac{1}{\alpha}\left\{\frac{(x+t)^2}{2}+c_1-c_2e^{-t}+(1-t)(1-c)\right\}. \tag{0.79}$$

Consider the one dimensional subalgebra

$$X=\beta\frac{\partial}{\partial t}+(x+t)\frac{\partial}{\partial u}. \tag{0.80}$$

The characteristic equation of (0.80) is given by

$$\frac{dx}{0}=\frac{du}{x+t}=\frac{dt}{\beta}. \tag{0.81}$$

From equation (0.81) the invariants are $I_1=\beta u-\frac{1}{2}(x+t)^2$, $I_2=x$.

If we let I_1 be a function of I_2, then

$$u(t,x) = \frac{1}{\alpha}\left\{\frac{(x+t)^2}{2} + \phi(x)\right\} \text{ where } \phi(x) = I_1 \text{ i.e } I_1 = \phi(I_2). \tag{0.82}$$

The substitution of (0.82) into (0.22) asserts that

$$\begin{aligned} u_t &= \tfrac{1}{\beta}(x+t) \\ u_{tt} &= \tfrac{1}{\beta} \\ u_x &= \tfrac{1}{\beta}\left\{(x+t) - \phi'(x)\right\} \\ u_{xx} &= \tfrac{1}{\beta}(1 - \phi''(x)) \end{aligned} \tag{0.83}$$

Hence

$u_{tt} + u_t - cu_{xx} - u_x = 0$, implies that

$\frac{1}{\beta}\left\{1 + (x+t) - c(1 - \phi''(x) - \left((x+t) - \phi'(x)\right)\right\} = 0,$

which simplifies to

$$c\phi''(x) + \phi'(x) = 1 - c. \tag{0.84}$$

Thus solving the equation (0.84) we get that

$$\phi(x) = c_1 - c_2 e^{-\frac{x}{c}} + x - cx + c^2 - c. \tag{0.85}$$

Hence the invariant solution of (0.22) is given by

$$u(t,x) = \frac{1}{\beta}\left\{\frac{(x+t)^2}{2} + c_1 - c_2 e^{-\frac{x}{c}} + x - cx + c^2 - c\right\}. \tag{0.86}$$

Consider the equation $u_{tt} + u_t = u_x^{-2}u_{xx} - \ln u_x$, whose the set of generators is given by $X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial t}$, $X_3 = \frac{\partial}{\partial u}$, $X_4 = e^{-t}\frac{\partial}{\partial u}$, $X_5 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial u}$.

Using the one dimensional subalgebras $X = X_1 + \beta X_4$, which translates to $X = \frac{\partial}{\partial x} + \beta e^{-t}\frac{\partial}{\partial u}$, its characteristic equation is given by

$$\frac{dx}{1} = \frac{du}{\beta e^{-t}} = \frac{dt}{0}. \tag{0.87}$$

From equation (0.87) the invariants are given by $I_1 = u - \beta e^{-t}x$, $I_2 = t$.

If we let I_1 be a function of I_2, then

$$u(t,x) = \beta e^{-t}x + \phi(t) \text{ where } \phi(t) = I_1 \text{ i.e } I_1 = \phi(I_2). \tag{0.88}$$

The substitution of (0.88) into (0.23) asserts that

$$\begin{aligned} u_t &= -\beta e^{-t}x + \phi'(t) \\ u_{tt} &= \beta e^{-t}x + \phi''(t) \\ u_x &= \beta e^{-t} \\ u_{xx} &= 0 \end{aligned} \tag{0.89}$$

Hence $u_{tt} + u_t - u_x^{-2}u_{xx} - \ln u_x = \phi''(t) + \phi'(t) - \ln \beta e^{-t} = 0$,

simplifies to $\phi''(t) + \phi'(t) = \ln \beta e^{-t}$.Solving the equation we obtain that $\phi(t) = c_1 - c_2e^{-t} + t\ln\beta - \frac{1}{2}t^2 + t - 1 - \ln\beta$.The invariant solution of (0.33) is given by

$$u(t,x) = \beta x e^{-t} + c_1 - c_2e^{-t} + t\ln\beta - \frac{1}{2}t^2 + t - 1 - \ln\beta. \tag{0.90}$$

0.5 CONCLUSION

From the present study, the methods of determining the principal Lie algebra, the equivalence Lie algebra have been gained. However, the technique and methods of finding optimal systems of one-dimensional subalgebras, the extension of the principal Lie algebra by one for a variety of differential equations has been acquired. We would like to explore the further and even for higher dimensional subalgebras. Furthermore one would look to taking similar projects further and extend the to determining the Lagrangians of the differential equations (if they exist) and apply the conservation laws to determine the conserved quantities for a variety of differential equations. The methods of calculating the invariant solutions have also been gained in the present work. Throughout the present study we have successfully used the Lie point symmetries. Future projects would include exploring various topics in approximate symmetries.

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