LIE SYMMETRY ANALYSIS OF CERTAIN NONLINEAR EVOLUTION EQUATIONS OF MATHEMATICAL PHYSICS

A R ADEM

LIE SYMMETRY ANALYSIS OF CERTAIN NONLINEAR EVOLUTION EQUATIONS OF MATHEMATICAL PHYSICS

by

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Contents

Declaration

I declare that the thesis for the degree of Doctor of Philosophy at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other university, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

Signed: ....

MR ABDULLAHI RASHID ADEM

Date: ....

This thesis has been submitted with my approval as a University supervisor and would certify that the requirements for the applicable Doctor of Philosophy degree rules and regulations have been fulfilled.

Signed:..

PROF C.M. KHALIQUE

Date: ....

Declaration of Publications

Details of contribution to publications that form part of this thesis.

Chapter 2

A. R. Adem, C. M. Khalique, M. Molati, Group classification, symmetry reductions and exact solutions of a generalized Korteweg-de Vries-Burgers equation, Submitted for publication to Pramana

Chapter 3

A. R. Adem, C. M. Khalique, Exact Solutions and Conservation Laws of a Two-dimensional Integrable Generalization of the Kaup-Kupershmidt Equation, J. Appl. Math. 2013, Article ID 647313 (2013)

Chapter 4

A. R. Adem, C. M. Khalique, On the solutions and conservation laws of a coupled KdV system, Appl. Math. Comput. 219 (2012) 959-969

Chapter 5

A. R. Adem, C. M. Khalique, Lie group analysis and conservation laws of a coupled variable-coefficient modified Korteweg-de Vries system, Submitted for publication to Ocean Engineering

Chapter 6

A. R. Adem, C. M. Khalique, Symmetry reductions, exact solutions and conservation laws of a new coupled KdV system, Commun. Nonlinear Sci. Numer. Simul. 17 (2012) 3465-3475

Chapter 7

A. R. Adem, C. M. Khalique, New exact solutions and conservation laws of a coupled Kadomtsev-Petviashvili system, Comput. & Fluids 81 (2013) 10-16

Dedication

To my parents

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Abstract

In this work we study the applications of Lie symmetry analysis to certain nonlinear evolution equations of mathematical physics. Exact solutions and conservation laws are obtained for such equations. The equations which are considered in this thesis are a generalized Korteweg-de Vries-Burgers equation, a two-dimensional integrable generalization of the Kaup-Kupershmidt equation, a coupled Korteweg-de Vries system, a generalized coupled variable-coefficient modified Korteweg-de Vries system, a new coupled Korteweg-de Vries system and a new coupled Kadomtsev-Petviashvili system.

The generalized Korteweg-de Vries-Burgers equation is investigated from the point of view of Lie group classification. We show that this equation admits a four-dimensional equivalence Lie algebra. It is also shown that the principal Lie algebra consists of a single translation symmetry. Several possible extensions of the principal Lie algebra are computed and their associated symmetry reductions and exact solutions are obtained.

The Lie symmetry method is performed on a two-dimensional integrable generalization of the Kaup-Kupershmidt equation. Exact solutions are obtained using the Lie symmetry method in conjunction with the extended tanh method and the extended Jacobi elliptic function method. In addition to exact solutions we also present conservation laws which are derived using the multiplier approach.

A coupled Korteweg-de Vries system and a generalized coupled variable-coefficient modified Korteweg-de Vries system are investigated using Lie symmetry analysis. The similarity reductions and exact solutions with the aid of simplest equations and Jacobi elliptic function methods are obtained for the coupled Korteweg-de Vries system and the generalized coupled variable-coefficient modified Korteweg-de Vries system. In addition to this, the conservation laws for the two systems are derived using the multiplier approach and the conservation theorem due to Ibragimov.

Finally, a new coupled Korteweg-de Vries system and a new coupled Kadomtsev-

Petviashvili system are analyzed using Lie symmetry method. Exact solutions are obtained using the Lie symmetry method in conjunction with the simplest equation, Jacobi elliptic function and (G'/G)-expansion methods. Conservation laws are also obtained for both the systems by employing the multiplier approach.

List of Acronyms

PDE: Partial differential equation

ODE: Ordinary differential equation

KdV: Korteweg-de Vries

CVCmKdV: Coupled variable-coefficient modified Korteweg-de Vries

KP: Kadomtsev-Petviashvili

Introduction

Nonlinear evolution equations describe a variety of physical phenomena in the fields such as physics, chemistry, biology, fluid dynamics, etc. Thus, it is important to investigate the exact explicit solutions and conservation laws of nonlinear evolution equations. Unfortunately, it is almost impossible to find all the solutions of a nonlinear evolution equation. Finding solutions of such an equation is an arduous task and only in certain special cases one can write down the solutions explicitly [1]. However, in recent years important progress has been made and many effective methods for obtaining exact solutions of nonlinear evolution equations have been proposed. Some of the most important methods found in the literature include the inverse scattering method [2], Hirota bilinear method [3], the Jacobi elliptic function expansion method [4], the Lie symmetry analysis [5–8], the tanh-function expansion method [9], the auxiliary ordinary differential equation method [10] and the F-expansion method [11].

Symmetries infuse many mathematical models, specifically those expressed in terms of differential equations. The mathematical field which exemplifies and manufactures symmetries of differential equations is Lie group theory. Lie group theory was initiated by Marius Sophus Lie (1842-1899), a Norwegian mathematician who made significant contributions to the theories of algebraic invariants and differential equations. It is based upon the study of the invariance under one parameter Lie group of point transformations [6–8]. It systematically unifies well known ad hoc techniques to construct explicit solutions for differential equations. In the last six decades there have been considerable developments in Lie symmetry methods for differential equations as can be seen by the number of research papers [12–21], books [5–8] and new symbolic softwares [22–28] devoted to the subject.

Many differential equations of physical interest involve parameters, arbitrary elements or functions, which need to be determined. Usually, these arbitrary parameters are determined experimentally. However, the Lie symmetry approach through the method of group classification has proven to be a versatile tool in specifying the forms of these parameters systematically [8, 12–17]. The first group classification problem was investigated by Sophus Lie [29] in 1881 for a linear second-order partial differential equations with two independent variables. The main idea of group classification of a differential equation involving arbitrary elements, say, for example, $g(u)$ and $f(x)$, consists of finding the Lie point symmetries of the differential equation with arbitrary functions $g(u)$ and $f(x)$, and then computing systematically all possible forms of $g(u)$ and $f(x)$ for which the principal Lie algebra can be extended.

Conservation laws play an important rôle in the solution process of differential equations. It is well known that finding the conservation laws of a system of differential equations is often the first step towards finding a solution [6]. Conservation laws are useful in the numerical integration of partial differential equations, for example, to control numerical errors [30]. Conservation laws also play an important role in the theory of non-classical transformations [31, 32], normal forms and asymptotic integrability [33]. Recently, conservation laws have been used to construct solutions of partial differential equations [34–37].

In this thesis certain nonlinear evolution equations will be studied. These are the generalized Korteweg-de Vries-Burgers, the coupled Korteweg-de Vries system, a new coupled Korteweg-de Vries system, the generalized coupled variable-coefficient modified Korteweg-de Vries system, a new coupled Kadomtsev-Petviashvili system and a two-dimensional integrable generalization of the Kaup-Kupershmidt equation.

The first equation that is analyzed from the point of view of Lie group classification in this thesis is the generalized Korteweg-de Vries-Burgers equation [38]

$$u_t + \delta u_{xxx} + g(u)u_x - \nu u_{xx} + \gamma u = f(x), \tag{1}$$

which contains two arbitrary functions $g(u)$ and $f(x)$. This equation arises from many physical scenarios such as the propagation of undular bores in shallow water, the flow of liquids containing gas bubbles, weakly nonlinear plasma waves with certain dissipative effect, theory of ferro electricity, nonlinear circuit and the propagation of waves in an elastic tube filled with a viscous fluid [39].

The second equation that is studied in this thesis is a two-dimensional integrable generalization of the Kaup-Kupershmidt equation [40, 41]

$$\begin{aligned} & u_t + u_{xxxxx} + \frac{25}{2}u_x u_{xx} + 5uu_{xxx} + 5u^2 u_x + 5u_{xxy} - 5\partial_x^{-1}u_{yy} + 5uu_y \\ & +5u_x\partial_x^{-1}u_y = 0, \end{aligned} \tag{2}$$

which arises in various problems in many areas of theoretical physics. The above equation occurs as special reduction of integrable nonlinear systems [42, 43]. It should be noted that the Zakharov-Manakov delta dressing method was used to obtain soliton and periodic solutions of (2) [42, 43].

The celebrated Korteweg-de Vries equation [44] is

$$u_t + u_{xxx} + 6uu_x = 0, \tag{3}$$

which governs the dynamics of solitary waves. Traveling wave solutions that do not change their form during propagation are called solitary waves. Solitary waves that retain their shape upon collision are called solitons [45]. Solitons are results of a delicate balance between dispersion and nonlinearity. The Korteweg-de Vries equation was originally derived to describe shallow water waves of long wavelength and small amplitude. It is an important equation from the view point of integrable systems as it has an infinite number of conservation laws, gives multiple-soliton solutions, and has many other physical properties (see for example [46] and references therein).

In recent years, the coupled Korteweg-de Vries equations have been the centre of attraction and extensive studies have been made by many authors, see for example, the Refs. [47–53]. In [47] a typical hydrodynamic model which describes a resonant interaction of two wave modes in a shallow stratified liquid is derived and is given by

$$u_t + u_{xxx} - \frac{7}{4}uu_x - vv_x + \frac{5}{4}(uv)_x = 0, \tag{4a}$$

$$v_t + v_{xxx} - \frac{5}{4}uu_x - \frac{7}{4}vv_x + 2(uv)_x = 0. \tag{4b}$$

This coupled Korteweg-de Vries system (4) was studied by Wang [47] for its integrability by using the prolongation technique and singularity analysis. Wazwaz [46] also considered the system (4) and employed the Hirota's bilinear method combined with Hereman et al [54] simplified approach to study the integrability of (4). Multiple-soliton solutions and multiple singular soliton solutions were obtained for (4). The system (4) will be the third system that will be studied in this thesis.

Next we study the generalized coupled variable-coefficient modified Korteweg-de Vries system [55]

$$u_t - \alpha(t)\left(u_{xxx} + 6(u^2 - v^2)u_x - 12uvv_x\right) - 4\beta(t)u_x = 0, \tag{5a}$$

$$v_t - \alpha(t)\left(v_{xxx} + 6(u^2 - v^2)v_x + 12uvu_x\right) - 4\beta(t)v_x = 0, \tag{5b}$$

which models a two-layer fluid and is applied to investigate the atmospheric and oceanic phenomena such as the atmospheric blockings, interactions between the atmosphere and ocean, oceanic circulations and hurricanes or typhoons. It should be noted that if $v = 0$, (5) reduces to a variable-coefficient modified Korteweg-de Vries equation

$$u_t - \alpha(t)\left(u_{xxx} + 6u^2u_x\right) - 4\beta(t)u_x = 0,$$

which has been investigated in [56–58].

In [59] a new coupled Korteweg-de Vries system

$$u_t + u_{xxx} + 3uu_x + 3ww_x = 0, \tag{6a}$$

$$v_t + v_{xxx} + 3vv_x + 3ww_x = 0, \tag{6b}$$

$$w_t + w_{xxx} + \frac{3}{2}(uw)_x + \frac{3}{2}(vw)_x = 0 \tag{6c}$$

was examined. The 4×4 matrix spectral problem with three potentials has been used in [59] to derive a hierarchy of nonlinear evolution equations, which includes the coupled Korteweg-de Vries equation (6). It is shown that the hierarchy possesses the generalized bi-Hamiltonian structures with the aid of the trace identity. Wazwaz [46] also considered the system (6) and employed Hirota's bilinear method combined with Hereman et al [54] simplified approach to study the integrability of (6). Multiple-soliton solutions and multiple singular soliton solutions were obtained for (6) in [46]. We will look for exact solutions and conservation laws for (6).

The Kadomtsev-Petviashvili equation given by

$$\left(u_t + 6uu_x + u_{xxx}\right)_x + u_{yy} = 0$$

originated from a 1970 paper [60] by two Russian physicists, Boris Kadomtsev (1928-1998) and Vladimir Petviashvili (1936-1993). The Kadomtsev-Petviashvili equation is a model for shallow long waves in the x-direction with some mild dispersion in the y-direction. It is completely integrable by the inverse scattering transform method and gives multiple-soliton solutions. The Kadomtsev-Petviashvili equation is actually an extension of the Korteweg-de Vries equation (3) that is commonly studied in the context of shallow water waves in fluid dynamics. This equation is used to study the shallow water waves on beaches as sea beaches can be comfortably treated as a two-dimensional plane. These two-dimensional waves leave a diamond pattern mark on the sandy beaches that is known as parting lineation. It is also studied in the context of plasma physics and it describes the dynamics of solitons and nonlinear waves in plasmas and superfluids [61].

The coupled Korteweg-de Vries system (6) formulated in the Kadomtsev-Petviashvili sense, is given by [62]

$$\left(u_t + u_{xxx} + 3uu_x + 3ww_x\right)_x + u_{yy} = 0, \tag{7a}$$

$$\left(v_t + v_{xxx} + 3vv_x + 3ww_x\right)_x + v_{yy} = 0, \tag{7b}$$

$$\left(w_t + w_{xxx} + \frac{3}{2}(uw)_x + \frac{3}{2}(vw)_x\right)_x + w_{yy} = 0 \tag{7c}$$

and it will be the subject of our study in this work.

The outline of this thesis is as follows.

In Chapter one, the basic definitions and theorems concerning the one-parameter groups of transformations and conservation laws are presented.

Chapter two deals with the Lie group classification of the generalized Korteweg-de Vries-Burgers equation (1).

Chapter three discusses the solutions and conservation laws of a two-dimensional integrable generalization of the Kaup-Kupershmidt equation (2).

Chapters four and five deals with the solutions and conservation laws of a coupled Korteweg-de Vries system (4) and the generalized coupled variable-coefficient modified Korteweg-de Vries system (5), respectively.

Chapter six discusses the solutions and conservation laws of the new coupled Korteweg-de Vries system (6).

In Chapter seven the solutions and conservation laws of the coupled Kadomtsev-Petviashvili system (7) are obtained.

The results in Chapter two and Chapter five have been sent for publication in [63,64] and the results of Chapters three, four, six and seven have been published in [65–68], respectively.

Finally, in Chapter eight, a summary of the results of the thesis is presented and future work is discussed.

Bibliography is given at the end.

Chapter 1

Lie symmetry methods and conservation laws for differential equations

In this chapter we give some basic methods of Lie symmetry analysis and conservation laws of partial differential equations (PDEs).

1.1 Introduction

In the late nineteenth century an outstanding mathematician Sophus Lie (1842-1899) developed a new method, known as Lie group analysis, for solving differential equations and showed that the majority of adhoc methods of integration of differential equations could be explained and deduced simply by means of his theory. Recently, many good books have appeared in the literature in this field. We mention a few here, Bluman and Kumei [50], Ovsiannikov [5], Olver [7], Stephani [69], Ibragimov [8, 70, 71], Cantwell [72] and Mahomed [73]. Definitions and results given in this Chapter are taken from the books mentioned above.

Conservation laws for PDEs are constructed using two different approaches; the multiplier method [74] and the new conservation theorem due to Ibragimov [75]. First we present some preliminaries which we will need later in the thesis. For details the reader is referred to [8, 74, 75].

1.2 Continuous one-parameter groups

Let $x = (x^1, ..., x^n)$ be the independent variables with coordinates x^i and $u = (u^1, ..., u^m)$ be the dependent variables with coordinates u^α (n and m finite). Consider a change of the variables x and u involving a real parameter a:

$$T_a : \overline{x^i} = f^i(x, u, a), \ \overline{u^\alpha} = \phi^\alpha(x, u, a), \tag{1.1}$$

where a continuously ranges in values from a neighborhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$, and f^i and ϕ^α are differentiable functions.

Definition 1.1 A set G of transformations (1.1) is called a *continuous one-parameter (local) Lie group of transformations* in the space of variables x and u if

(i) For $T_a, T_b \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then $T_b T_a = T_c \in G$, $c = \phi(a, b) \in \mathcal{D}$ (Closure)

(ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0 T_a = T_a T_0 = T_a$ (Identity)

(iii) For $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that $T_a T_{a^{-1}} = T_{a^{-1}} T_a = T_0$ (Inverse)

We note that the associativity property follows from (i). The group property (i) can be written as

$$\begin{aligned} \overline{\overline{x^i}} &\equiv f^i(\bar{x}, \bar{u}, b) = f^i(x, u, \phi(a, b)), \\ \overline{\overline{u^\alpha}} &\equiv \phi^\alpha(\bar{x}, \bar{u}, b) = \phi^\alpha(x, u, \phi(a, b)) \end{aligned} \tag{1.2}$$

and the function ϕ is called the *group composition law.* A group parameter a is called *canonical* if $\phi(a, b) = a + b$.

Theorem 1.1 For any $\phi(a,b)$, there exists the canonical parameter $\tilde{a}$ defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)}, \text{ where } w(s) = \left.\frac{\partial\,\phi(s,b)}{\partial b}\right|_{b=0}.$$

1.3 Prolongation of point transformations and Group generator

The derivatives of u with respect to x are defined as

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j D_i(u_i), \cdots, \tag{1.3}$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, ..., n \tag{1.4}$$

is the operator of total differentiation. The collection of all first derivatives u_i^α is denoted by $u_{(1)}$, i.e.,

$$u_{(1)} = \{u_i^\alpha\} \quad \alpha = 1, ..., m, \quad i = 1, ..., n.$$

Similarly

$$u_{(2)} = \{u_{ij}^\alpha\} \quad \alpha = 1, ..., m, \quad i, j = 1, ..., n$$

and $u_{(3)} = \{u_{ijk}^\alpha\}$ and likewise $u_{(4)}$ etc. Since $u_{ij}^\alpha = u_{ji}^\alpha$, $u_{(2)}$ contains only u_{ij}^α for $i \leq j$. In the same manner $u_{(3)}$ has only terms for $i \leq j \leq k$. There is natural ordering in $u_{(4)}, u_{(5)} \cdots$.

In group analysis all variables $x, u, u_{(1)} \cdots$ are considered functionally independent variables connected only by the differential relations (1.3). Thus the u_s^α are called differential variables [8].

We now consider a pth-order PDE(s), namely

$$E_\alpha(x, u, u_{(1)}, ..., u_{(p)}) = 0. \tag{1.5}$$

Prolonged or extended groups

If $z = (x, u)$, one-parameter group of transformations G is

$$\bar{x}^i = f^i(x, u, a), \quad f^i|_{a=0} = x^i,$$

$$\bar{u}^\alpha = \phi^\alpha(x, u, a), \quad \phi^\alpha|_{a=0} = u^\alpha. \tag{1.6}$$

According to the Lie's theory, the construction of the symmetry group G is equivalent to the determination of the corresponding *infinitesimal transformations* :

$$\bar{x}^i \approx x^i + a\,\xi^i(x, u), \quad \bar{u}^\alpha \approx u^\alpha + a\,\eta^\alpha(x, u) \tag{1.7}$$

obtained from (1.1) by expanding the functions f^i and ϕ^α into Taylor series in a about $a = 0$ and also taking into account the initial conditions

$$f^i\big|_{a=0} = x^i, \quad \phi^\alpha\big|_{a=0} = u^\alpha.$$

Thus, we have

$$\xi^i(x, u) = \left.\frac{\partial f^i}{\partial a}\right|_{a=0}, \quad \eta^\alpha(x, u) = \left.\frac{\partial \phi^\alpha}{\partial a}\right|_{a=0}. \tag{1.8}$$

One can now introduce the *symbol* of the infinitesimal transformations by writing (1.7) as

$$\bar{x}^i \approx (1 + a\,X)x, \quad \bar{u}^\alpha \approx (1 + a\,X)u,$$

where

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}. \tag{1.9}$$

This differential operator X is known as the infinitesimal operator or generator of the group G. If the group G is admitted by (1.5), we say that X is an *admitted operator* of (1.5) or X is an *infinitesimal symmetry* of equation (1.5).

We now see how the derivatives are transformed.

The D_i transforms as

$$D_i = D_i(f^j)\bar{D}_j, \tag{1.10}$$

where $\bar{D}_j$ is the total differentiations in transformed variables $\bar{x}^i$. So

$$\bar{u}_i^\alpha = \bar{D}_j(u^\alpha), \qquad \bar{u}_{ij}^\alpha = \bar{D}_j(\bar{u}_i^\alpha) = \bar{D}_i(\bar{u}_j^\alpha), \cdots .$$

Now let us apply (1.10) and (1.6)

$$\begin{aligned} D_i(\phi^\alpha) &= D_i(f^j)\bar{D}_j(\bar{u}^\alpha) \\ &= D_i(f^j)\bar{u}_j^\alpha. \end{aligned} \tag{1.11}$$

This

$$\left(\frac{\partial f^j}{\partial x^i} + u_i^\beta \frac{\partial f^j}{\partial u^\beta}\right)\bar{u}_j^\alpha = \frac{\partial \phi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \phi^\alpha}{\partial u^\beta}. \tag{1.12}$$

The quantities $\bar{u}_j^\alpha$ can be represented as functions of $x, u, u_{(i)}, a$ for small a, ie., (1.12) is locally invertible:

$$\bar{u}_i^\alpha = \psi_i^\alpha(x, u, u_{(1)}, a), \quad \psi^\alpha\big|_{a=0} = u_i^\alpha. \tag{1.13}$$

The transformations in $x, u, u_{(1)}$ space given by (1.6) and (1.13) form a one-parameter group (one can prove this but we do not consider the proof) called the first prolongation or just extension of the group G and denoted by $G^{[1]}$.

We let

$$\bar{u}_i^\alpha \approx u_i^\alpha + a\zeta_i^\alpha \tag{1.14}$$

be the infinitesimal transformation of the first derivatives so that the infinitesimal transformation of the group $G^{[1]}$ is (1.7) and (1.14).

Higher-order prolongations of G, viz. $G^{[2]}$, $G^{[3]}$ can be obtained by derivatives of (1.11).

Prolonged generators

Using (1.11) together with (1.7) and (1.14) we get

$$
\begin{aligned}
D_i(f^j)(\bar{u}_j^\alpha) &= D_i(\phi^\alpha) \\
D_i(x^j + a\xi^j)(u_j^\alpha + a\zeta_j^\alpha) &= D_i(u^\alpha + a\eta^\alpha) \\
(\delta_i^j + aD_i\xi^j)(u_j^\alpha + a\zeta_j^\alpha) &= u_i^\alpha + aD_i\eta^\alpha \\
u_i^\alpha + a\zeta_i^\alpha + au_j^\alpha D_i\xi^j &= u_i^\alpha + aD_i\eta^\alpha \\
\zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \quad \text{(sum on } j). \qquad (1.15)
\end{aligned}
$$

This is called the first prolongation formula. Likewise, one can obtain the second prolongation, viz.,

$$\zeta_{ij}^\alpha = D_j(\eta_i^\alpha) - u_{ik}^\alpha D_j(\xi^k), \quad \text{(sum on } k). \qquad (1.16)$$

By induction (recursively)

$$\zeta_{i_1,i_2,\ldots,i_p}^\alpha = D_{i_p}(\zeta_{i_1,i_2,\ldots,i_{p-1}}^\alpha) - u_{i_1,i_2,\ldots,i_{p-1}j}^\alpha D_{i_p}(\xi^j), \quad \text{(sum on } j). \qquad (1.17)$$

The first and higher prolongations of the group G form a group denoted by $G^{[1]}, \cdots, G^{[p]}$. The corresponding prolonged generators are

$$
\begin{aligned}
X^{[1]} &= X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} \quad \text{(sum on } i, \alpha), \\
&\cdot \\
&\cdot \\
&\cdot \\
X^{[p]} &= X^{[p-1]} + \zeta_{i_1,\ldots,i_p}^\alpha \frac{\partial}{\partial u_{i_1,\ldots,i_p}^\alpha} \quad p \geq 1,
\end{aligned}
$$

where

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}.$$

1.4 Group admitted by a PDE

Definition 1.2 The vector field

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}, \tag{1.18}$$

is a *point symmetry* of the pth-order PDE (1.5), if

$$X^{[p]}(E_\alpha) = 0 \tag{1.19}$$

whenever $E_\alpha = 0$. This can also be written as

$$X^{[p]}\, E_\alpha\big|_{E_\alpha=0} = 0, \tag{1.20}$$

where the symbol $|_{E_\alpha=0}$ means evaluated on the equation $E_\alpha = 0$.

Definition 1.3 Equation (1.19) is called the *determining equation* of (1.5) because it determines all the infinitesimal symmetries of (1.5).

Definition 1.4 (Symmetry group) A one-parameter group G of transformations (1.1) is called a symmetry group of equation (1.5) if (1.5) is form-invariant (has the same form) in the new variables $\bar{x}$ and $\bar{u}$, i.e.,

$$E_\alpha(\bar{x}, \bar{u}, \bar{u_{(1)}}, \cdots, \bar{u_{(p)}}) = 0, \tag{1.21}$$

where the function E_α is the same as in equation (1.5).

1.5 Group invariants

Definition 1.5 A function $F(x,u)$ is called an *invariant of the group of transformation* (1.1) if

$$F(\bar{x}, \bar{u}) \equiv F(f^i(x,u,a),\, \phi^\alpha(x,u,a)) = F(x,u), \tag{1.22}$$

identically in x, u and a.

Theorem 1.2 (Infinitesimal criterion of invariance) A necessary and sufficient condition for a function $F(x,u)$ to be an invariant is that

$$X\,F \equiv \xi^i(x,u)\frac{\partial F}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial F}{\partial u^\alpha} = 0\,. \tag{1.23}$$

It follows from the above theorem that every one-parameter group of point transformations (1.1) has $n-1$ functionally independent invariants, which can be taken to be the left-hand side of any first integrals

$$J_1(x,u)=c_1,\cdots,J_{n-1}(x,u)=c_n$$

of the characteristic equations

$$\frac{dx^1}{\xi^1(x,u)}=\cdots=\frac{dx^n}{\xi^n(x,u)}=\frac{du^1}{\eta^1(x,u)}=\cdots=\frac{du^n}{\eta^n(x,u)}.$$

Theorem 1.3 If the infinitesimal transformation (1.7) or its symbol X is given, then the corresponding one-parameter group G is obtained by solving the Lie equations

$$\frac{d\bar{x^i}}{da}=\xi^i(\bar{x},\bar{u}),\quad \frac{d\bar{u^\alpha}}{da}=\eta^\alpha(\bar{x},\bar{u}) \tag{1.24}$$

subject to the initial conditions

$$\bar{x^i}\big|_{a=0}=x,\quad \bar{u^\alpha}\big|_{a=0}=u\,.$$

1.6 Lie algebra

Let us consider two operators X_1 and X_2 defined by

$$X_1=\xi_1^i(x,u)\frac{\partial}{\partial x^i}+\eta_1^\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

and

$$X_2=\xi_2^i(x,u)\frac{\partial}{\partial x^i}+\eta_2^\alpha(x,u)\frac{\partial}{\partial u^\alpha}.$$

Definition 1.6 The *commutator* of X_1 and X_2, written as $[X_1,X_2]$, is defined by $[X_1,X_2]=X_1(X_2)-X_2(X_1)$.

Definition 1.7 A Lie algebra is a vector space L (over the field of real numbers) of operators $X=\xi^i(x,u)\frac{\partial}{\partial x^i}+\eta^\alpha(x,u)\frac{\partial}{\partial u}$ with the following property. If the operators

$$X_1 = \xi_1^i(x,u)\frac{\partial}{\partial x^i} + \eta_1^\alpha(x,u)\frac{\partial}{\partial u}, \quad X_2 = \xi_2^i(x,u)\frac{\partial}{\partial x^i} + \eta_2^\alpha(x,u)\frac{\partial}{\partial u}$$

are any elements of L, then their commutator

$$[X_1, X_2] = X_1(X_2) - X_2(X_1)$$

is also an element of L. It follows that the commutator is

1. Bilinear: for any $X, Y, Z \in L$ and $a, b \in \mathbb{R}$,

$$[aX + bY, Z] = a[X, Z] + b[Y, Z], [X, aY + bZ] = a[X, Y] + b[X, Z];$$

2. Skew-symmetric: for any $X, Y \in L$,

$$[X, Y] = -[Y, X];$$

3. and satisfies the Jacobi identity: for any $X, Y, Z \in L$,

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

1.7 Conservation laws

1.7.1 Fundamental operators and their relationship

Consider a kth-order system of PDEs of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$, namely

$$E_\alpha(x, u, u_{(1)}, \ldots, u_{(k)}) = 0, \quad \alpha = 1, \ldots, m. \tag{1.25}$$

The *Euler-Lagrange operator*, for each α, is given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1} \ldots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}, \qquad \alpha = 1, \ldots, m, \tag{1.26}$$

and the *Lie-Bäcklund operator* is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \; \xi^i, \eta^\alpha \in \mathcal{A}, \tag{1.27}$$

where $\mathcal{A}$ is the space of *differential functions* [8]. The operator (1.27) is an abbreviated form of infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} \zeta^\alpha_{i_1 i_2 \ldots i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}, \tag{1.28}$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} \zeta^\alpha_i &= D_i(W^\alpha) + \xi^j u^\alpha_{ij} \\ \zeta^\alpha_{i_1 \ldots i_s} &= D_{i_1} \ldots D_{i_s}(W^\alpha) + \xi^j u^\alpha_{j i_1 \ldots i_s}, \quad s > 1, \end{aligned} \tag{1.29}$$

in which W^α is the *Lie characteristic function* given by

$$W^\alpha = \eta^\alpha - \xi^i u^\alpha_j. \tag{1.30}$$

One can write the Lie-Bäcklund operator (1.28) in characteristic form as

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} D_{i_1} \ldots D_{i_s}(W^\alpha) \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}. \tag{1.31}$$

The *Noether operators* associated with a Lie-Bäcklund symmetry operator X are given by

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u^\alpha_i} + \sum_{s\geq 1} D_{i_1} \ldots D_{i_s}(W^\alpha) \frac{\delta}{\delta u^\alpha_{i i_1 i_2 \ldots i_s}}, \quad i = 1, \ldots, n, \tag{1.32}$$

where the Euler-Lagrange operators with respect to derivatives of u^α are obtained from (1.26) by replacing u^α by the corresponding derivatives. For example,

$$\frac{\delta}{\delta u^\alpha_i} = \frac{\partial}{\partial u^\alpha_i} + \sum_{s\geq 1} (-1)^s D_{j_1} \ldots D_{j_s} \frac{\partial}{\partial u^\alpha_{i j_1 j_2 \ldots j_s}}, \quad i = 1, \ldots, n, \quad \alpha = 1, \ldots, m, \tag{1.33}$$

and the Euler-Lagrange, Lie-Bäcklund and Noether operators are connected by the operator identity [75]

$$X + D_i(\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i N^i. \tag{1.34}$$

The n-tuple vector $T = (T^1, T^2, \ldots, T^n)$, $T^j \in \mathcal{A}$, $j = 1, \ldots, n$, is a *conserved vector* of (1.25) if T^i satisfies

$$D_i T^i|_{(1.25)} = 0. \tag{1.35}$$

The equation (1.35) defines a *local conservation law* of system (1.25).

1.7.2 Multiplier Method

A multiplier $\Lambda_\alpha(x, u, u_{(1)}, \ldots)$ has the property that [74]

$$\Lambda_\alpha E_\alpha = D_i T^i \tag{1.36}$$

holds identically. The right hand side of (1.36) is a divergence expression. The determining equations for the multiplier Λ_α is [74]

$$\frac{\delta(\Lambda_\alpha E_\alpha)}{\delta u^\alpha} = 0. \tag{1.37}$$

1.7.3 Variational method for a system and its adjoint

The system of *adjoint equations* for the system of kth-order differential equations (1.25) is defined by

$$E^*_\alpha(x, u, v, \ldots, u_{(k)}, v_{(k)}) = 0, \quad \alpha = 1, \ldots, m, \tag{1.38}$$

where $v = (v^1, v^2, \ldots, v^m)$ are new dependent variables [75].

Assume that the system of equations (1.25) admits the symmetry generator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}. \tag{1.39}$$

Then the system of adjoint equations (1.38) admits the operator [75]

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \eta^\alpha_* \frac{\partial}{\partial v^\alpha}, \quad \eta^\alpha_* = -[\lambda^\alpha_\beta v^\beta + v^\alpha D_i(\xi^i)], \tag{1.40}$$

where the operator (1.40) is an extension of (1.39) to the variable v^α and the λ^α_β are obtainable from

$$X(E_\alpha) = \lambda^\beta_\alpha E_\beta. \tag{1.41}$$

Theorem 1 [75] Every Lie point, Lie-Bäcklund and non local symmetry (1.39) admitted by the system of equations (1.25) give rise to a conservation law for the system consisting of the equation (1.25) and the adjoint equation (1.38), where the components T^i of the conserved vector $T = (T^1, \ldots, T^n)$ are determined by

$$T^i = \xi^i L + W^\alpha \frac{\delta L}{\delta u^\alpha_i} + \sum_{s \geq 1} D_{i_1} \ldots D_{i_s}(W^\alpha) \frac{\delta L}{\delta u^\alpha_{i i_1 i_2 \ldots i_s}}, \quad i = 1, \ldots, n, \tag{1.42}$$

with Lagrangian given by

$$L = v^{\alpha} E_{\alpha}(x, u, \ldots, u_{(k)}). \tag{1.43}$$

1.8 Conclusion

In this chapter we presented a brief introduction to the Lie group analysis and conservation laws of PDEs and gave some results which will be used throughout this project. We also gave the algorithm to determine the Lie point symmetries and conservation laws of PDEs.

Chapter 2

Group classification, symmetry reductions and exact solutions of a generalized Korteweg-de Vries-Burgers equation

2.1 Introduction

In this chapter we study the generalized Korteweg-de Vries-Burgers equation

$$u_t + \delta u_{xxx} + g(u)u_x - \nu u_{xx} + \gamma u = f(x), \tag{2.1}$$

which contains two arbitrary functions $g(u)$ and $f(x)$. We perform Lie group classification of (2.1) and then find symmetry reductions and exact solutions.

This work is new and has been submitted for publication. See [63].

2.2 Equivalence transformations

An equivalence transformation (see for example [8]) of (2.1) is an invertible transformation involving the variables t, x and u that map (2.1) into itself. The operator

$$\begin{aligned} Y &= \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u + \mu^1(t,x,u,f,g)\partial_f \\ &\quad + \mu^2(t,x,u,f,g)\partial_g, \end{aligned} \tag{2.2}$$

is the generator of the equivalence group for system (2.1) provided it is admitted by the extended system

$$u_t + \delta u_{xxx} + g(u)u_x - \nu u_{xx} + \gamma u = f(x), \tag{2.3a}$$

$$f_t = 0, \quad f_u = 0, \quad g_t = 0, \quad g_x = 0. \tag{2.3b}$$

The prolonged operator for the extended system (2.3) has the form

$$\widetilde{Y} = Y^{[3]} + \omega_1^1\partial_{f_t} + \omega_2^1\partial_{f_x} + \omega_3^1\partial_{f_u} + \omega_1^2\partial_{g_t} + \omega_2^2\partial_{g_x} + \omega_3^2\partial_{g_u}, \tag{2.4}$$

where $Y^{[3]}$ is the third-prolongation of (2.2) given by

$$\begin{aligned} Y^{[3]} &= \tau\partial_t + \xi\partial_x + \eta\partial_u + \mu^1\partial_f + \mu^2\partial_g + \zeta_1\partial_{u_t} + \zeta_2\partial_{u_x} \\ &\quad + \zeta_{22}\partial_{u_{xx}} + \zeta_{222}\partial_{u_{xxx}}. \end{aligned}$$

The variables ζ's and ω's are defined by the prolongation formulae

$$\begin{aligned} \zeta_1 &= D_t(\eta) - u_x D_t(\tau) - u_t D_t(\xi), \\ \zeta_2 &= D_x(\eta) - u_x D_x(\tau) - u_t D_x(\xi), \\ \zeta_{22} &= D_x(\zeta_2) - u_{xx} D_x(\tau) - u_{xt} D_x(\xi), \\ \zeta_{222} &= D_x(\zeta_{22}) - u_{xxx} D_x(\tau) - u_{xxt} D_x(\xi) \end{aligned}$$

and

$$\begin{aligned}
\omega_1^1 &= \widetilde{D}_t(\mu^1) - f_t\widetilde{D}_t(\tau) - f_x\widetilde{D}_t(\xi) - f_u\widetilde{D}_t(\eta),\\
\omega_2^1 &= \widetilde{D}_x(\mu^1) - f_t\widetilde{D}_x(\tau) - f_x\widetilde{D}_x(\xi) - f_u\widetilde{D}_x(\eta),\\
\omega_3^1 &= \widetilde{D}_u(\mu^1) - f_t\widetilde{D}_u(\tau) - f_x\widetilde{D}_u(\xi) - f_u\widetilde{D}_u(\eta),\\
\omega_1^2 &= \widetilde{D}_t(\mu^2) - g_t\widetilde{D}_t(\tau) - g_x\widetilde{D}_t(\xi) - g_u\widetilde{D}_t(\eta),\\
\omega_2^2 &= \widetilde{D}_x(\mu^2) - g_t\widetilde{D}_x(\tau) - g_x\widetilde{D}_x(\xi) - g_u\widetilde{D}_x(\eta),\\
\omega_3^2 &= \widetilde{D}_u(\mu^2) - g_t\widetilde{D}_u(\tau) - g_x\widetilde{D}_u(\xi) - g_u\widetilde{D}_u(\eta),
\end{aligned}$$

respectively, where

$$D_t = \partial_t + u_t\partial_u + \cdots\,, \qquad D_x = \partial_x + u_x\partial_u + \cdots$$

are the total derivative operators and

$$\begin{aligned}
\widetilde{D}_x &= \partial_x + f_x\partial_f + g_x\partial_g + \cdots\,,\\
\widetilde{D}_t &= \partial_t + f_t\partial_f + g_t\partial_g + \cdots\,,\\
\widetilde{D}_u &= \partial_u + f_u\partial_f + g_u\partial_g + \cdots
\end{aligned}$$

are the total derivative operators for the extended system. The application of the prolongation (2.4) and the invariance conditions of system (2.3) leads to following equivalent generators

$$\begin{aligned}
Y_1 &= \frac{\partial}{\partial t},\\
Y_2 &= \frac{\partial}{\partial x},\\
Y_3 &= \frac{\partial}{\partial u} + \gamma\frac{\partial}{\partial f},\\
Y_4 &= u\frac{\partial}{\partial u} + f\frac{\partial}{\partial f}.
\end{aligned}$$

Thus the four-parameter equivalence group is given by

$$\begin{aligned}
Y_1 &: \ \bar{t} = a_1 + t,\ \bar{x} = x,\ \bar{u} = u,\ \bar{f} = f,\ \bar{g} = g,\\
Y_2 &: \ \bar{t} = t,\ \bar{x} = a_2 + x,\ \bar{u} = u,\ \bar{f} = f,\ \bar{g} = g,\\
Y_3 &: \ \bar{t} = t,\ \bar{x} = x,\ \bar{u} = a_3 + u,\ \bar{f} = \gamma a_3 + f,\ \bar{g} = g,\\
Y_4 &: \ \bar{t} = t,\ \bar{x} = x,\ \bar{u} = e^{a_4}u,\ \bar{f} = e^{a_4}f,\ \bar{g} = g
\end{aligned}$$

and their composition gives

$$
\begin{aligned}
\bar{t} &= a_1 + t,\\
\bar{x} &= a_2 + x,\\
\bar{u} &= (a_3 + u)e^{a_4},\\
\bar{f} &= (\gamma a_3 + f)e^{a_4},\\
\bar{g} &= g.
\end{aligned}
$$

2.3 Principal Lie algebra

The symmetry group of equation (2.1) will be generated by the vector field of the form

$$\Gamma = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}. \tag{2.5}$$

Applying the third prolongation of Γ to (2.1) yields the following overdetermined system of linear partial differential equations (PDEs):

$$
\begin{aligned}
&\tau_u = 0, \tau_x = 0, \xi_u = 0, \eta_{uu} = 0,\\
&2\nu\xi_x - \nu\tau_t + 3\delta\eta_{xu} - 3\delta\xi_{x,x} = 0, 3\xi_x - \tau_t = 0,\\
&\eta g_u - \xi_t - g\xi_x + g\tau_t - 2\nu\eta_{xu} + \nu\xi_{xx} + 3\delta\eta_{xxu} - \delta\xi_{xxx} = 0,\\
&\gamma\eta - \xi f_x + \eta_t + f\eta_u - u\gamma\eta_u + g\eta_x - f\tau_t + u\gamma\tau_t - \nu\eta_{xx} + \delta\eta_{xxx} = 0. \quad (2.6)
\end{aligned}
$$

Solving the above system for arbitrary f and g we find that the principal Lie algebra consists of one translation symmetry, namely

$$\Gamma_1 = \frac{\partial}{\partial t}.$$

2.4 Lie group classification

Solving the system (2.6), we obtain the following classifying relations:

$$\begin{aligned}
&\frac{2ga_t}{3} - \frac{2a_t\nu^2}{9\delta} + \left(B + u\left(k + \frac{x\nu a_t}{9\delta}\right)\right) g_u - q_t - \frac{1}{3}xa_{tt} = 0,\\
&+u\gamma a_t - fa_t + f\left(k + \frac{x\nu a_t}{9\delta}\right) - u\gamma\left(k + \frac{x\nu a_t}{9\delta}\right) + \gamma\left(B + u\left(k + \frac{x\nu a_t}{9\delta}\right)\right) + B_t\\
&+g\left(\frac{u\nu a_t}{9\delta} + B_x\right) - \left(q + \frac{xa_t}{3}\right) f_x + u\left(k_t + \frac{x\nu a_{tt}}{9\delta}\right) - \nu B_{xx} + \delta B_{xxx} = 0.
\end{aligned}$$

These classifying relations lead to the following six cases for the functions g and f and for each case we also provide the associated extended symmetries.

Case (A): $f(x) = f_0$, $g(u) = g_0$, where f_0, g_0 are nonzero constants.

$$\begin{aligned}
\Gamma_2 &= 9\delta t\frac{\partial}{\partial t} - (9\delta\gamma tu + g_0\nu tu - \nu ux)\frac{\partial}{\partial u} - \left(2\nu^2 t - 6\delta g_0 t - 3\delta x\right)\frac{\partial}{\partial x},\\
\Gamma_3 &= u\frac{\partial}{\partial u},\\
\Gamma_4 &= \frac{\partial}{\partial x},\\
\Gamma_5 &= F(t,x)\frac{\partial}{\partial u},
\end{aligned}$$

where $F(t,x)$ is any solution of

$$\begin{aligned}
&\left(6561\delta^2 f_0\gamma t + 729\delta f_0 g_0\nu t - 729\delta f_0\nu x + 6561\delta^2 f_0\right)C_2 + 729\gamma\delta F\\
&+729g_0\delta F_x - 729\nu\delta F_{xx} + 729\delta^2 F_{xxx} + 729f_0C_3\delta + 729\delta F_t = 0.
\end{aligned}$$

Case (B): $f(x) = f_0 + f_1x$, $g(u) = g_0$, $f_1 = \gamma\rho$, where f_0, ρ are nonzero constants.

$$\begin{aligned}
\Gamma_2 &= 9t\delta\frac{\partial}{\partial t} - (9\delta\gamma tu + g_0\nu tu - \nu ux)\frac{\partial}{\partial u} - \left(2\nu^2 t - 6\delta g_0 t - 3\delta x\right)\frac{\partial}{\partial x},\\
\Gamma_3 &= u\frac{\partial}{\partial u},\\
\Gamma_4 &= \frac{\partial}{\partial x},\\
\Gamma_5 &= F(t,x)\frac{\partial}{\partial u},
\end{aligned}$$

and $F(t, x)$ is any solution of

$$\begin{aligned}
&(729\delta^2\gamma^2\rho t x + 81\delta g_0\gamma\nu\rho t x + 486\delta^2 g_0\gamma\rho t - 162\delta\gamma\nu^2\rho t - 81\delta\gamma\nu\rho x^2 \\
&+729\delta^2 f_0\gamma t + 972\delta^2\gamma\rho x + 81\delta f_0 g_0\nu t - 81\delta f_0\nu x + 729\delta^2 f_0)C_2 \\
&+ (81\delta\gamma\rho x + 81\delta f_0)\, C_3 - 81\gamma\rho C_4\delta + 81\gamma F\delta + 81 g_0\delta F_x \\
&-81\nu\delta F_{xx} + 81\delta^2 F_{xxx} + 81\delta F_t = 0.
\end{aligned}$$

Case (C): $f(x) = \bar{f}_0 + e^{\bar{f}_1 x}$, $g(u) = g_0$, $\bar{f}_1 = 1/\alpha$, $\bar{f}_0 = -\beta\gamma$, where g_0, α, β are nonzero constants.

$$\begin{aligned}
\Gamma_2 &= 9t\delta\frac{\partial}{\partial t} - (9\delta\gamma t u + u t\nu g_0 - \nu u x)\frac{\partial}{\partial u} - \left(2\nu^2 t - 6\delta g_0 t - 3\delta x\right)\frac{\partial}{\partial x}, \\
\Gamma_3 &= u\frac{\partial}{\partial u}, \\
\Gamma_4 &= \frac{\partial}{\partial x}, \\
\Gamma_5 &= F\left(t, x\right)\frac{\partial}{\partial u},
\end{aligned}$$

where $F(t, x)$ is any solution of

$$\begin{aligned}
&(27\delta(9\delta\gamma^2 t + g_0\gamma\nu t + 6\delta g_0 t - \gamma\nu x - 2\nu^2 t + 9\delta\gamma + 3\delta x)\mathrm{e}^{\frac{x}{\gamma}} \\
&+27\delta\left(-9\beta\delta\gamma^3 t - \beta g_0\gamma^2\nu t + \beta\gamma^2\nu x - 9\beta\delta\gamma^2)\right)C_2 + \left(-27\beta\delta\gamma^2 + 27\mathrm{e}^{\frac{x}{\gamma}}\gamma\delta\right)C_3 \\
&+27\gamma g_0\delta F_x - 27\gamma\nu\delta F_{xx} + 27\gamma\delta^2 F_{xxx} + 27\gamma^2\delta F + 27\gamma\delta F_t - 27\mathrm{e}^{\frac{x}{\gamma}} C_4\delta = 0.
\end{aligned}$$

Case (D): $f(x) = f_0$, $g(u) = g_0 - g_1 \ln u$, where f_0, g_0, g_1 are nonzero constants.

$$\Gamma_2 = \frac{\partial}{\partial x}.$$

Case (E): $f(x) = f_0$, $g(u) = u^2 + \bar{g}_0 u + \bar{g}_1$, where $\bar{g}_0 \neq 0$ is an arbitrary constant.

$$\Gamma_2 = \frac{\partial}{\partial x}.$$

Case (F): $f(x) = f_0 + f_1 x$, $g(u) = g_0 + \tilde{g}_1 u$, where $f_0, f_1, g_0, \tilde{g}_1$ are nonzero constants.

$$\begin{aligned}
\Gamma_2 &= e^{(-1/2)tR_1} R_1\frac{\partial}{\partial u} - 2\tilde{g}_1 e^{(-1/2)tR_1}\frac{\partial}{\partial x}, \\
\Gamma_3 &= e^{(-1/2)tR_2} R_2\frac{\partial}{\partial u} - 2\tilde{g}_1 e^{(-1/2)tR_2}\frac{\partial}{\partial x},
\end{aligned}$$

where

$$R_1 = \gamma - \sqrt{4f_1\tilde{g}_1 + \gamma^2} \neq 0, \quad R_2 = \gamma + \sqrt{4f_1\tilde{g}_1 + \gamma^2} \neq 0$$

are arbitrary constants.

2.5 Symmetry reductions and exact solutions

In order to obtain symmetry reductions and exact solutions, one has to solve the associated Lagrange equations

$$\frac{dt}{\tau(t,x,u,)} = \frac{dx}{\xi(t,x,u)} = \frac{du}{\eta(t,x,u)}.$$

For symmetry reductions purposes we consider only those cases in which the equation (2.1) is nonlinear.

2.5.1 Case (D).

The linear combination of $\Gamma_1 + c\Gamma_2$ gives rise to the group-invariant solution

$$u = F(z) \tag{2.7}$$

where c is a non-zero constant, $z = x - ct$ is an invariant of the symmetry $\Gamma_1 + c\Gamma_2$ and $F(z)$ satisfies the third-order nonlinear ordinary differential equation (ODE)

$$\delta\, F'''(z) - \nu\, F''(z) - cF'(z) + g_0 F'(z) - g_1 F'(z) \ln(F(z)) + \gamma\, F(z) - f_0 = 0.$$

2.5.2 Case (E).

The symmetry $\Gamma_1 + c\Gamma_2$ gives rise to the group-invariant solution

$$u = F(z) \tag{2.8}$$

where $z = x - ct$ is an invariant of $\Gamma_1 + c\Gamma_2$ and $F(z)$ satisfies

$$\delta F'''(z) - \nu F''(z) - cF'(z) + \left(g_0 F(z) + F(z)^2 + g_1\right) F'(z) + \gamma F(z) - f_0 = 0.$$

2.5.3 Case (F). One-dimensional optimal system of subalgebras

In this case we have three symmetries for the corresponding equation (2.1) and so we first obtain the optimal system of one-dimensional subalgebras and then present the optimal system of group-invariant solutions. We use the method given in [7]. The adjoint transformations are given by

$$\mathrm{Ad}(\exp(\epsilon\Gamma_i))\Gamma_j = \Gamma_j - \epsilon[\Gamma_i, \Gamma_j] + \frac{1}{2}\,\epsilon^2[\Gamma_i, [\Gamma_i, \Gamma_j]] - \cdots ,$$

where $[\Gamma_i, \Gamma_j]$ denotes the commutator of Γ_i and Γ_j defined as

$$[\Gamma_i, \Gamma_j] = \Gamma_i\Gamma_j - \Gamma_j\Gamma_i.$$

In Table 1 and Table 2, we give, respectively, the commutator table of the Lie point symmetries of the system (2.1) and the adjoint representations of the symmetry group of (2.1). These tables are then used to construct the optimal system of one-dimensional subalgebras for system (2.1).

Table 1. Commutator table of the Lie algebra of system (2.1)

	Γ_1	Γ_2	Γ_3
Γ_1	0	$-\frac{1}{2}R_1\Gamma_2$	$-\frac{1}{2}R_2\Gamma_3$
Γ_2	$\frac{1}{2}R_1\Gamma_2$	0	0
Γ_3	$\frac{1}{2}R_2\Gamma_3$	0	0

Table 2. Adjoint table of the Lie algebra of system (2.1)

Ad	Γ_1	Γ_2	Γ_3
Γ_1	Γ_1	$e^{(1/2)R_1\epsilon}\Gamma_2$	$e^{(1/2)R_2\epsilon}\Gamma_3$
Γ_2	$\Gamma_1 - \frac{1}{2}R_1\epsilon\Gamma_2$	Γ_2	Γ_3
Γ_3	$\Gamma_1 - \frac{1}{2}R_2\epsilon\Gamma_3$	Γ_3	Γ_3

Thus, from Tables 1 and 2 one can obtain an optimal system of one-dimensional subalgebras given by $\{\Gamma_1, \Gamma_3 + \Gamma_2, \Gamma_3 - \Gamma_2, \Gamma_3\}$.

Symmetry reductions and exact solutions based on the one-dimensional optimal system of subalgebras

Here we use the optimal system of one-dimensional subalgebras calculated above to obtain symmetry reductions that transform (2.1) into ordinary differential equations (ODEs). We then look for exact solutions of the ODEs.

Case (F.1) The symmetry Γ_1 gives rise to the group-invariant solution

$$u = F(z) \tag{2.9}$$

where $z = x$ is an invariant of the symmetry Γ_1 and $F(z)$ satisfies the ODE

$$\delta\, F'''(z) + g_0 F'(z) + \tilde{g}_1 F'(z)\, F(z) - \nu\, F''(z) + \gamma\, F(z) - f_1\, z - f_0 = 0.$$

Case (F.2) The symmetry $\Gamma_3 + \Gamma_2$ gives us the group-invariant solution

$$\begin{aligned} u(t,x) \;&=\; \frac{1}{2\tilde{g}_1(e^{(-1/2)tP_1} + e^{(1/2)tP_1})}\Big\{2F(z)\tilde{g}_1 e^{(-1/2)tP_1} + 2F(z)\tilde{g}_1 e^{(1/2)tP_1} \\ &\quad - e^{(-1/2)tP_1} P_1 x - e^{(-1/2)tP_1}\gamma x + P_1 e^{(1/2)tP_1} x - e^{(1/2)tP_1}\gamma x\Big\}, \end{aligned} \tag{2.10}$$

where $P_1 = \sqrt{4\, f_1\, \tilde{g}_1 + \gamma^2}$ is a non-zero arbitrary constant, $z = t$ is an invariant of $\Gamma_3 + \Gamma_2$ and the function $F(z)$ satisfies the ODE

$$\begin{aligned} &-F(z)e^{-(1/2)\,zP_1} P_1\, \tilde{g}_1 + \gamma\, F(z)e^{-(1/2)\,zP_1}\tilde{g}_1 + F(z)P_1\, e^{(1/2)\,zP_1}\tilde{g}_1 + \gamma\, F(z)e^{(1/2)\,zP_1}\tilde{g}_1 \\ &+2\,(F'(z))e^{-(1/2)\,zP_1}\tilde{g}_1 - g_0\, e^{-(1/2)\,zP_1} P_1 - g_0\, e^{-(1/2)\,zP_1}\gamma - 2\, e^{-(1/2)\,zP_1} f_0\, \tilde{g}_1 \\ &+2\,(F'(z))e^{(1/2)\,zP_1}\tilde{g}_1 + g_0\, P_1\, e^{(1/2)\,zP_1} - g_0\, e^{(1/2)\,zP_1}\gamma - 2\, e^{(1/2)\,zP_1} f_0\, \tilde{g}_1 = 0 \end{aligned}$$

whose solution is

$$\begin{aligned} F(z) = \Big\{[&\frac{(-P_1 g_0 + 2f_0\tilde{g}_1 + \gamma g_0)e^{(1/2)(\gamma-P_1)z}}{\tilde{g}_1(\gamma+P_1)} \\ &+\frac{(P_1 g_0 + 2f_0\tilde{g}_1 + \gamma g_0)e^{(1/2)(\gamma-3P_1)z}}{\tilde{g}_1(\gamma-P_1)}]e^{zP_1} + C_1\Big\}e^{-(1/2)z(\gamma+P_1)}(e^{-zP_1}+1)^{-1}, \end{aligned}$$

where $P_1 \neq \pm\gamma$ and C_1 is an arbitrary constant. Consequently the required group invariant solution is completed by (2.10).

Case (F.3) The symmetry $\Gamma_3 - \Gamma_2$ gives rise to the group-invariant solution of the form

$$u(t,x) = \frac{1}{2g_1\,(e^{-(1/2)\,tP_1} - e^{(1/2)\,tP_1})}\Big\{2\,F(z)g_1\,e^{-(1/2)\,tP_1} - 2\,F(z)g_1\,e^{(1/2)\,tP_1}$$
$$-e^{-(1/2)\,tP_1}P_1\,x - e^{-(1/2)\,tP_1}\gamma\,x - P_1\,e^{(1/2)\,tP_1}x + e^{(1/2)\,tP_1}\gamma\,x\Big\}, \tag{2.11}$$

where $z = t$ is an invariant of $\Gamma_3 - \Gamma_2$ and the function $F(z)$ satisfies

$$-F(z)e^{-(1/2)zP_1}P_1g_1 + \gamma F(z)e^{-(1/2)zP_1}g_1 - F(z)P_1e^{(1/2)zP_1}g_1 - \gamma F(z)e^{(1/2)zP_1}g_1$$
$$+2(F'(z))e^{-(1/2)zP_1}g_1 - g_0e^{-(1/2)zP_1}P_1 - g_0e^{-(1/2)zP_1}\gamma - 2e^{-(1/2)zP_1}f_0g_1$$
$$-2(F'(z))e^{(1/2)zP_1}g_1 - g_0P_1e^{(1/2)zP_1} + g_0e^{(1/2)zP_1}\gamma + 2e^{(1/2)zP_1}f_0g_1 = 0$$

whose solution is

$$F(z) = \Big\{[\frac{(g_0\,P_1 + 2\,f_0\,\tilde{g}_1 + \gamma\,g_0)\mathrm{e}^{(1/2)(\gamma-3P_1)z}}{\tilde{g}_1\,(\gamma - P_1)}$$
$$-\frac{(-g_0\,P_1 + 2\,f_0\,\tilde{g}_1 + \gamma\,g_0)\mathrm{e}^{(1/2)(\gamma-P_1)z}}{\tilde{g}_1\,(\gamma + P_1)}]\mathrm{e}^{zP_1} + B_1\Big\}\mathrm{e}^{-(1/2)\,z(\gamma+P_1)}(\mathrm{e}^{-\,zP_1} - 1)^{-1}$$

where $P_1 \neq \pm\gamma$ and B_1 is an arbitrary constant. Consequently the group-invariant solution is completed by (2.11).

Case (F.4) The symmetry Γ_3 gives the group-invariant solution

$$u(t,x) = \frac{2F\,(z)\,\tilde{g}_1 - P_1x - x\gamma}{2\tilde{g}_1} \tag{2.12}$$

where $z = t$ is an invariant of Γ_3 and the function $F(z)$ satisfies

$$F\,(z)\,\gamma\,\tilde{g}_1 - F\,(z)\,P_1\,\tilde{g}_1 + 2\,(F'\,(z))\,\tilde{g}_1 - g_0\,P_1 - g_0\,\gamma - 2\,f_0\,\tilde{g}_1 = 0$$

whose solution is given by

$$F\,(z) = e^{-(1/2)\,(\gamma-P_1)z}C_1 + \frac{g_0\,P_1 + 2\,f_0\,\tilde{g}_1 + g_0\,\gamma}{\tilde{g}_1\,(\gamma - P_1)}$$

and consequently the group-invariant solution is completed by (2.12).

2.6 Conclusion

In this chapter Lie group classification was performed on a generalized Korteweg-de Vries-Burgers equation (2.1). The functional forms of (2.1) of the type linear, quadratic, exponential and logarithmic were obtained. The Lie algebra obtained was of dimension two, three and infinite. For the case when the principal Lie algebra was extended by two symmetries, one-dimensional optimal system of subalgebras was obtained and the corresponding group-invariant solutions were derived.

Chapter 3

Exact solutions and conservation laws of a two-dimensional integrable generalization of the Kaup-Kupershmidt equation

In this chapter, we study the two-dimensional integrable generalization of the Kaup-Kupershmidt equation, namely,

$$\begin{aligned} & u_t + u_{xxxxx} + \frac{25}{2}u_x u_{xx} + 5uu_{xxx} + 5u^2u_x + 5u_{xxy} - 5\partial_x^{-1}u_{yy} + 5uu_y \\ & +5u_x\partial_x^{-1}u_y = 0, \end{aligned} \tag{3.1}$$

which arises in various problems in many areas of theoretical physics. The above equation occurs as a special reduction of integrable nonlinear systems [42, 43]. The Zakharov-Manakov delta dressing method was used to obtain soliton and periodic solutions of (3.1) [42, 43]. Here we use Lie symmetry method along with the extended tanh and the extended Jacobi elliptic function methods to obtain new solutions of (3.1). We also derive conservation laws using the multiplier approach [74].

However, to study the two-dimensional integrable generalization of the Kaup-Kupershmidt equation (3.1) we first introduce a new dependent variable v and set $v = \partial_x^{-1}u_y =$

$\int u_y dx$. This allows us to remove the integral terms from the equation and replace the equation (3.1) by a system given by

$$\begin{aligned}
&u_t + u_{xxxxx} + \frac{25}{2}\, u_x u_{xx} + 5\, u u_{xxx} + 5\, u^2 u_x + 5\, u_{xxy} - 5\, v_y \\
&+5\, u v_x + 5\, u_x v = 0, && (3.2a) \\
&u_y - v_x = 0. && (3.2b)
\end{aligned}$$

This work has been published in [65].

3.1 Exact solutions of (3.2)

The symmetry group of the system (3.2) will be generated by the vector field of the form

$$\begin{aligned}
X &= \xi^1(t,x,y,u,v)\frac{\partial}{\partial t} + \xi^2(t,x,y,u,v)\frac{\partial}{\partial x} + \xi^3(t,x,y,u,v)\frac{\partial}{\partial y} \\
&\quad +\eta^1(t,x,y,u,v)\frac{\partial}{\partial u} + \eta^2(t,x,y,u,v)\frac{\partial}{\partial v}.
\end{aligned}$$

The application of fifth prolongation, $\mathrm{pr}^{(5)}X$, to (3.2) results in an overdetermined system of linear partial differential equations given by

$$\begin{aligned}
&\xi^2_v = 0,\ \xi^2_u = 0,\ \xi^3_v = 0,\ \xi^3_u = 0,\ \xi^3_x = 0,\ \eta^1_v = 0,\ \xi^1_v = 0,\ \xi^1_u = 0,\ \xi^1_y = 0,\ \xi^1_x = 0, \\
&\eta^1_{uu} = 0,\ \eta^1_y - \eta^2_x = 0,\ \xi^2_y + \eta^2_u = 0,\ 2\xi^2_{xx} - \eta^1_{xu} = 0,\ \xi^3_y - 3\xi^2_x = 0, \\
&\xi^2_{xx} - 2\eta^1_{xu} = 0,\ \xi^1_t - 5\xi^2_x = 0,\ \eta^1_u + 2\xi^2_x = 0,\ \xi^2_{xx} - 2\eta^1_{xu} = 0, \\
&\eta^1_u + \xi^2_x - \xi^3_y - \eta^2_v = 0,\ \xi^3_y - 5\xi^2_x - \eta^2_v + 2u\eta^1_u\xi^2_x + \eta_1 + 2\eta^1_{xxu} - \xi^2_y - 2\xi^2_{xxx} = 0, \\
&6u\eta^1_{xu} - 6u\xi^2_{xx} + 5\eta^1_x + 2\eta^1_{yu} + 4\eta^1_{xxxu} - 4\xi^2_{xy} - 2\xi^2_{xxxx} = 0, \\
&5\eta^1_x u^2 + 5u\eta^1_{xxx} + 5u\eta^2_x + 5\eta^1_x v + \eta^1_t + 5\eta^1_{xxy} + \eta^1_{xxxxx} - 5\eta^2_y - 5\eta^1_u u + 5u\eta^2_v + 20u\xi^2_x \\
&+5\eta_1 + 5\eta^1_{xxu} - 5\eta^2_u + 5\xi^2_y - \xi^3_t 40\xi^2_x u^2 + 20u\eta_1 + 30u\eta^1_{xxu} + 10u\eta^2_u - 10u\xi^2_{xxx} + 40\xi^2_x v \\
&+10\eta_2 + 25\eta^1_{xx} + 20\eta^1_{xyu} + 10\eta^1_{xxxxu} - 2\xi^2_t - 10\xi^2_{xxy} - 2\xi^2_{xxxxx} = 0.
\end{aligned}$$

The general solution of the above overdetermined system of linear partial differential equations, using Maple, is given by

$$\begin{aligned}
\xi^1 &= 5F_2(t) + 5yF_3'(t) + 5y^2F_1''(t) + 50xF_1'(t),\\
\xi^2 &= 150yF_1'(t) + 75F_3(t),\\
\xi^3 &= 250F_1(t),\\
\eta^1 &= 5F_3'(t) + 10yF_1''(t) - 100uF_1'(t),\\
\eta^2 &= -5uF_3'(t) + yF_3''(t) + F_2'(t) - 200vF_1'(t) + (10x - 10yu)F_1''(t) + y^2F_1'''(t),
\end{aligned}$$

where $F_1(t), F_2(t)$ and $F_3(t)$ are arbitrary functions of t. We confine the arbitrary functions to be of the form $F_1(t) = C_1t + C_2, F_2(t) = C_3t + C_4, F_3(t) = C_5t + C_6$, where $C_1, \cdots, C_6$ are arbitrary constants. Consequently, we have the six-dimensional Lie algebra spanned by the following linearly independent operators:

$$\begin{aligned}
\Gamma_1 &= \frac{\partial}{\partial t},\\
\Gamma_2 &= \frac{\partial}{\partial x},\\
\Gamma_3 &= \frac{\partial}{\partial y},\\
\Gamma_4 &= 5t\frac{\partial}{\partial x} + \frac{\partial}{\partial v},\\
\Gamma_5 &= y\frac{\partial}{\partial x} + 15t\frac{\partial}{\partial y} + \frac{\partial}{\partial u} - u\frac{\partial}{\partial v},\\
\Gamma_6 &= x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y} + 5t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} - 4v\frac{\partial}{\partial v}.
\end{aligned}$$

3.1.1 Symmetry reduction of (3.2)

One of the main reasons for calculating symmetries of a differential equation is to use them for obtaining symmetry reductions and finding exact solutions. This can be achieved with the use of Lie point symmetries admitted by (3.2). It is a well known fact that the reduction of a partial differential equation with respect to r-dimensional (solvable) subalgebra of its Lie symmetry algebra leads to reducing the number of independent variables by r.

Consider the first three translation symmetries and let $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$. We use Γ to reduce (3.2) to a system of partial differential equations (PDEs) in two independent variables. The symmetry Γ yields the following invariants:

$$f = t - y, \quad g = x - y, \quad \phi = u, \quad \psi = v. \tag{3.3}$$

Considering ϕ, ψ as the new dependent variables and f and g as new independent variables, (3.2) transforms to

$$\begin{aligned}
&\phi_{ggggg} + 5\,\phi_g\phi^2 - 5\,\phi_{fgg} + 5\,\phi_g\psi + 5\,\psi_f + \phi_f + 5\,\psi_g\phi + \frac{25}{2}\,\phi_g\phi_{gg} \\
&+5\,\psi_g + 5\,\phi_{ggg}\phi - 5\,\phi_{ggg} = 0, \\
&\phi_f + \phi_g + \psi_g = 0,
\end{aligned}$$

which is a system of nonlinear PDEs in two independent variables f and g. We now further reduce this system by using its symmetries. This system has the two translation symmetries, namely

$$\Upsilon_1 = \frac{\partial}{\partial f}, \quad \Upsilon_2 = \frac{\partial}{\partial g}.$$

By taking a linear combination $\rho\Upsilon_1 + \Upsilon_2$ of the above symmetries, we see that it yields the invariants

$$z = f - \rho g, \quad \phi = F, \quad \psi = G.$$

Now treating F and G as new dependent variables and z as the new independent variable the above system transforms to the following system of nonlinear coupled ODEs:

$$\begin{aligned}
&\rho^5 F'''''(z) + 5\rho^3 F(z)F'''(z) - 5\rho^3 F'''(z) + 5\rho^2 F'''(z) \\
&+5\rho G(z)F'(z) + 5\rho F(z)^2 F'(z) - F'(z) + \frac{25}{2}\rho^3 F'(z)F''(z) \\
&+5\rho F(z)G'(z) + 5\rho G'(z) - 5G'(z) = 0, \qquad (3.4\text{a}) \\
&\rho F'(z) - F'(z) + \rho G'(z) = 0. \qquad (3.4\text{b})
\end{aligned}$$

3.1.2 Exact solutions using the extended tanh method

In this section we use the extended tanh function method which was introduced by Wazwaz [76]. The basic idea in this method is to assume that the solution of (3.4) can be written in the form

$$F(z) = \sum_{i=-M}^{M} A_i H(z)^i, \quad G(z) = \sum_{i=-M}^{M} B_i H(z)^i, \tag{3.5}$$

where $H(z)$ satisfies an auxiliary equation, say for example the Riccati equation

$$H'(z) = 1 - H^2(z), \tag{3.6}$$

whose solution is given by

$$H(z) = \tanh(z).$$

The positive integer M will be determined by the homogeneous balance method between the highest order derivative and highest order nonlinear term appearing in (3.4). A_i, B_i are parameters to be determined.

In our case, the balancing procedure gives $M = 2$ and so the solutions of (3.4) are of the form

$$F(z) = A_{-2}H^{-2} + A_{-1}H^{-1} + A_0 + A_1 H + A_2 H^2, \tag{3.7a}$$

$$G(z) = B_{-2}H^{-2} + B_{-1}H^{-1} + B_0 + B_1 H + B_2 H^2. \tag{3.7b}$$

Substituting (3.7) into (3.4) and making use of the Riccati equation (3.6) and then equating the coefficients of the functions H^i to zero, we obtain the following algebraic system of equations in terms of A_i and B_i $(i = -2, -1, 0, 1, 2)$:

$$\begin{aligned}
&2A_{-2} - 2\rho A_{-2} - 2\rho B_{-2} = 0,\ A_{-1} - \rho A_{-1} - \rho B_{-1} = 0,\ A_1 - \rho A_1 - \rho B_1 = 0,\\
&-2A_2 + 2\rho A_2 + 2\rho B_2 = 0,\ 2A_2 - 2\rho A_2 - 2\rho B_2 = 0,\ 2\rho A_{-2} + 2\rho B_{-2} - 2A_{-2} = 0,\\
&270\rho^3 {A_{-2}}^2 + 10\rho {A_{-2}}^3 + 720\rho^5 A_{-2} = 0,\ 270\rho^3 {A_2}^2 + 720\rho^5 A_2 + 10\rho {A_2}^3 = 0,\\
&25\rho {A_{-2}}^2 A_{-1} + 275\rho^3 A_{-2}A_{-1} + 120\rho^5 A_{-1} = 0,\ 275\rho^3 A_1 A_2 + 25\rho A_1 {A_2}^2 + 120\rho^5 A_1 = 0,\\
&-A_1 + \rho A_1 + \rho B_1 - A_{-1} + \rho A_{-1} + \rho B_{-1} = 0,\\
&-1680\rho^5 A_2 + 20\rho {A_2}^2 A_0 + 120\rho^3 A_2 A_0 - 10\rho {A_2}^3 + 120\rho^2 A_2 - 120\rho^3 A_2\\
&-550\rho^3 {A_2}^2 + 20\rho A_2 B_2 + 55\rho^3 {A_1}^2 + 20\rho {A_1}^2 A_2 = 0,
\end{aligned}$$

$$20\rho A_{-2}{}^2A_0+55\rho^3A_{-1}{}^2+20\rho A_{-2}B_{-2}+120\rho^3A_{-2}A_0+20\rho A_{-2}A_{-1}{}^2+120\rho^2A_{-2}$$
$$-120\rho^3A_{-2}-550\rho^3A_{-2}{}^2-1680\rho^5A_{-2}-10\rho A_{-2}{}^3=0,$$
$$15\rho A_{-1}A_2{}^2+45\rho^3A_{-1}A_2-515\rho^3A_1A_2-240\rho^5A_1+5\rho A_1{}^3+30\rho^2A_1+30\rho A_1A_0A_2$$
$$+30\rho^3A_1A_0-25\rho A_1A_2{}^2+15\rho A_1B_2-30\rho^3A_1+15\rho A_2B_1=0,$$

$$-515\rho^3A_{-2}A_{-1}+15\rho A_{-1}B_{-2}+30\rho A_{-2}A_{-1}A_0+45\rho^3A_{-2}A_1-30\rho^3A_{-1}$$
$$-240\rho^5A_{-1}+5\rho A_{-1}{}^3+15\rho A_{-2}{}^2A_1+15\rho A_{-2}B_{-1}+30\rho^2A_{-1}$$
$$-25\rho A_{-2}{}^2A_{-1}+30\rho^3A_{-1}A_0=0,$$
$$-10\rho A_{-2}B_0-5\rho^3A_1A_{-1}+80\rho^3A_{-2}A_0+35\rho^3A_{-1}{}^2-20\rho^3A_{-2}A_2$$
$$-10\rho B_{-2}-10\rho A_{-2}{}^2A_2-272\rho^5A_{-2}-10\rho B_{-2}A_0-50\rho^3A_{-2}{}^2$$
$$-10\rho A_{-1}{}^2A_0-80\rho^3A_{-2}+10B_{-2}-10\rho A_{-1}B_{-1}-10\rho A_{-2}A_0{}^2$$
$$+80\rho^2A_{-2}+2A_{-2}-20\rho A_{-2}A_{-1}A_1=0,$$

$$-5\rho^3A_1A_{-1}-20\rho^3A_{-2}A_2-10\rho B_2A_0+2A_2-10\rho A_1{}^2A_0$$
$$-10\rho A_{-2}A_2{}^2-10\rho A_2A_0{}^2-50\rho^3A_2{}^2-80\rho^3A_2+80\rho^2A_2-10\rho A_2B_0$$
$$-20\rho A_{-1}A_1A_2+35\rho^3A_1{}^2+80\rho^3A_2A_0+10B_2$$
$$-10\rho A_1B_1-10\rho B_2-272\rho^5A_2=0,$$

$$-200\rho^2A_2-200\rho^3A_2A_0+10\rho A_{-2}A_2{}^2+10\rho A_2B_0-2A_2+5\rho^3A_1A_{-1}$$
$$+1232\rho^5A_2-20\rho A_2B_2+10\rho A_1B_1-20\rho A_2{}^2A_0+20\rho^3A_{-2}A_2$$
$$+10\rho A_1{}^2A_0+20\rho A_{-1}A_1A_2+10\rho B_2A_0-10B_2+10\rho A_2A_0{}^2$$
$$-90\rho^3A_1{}^2-20\rho A_1{}^2A_2+330\rho^3A_2{}^2+10\rho B_2+200\rho^3A_2=0,$$

$$
\begin{aligned}
&200\rho^3A_{-2}+10\rho {A_{-1}}^2A_0+20\rho^3A_{-2}A_2+10\rho B_{-2}A_0+10\rho {A_{-2}}^2A_2\\
&-200\rho^3A_{-2}A_0-200\rho^2A_{-2}-20\rho A_{-2}{A_{-1}}^2+10\rho A_{-1}B_{-1}\\
&+20\rho A_{-2}A_{-1}A_1+10\rho B_{-2}-20\rho A_{-2}B_{-2}+10\rho A_{-2}B_0\\
&-10B_{-2}+330\rho^3{A_{-2}}^2+1232\rho^5A_{-2}-20\rho {A_{-2}}^2A_0+10\rho A_{-2}{A_0}^2\\
&-90\rho^3{A_{-1}}^2-2A_{-2}+5\rho^3A_1A_{-1}=0,\\
&-15\rho {A_{-2}}^2A_1-15\rho A_{-1}B_{-2}+5\rho {A_{-1}}^2A_1+265\rho^3A_{-2}A_{-1}\\
&+40\rho^3A_{-1}-15\rho A_{-2}B_{-1}-5B_{-1}-40\rho^3A_{-1}A_0+10\rho A_{-2}A_0A_1\\
&+5\rho A_{-1}{A_0}^2+10\rho A_{-2}A_{-1}A_2+5\rho A_{-1}B_0-65\rho^3A_{-2}A_1+136\rho^5A_{-1}\\
&+5\rho B_{-1}A_0-40\rho^2A_{-1}-30\rho A_{-2}A_{-1}A_0+5\rho^3A_{-1}A_2-5\rho {A_{-1}}^3\\
&+5\rho B_{-1}+5\rho A_1B_{-2}+5\rho A_{-2}B_1-A_{-1}=0,\\
&10\rho A_{-1}A_0A_2-5\rho {A_1}^3-40\rho^2A_1-15\rho A_{-1}{A_2}^2+136\rho^5A_1\\
&-15\rho A_2B_1+5\rho A_2B_{-1}-5B_1+5\rho A_1B_0+10\rho A_{-2}A_1A_2\\
&+265\rho^3A_1A_2-A_1+5\rho A_{-1}{A_1}^2+5\rho^3A_{-2}A_1-15\rho A_1B_2\\
&+40\rho^3A_1+5\rho B_1A_0-40\rho^3A_1A_0+5\rho A_1{A_0}^2+5\rho B_1-30\rho A_1A_0A_2\\
&+5\rho A_{-1}B_2-65\rho^3A_{-1}A_2=0,
\end{aligned}
$$

$$
\begin{aligned}
&-5\rho B_{-1}-5\rho B_1-10\rho^3A_1-16\rho^5A_1-10\rho^3A_{-1}+10\rho^2A_{-1}+10\rho^2A_1\\
&-16\rho^5A_{-1}+10\rho^3A_1A_0-25\rho^3A_{-2}A_{-1}+15\rho^3A_{-2}A_1+10\rho^3A_{-1}A_0\\
&+15\rho^3A_{-1}A_2-25\rho^3A_1A_2-5\rho A_{-1}{A_0}^2-5\rho A_{-1}{A_1}^2-5\rho {A_{-1}}^2A_1-5\rho A_1{A_0}^2\\
&-5\rho A_{-2}B_1-5\rho A_{-1}B_0-5\rho A_{-1}B_2-5\rho A_1B_{-2}-5\rho A_1B_0-5\rho A_2B_{-1}\\
&-5\rho B_{-1}A_0-5\rho B_1A_0+A_{-1}+A_1+5B_{-1}+5B_1-10\rho A_{-2}A_{-1}A_2\\
&-10\rho A_{-2}A_0A_1-10\rho A_{-2}A_1A_2-10\rho A_{-1}A_0A_2=0.
\end{aligned}
$$

Solving the above system of algebraic equations, with the aid of Mathematica, one possible set of values of A_i and B_i $(i = -2, -1, 0, 1, 2)$ are

$$\begin{aligned}
A_{-2} &= -24\,\rho^2,\\
A_{-1} &= 0,\\
A_0 &= \frac{-1+\rho+16\,\rho^3}{\rho},\\
A_1 &= 0,\\
A_2 &= -24\,\rho^2,\\
B_{-2} &= 24\,\rho\,(-1+\rho)\,,\\
B_{-1} &= 0,\\
B_0 &= -\frac{-5+9\,\rho-80\,\rho^3-5\,\rho^2+80\,\rho^4+2816\,\rho^6}{5\rho^2},\\
B_1 &= 0,\\
B_2 &= 24\,\rho\,(-1+\rho)
\end{aligned}$$

where ρ is any root of $2816\rho^6 + 320\rho^4 - 320\rho^3 - 5\rho^2 + 9\rho - 5 = 0$. As a result, a solution of (3.1) is

$$\begin{aligned}
u(t,x,y) \quad = \quad & A_{-2}\mathrm{coth}^2(z) + A_{-1}\mathrm{coth}(z) + A_0 + A_1\tanh(z)\\
& + A_2\tanh^2(z), \qquad (3.8)
\end{aligned}$$

where $z = t - \rho x + (\rho - 1)y$.

A profile of the solution (3.8) is given in Figure 3.1.

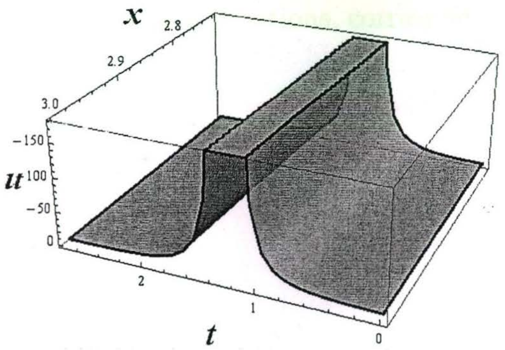

Figure 3.1: Evolution of travelling wave solution (3.8) with parameters $y = 0,\ \rho = 0.42$.

3.1.3 Exact solutions using extended Jacobi elliptic function method

In this subsection we obtain exact solutions of (3.1) in terms of the Jacobi elliptic functions. We note that the cosine-amplitude function, $\mathrm{cn}(z|\omega)$, and the sine-amplitude function, $\mathrm{sn}(z|\omega)$ are solutions of the first-order differential equations

$$H'(z) = -\left\{ \left(1 - H^2(z)\right)\left(1 - \omega + \omega H^2(z)\right) \right\}^{\frac{1}{2}} \tag{3.9}$$

and

$$H'(z) = \left\{ \left(1 - H^2(z)\right)\left(1 - \omega H^2(z)\right) \right\}^{\frac{1}{2}}, \tag{3.10}$$

respectively [77]. We recall the following facts:

(i) When $\omega \to 1$, the Jacobi elliptic functions degenerate to the hyperbolic functions, $\mathrm{cn}(z|\omega) \to \mathrm{sech}(z)$, $\mathrm{sn}(z|\omega) \to \tanh(z)$.

(ii) When $\omega \to 0$, the Jacobi elliptic functions degenerate to the trigonometric functions, $\mathrm{cn}(z|\omega) \to \cos(z)$, $\mathrm{sn}(z|\omega) \to \sin(z)$.

(iii) $\mathrm{nc}(z|\omega) = \frac{1}{\mathrm{cn}(z|\omega)}$, $\mathrm{ns}(z|\omega) = \frac{1}{\mathrm{sn}(z|\omega)}$.

We now treat the above ODEs as our auxillary equations and apply the procedure of the previous subsection to system (3.4). Leaving out the details, we obtain two solutions, the cnoidal and snoidal wave solutions, corresponding to the two equations (3.9) and (3.10) given by, respectively,

$$u(t,x,y) = A_{-2}\mathrm{nc}^2(z|\omega) + A_{-1}\mathrm{nc}(z|\omega) + A_0 + A_1\mathrm{cn}(z|\omega) + A_2\mathrm{cn}^2(z|\omega) \tag{3.11}$$

where

$$
\begin{aligned}
&A_{-2} = -3\,\rho^2 + 3\,\rho^2\omega,\\
&A_{-1} = 0,\\
&A_0 = -\frac{2\,\rho^3\omega + 1 - \rho - \rho^3}{\rho},\\
&A_1 = 0,\\
&A_2 = 3\,\rho^2\omega,\\
&B_{-2} = -3\,\rho\,(\omega - 1)\,(\rho - 1)\,,\\
&B_{-1} = 0,\\
&B_0 = -\frac{-5 + 9\,\rho - 5\,\rho^2 - 5\,\rho^3 + \rho^6 - 10\,\rho^4\omega + 16\,\rho^6\omega^2 - 16\,\rho^6\omega + 10\,\rho^3\omega + 5\,\rho^4}{5\rho^2},\\
&B_1 = 0,\\
&B_2 = -3\,\rho\,\omega\,(\rho - 1)\,,
\end{aligned}
$$

with ρ as any root of $(16\omega^2 - 16\omega + 1)\,\rho^6 + (5 - 10\omega)\rho^4 + (10\omega - 5)\rho^3 - 5\rho^2 + 9\rho - 5 = 0$ and

$$u(t,x,y) \;=\; A_{-2}\mathrm{ns}^2(z|\omega) + A_{-1}\mathrm{ns}(z|\omega) + A_0 + A_1\mathrm{sn}(z|\omega) + A_2\mathrm{sn}^2(z|\omega), \tag{3.12}$$

with

$$
\begin{aligned}
&A_{-2} = -3\,\rho^2,\\
&A_{-1} = 0,\\
&A_0 = \frac{-1 + \rho + \rho^3 + \rho^3\omega}{\rho},\\
&A_1 = 0,\\
&A_2 = -3\,\omega\,\rho^2,\\
&B_{-2} = 3\,\rho\,(-1 + \rho)\,,\\
&B_{-1} = 0,\\
&B_0 = -\frac{-5 + 9\,\rho - 5\,\rho^2 - 5\,\rho^3 + \rho^6\omega^2 + 14\,\rho^6\omega - 5\,\rho^3\omega + 5\,\rho^4\omega + 5\,\rho^4 + \rho^6}{5\rho^2},\\
&B_1 = 0,\\
&B_2 = 3\,\omega\,\rho\,(-1 + \rho)\,,
\end{aligned}
$$

where ρ is any root of $(\omega^2 + 14\omega + 1)\,\rho^6 + (5\omega + 5)\rho^4 - (5\omega + 5)\rho^3 - 5\rho^2 + 9\rho - 5 = 0$ and $z = t - \rho x + (\rho - 1)y$.

A profile of solutions (3.11), (3.12) are given in Figures 3.2 - 3.3.

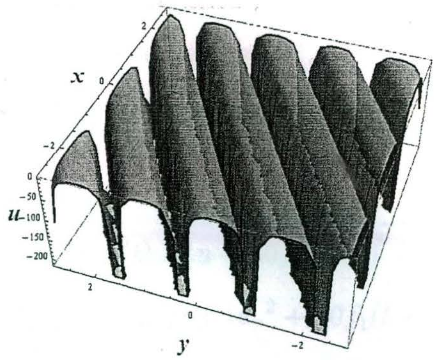

Figure 3.2: Evolution of travelling wave solution (3.11) with parameters $t = 0$, $\omega = 0.1$, $\rho = -1.53$.

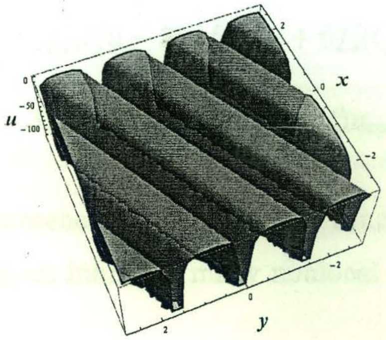

Figure 3.3: Evolution of travelling wave solution (3.12) with parameters $t = 0$, $\omega = 0.2$, $\rho = -1.12$.

3.2 Conservation laws of (3.1)

We now construct conservation laws for the two-dimensional integrable generalization of the Kaup-Kupershmidt equation (3.1) using the multiplier approach [74]. For the coupled system (3.2), we obtain multipliers of the form, $\Lambda_1 = \Lambda_1(t, x, y, u, v)$ and

$\Lambda_2 = \Lambda_2(t, x, y, u, v)$ that are given by

$$\begin{aligned}
\Lambda_1 &= yf_2(t) + f_1(t), \\
\Lambda_2 &= -5xf_2(t) + f_3(t) - \frac{1}{2}y(yf_2'(t) + 2f_1'(t)),
\end{aligned}$$

where f_i, $i = 1, 2, 3$ are arbitrary functions of t. Corresponding to the above multipliers we obtain the following nonlocal conserved vector of (3.1):

$$\begin{aligned}
T^t &= f_1(t)u + yf_2(t)u, \\
T^x &= \frac{1}{12}\Big\{60f_1(t)u_{xx}u + 60yf_2(t)u_{xx}u + 60f_1(t)u\int u_y\,dx + 60yf_2(t)u\int u_y\,dx \\
&+ 20f_1(t)u^3 + 20yf_2(t)u^3 + 45yf_2(t){u_x}^2 + 40f_1(t)u_{xy} + 40yf_2(t)u_{xy} + 12yf_2(t)u_{xxxx} \\
&+ 60xf_2(t)\int u_y\,dx - 12f_3(t)\int u_y\,dx + 45f_1(t){u_x}^2 - 20f_2(t)u_x + 12f_1(t)u_{xxxx} \\
&+ 6y^2f_2'(t)\int u_y\,dx + 12yf_1'(t)\int u_y\,dx\Big\}, \\
T^y &= \frac{1}{6}\Big\{-3y^2f_2'(t)u - 6yf_1'(t)u - 30xf_2(t)u + 6f_3(t)u + 10yf_2(t)u_{xx} \\
&- 30yf_2(t)\int u_y\,dx - 30f_1(t)\int u_y\,dx + 10f_1(t)u_{xx}\Big\}.
\end{aligned}$$

Remark 2 Due to the presence of the arbitrary functions, f_i, $i = 1, 2, 3$, in the multipliers, one can obtain an infinitely many nonlocal conservation laws.

3.3 Conclusion

In this chapter we studied the two-dimensional generalization of the Kaup-Kupershmidt equation (3.1). Lie point symmetries of this equation were obtained and the three translation symmetries were used to transform the equation into a system of ODEs. Then the extended tanh and the extended Jacobi elliptic function methods were employed to solve this ODEs system to obtain exact solutions of (3.1). Furthermore, conservation laws of (3.1) were also computed using the multiplier approach.

Chapter 4

On the solutions and conservation laws of a coupled KdV system

4.1 Introduction

In recent years, the coupled Korteweg-de Vries equations have been the centre of attraction and extensive studies have been made by many authors, see for example, the Refs. [47–53]. In [47] a typical hydrodynamic model which describes a resonant interaction of two wave modes in a shallow stratified liquid is derived and is given by

$$u_t + u_{xxx} - \frac{7}{4}uu_x - vv_x + \frac{5}{4}(uv)_x = 0, \tag{4.1a}$$

$$v_t + v_{xxx} - \frac{5}{4}uu_x - \frac{7}{4}vv_x + 2(uv)_x = 0. \tag{4.1b}$$

This coupled Korteweg-de Vries system (4.1) was studied by Wang [47] for its integrability by using the prolongation technique and singularity analysis. Wazwaz [46] also considered the system (4.1) and employed the Hirota's bilinear method combined with Hereman et al [54] simplified approach to study the integrability of (4). Multiple-soliton solutions and multiple singular soliton solutions were obtained for (4).

In this chapter we study the coupled KdV system (4.1) using Lie symmetry method.

The similarity reductions and exact solutions with the aid of simplest equation and Jacobi elliptic function methods are obtained based on the optimal systems of one-dimensional subalgebras for the KdV system. In addition, the conservation laws of the coupled KdV system are also derived using the multiplier approach [74] and the conservation theorem due to Ibragimov [75].

This work has been published and appears in [66].

4.2 Symmetry reductions and exact solutions of (4.1)

The symmetry group of the coupled KdV system (4.1) will be generated by the vector field of the form

$$X = \xi^1(t,x,u,v)\frac{\partial}{\partial t} + \xi^2(t,x,u,v)\frac{\partial}{\partial x} + \eta^1(t,x,u,v)\frac{\partial}{\partial u} + \eta^2(t,x,u,v)\frac{\partial}{\partial v}.$$

Applying the third prolongation $\mathrm{pr}^{(3)}X$ [7] to (4.1) gives the following overdetermined system of linear partial differential equations:

$$\begin{aligned}
&\xi^1_x = 0,\ \xi^2_u = 0,\ \xi^2_v = 0,\ \xi^1_u = 0,\ \xi^1_v = 0,\ \eta^2_{uv} = 0,\ \eta^1_{uu} = 0,\ \eta^2_{vv} = 0,\\
&\eta^1_{uv} = 0,\ \eta^2_{uu} = 0,\ \eta^1_{vv} = 0,\ \eta^2_{xu} = 0,\ \eta^1_{xv} = 0,\ \eta^1_{xu} - \xi^2_{xx} = 0,\ \eta^2_{xv} - \xi^2_{xx} = 0,\\
&\xi^1_t - 3\xi^2_x = 0,\ -7\eta^1_x u + 5\eta^2_x u + 5\eta^1_x v - 4\eta^2_x v + 4\eta^1_t + 4\eta^1_{xxx} = 0,\\
&-5\eta^1_x u + 8\eta^2_x u + 8\eta^1_x v - 7\eta^2_x v + 4\eta^2_t + 4\eta^2_{xxx} = 0,\\
&-5\eta^1_u u + 15\eta^2_u u + 5\eta^2_v u - 5\xi^1_t u + 5\xi^2_x u + 8\eta^1_u v - 12\eta^2_u v - 8\eta^2_v v + 8\xi^1_t v\\
&-8\xi^2_x v - 5\eta_1 + 8\eta_2 = 0,\\
&-5\eta^1_u u - 15\eta^1_v u + 5\eta^2_v u + 5\xi^1_t u - 5\xi^2_x u + 4\eta^1_u v + 12\eta^1_v v - 4\eta^2_v v - 4\xi^1_t v\\
&+4\xi^2_x v + 5\eta_1 - 4\eta_2 = 0,\\
&5\eta^1_v u + 5\eta^2_u u - 7\xi^1_t u + 7\xi^2_x u - 8\eta^1_v v - 4\eta^2_u v + 5\xi^1_t v - 5\xi^2_x v - 7\eta_1 + 5\eta_2\\
&+12\eta^1_{xxu} - 4\xi^2_t - 4\xi^2_{xxx} = 0,\\
&-5\eta^1_v u - 5\eta^2_u u + 8\xi^1_t u - 8\xi^2_x u + 8\eta^1_v v + 4\eta^2_u v - 7\xi^1_t v + 7\xi^2_x v + 8\eta_1 - 7\eta_2\\
&+12\eta^2_{xxv} - 4\xi^2_t - 4\xi^2_{xxx} = 0.
\end{aligned}$$

On solving the above system, one obtains the following three Lie point symmetries:

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial x}, \\
X_2 &= \frac{\partial}{\partial t}, \\
X_3 &= 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v}.
\end{aligned}$$

4.2.1 One-dimensional optimal system of subalgebras

In this subsection we present the optimal system of one-dimensional subalgebras for the system (4.1) to obtain the optimal system of group-invariant solutions. For this purpose we follow the procedure given in Section 2.5.3 of Chapter 2.

Table 1. Commutator table of the Lie algebra of system (4.1)

	X_1	X_2	X_3
X_1	0	0	X_1
X_2	0	0	$3X_2$
X_3	$-X_1$	$-3X_2$	0

Table 2. Adjoint table of the Lie algebra of system (4.1)

Ad	X_1	X_2	X_3
X_1	X_1	X_2	$X_3 - \epsilon X_1$
X_2	X_1	X_2	$X_3 - 3\epsilon X_2$
X_3	$e^{\epsilon}X_1$	$e^{3\epsilon}X_2$	X_3

From Tables 1 and 2 one can obtain an optimal system of one-dimensional subalgebras given by $\{\nu X_1 + X_2, X_2, X_3\}$, where ν is a nonzero constant.

4.2.2 Symmetry reductions of (4.1)

In this subsection we use the optimal system of one-dimensional subalgebras calculated above to obtain symmetry reductions that transform (4.1) into a system of ordinary differential equations (ODEs). Later in the next subsection we will look for exact solutions of (4.1).

Case 1. $\nu X_1 + X_2$

The symmetries $\nu X_1 + X_2$ gives rise to the group-invariant solution

$$u = F(z), \quad v = G(z), \tag{4.2}$$

where $z = x - \nu t$ is an invariant of $\nu X_1 + X_2$. Substitution of (4.2) into (4.1) results in the system of ODEs

$$\begin{aligned} &4F'''(z) + 5G(z)F'(z) - 4\nu F'(z) - 7F(z)F'(z) \\ &+5F(z)G'(z) - 4G(z)G'(z) = 0, \qquad (4.3a) \\ &4G'''(z) + 8G(z)F'(z) - 5F(z)F'(z) + 8F(z)G'(z) \\ &-4\nu G'(z) - 7G(z)G'(z) = 0. \qquad (4.3b) \end{aligned}$$

Case 2. X_2

For the symmetry X_2 we obtain the group-invariant solution of the form

$$u = F(z), \quad v = G(z), \tag{4.4}$$

where $z = x$ is an invariant of X_2 and the functions F and G satisfy the following system of ODEs:

$$\begin{aligned} &4F'''(z) + 5G(z)F'(z) - 7F(z)F'(z) + 5F(z)G'(z) - 4G(z)G'(z) = 0, \\ &4G'''(z) + 8G(z)F'(z) - 5F(z)F'(z) + 8F(z)G'(z) - 7G(z)G'(z) = 0. \end{aligned}$$

Case 3. X_3

By solving the corresponding Lagrange system for the symmetry X_3, one obtains an invariant $z = xt^{-1/3}$ and the group-invariant solution of the form

$$u = t^{-2/3}F(z), \quad v = t^{-2/3}G(z), \tag{4.5}$$

where the functions F and G satisfy the following system of ODEs:

$$\begin{aligned} &12F'''(z) + 15G(z)F'(z) - 21F(z)F'(z) - 4zF'(z) + 15F(z)G'(z) - 8F(z) - 12G(z)G'(z) = 0, \\ &24G(z)F'(z) - 15F(z)F'(z) + 24F(z)G'(z) + 12G'''(z) - 21G(z)G'(z) - 4zG'(z) - 8G(z) = 0. \end{aligned}$$

4.2.3 Exact solutions using simplest equation method

We now use the simplest equation method [78–81], to solve the system (4.3) and as a result we obtain the exact solutions of our coupled KdV system (4.1). The simplest equations that will be used are the Bernoulli and Riccati equations. To the best of our knowledge, it is for the first time that this method has been applied to systems of nonlinear ODEs.

We briefly recall the simplest equation method here. Let us consider the solutions of (4.3) in the form

$$F(z)=\sum_{i=0}^{M} A_i(H(z))^i, \quad G(z)=\sum_{i=0}^{M} B_i(H(z))^i, \tag{4.6}$$

where $H(z)$ satisfies the Bernoulli or Riccati equation, M is a positive integer that can be determined by balancing procedure as in [80] and $A_0,\cdots,A_M,B_0,\cdots,B_M$ are parameters to be determined. We note that the Bernoulli and Riccati equations are well-known nonlinear ODEs whose solutions can be expressed in terms of elementary functions.

We consider the Bernoulli equation

$$H'(z)=aH(z)+bH^2(z), \tag{4.7}$$

where a and b are constants. The solution of this equation is given by

$$H(z)=a\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\}.$$

For the Riccati equation

$$H'(z)=aH^2(z)+bH(z)+c, \tag{4.8}$$

where a, b and c are constants, we shall use the solutions

$$H(z)=-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]$$

and

$$H(z)=-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)},$$

where $\theta^2=b^2-4ac>0$ and C is a constant of integration.

Solutions of (4.1) using the Bernoulli equation as the simplest equation

The balancing procedure [80] yields $M = 2$ so the solutions of (4.3) are of the form

$$F(z) = A_0 + A_1 H + A_2 H^2, \quad G(z) = B_0 + B_1 H + B_2 H^2. \tag{4.9}$$

Substituting (4.9) into (4.3) and making use of the Bernoulli equation (4.7) and then equating all coefficients of the functions H^i to zero, we obtain the following algebraic system of equations in terms of A_0, A_1,A_2 and B_0, B_1,B_2:

$$\begin{aligned}
&5A_1B_0 + 5A_0B_1 - 4A_1\nu - 7A_0A_1 + 4A_1 - 4B_0B_1 = 0,\\
&15A_1B_0 + 10A_1B_1 + 10A_2B_0 + 15A_0B_1 + 10A_0B_2\\
&-12A_1\nu - 8A_2\nu - 7A_1^2 - 21A_0A_1 + 84A_1 - 14A_0A_2\\
&+32A_2 - 4B_1^2 - 12B_0B_1 - 8B_0B_2 = 0,\\
&30A_1B_1 + 15A_1B_2 + 30A_2B_0 + 15A_2B_1 + 30A_0B_2\\
&-24A_2\nu - 21A_1^2 - 21A_2A_1 + 432A_1 - 42A_0A_2 + 456A_2\\
&-12B_1^2 - 24B_0B_2 - 12B_1B_2 = 0,\\
&45A_2B_1 + 20A_2B_2 + 45A_1B_2 - 14A_2^2 - 63A_1A_2 + 1944A_2\\
&+648A_1 - 8B_2^2 - 36B_1B_2 = 0,\\
&60A_2B_2 - 42A_2^2 + 2592A_2 - 24B_2^2 = 0,\\
&8A_1B_0 + 8A_0B_1 - 5A_0A_1 - 4B_1\nu - 7B_0B_1 + 4B_1 = 0,\\
&24A_1B_0 + 16A_1B_1 + 16A_2B_0 + 24A_0B_1 + 16A_0B_2 - 5A_1^2\\
&-15A_0A_1 - 10A_0A_2 - 12B_1\nu - 8B_2\nu - 7B_1^2 - 21B_0B_1\\
&+84B_1 - 14B_0B_2 + 32B_2 = 0,\\
&48A_1B_1 + 24A_1B_2 + 48A_2B_0 + 24A_2B_1 + 48A_0B_2\\
&-15A_1^2 - 15A_2A_1 - 30A_0A_2 - 24B_2\nu - 21B_1^2 + 432B_1\\
&-42B_0B_2 - 21B_1B_2 + 456B_2 = 0,\\
&72A_2B_1 + 32A_2B_2 + 72A_1B_2 - 10A_2^2 - 45A_1A_2 - 14B_2^2\\
&+648B_1 - 63B_1B_2 + 1944B_2 = 0,\\
&96A_2B_2 - 30A_2^2 - 42B_2^2 + 2592B_2 = 0.
\end{aligned}$$

Solving the above system of algebraic equations, with the aid of Mathematica, we obtain

$$\begin{aligned}
&a = 1, \quad b = 3,\\
&A_0 = k(\nu - 1),\\
&A_1 = -\frac{36A_0}{\nu - 1},\\
&A_2 = 3A_1,\\
&B_0 = \frac{21456A_0 - 469A_0A_1 + 3456\nu - 3456}{10656},\\
&B_1 = \frac{-469A_1^3 - 30048A_1^2 + 1665792A_1}{1022976},\\
&B_2 = \frac{-469A_1^3 - 30048A_1^2 + 1665792A_1}{340992},
\end{aligned}$$

where k is any root of $469k^3 + 416k^2 + 304k + 256 = 0$. As a result, a solution of (4.1) is given by

$$\begin{aligned}
u(t,x) &= A_0 + A_1 a\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}\\
&\quad + A_2 a^2\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}^2, \qquad (4.10a)\\
v(t,x) &= B_0 + B_1 a\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}\\
&\quad + B_2 a^2\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}^2, \qquad (4.10b)
\end{aligned}$$

where $z = x - \nu t$ and C is a constant of integration.

Solutions of (4.1) using the Riccati equation as the simplest equation

The balancing procedure yields $M = 2$ so the solutions of (4.3) are of the form

$$F(z) = A_0 + A_1 H + A_2 H^2, \quad G(z) = B_0 + B_1 H + B_2 H^2. \qquad (4.11)$$

Substituting (4.11) into (4.3) and making use of the Riccati equation (4.8), we obtain the following algebraic system of equations in terms of $A_0, A_1, A_2, B_0, B_1, B_2$ by equating all coefficients of the functions H^i to zero:

$$
\begin{aligned}
&8aA_1c^2+4A_1b^2c+24A_2bc^2+5A_1B_0c+5A_0B_1c-4A_1c\nu-7A_0A_1c-4B_0B_1c=0,\\
&32aA_1bc+64aA_2c^2+4A_1b^3+56A_2b^2c+5A_1bB_0+5A_0bB_1-4A_1b\nu-7A_0A_1b\\
&+10A_2B_0c+10A_1B_1c+10A_0B_2c-8A_2c\nu-7A_1^2c-14A_0A_2c-4bB_0B_1\\
&-4B_1^2c-8B_0B_2c=0,\\
&32a^2A_1c+28aA_1b^2+208aA_2bc+5aA_1B_0+5aA_0B_1-4aA_1\nu-7aA_0A_1\\
&-4aB_0B_1+32A_2b^3+10A_2bB_0+10A_1bB_1+10A_0bB_2-8A_2b\nu-7A_1^2b\\
&-14A_0A_2b+15A_2B_1c+15A_1B_2c-21A_1A_2c-4bB_1^2-8bB_0B_2-12B_1B_2c=0,\\
&48a^2A_1b+160a^2A_2c+152aA_2b^2+10aA_2B_0+10aA_1B_1+10aA_0B_2-8aA_2\nu\\
&-7aA_1^2-14aA_0A_2-4aB_1^2-8aB_0B_2+15A_2bB_1+15A_1bB_2-21A_1A_2b\\
&+20A_2B_2c-14A_2^2c-12bB_1B_2-8B_2^2c=0,\\
&24a^3A_1+216a^2A_2b+15aA_2B_1+15aA_1B_2-21aA_1A_2-12aB_1B_2\\
&+20A_2bB_2-14A_2^2b-8bB_2^2=0,\\
&96a^3A_2+20aA_2B_2-14aA_2^2-8aB_2^2=0,\\
&8aB_1c^2+8A_1B_0c+8A_0B_1c-5A_0A_1c+4b^2B_1c+24bB_2c^2-4B_1c\nu-7B_0B_1c=0,\\
&32abB_1c+64aB_2c^2+8A_1bB_0+8A_0bB_1-5A_0A_1b+16A_2B_0c+16A_1B_1c\\
&+16A_0B_2c-5A_1^2c-10A_0A_2c+4b^3B_1+56b^2B_2c-4bB_1\nu-7bB_0B_1\\
&-8B_2c\nu-7B_1^2c-14B_0B_2c=0,\\
&32a^2B_1c+8aA_1B_0+8aA_0B_1-5aA_0A_1+28ab^2B_1+208abB_2c-4aB_1\nu\\
&-7aB_0B_1+16A_2bB_0+16A_1bB_1+16A_0bB_2-5A_1^2b-10A_0A_2b+24A_2B_1c\\
&+24A_1B_2c-15A_1A_2c+32b^3B_2-8bB_2\nu-7bB_1^2-14bB_0B_2-21B_1B_2c=0,\\
&48a^2bB_1+160a^2B_2c+16aA_2B_0+16aA_1B_1+16aA_0B_2-5aA_1^2\\
&-10aA_0A_2+152ab^2B_2-8aB_2\nu-7aB_1^2-14aB_0B_2+24A_2bB_1+24A_1bB_2\\
&-15A_1A_2b+32A_2B_2c-10A_2^2c-21bB_1B_2-14B_2^2c=0,\\
&24a^3B_1+216a^2bB_2+24aA_2B_1+24aA_1B_2-15aA_1A_2-21aB_1B_2\\
&+32A_2bB_2-10A_2^2b-14bB_2^2=0,\\
&96a^3B_2+32aA_2B_2-10aA_2^2-14aB_2^2=0.
\end{aligned}
$$

Solving the above algebraic equations one obtains

$$\begin{aligned}
A_0 &= k\left(8ac+b^2-\nu\right),\\
A_1 &= \frac{12aA_0b}{8ac+b^2-\nu},\\
A_2 &= \frac{aA_1}{b},\\
B_0 &= \frac{-9216a^3c+7152a^2A_0-1152a^2b^2+1152a^2\nu-469A_0A_2}{3552a^2},\\
B_1 &= \frac{A_1\left(185088a^4-10016a^2A_2-469A_2^2\right)}{113664a^4},\\
B_2 &= \frac{A_2\left(185088a^4-10016a^2A_2-469A_2^2\right)}{113664a^4},
\end{aligned}$$

where k is any root of $469k^3-416k^2+304k-256=0$ and hence solutions of (4.1) are given by

$$\begin{aligned}
u(t,x) &= A_0 + A_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}\\
&\quad + A_2\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2, \qquad (4.12a)\\
v(t,x) &= B_0 + B_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}\\
&\quad + B_2\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2 \qquad (4.12b)
\end{aligned}$$

and

$$\begin{aligned}
u(t,x) &= A_0 + A_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}\\
&\quad + A_2\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^2, \qquad (4.13a)\\
v(t,x) &= B_0 + B_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}\\
&\quad + B_2\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^2, \qquad (4.13b)
\end{aligned}$$

where $z = x-\nu t$ and C is a constant of integration.

A profile of the solution (4.12) is given in Figure 4.1.

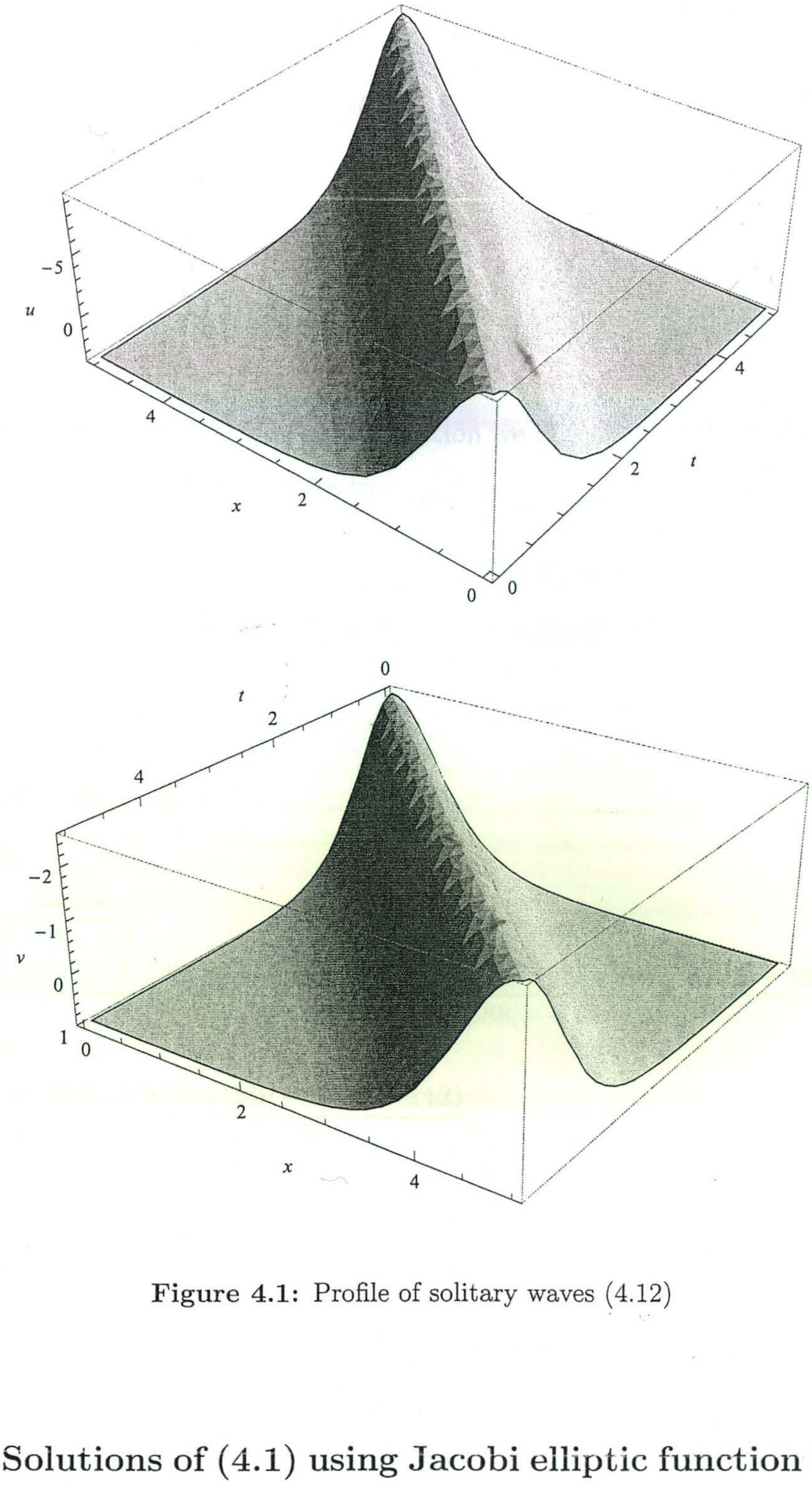

Figure 4.1: Profile of solitary waves (4.12)

4.2.4 Solutions of (4.1) using Jacobi elliptic function method

In this subsection we obtain exact solutions of the coupled KdV system (4.1) in terms of Jacobi elliptic functions. The cosine-amplitude function, $\mathrm{cn}(z|\omega)$, and the sine-amplitude function, $\operatorname{sn}(z|\omega)$ are solutions of the first-order differential equations

$$H'(z) = -\left\{ \left(1 - H^2(z)\right)\left(1 - \omega + \omega H^2(z)\right) \right\}^{\frac{1}{2}} \tag{4.14}$$

and

$$H'(z) = \left\{ \left(1 - H^2(z)\right)\left(1 - \omega H^2(z)\right) \right\}^{\frac{1}{2}}, \tag{4.15}$$

respectively [77].

The above ODEs are now treated as our simplest equations and consequently following the procedure of the previous subsection we obtain the following cnoidal and snoidal wave solutions of (4.1) given by

$$u(t,x) = A_0 + A_1\operatorname{cn}(z|\omega) + A_2\operatorname{cn}^2(z|\omega), \tag{4.16a}$$

$$v(t,x) = B_0 + B_1\operatorname{cn}(z|\omega) + B_2\operatorname{cn}^2(z|\omega), \tag{4.16b}$$

where

$$\begin{aligned}
&A_0 = k(\nu - 8\omega + 4),\\
&A_1 = 0,\\
&A_2 = \frac{12A_0\omega}{\nu - 8\omega + 4},\\
&B_0 = \frac{1788A_0\nu + 1407A_0^2 + 7152A_0 + 938A_0A_2 + 288\nu^2 - 2304\nu\omega + 2304\nu - 9216\omega + 4608}{888(\nu + 4)},\\
&B_1 = 0,\\
&B_2 = \frac{A_2(29A_0 - 6B_0 + 28\nu - 224\omega + 112)}{16(\nu - 8\omega + 4)},\\
&k \quad \text{is any root of} \quad 469k^3 + 416k^2 + 304k + 256 = 0
\end{aligned}$$

and

$$u(t,x) = A_0 + A_1\operatorname{sn}(z|\omega) + A_2\operatorname{sn}^2(z|\omega),$$

$$v(t,x) = B_0 + B_1\operatorname{sn}(z|\omega) + B_2\operatorname{sn}^2(z|\omega),$$

where

$$
\begin{aligned}
A_0 &= k(\nu + 4\omega + 4), \\
A_1 &= 0, \\
A_2 &= -\frac{12A_0\omega}{\nu + 4\omega + 4}, \\
B_0 &= \frac{1788A_0\nu + 1407A_0^2 + 7152A_0 + 469A_0A_2 + 288\nu^2 + 1152\nu\omega + 2304\nu + 4608\omega + 4608}{888(\nu + 4)}, \\
B_1 &= 0, \\
B_2 &= \frac{A_2(29A_0 - 6B_0 + 28\nu + 112\omega + 112)}{16(\nu + 4\omega + 4)}, \\
k &\quad \text{is any root of} \quad 469k^3 + 416k^2 + 304k + 256 = 0,
\end{aligned}
$$

and $z = x - \nu t$.

The profile of the solution (4.16) is given in Figure 4.2.

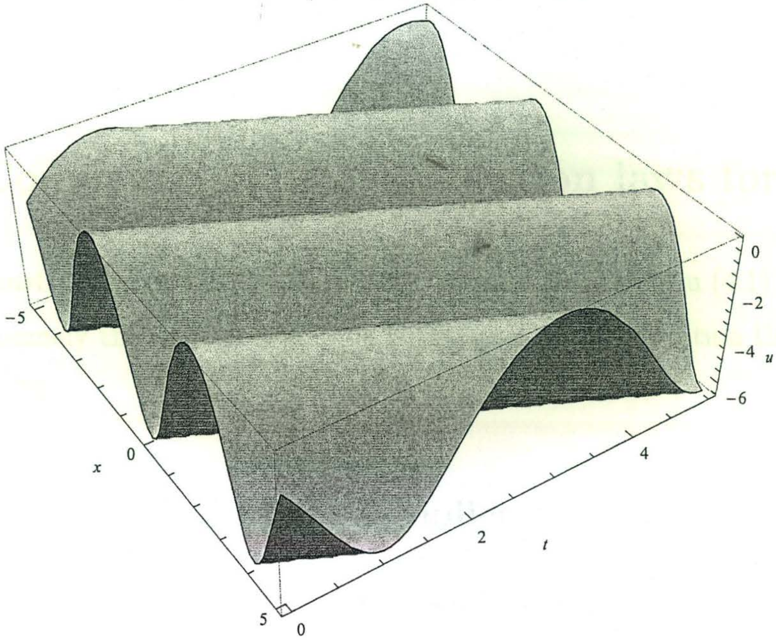

Figure 4.2: Profile of cnoidal waves (4.16)

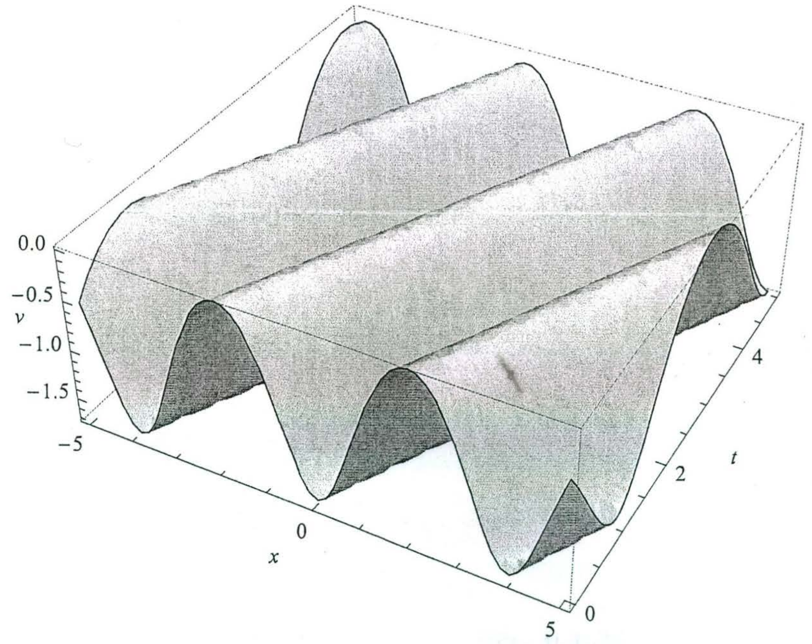

Figure 4.3: Profile of cnoidal waves (4.16)

4.3 Construction of conservation laws for (4.1)

We now construct conservation laws for the coupled KdV system (4.1) using the two methods, namely the multiplier approach [74] and the conservation theorem due to Ibragimov [75].

4.3.1 Application of the Multiplier Method

For the coupled KdV system (4.1), after some lengthy calculations, we obtain the second-order multipliers, $\Lambda_1 = \Lambda_1(t, x, u, v, u_x, v_x, u_{xx}, v_{xx})$ and $\Lambda_2 = \Lambda_2(t, x, u, v, u_x, v_x, u_{xx}, v_{xx})$ that are given by

$$\Lambda_1 = C_1 + C_3\left(-2\,v + 3\,u\right) + C_4\left(11\,u^2 - 14\,v^2 + 16\,v_{xx} - 24\,u_{xx} + 2\,vu\right), \quad (4.17)$$

$$\Lambda_2 = C_2 + C_3\left(-2\,u + 3\,v\right) + C_4\left(-28\,vu + 21\,v^2 + u^2 + 16\,u_{xx} - 24\,v_{xx}\right), \quad (4.18)$$

where C_i, $i = 1, 2, 3, 4$ are arbitrary constants. Corresponding to the above multipliers we obtain the following four local conserved vectors of (4.1):

$$\begin{aligned}
\Phi_1^t &= u, \\
\Phi_1^x &= \frac{1}{8}\Big\{10uv - 7u^2 - 4v^2 + 8u_{xx}\Big\};
\end{aligned}$$

$$\begin{aligned}
\Phi_2^t &= v, \\
\Phi_2^x &= \frac{1}{8}\Big\{16uv - 5u^2 - 7v^2 + 8v_{xx}\Big\};
\end{aligned}$$

$$\begin{aligned}
\Phi_3^t &= \frac{1}{2}\Big\{-4uv + 3u^2 + 3v^2\Big\}, \\
\Phi_3^x &= \frac{1}{12}\Big\{-24v_{xx}u - 24u_{xx}v + 36u_{xx}u + 36v_{xx}v - 3u^2v \\
&\quad +30uv^2 - 11u^3 - 13v^3 + 24u_xv_x - 18{u_x}^2 - 18{v_x}^2\Big\}
\end{aligned}$$

and

$$\begin{aligned}
\Phi_4^t &= \frac{1}{3}\Big\{24v_{xx}u + 24u_{xx}v - 36u_{xx}u - 36v_{xx}v + 3u^2v - 42uv^2 + 11u^3 + 21v^3\Big\}, \\
\Phi_4^x &= \frac{1}{16}\Big\{16v_{xx}u^2 - 128{v_x}^2u + 32u_{xx}uv - 576v_{xx}uv - 128uv_{tx} + 128u_xv_xv \\
&\quad -96u_{xx}v^2 - 128vu_{tx} + 176u_{xx}u^2 + 192uu_{tx} + 336v_{xx}v^2 + 192vv_{tx} \\
&\quad +252u^3v - 486u^2v^2 + 384uv^3 - 82u^4 - 91v^4 + 128u_tv_x + 128v_tu_x \\
&\quad -192u_tu_x - 192v_tv_x + 256u_{xx}v_{xx} - 192{u_{xx}}^2 - 192{v_{xx}}^2\Big\}.
\end{aligned}$$

Remark 1. Higher order conservation laws of (4.1) can be computed by increasing the order of the multipliers.

4.3.2 Application of the new Conservation Theorem

The adjoint equations of (4.1) are given by

$$\frac{5}{4}\psi_xu + \frac{7}{4}\phi_xu - 2\psi_xv - \frac{5}{4}\phi_xv - \phi_t - \phi_{xxx} = 0, \tag{4.19a}$$

$$-2\psi_xu - \frac{5}{4}\phi_xu + \frac{7}{4}\psi_xv + \phi_xv - \psi_t - \psi_{xxx} = 0, \tag{4.19b}$$

where $\phi = \phi(t, x)$ and $\psi = \psi(t, x)$ are the new dependent variables. The Lagrangian for the system of equations (4.1) and (4.19) is given by

$$\begin{aligned} L &= \phi(t,x)\left\{u_t + u_{xxx} - \frac{7}{4}uu_x - vv_x + \frac{5}{4}(uv)_x\right\} \\ &\quad + \psi(t,x)\left\{v_t + v_{xxx} - \frac{5}{4}uu_x - \frac{7}{4}vv_x + 2(uv)_x\right\}. \end{aligned} \tag{4.20}$$

We have the following three cases:

(i) We first consider the Lie point symmetry generator $X_1 = \partial_x$ of the coupled KdV system (4.1). Corresponding to this symmetry the Lie characteristic function $W = -(u_x + v_x)$. Thus using theorem (1) of chapter 1, the components of the conserved vector are given by

$$\begin{aligned} T_1^t &= u_x\phi - v_x\psi, \\ T_1^x &= u_t\phi + v_t\psi + u_{xx}\phi_x - u_x\phi_{xx} + v_{xx}\psi_x - v_x\psi_{xx}. \end{aligned}$$

(ii) The Lie point symmetry generator $X_2 = \partial_t$ has the Lie characteristic function $W = -(u_t + v_t)$. Hence one obtains the conserved vector whose components are

$$\begin{aligned} T_2^t &= \frac{1}{4}\Big\{8u_xv\psi + 8v_xu\psi + 5u_xv\phi + 5v_xu\phi - 5u_xu\psi - 7u_xu\phi \\ &\quad + 4u_{xxx}\phi - 7v_xv\psi + 4v_{xxx}\psi - 4v_xv\phi\Big\}, \\ T_2^x &= \frac{1}{4}\Big\{ - 8u_tv\psi - 8v_tu\psi - 5u_tv\phi - 5v_tu\phi + 5u_tu\psi + 7u_tu\phi - 4\phi u_{txx} \\ &\quad + 7v_tv\psi - 4\psi v_{txx} + 4v_tv\phi - 4u_t\phi_{xx} + 4\phi_xu_{tx} - 4v_t\psi_{xx} + 4\psi_xv_{tx}\Big\}. \end{aligned}$$

(iii) Finally, we consider the symmetry generator $X_3 = 3t\partial_t + x\partial_x - 2u\partial_u - 2v\partial_v$. For this case the Lie characteristic function $W = -2u - 2v - (3tu_t + 3tv_t + xu_x + xv_x)$

and as a result the components of the conserved vector are given by

$$\begin{aligned}
T_3^t &= \frac{1}{4}\Big\{24tu_xv\psi + 24tv_xu\psi + 15tu_xv\phi + 15tv_xu\phi - 15tu_xu\psi - 4xu_x\phi \\
&\quad -21tu_xu\phi + 12tu_{xxx}\phi - 4xv_x\psi - 21tv_xv\psi + 12tv_{xxx}\psi - 12tv_xv\phi - 8u\phi - 8v\psi\Big\}, \\
T_3^x &= \frac{1}{4}\Big\{ - 24tv_tu\psi - 24tu_tv\psi - 15tv_tu\phi - 15tu_tv\phi + 15tu_tu\psi - 8\phi_{xx}u + 21tu_tu\phi \\
&\quad -16u_{xx}\phi + 4xu_t\phi - 12t\phi u_{txx} - 16v_{xx}\psi - 8\psi_{xx}v + 4xv_t\psi + 21tv_tv\psi - 12t\psi v_{txx} \\
&\quad +12tv_tv\phi - 32uv\psi - 20uv\phi + 10u^2\psi + 14u^2\phi + 14v^2\psi + 8v^2\phi - 12tu_t\phi_{xx} + 12t\phi_xu_{tx} \\
&\quad -12tv_t\psi_{xx} + 12t\psi_xv_{tx} + 12u_x\phi_x + 4xu_{xx}\phi_x - 4xu_x\phi_{xx} + 12v_x\psi_x + 4xv_{xx}\psi_x - 4xv_x\psi_{xx}\Big\}.
\end{aligned}$$

Remark 2. The components of the conserved vectors contain the arbitrary solutions ϕ and ψ of the adjoint equations (4.19) and hence one can obtain an infinite number of conservation laws.

4.4 Conclusion

In this chapter we studied the coupled KdV system (4.1) from the point of view of Lie symmetry analysis. Similarity reductions and exact solutions with the aid of simplest equation and Jacobi elliptic function methods were obtained based on the optimal systems of one-dimensional subalgebras for the KdV system. The exact solutions obtained were solitary, cnoidal and snoidal waves. Finally, conservation laws for the coupled KdV system (4.1) were derived by employing the multiplier method and the new conservation theorem.

Chapter 5

Lie group analysis and conservation laws of a coupled variable-coefficient modified Korteweg-de Vries system

In this chapter Lie group analysis is performed on the generalized coupled variable-coefficient modified Korteweg-de Vries (CVCmKdV) system

$$u_t - \alpha(t)\left(u_{xxx} + 6(u^2 - v^2)u_x - 12uvv_x\right) - 4\beta(t)u_x = 0, \tag{5.1a}$$

$$v_t - \alpha(t)\left(v_{xxx} + 6(u^2 - v^2)v_x + 12uvu_x\right) - 4\beta(t)v_x = 0, \tag{5.1b}$$

which models a two-layer fluid, and is applied to investigate the atmospheric and oceanic phenomena such as the atmospheric blockings, interactions between the atmosphere and ocean, oceanic circulations and hurricanes or typhoons. It should be noted that if $v = 0$, (5.1) reduces to a variable-coefficient modified Korteweg-de Vries

$$u_t - \alpha(t)\left(u_{xxx} + 6u^2u_x\right) - 4\beta(t)u_x = 0,$$

which has been investigated in [56–58].

The similarity reductions and exact solutions with the aid of simplest equation and Jacobi elliptic function methods are computed. In addition to this, the conservation laws of the CVCmKdV system are also derived using the multiplier approach and a new conservation theorem.

This work has been submitted for publication [64].

5.1 Symmetry reductions and exact solutions of (5.1)

The symmetry group of the CVCmKdV system (5.1) will be generated by the vector field of the form

$$X = \xi^1(t,x,u,v)\frac{\partial}{\partial t} + \xi^2(t,x,u,v)\frac{\partial}{\partial x} + \eta^1(t,x,u,v)\frac{\partial}{\partial u} + \eta^2(t,x,u,v)\frac{\partial}{\partial v}.$$

Applying the third prolongation $\mathrm{pr}^{(3)}X$ to (5.1) results in the following overdetermined system of linear partial differential equations:

$$\begin{aligned}
&\xi^1_x = 0,\ \xi^2_u = 0,\ \xi^2_v = 0,\ \xi^1_u = 0,\ \xi^1_v = 0,\ \eta^2_{uu} = 0,\ \eta^2_{uv} = 0,\ \eta^2_{vv} = 0,\ \eta^1_{uu} = 0,\\
&\eta^1_{uv} = 0,\ \eta^1_{vv} = 0,\ \eta^2_{xu} = 0,\ \eta^1_{xv} = 0,\ \eta^2_{xv} - \xi^2_{xx} = 0,\ \xi^2_{xx} - \eta^1_{xu} = 0,\\
&-\alpha(t)\xi^1_t + 3\alpha(t)\xi^2_x - \xi^1\alpha'(t) = 0,\\
&-6\alpha(t)\eta^1_x u^2 + 6\alpha(t)\eta^1_x v^2 + 12\alpha(t)\eta^2_x uv - \alpha(t)\eta^1_{xxx} - 4\eta^1_x\beta(t) + \eta^1_t = 0,\\
&-12\alpha(t)\eta^1_x uv - 6\alpha(t)\eta^2_x u^2 + 6\alpha(t)\eta^2_x v^2 - \alpha(t)\eta^2_{xxx} - 4\eta^2_x\beta(t) + \eta^2_t = 0,\\
&-\alpha(t)\xi^1_t uv + \alpha(t)\eta^2_v uv + \alpha(t)\xi^2_x uv - \alpha(t)\eta^1_u uv - \eta_1\alpha(t)v - \eta_2\alpha(t)u - \xi^1(\alpha'(t))uv = 0,\\
&\alpha(t)\xi^1_t uv + \alpha(t)\eta^2_v uv - \alpha(t)\xi^2_x uv - \alpha(t)\eta^1_u uv + \eta_1\alpha(t)v + \eta_2\alpha(t)u + \xi_1(\alpha'(t))uv = 0,\\
&-12\eta_1\alpha(t)u + 12\eta_2\alpha(t)v - 6\alpha(t)\xi^1_t u^2 + 6\alpha(t)\xi^1_t v^2 + 6\alpha(t)\xi^2_x u^2 - 6\alpha(t)\xi^2_x v^2\\
&-12\alpha(t)\eta^2_u uv - 12\alpha(t)\eta^1_v uv - 3\alpha(t)\eta^2_{xxv} + \alpha(t)\xi^2_{xxx} - 4\beta(t)\xi^1_t + 4\beta(t)\xi^2_x - 6\xi^1(\alpha'(t))u^2\\
&+6\xi^1(\alpha'(t))v^2 - 4\xi^1\beta'(t) - \xi^2_t = 0,\\
&-12\eta_1\alpha(t)u + 12\eta_2\alpha(t)v - 6\alpha(t)\xi^1_t u^2 + 6\alpha(t)\xi^1_t v^2 + 6\alpha(t)\xi^2_x u^2 - 6\alpha(t)\xi^2_x v^2\\
&+12\alpha(t)\eta^2_u uv + 12\alpha(t)\eta^1_v uv + \alpha(t)\xi^2_{xxx} - 3\alpha(t)\eta^1_{xxu} - 4\beta(t)\xi^1_t + 4\beta(t)\xi^2_x - 6\xi^1(\alpha'(t))u^2\\
&+6\xi^1(\alpha'(t))v^2 - 4\xi^1\beta'(t) - \xi^2_t = 0.
\end{aligned}$$

Solving the above overdetermined system of linear partial differential equations using Maple, one obtains the following Lie point symmetries of the system (5.1):

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial x},\\
X_2 &= -4\left\{\frac{\beta(t)}{\alpha(t)}\right\}\frac{\partial}{\partial x} + \left\{\frac{1}{\alpha(t)}\right\}\frac{\partial}{\partial t},\\
X_3 &= \left\{4\int \frac{-2\,\beta\,(t)\,(\alpha\,(t))^2 + 3\,\beta\,(t)\,(\alpha'\,(t))\,[\int\alpha\,(t)\,dt] - 3\,\left(\beta'\,(t)\right)[\int\alpha\,(t)\,dt]\alpha\,(t)}{(\alpha\,(t))^2}dt\right.\\
&\left.+x\right\}\frac{\partial}{\partial x} + 3\left\{\frac{\int\alpha\,(t)\,dt}{\alpha\,(t)}\right\}\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}.
\end{aligned}
$$

We now use the symmetries X_1 and X_2 and transform the system (5.1) to a system of ordinary differential equations (ODEs). Consider the linear combination, $\rho_1 X_1 + \rho_2 X_2$, where ρ_1 and ρ_2 are constants. This symmetry gives rise to the group-invariant solution

$$u = F(z), \quad v = G(z), \tag{5.2}$$

where $z = x\rho_2 + \int_{t_0}^{t}[-\rho_1\alpha(\tau) + 4\rho_2\beta(\tau)]\,d\tau$ is an invariant of the symmetry $\rho_1 X_1 + \rho_2 X_2$ and t_0 is a constant. Substitution of (5.2) into (5.1) results into a nonlinear third-order system of ordinary differential equations

$$
\begin{aligned}
&\rho_2^3 F'''(z)\alpha(t) - 6\rho_2 G(z)^2\alpha(t)F'(z) + 6\rho_2 F(z)^2\alpha(t)F'(z)\\
&+\rho_1\alpha(t)F'(z) - 12\rho_2 F(z)G(z)\alpha(t)G'(z) = 0, \qquad (5.3\text{a})\\
&\rho_2^3 G'''(z)\alpha(t) + 12\rho_2 F(z)G(z)\alpha(t)F'(z) + 6\rho_2 F(z)^2\alpha(t)G'(z)\\
&-6\rho_2 G(z)^2\alpha(t)G'(z) + \rho_1\alpha(t)G'(z) = 0. \qquad (5.3\text{b})
\end{aligned}
$$

In the next two sections, we obtain exact solutions of the system (5.3) by using the simplest equation method and the Jacobi elliptic function method, respectively.

5.2 Exact solutions using simplest equation method

In this section we use the simplest equation method, which was used in Chapter 4, to solve the system (5.3) and as a result we obtain the exact solutions of our CVCmKdV system (5.1).

5.2.1 Solutions of (5.1) using the Bernoulli equation as the simplest equation

The balancing procedure yields $M = 1$, so the solutions of (5.3) are of the form

$$F(z) = \mathcal{A}_0 + \mathcal{A}_1 H, \quad G(z) = \mathcal{B}_0 + \mathcal{B}_1 H. \tag{5.4}$$

Substituting (5.4) into (5.3) and making use of the Bernoulli equation and then equating all coefficients of the functions H^i to zero, we obtain the following algebraic system of equations in terms of $\mathcal{A}_0, \mathcal{A}_1$, $\mathcal{B}_0$ and $\mathcal{B}_1$:

$$
\begin{aligned}
&-a^3\alpha\mathcal{A}_1{\rho_2}^3 - a\alpha\mathcal{A}_1\rho_1 - 6a\alpha\mathcal{A}_0^2\mathcal{A}_1\rho_2 + 6a\alpha\mathcal{A}_1\rho_2\mathcal{B}_0^2 + 12a\alpha\mathcal{A}_0\rho_2\mathcal{B}_0\mathcal{B}_1 = 0,\\
&-7a^2\alpha\mathcal{A}_1 b{\rho_2}^3 - 12a\alpha\mathcal{A}_0\mathcal{A}_1^2\rho_2 + 12a\alpha\mathcal{A}_0\rho_2\mathcal{B}_1^2 + 24a\alpha\mathcal{A}_1\rho_2\mathcal{B}_0\mathcal{B}_1\\
&-\alpha\mathcal{A}_1 b\rho_1 - 6\alpha\mathcal{A}_0^2\mathcal{A}_1 b\rho_2 + 6\alpha\mathcal{A}_1 b\rho_2\mathcal{B}_0^2 + 12\alpha\mathcal{A}_0 b\rho_2\mathcal{B}_0\mathcal{B}_1 = 0,\\
&-6a\alpha\mathcal{A}_1^3\rho_2 + 18a\alpha\mathcal{A}_1\rho_2\mathcal{B}_1^2 - 12a\alpha\mathcal{A}_1 b^2{\rho_2}^3 - 12\alpha\mathcal{A}_0\mathcal{A}_1^2 b\rho_2\\
&+12\alpha\mathcal{A}_0 b\rho_2\mathcal{B}_1^2 + 24\alpha\mathcal{A}_1 b\rho_2\mathcal{B}_0\mathcal{B}_1 = 0,\\
&-6\alpha\mathcal{A}_1 b^3{\rho_2}^3 - 6\alpha\mathcal{A}_1^3 b\rho_2 + 18\alpha\mathcal{A}_1 b\rho_2\mathcal{B}_1^2 = 0,\\
&-a^3\alpha{\rho_2}^3\mathcal{B}_1 - 12a\alpha\mathcal{A}_0\mathcal{A}_1\rho_2\mathcal{B}_0 - 6a\alpha\mathcal{A}_0^2\rho_2\mathcal{B}_1 - a\alpha\rho_1\mathcal{B}_1 + 6a\alpha\rho_2\mathcal{B}_0^2\mathcal{B}_1 = 0,\\
&-7a^2\alpha b{\rho_2}^3\mathcal{B}_1 - 12a\alpha\mathcal{A}_1^2\rho_2\mathcal{B}_0 - 24a\alpha\mathcal{A}_0\mathcal{A}_1\rho_2\mathcal{B}_1 + 12a\alpha\rho_2\mathcal{B}_0\mathcal{B}_1^2\\
&-12\alpha\mathcal{A}_0\mathcal{A}_1 b\rho_2\mathcal{B}_0 - 6\alpha\mathcal{A}_0^2 b\rho_2\mathcal{B}_1 - \alpha b\rho_1\mathcal{B}_1 + 6\alpha b\rho_2\mathcal{B}_0^2\mathcal{B}_1 = 0,\\
&-18a\alpha\mathcal{A}_1^2\rho_2\mathcal{B}_1 - 12a\alpha b^2{\rho_2}^3\mathcal{B}_1 + 6a\alpha\rho_2\mathcal{B}_1^3 - 12\alpha\mathcal{A}_1^2 b\rho_2\mathcal{B}_0\\
&-24\alpha\mathcal{A}_0\mathcal{A}_1 b\rho_2\mathcal{B}_1 + 12\alpha b\rho_2\mathcal{B}_0\mathcal{B}_1^2 = 0,\\
&-6\alpha b^3{\rho_2}^3\mathcal{B}_1 - 18\alpha\mathcal{A}_1^2 b\rho_2\mathcal{B}_1 + 6\alpha b\rho_2\mathcal{B}_1^3 = 0.
\end{aligned}
$$

Solving the above system of algebraic equations, with the aid of Mathematica, we obtain

$$
\begin{aligned}
a &= \frac{\sqrt{2}\sqrt{\rho_1}}{{\rho_2}^{3/2}},\\
\mathcal{A}_1 &= \frac{ib\rho_2}{2},\\
\mathcal{B}_0 &= \frac{\sqrt{a^2b{\rho_2}^2 + 8a\mathcal{A}_0\mathcal{A}_1 - 4\mathcal{A}_0^2 b}}{2\sqrt{b}},\\
\mathcal{B}_1 &= -\frac{\mathcal{A}_1 b\mathcal{B}_0}{\mathcal{A}_0 b - a\mathcal{A}_1}.
\end{aligned}
$$

As a result a solution of (5.1) is given by

$$u(t,x) = \mathcal{A}_0 + \mathcal{A}_1 a \left\{ \frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]} \right\}, \tag{5.5a}$$

$$v(t,x) = \mathcal{B}_0 + \mathcal{B}_1 a \left\{ \frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]} \right\}, \tag{5.5b}$$

where $z = x\rho_2 + \int_{t_0}^{t} [-\rho_1 \alpha(\tau) + 4\rho_2 \beta(\tau)]\, d\tau$ and C is a constant of integration.

5.2.2 Solutions of (5.1) using Riccati equation as the simplest equation

The balancing procedure yields $M = 1$, so the solutions of (5.3) are of the form

$$F(z) = \mathcal{A}_0 + \mathcal{A}_1 H, \quad G(z) = \mathcal{B}_0 + \mathcal{B}_1 H. \tag{5.6}$$

Substituting (5.6) into (5.3) and making use of the Riccati equation, we obtain the following algebraic system of equations in terms of $\mathcal{A}_0, \mathcal{A}_1, \mathcal{B}_0, \mathcal{B}_1$ by equating all coefficients of the functions H^i to zero:

$$-2a\alpha\mathcal{A}_1 c^2 {\rho_2}^3 - \alpha\mathcal{A}_1 b^2 c {\rho_2}^3 - \alpha\mathcal{A}_1 c\rho_1 - 6\alpha\mathcal{A}_0^2\mathcal{A}_1 c\rho_2 + 6\alpha\mathcal{A}_1 c\rho_2\mathcal{B}_0^2$$

$$+12\alpha\mathcal{A}_0 c\rho_2\mathcal{B}_0\mathcal{B}_1 = 0,$$

$$-8a\alpha\mathcal{A}_1 bc{\rho_2}^3 + \alpha\mathcal{A}_1\left(-b^3\right){\rho_2}^3 - \alpha\mathcal{A}_1 b\rho_1 - 6\alpha\mathcal{A}_0^2\mathcal{A}_1 b\rho_2 + 6\alpha\mathcal{A}_1 b\rho_2\mathcal{B}_0^2$$

$$+12\alpha\mathcal{A}_0 b\rho_2\mathcal{B}_0\mathcal{B}_1 - 12\alpha\mathcal{A}_0\mathcal{A}_1^2 c\rho_2 + 12\alpha\mathcal{A}_0 c\rho_2\mathcal{B}_1^2 + 24\alpha\mathcal{A}_1 c\rho_2\mathcal{B}_0\mathcal{B}_1 = 0,$$

$$-8a^2\alpha\mathcal{A}_1 c{\rho_2}^3 - a\alpha\mathcal{A}_1\rho_1 - 6a\alpha\mathcal{A}_0^2\mathcal{A}_1\rho_2 + 6a\alpha\mathcal{A}_1\rho_2\mathcal{B}_0^2 + 12a\alpha\mathcal{A}_0\rho_2\mathcal{B}_0\mathcal{B}_1$$

$$-7a\alpha\mathcal{A}_1 b^2{\rho_2}^3 - 12\alpha\mathcal{A}_0\mathcal{A}_1^2 b\rho_2 + 12\alpha\mathcal{A}_0 b\rho_2\mathcal{B}_1^2 + 24\alpha\mathcal{A}_1 b\rho_2\mathcal{B}_0\mathcal{B}_1$$

$$-6\alpha\mathcal{A}_1^3 c\rho_2 + 18\alpha\mathcal{A}_1 c\rho_2\mathcal{B}_1^2 = 0,$$

$$-12a^2\alpha\mathcal{A}_1 b{\rho_2}^3 - 12a\alpha\mathcal{A}_0\mathcal{A}_1^2\rho_2 + 12a\alpha\mathcal{A}_0\rho_2\mathcal{B}_1^2 + 24a\alpha\mathcal{A}_1\rho_2\mathcal{B}_0\mathcal{B}_1$$

$$-6\alpha\mathcal{A}_1^3 b\rho_2 + 18\alpha\mathcal{A}_1 b\rho_2\mathcal{B}_1^2 = 0,$$

$$-6a^3\alpha\mathcal{A}_1{\rho_2}^3 - 6a\alpha\mathcal{A}_1^3\rho_2 + 18a\alpha\mathcal{A}_1\rho_2\mathcal{B}_1^2 = 0,$$

$$
\begin{aligned}
&-2a\alpha c^2\rho_2{}^3\mathcal{B}_1 - \alpha b^2 c\rho_2{}^3\mathcal{B}_1 - 12\alpha\mathcal{A}_0\mathcal{A}_1 c\rho_2\mathcal{B}_0 - 6\alpha\mathcal{A}_0^2 c\rho_2\mathcal{B}_1\\
&-\alpha c\rho_1\mathcal{B}_1 + 6\alpha c\rho_2\mathcal{B}_0^2\mathcal{B}_1 = 0,\\
&-8a\alpha bc\rho_2{}^3\mathcal{B}_1 + \alpha\left(-b^3\right)\rho_2{}^3\mathcal{B}_1 - 12\alpha\mathcal{A}_0\mathcal{A}_1 b\rho_2\mathcal{B}_0 - 6\alpha\mathcal{A}_0^2 b\rho_2\mathcal{B}_1 - \alpha b\rho_1\mathcal{B}_1\\
&+6\alpha b\rho_2\mathcal{B}_0^2\mathcal{B}_1 - 12\alpha\mathcal{A}_1^2 c\rho_2\mathcal{B}_0 - 24\alpha\mathcal{A}_0\mathcal{A}_1 c\rho_2\mathcal{B}_1 + 12\alpha c\rho_2\mathcal{B}_0\mathcal{B}_1^2 = 0,\\
&-8a^2\alpha c\rho_2{}^3\mathcal{B}_1 - 12a\alpha\mathcal{A}_0\mathcal{A}_1\rho_2\mathcal{B}_0 - 6a\alpha\mathcal{A}_0^2\rho_2\mathcal{B}_1 - 7a\alpha b^2\rho_2{}^3\mathcal{B}_1 - a\alpha\rho_1\mathcal{B}_1\\
&+6a\alpha\rho_2\mathcal{B}_0^2\mathcal{B}_1 - 12\alpha\mathcal{A}_1^2 b\rho_2\mathcal{B}_0 - 24\alpha\mathcal{A}_0\mathcal{A}_1 b\rho_2\mathcal{B}_1 + 12\alpha b\rho_2\mathcal{B}_0\mathcal{B}_1^2\\
&-18\alpha\mathcal{A}_1^2 c\rho_2\mathcal{B}_1 + 6\alpha c\rho_2\mathcal{B}_1^3 = 0,\\
&-12a^2\alpha b\rho_2{}^3\mathcal{B}_1 - 12a\alpha\mathcal{A}_1^2\rho_2\mathcal{B}_0 - 24a\alpha\mathcal{A}_0\mathcal{A}_1\rho_2\mathcal{B}_1 + 12a\alpha\rho_2\mathcal{B}_0\mathcal{B}_1^2\\
&-18\alpha\mathcal{A}_1^2 b\rho_2\mathcal{B}_1 + 6\alpha b\rho_2\mathcal{B}_1^3 = 0,\\
&-6a^3\alpha\rho_2{}^3\mathcal{B}_1 - 18a\alpha\mathcal{A}_1^2\rho_2\mathcal{B}_1 + 6a\alpha\rho_2\mathcal{B}_1^3 = 0.
\end{aligned}
$$

Solving the above system of algebraic equations, with the aid of Mathematica, we obtain

$$
\begin{aligned}
a &= \frac{b^2\rho_2{}^3 - 2\rho_1}{4c\rho_2{}^3},\\
\mathcal{A}_1 &= \frac{ia\rho_2}{2},\\
\mathcal{B}_0 &= \frac{\sqrt{-4a\mathcal{A}_0^2 + ab^2\rho_2{}^2 + 8\mathcal{A}_0\mathcal{A}_1 b}}{2\sqrt{a}},\\
\mathcal{B}_1 &= -\frac{a\mathcal{A}_1\mathcal{B}_0}{a\mathcal{A}_0 - \mathcal{A}_1 b}.
\end{aligned}
$$

As a result two solutions of (5.1) are

$$u(t,x) = \mathcal{A}_0 + \mathcal{A}_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}, \tag{5.7a}$$

$$v(t,x) = \mathcal{B}_0 + \mathcal{B}_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\} \tag{5.7b}$$

and

$$\begin{aligned} u(t,x) &= \mathcal{A}_0 + \mathcal{A}_1\Bigg\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) \\ &\quad + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\Bigg\}, \end{aligned} \tag{5.8a}$$

$$\begin{aligned} v(t,x) &= \mathcal{B}_0 + \mathcal{B}_1\Bigg\{ -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) \\ &\quad + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\Bigg\}, \end{aligned} \tag{5.8b}$$

where $z = x\rho_2 + \int_{t_0}^{t}[-\rho_1\alpha(\tau) + 4\rho_2\beta(\tau)]\,d\tau$ and C is a constant of integration.

5.3 Solutions of (5.1) using Jacobi elliptic function method

In this subsection we obtain exact solutions of the CVCmKdV system (5.1) in terms of Jacobi elliptic functions (see Section 4.2.4 of Chapter 4) and as a result we obtain the following cnoidal and snoidal wave solutions of (5.1) given by, respectively,

$$u(t,x) = \mathcal{A}_0 + \mathcal{A}_1\mathrm{cn}(z|\omega), \tag{5.9a}$$

$$v(t,x) = \mathcal{B}_0 + \mathcal{B}_1\mathrm{cn}(z|\omega), \tag{5.9b}$$

where

$$\begin{aligned} \rho_1 &= {\rho_2}^3 - 2{\rho_2}^3\omega, \\ \mathcal{A}_0 &= 0, \\ \mathcal{A}_1 &= \frac{\rho_2\sqrt{\omega}}{2}, \\ \mathcal{B}_0 &= 0, \\ \mathcal{B}_1 &= \frac{1}{2}i\rho_2\sqrt{\omega} \end{aligned}$$

and

$$u(t,x) = \mathcal{A}_0 + \mathcal{A}_1\mathrm{sn}(z|\omega),$$

$$v(t,x) = \mathcal{B}_0 + \mathcal{B}_1\mathrm{sn}(z|\omega),$$

where

$$\begin{aligned}\rho_1 &= {\rho_2}^3(\omega+1),\\ \mathcal{A}_1 &= \frac{1}{2} i\rho_2\sqrt{\omega},\\ \mathcal{B}_0 &= i\mathcal{A}_0,\\ \mathcal{B}_1 &= -\frac{\mathcal{A}_1\mathcal{B}_0}{\mathcal{A}_0}\end{aligned}$$

and $z = x\rho_2 + \int_{t_0}^{t}[-\rho_1\alpha(\tau) + 4\rho_2\beta(\tau)]\,d\tau$.

The profile of the solution (5.9) is given in Figure 5.1.

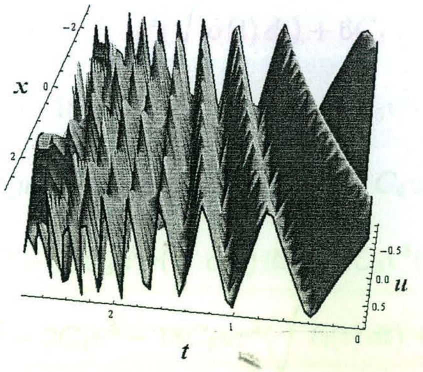

Figure 5.1: Profile of cnoidal waves (5.9)

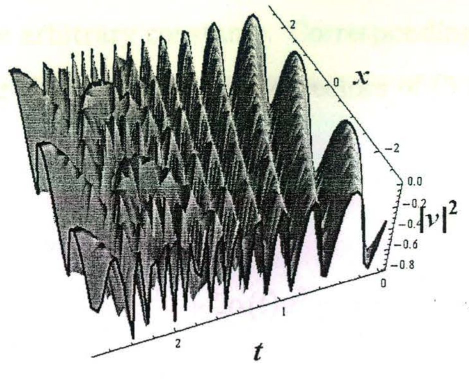

Figure 5.2: Profile of cnoidal waves (5.9)

5.4 Conservation laws of (5.1)

In this section we obtain conservation laws for the system (5.1) using two different approaches; the multiplier method and the new conservation theorem.

5.4.1 Application of the Multiplier Method

For the CVCmKdV system (5.1), we obtain the second-order multipliers , $\Lambda_1 = \Lambda_1(t, x, u, v, u_x, v_x, u_{xx}, v_{xx})$ and $\Lambda_2 = \Lambda_2(t, x, u, v, u_x, v_x, u_{xx}, v_{xx})$ that are given by

$$\begin{aligned}
\Lambda_1 &= -6C_2uv^2 + 2C_2u^3 - 18C_3uv^2(\int \alpha(t)\,dt) + 6C_3u^3(\int \alpha(t)\,dt) + 4C_3u(\int \beta(t)\,dt) + C_3xu \\
&\quad +6C_4u^2v - 2C_4v^3 + 18C_5u^2v(\int \alpha(t)\,dt) - 6C_5v^3(\int \alpha(t)\,dt) + 4C_5v(\int \beta(t)\,dt) + C_5xv \\
&\quad +C_7v + C_8u + C_2u_{,xx} + 3C_3u_{,xx}(\int \alpha(t)\,dt) + C_4v_{,xx} + 3C_5v_{,xx}(\int \alpha(t)\,dt) + C_6, \\
\Lambda_2 &= -6C_2u^2v + 2C_2v^3 - 18C_3u^2v(\int \alpha(t)\,dt) + 6C_3v^3(\int \alpha(t)\,dt) - 4C_3v(\int \beta(t)\,dt) \\
&\quad -C_3xv - 6C_4uv^2 + 2C_4u^3 - 18C_5uv^2(\int \alpha(t)\,dt) + 6C_5u^3(\int \alpha(t)\,dt) \\
&\quad +4C_5u(\int \beta(t)\,dt) + C_5xu + C_7u - C_8v - C_2v_{xx} - 3C_3v_{xx}(\int \alpha(t)\,dt) + C_4u_{xx} \\
&\quad +3C_5u_{xx}(\int \alpha(t)\,dt) + C_1,
\end{aligned}$$

where C_i, $i = 1 \cdots 8$ are arbitrary constants. Corresponding to the above multipliers we obtain the following eight local conserved vectors of (5.1):

$$\begin{aligned}
\Phi_1^t &= v, \\
\Phi_1^x &= -6\alpha(t)u^2v + 2\alpha(t)v^3 - 4\beta(t)v - \alpha(t)v_{xx};
\end{aligned}$$

$$\begin{aligned}
\Phi_2^t &= \frac{1}{2}\Big\{u_{xx}u - v_{xx}v - 6u^2v^2 + u^4 + v^4\Big\},\\
\Phi_2^x &= \frac{1}{2}\Big\{12\alpha(t)v_{xx}u^2v + 12\alpha(t)u_{xx}uv^2 - 4\alpha(t)u_{xx}u^3 - uu_{tx} - 4\alpha(t)v_{xx}v^3\\
&\quad +vv_{tx} + 60\alpha(t)u^4v^2 - 60\alpha(t)u^2v^4 + 24\beta(t)u^2v^2 - 4\alpha(t)u^6 - 4\beta(t)u^4\\
&\quad +4\alpha(t)v^6 - 4\beta(t)v^4 - \alpha(t){u_{xx}}^2 - 4\beta(t){u_x}^2 + u_tu_x + \alpha(t){v_{xx}}^2 + 4\beta(t){v_x}^2\\
&\quad -v_tv_x\Big\};
\end{aligned}$$

$$\begin{aligned}
\Phi_3^t &= \frac{1}{2}\Big\{3u_{xx}u(\int \alpha(t)\,dt) - 3v_{xx}v(\int \alpha(t)\,dt) - 18u^2v^2(\int \alpha(t)\,dt) + 3u^4(\int \alpha(t)\,dt)\\
&\quad +4u^2(\int \beta(t)\,dt) + xu^2 + 3v^4(\int \alpha(t)\,dt) - 4v^2(\int \beta(t)\,dt) - xv^2\Big\},
\end{aligned}$$

$$\begin{aligned}
\Phi_3^x &= \frac{1}{2}\Big\{36\alpha(t)v_{xx}u^2v(\int \alpha(t)\,dt) + 36\alpha(t)u_{xx}uv^2(\int \alpha(t)\,dt) - 8\alpha(t)u_{xx}u(\int \beta(t)\,dt)\\
&\quad -\alpha(t)u_xu - 2x\alpha(t)u_{xx}u - 12\alpha(t)u_{xx}u^3(\int \alpha(t)\,dt) - 3uu_{tx}(\int \alpha(t)\,dt)\\
&\quad +8\alpha(t)v_{xx}v(\int \beta(t)\,dt) + \alpha(t)v_xv + 2x\alpha(t)v_{xx}v - 12\alpha(t)v_{xx}v^3(\int \alpha(t)\,dt)\\
&\quad +3vv_{tx}(\int \alpha(t)\,dt) + 72\alpha(t)u^2v^2(\int \beta(t)\,dt) + 72\beta(t)u^2v^2(\int \alpha(t)\,dt) + 18x\alpha(t)u^2v^2\\
&\quad +180\alpha(t)u^4v^2(\int \alpha(t)\,dt) - 180\alpha(t)u^2v^4(\int \alpha(t)\,dt) - 12\alpha(t)u^4(\int \beta(t)\,dt)\\
&\quad -12\beta(t)u^4(\int \alpha(t)\,dt) - 3x\alpha(t)u^4 - 12\alpha(t)u^6(\int \alpha(t)\,dt) - 4x\beta(t)u^2\\
&\quad -16\beta(t)u^2(\int \beta(t)\,dt) - 12\alpha(t)v^4(\int \beta(t)\,dt) - 12\beta(t)v^4(\int \alpha(t)\,dt) - 3x\alpha(t)v^4\\
&\quad +12\alpha(t)v^6(\int \alpha(t)\,dt) + 4x\beta(t)v^2 + 16\beta(t)v^2(\int \beta(t)\,dt) + 4\alpha(t){u_x}^2(\int \beta(t)\,dt)\\
&\quad -12\beta(t){u_x}^2(\int \alpha(t)\,dt) + x\alpha(t){u_x}^2 - 3\alpha(t){u_{xx}}^2(\int \alpha(t)\,dt) + 3u_tu_x(\int \alpha(t)\,dt)\\
&\quad -4\alpha(t){v_x}^2(\int \beta(t)\,dt) + 12\beta(t){v_x}^2(\int \alpha(t)\,dt) - x\alpha(t){v_x}^2 + 3\alpha(t){v_{xx}}^2(\int \alpha(t)\,dt)\\
&\quad -3v_tv_x(\int \alpha(t)\,dt)\Big\};
\end{aligned}$$

$$
\begin{aligned}
\Phi_4^t &= \frac{1}{2}\Big\{ v_{xx}u + u_{xx}v + 4u^3v - 4uv^3 \Big\}, \\
\Phi_4^x &= \frac{1}{2}\Big\{ -4\alpha(t)v_{xx}u^3 - 12\alpha(t)u_{xx}u^2v + 12\alpha(t)v_{xx}uv^2 + 4\alpha(t)u_{xx}v^3 - uv_{tx} - vu_{tx} \\
&\quad -24\alpha(t)u^5v + 80\alpha(t)u^3v^3 - 24\alpha(t)uv^5 - 16\beta(t)u^3v + 16\beta(t)uv^3 - 2\alpha(t)u_{xx}v_{xx} \\
&\quad -8\beta(t)u_xv_x + u_tv_x + v_tu_x \Big\};
\end{aligned}
$$

$$
\begin{aligned}
\Phi_5^t &= \frac{1}{2}\Big\{ 3v_{xx}u(\int \alpha(t)\,dt) + 3u_{xx}v(\int \alpha(t)\,dt) + 12u^3v(\int \alpha(t)\,dt) \\
&\quad -12uv^3(\int \alpha(t)\,dt) + 8uv(\int \beta(t)\,dt) + 2xuv \Big\},
\end{aligned}
$$

$$
\begin{aligned}
\Phi_5^x &= \frac{1}{2}\Big\{ -8\alpha(t)v_{xx}u(\int \beta(t)\,dt) - 8\alpha(t)u_{xx}v(\int \beta(t)\,dt) - \alpha(t)v_xu - 2x\alpha(t)v_{xx}u \\
&\quad -\alpha(t)u_xv - 2x\alpha(t)u_{xx}v - 12\alpha(t)v_{xx}u^3(\int \alpha(t)\,dt) - 36\alpha(t)u_{xx}u^2v(\int \alpha(t)\,dt) \\
&\quad +36\alpha(t)v_{xx}uv^2(\int \alpha(t)\,dt) - 3uv_{tx}(\int \alpha(t)\,dt) + 12\alpha(t)u_{xx}v^3(\int \alpha(t)\,dt) \\
&\quad -3vu_{tx}(\int \alpha(t)\,dt) - 48\alpha(t)u^3v(\int \beta(t)\,dt) - 48\beta(t)u^3v(\int \alpha(t)\,dt) \\
&\quad +48\alpha(t)uv^3(\int \beta(t)\,dt) + 48\beta(t)uv^3(\int \alpha(t)\,dt) - 12x\alpha(t)u^3v \\
&\quad +12x\alpha(t)uv^3 - 72\alpha(t)u^5v(\int \alpha(t)\,dt) + 240\alpha(t)u^3v^3(\int \alpha(t)\,dt) \\
&\quad -72\alpha(t)uv^5(\int \alpha(t)\,dt) - 8x\beta(t)uv - 32\beta(t)uv(\int \beta(t)\,dt) \\
&\quad +8\alpha(t)u_xv_x(\int \beta(t)\,dt) - 24\beta(t)u_xv_x(\int \alpha(t)\,dt) + 2x\alpha(t)u_xv_x \\
&\quad -6\alpha(t)u_{xx}v_{xx}(\int \alpha(t)\,dt) + 3u_tv_x(\int \alpha(t)\,dt) + 3v_tu_x(\int \alpha(t)\,dt) \Big\};
\end{aligned}
$$

$$
\begin{aligned}
\Phi_6^t &= u, \\
\Phi_6^x &= 6\alpha(t)uv^2 - 2\alpha(t)u^3 - 4\beta(t)u - \alpha(t)u_{xx};
\end{aligned}
$$

$$
\begin{aligned}
\Phi_7^t &= uv, \\
\Phi_7^x &= -\alpha(t)v_{xx}u - \alpha(t)u_{xx}v - 6\alpha(t)u^3v + 6\alpha(t)uv^3 - 4\beta(t)uv + \alpha(t)u_xv_x;
\end{aligned}
$$

$$
\begin{aligned}
\Phi_8^t &= \frac{1}{2}\Big\{u^2 - v^2\Big\},\\
\Phi_8^x &= \frac{1}{2}\Big\{-2\alpha(t)u_{xx}u + 2\alpha(t)v_{xx}v + 18\alpha(t)u^2v^2 - 3\alpha(t)u^4 - 4\beta(t)u^2 - 3\alpha(t)v^4\\
&\quad +4\beta(t)v^2 + \alpha(t){u_x}^2 - \alpha(t){v_x}^2\Big\}.
\end{aligned}
$$

Remark 1. Higher order conservation laws of (5.1) can be computed by increasing the order of the multipliers.

5.4.2 Application of the new Conservation Theorem

The adjoint equations of (5.1) are given by

$$\phi_t - \alpha(t)\left(\phi_{xxx} + 6(u^2 - v^2)\phi_x + 12uv\phi_x\right) - 4\beta(t)\phi_x = 0, \tag{5.10a}$$

$$\psi_t - \alpha(t)\left(\psi_{xxx} + 6(u^2 - v^2)\psi_x - 12uv\psi_x\right) - 4\beta(t)\psi_x = 0. \tag{5.10b}$$

Corresponding to the three symmetries of (5.1) we have the following three cases:

(i) We first consider the Lie point symmetry generator $X_1 = \partial_x$ of (5.1). Corresponding to this symmetry the Lie characteristic function $W = -(u_x + v_x)$. Thus, the components of the conserved vector are given by

$$
\begin{aligned}
T_1^t &= -u_x\phi - v_x\psi,\\
T_1^x &= u_t\phi + v_t\psi - \alpha(t)u_{xx}\phi_x + \alpha(t)u_x\phi_{xx} - \alpha(t)v_{xx}\psi_x + \alpha(t)v_x\psi_{xx}.
\end{aligned}
$$

(ii) The Lie point symmetry generator $X_2 = -4\frac{\beta(t)}{\alpha(t)}\partial_x + \frac{1}{\alpha(t)}\partial_t$ has the Lie characteristic function $W = 4\frac{\beta(t)}{\alpha(t)}u_x + 4\frac{\beta(t)}{\alpha(t)}v_x - \frac{1}{\alpha(t)}u_t - \frac{1}{\alpha(t)}v_t$. Hence, one obtains the conserved vector whose components are

$$
\begin{aligned}
T_2^t &= -6v_xu^2\psi - 12u_xuv\psi + 12v_xuv\phi + 6u_xv^2\phi - 6u_xu^2\phi - u_{xxx}\phi + 6v_xv^2\psi - v_{xxx}\psi,\\
T_2^x &= 6v_tu^2\psi + 12u_tuv\psi - 12v_tuv\phi - 6u_tv^2\phi + 6u_tu^2\phi + \phi u_{txx} - 6v_tv^2\psi + \psi v_{txx}\\
&\quad +4\beta(t)u_{xx}\phi_x - 4\beta(t)u_x\phi_{xx} + u_t\phi_{xx} - \phi_xu_{tx} + 4\beta(t)v_{xx}\psi_x - 4\beta(t)v_x\psi_{xx}\\
&\quad +v_t\psi_{xx} - \psi_xv_{tx}.
\end{aligned}
$$

Finally, we consider the third symmetry generator

$$X_3 = \left\{4\int \frac{-2\,\beta\,(t)\,(\alpha\,(t))^2 + 3\,\beta\,(t)\,(\alpha'\,(t))\,[\int \alpha\,(t)\,dt] - 3\,(\beta'\,(t))\,[\int \alpha\,(t)\,dt]\alpha\,(t)}{(\alpha\,(t))^2}\,dt \right.$$
$$\left. + x\right\}\frac{\partial}{\partial x} + 3\left\{\frac{\int \alpha\,(t)\,dt}{\alpha\,(t)}\right\}\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}.$$

In a similar fashion the components of the conserved vector are given by

$$\begin{aligned}
T_3^t \;=\; & \frac{1}{\alpha(t)}\Big\{ - 18(\int \alpha(t)\,dt)\alpha(t)\phi u_x u^2 - 18(\int \alpha(t)\,dt)\alpha(t)\psi v_x u^2 \\
& -\alpha(t)\phi u - 36(\int \alpha(t)\,dt)v\alpha(t)\psi u_x u + 36(\int \alpha(t)\,dt)v\alpha(t)\phi v_x u \\
& -v\alpha(t)\psi + 18(\int \alpha(t)\,dt)v^2\alpha(t)\phi u_x - x\alpha(t)\phi u_x \\
& -4\left(\int\left(\beta(t)\left(\frac{3(\int \alpha(t)\,dt)\alpha'(t)}{\alpha(t)^2} - 2\right) - \frac{3(\int \alpha(t)\,dt)\beta'(t)}{\alpha(t)}\right)dt\right)\alpha(t)\phi u_x \\
& -12(\int \alpha(t)\,dt)\beta(t)\phi u_x + 18(\int \alpha(t)\,dt)v^2\alpha(t)\psi v_x - x\alpha(t)\psi v_x \\
& -4\left(\int\left(\beta(t)\left(\frac{3(\int \alpha(t)\,dt)\alpha'(t)}{\alpha(t)^2} - 2\right) - \frac{3(\int \alpha(t)\,dt)\beta'(t)}{\alpha(t)}\right)dt\right)\alpha(t)\psi v_x \\
& -12(\int \alpha(t)\,dt)\beta(t)\psi v_x - 3(\int \alpha(t)\,dt)\alpha(t)\phi u_{xxx} \\
& -3(\int \alpha(t)\,dt)\alpha(t)\psi v_{xxx}\Big\},
\end{aligned}$$

$$
\begin{aligned}
T_3^x \;=\; & \frac{1}{\alpha(t)}\Big\{ 6\alpha(t)^2\phi u^3 + 18v\alpha(t)^2\psi u^2 \\
& +18(\int \alpha(t)\,dt)\alpha(t)\phi u_t u^2 + 18(\int \alpha(t)\,dt)\alpha(t)\psi v_t u^2 \\
& -18v^2\alpha(t)^2\phi u + 4\alpha(t)\beta(t)\phi u + \alpha(t)^2\phi_{xx}u \\
& +36(\int \alpha(t)\,dt)v\alpha(t)\psi u_t u - 36(\int \alpha(t)\,dt)v\alpha(t)\phi v_t u \\
& -6v^3\alpha(t)^2\psi + 4v\alpha(t)\beta(t)\psi - 2\alpha(t)^2u_x\phi_x - 2\alpha(t)^2v_x\psi_x \\
& +3\alpha(t)^2\phi u_{xx} - x\alpha(t)^2\phi_x u_{xx} \\
& -4\left(\int \left(\beta(t)\left(\frac{3(\int \alpha(t)\,dt)\alpha'(t)}{\alpha(t)^2} - 2\right) - \frac{3(\int \alpha(t)\,dt)\beta'(t)}{\alpha(t)}\right) dt\right)\alpha(t)^2\phi_x u_{xx} \\
& +3\alpha(t)^2\psi v_{xx} - x\alpha(t)^2\psi_x v_{xx} \\
& -4\left(\int \left(\beta(t)\left(\frac{3(\int \alpha(t)\,dt)\alpha'(t)}{\alpha(t)^2} - 2\right) - \frac{3(\int \alpha(t)\,dt)\beta'(t)}{\alpha(t)}\right) dt\right)\alpha(t)^2\psi_x v_{xx} \\
& +x\alpha(t)^2u_x\phi_{xx} + 4\left(\int \left(\beta(t)\left(\frac{3(\int \alpha(t)\,dt)\alpha'(t)}{\alpha(t)^2} - 2\right) - \frac{3(\int \alpha(t)\,dt)\beta'(t)}{\alpha(t)}\right) dt\right)\alpha(t)^2u_x\phi_{xx} \\
& +v\alpha(t)^2\psi_{xx} + x\alpha(t)^2v_x\psi_{xx} \\
& +4\left(\int \left(\beta(t)\left(\frac{3(\int \alpha(t)\,dt)\alpha'(t)}{\alpha(t)^2} - 2\right) - \frac{3(\int \alpha(t)\,dt)\beta'(t)}{\alpha(t)}\right) dt\right)\alpha(t)^2v_x\psi_{xx}
\end{aligned}
$$

$$
\begin{aligned}
& -18(\int \alpha(t)\,dt)v^2\alpha(t)\phi u_t + x\alpha(t)\phi u_t \\
& +4\left(\int \left(\beta(t)\left(\frac{3(\int \alpha(t)\,dt)\alpha'(t)}{\alpha(t)^2} - 2\right) - \frac{3(\int \alpha(t)\,dt)\beta'(t)}{\alpha(t)}\right) dt\right)\alpha(t)\phi u_t \\
& +12(\int \alpha(t)\,dt)\beta(t)\phi u_t + 3(\int \alpha(t)\,dt)\alpha(t)\phi_{xx}u_t \\
& -18(\int \alpha(t)\,dt)v^2\alpha(t)\psi v_t + x\alpha(t)\psi v_t \\
& +4\left(\int \left(\beta(t)\left(\frac{3(\int \alpha(t)\,dt)\alpha'(t)}{\alpha(t)^2} - 2\right) - \frac{3(\int \alpha(t)\,dt)\beta'(t)}{\alpha(t)}\right) dt\right)\alpha(t)\psi v_t \\
& +12(\int \alpha(t)\,dt)\beta(t)\psi v_t + 3(\int \alpha(t)\,dt)\alpha(t)\psi_{xx}v_t - 3(\int \alpha(t)\,dt)\alpha(t)\phi_x u_{tx} \\
& -3(\int \alpha(t)\,dt)\alpha(t)\psi_x v_{tx} + 3(\int \alpha(t)\,dt)\alpha(t)\phi u_{txx} + 3(\int \alpha(t)\,dt)\alpha(t)\psi v_{txx}\Big\}.
\end{aligned}
$$

Remark 2. The components of the conserved vectors contain the arbitrary solutions ϕ and ψ of the adjoint equations (5.10) and hence one can obtain an infinite number of conservation laws.

5.5 Conclusion

In this chapter Lie group analysis was employed to study the generalized coupled variable-coefficient modified Korteweg-de Vries (CVCmKdV) system (5.1). We obtained exact solutions for this system with the aid of simplest equation and Jacobi elliptic function methods. The exact solutions obtained were solitary, cnoidal and snoidal waves. Furthermore, the conservation laws of the CVCmKdV system were constructed using the multiplier method and the new conservation theorem.

Chapter 6

Symmetry reductions, exact solutions and conservation laws of a new coupled KdV system

In [59] the new coupled Korteweg-de Vries system

$$u_t + u_{xxx} + 3uu_x + 3ww_x = 0, \tag{6.1a}$$

$$v_t + v_{xxx} + 3vv_x + 3ww_x = 0, \tag{6.1b}$$

$$w_t + w_{xxx} + \frac{3}{2}(uw)_x + \frac{3}{2}(vw)_x = 0 \tag{6.1c}$$

was examined. The 4 × 4 matrix spectral problem with three potentials has been used in [59] to derive a hierarchy of nonlinear evolution equations, which includes the coupled Korteweg-de Vries equation (6.1). It was shown that the hierarchy possesses the generalized bi-Hamiltonian structures with the aid of the trace identity. Wazwaz [46] also considered the system (6.1) and employed the Hirota's bilinear method combined with Hereman et al [54] simplified approach to study the integrability of (6.1). Multiple-soliton solutions and multiple singular soliton solutions were obtained for (6.1) in [46].

In this chapter Lie symmetry analysis is performed on (6.1) and similarity reductions and new exact solutions are obtained. In addition, we derive the conservation laws of (6.1) using the multiplier method

This work has been published in [67].

6.1 Some symmetry reductions and exact solutions of (6.1)

The symmetry group of the coupled KdV system (6.1) will be generated by the vector field of the form

$$\begin{aligned} X &= \xi^1(t,x,u,v,w)\frac{\partial}{\partial t} + \xi^2(t,x,u,v,w)\frac{\partial}{\partial x} + \eta^1(t,x,u,v,w)\frac{\partial}{\partial u} + \eta^2(t,x,u,v,w)\frac{\partial}{\partial v} \\ &\quad + \eta^3(t,x,u,v,w)\frac{\partial}{\partial w}. \end{aligned}$$

Applying the third prolongation $\mathrm{pr}^{(3)}X$ [7] to (6.1) results in an overdetermined system of linear partial differential equations,

$$\begin{aligned}
&\xi^1_x = 0,\ \xi^2_w = 0,\ \xi^2_u = 0,\ \xi^1_u = 0,\ \xi^1_w = 0,\ \xi^2_v = 0,\ \xi^1_v = 0,\ \eta^3_{xu} = 0,\ \eta^3_{xv} = 0,\ \eta^2_{xu} = 0,\\
&\eta^2_{xw} = 0,\ \eta^2_{uu} = 0,\ \eta^2_{uv} = 0,\ \eta^2_{uw} = 0,\ \eta^2_{vv} = 0,\ \eta^2_{vw} = 0,\ \eta^2_{ww} = 0,\ \eta^1_{xv} = 0,\ \eta^1_{xw} = 0,\\
&\eta^1_{uu} = 0,\ \eta^1_{uv} = 0,\ \eta^1_{uw} = 0,\ \eta^1_{ww} = 0,\ \eta^3_{ww} = 0,\ \eta^3_{uw} = 0,\ \eta^3_{vw} = 0,\ \eta^1_{vv} = 0,\ \eta^3_{uu} = 0,\\
&\eta^3_{uv} = 0,\ \eta^3_{vv} = 0,\ \eta^1_{vw} = 0,\ \eta^1_{xu} - \xi^2_{xx} = 0,\ \eta^2_{xv} - \xi^2_{xx} = 0,\ \eta^3_{xw} - \xi^2_{xx} = 0,\ \xi^1_t - 3\xi^2_x = 0,\\
&\xi^1_t - 3\xi^2_x = 0,\ 3u\eta^1_x + 3w\eta^3_x + \eta^1_t + \eta^1_{xxx} = 0,\ 3v\eta^2_x + 3w\eta^3_x + \eta^2_t + \eta^2_{xxx} = 0,\\
&2\eta^1_v u - 2\eta^1_v v - \eta^1_w w + 2w\eta^3_v = 0,\ -2\eta^2_u u + 2\eta^2_u v - \eta^2_w w + 2w\eta^3_u = 0,\\
&3\eta^3_x u + 3\eta^3_x v + 3w\eta^1_x + 3w\eta^2_x + 2\eta^3_t + 2\eta^3_{xxx} = 0,\\
&6\xi^1_t u - 6u\xi^2_x - 3\eta^1_w w + 6w\eta^3_u + 6\eta_1 + 6\eta^1_{xxu} - 2\xi^2_t - 2\xi^2_{xxx} = 0,\\
&6\xi^1_t v - 6v\xi^2_x - 3\eta^2_w w + 6w\eta^3_v + 6\eta_2 + 6\eta^2_{xxv} - 2\xi^2_t - 2\xi^2_{xxx} = 0,\\
&\eta^3_v u - \eta^3_v v + \eta^1_v w + \eta^2_v w - w\eta^3_w + \xi^1_t w - w\xi^2_x + \eta_3 = 0,\\
&\eta^1_w u - \eta^1_w v - 2\eta^1_u w - 2\eta^1_v w + 2w\eta^3_w + 2\xi^1_t w - 2w\xi^2_x + 2\eta_3 = 0,\\
&-\eta^3_u u + \eta^3_u v + \eta^1_u w + \eta^2_u w - w\eta^3_w + \xi^1_t w - w\xi^2_x + \eta_3 = 0,\\
&-\eta^2_w u + \eta^2_w v - 2\eta^2_u w - 2\eta^2_v w + 2w\eta^3_w + 2\xi^1_t w - 2w\xi^2_x + 2\eta_3 = 0,
\end{aligned}$$

$$3\xi^1_t u - 3u\xi^2_x + 3\xi^1_t v - 3v\xi^2_x + 3\eta^1_w w + 3\eta^2_w w - 6w\eta^3_u - 6w\eta^3_v + 3\eta_1 + 3\eta_2$$
$$+6\eta^3_{xxw} - 2\xi^2_t - 2\xi^2_{xxx} = 0.$$

The general solution of the overdetermined system of linear partial differential equations is given by

$$\xi^1(t,x,u,v,w) = C_1 + 3tC_5,$$
$$\xi^2(t,x,u,v,w) = C_3 + 3tC_4 + xC_5,$$
$$\eta^1(t,x,u,v,w) = -2wC_2 + C_4 - 2uC_5,$$
$$\eta^2(t,x,u,v,w) = 2wC_2 + C_4 - 2vC_5,$$
$$\eta^3(t,x,u,v,w) = (u-v)C_2 - 2wC_5.$$

The above general solution contains five arbitrary constants. Hence the infinitesimal symmetries of (6.1) form the five-dimensional Lie algebra spanned by the following linearly independent operators:

$$X_1 = \frac{\partial}{\partial t},$$
$$X_2 = -2w\frac{\partial}{\partial u} + 2w\frac{\partial}{\partial v} + (u-v)\frac{\partial}{\partial w},$$
$$X_3 = \frac{\partial}{\partial x},$$
$$X_4 = 3t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} + \frac{\partial}{\partial v},$$
$$X_5 = 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v} - 2w\frac{\partial}{\partial w}.$$

6.1.1 Some symmetry reductions of (6.1)

In order to obtain symmetry reductions and exact solutions, one has to solve the associated Lagrange equations

$$\frac{dt}{\xi^1(t,x,u,v,w)} = \frac{dx}{\xi^2(t,x,u,v,w)} = \frac{du}{\eta^1(t,x,u,v,w)} = \frac{dv}{\eta^2(t,x,u,v,w)} = \frac{dw}{\eta^3(t,x,u,v,w)}.$$

We consider the following cases:

Case 1. $C_1 = \alpha, C_2 = 0, C_3 = 0, C_4 = 1, C_5 = 0$

The symmetry $\alpha X_1 + X_4$ gives rise to the group-invariant solution

$$u = E(z) + \frac{t}{\alpha}, \quad v = F(z) + \frac{t}{\alpha}, \quad w = G(z), \tag{6.2}$$

where $z = x - \dfrac{3t^2}{2\alpha}$ is an invariant of the symmetry $\alpha X_1 + X_4$. Substitution of (6.2) into (6.1) results in the system of ordinary differential equations (ODEs) where E, F and G satisfy

$$\begin{aligned}
&E'''(z) + 3E(z)E'(z) + 3G(z)G'(z) + \frac{1}{\alpha} = 0,\\
&F'''(z) + 3F(z)F'(z) + 3G(z)G'(z) + \frac{1}{\alpha} = 0,\\
&G'''(z) + \frac{3}{2}G(z)E'(z) + \frac{3}{2}E(z)G'(z) + \frac{3}{2}G(z)F'(z) + \frac{3}{2}F(z)G'(z) = 0.
\end{aligned}$$

Case 2. $C_1 = 0, C_2 = 0, C_3 = 0, C_4 = 1, C_5 = 0$

The symmetry X_4 gives rise to the group-invariant solution of the form

$$u = E(z) + \frac{x}{3t}, \quad v = F(z) + \frac{x}{3t}, \quad w = G(z), \tag{6.3}$$

where $z = t$ is an invariant of X_4 and the functions E, F and G satisfy the following system of ODEs:

$$\begin{aligned}
E'(z) + \frac{1}{z}E(z) &= 0\\
F'(z) + \frac{1}{z}F(z) &= 0,\\
G'(z) + \frac{1}{z}G(z) &= 0.
\end{aligned}$$

Solving the above system of ODEs one obtains the rational solution

$$\begin{aligned}
u(t,x) &= \frac{C_1}{t} + \frac{x}{3t},\\
v(t,x) &= \frac{C_2}{t} + \frac{x}{3t},\\
w(t,x) &= \frac{C_3}{t}.
\end{aligned}$$

Case 3. $C_1 = a_1, C_2 = a_2, C_3 = 0, C_4 = 0, C_5 = 1$

By solving the corresponding Lagrange system for the symmetry $a_1X_1+a_2X_2+X_5$, one obtains an invariant $z=xt^{-1/3}$ and the group-invariant solution of the form

$$u=\frac{E(z)}{(a_1+3t)^{2/3}}+\frac{a_2}{2},\quad v=\frac{F(z)}{(a_1+3t)^{2/3}}+\frac{a_2}{2},\quad w=\frac{G(z)}{(a_1+3t)^{2/3}},\tag{6.4}$$

where the functions E, F and G satisfy the following system of ODEs:

$$\begin{aligned}
&E'''(z)+3E(z)E'(z)-zE'(z)-2E(z)+3G(z)G'(z)=0,\\
&F'''(z)+3F(z)F'(z)-zF'(z)-2F(z)+3G(z)G'(z)=0,\\
&2G'''(z)+3G(z)E'(z)+3E(z)G'(z)+3G(z)F'(z)\\
&+3F(z)G'(z)-2zG'(z)-4G(z)=0.
\end{aligned}$$

Case 4. $C_1=1, C_2=0, C_3=\nu, C_4=0, C_5=0$

In this case by solving the corresponding Lagrange system for the symmetry $X_1+\nu X_3$, one obtains an invariant $z=x-\nu t$, and the group-invariant solution of the form

$$u=E(z),\quad v=F(z),\quad w=G(z),\tag{6.5}$$

where the functions E, F and G satisfy

$$E'''(z)-\nu E'(z)+3E(z)E'(z)+3G(z)G'(z)=0,\tag{6.6a}$$

$$F'''(z)-\nu F'(z)+3F(z)F'(z)+3G(z)G'(z)=0,\tag{6.6b}$$

$$\begin{aligned}
&G'''(z)+\frac{3}{2}G(z)E'(z)+\frac{3}{2}E(z)G'(z)+\frac{3}{2}G(z)F'(z)\\
&+\frac{3}{2}F(z)G'(z)-\nu G'(z)=0.
\end{aligned}\tag{6.6c}$$

6.1.2 Exact solutions using simplest equation method

Let us consider the solutions of (6.6) in the form

$$E(z)=\sum_{i=0}^{M}\mathcal{A}_i(H(z))^i,\quad F(z)=\sum_{i=0}^{M}\mathcal{B}_i(H(z))^i,\quad G(z)=\sum_{i=0}^{M}\mathcal{C}_i(H(z))^i,\tag{6.7}$$

Solutions of (6.1) using the Bernoulli equation as the simplest equation

The balancing procedure gives $M = 2$, so the solutions of (6.6) are of the form

$$E(z) = \mathcal{A}_0 + \mathcal{A}_1 H + \mathcal{A}_2 H^2, \tag{6.8a}$$

$$F(z) = \mathcal{B}_0 + \mathcal{B}_1 H + \mathcal{B}_2 H^2, \tag{6.8b}$$

$$G(z) = \mathcal{C}_0 + \mathcal{C}_1 H + \mathcal{C}_2 H^2. \tag{6.8c}$$

Substituting (6.8) into (6.6) and making use of the Bernoulli equation and then equating the coefficients of the functions H^i to zero, we obtain the following algebraic system of equations in terms of $\mathcal{A}_i$, $\mathcal{B}_i$ and $\mathcal{C}_i$ $(i = 0, 1, 2)$:

$$a^3\mathcal{A}_1 - a\mathcal{A}_1\nu + 3a\mathcal{A}_0\mathcal{A}_1 + 3a\mathcal{C}_0\mathcal{C}_1 = 0,$$

$$8a^3\mathcal{A}_2 + 7a^2\mathcal{A}_1 b - 2a\mathcal{A}_2\nu + 3a\mathcal{A}_1^2 + 6a\mathcal{A}_0\mathcal{A}_2 + 3a\mathcal{C}_1^2 + 6a\mathcal{C}_0\mathcal{C}_2 - \mathcal{A}_1 b\nu + 3\mathcal{A}_0\mathcal{A}_1 b + 3b\mathcal{C}_0\mathcal{C}_1 = 0,$$

$$38a^2\mathcal{A}_2 b + 9a\mathcal{A}_1\mathcal{A}_2 + 12a\mathcal{A}_1 b^2 + 9a\mathcal{C}_1\mathcal{C}_2 - 2\mathcal{A}_2 b\nu + 3\mathcal{A}_1^2 b + 6\mathcal{A}_0\mathcal{A}_2 b + 3b\mathcal{C}_1^2 + 6b\mathcal{C}_0\mathcal{C}_2 = 0,$$

$$6a\mathcal{A}_2^2 + 54a\mathcal{A}_2 b^2 + 6a\mathcal{C}_2^2 + 6\mathcal{A}_1 b^3 + 9\mathcal{A}_1\mathcal{A}_2 b + 9b\mathcal{C}_1\mathcal{C}_2 = 0,$$

$$24\mathcal{A}_2 b^3 + 6\mathcal{A}_2^2 b + 6b\mathcal{C}_2^2 = 0,$$

$$a^3\mathcal{B}_1 + 3a\mathcal{C}_0\mathcal{C}_1 - a\nu\mathcal{B}_1 + 3a\mathcal{B}_0\mathcal{B}_1 = 0,$$

$$8a^3\mathcal{B}_2 + 7a^2 b\mathcal{B}_1 + 3a\mathcal{C}_1^2 + 6a\mathcal{C}_0\mathcal{C}_2 - 2a\nu\mathcal{B}_2 + 3a\mathcal{B}_1^2 + 6a\mathcal{B}_0\mathcal{B}_2 + 3b\mathcal{C}_0\mathcal{C}_1 - b\nu\mathcal{B}_1 + 3b\mathcal{B}_0\mathcal{B}_1 = 0,$$

$$38a^2 b\mathcal{B}_2 + 12ab^2\mathcal{B}_1 + 9a\mathcal{C}_1\mathcal{C}_2 + 9a\mathcal{B}_1\mathcal{B}_2 + 3b\mathcal{C}_1^2 + 6b\mathcal{C}_0\mathcal{C}_2 - 2b\nu\mathcal{B}_2 + 3b\mathcal{B}_1^2 + 6b\mathcal{B}_0\mathcal{B}_2 = 0,$$

$$54ab^2\mathcal{B}_2 + 6a\mathcal{C}_2^2 + 6a\mathcal{B}_2^2 + 6b^3\mathcal{B}_1 + 9b\mathcal{C}_1\mathcal{C}_2 + 9b\mathcal{B}_1\mathcal{B}_2 = 0,$$

$$24b^3\mathcal{B}_2 + 6b\mathcal{C}_2^2 + 6b\mathcal{B}_2^2 = 0,$$

$$a^3\mathcal{C}_1 + \frac{3}{2}a\mathcal{A}_1\mathcal{C}_0 + \frac{3}{2}a\mathcal{A}_0\mathcal{C}_1 - a\mathcal{C}_1\nu + \frac{3}{2}a\mathcal{C}_0\mathcal{B}_1 + \frac{3}{2}a\mathcal{C}_1\mathcal{B}_0 = 0,$$

$$8a^3\mathcal{C}_2 + 7a^2 b\mathcal{C}_1 + 3a\mathcal{A}_2\mathcal{C}_0 + 3a\mathcal{A}_1\mathcal{C}_1 + 3a\mathcal{A}_0\mathcal{C}_2 - 2a\mathcal{C}_2\nu + 3a\mathcal{C}_0\mathcal{B}_2 + 3a\mathcal{C}_1\mathcal{B}_1$$
$$+3a\mathcal{C}_2\mathcal{B}_0 + \frac{3}{2}\mathcal{A}_1 b\mathcal{C}_0 + \frac{3}{2}\mathcal{A}_0 b\mathcal{C}_1 - b\mathcal{C}_1\nu + \frac{3}{2}b\mathcal{C}_0\mathcal{B}_1 + \frac{3}{2}b\mathcal{C}_1\mathcal{B}_0 = 0,$$

$$38a^2 b\mathcal{C}_2 + \frac{9}{2}a\mathcal{A}_2\mathcal{C}_1 + \frac{9}{2}a\mathcal{A}_1\mathcal{C}_2 + 12ab^2\mathcal{C}_1 + \frac{9}{2}a\mathcal{C}_1\mathcal{B}_2 + \frac{9}{2}a\mathcal{C}_2\mathcal{B}_1 + 3\mathcal{A}_2 b\mathcal{C}_0$$
$$+3\mathcal{A}_1 b\mathcal{C}_1 + 3\mathcal{A}_0 b\mathcal{C}_2 - 2b\mathcal{C}_2\nu + 3b\mathcal{C}_0\mathcal{B}_2 + 3b\mathcal{C}_1\mathcal{B}_1 + 3b\mathcal{C}_2\mathcal{B}_0 = 0,$$

$$6a\mathcal{A}_2\mathcal{C}_2 + 54ab^2\mathcal{C}_2 + 6a\mathcal{C}_2\mathcal{B}_2 + 6b^3\mathcal{C}_1 + \frac{9}{2}\mathcal{A}_2 b\mathcal{C}_1 + \frac{9}{2}\mathcal{A}_1 b\mathcal{C}_2 + \frac{9}{2}b\mathcal{C}_1\mathcal{B}_2 + \frac{9}{2}b\mathcal{C}_2\mathcal{B}_1 = 0,$$

$$24b^3\mathcal{C}_2 + 6\mathcal{A}_2 b\mathcal{C}_2 + 6b\mathcal{C}_2\mathcal{B}_2 = 0.$$

Solving the resultant system of algebraic equations with the aid of Mathematica, one possible set of values of $\mathcal{A}_i$, $\mathcal{B}_i$ and $\mathcal{C}_i$ $(i = 0, 1, 2)$ are

$$\begin{aligned}
\mathcal{A}_0 &= \mathcal{A}_0, \\
\mathcal{A}_1 &= \mathcal{A}_1, \\
\mathcal{A}_2 &= \frac{\mathcal{A}_1 b}{a}, \\
\mathcal{B}_0 &= \frac{-4a^2b^2 - 2a\mathcal{A}_1 b + 2\mathcal{A}_2\nu - 3\mathcal{A}_0\mathcal{A}_2 + 4b^2\nu}{3\left(\mathcal{A}_2 + 4b^2\right)}, \\
\mathcal{B}_1 &= -4ab - \mathcal{A}_1, \\
\mathcal{B}_2 &= -\mathcal{A}_2 - 4b^2, \\
\mathcal{C}_0 &= \frac{1}{3}\sqrt{\left(a^2 + 3\mathcal{A}_0 - \nu\right)\left(a^2 - \nu + 3\mathcal{B}_0\right)}, \\
\mathcal{C}_1 &= -\frac{3\mathcal{A}_1\mathcal{C}_0}{a^2 - \nu + 3\mathcal{B}_0}, \\
\mathcal{C}_2 &= \frac{b\mathcal{C}_1}{a}.
\end{aligned}$$

As a result, a solution of (6.1) is

$$\begin{aligned}
u(t,x) = \mathcal{A}_0 + \mathcal{A}_1 a &\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\} \\
+\mathcal{A}_2 a^2 &\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}^2, \qquad (6.9a) \\
v(t,x) = \mathcal{B}_0 + \mathcal{B}_1 a &\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\} \\
+\mathcal{B}_2 a^2 &\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}^2, \qquad (6.9b) \\
w(t,x) = \mathcal{C}_0 + \mathcal{C}_1 a &\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\} \\
+\mathcal{C}_2 a^2 &\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}^2, \qquad (6.9c)
\end{aligned}$$

where $z = x - \nu t$.

Solutions of (6.1) using Riccati equation as the simplest equation

The balancing procedure gives $M = 2$, so the solutions of (6.6) are of the form

$$E(z) = \mathcal{A}_0 + \mathcal{A}_1 H + \mathcal{A}_2 H^2, \qquad (6.10a)$$

$$F(z) = \mathcal{B}_0 + \mathcal{B}_1 H + \mathcal{B}_2 H^2, \qquad (6.10b)$$

$$G(z) = \mathcal{C}_0 + \mathcal{C}_1 H + \mathcal{C}_2 H^2. \qquad (6.10c)$$

Substituting (6.10) into (6.6) and making use of the Riccati equation, we obtain the following algebraic system of equations in terms of $\mathcal{A}_i$, $\mathcal{B}_i$ and $\mathcal{C}_i$ $(i = 0, 1, 2)$ by equating all coefficients of the functions H^i to zero:

$$\begin{aligned}
&2a\mathcal{A}_1c^2 + \mathcal{A}_1b^2c + 6\mathcal{A}_2bc^2 - \mathcal{A}_1c\nu + 3\mathcal{A}_0\mathcal{A}_1c + 3c\mathcal{C}_0\mathcal{C}_1 = 0,\\
&8a\mathcal{A}_1bc + 16a\mathcal{A}_2c^2 + \mathcal{A}_1b^3 + 14\mathcal{A}_2b^2c - \mathcal{A}_1b\nu + 3\mathcal{A}_0\mathcal{A}_1b\\
&+3b\mathcal{C}_0\mathcal{C}_1 - 2\mathcal{A}_2c\nu + 3\mathcal{A}_1^2c + 6\mathcal{A}_0\mathcal{A}_2c + 3c\mathcal{C}_1^2 + 6c\mathcal{C}_0\mathcal{C}_2 = 0,\\
&8a^2\mathcal{A}_1c - a\mathcal{A}_1\nu + 3a\mathcal{A}_0\mathcal{A}_1 + 7a\mathcal{A}_1b^2 + 52a\mathcal{A}_2bc + 3a\mathcal{C}_0\mathcal{C}_1 + 8\mathcal{A}_2b^3 - 2\mathcal{A}_2b\nu\\
&+3\mathcal{A}_1^2b + 6\mathcal{A}_0\mathcal{A}_2b + 3b\mathcal{C}_1^2 + 6b\mathcal{C}_0\mathcal{C}_2 + 9\mathcal{A}_1\mathcal{A}_2c + 9c\mathcal{C}_1\mathcal{C}_2 = 0,\\
&12a^2\mathcal{A}_1b + 40a^2\mathcal{A}_2c - 2a\mathcal{A}_2\nu + 3a\mathcal{A}_1^2 + 6a\mathcal{A}_0\mathcal{A}_2 + 38a\mathcal{A}_2b^2\\
&+3a\mathcal{C}_1^2 + 6a\mathcal{C}_0\mathcal{C}_2 + 9\mathcal{A}_1\mathcal{A}_2b + 9b\mathcal{C}_1\mathcal{C}_2 + 6\mathcal{A}_2^2c + 6c\mathcal{C}_2^2 = 0,\\
&6a^3\mathcal{A}_1 + 54a^2\mathcal{A}_2b + 9a\mathcal{A}_1\mathcal{A}_2 + 9a\mathcal{C}_1\mathcal{C}_2 + 6\mathcal{A}_2^2b + 6b\mathcal{C}_2^2 = 0,\\
&24a^3\mathcal{A}_2 + 6a\mathcal{A}_2^2 + 6a\mathcal{C}_2^2 = 0,\\
&2ac^2\mathcal{B}_1 + b^2c\mathcal{B}_1 + 6bc^2\mathcal{B}_2 + 3c\mathcal{C}_0\mathcal{C}_1 - c\nu\mathcal{B}_1 + 3c\mathcal{B}_0\mathcal{B}_1 = 0,\\
&8abc\mathcal{B}_1 + 16ac^2\mathcal{B}_2 + b^3\mathcal{B}_1 + 14b^2c\mathcal{B}_2 + 3b\mathcal{C}_0\mathcal{C}_1 - b\nu\mathcal{B}_1 + 3b\mathcal{B}_0\mathcal{B}_1\\
&+3c\mathcal{C}_1^2 + 6c\mathcal{C}_0\mathcal{C}_2 - 2c\nu\mathcal{B}_2 + 3c\mathcal{B}_1^2 + 6c\mathcal{B}_0\mathcal{B}_2 = 0,\\
&8a^2c\mathcal{B}_1 + 7ab^2\mathcal{B}_1 + 52abc\mathcal{B}_2 + 3a\mathcal{C}_0\mathcal{C}_1 - a\nu\mathcal{B}_1 + 3a\mathcal{B}_0\mathcal{B}_1 + 8b^3\mathcal{B}_2\\
&+3b\mathcal{C}_1^2 + 6b\mathcal{C}_0\mathcal{C}_2 - 2b\nu\mathcal{B}_2 + 3b\mathcal{B}_1^2 + 6b\mathcal{B}_0\mathcal{B}_2 + 9c\mathcal{C}_1\mathcal{C}_2 + 9c\mathcal{B}_1\mathcal{B}_2 = 0,\\
&12a^2b\mathcal{B}_1 + 40a^2c\mathcal{B}_2 + 38ab^2\mathcal{B}_2 + 3a\mathcal{C}_1^2 + 6a\mathcal{C}_0\mathcal{C}_2 - 2a\nu\mathcal{B}_2\\
&+3a\mathcal{B}_1^2 + 6a\mathcal{B}_0\mathcal{B}_2 + 9b\mathcal{C}_1\mathcal{C}_2 + 9b\mathcal{B}_1\mathcal{B}_2 + 6c\mathcal{C}_2^2 + 6c\mathcal{B}_2^2 = 0,\\
&6a^3\mathcal{B}_1 + 54a^2b\mathcal{B}_2 + 9a\mathcal{C}_1\mathcal{C}_2 + 9a\mathcal{B}_1\mathcal{B}_2 + 6b\mathcal{C}_2^2 + 6b\mathcal{B}_2^2 = 0,\\
&24a^3\mathcal{B}_2 + 6a\mathcal{C}_2^2 + 6a\mathcal{B}_2^2 = 0,\\
&2ac^2\mathcal{C}_1 + b^2c\mathcal{C}_1 + 6bc^2\mathcal{C}_2 + \frac{3}{2}\mathcal{A}_1c\mathcal{C}_0 + \frac{3}{2}\mathcal{A}_0c\mathcal{C}_1 - c\mathcal{C}_1\nu + \frac{3}{2}c\mathcal{C}_0\mathcal{B}_1 + \frac{3}{2}c\mathcal{C}_1\mathcal{B}_0 = 0,\\
&8abc\mathcal{C}_1 + 16ac^2\mathcal{C}_2 + b^3\mathcal{C}_1 + 14b^2c\mathcal{C}_2 + \frac{3}{2}\mathcal{A}_1b\mathcal{C}_0 + \frac{3}{2}\mathcal{A}_0b\mathcal{C}_1 - b\mathcal{C}_1\nu\\
&+\frac{3}{2}b\mathcal{C}_0\mathcal{B}_1 + \frac{3}{2}b\mathcal{C}_1\mathcal{B}_0 + 3\mathcal{A}_2c\mathcal{C}_0 + 3\mathcal{A}_1c\mathcal{C}_1 + 3\mathcal{A}_0c\mathcal{C}_2 - 2c\mathcal{C}_2\nu + 3c\mathcal{C}_0\mathcal{B}_2 + 3c\mathcal{C}_1\mathcal{B}_1 + 3c\mathcal{C}_2\mathcal{B}_0 = 0,
\end{aligned}$$

$$
\begin{aligned}
&8a^2c\mathcal{C}_1+\frac{3}{2}a\mathcal{A}_1\mathcal{C}_0+\frac{3}{2}a\mathcal{A}_0\mathcal{C}_1+7ab^2\mathcal{C}_1+52abc\mathcal{C}_2-a\mathcal{C}_1\nu+\frac{3}{2}a\mathcal{C}_0\mathcal{B}_1\\
&+\frac{3}{2}a\mathcal{C}_1\mathcal{B}_0+8b^3\mathcal{C}_2+3\mathcal{A}_2b\mathcal{C}_0+3\mathcal{A}_1b\mathcal{C}_1+3\mathcal{A}_0b\mathcal{C}_2-2b\mathcal{C}_2\nu+3b\mathcal{C}_0\mathcal{B}_2\\
&+3b\mathcal{C}_1\mathcal{B}_1+3b\mathcal{C}_2\mathcal{B}_0+\frac{9}{2}\mathcal{A}_2c\mathcal{C}_1+\frac{9}{2}\mathcal{A}_1c\mathcal{C}_2+\frac{9}{2}c\mathcal{C}_1\mathcal{B}_2+\frac{9}{2}c\mathcal{C}_2\mathcal{B}_1=0,\\
&12a^2b\mathcal{C}_1+40a^2c\mathcal{C}_2+3a\mathcal{A}_2\mathcal{C}_0+3a\mathcal{A}_1\mathcal{C}_1+3a\mathcal{A}_0\mathcal{C}_2+38ab^2\mathcal{C}_2\\
&-2a\mathcal{C}_2\nu+3a\mathcal{C}_0\mathcal{B}_2+3a\mathcal{C}_1\mathcal{B}_1+3a\mathcal{C}_2\mathcal{B}_0+\frac{9}{2}\mathcal{A}_2b\mathcal{C}_1\\
&+\frac{9}{2}\mathcal{A}_1b\mathcal{C}_2+\frac{9}{2}b\mathcal{C}_1\mathcal{B}_2+\frac{9}{2}b\mathcal{C}_2\mathcal{B}_1+6\mathcal{A}_2c\mathcal{C}_2+6c\mathcal{C}_2\mathcal{B}_2=0,\\
&6a^3\mathcal{C}_1+54a^2b\mathcal{C}_2+\frac{9}{2}a\mathcal{A}_2\mathcal{C}_1+\frac{9}{2}a\mathcal{A}_1\mathcal{C}_2+\frac{9}{2}a\mathcal{C}_1\mathcal{B}_2+\frac{9}{2}a\mathcal{C}_2\mathcal{B}_1+6\mathcal{A}_2b\mathcal{C}_2+6b\mathcal{C}_2\mathcal{B}_2=0,\\
&24a^3\mathcal{C}_2+6a\mathcal{A}_2\mathcal{C}_2+6a\mathcal{C}_2\mathcal{B}_2=0.
\end{aligned}
$$

Solving the above system, one possible set of values are

$$
\begin{aligned}
\mathcal{A}_0&=\mathcal{A}_0,\\
\mathcal{A}_1&=\mathcal{A}_1,\\
\mathcal{A}_2&=\frac{a\mathcal{A}_1}{b},\\
\mathcal{B}_0&=\frac{-32a^3c-4a^2b^2+4a^2\nu-2a\mathcal{A}_1b-16a\mathcal{A}_2c+2\mathcal{A}_2\nu-3\mathcal{A}_0\mathcal{A}_2}{3\left(4a^2+\mathcal{A}_2\right)},\\
\mathcal{B}_1&=-4ab-\mathcal{A}_1,\\
\mathcal{B}_2&=-4a^2-\mathcal{A}_2,\\
\mathcal{C}_0&=\frac{1}{3}\sqrt{\left(8ac+3\mathcal{A}_0+b^2-\nu\right)\left(8ac+b^2-\nu+3\mathcal{B}_0\right)},\\
\mathcal{C}_1&=-\frac{3\mathcal{A}_1\mathcal{C}_0}{8ac+b^2-\nu+3\mathcal{B}_0},\\
\mathcal{C}_2&=\frac{a\mathcal{C}_1}{b}.
\end{aligned}
$$

Hence solutions of (6.1) are

$$\begin{aligned} u(t,x) &= \mathcal{A}_0 + \mathcal{A}_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\} \\ &\quad +\mathcal{A}_2\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2, \qquad (6.11a) \\ v(t,x) &= \mathcal{B}_0 + \mathcal{B}_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\} \\ &\quad +\mathcal{B}_2\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2, \qquad (6.11b) \\ w(t,x) &= \mathcal{C}_0 + \mathcal{C}_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\} \\ &\quad +\mathcal{C}_2\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2 \qquad (6.11c) \end{aligned}$$

and

$$\begin{aligned} u(t,x) &= \mathcal{A}_0 + \mathcal{A}_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\} \\ &\quad +\mathcal{A}_2\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^2, \qquad (6.12a) \\ v(t,x) &= \mathcal{B}_0 + \mathcal{B}_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\} \\ &\quad +\mathcal{B}_2\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^2, \qquad (6.12b) \\ w(t,x) &= \mathcal{C}_0 + \mathcal{C}_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\} \\ &\quad +\mathcal{C}_2\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^2, \qquad (6.12c) \end{aligned}$$

where $z = x - \nu t$.

A profile of the solution (6.9) is given in Figure 6.1 - 6.3.

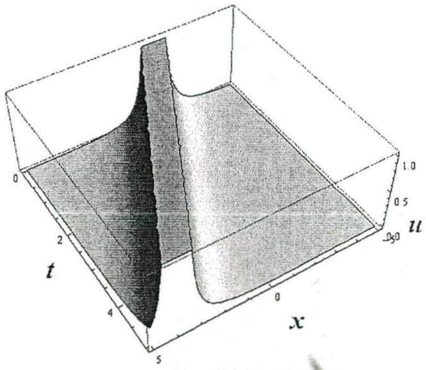

Figure 6.1: Evolution of solitary wave solution (6.9) with parameters $C_1 = 0,\ \nu = 1,\ a = 1,\ b = 3,\ c = 1,\ \mathcal{A}_0 = 0,\ \mathcal{A}_1 = 1$.

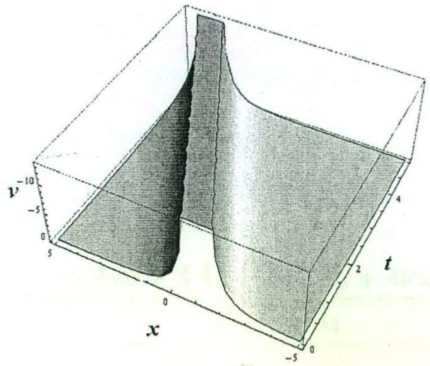

Figure 6.2: Evolution of solitary wave solution (6.9) with parameters $C_1 = 0,\ \nu = 1,\ a = 1,\ b = 3,\ c = 1,\ \mathcal{A}_0 = 0,\ \mathcal{A}_1 = 1$.

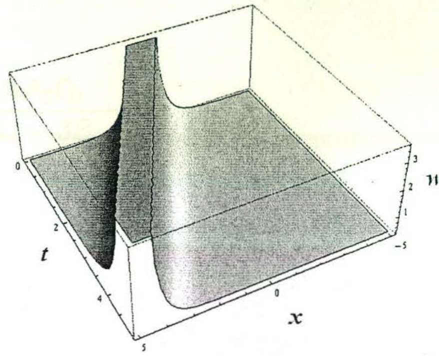

Figure 6.3: Evolution of solitary wave solution (6.9) with parameters $C_1 = 0,\ \nu = 1,\ a = -2,\ b = 1,\ c = -1,\ \mathcal{A}_0 = 0,\ \mathcal{A}_1 = 1$.

6.1.3 Solutions of (6.1) using Jacobi elliptic function method

In this subsection we use the Jacobi elliptic function method, which was used in Chapter 4, to obtain exact solutions of (6.1) and as a result we obtain the following cnoidal and snoidal wave solutions given by

$$u(t,x) = \mathcal{A}_0 + \mathcal{A}_1 \mathrm{cn}(z|\omega) + \mathcal{A}_2 \mathrm{cn}^2(z|\omega), \tag{6.13a}$$

$$v(t,x) = \mathcal{B}_0 + \mathcal{B}_1 \mathrm{cn}(z|\omega) + \mathcal{B}_2 \mathrm{cn}^2(z|\omega), \tag{6.13b}$$

$$w(t,x) = \mathcal{C}_0 + \mathcal{C}_1 \mathrm{cn}(z|\omega) + \mathcal{C}_2 \mathrm{cn}^2(z|\omega), \tag{6.13c}$$

where

$$\begin{aligned}
\mathcal{A}_0 &= \mathcal{A}_0, \\
\mathcal{A}_1 &= 0, \\
\mathcal{A}_2 &= \mathcal{A}_2, \\
\mathcal{B}_0 &= \frac{-2\mathcal{A}_2\nu + 16\mathcal{A}_2\omega + 3\mathcal{A}_0\mathcal{A}_2 - 8\mathcal{A}_2 + 4\nu\omega - 32\omega^2 + 16\omega}{12\omega - 3\mathcal{A}_2}, \\
\mathcal{B}_1 &= 0, \\
\mathcal{B}_2 &= 4\omega - \mathcal{A}_2, \\
\mathcal{C}_0 &= \frac{1}{3}\sqrt{(-3\mathcal{A}_0 + \nu - 8\omega + 4)(\nu - 8\omega - 3\mathcal{B}_0 + 4)}, \\
\mathcal{C}_1 &= 0, \\
\mathcal{C}_2 &= \frac{3\mathcal{A}_2\mathcal{C}_0}{\nu - 8\omega - 3\mathcal{B}_0 + 4}
\end{aligned}$$

and

$$u(t,x) = \mathcal{A}_0 + \mathcal{A}_1 \mathrm{sn}(z|\omega) + \mathcal{A}_2 \mathrm{sn}^2(z|\omega), \tag{6.14a}$$

$$v(t,x) = \mathcal{B}_0 + \mathcal{B}_1 \mathrm{sn}(z|\omega) + \mathcal{B}_2 \mathrm{sn}^2(z|\omega), \tag{6.14b}$$

$$w(t,x) = \mathcal{C}_0 + \mathcal{C}_1 \mathrm{sn}(z|\omega) + \mathcal{C}_2 \mathrm{sn}^2(z|\omega) \tag{6.14c}$$

with

$$
\begin{aligned}
\mathcal{A}_0 &= \mathcal{A}_0, \\
\mathcal{A}_1 &= 0, \\
\mathcal{A}_2 &= \mathcal{A}_2, \\
\mathcal{B}_0 &= \frac{2\mathcal{A}_2\nu + 8\mathcal{A}_2\omega - 3\mathcal{A}_0\mathcal{A}_2 + 8\mathcal{A}_2 + 4\nu\omega + 16\omega^2 + 16\omega}{3\left(\mathcal{A}_2 + 4\omega\right)}, \\
\mathcal{B}_1 &= 0, \\
\mathcal{B}_2 &= -\mathcal{A}_2 - 4\omega, \\
\mathcal{C}_0 &= \frac{1}{3}\sqrt{\left(-3\mathcal{A}_0 + \nu + 4\omega + 4\right)\left(\nu + 4\omega - 3\mathcal{B}_0 + 4\right)}, \\
\mathcal{C}_1 &= 0, \\
\mathcal{C}_2 &= \frac{3\mathcal{A}_2\mathcal{C}_0}{\nu + 4\omega - 3\mathcal{B}_0 + 4}
\end{aligned}
$$

and $z = x - \nu t$.

The profile of the solution (6.14) is given in Figure 6.4 - 6.6.

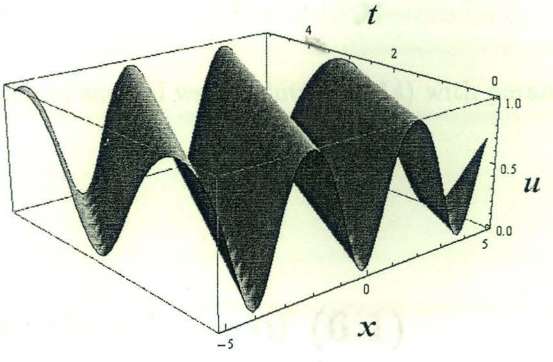

Figure 6.4: Evolution of snoidal wave solution (6.14) with parameters $\mathcal{A}_0 = 0$, $\mathcal{A}_2 = 1$, $\nu = 1$, $\omega = 0.6$.

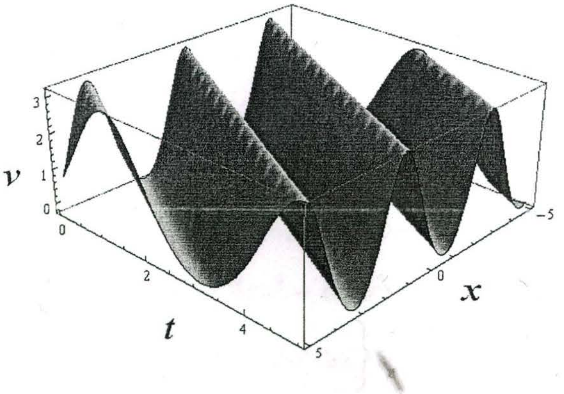

Figure 6.5: Evolution of snoidal wave solution (6.14) with parameters $\mathcal{A}_0 = 0,\ \mathcal{A}_2 = 1,\ \nu = 1,\ \omega = 0.6$.

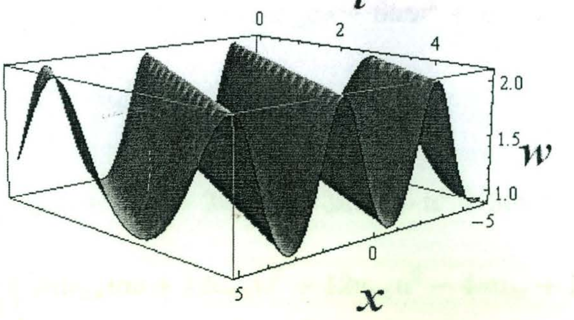

Figure 6.6: Evolution of snoidal wave solution (6.14) with parameters $\mathcal{A}_0 = 0,\ \mathcal{A}_2 = -1,\ \nu = 1,\ \omega = 0.6$.

6.2 Conservation laws of (6.1)

For the coupled KdV system (6.1), we obtain the second-order multipliers, $\Lambda_1(t, x, u, v, w, u_x, v_x, w_x, u_{xx}, v_{xx}, w_{xx})$, $\Lambda_2(t, x, u, v, w, u_x, v_x, w_x, u_{xx}, v_{xx}, w_{xx})$ and $\Lambda_3(t, x, u, v, w, u_x, v_x, w_x, u_{xx}, v_{xx}, w_{xx})$ that are given by

$$\begin{aligned} \Lambda_1 &= \frac{1}{6}C_3\left(3u^2 + 3w^2 + 2u_{xx}\right) + \frac{1}{6}(3C_1t + 3C_2)u - \frac{C_1x}{6} + C_6, && (6.15)\\ \Lambda_2 &= \frac{1}{6}C_3\left(3v^2 + 3w^2 + 2v_{xx}\right) + \frac{1}{6}(3C_1t + 3C_2)v - \frac{C_1x}{6} + C_5, && (6.16)\\ \Lambda_3 &= \frac{1}{3}w(C_3(3u + 3v) + 3C_1t + 3C_2) + \frac{2}{3}C_3w_{xx} + C_4, && (6.17) \end{aligned}$$

where C_i, $i = 1, 2, 3, 4, 5, 6$ are arbitrary constants.

Corresponding to the above multipliers we have the following first six (of infinitely many) conserved vectors of (6.1):

$$
\begin{aligned}
T_1^t &= \frac{1}{12}\Big\{3tu^2 - 2xu + 3tv^2 - 2xv + 6tw^2\Big\},\\
T_1^x &= \frac{1}{12}\Big\{6tu_{xx}u + 6tv_{xx}v + 12tw_{xx}w + 18tuw^2 + 6tu^3 - 3xu^2\\
&\quad +18tvw^2 + 6tv^3 - 3xv^2 - 6xw^2 - 3tu_x{}^2 - 3tv_x{}^2 - 6tw_x{}^2\\
&\quad +2u_x - 2xu_{xx} + 2v_x - 2xv_{xx}\Big\};
\end{aligned}
$$

$$
\begin{aligned}
T_2^t &= \frac{1}{4}\Big\{u^2 + v^2 + 2w^2\Big\},\\
T_2^x &= \frac{1}{4}\Big\{2u_{xx}u + 2v_{xx}v + 4w_{xx}w + 6uw^2 + 2u^3 + 6vw^2 + 2v^3\\
&\quad -u_x{}^2 - v_x{}^2 - 2w_x{}^2\Big\};
\end{aligned}
$$

$$
\begin{aligned}
T_3^t &= \frac{1}{6}\Big\{u_{xx}u + v_{xx}v + 2w_{xx}w + 3uw^2 + u^3 + 3vw^2 + v^3\Big\},\\
T_3^x &= \frac{1}{24}\Big\{24w_{xx}uw + 12u_{xx}w^2 + 12u_{xx}u^2 - 4uu_{tx} + 12v_{xx}w^2\\
&\quad +24w_{xx}vw + 12v_{xx}v^2 - 4vv_{tx} - 8ww_{tx} + 36uvw^2 + 36u^2w^2\\
&\quad +9u^4 + 36v^2w^2 + 9v^4 + 18w^4 + 4u_tu_x + 4v_tv_x + 8w_tw_x\\
&\quad +4u_{xx}{}^2 + 4v_{xx}{}^2 + 8w_{xx}{}^2\Big\};
\end{aligned}
$$

$$
\begin{aligned}
T_4^t &= w,\\
T_4^x &= \frac{1}{2}\Big\{3uw + 3vw + 2w_{xx}\Big\};
\end{aligned}
$$

$$
\begin{aligned}
T_5^t &= v,\\
T_5^x &= \frac{1}{2}\Big\{3v^2 + 3w^2 + 2v_{xx}\Big\};
\end{aligned}
$$

$$
\begin{aligned}
T_6^t &= u,\\
T_6^x &= \frac{1}{2}\Big\{3u^2 + 3w^2 + 2u_{xx}\Big\}.
\end{aligned}
$$

Remark We note that the last three conserved quantities can be obtained directly from the equations (6.1) by inspection and also higher order conservation laws of (6.1) can be computed by increasing the order of the multipliers.

6.3 Conclusion

In this chapter Lie symmetry analysis was employed to study a new coupled KdV system. We obtained similarity reductions and exact solutions with the aid of simplest equation and Jacobi elliptic function methods. The exact solutions obtained were solitary, cnoidal and snoidal waves. The conservation laws of the coupled KdV system were constructed using the multiplier method.

Chapter 7

New exact solutions and conservation laws of a coupled Kadomtsev-Petviashvili system

7.1 Introduction

In Chapter 6 we studied the coupled KdV system

$$u_t + u_{xxx} + 3uu_x + 3ww_x = 0, \tag{7.1a}$$

$$v_t + v_{xxx} + 3vv_x + 3ww_x = 0, \tag{7.1b}$$

$$w_t + w_{xxx} + \frac{3}{2}(uw)_x + \frac{3}{2}(vw)_x = 0. \tag{7.1c}$$

In this chapter we consider the above system formulated in the KP sense, which is given by [62]

$$\left(u_t + u_{xxx} + 3uu_x + 3ww_x\right)_x + u_{yy} = 0, \tag{7.2a}$$

$$\left(v_t + v_{xxx} + 3vv_x + 3ww_x\right)_x + v_{yy} = 0, \tag{7.2b}$$

$$\left(w_t + w_{xxx} + \frac{3}{2}(uw)_x + \frac{3}{2}(vw)_x\right)_x + w_{yy} = 0 \tag{7.2c}$$

and obtain exact solutions of the system. We also derive conservation laws for (7.2).

This work has been published. See [68].

7.2 Symmetries and exact solutions of (7.2)

The symmetry group of the coupled KP system (7.2) will be generated by the vector field of the form

$$\begin{aligned} X &= \xi^1(t,x,y,u,v,w)\frac{\partial}{\partial t} + \xi^2(t,x,y,u,v,w)\frac{\partial}{\partial x} + \xi^3(t,x,y,u,v,w)\frac{\partial}{\partial y} \\ &\quad + \eta^1(t,x,y,u,v,w)\frac{\partial}{\partial u} + \eta^2(t,x,y,u,v,w)\frac{\partial}{\partial v} + \eta^3(t,x,y,u,v,w)\frac{\partial}{\partial w}. \end{aligned}$$

By applying the fourth prolongation $\mathrm{pr}^{(4)}X$ [7] to (7.2), we obtain an overdetermined system of linear partial differential equations (PDEs),

$$\begin{aligned}
&\eta^2_u = 0,\ \eta^1_v = 0,\ \xi^1_y = 0,\ \xi^3_x = 0,\ \xi^2_u = 0,\ \xi^2_v = 0,\ \xi^1_x = 0,\ \xi^3_u = 0,\ \xi^3_w = 0,\\
&\xi^3_v = 0,\ \xi^1_u = 0,\ \xi^1_w = 0,\ \xi^1_v = 0,\ \xi^2_w = 0,\ \eta^3_{uw} = 0,\ \eta^3_{uu} = 0,\ \eta^3_{uv} = 0,\ \eta^3_{vv} = 0,\\
&\eta^3_{vw} = 0,\ \eta^3_{ww} = 0,\ \eta^2_{xw} = 0,\ \eta^2_{vv} = 0,\ \eta^2_{vw} = 0,\ \eta^2_{ww} = 0,\ \eta^1_{xw} = 0,\ \eta^1_{uu} = 0,\\
&\eta^1_{uw} = 0,\ \eta^2_{xv} = 0,\ \eta^1_{xu} = 0,\ \xi^2_{xx} = 0,\ \eta^3_{xw} = 0,\ \eta^3_{yu} = 0,\ \eta^3_{yv} = 0,\ \eta^2_{yw} = 0,\\
&\eta^1_{yw} = 0,\ \eta^3_{xu} = 0,\ \eta^3_{xv} = 0,\ \eta^1_{ww} = 0,\ 2\eta^3_v - \eta^1_w = 0,\ \eta^2_w - 2\eta^3_u = 0,\ 2\eta^3_v - \eta^1_w = 0,\\
&2\eta^3_v + \eta^2_w = 0,\ \eta^3_u + \eta^3_v = 0,\ 2\eta^3_u - \eta^2_w = 0,\ 2\eta^3_u + \eta^1_w = 0,\ 6\eta^3_x + \eta^1_{tw} = 0,\\
&6\eta^3_x + \eta^2_{tw} = 0,\ 3\eta^3_x + \eta^3_{tv} = 0,\ 3\eta^3_x + \eta^3_{tu} = 0,\ -\xi^1_t + 3\xi^2_x = 0,\ 2\xi^2_x + \eta^1_u = 0,\\
&2\xi^2_x + \eta^2_v = 0,\ -\xi^3_y + 2\xi^2_x = 0,\ -\xi^3_{yy} + 2\eta^3_{yw} = 0,\ -\xi^3_{yy} + 2\eta^1_{yu} = 0,\\
&-\xi^3_{yy} + 2\eta^2_{yv} = 0,\ -\xi^3_t - 2\xi^2_y = 0,\ 2\xi^2_x - \eta^1_u + 2\eta^3_w = 0,\ 2\xi^2_x - \eta^2_v + 2\eta^3_w = 0,\\
&6\eta^1_x - \xi^2_{tx} + \eta^1_{tu} - \xi^2_{yy} = 0,\ 6\eta^2_x - \xi^2_{tx} + \eta^2_{tv} - \xi^2_{yy} = 0,\ -\eta^3_u - \eta^3_v + \eta^1_w + \eta^2_w = 0,\\
&12v\xi^2_x - 3\eta^2_w w + 6w\eta^3_v + 6\eta_2 - 2\xi^2_t = 0,\ 12u\xi^2_x - 3\eta^1_w w + 6w\eta^3_u + 6\eta_1 - 2\xi^2_t = 0,\\
&3u\eta^1_{xx} + 3w\eta^3_{xx} + \eta^1_{tx} + \eta^1_{yy} + \eta^1_{xxxx} = 0,\ 3v\eta^2_{xx} + 3w\eta^3_{xx} + \eta^2_{tx} + \eta^2_{yy} + \eta^2_{xxxx} = 0,\\
&3\eta^1_x + 3\eta^2_x - \xi^2_{tx} + \eta^3_{tw} - \xi^2_{yy} = 0,\ -\eta^3_u u + \eta^3_u v + \eta^1_u w - w\eta^3_w + 2w\xi^2_x + \eta_3 = 0,\\
&\eta^3_v u - \eta^3_v v + \eta^2_v w - w\eta^3_w + 2w\xi^2_x + \eta_3 = 0,\\
&\eta^1_w u - \eta^1_w v - 2\eta^1_u w + 2w\eta^3_w + 4w\xi^2_x + 2\eta_3 = 0,\\
&-\eta^2_w u + \eta^2_w v - 2\eta^2_v w + 2w\eta^3_w + 4w\xi^2_x + 2\eta_3 = 0,
\end{aligned}$$

$$3\eta^3_{xx}u + 3\eta^3_{xx}v + 3w\eta^1_{xx} + 3w\eta^2_{xx} + 2\eta^3_{tx} + 2\eta^3_{yy} + 2\eta^3_{xxxx} = 0,$$
$$6u\xi^2_x + 6v\xi^2_x + 3\eta^1_w w + 3\eta^2_w w - 6w\eta^3_u - 6w\eta^3_v + 3\eta_1 + 3\eta_2 - 2\xi^2_t = 0.$$

The general solution of the above overdetermined system of linear PDEs is given by

$$\begin{aligned}
\xi^1(t,x,y,u,v,w) &= 18F_1(t),\\
\xi^2(t,x,y,u,v,w) &= -3y^2F_1''(t) + 3F_3(t) - 3yF_2'(t) + 6xF_1'(t),\\
\xi^3(t,x,y,u,v,w) &= 12yF_1'(t) + 6F_2(t),\\
\eta^1(t,x,y,u,v,w) &= -2C_1w - y^2F_1'''(t) + F_3(t) - yF_2''(t) - 12uF_1'(t) + 2xF_1''(t),\\
\eta^2(t,x,y,u,v,w) &= -y^2F_1''''(t) + F_3'(t) - yF_2''(t) + 2wC_1 - 12vF_1'(t) + 2xF_1''(t),\\
\eta^3(t,x,y,u,v,w) &= (u-v)C_1 - 12wF_1'(t),
\end{aligned}$$

where C_1 is an arbitrary constant and F_1, F_2 and F_3 are arbitrary functions of t. For simplicity we restrict these arbitrary functions to arbitrary constants C_2, C_3 and C_4 respectively. As a result we obtain the 4-dimensional Lie algebra spanned by the following linearly independent operators:

$$\begin{aligned}
\Gamma_1 &= \frac{\partial}{\partial t},\\
\Gamma_2 &= \frac{\partial}{\partial x},\\
\Gamma_3 &= \frac{\partial}{\partial y},\\
\Gamma_4 &= -2w\frac{\partial}{\partial u} + 2w\frac{\partial}{\partial v} + (u-v)\frac{\partial}{\partial w}.
\end{aligned}$$

7.2.1 Symmetry reductions of (7.2)

We now make use of the symmetry $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$ and reduce (7.2) to a system of nonlinear PDEs in two independent variables. The symmetry Γ yields the following five invariants:

$$f = t - y, \quad g = x - y, \quad \phi = u, \quad \psi = v, \quad \theta = w. \tag{7.3}$$

Treating ϕ, ψ, θ as the new dependent variables and f, g as new independent variables, (3) transforms to

$$\begin{aligned}
&3\phi_{fg} + \phi_{gggg} + 3{\phi_g}^2 + \phi_{gg} + 3\theta_{gg}\theta + \phi_{ff} + 3\phi_{gg}\phi + 3{\theta_g}^2 = 0,\\
&3\psi_{fg} + \psi_{gggg} + 3{\psi_g}^2 + \psi_{gg} + 3\theta_{gg}\theta + \psi_{ff} + 3\psi_{gg}\psi + 3{\theta_g}^2 = 0,\\
&3\theta_{fg} + \theta_{gggg} + 3\psi_g\theta_g + \frac{3}{2}\phi_{gg}\theta + \theta_{ff} + 3\phi_g\theta_g + \frac{3}{2}\psi_{gg}\theta\\
&+\frac{3}{2}\theta_{gg}\phi + \frac{3}{2}\theta_{gg}\psi + \theta_{gg} = 0,
\end{aligned}$$

which is a system of nonlinear PDEs in two independent variables f and g.

We now find Lie point symmetries of the above system and use them to reduce it to a system of ordinary differential equations (ODEs). The above system has two translation symmetries, viz.,

$$\begin{aligned}
\Upsilon_1 &= \frac{\partial}{\partial g}\\
\Upsilon_2 &= \frac{\partial}{\partial f}.
\end{aligned}$$

By taking a linear combination $\Upsilon_1 + \rho\Upsilon_2$ (ρ is a constant) of the above symmetries, we see that it yields the four invariants

$$z = f - \rho g, \quad \phi = E, \quad \psi = F, \quad \theta = H.$$

Now treating E,F,H as new dependent variables and z as the new independent variable the above system transforms to the following system of nonlinear coupled ODEs:

$$\begin{aligned}
&\rho^4 E''''(z) + 3\rho^2 E(z)E''(z) + \rho^2 E''(z) - 3\rho E''(z) + E''(z)\\
&+3\rho^2 E'(z)^2 + 3\rho^2 H(z)H''(z) + 3\rho^2 H'(z)^2 = 0, \qquad (7.4\text{a})\\
&\rho^4 F''''(z) + 3\rho^2 F(z)F''(z) + \rho^2 F''(z) - 3\rho F''(z)\\
&+F''(z) + 3\rho^2 F'(z)^2 + 3\rho^2 H(z)H''(z) + 3\rho^2 H'(z)^2 = 0, \qquad (7.4\text{b})\\
&\rho^4 H''''(z) + \frac{3}{2}\rho^2 H(z)E''(z) + 3\rho^2 E'(z)H'(z) + \frac{3}{2}\rho^2 E(z)H''(z)\\
&+\frac{3}{2}\rho^2 H(z)F''(z) + 3\rho^2 F'(z)H'(z) + \frac{3}{2}\rho^2 F(z)H''(z) + \rho^2 H''(z)\\
&-3\rho H''(z) + H''(z) = 0. \qquad (7.4\text{c})
\end{aligned}$$

In the next two subsections we shall solve the above system of ODEs.

7.2.2 Solutions of (7.2) using (G'/G)-expansion method

In this subsection we use the $(G'/G)-$expansion method [82] and obtain some exact solutions of the system of ODEs (7.4). This will result in the exact solutions of the coupled KP system (7.2).

Let us consider the solution of (7.4) in the form

$$E(z)=\sum_{i=0}^{M}\mathcal{A}_i\left(\frac{G'(z)}{G(z)}\right)^i, F(z)=\sum_{i=0}^{M}\mathcal{B}_i\left(\frac{G'(z)}{G(z)}\right)^i, H(z)=\sum_{i=0}^{M}\mathcal{C}_i\left(\frac{G'(z)}{G(z)}\right)^i, \quad (7.5)$$

where $G(z)$ satisfies the linear second-order ODE with constant coefficients, viz.,

$$G''+\lambda G'+\mu G=0, \quad (7.6)$$

where λ and μ are constants. The positive integer M will be determined by the homogeneous balance method between the highest order derivative and highest order nonlinear term appearing in (7.4). $\mathcal{A}_0,\cdots,\mathcal{A}_M$, $\mathcal{B}_0,\cdots,\mathcal{B}_M$ and $\mathcal{C}_0,\cdots,\mathcal{C}_M$, are parameters to be determined.

Application of the balancing procedure to the system of ODEs, yields $M=2$, so the solutions of (7.4) are of the form

$$E(z)=\mathcal{A}_0+\mathcal{A}_1\left(\frac{G'(z)}{G(z)}\right)+\mathcal{A}_2\left(\frac{G'(z)}{G(z)}\right)^2, \quad (7.7a)$$

$$F(z)=\mathcal{B}_0+\mathcal{B}_1\left(\frac{G'(z)}{G(z)}\right)+\mathcal{B}_2\left(\frac{G'(z)}{G(z)}\right)^2, \quad (7.7b)$$

$$H(z)=\mathcal{C}_0+\mathcal{C}_1\left(\frac{G'(z)}{G(z)}\right)+\mathcal{C}_2\left(\frac{G'(z)}{G(z)}\right)^2. \quad (7.7c)$$

Substituting (7.7) into (7.4) and making use of (7.6) leads to the following overdetermined system of algebraic equations:

$$
\begin{aligned}
&\mathcal{A}_1\lambda^3\mu\rho^4 + 14\mathcal{A}_2\lambda^2\mu^2\rho^4 + 8\mathcal{A}_1\lambda\mu^2\rho^4 + \mathcal{A}_1\lambda\mu\rho^2 + 3\mathcal{A}_0\mathcal{A}_1\lambda\mu\rho^2 - 3\mathcal{A}_1\lambda\mu\rho \\
&+\mathcal{A}_1\lambda\mu + 16\mathcal{A}_2\mu^3\rho^4 + 3\mathcal{A}_1^2\mu^2\rho^2 + 2\mathcal{A}_2\mu^2\rho^2 + 6\mathcal{A}_0\mathcal{A}_2\mu^2\rho^2 \\
&-6\mathcal{A}_2\mu^2\rho + 2\mathcal{A}_2\mu^2 + 3\mathcal{C}_0\mathcal{C}_1\lambda\mu\rho^2 + 3\mathcal{C}_1^2\mu^2\rho^2 + 6\mathcal{C}_0\mathcal{C}_2\mu^2\rho^2 = 0, \\
&\mathcal{A}_1\lambda^4\rho^4 + 30\mathcal{A}_2\lambda^3\mu\rho^4 + 22\mathcal{A}_1\lambda^2\mu\rho^4 + \mathcal{A}_1\lambda^2\rho^2 + 3\mathcal{A}_0\mathcal{A}_1\lambda^2\rho^2 - 3\mathcal{A}_1\lambda^2\rho \\
&+\mathcal{A}_1\lambda^2 + 120\mathcal{A}_2\lambda\mu^2\rho^4 + 9\mathcal{A}_1^2\lambda\mu\rho^2 + 6\mathcal{A}_2\lambda\mu\rho^2 + 18\mathcal{A}_0\mathcal{A}_2\lambda\mu\rho^2 - 18\mathcal{A}_2\lambda\mu\rho \\
&+6\mathcal{A}_2\lambda\mu + 16\mathcal{A}_1\mu^2\rho^4 + 18\mathcal{A}_1\mathcal{A}_2\mu^2\rho^2 + 2\mathcal{A}_1\mu\rho^2 + 6\mathcal{A}_0\mathcal{A}_1\mu\rho^2 - 6\mathcal{A}_1\mu\rho \\
&+2\mathcal{A}_1\mu + 3\mathcal{C}_0\mathcal{C}_1\lambda^2\rho^2 + 9\mathcal{C}_1^2\lambda\mu\rho^2 + 18\mathcal{C}_0\mathcal{C}_2\lambda\mu\rho^2 + 18\mathcal{C}_1\mathcal{C}_2\mu^2\rho^2 + 6\mathcal{C}_0\mathcal{C}_1\mu\rho^2 = 0, \\
&16\mathcal{A}_2\lambda^4\rho^4 + 15\mathcal{A}_1\lambda^3\rho^4 + 232\mathcal{A}_2\lambda^2\mu\rho^4 + 6\mathcal{A}_1^2\lambda^2\rho^2 + 4\mathcal{A}_2\lambda^2\rho^2 + 12\mathcal{A}_0\mathcal{A}_2\lambda^2\rho^2 \\
&-12\mathcal{A}_2\lambda^2\rho + 4\mathcal{A}_2\lambda^2 + 60\mathcal{A}_1\lambda\mu\rho^4 + 45\mathcal{A}_1\mathcal{A}_2\lambda\mu\rho^2 + 3\mathcal{A}_1\lambda\rho^2 + 9\mathcal{A}_0\mathcal{A}_1\lambda\rho^2 \\
&-9\mathcal{A}_1\lambda\rho + 3\mathcal{A}_1\lambda + 136\mathcal{A}_2\mu^2\rho^4 + 18\mathcal{A}_2^2\mu^2\rho^2 + 12\mathcal{A}_1^2\mu\rho^2 + 8\mathcal{A}_2\mu\rho^2 + 24\mathcal{A}_0\mathcal{A}_2\mu\rho^2 \\
&-24\mathcal{A}_2\mu\rho + 8\mathcal{A}_2\mu + 6\mathcal{C}_1^2\lambda^2\rho^2 + 12\mathcal{C}_0\mathcal{C}_2\lambda^2\rho^2 + 45\mathcal{C}_1\mathcal{C}_2\lambda\mu\rho^2 + 9\mathcal{C}_0\mathcal{C}_1\lambda\rho^2 \\
&+18\mathcal{C}_2^2\mu^2\rho^2 + 12\mathcal{C}_1^2\mu\rho^2 + 24\mathcal{C}_0\mathcal{C}_2\mu\rho^2 = 0, \\
&130\mathcal{A}_2\lambda^3\rho^4 + 50\mathcal{A}_1\lambda^2\rho^4 + 27\mathcal{A}_1\mathcal{A}_2\lambda^2\rho^2 + 440\mathcal{A}_2\lambda\mu\rho^4 + 42\mathcal{A}_2^2\lambda\mu\rho^2 + 15\mathcal{A}_1^2\lambda\rho^2 \\
&+10\mathcal{A}_2\lambda\rho^2 + 30\mathcal{A}_0\mathcal{A}_2\lambda\rho^2 - 30\mathcal{A}_2\lambda\rho + 10\mathcal{A}_2\lambda + 40\mathcal{A}_1\mu\rho^4 + 54\mathcal{A}_1\mathcal{A}_2\mu\rho^2 + 6\mathcal{A}_0\mathcal{A}_1\rho^2 \\
&+2\mathcal{A}_1\rho^2 - 6\mathcal{A}_1\rho + 2\mathcal{A}_1 + 27\mathcal{C}_1\mathcal{C}_2\lambda^2\rho^2 + 42\mathcal{C}_2^2\lambda\mu\rho^2 + 15\mathcal{C}_1^2\lambda\rho^2 + 30\mathcal{C}_0\mathcal{C}_2\lambda\rho^2 \\
&+54\mathcal{C}_1\mathcal{C}_2\mu\rho^2 + 6\mathcal{C}_0\mathcal{C}_1\rho^2 = 0, \\
&330\mathcal{A}_2\lambda^2\rho^4 + 24\mathcal{A}_2^2\lambda^2\rho^2 + 60\mathcal{A}_1\lambda\rho^4 + 63\mathcal{A}_1\mathcal{A}_2\lambda\rho^2 + 240\mathcal{A}_2\mu\rho^4 + 48\mathcal{A}_2^2\mu\rho^2 + 9\mathcal{A}_1^2\rho^2 \\
&+18\mathcal{A}_0\mathcal{A}_2\rho^2 + 6\mathcal{A}_2\rho^2 - 18\mathcal{A}_2\rho + 6\mathcal{A}_2 + 24\mathcal{C}_2^2\lambda^2\rho^2 + 63\mathcal{C}_1\mathcal{C}_2\lambda\rho^2 + 48\mathcal{C}_2^2\mu\rho^2 \\
&+9\mathcal{C}_1^2\rho^2 + 18\mathcal{C}_0\mathcal{C}_2\rho^2 = 0, \\
&336\mathcal{A}_2\lambda\rho^4 + 54\mathcal{A}_2^2\lambda\rho^2 + 24\mathcal{A}_1\rho^4 + 36\mathcal{A}_1\mathcal{A}_2\rho^2 + 54\mathcal{C}_2^2\lambda\rho^2 + 36\mathcal{C}_1\mathcal{C}_2\rho^2 = 0, \\
&120\mathcal{A}_2\rho^4 + 30\mathcal{A}_2^2\rho^2 + 30\mathcal{C}_2^2\rho^2 = 0,
\end{aligned}
$$

$$3\mathcal{C}_0\mathcal{C}_1\lambda\mu\rho^2 + 3\mathcal{C}_1^2\mu^2\rho^2 + 6\mathcal{C}_0\mathcal{C}_2\mu^2\rho^2 + \lambda^3\mu\rho^4\mathcal{B}_1 + 14\lambda^2\mu^2\rho^4\mathcal{B}_2 + 8\lambda\mu^2\rho^4\mathcal{B}_1$$

$$+\lambda\mu\rho^2\mathcal{B}_1 + 3\lambda\mu\rho^2\mathcal{B}_0\mathcal{B}_1 - 3\lambda\mu\rho\mathcal{B}_1 + \lambda\mu\mathcal{B}_1 + 16\mu^3\rho^4\mathcal{B}_2 + 3\mu^2\rho^2\mathcal{B}_1^2$$

$$+2\mu^2\rho^2\mathcal{B}_2 + 6\mu^2\rho^2\mathcal{B}_0\mathcal{B}_2 - 6\mu^2\rho\mathcal{B}_2 + 2\mu^2\mathcal{B}_2 = 0,$$

$$3\mathcal{C}_0\mathcal{C}_1\lambda^2\rho^2 + 9\mathcal{C}_1^2\lambda\mu\rho^2 + 18\mathcal{C}_0\mathcal{C}_2\lambda\mu\rho^2 + 18\mathcal{C}_1\mathcal{C}_2\mu^2\rho^2 + 6\mathcal{C}_0\mathcal{C}_1\mu\rho^2 + \lambda^4\rho^4\mathcal{B}_1$$

$$+30\lambda^3\mu\rho^4\mathcal{B}_2 + 22\lambda^2\mu\rho^4\mathcal{B}_1 + \lambda^2\rho^2\mathcal{B}_1 + 3\lambda^2\rho^2\mathcal{B}_0\mathcal{B}_1 - 3\lambda^2\rho\mathcal{B}_1 + \lambda^2\mathcal{B}_1 + 120\lambda\mu^2\rho^4\mathcal{B}_2$$

$$+9\lambda\mu\rho^2\mathcal{B}_1^2 + 6\lambda\mu\rho^2\mathcal{B}_2 + 18\lambda\mu\rho^2\mathcal{B}_0\mathcal{B}_2 - 18\lambda\mu\rho\mathcal{B}_2 + 6\lambda\mu\mathcal{B}_2 + 16\mu^2\rho^4\mathcal{B}_1 + 18\mu^2\rho^2\mathcal{B}_1\mathcal{B}_2$$

$$+2\mu\rho^2\mathcal{B}_1 + 6\mu\rho^2\mathcal{B}_0\mathcal{B}_1 - 6\mu\rho\mathcal{B}_1 + 2\mu\mathcal{B}_1 = 0,$$

$$6\mathcal{C}_1^2\lambda^2\rho^2 + 12\mathcal{C}_0\mathcal{C}_2\lambda^2\rho^2 + 45\mathcal{C}_1\mathcal{C}_2\lambda\mu\rho^2 + 9\mathcal{C}_0\mathcal{C}_1\lambda\rho^2 + 18\mathcal{C}_2^2\mu^2\rho^2 + 12\mathcal{C}_1^2\mu\rho^2$$

$$+24\mathcal{C}_0\mathcal{C}_2\mu\rho^2 + 16\lambda^4\rho^4\mathcal{B}_2 + 15\lambda^3\rho^4\mathcal{B}_1 + 232\lambda^2\mu\rho^4\mathcal{B}_2 + 6\lambda^2\rho^2\mathcal{B}_1^2 + 4\lambda^2\rho^2\mathcal{B}_2$$

$$+12\lambda^2\rho^2\mathcal{B}_0\mathcal{B}_2 - 12\lambda^2\rho\mathcal{B}_2 + 4\lambda^2\mathcal{B}_2 + 60\lambda\mu\rho^4\mathcal{B}_1 + 45\lambda\mu\rho^2\mathcal{B}_1\mathcal{B}_2 + 3\lambda\rho^2\mathcal{B}_1$$

$$+9\lambda\rho^2\mathcal{B}_0\mathcal{B}_1 - 9\lambda\rho\mathcal{B}_1 + 3\lambda\mathcal{B}_1 + 136\mu^2\rho^4\mathcal{B}_2 + 18\mu^2\rho^2\mathcal{B}_2^2 + 12\mu\rho^2\mathcal{B}_1^2$$

$$+8\mu\rho^2\mathcal{B}_2 + 24\mu\rho^2\mathcal{B}_0\mathcal{B}_2 - 24\mu\rho\mathcal{B}_2 + 8\mu\mathcal{B}_2 = 0,$$

$$27\mathcal{C}_1\mathcal{C}_2\lambda^2\rho^2 + 42\mathcal{C}_2^2\lambda\mu\rho^2 + 15\mathcal{C}_1^2\lambda\rho^2 + 30\mathcal{C}_0\mathcal{C}_2\lambda\rho^2 + 54\mathcal{C}_1\mathcal{C}_2\mu\rho^2 + 6\mathcal{C}_0\mathcal{C}_1\rho^2$$

$$+130\lambda^3\rho^4\mathcal{B}_2 + 50\lambda^2\rho^4\mathcal{B}_1 + 27\lambda^2\rho^2\mathcal{B}_1\mathcal{B}_2 + 440\lambda\mu\rho^4\mathcal{B}_2 + 42\lambda\mu\rho^2\mathcal{B}_2^2 + 15\lambda\rho^2\mathcal{B}_1^2$$

$$+10\lambda\rho^2\mathcal{B}_2 + 30\lambda\rho^2\mathcal{B}_0\mathcal{B}_2 - 30\lambda\rho\mathcal{B}_2 + 10\lambda\mathcal{B}_2 + 40\mu\rho^4\mathcal{B}_1 + 54\mu\rho^2\mathcal{B}_1\mathcal{B}_2 + 6\rho^2\mathcal{B}_0\mathcal{B}_1$$

$$+2\rho^2\mathcal{B}_1 - 6\rho\mathcal{B}_1 + 2\mathcal{B}_1 = 0,$$

$$24\mathcal{C}_2^2\lambda^2\rho^2 + 63\mathcal{C}_1\mathcal{C}_2\lambda\rho^2 + 48\mathcal{C}_2^2\mu\rho^2 + 9\mathcal{C}_1^2\rho^2 + 18\mathcal{C}_0\mathcal{C}_2\rho^2 + 330\lambda^2\rho^4\mathcal{B}_2 + 24\lambda^2\rho^2\mathcal{B}_2^2$$

$$+60\lambda\rho^4\mathcal{B}_1 + 63\lambda\rho^2\mathcal{B}_1\mathcal{B}_2 + 240\mu\rho^4\mathcal{B}_2 + 48\mu\rho^2\mathcal{B}_2^2 + 9\rho^2\mathcal{B}_1^2 + 18\rho^2\mathcal{B}_0\mathcal{B}_2$$

$$+6\rho^2\mathcal{B}_2 - 18\rho\mathcal{B}_2 + 6\mathcal{B}_2 = 0,$$

$$54\mathcal{C}_2^2\lambda\rho^2 + 36\mathcal{C}_1\mathcal{C}_2\rho^2 + 336\lambda\rho^4\mathcal{B}_2 + 54\lambda\rho^2\mathcal{B}_2^2 + 24\rho^4\mathcal{B}_1 + 36\rho^2\mathcal{B}_1\mathcal{B}_2 = 0,$$

$$30\mathcal{C}_2^2\rho^2 + 120\rho^4\mathcal{B}_2 + 30\rho^2\mathcal{B}_2^2 = 0,$$

$$\frac{3}{2}\mathcal{A}_1\mathcal{C}_0\lambda\mu\rho^2 + \frac{3}{2}\mathcal{A}_0\mathcal{C}_1\lambda\mu\rho^2 + 3\mathcal{A}_2\mathcal{C}_0\mu^2\rho^2 + 3\mathcal{A}_1\mathcal{C}_1\mu^2\rho^2 + 3\mathcal{A}_0\mathcal{C}_2\mu^2\rho^2 + \mathcal{C}_1\lambda^3\mu\rho^4$$

$$+14\mathcal{C}_2\lambda^2\mu^2\rho^4 + 8\mathcal{C}_1\lambda\mu^2\rho^4 + \mathcal{C}_1\lambda\mu\rho^2 - 3\mathcal{C}_1\lambda\mu\rho + \mathcal{C}_1\lambda\mu + 16\mathcal{C}_2\mu^3\rho^4 + 2\mathcal{C}_2\mu^2\rho^2$$

$$-6\mathcal{C}_2\mu^2\rho + 2\mathcal{C}_2\mu^2 + \frac{3}{2}\mathcal{C}_0\lambda\mu\rho^2\mathcal{B}_1 + \frac{3}{2}\mathcal{C}_1\lambda\mu\rho^2\mathcal{B}_0 + 3\mathcal{C}_0\mu^2\rho^2\mathcal{B}_2 + 3\mathcal{C}_1\mu^2\rho^2\mathcal{B}_1$$

$$+3\mathcal{C}_2\mu^2\rho^2\mathcal{B}_0 = 0,$$

$$\frac{3}{2}\mathcal{A}_1\mathcal{C}_0\lambda^2\rho^2+\frac{3}{2}\mathcal{A}_0\mathcal{C}_1\lambda^2\rho^2+9\mathcal{A}_2\mathcal{C}_0\lambda\mu\rho^2+9\mathcal{A}_1\mathcal{C}_1\lambda\mu\rho^2+9\mathcal{A}_0\mathcal{C}_2\lambda\mu\rho^2+9\mathcal{A}_2\mathcal{C}_1\mu^2\rho^2$$

$$+9\mathcal{A}_1\mathcal{C}_2\mu^2\rho^2+3\mathcal{A}_1\mathcal{C}_0\mu\rho^2+3\mathcal{A}_0\mathcal{C}_1\mu\rho^2+\mathcal{C}_1\lambda^4\rho^4+30\mathcal{C}_2\lambda^3\mu\rho^4+22\mathcal{C}_1\lambda^2\mu\rho^4$$

$$+\mathcal{C}_1\lambda^2\rho^2-3\mathcal{C}_1\lambda^2\rho+\mathcal{C}_1\lambda^2+120\mathcal{C}_2\lambda\mu^2\rho^4+6\mathcal{C}_2\lambda\mu\rho^2-18\mathcal{C}_2\lambda\mu\rho+6\mathcal{C}_2\lambda\mu$$

$$+16\mathcal{C}_1\mu^2\rho^4+2\mathcal{C}_1\mu\rho^2-6\mathcal{C}_1\mu\rho+2\mathcal{C}_1\mu+\frac{3}{2}\mathcal{C}_0\lambda^2\rho^2\mathcal{B}_1+\frac{3}{2}\mathcal{C}_1\lambda^2\rho^2\mathcal{B}_0+9\mathcal{C}_0\lambda\mu\rho^2\mathcal{B}_2$$

$$+9\mathcal{C}_1\lambda\mu\rho^2\mathcal{B}_1+9\mathcal{C}_2\lambda\mu\rho^2\mathcal{B}_0+9\mathcal{C}_1\mu^2\rho^2\mathcal{B}_2+9\mathcal{C}_2\mu^2\rho^2\mathcal{B}_1+3\mathcal{C}_0\mu\rho^2\mathcal{B}_1+3\mathcal{C}_1\mu\rho^2\mathcal{B}_0=0,$$

$$6\mathcal{A}_2\mathcal{C}_0\lambda^2\rho^2+6\mathcal{A}_1\mathcal{C}_1\lambda^2\rho^2+6\mathcal{A}_0\mathcal{C}_2\lambda^2\rho^2+\frac{45}{2}\mathcal{A}_2\mathcal{C}_1\lambda\mu\rho^2+\frac{45}{2}\mathcal{A}_1\mathcal{C}_2\lambda\mu\rho^2$$

$$+\frac{9}{2}\mathcal{A}_1\mathcal{C}_0\lambda\rho^2+\frac{9}{2}\mathcal{A}_0\mathcal{C}_1\lambda\rho^2+18\mathcal{A}_2\mathcal{C}_2\mu^2\rho^2+12\mathcal{A}_2\mathcal{C}_0\mu\rho^2+12\mathcal{A}_1\mathcal{C}_1\mu\rho^2+12\mathcal{A}_0\mathcal{C}_2\mu\rho^2$$

$$+16\mathcal{C}_2\lambda^4\rho^4+15\mathcal{C}_1\lambda^3\rho^4+232\mathcal{C}_2\lambda^2\mu\rho^4+4\mathcal{C}_2\lambda^2\rho^2-12\mathcal{C}_2\lambda^2\rho+4\mathcal{C}_2\lambda^2+60\mathcal{C}_1\lambda\mu\rho^4$$

$$+3\mathcal{C}_1\lambda\rho^2-9\mathcal{C}_1\lambda\rho+3\mathcal{C}_1\lambda+136\mathcal{C}_2\mu^2\rho^4+8\mathcal{C}_2\mu\rho^2-24\mathcal{C}_2\mu\rho+8\mathcal{C}_2\mu+6\mathcal{C}_0\lambda^2\rho^2\mathcal{B}_2$$

$$+6\mathcal{C}_1\lambda^2\rho^2\mathcal{B}_1+6\mathcal{C}_2\lambda^2\rho^2\mathcal{B}_0+\frac{45}{2}\mathcal{C}_1\lambda\mu\rho^2\mathcal{B}_2+\frac{45}{2}\mathcal{C}_2\lambda\mu\rho^2\mathcal{B}_1+\frac{9}{2}\mathcal{C}_0\lambda\rho^2\mathcal{B}_1+\frac{9}{2}\mathcal{C}_1\lambda\rho^2\mathcal{B}_0$$

$$+18\mathcal{C}_2\mu^2\rho^2\mathcal{B}_2+12\mathcal{C}_0\mu\rho^2\mathcal{B}_2+12\mathcal{C}_1\mu\rho^2\mathcal{B}_1+12\mathcal{C}_2\mu\rho^2\mathcal{B}_0=0,$$

$$\frac{27}{2}\mathcal{A}_2\mathcal{C}_1\lambda^2\rho^2+\frac{27}{2}\mathcal{A}_1\mathcal{C}_2\lambda^2\rho^2+42\mathcal{A}_2\mathcal{C}_2\lambda\mu\rho^2+15\mathcal{A}_2\mathcal{C}_0\lambda\rho^2+15\mathcal{A}_1\mathcal{C}_1\lambda\rho^2+15\mathcal{A}_0\mathcal{C}_2\lambda\rho^2$$

$$+27\mathcal{A}_2\mathcal{C}_1\mu\rho^2+27\mathcal{A}_1\mathcal{C}_2\mu\rho^2+3\mathcal{A}_1\mathcal{C}_0\rho^2+3\mathcal{A}_0\mathcal{C}_1\rho^2+130\mathcal{C}_2\lambda^3\rho^4+50\mathcal{C}_1\lambda^2\rho^4+440\mathcal{C}_2\lambda\mu\rho^4$$

$$+10\mathcal{C}_2\lambda\rho^2-30\mathcal{C}_2\lambda\rho+10\mathcal{C}_2\lambda+40\mathcal{C}_1\mu\rho^4+2\mathcal{C}_1\rho^2-6\mathcal{C}_1\rho+2\mathcal{C}_1+\frac{27}{2}\mathcal{C}_1\lambda^2\rho^2\mathcal{B}_2$$

$$+\frac{27}{2}\mathcal{C}_2\lambda^2\rho^2\mathcal{B}_1+42\mathcal{C}_2\lambda\mu\rho^2\mathcal{B}_2+15\mathcal{C}_0\lambda\rho^2\mathcal{B}_2+15\mathcal{C}_1\lambda\rho^2\mathcal{B}_1+15\mathcal{C}_2\lambda\rho^2\mathcal{B}_0+27\mathcal{C}_1\mu\rho^2\mathcal{B}_2$$

$$+27\mathcal{C}_2\mu\rho^2\mathcal{B}_1+3\mathcal{C}_0\rho^2\mathcal{B}_1+3\mathcal{C}_1\rho^2\mathcal{B}_0=0,$$

$$24\mathcal{A}_2\mathcal{C}_2\lambda^2\rho^2+\frac{63}{2}\mathcal{A}_2\mathcal{C}_1\lambda\rho^2+\frac{63}{2}\mathcal{A}_1\mathcal{C}_2\lambda\rho^2+48\mathcal{A}_2\mathcal{C}_2\mu\rho^2+9\mathcal{A}_2\mathcal{C}_0\rho^2+9\mathcal{A}_1\mathcal{C}_1\rho^2$$

$$+9\mathcal{A}_0\mathcal{C}_2\rho^2+330\mathcal{C}_2\lambda^2\rho^4+60\mathcal{C}_1\lambda\rho^4+240\mathcal{C}_2\mu\rho^4+6\mathcal{C}_2\rho^2-18\mathcal{C}_2\rho+6\mathcal{C}_2+24\mathcal{C}_2\lambda^2\rho^2\mathcal{B}_2$$

$$+\frac{63}{2}\mathcal{C}_1\lambda\rho^2\mathcal{B}_2+\frac{63}{2}\mathcal{C}_2\lambda\rho^2\mathcal{B}_1+48\mathcal{C}_2\mu\rho^2\mathcal{B}_2+9\mathcal{C}_0\rho^2\mathcal{B}_2+9\mathcal{C}_1\rho^2\mathcal{B}_1+9\mathcal{C}_2\rho^2\mathcal{B}_0=0,$$

$$54\mathcal{A}_2\mathcal{C}_2\lambda\rho^2+18\mathcal{A}_2\mathcal{C}_1\rho^2+18\mathcal{A}_1\mathcal{C}_2\rho^2+336\mathcal{C}_2\lambda\rho^4+24\mathcal{C}_1\rho^4+54\mathcal{C}_2\lambda\rho^2\mathcal{B}_2$$

$$+18\mathcal{C}_1\rho^2\mathcal{B}_2+18\mathcal{C}_2\rho^2\mathcal{B}_1=0,$$

$$30\mathcal{A}_2\mathcal{C}_2\rho^2+120\mathcal{C}_2\rho^4+30\mathcal{C}_2\rho^2\mathcal{B}_2=0.$$

Solving this system of algebraic equations, with the aid of Mathematica, we obtain

$$
\begin{aligned}
\mathcal{A}_1 &= -2\lambda\rho^2,\\
\mathcal{A}_2 &= -2\rho^2,\\
\mathcal{B}_0 &= \mathcal{A}_0,\\
\mathcal{B}_1 &= -2\lambda\rho^2,\\
\mathcal{B}_2 &= -2\rho^2,\\
\mathcal{C}_0 &= \frac{3\mathcal{A}_0\rho^2+\lambda^2\rho^4+8\mu\rho^4+\rho^2-3\rho+1}{3\rho^2},\\
\mathcal{C}_1 &= \frac{2\left(3\mathcal{A}_0\lambda\rho^2+\lambda^3\rho^4+8\lambda\mu\rho^4+\lambda\rho^2-3\lambda\rho+\lambda\right)}{3\mathcal{C}_0},\\
\mathcal{C}_2 &= \frac{\mathcal{C}_1}{\lambda}.
\end{aligned}
$$

Now using the general solution of (7.6) in (7.7), we have the following three types of travelling wave solutions of the coupled KP equation (7.2):

When $\lambda^2-4\mu>0$, we obtain the hyperbolic function solutions

$$
\begin{aligned}
u_1(t,x,y) &= \mathcal{A}_0+\mathcal{A}_1\bigg(-\frac{\lambda}{2}+\delta_1\frac{C_1\sinh(\delta_1 z)+C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z)+C_2\sinh(\delta_1 z)}\bigg)\\
&\quad+\mathcal{A}_2\bigg(-\frac{\lambda}{2}+\delta_1\frac{C_1\sinh(\delta_1 z)+C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z)+C_2\sinh(\delta_1 z)}\bigg)^2, \qquad (7.8a)\\
v_1(t,x,y) &= \mathcal{B}_0+\mathcal{B}_1\bigg(-\frac{\lambda}{2}+\delta_1\frac{C_1\sinh(\delta_1 z)+C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z)+C_2\sinh(\delta_1 z)}\bigg)\\
&\quad+\mathcal{B}_2\bigg(-\frac{\lambda}{2}+\delta_1\frac{C_1\sinh(\delta_1 z)+C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z)+C_2\sinh(\delta_1 z)}\bigg)^2, \qquad (7.8b)\\
w_1(t,x,y) &= \mathcal{B}_0+\mathcal{B}_1\bigg(-\frac{\lambda}{2}+\delta_1\frac{C_1\sinh(\delta_1 z)+C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z)+C_2\sinh(\delta_1 z)}\bigg)\\
&\quad+\mathcal{B}_2\bigg(-\frac{\lambda}{2}+\delta_1\frac{C_1\sinh(\delta_1 z)+C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z)+C_2\sinh(\delta_1 z)}\bigg)^2, \qquad (7.8c)
\end{aligned}
$$

where $z=t-\rho x+(\rho-1)y$, $\delta_1=\frac{1}{2}\sqrt{\lambda^2-4\mu}$, C_1 and C_2 are arbitrary constants.

When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function solutions

$$\begin{aligned}
u_2(t,x,y) &= \mathcal{A}_0 + \mathcal{A}_1\left(-\frac{\lambda}{2} + \delta_2\frac{-C_1\sin(\delta_2 z) + C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z) + C_2\sin(\delta_2 z)}\right) \\
&\quad + \mathcal{A}_2\left(-\frac{\lambda}{2} + \delta_2\frac{-C_1\sin(\delta_2 z) + C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z) + C_2\sin(\delta_2 z)}\right)^2, \\
v_2(t,x,y) &= \mathcal{B}_0 + \mathcal{B}_1\left(-\frac{\lambda}{2} + \delta_2\frac{-C_1\sin(\delta_2 z) + C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z) + C_2\sin(\delta_2 z)}\right) \\
&\quad + \mathcal{B}_2\left(-\frac{\lambda}{2} + \delta_2\frac{-C_1\sin(\delta_2 z) + C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z) + C_2\sin(\delta_2 z)}\right)^2, \\
w_2(t,x,y) &= \mathcal{B}_0 + \mathcal{B}_1\left(-\frac{\lambda}{2} + \delta_2\frac{-C_1\sin(\delta_2 z) + C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z) + C_2\sin(\delta_2 z)}\right) \\
&\quad + \mathcal{B}_2\left(-\frac{\lambda}{2} + \delta_2\frac{-C_1\sin(\delta_2 z) + C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z) + C_2\sin(\delta_2 z)}\right)^2,
\end{aligned}$$

where $z = t - \rho x + (\rho - 1)y$, $\delta_2 = \frac{1}{2}\sqrt{4\mu - \lambda^2}$, C_1 and C_2 are arbitrary constants.

When $\lambda^2 - 4\mu = 0$, we obtain the rational function solutions

$$\begin{aligned}
u_3(t,x,y) &= \mathcal{A}_0 + \mathcal{A}_1\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}\right) + \mathcal{A}_2\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}\right)^2, \\
v_3(t,x,y) &= \mathcal{B}_0 + \mathcal{B}_1\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}\right) + \mathcal{B}_2\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}\right)^2, \\
w_3(t,x,y) &= \mathcal{B}_0 + \mathcal{B}_1\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}\right) + \mathcal{B}_2\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}\right)^2,
\end{aligned}$$

where $z = t - \rho x + (\rho - 1)y$, C_1 and C_2 are arbitrary constants.

A profile of the solution (7.8) is given in Figures 7.1 - 7.3.

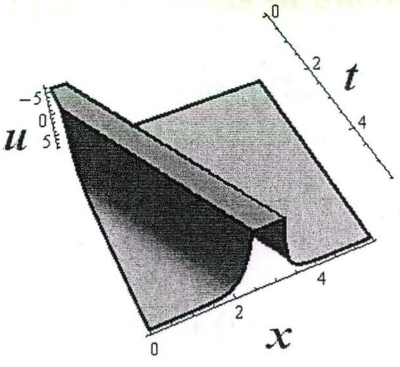

Figure 7.1: Evolution of travelling wave solution (7.8) with parameters $C_1 = 0$, $C_2 = 1$, $\lambda = 3$, $\mu = 1$, $\rho = 2$, $y = 0$, $\mathcal{A}_0 = 0$.

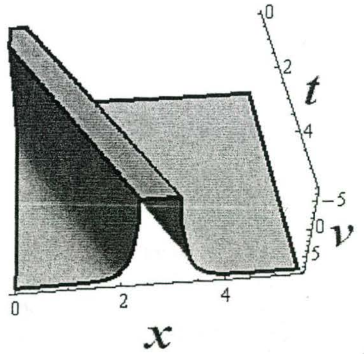

Figure 7.2: Evolution of travelling wave solution (7.8) with parameters $C_1 = 0,\ C_2 = 1,\ \lambda = 3,\ \mu = 1,\ \rho = 2,\ y = 0,\ \mathcal{A}_0 = 0.$

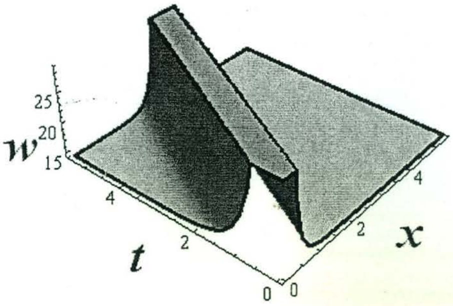

Figure 7.3: Evolution of travelling wave solution (7.8) with parameters $C_1 = 0,\ C_2 = 1,\ \lambda = 3,\ \mu = 1,\ \rho = 2,\ y = 0,\ \mathcal{A}_0 = 0.$

7.2.3 Solutions of (7.2) in terms of Jacobi elliptic functions

In this subsection we obtain exact solutions of (7.2) in terms of Jacobi elliptic functions. We note that the cosine-amplitude function, $\mathrm{cn}(z|m)$, the delta amplitude function, $\mathrm{dn}(z|m)$, and the sine-amplitude function, $\mathrm{sn}(z|m)$ satisfy the following ordinary differential equations:

$$G''(z) + 2mG(z)^3 + (1 - 2m)G(z) = 0, \tag{7.9}$$

$$G''(z) - (2 - m)G(z) + 2G(z)^3 = 0 \tag{7.10}$$

and

$$G''(z) - 2mG(z)^3 + (m + 1)G(z) = 0, \tag{7.11}$$

respectively [77].

We now use the above ODEs in place of (7.6) in the (G'/G)-expansion method (The ODEs are used here for the first time in (G'/G)-expansion method). Consequently, following the same procedure of the previous subsection we obtain new exact solutions of the coupled KP equation (7.2) in terms of the Jacobi elliptic functions, namely

$$\begin{aligned}
u_4(t,x,y) &= \mathcal{A}_0 - \mathcal{A}_1\left(\frac{\mathrm{dn}(z|m)\mathrm{sn}(z|m)}{\mathrm{cn}(z|m)}\right) + \mathcal{A}_2\left(\frac{\mathrm{dn}(z|m)\mathrm{sn}(z|m)}{\mathrm{cn}(z|m)}\right)^2,\\
v_4(t,x,y) &= \mathcal{B}_0 - \mathcal{B}_1\left(\frac{\mathrm{dn}(z|m)\mathrm{sn}(z|m)}{\mathrm{cn}(z|m)}\right) + \mathcal{B}_2\left(\frac{\mathrm{dn}(z|m)\mathrm{sn}(z|m)}{\mathrm{cn}(z|m)}\right)^2,\\
w_4(t,x,y) &= \mathcal{C}_0 - \mathcal{C}_1\left(\frac{\mathrm{dn}(z|m)\mathrm{sn}(z|m)}{\mathrm{cn}(z|m)}\right) + \mathcal{C}_2\left(\frac{\mathrm{dn}(z|m)\mathrm{sn}(z|m)}{\mathrm{cn}(z|m)}\right)^2,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{A}_1 &= 0,\\
\mathcal{A}_2 &= -2\rho^2,\\
\mathcal{B}_0 &= \mathcal{A}_0,\\
\mathcal{B}_1 &= 0,\\
\mathcal{B}_2 &= -2\rho^2,\\
\mathcal{C}_0 &= \frac{3\mathcal{A}_0\rho^2 - 16m\rho^4 + 8\rho^4 + \rho^2 - 3\rho + 1}{3\rho^2},\\
\mathcal{C}_1 &= 0,\\
\mathcal{C}_2 &= -\frac{2\left(-3\mathcal{A}_0\rho^2 + 16m\rho^4 - 8\rho^4 - \rho^2 + 3\rho - 1\right)}{3\mathcal{C}_0},\\
z &= t - \rho x + (\rho - 1)y;
\end{aligned}$$

$$\begin{aligned}
u_5(t,x,y) &= \mathcal{A}_0 - \mathcal{A}_1\left(\frac{m\mathrm{cn}(z|m)\mathrm{sn}(z|m)}{\mathrm{dn}(z|m)}\right) + \mathcal{A}_2\left(\frac{m\mathrm{cn}(z|m)\mathrm{sn}(z|m)}{\mathrm{dn}(z|m)}\right)^2,\\
v_5(t,x,y) &= \mathcal{B}_0 - \mathcal{B}_1\left(\frac{m\mathrm{cn}(z|m)\mathrm{sn}(z|m)}{\mathrm{dn}(z|m)}\right) + \mathcal{B}_2\left(\frac{m\mathrm{cn}(z|m)\mathrm{sn}(z|m)}{\mathrm{dn}(z|m)}\right)^2,\\
w_5(t,x,y) &= \mathcal{C}_0 - \mathcal{C}_1\left(\frac{m\mathrm{cn}(z|m)\mathrm{sn}(z|m)}{\mathrm{dn}(z|m)}\right) + \mathcal{C}_2\left(\frac{m\mathrm{cn}(z|m)\mathrm{sn}(z|m)}{\mathrm{dn}(z|m)}\right)^2,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{A}_1 &= 0,\\
\mathcal{B}_0 &= \frac{32\mathcal{A}_2\rho^4 - 3\mathcal{A}_0\mathcal{A}_2\rho^2 - 2\mathcal{A}_2\rho^2 + 6\mathcal{A}_2\rho - 2\mathcal{A}_2 - 16\mathcal{A}_2 m\rho^4 - 32m\rho^6 + 64\rho^6 - 4\rho^4 + 12\rho^3 - 4\rho^2}{3\rho^2\left(\mathcal{A}_2 + 4\rho^2\right)},\\
\mathcal{B}_1 &= 0,\\
\mathcal{B}_2 &= -\mathcal{A}_2 - 4\rho^2,\\
\mathcal{C}_0 &= \frac{\sqrt{3\mathcal{A}_0\rho^2 + 8m\rho^4 - 16\rho^4 + \rho^2 - 3\rho + 1}\sqrt{8m\rho^4 - 16\rho^4 + \rho^2 - 3\rho + 3\rho^2\mathcal{B}_0 + 1}}{3\rho^2},\\
\mathcal{C}_1 &= 0,\\
\mathcal{C}_2 &= -\frac{2\left(\mathcal{A}_2\mathcal{C}_0 + 2\mathcal{C}_0\rho^2\right)}{\mathcal{B}_0 - \mathcal{A}_0},\\
z &= t - \rho x + (\rho - 1)y;
\end{aligned}$$

and

$$u_6(t,x,y) = \mathcal{A}_0 + \mathcal{A}_1\left(\frac{\mathrm{cn}(z|m)\mathrm{dn}(z|m)}{\mathrm{sn}(z|m)}\right) + \mathcal{A}_2\left(\frac{\mathrm{cn}(z|m)\mathrm{dn}(z|m)}{\mathrm{sn}(z|m)}\right)^2, \tag{7.12a}$$

$$v_6(t,x,y) = \mathcal{B}_0 + \mathcal{B}_1\left(\frac{\mathrm{cn}(z|m)\mathrm{dn}(z|m)}{\mathrm{sn}(z|m)}\right) + \mathcal{B}_2\left(\frac{\mathrm{cn}(z|m)\mathrm{dn}(z|m)}{\mathrm{sn}(z|m)}\right)^2, \tag{7.12b}$$

$$w_6(t,x,y) = \mathcal{C}_0 + \mathcal{C}_1\left(\frac{\mathrm{cn}(z|m)\mathrm{dn}(z|m)}{\mathrm{sn}(z|m)}\right) + \mathcal{C}_2\left(\frac{\mathrm{cn}(z|m)\mathrm{dn}(z|m)}{\mathrm{sn}(z|m)}\right)^2, \tag{7.12c}$$

where

$$\begin{aligned}
\mathcal{A}_0 &= \frac{-8m\rho^4 - 8\rho^4 - \rho^2 + 3\rho - 1}{3\rho^2},\\
\mathcal{A}_1 &= 0,\\
\mathcal{B}_0 &= \mathcal{A}_0,\\
\mathcal{B}_1 &= 0,\\
\mathcal{B}_2 &= -\mathcal{A}_2 - 4\rho^2,\\
\mathcal{C}_0 &= 0,\\
\mathcal{C}_1 &= 0,\\
\mathcal{C}_2 &= \sqrt{-4\mathcal{A}_2\rho^2 - \mathcal{A}_2^2},\\
z &= t - \rho x + (\rho - 1)y.
\end{aligned}$$

A profile of the solution (7.12) is given in Figures 7.4 - 7.6.

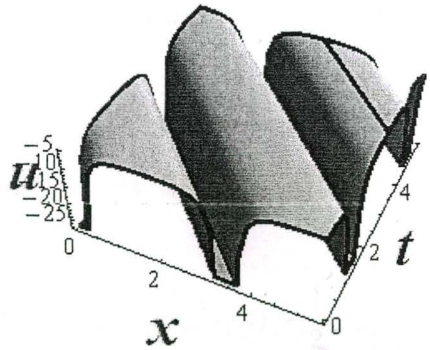

Figure 7.4: Evolution of periodic solution (7.12) with parameters $y = 0$, $m = 0.3$, $\rho = -1$, $\mathcal{A}_2 = -1$.

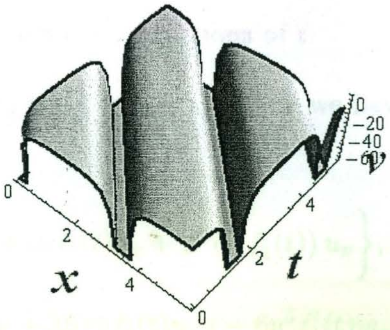

Figure 7.5: Evolution of periodic solution (7.12) with parameters $y = 0$, $m = 0.3$, $\rho = -1$, $\mathcal{A}_2 = -1$.

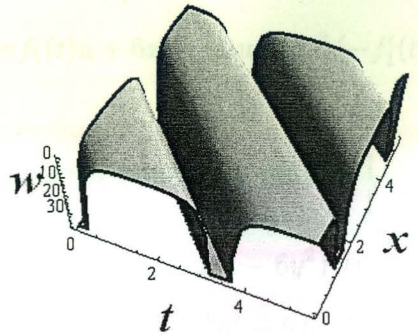

Figure 7.6: Evolution of periodic solution (7.12) with parameters $y = 0$, $m = 0.3$, $\rho = -1$, $\mathcal{A}_2 = -1$.

7.3 Conservation laws of (7.2)

For the coupled KP system (7.2), we see that the three zeroth-order multipliers, namely $\Lambda_1 = \Lambda_1(t,x,y,u,v,w)$, $\Lambda_2 = \Lambda_2(t,x,y,u,v,w)$ and $\Lambda_3 = \Lambda_3(t,x,y,u,v,w)$ are given by

$$\begin{aligned}
\Lambda_1 &= xyf_1(t) + xf_2(t) - \frac{1}{6}y^3 f_1'(t) - \frac{1}{2}y^2 f_2'(t) + yf_7(t) + f_8(t),\\
\Lambda_2 &= xyf_3(t) + xf_4(t) - \frac{1}{6}y^3 f_3'(t) - \frac{1}{2}y^2 f_4'(t) + yf_{11}(t) + f_{12}(t),\\
\Lambda_3 &= xyf_5(t) + xf_6(t) - \frac{1}{6}y^3 f_5'(t) - \frac{1}{2}y^2 f_6'(t) + yf_9(t) + f_{10}(t),
\end{aligned}$$

where f_i, $i = 1, \cdots, 12$ are arbitrary functions of t.

Therefore, corresponding to the above multipliers we have the following twelve conserved vectors of (7.2):

$$\begin{aligned}
T_1^t &= \frac{1}{12}\Big\{ -6yf_1(t)u + 6xyf_1(t)u_x + y^3\left(-f_1'(t)\right)u_x \Big\},\\
T_1^x &= \frac{1}{12}\Big\{ -6y^3 f_1'(t)u_x u + 36xyf_1(t)u_x u - 6y^3 f_1'(t)w_x w + 36xyf_1(t)w_x w + y^3 f_1{}''(t)u\\
&\quad -6xyf_1'(t)u - 18yf_1(t)u^2 - 18yf_1(t)w^2 - 12yf_1(t)u_{xx} + 12xyf_1(t)u_{xxx} + 6xyf_1(t)u_t\\
&\quad -y^3 f_1'(t)u_t - 2y^3 f_1'(t)u_{xxx} \Big\},\\
T_1^y &= \frac{1}{6}\Big\{ 3y^2 f_1'(t)u - 6xf_1(t)u + 6xyf_1(t)u_y + y^3\left(-f_1'(t)\right)u_y \Big\};
\end{aligned}$$

$$\begin{aligned}
T_2^t &= \frac{1}{4}\Big\{ -2f_2(t)u + 2xf_2(t)u_x + y^2\left(-f_2'(t)\right)u_x \Big\},\\
T_2^x &= \frac{1}{4}\Big\{ -6y^2 f_2'(t)u_x u + 12xf_2(t)u_x u - 6y^2 f_2'(t)w_x w + 12xf_2(t)w_x w\\
&\quad +y^2 f_2{}''(t)u - 2xf_2'(t)u - 6f_2(t)u^2 - 6f_2(t)w^2 - 4f_2(t)u_{xx} + 4xf_2(t)u_{xxx}\\
&\quad +2xf_2(t)u_t - y^2 f_2'(t)u_t - 2y^2 f_2'(t)u_{xxx} \Big\},\\
T_2^y &= \frac{1}{2}\Big\{ 2yf_2'(t)u + 2xf_2(t)u_y + y^2\left(-f_2'(t)\right)u_y \Big\};
\end{aligned}$$

$$
\begin{aligned}
T_3^t &= \frac{1}{12}\Big\{ -6yf_3(t)v + 6xyf_3(t)v_x + y^3\left(-f_3'(t)\right)v_x \Big\},\\
T_3^x &= \frac{1}{12}\Big\{ -6y^3f_3'(t)v_xv + 36xyf_3(t)v_xv - 6y^3f_3'(t)w_xw + 36xyf_3(t)w_xw\\
&\quad +y^3f_3''(t)v - 6xyf_3'(t)v - 18yf_3(t)v^2 - 18yf_3(t)w^2 - 12yf_3(t)v_{xx}\\
&\quad +12xyf_3(t)v_{xxx} + 6xyf_3(t)v_t - y^3f_3'(t)v_t - 2y^3f_3'(t)v_{xxx} \Big\},\\
T_3^y &= \frac{1}{6}\Big\{ 3y^2f_3'(t)v - 6xf_3(t)v + 6xyf_3(t)v_y + y^3\left(-f_3'(t)\right)v_y \Big\};
\end{aligned}
$$

$$
\begin{aligned}
T_4^t &= \frac{1}{4}\Big\{ -2f_4(t)v + 2xf_4(t)v_x + y^2\left(-f_4'(t)\right)v_x \Big\},\\
T_4^x &= \frac{1}{4}\Big\{ -6y^2f_4'(t)v_xv + 12xf_4(t)v_xv - 6y^2f_4'(t)w_xw + 12xf_4(t)w_xw\\
&\quad +y^2f_4''(t)v - 2xf_4'(t)v - 6f_4(t)v^2 - 6f_4(t)w^2 - 4f_4(t)v_{xx} + 4xf_4(t)v_{xxx}\\
&\quad +2xf_4(t)v_t - y^2f_4'(t)v_t - 2y^2f_4'(t)v_{xxx} \Big\},\\
T_4^y &= \frac{1}{2}\Big\{ 2yf_4'(t)v + 2xf_4(t)v_y + y^2\left(-f_4'(t)\right)v_y \Big\};
\end{aligned}
$$

$$
\begin{aligned}
T_5^t &= \frac{1}{12}\Big\{ -6yf_5(t)w + 6xyf_5(t)w_x + y^3\left(-f_5'(t)\right)w_x \Big\},\\
T_5^x &= \frac{1}{12}\Big\{ -3y^3f_5'(t)u_xw - 3y^3f_5'(t)w_xu + 18xyf_5(t)u_xw + 18xyf_5(t)w_xu\\
&\quad -3y^3f_5'(t)v_xw - 3y^3f_5'(t)w_xv + 18xyf_5(t)v_xw + 18xyf_5(t)w_xv - 18yf_5(t)uw\\
&\quad -18yf_5(t)vw + y^3f_5''(t)w - 6xyf_5'(t)w - 12yf_5(t)w_{xx} + 12xyf_5(t)w_{xxx}\\
&\quad +6xyf_5(t)w_t - y^3f_5'(t)w_t - 2y^3f_5'(t)w_{xxx} \Big\},\\
T_5^y &= \frac{1}{6}\Big\{ 3y^2f_5'(t)w - 6xf_5(t)w + 6xyf_5(t)w_y + y^3\left(-f_5'(t)\right)w_y \Big\};
\end{aligned}
$$

$$
\begin{aligned}
T_6^t &= \frac{1}{4}\Big\{ -2f_6(t)w + 2xf_6(t)w_x + y^2\left(-f_6'(t)\right)w_x \Big\},\\
T_6^x &= \frac{1}{4}\Big\{ -3y^2f_6'(t)u_xw - 3y^2f_6'(t)w_xu + 6xf_6(t)u_xw + 6xf_6(t)w_xu\\
&\quad -3y^2f_6'(t)v_xw - 3y^2f_6'(t)w_xv + 6xf_6(t)v_xw + 6xf_6(t)w_xv - 6f_6(t)uw\\
&\quad -6f_6(t)vw + y^2f_6''w - 2xf_6'(t)w - 4f_6(t)w_{xx} + 4xf_6(t)w_{xxx} + 2xf_6(t)w_t\\
&\quad -y^2f_6'(t)w_t - 2y^2f_6'(t)w_{xxx} \Big\},\\
T_6^y &= \frac{1}{2}\Big\{ 2yf_6'(t)w + 2xf_6(t)w_y + y^2\left(-f_6'(t)\right)w_y \Big\};
\end{aligned}
$$

$$\begin{aligned}
T_7^t &= \frac{1}{2}yf_7(t)u_x,\\
T_7^x &= \frac{1}{2}\left\{6yf_7(t)u_xu + 6yf_7(t)w_xw - yf_7'(t)u + 2yf_7(t)u_{xxx} + yf_7(t)u_t\right\},\\
T_7^y &= yf_7(t)u_y - f_7(t)u;
\end{aligned}$$

$$\begin{aligned}
T_8^t &= \frac{1}{2}f_8(t)u_x,\\
T_8^x &= \frac{1}{2}\left\{6f_8(t)u_xu + 6f_8(t)w_xw - f_8'(t)u + 2f_8(t)u_{xxx} + f_8(t)u_t\right\},\\
T_8^y &= f_8(t)u_y;
\end{aligned}$$

$$\begin{aligned}
T_9^t &= \frac{1}{2}yf_9(t)w_x,\\
T_9^x &= \frac{1}{2}\left\{3yf_9(t)u_xw + 3yf_9(t)w_xu + 3yf_9(t)v_xw + 3yf_9(t)w_xv - yf_9'(t)w\right.\\
&\qquad \left. +2yf_9(t)w_{xxx} + yf_9(t)w_t\right\},\\
T_9^y &= yf_9(t)w_y - f_9(t)w;
\end{aligned}$$

$$\begin{aligned}
T_{10}^t &= \frac{1}{2}f_{10}(t)w_x,\\
T_{10}^x &= \frac{1}{2}\left\{3f_{10}(t)u_xw + 3f_{10}(t)w_xu + 3f_{10}(t)v_xw + 3f_{10}(t)w_xv - f_{10}'(t)w\right.\\
&\qquad \left. +2f_{10}(t)w_{xxx} + f_{10}(t)w_t\right\},\\
T_{10}^y &= f_{10}(t)w_y;
\end{aligned}$$

$$\begin{aligned}
T_{11}^t &= \frac{1}{2}yf_{11}(t)v_x,\\
T_{11}^x &= \frac{1}{2}\left\{6yf_{11}(t)v_xv + 6yf_{11}(t)w_xw - yf_{11}'(t)v + 2yf_{11}(t)v_{xxx} + yf_{11}(t)v_t\right\},\\
T_{11}^y &= yf_{11}(t)v_y - f_{11}(t)v;
\end{aligned}$$

$$\begin{aligned}
T_{12}^t &= \frac{1}{2}f_{12}(t)v_x,\\
T_{12}^x &= \frac{1}{2}\left\{6f_{12}(t)v_xv + 6f_{12}(t)w_xw - f_{12}'(t)v + 2f_{12}(t)v_{xxx} + f_{12}(t)v_t\right\},\\
T_{12}^y &= f_{12}(t)v_y.
\end{aligned}$$

Remark Due to the presence of the arbitrary functions, f_i, $i = 1, \cdots, 12$, in the multipliers, one can obtain an infinitely many conservation laws for the coupled KP system.

7.4 Conclusion

In this chapter we obtained exact solutions of the coupled KP system (7.2) by the aid of Lie symmetries as well as the (G'/G)-expansion method. The solutions obtained were in terms of hyperbolic, trigonometric, rational and Jacobi elliptic functions. Conservation laws of the coupled KP system were also computed using the multiplier approach.

Chapter 8

Concluding remarks

In this research project we first recalled some important definitions and results from Lie group theory and conservation laws, which were later used in the thesis. In Chapter two, Lie group classification was performed on the generalized Korteweg-de Vries-Burgers equation. The generalized Korteweg-de Vries-Burgers equation admitted a four-dimensional equivalence Lie algebra. It was also shown that the principal Lie algebra consists of a single translation symmetry and several possible extensions of the principal Lie algebra were computed. The associated symmetry reductions and exact solutions were obtained. Also, a one-dimensional optimal system of subalgebras was obtained for the case when the principal Lie algebra was extended by two symmetries.

In Chapter three we studied the two-dimensional generalization of the Kaup-Kupershmidt equation. Lie point symmetries of this equation were obtained and the three translation symmetries were used to transform the equation into a system of ODEs. Then the extended tanh method and the extended Jacobi elliptic function method were employed to solve this ODEs system to obtain exact solutions. Furthermore, conservation laws were also computed using the multiplier approach. The conservation laws consisted of an infinite number of nonlocal conserved vectors.

In Chapter four we studied the coupled KdV system from the point of view of Lie symmetry analysis. Similarity reductions and exact solutions with the aid of simplest equation method and Jacobi elliptic function method were obtained based on the optimal systems of one-dimensional subalgebras for the KdV system. The exact solutions found were solitary, cnoidal and snoidal waves. Finally, local conservation laws for the coupled KdV system were derived by employing the multiplier method and the new conservation theorem.

In Chapter five Lie group analysis was utilized to study the generalized coupled variable-coefficient modified Korteweg-de Vries (CVCmKdV) system. We obtained exact solutions for this system with the aid of the simplest equation method and Jacobi elliptic function method. The exact solutions derived were solitary, cnoidal and snoidal waves. Furthermore, the local conservation laws of the CVCmKdV system were constructed using the multiplier method and the new conservation theorem.

In Chapter six Lie symmetry analysis was employed to study a new coupled KdV system. We obtained similarity reductions and exact solutions with the aid of simplest equation method and Jacobi elliptic function method. The exact solutions deduced were solitary, cnoidal and snoidal waves. The local conservation laws of the coupled KdV system were constructed using the multiplier method.

In Chapter seven we determined exact solutions of the coupled KP system (7.2) by the aid of Lie symmetries as well as the (G'/G)-expansion method. The solutions obtained were in terms of hyperbolic, trigonometric, rational and Jacobi elliptic functions. Local conservation laws of the coupled KP system were also computed using the multiplier approach.

In the future we intend to use the exact solutions found in this thesis as benchmarks against the numerical simulations in theoretical physics and fluid mechanics and also the conserved vectors will be used to construct solutions [37].

Chapter 9

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