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A permutation entropy analysis of Bitcoin volatility

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ABSTRACT

Cryptocurrencies are widely regarded as volatile and less predictable assets by financial participants. The behaviour and dynamics of Bitcoin's daily volatility, obtained by fitting GARCH models, are investigated for a period of 8 years using permutation entropy which is represented by the variable H for calculations. The best fitting GARCH models selected are the FIGARCH(1,0.7,1) and SGARCH(1,1) models based on maximum likelihood estimation, Akaike Information Criterion and Bayesian Information Criterion. Simulated volatilities are also obtained from the best fitting GARCH models using their respective parameters, to confirm how well the models fit. The results obtained show that the H values of Bitcoin are generally low and that the dynamics of Bitcoin's volatility is quite predictable, as Bitcoin's volatility is most likely to decline over time than increase or have an alternating movement. Also, the simulated volatilities show good agreement with the real-world volatility, confirming the models as good fits.

1. Introduction

According to CoinMarketCap [1], Bitcoin is the biggest cryptocurrency by market capitalization of about $584 trillion on 26,214 listed cryptocurrencies. Bitcoin was introduced by Nakamoto [2] and its usage began in 2009. Ever since, there has been the emergence of other cryptocurrencies such as Ethereum, Binance coin and Dogecoin, to name a few. However, Bitcoin remains the most traded cryptocurrency. Volatility, a measurement of risk, is an important feature of any financial asset, and its assessment is very important in order for investors to estimate potential threats. According to Bouoiyour and Selmi [3], one of the major criticisms of Bitcoin is its high volatility deeming it unsuitable to be used as money. Bukovina and Marticek [4] also stated that research has shown that Bitcoin is tremendously volatile in comparison to currencies such as the dollar, pound, euro or yen. The authors suggested that its high volatility threatens its chances of being a successful currency. According to Baur and Dimpfl [5], the term currency cannot be applied to Bitcoin due to its high volatility which makes it impossible for it to be used for a reliable exchange. Baur and Dimpfl [5] stated that Bitcoin should rather be classified as an investment. Bitcoin's volatility is a function of speculation, attention from investors, the interoperability and interaction of the market [6]. Despite the doubts surrounding Bitcoin, high volumes of trade have continued, while it is still mainly used as an asset.

In assessing volatilities of the financial markets, generalized autoregressive conditional heteroskedastic (GARCH) models are most appropriate due to their ability to successfully account for assumptions such as volatility clustering in the financial stock market [7,8]. Variations of the standard GARCH (SGARCH) model [9,10] include but are not limited to the integrated generalized autoregressive conditional heteroskedastic (IGARCH) model [11], the exponential generalized autoregressive conditional

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heteroskedastic (EGARCH) model [12], the nonlinear generalized autoregressive conditional heteroskedastic (NGARCH) model [13], the asymmetric power autoregressive conditional heteroskedastic (APARCH) model [14], the threshold generalized autoregressive conditional heteroskedastic (TGARCH) model [15], the fractionally integrated generalized autoregressive conditional heteroskedastic (FIGARCH) model [16] and the component generalized autoregressive conditional heteroskedastic (CGARCH) model [17].

When choosing a GARCH model, some major selection criteria include the maximum likelihood estimation (MLE), the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) [18]. The larger the MLE, the better the model fit. On the other hand, a smaller AIC and BIC indicate a better model fit. Several authors have worked on assessing the volatility of Bitcoin using different GARCH models. Bouoiyour and Selmi [3] concluded that the TGARCH model was the optimal model for estimating the volatility of Bitcoin for the period December 2010–June 2015. The authors noted that Bitcoin had a period of lower volatility between January and June 2015. A different GARCH model was selected by Katsiampa [19] who identified the autoregressive component generalized autoregressive conditional heteroskedastic (AR(1)-CGARCH(1,1)) model as the best model for the Bitcoin data that spanned from 18th of July, 2010 to 1st of October, 2016. The author's choice was based on three information criteria — AIC, BIC and Hannan–Quinn criterion (HQC). Agyarko et al. [20] opined that the best fit model for the estimation of the Bitcoin volatility for data that spanned from January 1, 2017 to September 12, 2018 was the SGARCH model for a generalized error distribution based on AIC.

There have also been investigations carried out regarding the predictability of the volatility of Bitcoin. Using a quantiles-based approach, Balcilar et al. [21] determined whether or not the returns and volatility of Bitcoin can be predicted using volume. The authors made use of data from 19th of December 2011 to 25th of April 2016. It was concluded that volumes can indeed predict returns except when the market is bearish or bullish. On the other hand, volatility of Bitcoin returns cannot be predicted at any point using volumes. Dias et al. [22] also considered the role investor sentiment plays in predicting Bitcoin return and volatility. The authors applied the quantile regression approach. Dias et al. [22] concluded that the interest and emotions of investors are significant predictors of the returns and volatility of Bitcoin. They also stated that the predictability power is subject to changes based on market conditions.

A statistical tool that can be used to measure predictability of a system is permutation entropy (PE). PE which is represented by the variable H for calculations was introduced by Bandt and Pompe [23] as a measure of the complexity of a given time series. Bandt [24] states that PE only makes use of the order of values, can be applied to nonlinear series and is computationally efficient. PE has been successfully applied in biomedicine [25,26], physical science [27–29] and finance [30–34]. In Finance, PE has gained traction in the analysis of market efficiency [30,34]. However, the use of permutation entropy to assess volatility of financial markets has received less attention. Ruiz et al. [35] proposed PE, topological entropy and modified PE as alternative volatility measures. Using simulated data, the authors concluded that the proposed measures are suitable for the quantification of the degree of randomness, so as to measure volatility.

In this paper, a comparative study is done between different GARCH models to estimate the daily volatility of Bitcoin for a period of time. In order to estimate the daily volatility, different GARCH models are fitted to Bitcoin daily returns. The best GARCH models are selected based on three selection criteria — AIC, BIC and MLE. The daily volatility of Bitcoin is obtained from the selected GARCH models. Using PE, we study the dynamics of the obtained daily volatility of Bitcoin to determine its degree of predictability. In addition, the selected models are used to simulate volatilities. Analysis is also carried out on the simulated volatilities using PE to check whether or not their dynamics resemble that of the real-world Bitcoin volatility in order to test how well the selected models fit the real-world data.

This paper is organized as follows: Section 2 deals with the tools used for research analysis. In Section 3, a descriptive analysis of the data used is presented. Section 4 presents the methodology of the research. Sections 5 and 6 cover results obtained. The conclusions of the study are briefly discussed in Section 7.

2. Modelling and analysis tools

We analyse the Bitcoin daily price in United States dollar from January 1, 2015–December 31, 2022. In obtaining the daily volatilities, the best fit GARCH models are chosen. The resulting daily volatilities are further analysed by means of PE. In this section, an overview of GARCH models and PE is provided.

2.1. GARCH model

The GARCH model is an improvement of the autoregressive conditional heteroskedastic (ARCH) model created by Engle [36]. In general, the GARCH model is more parsimonious than the ARCH model because it has fewer parameters when fitted to the same data [37]. The GARCH model for the value of the time series u_t at timestep t, is denoted as:

$$u_t = \epsilon_t \times \sigma_t, \tag{1}$$

where the innovations ϵ_t are usually assumed to be independent and identically distributed (i.i.d) with expected value of zero and variance of one. The innovations are assumed to be independent from the σ_k (volatility), for all $k \leq t$. However, to account for heavy tails, kurtosis, and asymmetric distributions of the innovations, this study assumes skewed normal, skewed student-t and skewed generalized error distributions in the analysis carried out. The selected GARCH models used in this paper are the SGARCH and FIGARCH models. The GARCH(1,1) approach is used for the selection of the SGARCH and FIGARCH models. Hence, the order of the GARCH terms σ^2 and the ARCH terms u^2 is 1. The choice of the GARCH(1,1) model as used in this paper is due to its high robustness and parsimony. This assertion is supported by Hansen and Lunde [38] who argue that finding a model that consistently outperforms the GARCH(1,1) model is an arduous task.

2.1.1. SGARCH model

The SGARCH model was developed by Bollerslev [9]. The conditional variance of SGARCH model σ_t^2 in (1) is expressed as follows:

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta\sigma_{t-1}^2, \tag{2}$$

where α_0, α_1, β are strictly positive and $\alpha_1 + \beta < 1$. This implies that the conditional variance has a stationary solution with finite expected value, i.e., $\mathrm{E}(\sigma_t^2) = \frac{\alpha_0}{1-\alpha_1-\beta}$, if and only if $\alpha_1 + \beta < 1$. The GARCH model makes use of the value of the squared past return (u_{t-1}^2) and the squared past conditional variance (σ_{t-1}^2) to estimate the current volatility (σ_t). The autoregressive moving average (ARMA) infinite representative of the GARCH model is given by [20,39]:

$$u_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i u_{t-i}^2 + \sum_{j=1}^{p} \beta_j u_{t-j}^2 - \sum_{j=1}^{p} \beta_j \upsilon_{t-j} + \upsilon_t, \tag{3}$$

where, $\alpha_0 > 1$, $\alpha_i, \beta_j \geq 0$, and $\upsilon_t = u_t^2 - \sigma_t^2$.

The SGARCH model works on the assumption that the effect of good and bad news is the same on the volatility of a market. Hence, it assumes that positive and negative error terms are symmetric.

2.1.2. FIGARCH model

The FIGARCH model developed by Baillie et al. [16] allows the capture of slowly decaying absolute or squared autocorrelations within a time series [40,41]. Eq. (3) can be expressed as an ARMA process in u_t^2 as follows [39]:

$$[1 - \alpha(L) - \beta(L)]u_t^2 = \alpha_0 + [1 - \beta(L)]\upsilon_t, \tag{4}$$

where the $\{\upsilon_t\}$ process is interpreted as the innovations for the conditional variance, $\alpha(L)$ and $\beta(L)$ are lag operators such that $\alpha(L) = \alpha_1(L) + \alpha_2(L^2) + \cdots + \alpha_q(L^q)$ and $\beta(L) = \beta_1(L) + \beta_2(L^2) + \cdots + \beta_p(L^p)$, respectively. Hence, an IGARCH(p, q) process can be written as:

$$[1 - \alpha(L) - \beta(L)](1 - L)u_t^2 = \alpha_0 + [1 - \beta(L)]\upsilon_t. \tag{5}$$

To obtain the FIGARCH model, the $(1 - L)$ in (5) is replaced with the fractional differencing operator $(1 - L)^d$ where $0 < d < 1$. Hence, FIGARCH(1, d, 1) model is obtained by considering:

$$[1 - \alpha(L) - \beta(L)](1 - L)^d u_t^2 = \alpha_0 + [1 - \beta(L)]\upsilon_t. \tag{6}$$

Thus, FIGARCH(1, d, 1) is given as:

$$\sigma_t^2 = \alpha_0 \left[1 - \beta(L)\right]^{-1} + [1 - [1 - \beta(L)]^{-1}[1 - \alpha_1(L)][1 - L]^d \big] u_t^2. \tag{7}$$

The FIGARCH model has a long-term memory and thus making it a more suitable model than other conditional heteroscedastic volatility models.

2.2. Permutation entropy H and forbidden patterns

H, the variable representing PE is a statistical measure for time series analysis that measures the degree of complexity and level of ordinal pattern predictability within a dynamic system. It is a pattern recognition tool that captures the order of relations between the values of a given time series and provides the probability distribution for the ordinal patterns. PE provides insights into the underlying driving mechanism of a time series, therefore informing us about the behaviour of the time series. H is defined on the interval [0,1]. When the H of a time series is 0, it implies that the ordinal patterns within the system are completely predictable. On the other hand, if the H of a given time series is 1, then the ordinal patterns within the system are completely unpredictable. Thus, a higher H indicates more unpredictability within a system.

For a time series of length N and an embedded dimension $d > 2$, the total number of subsets obtained from N for a time delay τ, is given by $N - \tau(d - 1)$. The subsets obtained are respectively arranged in ascending order and each of the values is given a rank number from the set $[1, \dots, d]$. Each subset has $d!$ ranking possibilities. A probability vector P is obtained on the basis of the ranking that exists in each subset in comparison to the total number of subsets.

For an obtained probability vector P of dimension $d!$ with members p_j as the occurrence for one of the possible categories, H is given by

$$H(P) = S(P)/\ln(d!), \tag{8}$$

where

$$S(P) = -\sum_{\substack{j=1 \\ p_j \neq 0}}^{d!} p_j \ln\left(p_j\right). \tag{9}$$

In the application of PE, certain ordinal patterns could be absent within the system, i.e., in ranking the subsets obtained from N, the probability of such ranking is zero and thus p_j for such patterns is 0. These patterns that are absent are referred to as forbidden patterns. While forbidden patterns and PE are related, they are interpreted differently in terms of the predictability of a system.

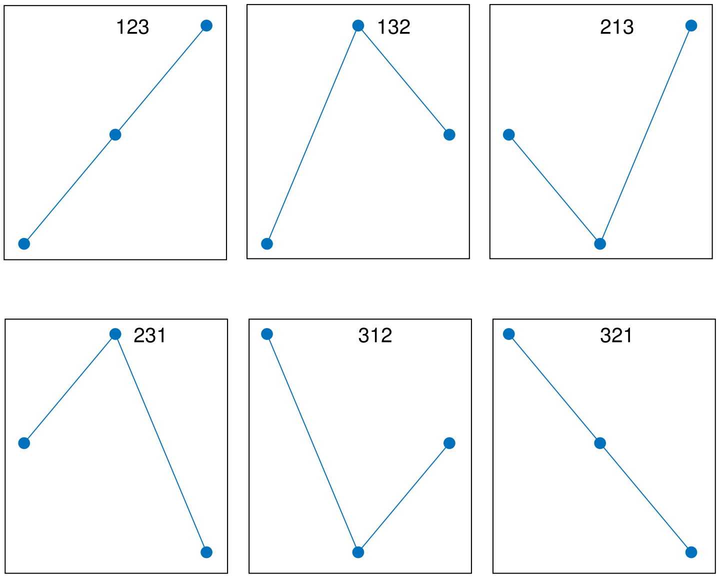
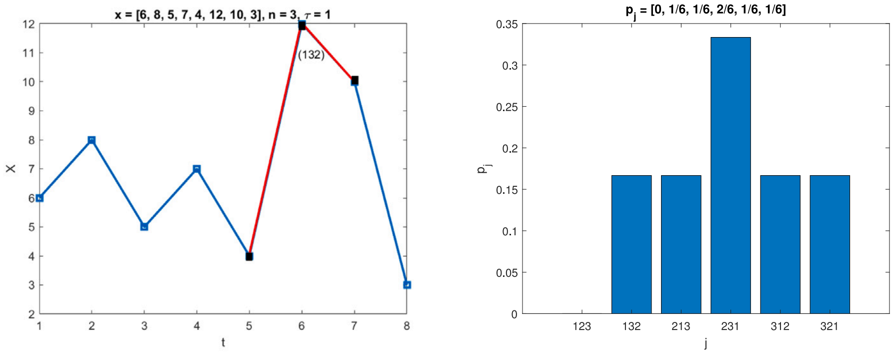


Fig. 1. Possible categories for $d = 3$: 123, 132, 213, 231, 312 and 321.



(a) (b)

Fig. 2. Illustration of the encoding for the calculation of permutation entropy. Panel (a): Dynamics between successive values. Lines coloured red and markers coloured black show the dynamics between the 5th–7th time entries — 4, 12, 10 as 132. Panel (b): Relative frequencies of all possible patterns.

As stated earlier, a higher value of H implies more unpredictability within a system. On the other hand, the larger the number of forbidden patterns (NFP) that exists within a system, the more predictable the system is. Summarily, NFP is positively correlated with level of predictability of a system, while H is negatively correlated with the level of predictability of a system.

Below is a simple example that demonstrates the process of obtaining H and explains the instances of forbidden patterns.

Consider the series X_t: {6, 8, 5, 7, 4, 12, 10, 3} of sample size $N = 8$. For $d = 3$ and $\tau = 1$, the subsets obtained from X_t are (6, 8, 5), (8, 5, 7), (5, 7, 4), (7, 4, 12), (4, 12, 10) and (12, 10, 3). The ranking of the first set is given by 231 — The smallest value (5) is assigned 1, the largest value (8) is assigned 3 and the intermediate value (6) is assigned 2. Similarly, the other subsets are ranked as follows: 312, 231, 213, 132 and 321. For $d = 3$, there are $d! = 6$ possible categories — 123, 132, 213, 231, 312 and 321 as shown in Fig. 1.

The probability vector P obtained based on the fraction of occurrences of each category is given by $P = \left[0, \frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right]$. H can be obtained using (8) and (9). In calculating H, the zero value(s) of vector P which indicate the forbidden pattern(s) is excluded in the application of (9) as the logarithm of 0 is undefined. The series X_t has a H value of 0.8710. For this considered series, 123 is identified as a forbidden pattern, as none of the subsets of X_t has a ranking of 123.

The visual representation of this illustration is shown in Fig. 2. Fig. 2(a) shows the dynamics between successive values of the series. From Fig. 2(b) which shows the relative frequencies of all possible patterns, it can be seen that the probability of obtaining pattern 123 is zero, thus affirming it as the forbidden pattern of the time series considered.

The ability of PE to identify unusual patterns, as well as forbidden patterns and its fast computation process makes it suitable for large sets of financial data.

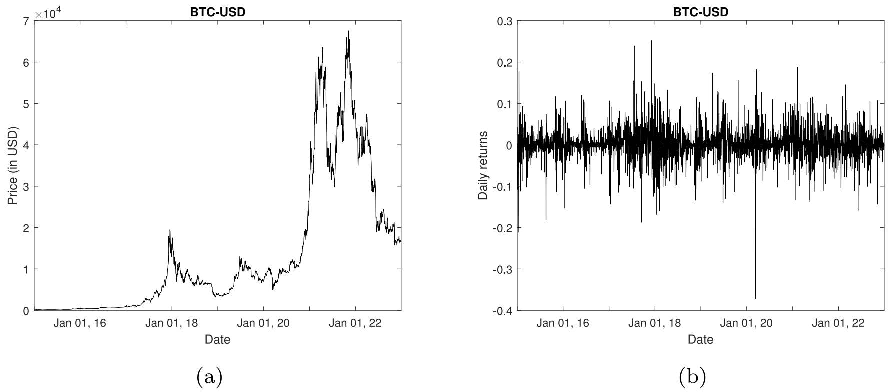


Fig. 3. Panel (a): Price trend of Bitcoin in USD from 2015–2022. Panel (b) Daily returns of Bitcoin from 2015–2022.

Table 1
Descriptive statistics of the BTC-USD data.

Statistic	Value
Mean	13 315.58
Median	7388.87
Trimmed mean	10 149.18
Standard deviation	16 297.17
Mean absolute deviation	9968.71
Minimum	178.10
Maximum	67 566.83
Range	67 388.73
Skew	1.5
Kurtosis	1.15
Standard error	301.49

3. Data

The daily price of Bitcoin expressed in terms of the United States dollar (BTC-USD) from January 1, 2015–December 31, 2022 spanning 2922 days is analysed. Fig. 3 shows the price trend of Bitcoin for the time period considered. The figure also shows the daily returns, u_t of Bitcoin for the same time period considered, which is calculated as follows:

$$u_t = \frac{C_t - C_{t-1}}{C_{t-1}}, \tag{10}$$

where

C_t = current closing price at day t

C_{t-1} = previous closing price.

From Fig. 3(a), it can be observed that the range of Bitcoin price over the period considered is large, with the lowest price being recorded at \$178.10 on the 14th of January 2015 and the highest price recorded at \$67,566.83 on the 8th of November 2021. Fig. 3(b) shows the real-world returns of Bitcoin. Notice that the daily returns range from −0.37 to 0.25.

The basic statistics of Bitcoin price over the time frame considered are summarized in Table 1. Table 1 reveals that the mean is greater than the median, hence the data is right skewed. This is also supported by the skew statistic being a positive value. A skew of 1.5 indicates that the data is highly skewed. A positive kurtosis of 1.15 indicates that the distribution of the data has thick tails.

4. Methodology

In order to estimate the daily volatility of Bitcoin, several GARCH models are built and a comparative study is made between the GARCH models. We select two GARCH models that best fit the daily returns of Bitcoin for the 8 years under consideration, based on their MLE, AIC, BIC. This yields two sets of daily volatilities of Bitcoin — one set for each of the optimal models. Taking each set of the BTC-USD daily volatility obtained from the selected models as a system, the behaviour of each set of daily volatility is investigated using PE. To capture sufficient periods, monthly and annual time frames are used for the analysis.

In analysing monthly volatilities, the sample size N is equal to 30 or 31 depending on the month, with the exception of February which has 28 days, except for the leap years in 2016 and 2020. In calculating H, the time delay τ is taken as 1 as the daily data of Bitcoin is used. The dimension d is taken as 3, satisfying the condition $N \geq 5d!$ [42,43]. It is important to note that while the sample size of February does not satisfy this condition, which is in fact a rule of thumb, it is only 1 or 2 less than the prescribed minimum of 30 for $d = 3$. For each month, a single H is obtained, therefore yielding 12 H values per annum and 96 H values for the 8 years under consideration. Also, for the monthly analysis, we do not only consider the H values for each month over the 8 years, but the patterns that exists within each month. The existence or non-existence of forbidden patterns within each month is also investigated. Since, $d = 3$ is used, there are 6 possible patterns — 123, 132, 213, 231, 312 and 321. The percentage return of each possible pattern $\%(\mathcal{P}_i)$ for $i \in [1, 6]$ in a given month, is calculated as follows:

$$\%(\mathcal{P}_i) = \frac{n(\mathcal{P}_i)}{\mathcal{R}} \times 100, \tag{11}$$

where

$\mathcal{R}$ = number of ranked subsets obtained for the month, and

$n(\mathcal{P}_i)$ = number of occurrences of each pattern for the month.

Eq. (11) essentially converts the elements of the probability vector P to percentages.

For the annual analysis, the sample size is 365 with the exception of the leap years 2016 and 2020 that have 366 days. The values of the parameters used for the monthly analysis are also used for the annual analysis for consistency. Unlike what is done while analysing monthly, for the annual analysis, we consider only the H values for each of the 8 years. The NFP do not appear because a year is large to ensure that all $d! = 6$ patterns are present.

To investigate the goodness of each of the optimal models, 100 sets of simulated volatilities for the 2922 days are generated using each of the optimal models. The behaviour of the simulated daily volatilities is compared to the daily volatility of the real-world data to investigate whether or not they exhibit similar behavioural patterns. These comparisons are also done on a monthly and annual basis.

The monthly comparison is done using the NFP has the basis for comparison. On the other hand, for the annual comparison, H values are the basis for comparison. A single H value is obtained for each year for each set of simulations. Hence, there are 100 H values from the simulated data for each year. The position the annual H values obtained from the real-world daily volatility lie on the range of annual H values from the simulated daily volatilities is shown for each year. In addition, the annual respective standard normal scores are calculated as follows:

$$z = \frac{x - \mu}{\sigma} \tag{12}$$

where, x is the H value from the real-world data on a given year and μ and σ are the mean and standard deviations of the 100 H values from the simulated data of that year. The z describes the distance of the annual H values of the real-world volatilities from the mean in units of the standard deviation of the annual H values of the simulated volatilities. The following criteria are used in assessing the z – value [44,45]:

- $|z| \leq 2.0$ is normal or satisfactory.
- $2.0 < |z| < 3.0$ is questionable or an unusual observation.
- $|z| \geq 3.0$ is highly unusual and unsatisfactory.

The description of Fig. 4 is outlined by the following steps:

Step 1: Calculate the daily returns of Bitcoin.

Step 2: Build GARCH(1,1) models with errors following either skewed normal, skewed student-t or skewed generalized error distributions and select the best fit models.

Step 3: Obtain the daily volatilities based on the two best fit models.

Step 4: Analyse the daily volatility of Bitcoin using PE.

Step 5: Draw conclusions on the behaviour of Bitcoin volatility.

To evaluate the goodness of the models selected in Step 2, the following additional steps are followed:

Step 6: Generate 100 simulations using optimal two GARCH models and extract the daily volatilities.

Step 7: Analyse the simulated volatilities using PE.

Step 8: Compare the results from the real-world data obtained in Step 4 to those from the simulated data obtained in Step 7.

5. Model selection

The GARCH models tested in this research were the SGARCH, EGARCH, IGARCH, APARCH and FIGARCH models for a skewed normal distribution, a skewed student's t-distribution and a skewed generalized error distribution. Based on MLE, AIC and BIC, the GARCH models that best fit the Bitcoin data are the FIGARCH and the SGARCH models for a skewed generalized error distribution. The FIGARCH model is a slightly better fit to the Bitcoin data than the SGARCH model. For the non-optimal models EGARCH, IGARCH and APARCH, the distributions that yielded the best results are the skewed student's t-distribution for the IGARCH model and the skewed normal distribution for the EGARCH and APARCH models.

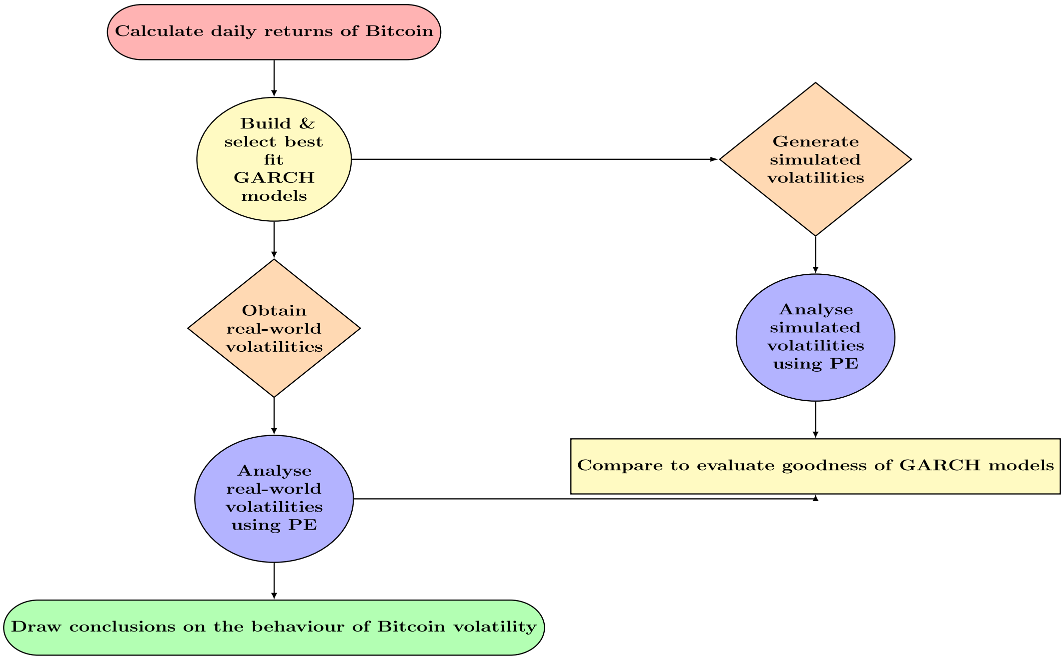


Fig. 4. Flow chart of analysis for analysing volatilities.

Table 2
Summary of models' estimated parameters and information selection criteria.

Model	FIGARCH(1,0.7,1)	SGARCH(1,1)	IGARCH(1,1)	EGARCH(1,1)	APARCH(1,1)
α_1	0.2	0.11892	0.10521	−0.03876	0.13814
β	0.79244	0.88008	0.89479	0.92780	0.84459
α_0	0.000021	0.00002	0.000014	−0.448851	0.000881
Skew	0.96500	0.96240	0.96492	0.96939	0.96506
Shape	0.89668	0.88376	3.26048	–	–
MLE	6008.791	6003.957	5999.034	5627.57	5622.29
AIC	−4.1101	−4.1075	−4.1048	−3.8498	−3.8455
BIC	−4.0978	−4.0972	−4.0966	−3.8395	−3.8332

Table 2 shows the summary of the estimated parameters, information selection criterion and the measures for good fit of the tested models, including the chosen FIGARCH(1,0.7,1) and SGARCH(1,1) models that best fit the Bitcoin data for the eight years under consideration. From Table 2, it is seen that the MLE of the FIGARCH(1,0.7,1) and SGARCH(1,1) models are the highest and second highest, respectively. All the estimated parameters of the two models were statistically significant at a 1% significance level. These findings are attesting to both models being the best fits. Also, the AIC and BIC of the FIGARCH(1,0.7,1) and SGARCH(1,1) models are the smallest in comparison to the other three tested models, further confirming the FIGARCH(1,0.7,1) and SGARCH(1,1) models as the respective best fits. The daily volatilities used for the analysis are obtained from the FIGARCH(1,0.7,1) and SGARCH(1,1) models. The estimated parameters of the FIGARCH(1,0.7,1) and SGARCH(1,1) model are respectively used to obtain simulated daily volatilities which are analysed using PE to examine how well the optimal models fit the Bitcoin data.

6. Volatility analysis

In this section, we analyse the behaviour of the daily volatility of Bitcoin using monthly and annual time frames. In addition, we test for how well the selected GARCH models fit the Bitcoin data. This test is done based on a comparison of the monthly and annual results obtained from the real-world data to those obtained from the simulated data.

6.1. Dynamics of Bitcoin volatility

Here, we investigate the dynamics of Bitcoin volatility on an annual and monthly basis. For the annual analysis, a single H value is obtained for each year yielding 8 H values, while for the monthly analysis, a single H value is obtained for each month per year yielding 96 (12 months by 8 years) H values. The analysis is carried out for the daily volatilities of Bitcoin obtained from the FIGARCH(1,0.7,1) model and the SGARCH(1,1) model, respectively.

Table 3
Annual H values of Bitcoin volatility.

Year	Annual H values	
	FIGARCH	SGARCH
2015	0.8685	0.7220
2016	0.8251	0.6787
2017	0.9145	0.8177
2018	0.9112	0.7917
2019	0.8878	0.7541
2020	0.8913	0.7842
2021	0.9444	0.8786
2022	0.867	0.7544

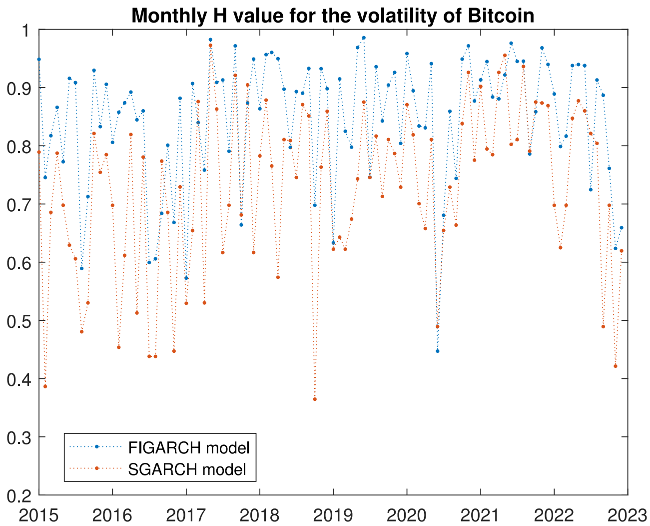


Fig. 5. Monthly H values of Bitcoin volatility.

6.1.1. Annual analysis

Table 3 shows the annual H values of Bitcoin volatility obtained from the selected GARCH models from 2015–2022. From Table 3, it can be seen that the annual H values of Bitcoin volatility obtained from the FIGARCH model, which is the better fit model based on MLE, AIC and BIC, are higher than the annual H values from the SGARCH model over the 8 years under consideration, i.e., the better fit model to the data yields higher H values. Therefore, the variable H might be useful in determining better fit GARCH models. Furthermore, since a higher H corresponds to less order in a system making it less predictable from an ordinal perspective, the results imply that the daily volatility of Bitcoin extracted from the SGARCH model has more order and a higher level of predictability from an ordinal perspective to that obtained from the FIGARCH model. However, the annual H values of Bitcoin volatility are generally low, even for the volatilities obtained from the FIGARCH model. It can be seen from Table 3 that the annual H value of Bitcoin is lower than 0.9 with the exception of the H value for 2021 for the FIGARCH model. The lowest and highest H values were recorded in 2016 and 2021, respectively for both the FIGARCH and SGARCH models as shown in Table 3. Therefore, the volatility of Bitcoin was most predictable in 2016 and least predictable in 2021 from an ordinal perspective.

6.1.2. Monthly analysis

We begin the monthly analysis by considering the monthly H values of Bitcoin volatility. Fig. 5 shows the H values of Bitcoin volatility obtained from the FIGARCH and SGARCH models for each month from 2015–2022. Fig. 5 clearly shows that the H values of Bitcoin volatility obtained from the FIGARCH model are higher than those obtained from the SGARCH model. The monthly H values of the volatility of Bitcoin obtained from the FIGARCH model range from 0.4472 observed in June 2020 to 0.9857 in June 2019, while those obtained from the SGARCH model range from 0.3646 observed in October 2018 to 0.9726 observed in May 2017. In comparison to the annual H values of Bitcoin volatility obtained in Section 6.1.1., the monthly H values of Bitcoin volatility have a bigger range. This shows that as the time frame is increased, H becomes more stable and less varied.

Next, we consider the dynamics of the daily volatility of Bitcoin obtained from the FIGARCH(1,0.7,1) and SGARCH(1,1) models. The application of (11) gives the percentage return of each of the possible patterns – 123, 132, 213, 231, 312 and 321 – for each month over the 8 years. The range of the percentage returns for the 6 patterns are recorded in Table 4 for the daily volatilities obtained from both the FIGARCH(1,0.7,1) and SGARCH(1,1) models. It can be observed that pattern 321 is the dominant pattern present in the dynamics of Bitcoin volatility. Table 4 also shows that some of the patterns were absent for some months having a

Table 4
Range of $\%(\mathcal{P}_i)$ for Bitcoin volatility obtained from the FIGARCH(1,0.7,1) and SGARCH(1,1) models.

Pattern	FIGARCH $\%(\mathcal{P}_i)$ range	SGARCH $\%(\mathcal{P}_i)$ range
123	0%–24%	0%–34%
132	3%–24%	3%–21%
213	3%–25%	0%–21%
231	0%–28%	0%–19%
312	0%–24%	0%–21%
321	17%–75%	14%–81%

Table 5
Number of forbidden patterns.

Pattern	Number of months pattern is forbidden/absent	
	FIGARCH	SGARCH
123	13	15
132	0	0
213	0	2
231	2	16
312	4	12
321	0	0

range starting from 0%. Patterns 123, 231 and 312 were absent during some months for Bitcoin volatility obtained from both selected models. In addition, pattern 213 was absent in some months for Bitcoin volatility obtained from the SGARCH model. Furthermore, it can be seen that patterns 132 and 321 is not forbidden in any month for both selected models, with pattern 321 having higher values, hence confirming its dominance. Table 5 shows the number of months (from a total of 96 months) each pattern is forbidden. From Table 5, it can be observed that there are more forbidden patterns present in the Bitcoin volatility obtained from the SGARCH model than those obtained from the FIGARCH model. This is expected as the H values of Bitcoin volatility obtained from the FIGARCH model are higher than those obtained from the SGARCH model. Although there is a significant discrepancy between the number of months patterns 231 and 312 are absent for the selected models, pattern 123 is an exception. The pattern 123 is absent for 13 months for Bitcoin volatility obtained from the FIGARCH model and 15 months for that obtained from the SGARCH model. It is important to note that for the Bitcoin volatility obtained from the SGARCH model, pattern 231 is absent for the greatest number of months — 16 months, a sharp contrast to the Bitcoin volatility obtained from the FIGARCH model where pattern 213 is only absent in 2 months. On the other hand, pattern 321 is not forbidden for any month for both models. Hence, the probability of the volatility of Bitcoin increasing steadily is far lower than the probability of its volatility decreasing steadily.

From the monthly analysis of Bitcoin volatility, it is clear that the volatility of Bitcoin is most likely to fall steadily than to rise or have an alternating movement. This is further confirmed by the pattern 123 being absent for a large number of months. Fig. 6 presents a visual representation of the patterns of movement of the volatility of Bitcoin obtained from the FIGARCH model. Two periods are shown on the figure — July 19, 2017 to September 4, 2017 and March 10, 2020–April 26, 2020. It is observed that Bitcoin volatility experiences large or smaller jumps at intervals. However, those increases in volatility are not usually followed by a consecutive increase, hence, the minimal appearance of the pattern 123. On the other hand, it is observed that the increases are followed by a steady decrease in the volatility of Bitcoin for a significant number of days before another increase in volatility might occur. This is responsible for the dominance of the pattern 321 in the volatility of Bitcoin. This behaviour observed in the Bitcoin volatility obtained from the FIGARCH model and illustrated in Fig. 6 is also exhibited by the Bitcoin volatility obtained from the SGARCH model. The existence of forbidden patterns and the overall low H values of the real-world volatility of Bitcoin indicate its high predictability. The results of consistent declines in the volatility of Bitcoin are corroborated by Bouoiyour and Selmi [3] and Fulton [46].

6.2. Test of selected GARCH models

In this subsection, we evaluate the goodness of the selected FIGARCH(1,0.7,1) and SGARCH(1,1) models. This is done by analysing the simulated volatilities generated from the parameters of the respective chosen model. We use PE and NFP for the analysis and check how well the results obtained reflect the results obtained from the analysis of Bitcoin volatility. For simplicity and easy comparison, the simulated volatilities are referred to as FIGARCH simulated volatilities and SGARCH simulated volatilities, respectively. Likewise, the volatilities obtained from the real world returns of Bitcoin which have been analysed in Section 6.1 are referred to as FIGARCH and SGARCH volatilities, respectively.

6.2.1. Annual analysis

Fig. 7 shows how the annual H values of the FIGARCH volatilities compares to the range of annual H values of the FIGARCH simulated volatilities. It can be observed from Figs. 7(c)–7(g) that the annual H values of FIGARCH volatilities for 2017–2021 lie to the right of the average of the annual H values of FIGARCH simulated volatilities. Hence, the positive z – values. On the other

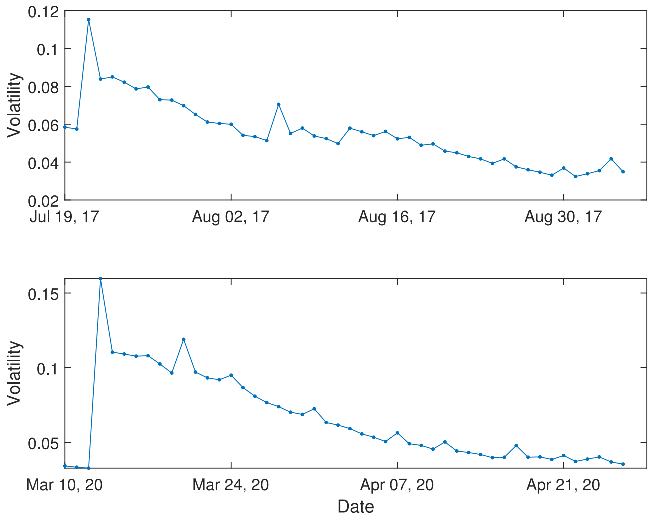


Fig. 6. Visual representation of the dynamics of Bitcoin volatility.

hand, Figs. 7(a), 7(b) and 7(h) show that the annual H values of FIGARCH volatilities for 2015, 2016 and 2022 are to the left of the H values of the simulated volatilities for those years. This is confirmed by the negative $z-$ values for those 3 years. The H value of FIGARCH volatilities for 2015 is the closest to the mean value of the H values of FIGARCH simulated volatilities for the same year, having a value of −0.18 standard deviations from the mean. On the other hand, 2016 is the furthest with a $z-$ value of −2.79. The H value of FIGARCH volatilities for 2021 also lie far away from the mean of the H values of FIGARCH simulated volatilities for the corresponding year, with a $z-$ value of 2.68.

Similarly, Fig. 8 shows that while the H values of the SGARCH volatilities are between −2 and 2 standard deviations away from the mean of the H values of the SGARCH simulated volatilities for all the years, 2016 and 2021 have high $z-$ values of −3.83 and 2.94, respectively. Fig. 8(b) shows that 2016 H value for the SGARCH volatilities do not lie on the range of values from the simulated data, hence it is an outlier. The H value for 2020 SGARCH volatilities is the closest to the mean of the H values from the simulations of the same year with a $z-$value of −0.10 as shown in Fig. 8(f). In addition, with the exceptions of 2017, 2018 and 2021 shown in Figs. 8(c), 8(d) and 8(g) respectively, the H values of the SGARCH volatilities lie to the left of the mean of the H values of their corresponding SGARCH simulated volatilities, thus having negative $z-$ values.

From the annual analysis of the goodness of fit of the selected FIGARCH and the SGARCH models, it can be observed that the simulated volatilities reflect the real-world Bitcoin volatility reasonably well. This is confirmed by the $z-$ values of the annual H values for the real data lying within the range of −2 and +2 with the exceptions of 2016 and 2021 for both the FIGARCH and SGARCH volatilities. The high $z-$ values of the annual H values for both years may mean that the simulated volatilities for those years may not reflect their corresponding real-world volatilities. However, only the $z-$value for the 2016 H value for the SGARCH volatility is an outright outlier as it falls outside the acceptable range of −3 and +3. The presence of a complete outlier in the case of SGARCH volatility proves that although both the FIGARCH and SGARCH models fit the Bitcoin volatility reasonably well, the FIGARCH model is indeed the better fit for the time period considered.

6.2.2. *Monthly analysis*

The real-world data consists of 96 months as used in Section 6.1.1, while the simulated data consists of 9600 (96 months per simulation by 100 simulations) months. We consider the percentage of months in which each pattern is forbidden over the 8 years. As observed with the real-world data, Table 6 shows that the simulated data also has pattern 123 being absent in the most number of months and pattern 321 not being absent in any month. The monthly comparison confirms the results obtained from the annual comparison, as the results obtained based on NFP, also shows that there is good agreement between the real-world data and the simulated data.

From the comparison between the results obtained from the analysis of the real-world Bitcoin data and the simulated data, it is observed that the simulated data is a good reflection of the real-world data. Hence, the selected FIGARCH and SGARCH models are good fits for the Bitcoin data. The results further confirm that of the two selected models, the FIGARCH model is the better fit.

7. Conclusion

This paper modelled the volatility of the daily returns of Bitcoin for 8 years — January 1, 2015–December 31, 2022. A comparative study was conducted between different GARCH models with the errors following three distributions — skewed normal, skewed student-t or skewed generalized error distributions. The FIGARCH(1,0.7,1) and SGARCH(1,1) models for a skewed

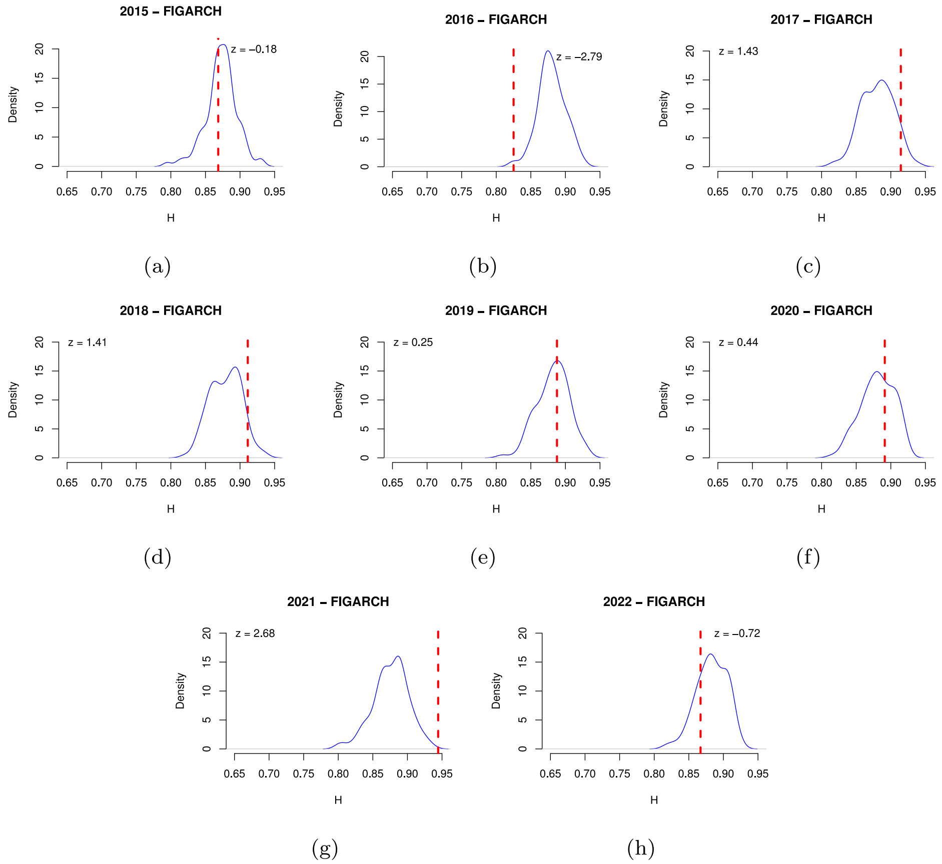


Fig. 7. Density plots of annual H values of the volatilities obtained from 100 simulations generated from the FIGARCH model parameters. The red broken lines represent the annual H values of the volatilities of the real-world BTC-USD data.

Table 6
Percentage of months each pattern is forbidden.

Pattern	Number of months pattern is forbidden/absent			
	FIGARCH	FIGARCH simulation	SGARCH	SGARCH simulation
123	14%	17%	16%	18%
132	0%	1%	0%	1%
213	0%	0%	2%	1%
231	2%	1%	17%	11%
312	4%	2%	13%	11%
321	0%	0%	0%	0%

generalized error distribution were the best fit models for the Bitcoin data. These models were chosen on the basis of three selection criteria — MLE, AIC and BIC.

PE, a pattern recognition tool which is represented by the variable H for calculations, was used to investigate the dynamics and level of randomness of Bitcoin daily volatility using annual and monthly time frames. The results showed that the annual and monthly H values of Bitcoin daily volatility are generally low, implying that the direction of movements of Bitcoin daily volatility are quite predictable from an ordinal perspective. Further investigations into the pattern of movements and dynamics of the daily volatility of Bitcoin showed that there was the absence of an increase in Bitcoin volatility for many months, attesting to the presence of forbidden patterns. On the other hand, the results showed considerable periods of steady declines in Bitcoin volatility. These results confirm that the direction of the volatility of Bitcoin is somewhat predictable as the probability of a steady decrease is superior to the probability of an alternate movement or a steady increase.

Furthermore, to evaluate the goodness of the two selected GARCH models, volatilities were simulated for the length of period considered based on the parameters of the selected FIGARCH(1,0.7,1) and SGARCH(1,1) models with the errors following a skewed generalized error distribution. Comparisons were made between the H values of the real-world Bitcoin daily volatilities and the simulated volatilities to check for how well the simulated volatilities generated from the parameters of the selected models reflect the real-world daily volatilities. The comparisons were made using standard normal (z) test statistic. The results showed that the

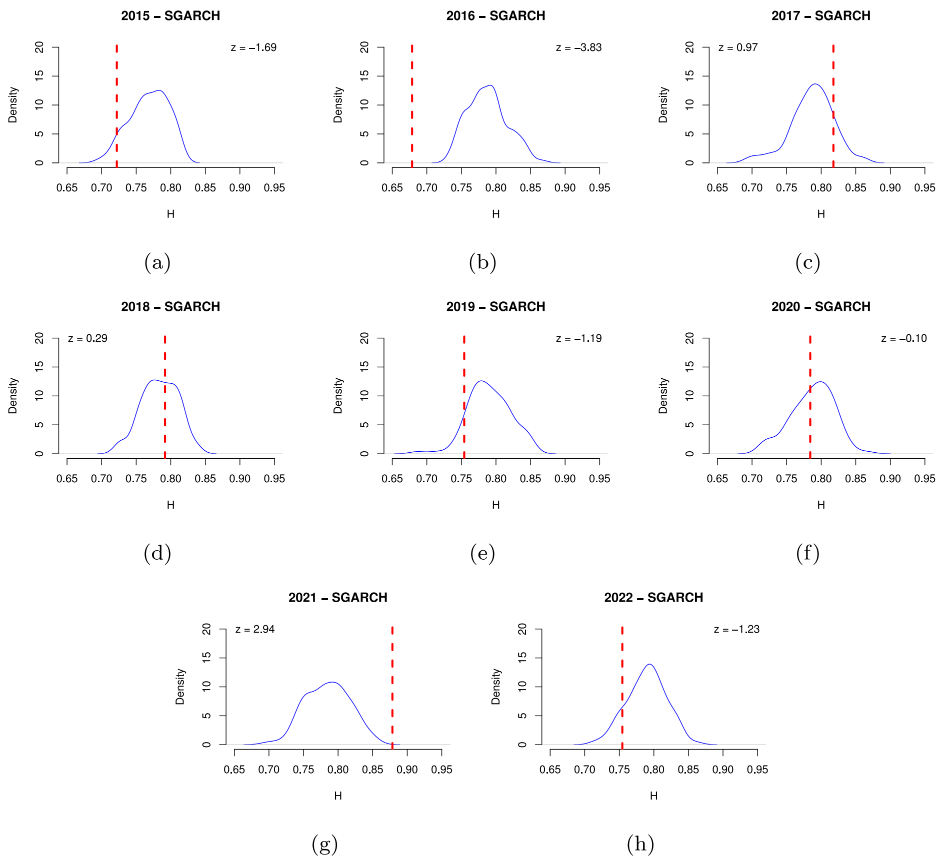


Fig. 8. Density plots of annual H values of the volatilities obtained from 100 simulations generated from the SGARCH model parameters. The red broken lines represent the annual H values of the volatilities of the real-world BTC-USD data.

dynamics of the simulated volatilities pretty much reflects the dynamics of the real-world volatility of Bitcoin as the z-values of the H values obtained from the real-world data lie within the range of −2 and +2 of the corresponding simulated data, with the exceptions of 2016 and 2017. In addition, a comparison was made between the NFP that exist within the real-world data and the simulated data, which further confirmed that the simulated data is a good reflection of the real-world data. The results obtained from the comparison between the simulated and real-world volatilities of Bitcoin further showed that the simulated volatilities are as predictable as the real-world volatility of Bitcoin with the simulated volatilities also having a greater probability of steady decreases than alternating movements or steady increases. Hence, confirming that the selected FIGARCH(1,0.7,1) and SGARCH(1,1) models are good fits for the Bitcoin data.

Summarily, analysis carried out on the volatility of Bitcoin data using PE shows that it is quite predictable from an ordinal perspective. It is however noteworthy that this study focuses on a specific time period. Hence, the results might change if a different time period is used due to the possibility of obtaining a different GARCH model that best fit the Bitcoin data for a different time period than the one considered in this paper. Also, the use of high-volume data such as minute or hourly data could provide better insight into the dynamics of the volatility of Bitcoin. However, daily data was used for this analysis due to data availability.

CRediT authorship contribution statement

Praise Otito Obanya: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Resources, Software, Validation, Visualization, Writing – original draft, Writing – review & editing. **Modisane Seitshiro:** Conceptualization, Funding acquisition, Methodology, Project administration, Resources, Supervision, Validation, Visualization, Writing – review & editing. **Carel Petrus Olivier:** Conceptualization, Funding acquisition, Methodology, Project administration, Supervision, Validation, Visualization, Writing – review & editing. **Tanja Verster:** Conceptualization, Funding acquisition, Methodology, Project administration, Supervision, Validation, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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