

A study of nonlinear integrable fifth-order equation and a longitudinal wave equation in a magnetoelectro-elastic circular rod

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**A STUDY OF NONLINEAR FIFTH-ORDER
EQUATION AND LONGITUDINAL WAVE
EQUATION IN A
MAGNETOELECTRO-ELASTIC CIRCULAR
ROD**

by

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Declaration

I, MAJOLA CHRISTINA, student number 25278045, declare that this dissertation for the degree of Master of Science in Applied Mathematics at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other university, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

Signed:

Miss C. Majola

Date:

This dissertation has been submitted with my approval as a University supervisor and would certify that the requirements applicable for the Master of Science degree rules and regulations have been fulfilled.

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Declaration of Publications

Details of contribution to publications that form part of this dissertation.

Chapter 1

C. Majola, B. Muatjetjea, A.R. Adem, A symmetry reduction, an exact solution and conservation laws of a two-wave mode Korteweg-de Vries equation. Submitted for publication.

Chapter 2

C. Majola, B. Muatjetjea, A.R. Adem, Conservation laws and exact solutions for a nonlinear integrable fifth-order equation. Submitted for publication.

Chapter 3

C. Majola, B. Muatjetjea, A.R. Adem, Conservation laws and exact solutions for new optical solitons of the Longitudinal wave equation in a Magneto-electro-Elastic circular rod. Submitted for publication.

Dedication

To my loving family and everyone who showed me support throughout my studies.

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List of Acronyms

ODE: Ordinary Differential equation

PDE: Partial differential equation

NLPDE: Nonlinear partial differential equation

TKdV: Two-wave mode Korteweg-de Vries

Abstract

In this dissertation we study two nonlinear partial differential equations namely, the longitudinal wave equation in a magnetoelectro-elastic circular rod and a new integrable fifth-order equation. We employ the multiplier method to find conservation laws and the Lie symmetry method to find symmetry reductions and exact solutions for the two underlying equations.

Introduction

Nonlinear partial differential equations (NLPDEs) depict a range of physical occurrences in the areas such as physics, chemistry, biology and fluid dynamics. Thus computing solutions of such NLPDEs is unavoidable task. However, determining solutions of NLPDEs is a very challenging task and only in certain cases one can obtain exact solutions. The most important methods for construction of exact solutions found in the literature are the inverse scattering transform, Hirota's bilinear method, homogeneous balance method, Bäcklund transformation, Darboux transformation method, the Fan sub-equation method and Lie symmetry analysis method [1–10].

The Lie symmetry method, also called the Lie group analysis method, is one of the most resourceful methods to extract solutions of NLPDEs. It is grounded upon the analysis of the invariance under one parameter Lie group of point transformations [8–10]. Established by Sophus Lie in the latter half of the nineteenth century, this technique methodically merges well known ad hoc methods to construct solutions for differential equations. There has been a considerable developments in Lie symmetry methods for differential equations [11–17].

Conservation laws [18–21] play an important role in reducing and finding the solution of differential equations. Computing conservation laws of different equations is the first step in determining exact solutions. The number of conservation laws determine how close one is to the complete solution. Thus the more conservation laws one finds, the closer one gets to the solution.

In this dissertation we study a new nonlinear integrable fifth-order equation [22] and a longitudinal wave equation in a magnetoelectro-elastic circular rod [23]. Firstly,

we study a new nonlinear integrable fifth-order equation that is given by

$$u_{ttt} - u_{txxxx} + au_{xxx} - 4(u_x u_t)_{xx} - 4(u_x u_{tx})_x = 0, \quad (1)$$

where $u(t, x)$ is a real valued function and a is a non-zero arbitrary constant. The above equation has a third-order term with respect to t , two dispersive terms and two nonlinear terms. Wazwaz and Kaur [22] computed the multiple soliton solutions of (1).

Finally, we consider the longitudinal wave equation in a magnetoelectro-elastic circular rod [23]

$$u_{tt} - r^2 u_{xx} - \left(\frac{1}{2}r^2 u^2 + nu_{tt}\right)_{xx} = 0.$$

Here r denotes the linear longitudinal wave velocity for a magnetoelectro-elastic circular rod and n is the dispersion parameter.

The outline of this dissertation is as follows:

In Chapter one, we give some basic methods of Lie symmetry analysis and conservation laws of partial differential equations.

In Chapter two, the multiplier method and Lie symmetry method are used in the analysis of a nonlinear integrable fifth-order equation.

In Chapter three, the conservation laws and exact solutions for the longitudinal wave equation in a magnetoelectro-elastic circular rod are computed.

In Chapter four, we discuss and conclude what we have done in this dissertation.

A bibliography is given at the end of this dissertation.

Chapter 1

Preliminaries

In this chapter we give some basic methods of Lie symmetry analysis and conservation laws of partial differential equations (PDEs).

1.1 Continuous one-parameter groups

Let $x = (x^1, \dots, x^n)$ be the independent variables with coordinates x^i and $u = (u^1, \dots, u^m)$ be the dependent variables with coordinates u^α (n and m finite). Consider a change of the variables x and u involving a real parameter a :

$$T_a : \bar{x}^i = f^i(x, u, a), \quad \bar{u}^\alpha = \phi^\alpha(x, u, a), \quad (1.1)$$

where a continuously ranges in values from a neighborhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$, and f^i and ϕ^α are differentiable functions.

Definition 1.1 A set G of transformations (1.1) is called a *continuous one-parameter (local) Lie group of transformations* in the space of variables x and u if

- (i) For $T_a, T_b \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then $T_b T_a = T_c \in G$, $c = \phi(a, b) \in \mathcal{D}$
(Closure)
- (ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0 T_a = T_a T_0 = T_a$ (Identity)

- (iii) For $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that
 $T_a T_{a^{-1}} = T_{a^{-1}} T_a = T_0$ (Inverse)

We note that the associativity property follows from (i). The group property (i) can be written as

$$\begin{aligned}\bar{x}^i &\equiv f^i(\bar{x}, \bar{u}, b) = f^i(x, u, \phi(a, b)), \\ \bar{u}^\alpha &\equiv \phi^\alpha(\bar{x}, \bar{u}, b) = \phi^\alpha(x, u, \phi(a, b))\end{aligned}\tag{1.2}$$

and the function ϕ is called the *group composition law*. A group parameter a is called *canonical* if $\phi(a, b) = a + b$.

Theorem 1.1 For any $\phi(a, b)$, there exists the canonical parameter $\tilde{a}$ defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)}, \text{ where } w(s) = \left. \frac{\partial \phi(s, b)}{\partial b} \right|_{b=0}.$$

1.2 Prolongation of point transformations and Group generator

The derivatives of u with respect to x are defined as

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j D_i(u_i), \dots, \tag{1.3}$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots, \quad i = 1, \dots, n \tag{1.4}$$

is the operator of total differentiation. The collection of all first derivatives u_i^α is denoted by $u_{(1)}$, i.e.,

$$u_{(1)} = \{u_i^\alpha\} \quad \alpha = 1, \dots, m, \quad i = 1, \dots, n.$$

Similarly

$$u_{(2)} = \{u_{ij}^\alpha\} \quad \alpha = 1, \dots, m, \quad i, j = 1, \dots, n$$

and $u_{(3)} = \{u_{ijk}^\alpha\}$ and likewise $u_{(4)}$ etc. Since $u_{ij}^\alpha = u_{ji}^\alpha$, $u_{(2)}$ contains only u_{ij}^α for $i \leq j$. In the same manner $u_{(3)}$ has only terms for $i \leq j \leq k$. There is natural ordering in $u_{(4)}$, $u_{(5)}$ $\dots$.

In group analysis all variables $x, u, u_{(1)} \dots$ are considered functionally independent variables connected only by the differential relations (1.3). Thus the u_s^α are called differential variables.

We now consider a p th-order PDE(s), namely

$$E_\alpha(x, u, u_{(1)}, \dots, u_{(p)}) = 0. \quad (1.5)$$

Prolonged or extended groups

If $z = (x, u)$, one-parameter group of transformations G is

$$\begin{aligned} \bar{x}^i &= f^i(x, u, a), \quad f^i|_{a=0} = x^i, \\ \bar{u}^\alpha &= \phi^\alpha(x, u, a), \quad \phi^\alpha|_{a=0} = u^\alpha. \end{aligned} \quad (1.6)$$

According to the Lie's theory, the construction of the symmetry group G is equivalent to the determination of the corresponding *infinitesimal transformations* :

$$\bar{x}^i \approx x^i + a \xi^i(x, u), \quad \bar{u}^\alpha \approx u^\alpha + a \eta^\alpha(x, u) \quad (1.7)$$

obtained from (1.1) by expanding the functions f^i and ϕ^α into Taylor series in a about $a = 0$ and also taking into account the initial conditions

$$f^i|_{a=0} = x^i, \quad \phi^\alpha|_{a=0} = u^\alpha.$$

Thus, we have

$$\xi^i(x, u) = \left. \frac{\partial f^i}{\partial a} \right|_{a=0}, \quad \eta^\alpha(x, u) = \left. \frac{\partial \phi^\alpha}{\partial a} \right|_{a=0}. \quad (1.8)$$

One can now introduce the *symbol* of the infinitesimal transformations by writing (1.7) as

$$\bar{x}^i \approx (1 + a X)x, \quad \bar{u}^\alpha \approx (1 + a X)u,$$

where

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}. \quad (1.9)$$

This differential operator X is known as the infinitesimal operator or generator of the group G . If the group G is admitted by (1.5), we say that X is an *admitted operator* of (1.5) or X is an *infinitesimal symmetry* of equation (1.5).

We now see how the derivatives are transformed.

The D_i transforms as

$$D_i = D_i(f^j) \bar{D}_j, \quad (1.10)$$

where $\bar{D}_j$ is the total differentiations in transformed variables $\bar{x}^i$. So

$$\bar{u}_i^\alpha = \bar{D}_j(u^\alpha), \quad \bar{u}_{ij}^\alpha = \bar{D}_j(\bar{u}_i^\alpha) = \bar{D}_i(\bar{u}_j^\alpha), \dots$$

Now let us apply (1.10) and (1.6)

$$\begin{aligned} D_i(\phi^\alpha) &= D_i(f^j) \bar{D}_j(\bar{u}^\alpha) \\ &= D_i(f^j) \bar{u}_j^\alpha. \end{aligned} \quad (1.11)$$

This

$$\left(\frac{\partial f^j}{\partial x^i} + u_i^\beta \frac{\partial f^j}{\partial u^\beta} \right) \bar{u}_j^\alpha = \frac{\partial \phi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \phi^\alpha}{\partial u^\beta}. \quad (1.12)$$

The quantities $\bar{u}_j^\alpha$ can be represented as functions of $x, u, u_{(i)}$, a for small a , i.e., (1.12) is locally invertible:

$$\bar{u}_i^\alpha = \psi_i^\alpha(x, u, u_{(1)}, a), \quad \psi^\alpha|_{a=0} = u_i^\alpha. \quad (1.13)$$

The transformations in $x, u, u_{(1)}$ space given by (1.6) and (1.13) form a one-parameter group (one can prove this but we do not consider the proof) called the first prolongation or just extension of the group G and denoted by $G^{[1]}$.

We let

$$\bar{u}_i^\alpha \approx u_i^\alpha + a \zeta_i^\alpha \quad (1.14)$$

be the infinitesimal transformation of the first derivatives so that the infinitesimal transformation of the group $G^{[1]}$ is (1.7) and (1.14).

Higher-order prolongations of G , viz. $G^{[2]}$, $G^{[3]}$ can be obtained by derivatives of (1.11).

Prolonged generators

Using (1.11) together with (1.7) and (1.14) we get

$$\begin{aligned}
D_i(f^j)(\bar{u}_j^\alpha) &= D_i(\phi^\alpha) \\
D_i(x^j + a\xi^j)(u_j^\alpha + a\zeta_j^\alpha) &= D_i(u^\alpha + a\eta^\alpha) \\
(\delta_i^j + aD_i\xi^j)(u_j^\alpha + a\zeta_j^\alpha) &= u_i^\alpha + aD_i\eta^\alpha \\
u_i^\alpha + a\zeta_i^\alpha + au_j^\alpha D_i\xi^j &= u_i^\alpha + aD_i\eta^\alpha \\
\zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \quad (\text{sum on } j). \quad (1.15)
\end{aligned}$$

This is called the first prolongation formula. Likewise, one can obtain the second prolongation, viz.,

$$\zeta_{ij}^\alpha = D_j(\eta_i^\alpha) - u_{ik}^\alpha D_j(\xi^k), \quad (\text{sum on } k). \quad (1.16)$$

By induction (recursively)

$$\zeta_{i_1, i_2, \dots, i_p}^\alpha = D_{i_p}(\zeta_{i_1, i_2, \dots, i_{p-1}}^\alpha) - u_{i_1, i_2, \dots, i_{p-1}j}^\alpha D_{i_p}(\xi^j), \quad (\text{sum on } j). \quad (1.17)$$

The first and higher prolongations of the group G form a group denoted by $G^{[1]}, \dots, G^{[p]}$.

The corresponding prolonged generators are

$$\begin{aligned}
X^{[1]} &= X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} \quad (\text{sum on } i, \alpha), \\
&\cdot \\
&\cdot \\
&\cdot \\
X^{[p]} &= X^{[p-1]} + \zeta_{i_1, \dots, i_p}^\alpha \frac{\partial}{\partial u_{i_1, \dots, i_p}^\alpha} \quad p \geq 1,
\end{aligned}$$

where

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}.$$

1.3 Group admitted by a PDE

Definition 1.2 The vector field

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}, \quad (1.18)$$

is a *point symmetry* of the p th-order PDE (1.5), if

$$X^{[p]}(E_\alpha) = 0 \quad (1.19)$$

whenever $E_\alpha = 0$. This can also be written as

$$X^{[p]} E_\alpha \big|_{E_\alpha=0} = 0, \quad (1.20)$$

where the symbol $\big|_{E_\alpha=0}$ means evaluated on the equation $E_\alpha = 0$.

Definition 1.3 Equation (1.19) is called the *determining equation* of (1.5) because it determines all the infinitesimal symmetries of (1.5).

Definition 1.4 (Symmetry group) A one-parameter group G of transformations (1.1) is called a symmetry group of equation (1.5) if (1.5) is form-invariant (has the same form) in the new variables $\bar{x}$ and $\bar{u}$, i.e.,

$$E_\alpha(\bar{x}, \bar{u}, u_{(1)}^-, \dots, u_{(p)}^-) = 0, \quad (1.21)$$

where the function E_α is the same as in equation (1.5).

1.4 Group invariants

Definition 1.5 A function $F(x, u)$ is called an *invariant of the group of transformation* (1.1) if

$$F(\bar{x}, \bar{u}) \equiv F(f^i(x, u, a), \phi^\alpha(x, u, a)) = F(x, u), \quad (1.22)$$

identically in x, u and a .

Theorem 1.2 (Infinitesimal criterion of invariance) A necessary and sufficient condition for a function $F(x, u)$ to be an invariant is that

$$X F \equiv \xi^i(x, u) \frac{\partial F}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial F}{\partial u^\alpha} = 0. \quad (1.23)$$

It follows from the above theorem that every one-parameter group of point transformations (1.1) has $n - 1$ functionally independent invariants, which can be taken to be the left-hand side of any first integrals

$$J_1(x, u) = c_1, \dots, J_{n-1}(x, u) = c_n$$

of the characteristic equations

$$\frac{dx^1}{\xi^1(x, u)} = \dots = \frac{dx^n}{\xi^n(x, u)} = \frac{du^1}{\eta^1(x, u)} = \dots = \frac{du^n}{\eta^n(x, u)}.$$

Theorem 1.3 If the infinitesimal transformation (1.7) or its symbol X is given, then the corresponding one-parameter group G is obtained by solving the Lie equations

$$\frac{d\bar{x}^i}{da} = \xi^i(\bar{x}, \bar{u}), \quad \frac{d\bar{u}^\alpha}{da} = \eta^\alpha(\bar{x}, \bar{u}) \quad (1.24)$$

subject to the initial conditions

$$\bar{x}^i|_{a=0} = x, \quad \bar{u}^\alpha|_{a=0} = u.$$

1.5 Lie algebra

Let us consider two operators X_1 and X_2 defined by

$$X_1 = \xi_1^i(x, u) \frac{\partial}{\partial x^i} + \eta_1^\alpha(x, u) \frac{\partial}{\partial u^\alpha}$$

and

$$X_2 = \xi_2^i(x, u) \frac{\partial}{\partial x^i} + \eta_2^\alpha(x, u) \frac{\partial}{\partial u^\alpha}.$$

Definition 1.6 The *commutator* of X_1 and X_2 , written as $[X_1, X_2]$, is defined by $[X_1, X_2] = X_1(X_2) - X_2(X_1)$.

Definition 1.7 A Lie algebra is a vector space L (over the field of real numbers) of operators $X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}$ with the following property. If the operators

$$X_1 = \xi_1^i(x, u) \frac{\partial}{\partial x^i} + \eta_1^\alpha(x, u) \frac{\partial}{\partial u^\alpha}, \quad X_2 = \xi_2^i(x, u) \frac{\partial}{\partial x^i} + \eta_2^\alpha(x, u) \frac{\partial}{\partial u^\alpha}$$

are any elements of L , then their commutator

$$[X_1, X_2] = X_1(X_2) - X_2(X_1)$$

is also an element of L . It follows that the commutator is

1. Bilinear: for any $X, Y, Z \in L$ and $a, b \in \mathbb{R}$,

$$[aX + bY, Z] = a[X, Z] + b[Y, Z], \quad [X, aY + bZ] = a[X, Y] + b[X, Z];$$

2. Skew-symmetric: for any $X, Y \in L$,

$$[X, Y] = -[Y, X];$$

3. and satisfies the Jacobi identity: for any $X, Y, Z \in L$,

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

1.6 Conservation laws

1.6.1 Fundamental operators and their relationship

Consider a k th-order system of PDEs of n independent variables $x = (x^1, x^2, \dots, x^n)$ and m dependent variables $u = (u^1, u^2, \dots, u^m)$, namely

$$E_\alpha(x, u, u_{(1)}, \dots, u_{(k)}) = 0, \quad \alpha = 1, \dots, m. \quad (1.25)$$

The *Euler-Lagrange operator*, for each α , is given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial u_{i_1 i_2 \dots i_s}^\alpha}, \quad \alpha = 1, \dots, m, \quad (1.26)$$

The n -tuple vector $T = (T^1, T^2, \dots, T^n)$, $T^j \in \mathcal{A}$, $j = 1, \dots, n$, is a *conserved vector* of (1.25) if T^i satisfies

$$D_i T^i|_{(1.25)} = 0. \quad (1.27)$$

The equation (1.27) defines a *local conservation law* of system (1.25).

A multiplier $\Lambda_\alpha(x, u, u_{(1)}, \dots)$ has the property that

$$\Lambda_\alpha E_\alpha = D_i T^i \quad (1.28)$$

holds identically. The right hand side of (1.28) is a divergence expression. The determining equations for the multiplier Λ_α is

$$\frac{\delta(\Lambda_\alpha E_\alpha)}{\delta u^\alpha} = 0. \quad (1.29)$$

1.7 An illustrative example on finding symmetry reduction, exact solution and conservation laws

In this section we will employ the Lie symmetry method and the multiplier method to construct a symmetry reduction, an exact solution and conservations laws of a two-wave mode Korteweg-de Vries equation (TKdV) [24]

$$u_{tt} - s^2 u_{xx} + u_t u_x + u u_{tx} - \alpha s (u u_{xx} + u_x^2) + u_{txx} - \beta s u_{xxxx} = 0. \quad (1.30)$$

The above equation [24] was proposed as a nonlinear multi-mode dispersive wave equation that describes the wave propagation in one space dimension in a weakly nonlinear and weakly dispersive system.

1.7.1 Symmetry reductions and exact solutions of (1.30)

The vector field

$$X = \xi^1(t, x, u) \frac{\partial}{\partial t} + \xi^2(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u} \quad (1.31)$$

is a Lie point symmetry of (1.30) if

$$X^{[4]} \left\{ u_{tt} - s^2 u_{xx} + u_t u_x + u u_{tx} - \alpha s (u u_{xx} + u_x^2) + u_{txx} - \beta s u_{xxx} \right\} \Big|_{(1.30)} = 0,$$

where $X^{[4]}$ is the fourth prolongation of (1.31). Expanding the above equation and splitting on the derivatives of u leads to the following overdetermined system of linear partial differential equations:

$$\xi_x^1 = 0, \quad (1.32)$$

$$\xi_u^2 = 0, \quad (1.33)$$

$$\xi_u^1 = 0, \quad (1.34)$$

$$\eta_{uu} = 0, \quad (1.35)$$

$$\eta_{xu} - \xi_{xx}^2 = 0, \quad (1.36)$$

$$2\beta s \xi_t^1 - 4\beta s \xi_x^2 + \xi_t^2 = 0, \quad (1.37)$$

$$\beta s \xi_t^1 - \beta s \xi_x^2 + \xi_t^2 = 0, \quad (1.38)$$

$$-\alpha \beta s \eta_u - 2\alpha \beta s \xi_x^2 + \alpha \xi_t^2 - \beta \xi_t^2 = 0, \quad (1.39)$$

$$4\beta s \eta_{xu} - 6\beta s \xi_{xx}^2 - \eta_{tu} + 3\xi_{tx}^2 = 0, \quad (1.40)$$

$$-\beta s \eta_u + \beta s \xi_t^1 - 3\beta s \xi_x^2 + \xi_t^2 = 0, \quad (1.41)$$

$$u \eta_{xu} + \eta_x + 2\eta_{tu} + \eta_{xxu} - \xi_{tt}^1 = 0, \quad (1.42)$$

$$-\alpha s u \eta_{xx} - \beta s \eta_{xxx} - s^2 \eta_{xx} + u \eta_{tx} + \eta_{tt} + \eta_{txx} = 0, \quad (1.43)$$

$$u \beta s \xi_t^1 - 3u \beta s \xi_x^2 - \beta s \eta_1 - 3\beta s \eta_{xxu} + 2\beta s \xi_t^2 + \beta s \xi_{xxx}^2 + u \xi_t^2 = 0, \quad (1.44)$$

$$\begin{aligned} & -2\alpha \beta s u \xi_x^2 - \alpha \beta s \eta_1 - 6\beta^2 s \eta_{xxu} + 4\beta^2 s \xi_{xxx}^2 - 2\beta s^2 \xi_x^2 + \alpha u \xi_t^2 - \beta u \xi_t^2 \\ & + 3\beta \eta_{txu} - 3\beta \xi_{txx}^2 + s \xi_t^2 = 0, \end{aligned} \quad (1.45)$$

$$\begin{aligned} & 2\alpha s u \eta_{xu} - \alpha s u \xi_{xx}^2 + 2\alpha s \eta_x + 4\beta s \eta_{xxu} - \beta s \xi_{xxx}^2 + 2s^2 \eta_{xu} - s^2 \xi_{xx}^2 \\ & - u \eta_{tu} + u \xi_{tx}^2 - \eta_t - 3\eta_{txu} + \xi_{tt}^2 + \xi_{txx}^2 = 0. \end{aligned} \quad (1.46)$$

Solving the above system of equations leads to the two dimensional Lie algebra spanned by the following linearly independent operators:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}. \end{aligned}$$

The linear combination of the translation symmetries $\omega X_1 + X_2$ gives rise to the group-invariant solution

$$u(x, t) = F(z), \quad (1.47)$$

where $z = x - \omega t$ is an invariant of the symmetry $\omega X_1 + X_2$. Substitution of (1.47) into (1.30) results in

$$\begin{aligned} s\beta F_{zzzz} + \omega F_{zzzz} + s^2 F_{zz} - \omega^2 F_{zz} + \omega F F_{zz} - \omega F_z^2 \\ + s\alpha (F F_{zz} + F_z^2) = 0, \end{aligned} \quad (1.48)$$

whose solution is given by

$$F(z) = \frac{3(\omega^2 - s^2) \operatorname{sech}^2\left(\frac{z\sqrt{\omega^2 - s^2}}{2\sqrt{s\beta + \omega}}\right)}{s\alpha + \omega}. \quad (1.49)$$

Consequently the required group invariant solution is completed by (1.47).

1.7.2 Conservation laws of (1.30)

A local conservation law for equation (1.30)

$$G = u_{tt} - s^2 u_{xx} + u_t u_x + u u_{tx} - \alpha s (u u_{xx} + u_x^2) + u_{txxx} - \beta s u_{xxxx} \quad (1.50)$$

is a space-time divergence

$$D_t T^t + D_x T^x|_{(1.30)} = 0, \quad (1.51)$$

which holds for all formal solutions $u(t, x)$ of equation (1.30), where the conserved density T^t and the spatial flux T^x are functions of t, x, u and derivatives of u .

We obtain a multiplier Λ , that is given by

$$\begin{aligned}\Lambda(t, x, u) = & 2s^2\alpha^3 A_2 xt + A_2 s^3 \alpha^2 t^2 + sA_2 \alpha^2 x^2 - 2s^2\alpha A_2 xt - A_2 s^3 t^2 + sA_4 \alpha t \\ & - sA_2 x^2 - 2\alpha A_2 + 2\beta A_2 + A_1 x + A_4 x + A_3 - \left\{ A_6 \cos \left(\frac{\alpha s^{3/2} \sqrt{\alpha^2 - 1} t}{\sqrt{\alpha - \beta}} \right) \right. \\ & - A_5 \sin \left(\frac{\alpha s^{3/2} \sqrt{\alpha^2 - 1} t}{\sqrt{\alpha - \beta}} \right) \left. \right\} \cos \left(\frac{\sqrt{s} \sqrt{\alpha^2 - 1} x}{\sqrt{\alpha - \beta}} \right) + \sin \left(\frac{\sqrt{s} \sqrt{\alpha^2 - 1} x}{\sqrt{\alpha - \beta}} \right) \\ & \left\{ A_5 \cos \left(\frac{\alpha s^{3/2} \sqrt{\alpha^2 - 1} t}{\sqrt{\alpha - \beta}} \right) + A_6 \sin \left(\frac{\alpha s^{3/2} \sqrt{\alpha^2 - 1} t}{\sqrt{\alpha - \beta}} \right) \right\}.\end{aligned}$$

Thus, corresponding to the above multiplier we have the following conservation laws:

$$D_t T_1^t + D_x T_1^x|_{(1.30)} = 0,$$

$$\begin{aligned}T_1^t &= \frac{1}{4} (2xu_x u - u^2 + 4xu_t - u_{xx} + xu_{xxx}), \\ T_1^x &= \frac{1}{4} (-4sx\alpha u_x u + 2xu_t u + 4s^2 u + 2s\alpha u^2 - 4s^2 x u_x + 4s\beta u_{xx} - 4sx\beta u_{xxx} \\ &\quad - 2u_{tx} + 3xu_{txx});\end{aligned}$$

$$D_t T_2^t + D_x T_2^x|_{(1.30)} = 0,$$

$$\begin{aligned}
T_2^t = & \frac{1}{4}(-8t\alpha^2us^3 + 8tus^3 - 2t^2uu_xs^3 + 2t^2\alpha^2uu_xs^3 - t^2u_{xxx}s^3 + t^2\alpha^2u_{xxx}s^3 \\
& - 4t^2u_t s^3 + 4t^2\alpha^2u_t s^3 - 2t\alpha^3u^2s^2 + 2t\alpha u^2s^2 - 8x\alpha^3us^2 + 8x\alpha us^2 \\
& + 4tx\alpha^3uu_xs^2 - 4tx\alpha uu_xs^2 - 2t\alpha^3u_{xx}s^2 + 2t\alpha u_{xx}s^2 + 2tx\alpha^3u_{xxx}s^2 \\
& - 2tx\alpha u_{xxx}s^2 + 8tx\alpha^3u_t s^2 - 8tx\alpha u_t s^2 - 2x\alpha^2u^2s + 2xu^2s + 2\alpha^2u_xs \\
& - 2x^2uu_xs + 2x^2\alpha^2uu_xs - 2u_xs - 2x\alpha^2u_{xx}s + 2xu_{xx}s - x^2u_{xxx}s \\
& + x^2\alpha^2u_{xxx}s - 4x^2u_t s + 4x^2\alpha^2u_t s - 4\alpha uu_x + 4\beta uu_x - 2\alpha u_{xxx} + 2\beta u_{xxx} \\
& - 8\alpha u_t + 8\beta u_t), \\
T_2^x = & \frac{1}{4}(4t^2u_xs^5 - 4t^2\alpha^2u_xs^5 + 8t\alpha^3us^4 - 8t\alpha us^4 - 8tx\alpha^3u_xs^4 + 8tx\alpha u_xs^4 \\
& - 4t^2\alpha^3uu_xs^4 + 4t^2\alpha uu_xs^4 + 4t^2\beta u_{xxx}s^4 - 4t^2\alpha^2\beta u_{xxx}s^4 + 4t\alpha^4u^2s^3 \\
& - 6t\alpha^2u^2s^3 + 2tu^2s^3 + 8x\alpha^2us^3 - 8xus^3 + 4x^2u_xs^3 - 4x^2\alpha^2u_xs^3 \\
& - 8tx\alpha^4uu_xs^3 + 8tx\alpha^2uu_xs^3 - 2t\alpha^2u_{xx}s^3 + 2tu_{xx}s^3 + 8t\alpha^3\beta u_{xx}s^3 \\
& - 8t\alpha\beta u_{xx}s^3 - 8tx\alpha^3\beta u_{xxx}s^3 + 8tx\alpha\beta u_{xxx}s^3 - 2t^2uu_t s^3 + 2t^2\alpha^2uu_t s^3 \\
& - 3t^2u_{txx}s^3 + 3t^2\alpha^2u_{txx}s^3 + 2x\alpha^3u^2s^2 - 2x\alpha u^2s^2 + 4\alpha^3u_xs^2 + 4\alpha u_xs^2 \\
& - 8\alpha^2\beta u_xs^2 - 4x^2\alpha^3uu_xs^2 + 4x^2\alpha uu_xs^2 - 2x\alpha^3u_{xx}s^2 + 2x\alpha u_{xx}s^2 \\
& + 8x\alpha^2\beta u_{xx}s^2 - 8x\beta u_{xx}s^2 + 4x^2\beta u_{xxx}s^2 - 4x^2\alpha^2\beta u_{xxx}s^2 + 4tx\alpha^3uu_t s^2 \\
& - 4tx\alpha uu_t s^2 - 4t\alpha^3u_{tx}s^2 + 4t\alpha u_{tx}s^2 + 6tx\alpha^3u_{txx}s^2 - 6tx\alpha u_{txx}s^2 + 8\alpha^2uu_xs \\
& - 8\alpha\beta uu_xs - 8\beta^2u_{xxx}s + 8\alpha\beta u_{xxx}s + 2\alpha^2u_t s - 2x^2uu_t s + 2x^2\alpha^2uu_t s - 2u_t s \\
& - 4x\alpha^2u_{tx}s + 4xu_{tx}s - 3x^2u_{txx}s + 3x^2\alpha^2u_{txx}s - 4\alpha uu_t + 4\beta uu_t - 6\alpha u_{txx} \\
& + 6\beta u_{txx});
\end{aligned}$$

$$D_t T_3^t + D_x T_3^x|_{(1.30)} = 0,$$

$$\begin{aligned}
T_3^t &= \frac{1}{4}(2u_xu + 4u_t + u_{xxx}), \\
T_3^x &= \frac{1}{4}(-4s\alpha u_xu + 2u_tu - 4s^2u_x - 4s\beta u_{xxx} + 3u_{txx});
\end{aligned}$$

$$D_t T_4^t + D_x T_4^x|_{(1.30)} = 0,$$

$$\begin{aligned}
T_4^t &= \frac{1}{4}(2st\alpha u_x u + 2xu_x u - 4s\alpha u - u^2 + st\alpha u_{xxx} + 4st\alpha u_t + 4xu_t - u_{xx} + xu_{xxx}), \\
T_4^x &= \frac{1}{4}(-4s^2 t \alpha^2 u_x u - 4sx\alpha u_x u + 2st\alpha u_t u + 2xu_t u + 4s^2 u + s\alpha u^2 - 4s^3 t \alpha u_x \\
&\quad - 4s^2 t \alpha \beta u_{xxx} - 4s^2 x u_x + 3st\alpha u_{txx} - s\alpha u_{xx} + 4s\beta u_{xx} - 4sx\beta u_{xxx} - 2u_{tx} \\
&\quad + 3xu_{txx});
\end{aligned}$$

$$\begin{aligned}
D_t T_5^t + D_x T_5^x|_{(1.30)} &= 0, \\
\Gamma &= \frac{\sqrt{\alpha^2 - 1}\sqrt{s}(\alpha st + x)}{\sqrt{\alpha - \beta}},
\end{aligned}$$

$$\begin{aligned}
T_5^t &= \frac{1}{4(\alpha - \beta)^{3/2}} \left\{ -3s^{3/2}\sqrt{\alpha^2 - 1}\cos(\Gamma)u\alpha^2 - s\sqrt{\alpha - \beta}\sin(\Gamma)u_x\alpha^2 \right. \\
&\quad - \sqrt{s}\sqrt{\alpha^2 - 1}\cos(\Gamma)u^2\alpha + 4s^{3/2}\sqrt{\alpha^2 - 1}\beta\cos(\Gamma)u\alpha + 2\sqrt{\alpha - \beta}\sin(\Gamma)uu_x\alpha \\
&\quad - \sqrt{s}\sqrt{\alpha^2 - 1}\cos(\Gamma)u_{xx}\alpha + \sqrt{\alpha - \beta}\sin(\Gamma)u_{xxx}\alpha + 4\sqrt{\alpha - \beta}\sin(\Gamma)u_t\alpha \\
&\quad + \sqrt{s}\sqrt{\alpha^2 - 1}\beta\cos(\Gamma)u^2 - s^{3/2}\sqrt{\alpha^2 - 1}\cos(\Gamma)u + s\sqrt{\alpha - \beta}\sin(\Gamma)u_x \\
&\quad - 2\sqrt{\alpha - \beta}\beta\sin(\Gamma)uu_x + \sqrt{s}\sqrt{\alpha^2 - 1}\beta\cos(\Gamma)u_{xx} - \sqrt{\alpha - \beta}\beta\sin(\Gamma)u_{xxx} \\
&\quad \left. - 4\sqrt{\alpha - \beta}\beta\sin(\Gamma)u_t \right\}, \\
T_5^x &= \frac{1}{4(\alpha - \beta)^{3/2}} \left\{ 3s^{5/2}\sqrt{\alpha^2 - 1}\cos(\Gamma)u\alpha^3 - 2s^2\sqrt{\alpha - \beta}\sin(\Gamma)u_x\alpha^3 \right. \\
&\quad + s^{3/2}\sqrt{\alpha^2 - 1}\cos(\Gamma)u^2\alpha^2 - 4s^{5/2}\sqrt{\alpha^2 - 1}\beta\cos(\Gamma)u\alpha^2 + 4s^2\sqrt{\alpha - \beta}\beta\sin(\Gamma)u_x\alpha^2 \\
&\quad - 4s\sqrt{\alpha - \beta}\sin(\Gamma)uu_x\alpha^2 - s^{3/2}\sqrt{\alpha^2 - 1}\cos(\Gamma)u_{xx}\alpha^2 - s\sqrt{\alpha - \beta}\sin(\Gamma)u_t\alpha^2 \\
&\quad - s^{3/2}\sqrt{\alpha^2 - 1}\beta\cos(\Gamma)u^2\alpha + s^{5/2}\sqrt{\alpha^2 - 1}\cos(\Gamma)u\alpha - 2s^2\sqrt{\alpha - \beta}\sin(\Gamma)u_x\alpha \\
&\quad + 4s\sqrt{\alpha - \beta}\beta\sin(\Gamma)uu_x\alpha + 5s^{3/2}\sqrt{\alpha^2 - 1}\beta\cos(\Gamma)u_{xx}\alpha - 4s\sqrt{\alpha - \beta}\beta\sin(\Gamma)u_{xxx}\alpha \\
&\quad + 2\sqrt{\alpha - \beta}\sin(\Gamma)uu_t\alpha - 2\sqrt{s}\sqrt{\alpha^2 - 1}\cos(\Gamma)u_{tx}\alpha + 3\sqrt{\alpha - \beta}\sin(\Gamma)u_{txx}\alpha \\
&\quad - 4s^{3/2}\sqrt{\alpha^2 - 1}\beta^2\cos(\Gamma)u_{xx} + 4s\sqrt{\alpha - \beta}\beta^2\sin(\Gamma)u_{xxx} + s\sqrt{\alpha - \beta}\sin(\Gamma)u_t \\
&\quad \left. - 2\sqrt{\alpha - \beta}\beta\sin(\Gamma)uu_t + 2\sqrt{s}\sqrt{\alpha^2 - 1}\beta\cos(\Gamma)u_{tx} - 3\sqrt{\alpha - \beta}\beta\sin(\Gamma)u_{txx} \right\};
\end{aligned}$$

$$D_t T_6^t + D_x T_6^x|_{(1.30)} = 0,$$

$$\Gamma = \frac{\sqrt{\alpha^2 - 1}\sqrt{s}(\alpha st + x)}{\sqrt{\alpha - \beta}},$$

$$\begin{aligned} T_6^t = & \frac{1}{4(\alpha - \beta)^{3/2}} \left\{ -3s^{3/2}\sqrt{\alpha^2 - 1}\sin(\Gamma)u\alpha^2 + s\sqrt{\alpha - \beta}\cos(\Gamma)u_x\alpha^2 \right. \\ & -\sqrt{s}\sqrt{\alpha^2 - 1}\sin(\Gamma)u^2\alpha + 4s^{3/2}\sqrt{\alpha^2 - 1}\beta\sin(\Gamma)u\alpha - 2\sqrt{\alpha - \beta}\cos(\Gamma)uu_x\alpha \\ & -\sqrt{s}\sqrt{\alpha^2 - 1}\sin(\Gamma)u_{xx}\alpha - \sqrt{\alpha - \beta}\cos(\Gamma)u_{xxx}\alpha - 4\sqrt{\alpha - \beta}\cos(\Gamma)u_t\alpha \\ & +\sqrt{s}\sqrt{\alpha^2 - 1}\beta\sin(\Gamma)u^2 - s^{3/2}\sqrt{\alpha^2 - 1}\sin(\Gamma)u - s\sqrt{\alpha - \beta}\cos(\Gamma)u_x \\ & +2\sqrt{\alpha - \beta}\beta\cos(\Gamma)uu_x + \sqrt{s}\sqrt{\alpha^2 - 1}\beta\sin(\Gamma)u_{xx} + \sqrt{\alpha - \beta}\beta\cos(\Gamma)u_{xxx} \\ & \left. +4\sqrt{\alpha - \beta}\beta\cos(\Gamma)u_t \right\}, \\ T_6^x = & \frac{1}{4(\alpha - \beta)^{3/2}} \left\{ 3s^{5/2}\sqrt{\alpha^2 - 1}\sin(\Gamma)u\alpha^3 + 2s^2\sqrt{\alpha - \beta}\cos(\Gamma)u_x\alpha^3 \right. \\ & +s^{3/2}\sqrt{\alpha^2 - 1}\sin(\Gamma)u^2\alpha^2 - 4s^{5/2}\sqrt{\alpha^2 - 1}\beta\sin(\Gamma)u\alpha^2 - 4s^2\sqrt{\alpha - \beta}\beta\cos(\Gamma)u_x\alpha^2 \\ & +4s\sqrt{\alpha - \beta}\cos(\Gamma)uu_x\alpha^2 - s^{3/2}\sqrt{\alpha^2 - 1}\sin(\Gamma)u_{xx}\alpha^2 + s\sqrt{\alpha - \beta}\cos(\Gamma)u_t\alpha^2 \\ & -s^{3/2}\sqrt{\alpha^2 - 1}\beta\sin(\Gamma)u^2\alpha + s^{5/2}\sqrt{\alpha^2 - 1}\sin(\Gamma)u\alpha + 2s^2\sqrt{\alpha - \beta}\cos(\Gamma)u_x\alpha \\ & -4s\sqrt{\alpha - \beta}\beta\cos(\Gamma)uu_x\alpha + 5s^{3/2}\sqrt{\alpha^2 - 1}\beta\sin(\Gamma)u_{xx}\alpha + 4s\sqrt{\alpha - \beta}\beta\cos(\Gamma)u_{xxx}\alpha \\ & -2\sqrt{\alpha - \beta}\cos(\Gamma)uu_t\alpha - 2\sqrt{s}\sqrt{\alpha^2 - 1}\sin(\Gamma)u_{tx}\alpha - 3\sqrt{\alpha - \beta}\cos(\Gamma)u_{txx}\alpha \\ & -4s^{3/2}\sqrt{\alpha^2 - 1}\beta^2\sin(\Gamma)u_{xx} - 4s\sqrt{\alpha - \beta}\beta^2\cos(\Gamma)u_{xxx} - s\sqrt{\alpha - \beta}\cos(\Gamma)u_t \\ & \left. +2\sqrt{\alpha - \beta}\beta\cos(\Gamma)uu_t + 2\sqrt{s}\sqrt{\alpha^2 - 1}\beta\sin(\Gamma)u_{tx} + 3\sqrt{\alpha - \beta}\beta\cos(\Gamma)u_{txx} \right\}. \end{aligned}$$

1.8 Conclusion

In this chapter we presented a brief introduction to the Lie group analysis and conservation laws of PDEs and gave some results which will be used throughout this project. We also provided the algorithm to determine the Lie point symmetries and conservation laws of PDEs. Finally an illustrative example was shown that captured the underlying concepts that would be employed in this dissertation.

Chapter 2

Conservation laws and exact solutions for a nonlinear integrable fifth-order equation

In this chapter, we consider an integrable fifth-order equation [22]

$$u_{ttt} - u_{txxxx} + au_{xxx} - 4(u_x u_t)_{xx} - 4(u_x u_{xt})_x = 0. \quad (2.1)$$

The above integrable fifth-order equation has been demonstrated to pass Painlevé test along with the required number of arbitrary functions to confirm its complete integrability. Multiple soliton solutions and multiple complex soliton solutions were computed by Hirota's direct method.

We will employ the multiplier method to derive the conservation laws and compute the Lie point symmetries of this underlying equation to extract symmetry reductions and exact solutions.

2.1 Conservation laws for a nonlinear integrable fifth-order equation

In this section we derive the conservation laws for equation (2.1). Here we will consider multipliers of the zeroth order $\Lambda(t, x, u)$ defined by

$$\frac{\delta}{\delta u}[\Lambda(t, x, u)(u_{ttt} - u_{txxxx} + au_{xxx} - 4(u_x u_t)_{xx} - 4(u_x u_{xt})_x) = 0, \quad (2.2)$$

where the Euler-Lagrange Operator $\delta/\delta u$ is defined by

$$\begin{aligned} \frac{\delta}{\delta u} = & \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} + D_t^2 \frac{\partial}{\partial u_{tt}} + D_x^2 \frac{\partial}{\partial u_{xx}} \\ & + D_x D_t \frac{\partial}{\partial u_{xt}} + \dots \end{aligned} \quad (2.3)$$

and the total differential operators are given by

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \dots,$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + \dots.$$

Expanding equation (2.2) leads to

$$\begin{aligned} & \left[\frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xx}} + D_x D_t \frac{\partial}{\partial u_{xt}} - D_x^3 \frac{\partial}{\partial u_{xxx}} - D_x^2 D_t \frac{\partial}{\partial u_{xxt}} \right. \\ & \left. - D_t^3 \frac{\partial}{\partial u_{ttt}} - D_x^4 D_t \frac{\partial}{\partial u_{xxxxt}} \right] \Lambda(u_{ttt} - u_{txxxx} + au_{xxx} - 4(u_x u_t)_{xx} - 4(u_x u_{xt})_x) = 0, \\ & \Lambda_u(u_{ttt} - u_{txxxx} + au_{xxx} - 4(u_x u_t)_{xx} - 4(u_x u_{xt})_x) + 4D_t u_{xxx} \Lambda + 8D_x u_{xxt} \Lambda \\ & - 12D_x^2 u_{xt} \Lambda - 12D_x D_t u_{xx} \Lambda - aD_x^3 \Lambda + 4D_x^3 \Lambda + 8D_x^3 u_t \Lambda + 8D_x^2 D_t u_x \Lambda - D_t^3 \Lambda \\ & + D_x^4 D_t \Lambda = 0. \end{aligned}$$

Since Λ depends only on t , x and u , the coefficients of the like derivatives of u can be equated to zero to yield the following system of over-determined linear partial

differential equations:

$$\Lambda_{ttt} = 0, \quad (2.4)$$

$$\Lambda_{tx} = 0, \quad (2.5)$$

$$\Lambda_{ut} = 0, \quad (2.6)$$

$$\Lambda_{xx} = 0, \quad (2.7)$$

$$\Lambda_{ux} = 0, \quad (2.8)$$

$$\Lambda_{uu} = 0. \quad (2.9)$$

We integrate equation (2.9) with respect to u obtain

$$\Lambda = A(t, x)u + B(t, x), \quad (2.10)$$

where $A(t, x)$ and $B(t, x)$ are arbitrary functions of t and x . Inserting equation (2.10) into equation (2.8) and solving the resulting equation yields

$$\Lambda = C(t)u + B(x, t). \quad (2.11)$$

Inserting equation (2.11) into equation (2.6) and solving the resulting equation yields

$$\Lambda = A_1 u + B(t, x). \quad (2.12)$$

We insert equation (2.12) in equation (2.7) to obtain

$$\Lambda = A_1 u + D(t)x + E(t), \quad (2.13)$$

where $D(t)$ and $E(t)$ are arbitrary functions of t . Inserting equation (2.13) into equation (2.5) we obtain

$$\Lambda = A_1 u + A_2 x + E(t). \quad (2.14)$$

The insertion of equation (2.14) into equation (2.4) yields the multiplier of the form

$$\Lambda = A_1 u + A_2 x + A_3 \frac{t^2}{2} + A_4 t + A_5, \quad (2.15)$$

where A_1, A_2, A_3, A_4 and A_5 are arbitrary constants of integration.

Corresponding to the above multiplier we obtain the following five conserved vectors of (2.1):

$$\begin{aligned}
T_1^t &= -\frac{1}{10}u_{xx}^2 + \frac{1}{5}u_{xxx}u_x + uu_{tt} - \frac{1}{5}uu_{xxxx} - \frac{1}{2}u_t^2 + \frac{8}{9}u_x^2 - \frac{8}{3}uu_xu_{xx} - \frac{2}{9}u^2u_{xxx}, \\
T_1^x &= \frac{3}{5}u_{txx}u_x - \frac{2}{5}u_{xx}u_{tx} + \frac{1}{5}u_{xxx}u_t + \frac{28}{9}u_tu_x^2 - \frac{1}{2}au_x^2 + \frac{2}{9}u^2u_{txx} - \frac{4}{5}uu_{txxx} + auu_{xx} \\
&\quad - \frac{32}{9}uu_tu_{xx} - \frac{52}{9}uu_xu_{tx}; \\
T_2^t &= \frac{1}{3}uu_{xx} + \frac{1}{5}u_{xxx} + \frac{4}{3}u_x^2 - 3xu_xu_{xxx} - \frac{1}{3}xuu_{xxx} - \frac{1}{5}xu_{xxxx} + xu_{tt}, \\
T_2^x &= -\frac{2}{3}uu_{tx} + \frac{10}{3}u_xu_t + \frac{3}{5}u_{txx} + axu_{xx} - \frac{4}{3}xu_{txxx} - au_x - \frac{11}{3}xu_{xx}u_t - \frac{16}{3}xu_xu_{tx} \\
&\quad + \frac{1}{3}xuu_{txx}; \\
T_3^t &= u - \frac{3}{2}t^2u_xu_{xx} - \frac{1}{6}t^2uu_{xxx} - \frac{1}{10}t^2u_{xxxx} + \frac{1}{2}t^2u_{tt} - tu_t, \\
T_3^x &= -\frac{2}{5}t^2u_{txxx} + \frac{1}{5}tu_{xxx} + \frac{4}{3}tu_x^2 - \frac{11}{6}t^2u_{xx}u_t - \frac{8}{3}t^2u_{tx}u_x + \frac{1}{3}tuu_{xx} + \frac{1}{6}t^2uu_{txx} \\
&\quad + \frac{1}{2}at^2u_{xx}; \\
T_4^t &= -u_t - 3tu_xu_{xx} - \frac{1}{3}tuu_{xxx} - \frac{1}{5}tu_{xxxx} + tu_{tt}, \\
T_4^x &= \frac{1}{5}u_{xxx} + \frac{4}{3}u_x^2 - \frac{4}{5}tu_{txxx} + \frac{1}{3}uu_{xx} - \frac{11}{3}tu_{xx}u_t - \frac{16}{3}tu_{tx}u_x + \frac{1}{3}tuu_{txx} + atu_{xx}; \\
T_5^t &= -\frac{1}{5}u_{xxxx} + u_{tt} - 3u_xu_{xx} - \frac{1}{3}uu_{xxx}, \\
T_5^x &= -\frac{4}{5}u_{txxx} + au_{xx} - \frac{11}{3}u_{xx}u_t - \frac{16}{3}u_{tx}u_x + \frac{1}{3}uu_{txx}.
\end{aligned}$$

2.2 Symmetry reductions of (2.1)

The vector field

$$X = \xi^1(t, x, u) \frac{\partial}{\partial t} + \xi^2(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u} \quad (2.16)$$

is a Lie point symmetry of (2.1) if

$$X^{[5]} \left\{ u_{ttt} - u_{txxxx} + au_{xxx} - 4(u_xu_t)_{xx} - 4(u_xu_{xt})_x \right\} \Big|_{(2.1)} = 0,$$

where $X^{[5]}$ is the fifth prolongation of (2.16). Expanding the above equation and splitting the derivatives of u leads to the following overdetermined system of linear

partial differential equations:

$$\xi_t^2 = 0, \quad (2.17)$$

$$\xi_u^2 = 0, \quad (2.18)$$

$$\xi_x^1 = 0, \quad (2.19)$$

$$\xi_u^1 = 0, \quad (2.20)$$

$$\xi_{tt}^1 = 0, \quad (2.21)$$

$$\eta_{tu} = 0, \quad (2.22)$$

$$\eta_{uu} = 0, \quad (2.23)$$

$$4\eta_{xxx} + \eta_{xxxxx} = 0, \quad (2.24)$$

$$-2\xi_x^2 + \xi_t^1 = 0, \quad (2.25)$$

$$\xi_x^2 + \eta_u = 0, \quad (2.26)$$

$$\xi_x^2 + \eta_u = 0, \quad (2.27)$$

$$-\xi_{xx}^2 + 2\eta_{xu} = 0, \quad (2.28)$$

$$-\xi_{xx}^2 + 2\eta_{xu} = 0, \quad (2.29)$$

$$-3\xi_{xx}^2 + 2\eta_{xu} = 0, \quad (2.30)$$

$$-\xi_{xxx}^2 + 5\eta_{xxu} = 0, \quad (2.31)$$

$$-\eta_{xu}a + \xi_{xx}^2a + 4\eta_{tx} = 0, \quad (2.32)$$

$$-3a\eta_{xxu} + a\xi_{xxx}^2 + 8\eta_{txx} = 0, \quad (2.33)$$

$$-a\xi_t^1 - a\xi_x^2 + 4\eta_t = 0, \quad (2.34)$$

$$4\eta_x - 2\xi_{xxx}^2 + 3\eta_{xxu} = 0, \quad (2.35)$$

$$12\eta_{xx} - \xi_{xxxx}^2 + 4\eta_{xxxu}, \quad (2.36)$$

$$-a\eta_{xxx} - \eta_{ttt} + \eta_{txxx} = 0. \quad (2.37)$$

Solving the above system leads to the following four linearly independent operators:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= \frac{\partial}{\partial u}, \\ X_4 &= (-3at + 4u) \frac{\partial}{\partial u} - 8t \frac{\partial}{\partial t} - 4x \frac{\partial}{\partial x}. \end{aligned}$$

Case 1. $b_1X_1 + b_2X_2 + b_3X_3$

The symmetries $b_1X_1 + b_2X_2 + b_3X_3$ gives rise to the group-invariant solution

$$u(t, x) = \frac{b_2 F(z) + b_3 x}{b_2}, \quad (2.38)$$

where $z = -\frac{b_1x - b_2t}{b_2}$ is an invariant of $b_1X_1 + b_2X_2 + b_3X_3$. Substitution of (2.38) into (2.1) results in the function $F(z)$ satisfying the following ODE:

$$\begin{aligned} &ab_1^3b_2F_{zzz} - 12b_1^3b_2F_{zzz}F_z - 12b_1^3b_2F_{zz}^2 + b_1^4F_{zzzz} + 8b_1^2b_2b_3F_{zzz} \\ &- b_2^4F_{zzz} = 0. \end{aligned} \quad (2.39)$$

Case 2. X_4

For the symmetry X_4 we obtain the group-invariant solution of the form

$$u(t, x) = \frac{atx + 4F(z)}{4x}, \quad (2.40)$$

where $z = \frac{t}{x^2}$ is an invariant of X_4 and the function F satisfies the following ODE:

$$\begin{aligned} &16z^4F_{zzzz} - 96z^3F_zF_{zzz} - 96z^3(F_{zz})^2 - 32z^2FF_{zzz} \\ &+ 240z^3F_{zzzz} - 864z^2F_zF_{zz} - 192zFF_{zz} + 1020z^2F_{zzz} \\ &- 768zF_z^2 - 192FF_z + 1320zF_{zz} + 360F_z - F_{zzz} = 0. \end{aligned} \quad (2.41)$$

Case 3. $k_1X_1 - k_2X_2$

By solving the corresponding Lagrange system for the symmetry $k_1X_1 - k_2X_2$, one obtains an invariant $z = k_1x + k_2t + k_3$ and the group-invariant solution of (2.1) is

the form

$$u(t, x) = F(z), \quad (2.42)$$

where the function $F(z)$ satisfies

$$12 k_1^3 k_2 F_z F_{zzz} + 12 k_1^3 k_2 (F_{zz})^2 + k_1^4 k_2 F_{zzzzz} - a k_1 (F_{zzz})^3 - k_2^3 F_{zzz} = 0. \quad (2.43)$$

2.3 Exact solutions using Kudryashov method

Let us recall the basic idea of the Kudryashov method. Consider the nonlinear partial differential equation in the form

$$E_1[u_t, u_x, \dots] = 0. \quad (2.44)$$

We use the following ansatz:

$$u(x, t) = F(z), \quad z = k_1 x + k_2 t + k_3. \quad (2.45)$$

From equation (2.44), we obtain the ordinary nonlinear differential equation

$$E_2[k_1 F'(z), k_2 F'(z), k_1^2 F''(z), k_2^2 F''(z), \dots] = 0, \quad (2.46)$$

which has a solution of the form

$$F(z) = \sum_{i=0}^M A_i (H(z))^i, \quad (2.47)$$

where

$$H(z) = \frac{1}{1 + \cosh(z) + \sinh(z)}$$

satisfies the equation

$$H'(z) = H(z)^2 - H(z), \quad (2.48)$$

and M is a positive integer while $A_0, \dots, A_M$ are parameters to be determined. The positive integer M will be determined by the homogeneous balance method between the highest order derivative and highest order nonlinear term appearing in (2.46).

2.3.1 Application of the Kudryashov method

Making use of anstaz (2.45), the balancing procedure gives $M = 1$ for (2.43). The solution of equation (2.43) are of the form

$$F(z) = A_0 + A_1 H. \quad (2.49)$$

Substituting equation (2.49) into equation (2.43) and making use of equation (2.48) and then equating all coefficients of the functions H^i to zero, we obtain the following overdetermined system of algebraic equations in terms of A_0, A_1 :

$$\begin{aligned} 120 A_1^2 k_1^3 k_2 + 120 A_1 k_1^4 k_2 &= 0, \\ 360 A_1^2 k_1^3 k_2 + 360 A_1 k_1^4 k_2 &= 0, \\ A_1 k_1^4 k_2 - a A_1 k_1^3 - A_1 k_2^3 &= 0, \\ 384 A_1^2 k_1^3 k_2 + 390 A_1 k_1^4 k_2 - 6 a A_1 k_1^3 - 6 A_1 k_2^3 &= 0, \\ 24 A_1^2 k_1^3 k_2 + 31 A_1 k_1^4 k_2 - 7 a A_1 k_1^3 - 7 A_1 k_2^3 &= 0, \\ 168 A_1^2 k_1^3 k_2 + 180 A_1 k_1^4 k_2 - 12 a A_1 k_1^3 - 12 A_1 k_2^3 &= 0. \end{aligned}$$

On solving the resultant system of algebraic equations, we obtain

$$\begin{aligned} a &= \frac{k_2 (k_1^4 - k_2^2)}{k_1^3}, \\ A_1 &= -k_1. \end{aligned}$$

Consequently a solution of equation (2.1) is

$$u(x, t) = A_0 - A_1 \left\{ \frac{1}{1 + \cosh(z) + \sinh(z)C} \right\}, \quad (2.50)$$

where $z = k_1 x + k_2 t + k_3$.

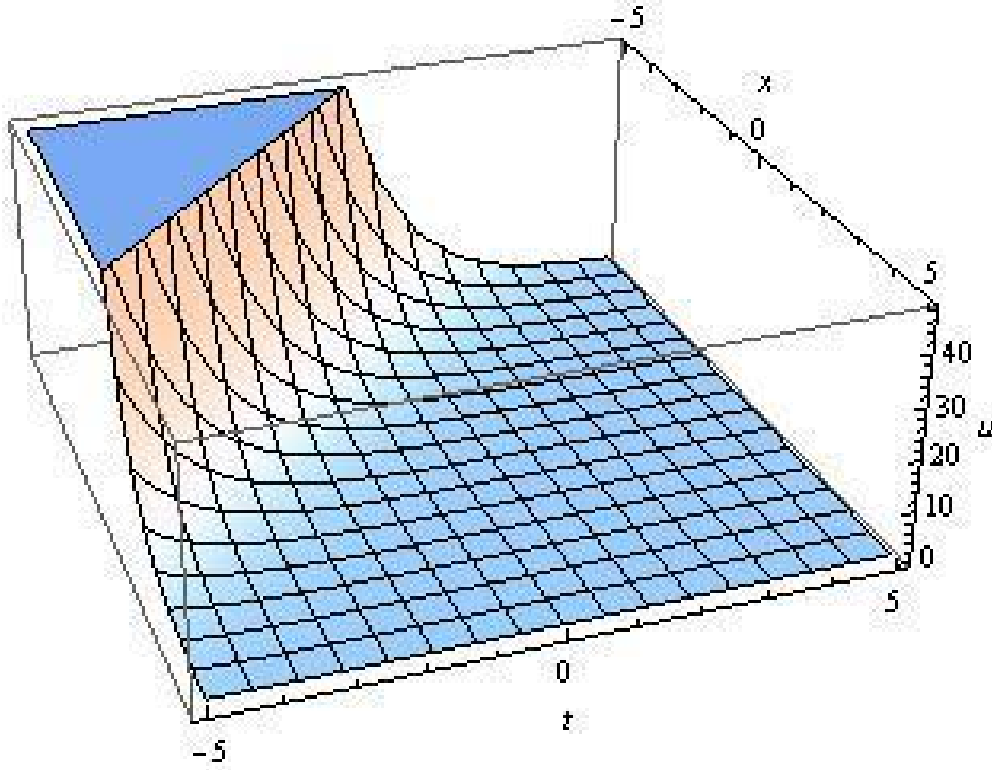


Figure 2.1: Evolution of travelling wave solution (2.50).

2.4 Concluding remarks

Symmetry reductions, exact solutions and conservation laws of a nonlinear integrable fifth-order equation were computed. The Lie symmetry method gave symmetry reductions and in conjunction with the Lie symmetry method, the Kudryashov method gave a travelling wave solution. Finally conservation laws were computed via the multiplier approach.

Chapter 3

Conservation laws and exact solutions of the longitudinal wave equation in a magnetoelectro-elastic rod

The nonlinear longitudinal wave equation in a magnetoelectro-elastic circular rod is given by

$$u_{tt} - r^2 u_{xx} - \left(\frac{1}{2} r^2 u^2 + n u_{tt} \right)_{xx} = 0. \quad (3.1)$$

The above equation was proposed in [23] and the Jacobi elliptic function expansion method was used in the construction of exact solutions.

In this chapter, we aim to implement the multiplier approach to construct conservation laws of (3.1) and secondly we aim to employ the Lie symmetry method to construct symmetry reductions. Finally travelling wave solutions are computed via the simplest equation method.

3.1 Conservation laws of the longitudinal wave equation in a magnetoelectro-elastic rod

In this section we derive the conservation laws for equation (3.1). Here we will consider multipliers of the second order $\Lambda(t, x, u, u_t, u_x, u_{tt}, u_{xx}, u_{tx})$ defined by

$$\frac{\delta}{\delta u} [\Lambda(t, x, u, u_t, u_x, u_{tt}, u_{xx}, u_{tx}) (u_{tt} - r^2 u_{xx} - (\frac{1}{2} r^2 u^2 + n u_{tt})_{xx})] = 0, \quad (3.2)$$

where the Euler-Langrage Operator $\delta/\delta u$ is defined by

$$\begin{aligned} \frac{\delta}{\delta u} = & \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} + D_t^2 \frac{\partial}{\partial u_{tt}} + D_x^2 \frac{\partial}{\partial u_{xx}} \\ & + D_x D_t \frac{\partial}{\partial u_{xt}} + \dots \end{aligned} \quad (3.3)$$

and the total differential operators are given by

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \dots,$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + \dots.$$

Expanding equation (3.2) leads to

$$\begin{aligned} & [\frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xx}} + D_t^2 \frac{\partial}{\partial u_{tt}} + D_x^2 D_t^2 \frac{\partial}{\partial u_{xxtt}}] (\Lambda (u_{tt} - r^2 u_{xx} \\ & - (\frac{1}{2} r^2 u^2 + n u_{tt})_{xx}) = 0, \\ & \Lambda_u (u_{tt} - r^2 u_{xx} - (\frac{1}{2} r^2 u^2 + n u_{tt})_{xx}) - r^2 u_{xx} \Lambda + r^2 D_x u_x \Lambda - r^2 D_x^2 \Lambda + D_t^2 \Lambda \\ & - n D_x^2 D_t^2 \Lambda = 0. \end{aligned}$$

Since Λ depends only on $t, x, u, u_t, u_x, u_{tt}, u_{xx}$ and u_{tx} , the coefficients of the derivatives of u can be equated to zero to yield the following system of over-determined

linear partial differential equations:

$$\Lambda_u = 0, \quad (3.4)$$

$$\Lambda_{u_t} = 0, \quad (3.5)$$

$$\Lambda_{u_x} = 0, \quad (3.6)$$

$$\Lambda_{u_{tt}} = 0, \quad (3.7)$$

$$\Lambda_{u_{xx}} = 0, \quad (3.8)$$

$$\Lambda_{u_{xt}} = 0, \quad (3.9)$$

$$\Lambda_{tt} = 0, \quad (3.10)$$

$$\Lambda_{xx} = 0. \quad (3.11)$$

Solving the above system of linear partial differential equations for Λ prompts the multiplier

$$\Lambda = (C_1x + C_2)t + C_3x + C_4, \quad (3.12)$$

where C_1, C_2, C_3 and C_4 are arbitrary constants of integration.

Corresponding to the above multiplier we obtain the following four conserved vectors of (3.1):

$$\begin{aligned} T_1^t &= -u_x - \frac{1}{3}nu_x + \frac{1}{6}nxu_{xx} + txu_t + \frac{1}{3}ntu_{tx} - \frac{1}{2}ntu_{txx}, \\ T_1^x &= \frac{1}{2}r^2tu^2 + rtu - r^2txuu_x - rtxu_x - \frac{1}{3}nu_t + \frac{1}{3}nxu_{tx} + \frac{1}{6}ntu_{tt} - \frac{1}{2}ntxu_{ttx}; \\ T_2^t &= -u + \frac{1}{6}nu_{xx} + tu_t - \frac{1}{2}ntu_{txx}, \\ T_2^x &= -r^2tuu_x - rtu_x + \frac{1}{3}nu_{tx} - \frac{1}{2}ntu_{txx}, \\ T_3^t &= xu_t + \frac{1}{3}nu_{tx} - \frac{1}{2}nxu_{txx}, \\ T_3^x &= \frac{1}{2}r^2u^2 + ru - r^2xuu_x - rxu_x + \frac{1}{6}nu_{tt} - \frac{1}{2}nxu_{ttx}, \\ T_4^t &= u_t - \frac{1}{2}nu_{txx}, \\ T_4^x &= -r^2uu_x - ru_x - \frac{1}{2}nu_{txx}. \end{aligned}$$

3.2 Symmetry reductions of (3.1)

The vector field

$$X = \xi^1(t, x, u) \frac{\partial}{\partial t} + \xi^2(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u} \quad (3.13)$$

is a Lie point symmetry of (3.1) if

$$X^{[4]} \left\{ u_{tt} - r^2 u_{xx} - \left(\frac{1}{2} r^2 u^2 + n u_{tt} \right)_{xx} \right\} \Big|_{(3.1)} = 0,$$

where $X^{[4]}$ is the fourth prolongation of (1.31). Expanding the above equation and splitting the derivatives of u leads to the following overdetermined system of linear partial differential equations:

$$\begin{aligned} \xi_t^2 &= 0, \\ \xi_u^2 &= 0, \\ \xi_x^1 &= 0, \\ \xi_u^1 &= 0, \\ \eta_{uu} &= 0, \\ \eta_{txu} &= 0, \\ 2\xi_t^1 + \eta_u &= 0, \\ \xi_{tt}^1 - 2\eta_{tu} &= 0, \\ \eta_{xxu}n - 2\xi_x^2 &= 0, \\ \xi_{xx}^2 - 2\eta_{xu} &= 0, \\ \eta_{xx}r^2u + \eta_{xx}r^2 + \eta_{ttxx}n - \eta_{tt} &= 0, \\ 2\xi_t^1r^2u + r^2\eta_1 + 2\xi_t^1r^2 + \eta_{ttu}n &= 0, \\ 2\eta_{xu}u - \xi_{xx}^2u + 2\eta_x + 2\eta_{xu} - \xi_{xx}^2 &= 0. \end{aligned}$$

Solving the above system leads to the infinitesimal symmetries of (3.1) that constitute the three dimensional Lie algebra spanned by the following linearly independent

operators:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= t \frac{\partial}{\partial t} - 2(u+1) \frac{\partial}{\partial u}. \end{aligned}$$

Case 1. $b_1 X_1 + b_2 X_2 + b_3 X_3$

The symmetries $b_1 X_1 + b_2 X_2 + b_3 X_3$ gives rise to the group-invariant solution

$$u(t, x) = \frac{-b_3^2 t^2 + 2 b_3 b_1 t + F(z)}{b_3^2 t^2 - 2 b_3 b_1 t + b_1^2}, \quad (3.14)$$

where $z = x + \frac{b_2 \ln(-b_3 t + b_1)}{b_3}$ is an invariant of $b_1 X_1 + b_2 X_2 + b_3 X_3$. Substitution of (3.14) into (3.1) results in function $F(z)$ satisfies the ODE

$$\begin{aligned} &b_1^2 r^2 F_{zz} + r^2 F F_{zz} - 6 b_1^2 b_3^2 + b_2^2 n F_{zzzz} - 5 b_2 b_3 n F_{zzz} \\ &+ 6 b_3^2 n F_{zz} + r^2 F_z^2 - 6 b_3^2 F - b_2^2 F_{zz} + 5 b_3 b_2 F_z = 0. \end{aligned} \quad (3.15)$$

Case 2. $k_1 X_1 - k_2 X_2$

By solving the corresponding Lagrange system for the symmetry $k_1 X_1 - k_2 X_2$, one obtains an invariant $z = k_1 x + k_2 t + k_3$ and the group-invariant solution of (3.1) is of the form

$$u(t, x) = F(z), \quad (3.16)$$

where the function $F(z)$ satisfies

$$r^2 k_1^2 F F_{zz} + n k_1^2 k_2^2 F_{zzzz} + r^2 k_1^2 F_z^2 + r^2 k_1^2 F_{zz} - k_2^2 F_{zz} = 0. \quad (3.17)$$

3.3 Exact solutions using the simplest equation method

We now use the simplest equation method, to solve (3.17) and as a result we obtain the exact solutions system (3.1). The simplest equations that will be used are the Bernoulli and Riccati equations.

We briefly recall the simplest equation method here. Let us consider the solutions of (3.1) in the form

$$F(z) = \sum_{i=0}^M A_i (H(z))^i, \quad (3.18)$$

where $H(z)$ satisfies the Bernoulli or Riccati equation, M is a positive integer that can be determined by balancing procedure and $A_0, \dots, A_M$ are parameters to be determined. We note that the Bernoulli and Riccati equations are well-known nonlinear equations whose solutions can be expressed in terms of elementary functions.

We consider the Riccati equation

$$H'(z) = aH^2(z) + bH(z) + c, \quad (3.19)$$

where a, b and c are constants, we shall use the solutions

$$H(z) = -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left[\frac{1}{2} \theta (z + C) \right]$$

and

$$H(z) = -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left(\frac{1}{2} \theta z \right) + \frac{\operatorname{sech} \left(\frac{\theta z}{2} \right)}{C \cosh \left(\frac{\theta z}{2} \right) - \frac{2a}{\theta} \sinh \left(\frac{\theta z}{2} \right)},$$

where $\theta^2 = b^2 - 4ac > 0$ and C is a constant of integration.

For Bernoulli equation

$$H'(z) = eH(z) + dH^2(z), \quad (3.20)$$

where a and b are constants. The solution of this equation is given by

$$H(z) = e \left\{ \frac{\cosh[e(z + C)] + \sinh[e(z + C)]}{1 - d \cosh[e(z + C)] - d \sinh[e(z + C)]} \right\}.$$

3.3.1 Application of the simplest equation method: Riccati equation

The balancing procedure yields $M = 2$ so the solutions of (3.17) are of the form

$$F(z) = A_0 + A_1 H + A_2 H^2. \quad (3.21)$$

Substituting equation (3.21) into equation (3.17) and making use of equation (3.19) and then equating all coefficients of the functions H^i to zero, we obtain the following overdetermined system of algebraic equations in terms of A_0, A_1, A_2 :

$$\begin{aligned}
& -120 a^4 n A_2 k_1^2 k_2^2 - 10 a^2 r^2 A_2^2 k_1^2 = 0, \\
& -24 a^4 n A_1 k_1^2 k_2^2 - 336 a^2 b n A_2 k_1^2 k_2^2 - 12 a^2 r^2 A_1 A_2 k_1^2 - 18 a b r^2 A_2^2 k_1^2 = 0, \\
& -60 a^3 b n A_1 k_1^2 k_2^2 - 240 a^3 c n A_2 k_1^2 k_2^2 - 330 a^2 b^2 n A_2 k_1^2 k_2^2 - 6 a^2 r^2 A_0 A_2^2 k_1^2 \\
& -3 a^2 r^2 A_1^2 k_1^2 - 21 a b r^2 A_1 A_2 k_1^2 - 16 a c r^2 A_2^2 k_1^2 - 8 b^2 r^2 A_2^2 k_1^2 \\
& -6 a^2 r^2 A_2 k_1^2 = 0, \\
& -8 a b c^2 n A_1 k_1^2 k_2^2 - 16 a c^3 n A_1 k_1^2 k_2^2 - b^3 c n A_1 k_1^2 k_2^2 - 14 b^2 c^2 n A_2 k_1^2 k_2^2 \\
& -b c r^2 A_0 A_1 k_1^2 - 2 c^2 r^2 A_0 A_2 k_1^2 - c^2 r^2 A_1^2 k_1^2 - b c r A_1 k_1^2 - 2 c^2 r A_2 k_1^2 \\
& + b c A_1 k_2^2 + 2 c^2 A_2 k_2^2 = 0, \\
& -40 a^3 c n A_1 k_1^2 k_2^2 - 50 a^2 b^2 n A_1 k_1^2 k_2^2 - 44 a^2 b c n A_2 k_1^2 k_2^2 - 130 a b^3 n A_2 k_1^2 k_2^2 \\
& -2 a^2 r^2 A_0 A_1 k_1^2 - 10 a b r^2 A_0 A_2 k_1^2 - 5 a b r^2 A_1^2 k_1^2 - 18 a c r^2 A_1 A_2 k_1^2 \\
& -9 b^2 r^2 A_1 A_2 k_1^2 - 14 b c r^2 A_2^2 k_1^2 - 2 a^2 r^2 A_1 k_1^2 - 10 a b c A_2 k_1^2 + 2 a^2 A_1 k_2^2 \\
& + 10 a b A_2 k_2^2 = 0, \\
& -16 a^2 c^2 n A_1 k_1^2 k_2^2 - 22 a b^2 c n A_1 k_1^2 k_2^2 - 120 a b c^2 n A_2 k_1^2 k_2^2 - b^4 n A_1 k_1^2 k_2^2 \\
& -30 b^3 c n A_2 k_1^2 k_2^2 - 2 a c r^2 A_0 A_1 k_1^2 - b^2 r^2 A_0 A_1 K_1^2 - 6 b c r^2 A_0 A_2 k_1^2 \\
& -3 b c r^2 A_1^2 k_1^2 - b c^2 r^2 A_1 A_2 k_1^2 - 2 a c r A_1 k_1^2 - b^2 r A_1 k_1^2 - 6 b c r A_2 k_1^2 \\
& + 2 a c A_1 k_2^2 + b^2 A_1 k_2^2 - 6 b c A_2 k_2^2 = 0, \\
& -60 a^2 b c n A_1 k_1^2 k_2^2 - 136 a^2 c^2 n A_2 k_1^2 k_2^2 - 15 a b^3 n A_1 k_1^2 k_2^2 - 232 a b^2 c n A_2 k_1^2 k_2^2 \\
& -16 b^4 n A_2 k_1^2 k_2^2 - 3 a b r^2 A_0 A_1 k_1^2 - 8 a c r^2 A_0 A_2 k_1^2 - 4 a c r^2 A_1^2 k_1^2 \\
& -4 b^2 r^2 A_0 A_2 k_1^2 - 2 b^2 r^2 A_1^2 k_1^2 - 15 b c r^2 A_1 A_2 k_1^2 - 6 c^2 r^2 A_2^2 k_1^2 - 3 a b r A_1 k_1^2 \\
& -8 a c r A_2 k_1^2 - 4 b^2 r A_2 k_1^2 + 3 a b A_1 k_2^2 + 8 a c A_2 k_2^2 + 4 b^2 A_2 k_2^2 = 0.
\end{aligned}$$

On solving the resultant system of algebraic equations, we obtain

$$\begin{aligned} A_0 &= -\frac{8acnk_1^2k_2^2 + b^2nk_1^2k_2^2 + rk_1^2 - k_2^2}{r^2k_1^2}, \\ A_1 &= -12\frac{abnk_2^2}{r^2}, \\ A_2 &= -12\frac{a^2nk_2^2}{r^2}. \end{aligned}$$

Consequently a solutions of equation (3.1) are

$$\begin{aligned} u(x, t) &= A_0 + A_1 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left(\frac{1}{2}\theta(z+2) \right) \right\} \\ &\quad A_2 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left(\frac{1}{2}\theta(z+2) \right) \right\}^2 \end{aligned} \quad (3.22)$$

and

$$\begin{aligned} u(x, t) &= A_0 + A_1 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left(\frac{1}{2}\theta z \right) + \frac{\operatorname{sech} \left(\frac{\theta z}{2} \right)}{C \cosh \left(\frac{\theta z}{2} \right) - \frac{2a}{\theta} \sinh \left(\frac{\theta z}{2} \right)} \right\} \\ &\quad A_2 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left(\frac{1}{2}\theta z \right) + \frac{\operatorname{sech} \left(\frac{\theta z}{2} \right)}{C \cosh \left(\frac{\theta z}{2} \right) - \frac{2a}{\theta} \sinh \left(\frac{\theta z}{2} \right)} \right\}^2, \end{aligned} \quad (3.23)$$

where $z = k_1x + k_2t + k_3$ and $\theta^2 = b^2 - 4ac > 0$.

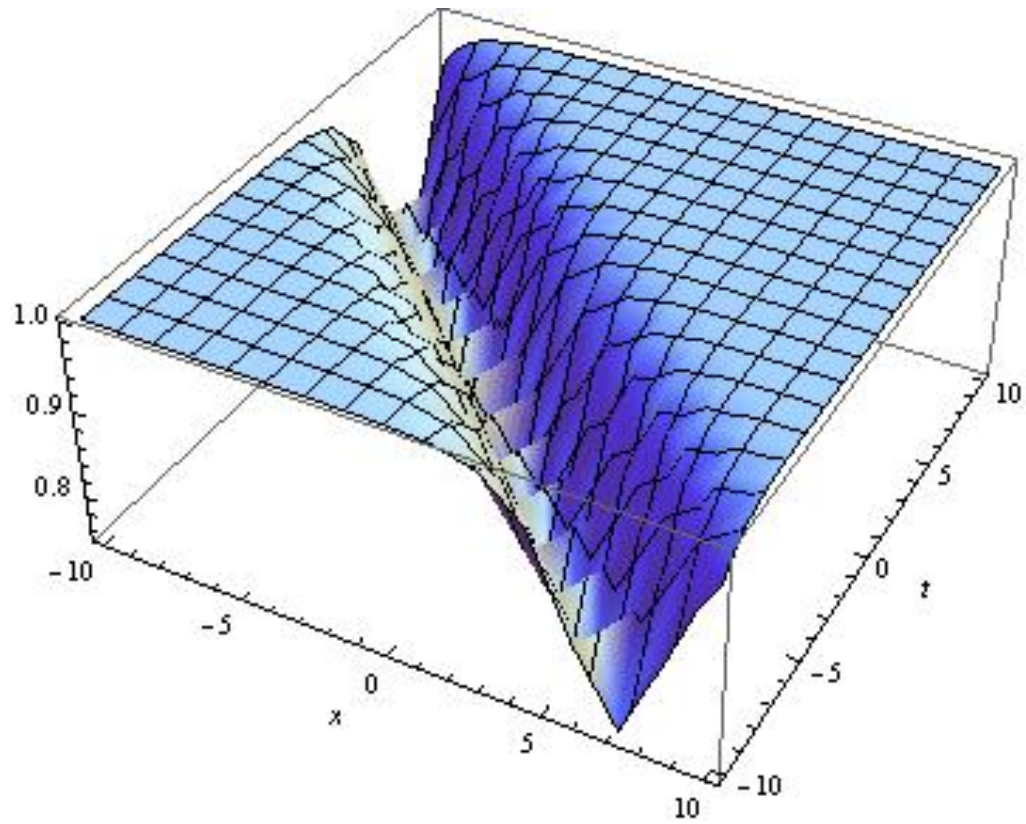


Figure 3.1: Evolution of travelling wave solution (3.22).

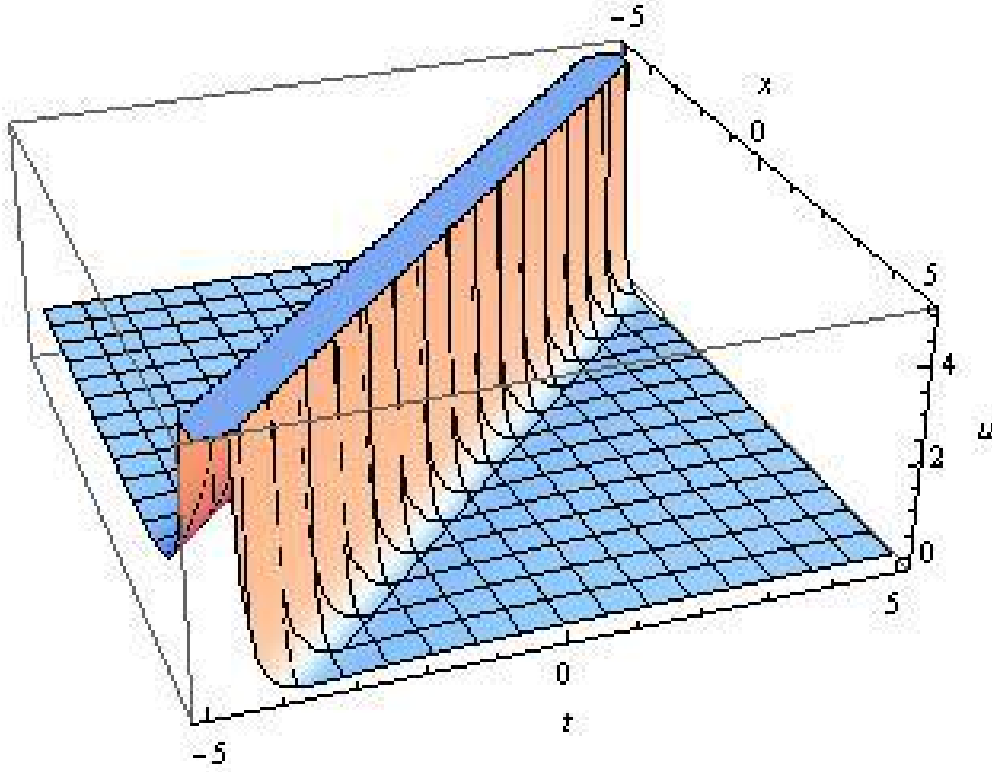


Figure 3.2: Evolution of travelling wave solution (3.23).

3.3.2 Application of the simplest equation method: Bernoulli equation

The balancing procedure yields $M = 2$ so the solutions of (3.17) are of the form

$$F(z) = A_0 + A_1 H + A_2 H^2. \quad (3.24)$$

Substituting equation (3.24) into equation (3.17) and making use of equation (3.20) and then equating all coefficients of the functions H^i to zero, we obtain the following

overdetermined system of algebraic equations in terms of A_0, A_1, A_2 :

$$\begin{aligned}
& -120 e^4 n A_2 k_1^2 k_2^2 - 10 e^2 r^2 A_2^2 k_1^2 = 0, \\
& -d^4 n A_1 k_1^2 k_2^2 - d^2 r^2 A_0 A_1 k_1^2 - d^2 r A_1 k_1^2 + d^2 A_1 k_2^2 = 0, \\
& -336 d e^3 n A_2 k_1^2 k_2^2 - 24 e^4 n A_1 k_1^2 k_2^2 - 18 d e r^2 A_2^2 k_1^2 - 12 e^2 r^2 A_1 A_2 k_1^2 = 0, \\
& -330 d^2 e^2 n A_2 k_1^2 k_2^2 - 60 d e^3 n A_1 k_1^2 k_2^2 - 8 d^2 r^2 A_2^2 k_1^2 - 21 d e r^2 A_1 A_2 k_1^2 \\
& -6 e^2 r^2 A_0 A_2 k_1^2 - 3 e^2 r^2 A_1^2 k_1^2 - 6 e^2 r A_2 k_1^2 + 6 e^2 A_2 k_2^2 = 0, \\
& -16 d^4 n A_2 k_1^2 k_2^2 - 15 d^3 e n A_1 k_1^2 k_2^2 - 4 d^2 r^2 A_0 A_2 k_1^2 - 2 d^2 r^2 A_1^2 k_1^2 \\
& -3 d e r^2 A_0 A_1 k_1^2 - 4 d^2 r A_2 k_1^2 - 3 d e r A_1 k_1^2 + 4 d^2 A_2 k_2^2 + 3 d e A_1 k_2^2 = 0, \\
& -130 d^3 e n A_2 k_1^2 k_2^2 - 50 d^2 e^2 n A_1 k_1^2 k_2^2 - 9 d^2 r^2 A_1 A_2 k_1^2 - 10 d e r^2 A_0 A_2 k_1^2 \\
& -5 d e r^2 A_1^2 k_1^2 - 2 e^2 r^2 A_0 A_1 k_1^2 - 10 d e r A_2 k_1^2 - 2 e^2 r A_1 k_1^2 + 10 d e A_2 k_2^2 \\
& + 2 e^2 A_1 k_2^2 = 0.
\end{aligned}$$

On solving the resultant system of algebraic equations, we obtain

$$\begin{aligned}
A_0 &= -\frac{d^2 n k_1^2 k_2^2 + r k_1^2 - k_2^2}{r^2 k_1^2}, \\
A_1 &= -12 \frac{d e n k_2^2}{r^2}, \\
A_2 &= -12 \frac{e^2 n k_2^2}{r^2}.
\end{aligned}$$

Consequently a solution of equation (3.1) is

$$\begin{aligned}
u(x, t) &= A_0 + A_1 e \left\{ \frac{\cosh[e(z + C)] + \sinh[e(z + C)]}{1 - d \cosh[e(z + C)] - d \sinh[e(z + C)]} \right\} \\
&\quad A_2 \left\{ e \frac{\cosh[e(z + C)] + \sinh[e(z + C)]}{1 - d \cosh[e(z + C)] - d \sinh[e(z + C)]} \right\}^2, \tag{3.25}
\end{aligned}$$

where $z = k_1 x + k_2 t + k_3$ and C is an arbitrary constant of integration.

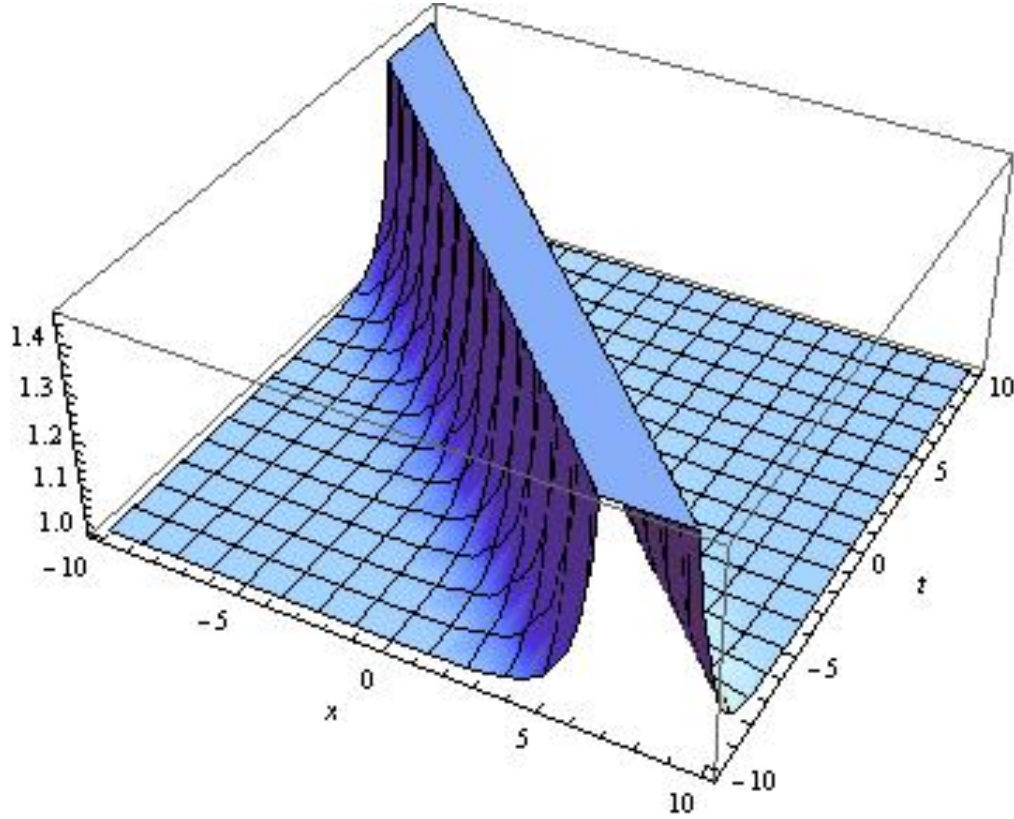


Figure 3.3: Evolution of travelling wave solution (3.25).

3.4 Concluding remarks

In this chapter, we studied the nonlinear longitudinal wave equation in a magnetoelectro-elastic circular rod via multiplier approach and Lie symmetry method. Conservation laws and Lie point symmetries of this equation were obtained. Finally the simplest equation method generated travelling wave solutions.

Chapter 4

Conclusions and Discussions

In this dissertation, we presented a brief introduction to the Lie group analysis and conservation laws of PDEs and gave some results which were employed throughout the dissertation.

In Chapter two symmetry reductions, exact solutions and conservation laws of a nonlinear integrable fifth-order equation were computed. The Lie symmetry method gave symmetry reductions and in conjunction with the Lie symmetry method, the Kudryashov method gave a travelling wave solution.

In Chapter three we studied the nonlinear longitudinal wave equation in a magnetoelectro-elastic circular rod via multiplier approach and Lie symmetry method. Conservation laws and Lie point symmetries of this equation were obtained. Finally the simplest equation method generated travelling wave solutions.

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