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Convexity structures in T_0-quasi-metric spaces and fixed point theorems

by

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Abstract

In this MSc dissertation, we generalize the concept of convexity structures in metric spaces to the framework of quasi-pseudometric spaces. In the T_0-quasi-metric setting we explore various interesting additional conditions that convexity structures in the sense of Takahashi can satisfy. Some interesting examples of convexity structures in both pseudometric and quasi-pseudometric spaces are discussed. Furthermore we look at fixed point theorems for maps which do not increase distance: so-called nonexpansive maps in convex metric spaces. Our results generalize and extend the well-known results in the literature. In this MSc dissertation, we generalize and extend the classical fixed point theorems obtained by F. Browder in Banach spaces. Moreover, we introduce the concept of fixed point theorems for nonexpansive mappings on convex T_0-quasi-metric spaces. We obtain necessary conditions for the existence of a fixed point of nonexpansive maps defined on a convex T_0-quasi-metric space.

Preface

The work described in this Masters Project was carried out under the supervision of Prof. Olivier Olela Otafudu, Department of Mathematical Sciences, North-West University, Mafikeng.

The project represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any other University. Where use has been made of the work of others it is duly acknowledged in the text.

Signed:

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Mcedisi Sphiwe Zweni (Student)

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Dedication

To my caring parents Elsie Zweni and Samuel Zweni, not forgetting my supporting brother Thembile Walter Zweni.

Acknowledgements

First of all, I would like to thank God for being with me since the beginning of my studies till now. I thank my supervisor Prof. Olivier Olela Otafudu for the guidance, support and patience he has shown. I thank North West university for their financial support.

Notation and Conventions

(S, m)	Pseudometric space
(S, q)	Quasi-pseudometric space
(S, m, W)	Convex metric space
(S, q, W)	Convex quasi-metric space
$W(u, v, t)$	Convex structure
$B_q(u, \epsilon)$	Open ball in quasi-pseudometric space (S, q)
$C_q(u, \epsilon)$	Closed ball in quasi-pseudometric space (S, q)
$P_0(S)$	The power set of S
$diam(S)$	Diameter of the set S

Contents

1

Introduction

In 1970, Takahashi [20] introduced the notion of convexity in metric spaces and studied some fixed point theorems for nonexpansive mappings in such spaces. A convex metric space is a generalized space. Takahashi calls a metric (S, m) convex if there exists a map W from $S \times S \times [0,1]$ into S satisfying, for all $u, v \in S$ and $t \in [0,1], m(z, W(u, v, t)) \leq tm(z, u) + (1-t)m(z, v)$ for all $z \in S$. Theorems obtained by F. Browder [6] in Banach spaces are extended to metric spaces which possess this convex structure. A fundamental problem in fixed point theory of nonexpansive mappings is to find conditions under which a set has the fixed point property for these mappings. Some interesting results in fixed point theory have been proved in convex metric spaces. Takahashi established that all normed spaces and their convex subsets are convex metric spaces. In addition, he also gave several examples of convex metric spaces which are not imbedded in any normed space or Banach space. Shimizu and Takahashi [19] introduced the notion of uniform convexity in convex metric spaces and generalized a fixed point theorem for set-valued nonexpansive mappings in uniformly convex metric spaces.

We point out that many authors studied convex metric spaces in the sense of Takahashi, for instance Talman in [21]. Consider a metric space (S, m) and let $P = \{(t_1, t_2, t_3) \in [0,1] \times [0.1] \times [0,1] : t_1 + t_2 + t_3 = 1\}$. Developing the notion of a convex metric space Talman calls (S, m) strongly convex if there exists a continuous function $K : S \times S \times S \times P \to S$ with the property that for each $(u_1, u_2, u_3, t_1, t_2, t_3) \in S \times S \times S \times P, K(u_1, u_2, u_3, t_1, t_2, t_3)$ is the unique point of S which satisfies $m(v, K(u_1, u_2, u_3, t_1, t_2, t_3)) \leq \sum_1^k t_k m(v, u_k)$ for every $v \in S$. The purpose of this study is to generalize the notion of convexity in metric spaces into the framework of quasi-metric spaces. Recently Kunzi and Yildiz [13] extended the concept of convexity structures from metric point of view to quasi-metric settings. They proved that numerous important results about convexity structures in metric spaces can be generalized to the quasi-metric setting. The main purpose of this study is for us to introduce the concept of fixed point theorems in convex T_0-quasi-metric spaces. In particular, we consider doubly closed convex bounded subsets of convex T_0-quasimetric spaces and moreover, we extend some important fixed point theorems for nonexpansive maps on these structures. Our results will extend previous results of Takahashi and many other authors

([4],[6],[13]) who studied convexity structures in metric spaces.

Outline of chapters

Chapter 1
In the first chapter we give a brief overview of certain well-known basic concepts from the theory of quasi-pseudometric spaces. We also give some interesting examples of T_0-quasi-metric spaces to be used throughout this MSc dissertation. Lastly we discuss the notion of convexity.

Chapter 2
In the second chapter we present and extend previous results of Takahashi [20] on convexity in metric spaces and nonexpansive mappings. We discuss convexity structures in a metric spaces. Furthermore, we summarize some fixed point theorems in convex metric spaces for nonexpansive mappings.

Chapter 3
In the third chapter we investigate the notion of convexity structure in the sense of Takahashi on T_0-quasi-metric spaces. We generalize well-known results about convexity structures in metric setting to quasi-metric point of view. Our generalization is based more on recent results of Kunzi and Yilzid [13] on convexity structures in T_0-quasi-metric spaces.

Chapter 4
In the forth chapter we start our own investigations on fixed point theorems in convex T_0-quasi-metric spaces. We make use of generalized results in chapter 2 for our own investigations. In this chapter we relate previous results on fixed point theorems in convex metric spaces to the framework of T_0-quasi-metric spaces.

Chapter 5
In this last chapter we conclude this MSc dissertation by reflecting on the main results of the dissertation. We summarize the results obtained and present some open problems to be studied in future.

2

Preliminaries

In this chapter, for the convenience of the reader we present some of important concepts and some basic results that are going to be useful in this MSc dissertation. We first recall the definitions of both pseudometric and quasi-pseudometric spaces and provide some examples.

2.1. Quasi-pseudometric spaces

Quasi-pseudometrics were first introduced in 1930s and are a subject of intensive research in the context of topology and theoretical computer science. The difference between a pseudometric and a quasi-pseudometric is that a quasi-pseudometric does not have to satisfy the symmetry axiom.

Definition 2.1.1. *[3, Definition 1] Consider a set S and let m be a function that maps $S \times S$ into $[0, \infty)$, that is $m : S \times S \to [0, \infty)$. Then m is called a pseudometric on S if*

(i) $m(u, u) = 0$ *for all* $u \in S$,

(ii) $m(u, v) = m(v, u)$ *for all* $u, v \in S$,

(iii) $m(u, v) \leq m(u, z) + m(z, v)$ *for all* $u, v, z \in S$.
The pair (S, m) *is called a* pseudometric space.
If in addition, for $u \neq v$ *we have*

$$m(u, v) > 0,$$

then m *is a* metric *on* S *and the pair* (S, m) *is called a* metric space.

If we delete the symmetry condition, that is, condition (ii) in Definition 1.1.1, we are led to the following concept of a quasi-pseudometric space.

Definition 2.1.2. *[15, Definition 0.1.1] Let S be a set and let $q : S \times S \to [0, \infty)$ be a function. Then q is called a* quasi-pseudometric *on S if*

(i) $q(u, u) = 0$ *for all* $u \in S$

(ii) $q(u,v) \leq q(u,z) + q(z,v)$ *for all* $u, v, z \in S$

Furthermore, q *is a* T_0-quasi-metric *if*

$$q(u,v) = 0 = q(v,u) \textit{ implies that } u = v,$$

for each $u, v \in S$.

Remark 2.1.3. [*13, Remark 1*] *If* q *is a quasi-pseudometric on a set* S*, then* $q^{-1} : S \times S \to [0, \infty)$ *on* S *defined by* $q^{-1}(u,v) = q(v,u)$ *for every* $u, v \in S$*, is called the* conjugate quasi-pseudometric. *A quasi-pseudometric on a set* S *such that* $q = q^{-1}$ *is a pseudometric. Note that if* (S,q) *is a* T_0*-quasi-metric space, then* $q^s = \sup\{q, q^{-1}\} = q \vee q^{-1}$*is also a metric.*

For any $u, v \in [0, \infty)$, let $u - v = \max\{u - v, 0\}$. Setting $q(u,v) = u - v$ whenever $u, v \in [0,1)$, we obtain a natural example of a T_0-quasi-metric space $([0,1), u)$ that we call the *standard* T_0*-quasi-metric* of $[0,1)$.

Given a quasi-pseudometric (S,q) then the set of *open balls* $B_q(u,\epsilon) = \{v \in S : q(u,v) < \epsilon\}$ with radius ϵ >0 and center $u \in S$ provides a base of the so-called quasi-pseudometric *topology* $\tau(q)$ on S. For each $u \in S$ and $\epsilon > 0$ the closed ball $C_q(u,\epsilon) = \{v \in S : q(u,v) \leq \epsilon\}$ it is $\tau(q^{-1})$*-closed.* Where $\tau(q)$ is a topology induced by the quasi-pseudometric (S,q).

Definition 2.1.4. [*15*] *For any quasi-pseudometric space* (S,q)*. If* A *is a subset of* S *and* $\epsilon \geq 0$*, we denote* $C_q(A,\epsilon)$ *by*

$$C_q(A,\epsilon) = \{u \in S : dist_q(u,A) \leq \epsilon\},$$

where

$$dist_q(u,A) = \inf\{q(u,a) : a \in A\}$$

and

$$dist_{q^{-1}}(u,A) = dist_q(A,u) = \inf\{q(a,u) : a \in A\}.$$

A subset A of a quasi-pseudometric space (S,q) will be called *bounded* provided that there is a positive real constant M such that $q(u,v) < M$ whenever $u, v \in A$.

Consider a T_0-quasi-metric space (S,q). If $A \subset S$ where A is bounded, then

$$r_u(A)_q = \sup\{q(u,v) : u \in S, v \in A\}$$

and

$$r_u(A)_{q^{-1}} = \sup\{q(v,u) : u \in S, v \in A\}.$$

We call

$$\delta(A) = \sup\{q(u,v) : u, v \in A\}$$

the *diameter* of A.
A point $u \in A$ is a *diametral point* of A if and only if

$$\sup q(u, v) = \delta(A).$$

The set $C(A) = \{u \in A : r_u(A) = r(A)\}$ is called the *Chebychev center* of A.

Remark 2.1.5. [15] *Given a T_0-quasi-metric space (S, q) and a bounded subset A of S. Values of $\delta(A), r_u(A)$ and $r(A)$ do not change when defined for the metric space (S, q^s) instead (S, q). Moreover the chebychev center of the metric space (S, q^s) is the same of the chebychev center of (S, q).*

A map $f : (S, q) \to (Y, q^*)$ between two quasi-pseudometric spaces (S, q) and (Y, q^*) is called *nonexpansive* provided that $q^*(f(u), f(v)) \le q(u, v)$ whenever $u, v \in S$.

Definition 2.1.6. [15, *Definition 4.1.2*] *Consider a T_0-quasi-metric space (S, q) and let a family of nonexpansive maps $\{T_i\}_{i \in I}$, with $T_i : S \to S$ be given. We say that $\{T_i\}_{i \in I}$ is a* commuting family *if $T_iT_j = T_jT_i$ whenever $i, j \in I$.*

Definition 2.1.7. [10] *If (S, q) is a quasi-pseudometric space, a sequence (u_n) is called:*
(i) Left-Cauchy *if for each $\epsilon > 0$. there exists $k \in \mathbb{N}$ such that $q(u_n, u_m) < \epsilon$ for all $m \ge n \ge k$.*
(ii) Right-Cauchy *if for each $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $q(u_n, u_m) < \epsilon$ for all $n \ge m \ge k$.*

Definition 2.1.8. [10] *Let (S, q) be a quasi-pseudometric space. The sequence (u_n) in S is a* Cauchy sequence *if $\lim_{n,m\to\infty} q(u_n, u_m) = 0$.*

Definition 2.1.9. [17] *A quasi-pseudometric space (S, q) is* bicomplete *if every Cauchy sequence (u_n) converges with respect to $\tau(q)$ and $\tau(q^{-1})$ to a point $u_0 \in S$.*

Example 2.1.1. Let S be a set and q be defined by

$$q(u, v) = \begin{cases} u - v & \text{if } u \ge v \\ 1 & \text{if } u < v. \end{cases}$$

Then q is a T_0-quasi-metric on S.

Example 2.1.2. Let (S, q) be a T_0-quasi-metric space. For any $u, v \in S$ with $u \ne v$ and $q(u, v) + q(v, u) \ne 0$. Then the function $w_{q(u,v).q(v.u)} : S \times S \to S$ defined by

$$w_{q(u,v),q(v.u)}(t, \acute{t}) = \begin{cases} (t - \acute{t})q(u, v) & \text{if } t \ge \acute{t} \\ (\acute{t} - t)q(v, u) & \text{if } t < \acute{t} \end{cases}$$

is a T_0-quasi-metric.

Remark 2.1.10. *For any T_0-quasi-metric space (S, q). If $q(u, v) = 1$ and $q(v, u) = 0$ whenever $u, v \in S$, then the T_0-quasi-metric $w_{q(u.v),q(v,u)}$ is the standard T_0-quasi-metric w on S.*

Definition 2.1.11. [*13, Definition 7*] *Consider a T_0-quasi-metric space (S, q). For any subset P of S, we call $cl_{\tau(q)}P \cap cl_{q^{-1}}P$ the* double closure *of P. Moreover if $P = cl_{\tau(q)}P \cap cl_{q^{-1}}P$, we say that P is* doubly closed.

Proposition 2.1.12. [*10, Proposition 5*] *Consider a quasi-metric space (S, q). For each $v \in S$, if $\lim_{n\to\infty} u_n = u$, then $\lim_{n\to\infty} q(v, u_n) = q(v, u)$.*

Proof. **Case 1.** $v = u$. Then we have

$$q(u, v) = q(u, u) = 0 = \lim_{n\to\infty} q(u, u_n) = \lim_{n\to\infty} q(v, u_n).$$

Case 2. $v \neq u$. If $u_n = u$ for infinitely many n, then

$$\lim_{n\to\infty} q(v, u_n) = q(v, u).$$

So, we may assume that $u_n \neq u$ also, $v \neq u_n$. Then we have, for all $n \in \mathbb{N}$,

$$q(v, u) \leq q(v, u_n) + q(u, u_n) \leq q(v, u) + q(u_n, u) + q(u, u_n).$$

Taking the limit as $n \to \infty$, we get $\lim_{n\to\infty} q(v, u_n) = q(v, u)$. □

A function f defined on a set S is continuous when, for every constant value c its domain $f(c)$ is defined and the limit of $f(u)$ as u approaches c equals $f(c)$ for all $u \in S$

Proposition 2.1.13. [*10, Proposition 2*] *Consider a quasi-metric space (S, q). Then q is a continuous function.*

Proof. Suppose that $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} v_n = v$ in (S, q). We have

$$q(u_n, v_n) \leq q(u_n, u) + q(u, v) + q(v, v_n).$$

It implies that

$$q(u_n, v_n) - q(u, v) \leq q(v_n, v) + q(v, v_n).$$

Also, we have

$$q(u, v) \leq q(u, u_n) + q(u_n, v_n) + q(v_n, v).$$

It implies that

$$q(u, v) - q(u_n, v_n) \leq q(u, u_n) + q(v_n, v).$$

We then have

$$0 \leq |q(u, v) - q(u_n, v_n)| \leq \max\{q(u_n, u) + q(v, v_n), q(u, u_n) + q(v_n, v)\}.$$

Taking the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} |q(u,v) - q(u_n, v_n)| = 0.$$

That is,

$$\lim_{n\to\infty} q(u_n, v_n) = q(u,v).$$

This proves that q is a continuous function. □

Definition 2.1.14. [13] *Consider a real vector space S and let $\| \cdot |: S \to [0,\infty)$ be a map such that*
(1) $\| 0 |= 0$.
(2) $\| u+v |=\| u | + \| v |$ *whenever* $u, v \in S$.
(3) $\| \alpha u |= \alpha \| u |$ *whenever* $u \in S$ *and* $\alpha \geq 0$.

Furthermore suppose that $\| u |=\| -u |= 0$ implies that $u = 0$.
The function $\| \cdot |$ is called an *asymmetric norm* on S. Obviously each asymmetrically normed vector space S induces a T_0-quasi-metric q on S by setting $q(u,v) =\| u - v |$ whenever $u, v \in S$.

A function defined on a set S with real values is called *bounded real-valued function*, if the set of its values is bounded.

Example 2.1.3. Consider a T_0-quasi-metric space (S, q), $B(S)$ be the real vector space of bounded real-valued functions on S and $\| f |= \sup_{u\in S}(f(u) - 0)$ whenever $f \in B(S)$. Clearly $\| f |$ is an asymmetric norm on BS. Consequently D defined by the condition that $D(f,g) =\| f-g |$ whenever $f, g \in B(S)$ is a T_0-quasi-metric on $B(S)$.

Definition 2.1.15. *Let S be a semigroup, and let $B(S)$ denote the set of all bounded, real valued functions on S. For each $f \in B(S)$ and for each $a \in S$, define $f_a \in B(S)$ by $f_a(u) = f(au)$ for each $u \in S$. We say that S is* left amenable *if there exists a function $\mu : B(S) \to \mathbb{R}$ such that for all $f, g \in S$, for each $a \in S$, and for all $r \in \mathbb{R}$:*

(1) $\sup_{u\in S} f(u) \geq \mu(f) \geq \inf_{u\in S} f(u)$,

(2) $\mu(f_a) = \mu(f)$,

(3) $\mu(f+g) = \mu(f) + \mu(g)$,

(4) $\mu(rf) = r\mu(f)$.

2.2. Convexity

In this section we discuss convexity. The notion of convexity is central in applied mathematics. It is also used in everyday life in connection with curvature properties of a surface. For example an optical lens is said to be convex if it is bulging outwards. Convexity appears in ancient Greek geometry, for example in the description of the five regular convex space polyhedra.

Definition 2.2.1. [*12, definition 1*] *By a convexity on a set S we mean a collection $\mathcal{G} \in \mathcal{P}(S)$ satisfying the conditions*

(1) $\emptyset, S \in \mathcal{G}$,

(2) $\bigcap \mathcal{A} \in \mathcal{G}$ *for nonempty* $\mathcal{A} \subset \mathcal{G}$,

(3) $\bigcup \mathcal{A} \in \mathcal{G}$ *whenever* $\mathcal{A} \subset \mathcal{G}$.

Geometrically we define convexity as follows:

Definition 2.2.2. [*11*] *Let S be a vector space. Then S is convex if, for $u.v \in S$ and $t \in [0,1]$, the line segment $tu + (1-t)v \in S$.*

Geometrically, this means that the line segment between any two points of S belongs to S.

Definition 2.2.3. [*11*] *A half-space is a convex set with the convex complement. The complements of convex sets are called concave.*

A function f that maps S into $\mathbb{R}$ is convex if, for all $u, v \in S$ and $t \in [0,1]$, it holds $f(tu + (1-t)v) \leq tf(u) + (1-t)f(v)$. Geometrically, this means that the line segment between any two points on the graph of the function lies above the graph. If the function f is convex then $f(t_1u_1 + \cdots + t_mu_m) \leq t_1f(u_1) + \cdots + t_mf(u_m)$ for all $u_1, \cdots, u_m \in S$ and $t \in [0,1]$ in the m-dimensional unit simplex $\{t \in \mathbb{R}^m : t_1 + \cdots + t_m = 1. t_1 \geq 0, \cdots, t_m \geq 0\}$. This is called *Jensens inequality*.

Example 2.2.1. Consider the interval $[a,b] \in \mathbb{R}$. We show that this interval is a convex set. Let $u, v \in [a,b]$ be two arbitrary elements. We need to prove that $tu + (1-t)v \in [a.b]$ for all $t \in [0,1]$. Since $u, v \in [a,b]$, then $u, v \leq b$. As $t \in [0,1]$, it follows that $tu + (1-t)v \leq b$. Using a similar argument $tu + (1-t)v \geq a$. As u and v were arbitrarily chosen, then $tu + (1-t)v \in [a,b]$ for all $u, v \in [a,b]$ and $t \in [0,1]$.

Theorem 2.2.4. [*11*] *If V and T are two convex sets in $\mathbb{R}^n$, then the intersection $V \cap T$ is a convex set.*

Proof. Let $u, v \in V \cap T$. Then $u, v \in V$ and $u, v \in T$. Since V and T are convex sets it follows that $z \in V$ and $z \in T$ where

$$z = tu + (1-t)v \quad \text{and} \quad t \in [0,1].$$

Hence $z \in V \cap T$. Since this is true for any $u, v \in V \cap T$ and any $t \in [0,1]$, $V \cap T$ is convex. □

Convex sets are closed under scalar multiplication, if $\alpha \in \mathbb{R}$ and C is convex then,

$$\alpha C = \{\alpha u : u \in C\}$$

is convex. The *Minkowski sum* of two convex sets defined by

$$C_1 + C_2 = \{u_1 + u_2 : u_1 \in C_1, u_2 \in C_2\},$$

is also convex.

Definition 2.2.5. [*11*] *The convex hull of a set S is the smallest closed convex set containing S, sometimes denoted $convS$.*

The convex hull of a finite set $u_1, \cdots, u_n$ is called an *n-polytope* and is denoted by $[u_1, \cdots, u_n]$. A *2-polytope* $[a, b]$ is called the segment joining a and b.

Proposition 2.2.6. [*12*] *For every subset A of a convexity space S the following holds:*

$$convA = \bigcup_{F \in [A]^{<\omega}} convF.$$

Consequently, the union of an up-directed collection of convex sets is convex.

Proposition 2.2.7. [*12, Definition 1.2*] *If A, B are two disjoint sets and A is convex then there exists a maximal, with respect to the inclusion, convex set C with $A \subset C$ and $C \cap B = \emptyset$.*

Let $N \in \omega$. A convexity $\mathcal{G}$ is called *N-ary* if $A \in \mathcal{G}$ whenever $convF \subset A$ for all $F \in [A]^{\leq N}$. A 2-ary convexity is called an *interval convexity* a space with 2-ary convexity will be called briefly *geometrical*.

Definition 2.2.8. [*12*] *A convexity space S is called* join-hull commutative *provided for each finite set $F \subset S$ and for each $u \in S$ we have*

$$conv(F \cup \{u\}) = \bigcup_{v \in convF} [u, v].$$

Proposition 2.2.9. [*12*] *Every join-hull commutative convexity space is geometrical.*

Proof. Suppose that S is join-hull commutative and $A \subset S$ is such that $[u, v] \subset A$ for all $u, v \in A$. By induction we show that for each $n \in \omega$ and for every $u_1, \cdots, u_n \in A$ it holds $[u_1, \cdots, u_n] \in A$. Indeed, if this is true for some n then for $u_1, \cdots, u_{n+1} \in A$ we have

$$[u_1, \cdots, u_{n+1}] = \bigcup_{v \in [u_1, \cdots, u_n]} [v, u_{n+1}] \subset A.$$

By Proposition 2.2.6, the set A is convex. □

3

Convex metric spaces

In this chapter, we present and extend the results of Takahashi [20] on convexity in metric spaces and nonexpansive mappings. We shall discuss convexity in a metric space and study the properties of the space which we call a convex metric space. Furthermore, we formulate some fixed point theorems for nonexpansive mappings.

3.1. Convex structure in metric spaces

In this section, we discuss important results on convex metric spaces. We will also look at some examples and characterizations on this theory.

Definition 3.1.1. [*9, Definition 1*] *Consider a metric space (S, m). A mapping $W : S \times S \times [0,1] \longrightarrow S$ is said to be a* convex structure *on S if for all $u, v \in S$ and $t \in [0,1]$,*

$$m(z, W(u, v, t)) \leq tm(z, u) + (1-t)m(z, v)$$

whenever $z \in S$.
Then (S, m) equipped with a Takahashi convex structure is said to be a convex metric space *denoted by (S, m, W).*

Lemma 3.1.2. [*20*] *Consider a convex metric space (S, m, W). For any $u, v \in S$ and $t \in [0,1]$ we have*

$$m(u, W(u, v, t)) = (1-t)m(u, v) \quad \text{and} \quad m(v, W(u, v, t)) = tm(u, v).$$

Proof. For any $u, v \in S$ and $t \in [0,1]$

$$\begin{aligned} m(u, W(u, v, t)) &\leq tm(u, u) + (1-t)m(u, v) \\ &= (1-t)m(u, v). \end{aligned}$$

Similarly

$$\begin{aligned} m(v, W(u, v, t)) &\leq tm(v, u) + (1-t)m(v, v) \\ &= tm(u, v). \end{aligned}$$

□

Example 3.1.1. Assume that $(S, ||.||)$ is a normed linear space. Then a map W that maps $S \times S \times [0,1]$ into S given by $W(u,v,t) = tu + (1-t)v$ for every $u, v \in S$, and $t \in [0,1]$, defines a convex structure on S. Indeed, if m is the metric induced by the norm $||.||$ then

$$\begin{aligned} m(z, W(u,v,t)) &= ||z - (tu + (1-t)v)|| \\ &= ||z - tz + tz - tu - (1-t)|| \\ &\leq t||z-u|| + (1-t)||z-v|| \\ &= tm(z,u) + (1-t)m(z,v). \end{aligned}$$

Definition 3.1.3. [9, *Definition 2*] *Suppose (S, m, W) is a convex metric space. A nonempty subset K of S is said to be convex if and only if $W(u,v,t) \in K$ whenever $u, v \in K$ and $t \in [0,1]$.*

Definition 3.1.4. [14, *Definition 2.4*] *Suppose (S, m, W) is a convex metric space and C be a convex subset of S. A self-mapping f on C has a property (I) if $f(W(u,v,t)) = W(f(u), f(v), t)$ for each $u, v \in C$ and $t \in [0,1]$.*

Let S be a set, then a function g defined on S is said to be *affine* if $g(u) = f(u) + c$ for some linear function f and constant c for all $u \in S$

Example 3.1.2. If $(S, ||||)$ is a normed space, then every affine mapping on a convex subset of S has the property (I).

Proposition 3.1.5. [20, *Proposition 1*] *Let $\{K_\alpha : \alpha \in A\}$ be a family of convex subsets of S, then $\bigcap_{\alpha \in A} K_\alpha$ is also a convex subset of S.*

Proposition 3.1.6. [20, *Proposition 2*] *Consider a convex metric space (S, m, W), both open $B_m(u,r)$ and closed $C_m(u,r)$ balls in (S, m, W) are convex subsets of S.*

Proof. Let $v, z \in B_m(u,r) = v \in S : m(u,v) < r$ and $t \in [0,1]$, then there exists $W(v,z,t) \in S$. We want to show that $W(v,z,t) \in B_m(u,r)$.

$$\begin{aligned} m(u, W(v,z,t)) &\leq tm(u,v) + (1-t)m(u,z) \\ &< tr + (1-t)r \\ &= tr + r - tr \\ &= r. \end{aligned}$$

Therefore $W(v,z,t) \in B_m(u,r)$. Similarly, $C_m(u,r)$ is also a convex subset of S. □

Proposition 3.1.7. [20, *Proposition 3*] *Suppose (S, m, W) is a convex metric space. For $u, v \in S$ and $t \in [0,1]$,*

$$m(u,v) = m(u, W(u,v,t)) + m(W(u,v,t), v).$$

Proof. Since (S, m, W) is a convex metric space, we obtain

$$\begin{aligned} m(u,v) &\leq m(u, W(u,v,t)) + m(W(u,v,t),v) \quad \text{triangle inequality} \\ &\leq tm(u,u) + (1-t)m(u,v) + tm(u,v) + (1-t)m(v.v) \\ &= tm(u,v) + (1-t)m(u,v) \\ &= m(u,v) \end{aligned}$$

for $u, v \in S$ and $t \in [0,1]$. Therefore, $m(u,v) = m(u, W(u,v.t)) + m(W(u,v,t),v)$. □

Definition 3.1.8. [9, *Definition 3*] *A convex metric space (S, m, W) will be said to have property (C) if every bounded decreasing set of nonempty closed convex subsets of S has a nonempty intersection.*

Remark 3.1.9. [9, *Remark 1*] *Every weakly compact convex subset of a Banach space has property (C).*

Proposition 3.1.10. [20, *Proposition 5*] *Consider a convex metric space (S, m, W). M be a nonempty compact subset of S and let K be the least closed convex set containing M. If the diameter $\delta(M)$ is positive, then there exists an element $z \in K$ such that $\sup\{m(u,z) : u \in M\} \leq \delta(M)$.*

Proof. Since M is compact, we may find $u_1, u_2 \in M$ such that $m(u_1, u_2) = \delta(M)$. Let $M_0 \in M$ be maximal so that $M_0 \in \{u_1, u_2\}$ and $m(u,v) = 0$ or $\delta(M)$ for $u, v \in M_0$. It is obvious that M_0 is finite. Let us assume $M_0 = \{u_1, u_2, \cdots, u_n\}$. We define

$$\begin{aligned} v_1 &= W(u_1, u_2, \frac{1}{2}), \\ v_2 &= W(u_3, v_1, \frac{1}{3}), \\ &\dots, \\ v_{n-2} &= W(u_{n-1}, v_{n-3}, \frac{1}{n-1}), \\ v_{n-1} &= W(u_n, v_{n-2}, \frac{1}{n}) = u. \end{aligned}$$

Since M is compact, we can find $y_0 \in M$ such that

$$m(v_0, z) = \sup\{m(u,z) : u \in M\}.$$

Now, by using the condition of convex metric space, we obtain

$$\begin{aligned}
m(v_0, z) &\leq \frac{1}{n}m(v_0, u_n) + \frac{n-1}{n}m(v_0, v_{n-2}) \\
&\leq \frac{1}{n}m(v_0, u_n) + \frac{n-1}{n}\Big(\frac{1}{n-1}m(v_0, u_{n-1}) + \frac{n-2}{n-1}m(v_0. v_{n-3})\Big) \\
&= \frac{1}{n}m(v_0, u_n) + \frac{1}{n}m(v_0, u_{n-1}) + \frac{n-2}{n}m(v_0, v_{n-3}) \\
&\quad \cdots, \\
&\leq \frac{1}{n}\sum_{k=1}^{n} m(v_0, u_k) \\
&\leq \delta(M).
\end{aligned}$$

Therefore if $m(v_0, z) = \delta(M)$, then we must have $m(v_0, u_k) = \delta(M) > 0$ for all $k = 1, 2, \cdots, n$, which means that $v_0 \in M_0$ by the definition of M_0. But, then we must have $v_0 = u_k$ for some $k = 1, 2, \cdots, n$. This is a contradiction. Therefore

$$\sup\{m(u, z) : u \in M\} = m(v_0, z) \leq \delta(M).$$

□

The above Proposition gives us the following definition.

Definition 3.1.11. [9, *Definition 6*] *A convex metric space (S, m, W) is said to have* normal structure *if for each closed bounded convex subset E of S containing at least two points, then there exists $u \in E$ which is not a diametral point of E.*

Remark 3.1.12. [9] *Any compact convex metric space has a normal structure.*

Definition 3.1.13. [21, *Definition 1.3*] *Let W be a convex structure on a metric space (S, m). We say that W is a* strict *convex structure if it has the property what whenever $w \in S$ and there is $(u, v, t) \in S \times S \times [0, 1]$ for which*

$$m(z, w) \leq tm(z, u) + (1 - t)m(z, v),$$

for every $z \in S$, then $w = W(u, v, t)$. If W is a strict convex structure on a metric space (S, m), we then call (S, m, W) a strictly convex metric space.

Lemma 3.1.14. [21, *Lemma 1.4*] *Let W be a strict convex structure on a metric space $(S. m)$. Then for every $u, v \in S$ and $t, \alpha \in [0, 1]$. we have*

$$W(W(u, v, t), v, \alpha) = W(u, v, t\alpha).$$

Proof. Let $z \in S$. Then

$$\begin{aligned}
m(z, W(W(u, v, t), y, \alpha)) &\leq \alpha m(z, W(u, v, t)) + (1 - \alpha)m(z, v) \\
&\leq \alpha[tm(z, u) + (1 - t)m(z, v)] + (1 - \alpha)m(z, v) \\
&\leq \alpha tm(z, u) + (1 - \alpha t)m(z, v),
\end{aligned}$$

hence by strictness, $W(W(u,v,t),y,\alpha) = W(u,v,t\alpha)$. $\square$

Definition 3.1.15. [7, *Definition 3.1*] *A strictly convex metric space (S,m) is said to be strictly convex space with convex round balls if*

$$\text{for all } \ a,b,c \in S (a \neq b) \ \text{ and } \ t \in [0,1] \ \text{ there exists } \ z \in S \ \text{ such that}$$

$$m(a,z) = tm(a,b) \ \text{ and } \ m(z,b) = (1-t)m(a,b),$$
$$m(c,z) < \max\{m(c,a), m(c,b)\}.$$

Lemma 3.1.16. [7] *Let (S,m) be a strictly convex metric space with convex round balls and K be a convex compact subset of S and $v \in S$. Then there exists unique $v_0 \in K$ such that*

$$m(v,v_0) = \inf\{m(u,v) : u \in K\}.$$

Proof. In K we define functional f with identity $f(u) = m(u,v)$, for all $u \in K$. Since K is a compact set then by Weierstrass theorem there exists $v_0 \in K$ such that

$$f(v_0) = \inf\{f(u) : u \in K\} \ \text{ or } \ m(v,v_0) = \inf\{m(u,v) : u \in K\}.$$

The uniqueness we prove from contrary. We suppose that there exists another point $v_0' \in K$ such that $m(v,v_0') = \inf\{m(u,v) : u \in K\}$. The set K is convex therefore for fixed $t \in [0,1]$ there exists $v_0'' \in K$ such that

$$m(v_0,v_0'') = tm(v_0,v_0') \ \text{ and } \ m(v_0'',v_0') = (1-t)m(v_0,v_0').$$

From the condition of convex round balls we have the following:

$$m(v,v_0'') < \max\{m(v,v_0), m(v,v_0')\} = \inf\{m(u,v) : u \in K\}.$$

We have obtained v_0'' to be closer than v_0 and v_0'. The contradiction completes the proof. $\square$

Continuity properties of a strict convex structure are investigated as follows:

Theorem 3.1.17. [25, *Theorem 1.5*] *If W is a strict convex structure on a metric space (S,m), then for every pair $u,v \in S$ with $u \neq v$ the function $t \to W(u,v,t)$ is an embedding of $[0,1]$ into S.*

Proof. Let $t_1, t_2 \in [0,1]$, and assume that $t_1 < t_2$. Then

$$\begin{aligned} m(W(u,v,t_1), W(u,v,t_2)) &= m(W(u,v,t_2\frac{t_1}{t_2}), W(u,v,t_2)) \\ &= m(W(W(u,v,t_2),v,\frac{t_1}{t_2}), W(u,v,t_2)) \\ &= (1-\frac{t_1}{t_2})m(W(u,v,t_2),v) \\ &= (t_2-t_1)m(u,v), \end{aligned}$$

which forms the theorem. $\square$

Theorem 3.1.18. [*21, Theorem 1.7*] *Consider a convex structure W on a metric space (S, m). Then W is continuous at each point (u, u, t) of $S \times S \times [0, 1]$.*

Theorem 3.1.19. [*21, Theorem 1.8*] *Let W be a strict convex structure on a compact metric space (S, m). Then W is continuous as a function from $S \times S \times [0, 1]$ to S.*

Proof. Let $\{(u_n, v_n, t_n)\}_{n=1}^{\infty}$ be a sequence in $S \times S \times [0, 1]$ which converges to (u, v, t), and let w be a limit point of the sequence $\{W(u_n, v_n, t_n)\}_{n=1}^{\infty}$. Select a subsequence $\{W(u_{n_k}, v_{n_k}, t_{n_k})\}_{k=1}^{\infty}$ which converges to w. Then for any $z \in S$, we have $m(z, W(u_{n_k}, v_{n_k}, t_{n_k})) \leq t_{n_k} m(z, u_{n_k}) + (1 - t_{n_k}) m(z, v_{n_k})$ for $k = 1, 2, \cdots$. By continuity of a metric, we conclude that $m(z, w) \leq t m(z, u) + (1 - t) m(z, v)$. Strictness now guarantees that $w = W(u, v, t)$. It follows that $W(u, v, t)$ is the only limit point of the sequence $\{W(u_n, v_n, t_n)\}_{n=1}^{\infty}$. Since S is compact, $\{W(u_n, v_n, t_n)\}_{n=1}^{\infty}$ must converge to $W(u, v, t)$, this shows that W is continuous as a function from $S \times S \times [0, 1]$ to S. □

In [21] Talman gave the following de?nition to generalize the results of Takahashi.

Definition 3.1.20. [*21, Definition 2.1*] *Let (S, m) be a metric space, and let $P = \{(t_1, t_2, t_3) \in [0, 1] \times [0, 1] \times [0, 1] : t_1 + t_2 + t_3 = 1\}$. A* strong convex structure *on S is a continuous function $K : S \times S \times S \times P \to S$ with the property that for each $(u_1, u_2, u_3, t_1, t_2, t_3) \in S \times S \times S \times P$, $K(u_1, u_2, u_3, t_1, t_2, t_3)$ is the unique point of S which satisfies*

$$m(v, K(u_1, u_2, u_3, t_1, t_2, t_3)) \leq \sum_{k=1}^{3} t_k m(v, u_k)$$

for every $v \in S$. A metric space with a strong convex structure will be called strongly convex.

Lemma 3.1.21. [*21, Lemma 2.3*] *Let (S, m) be a strongly convex metric space, and K be a strong convex structure. Define $W_K : S \times S \times [0, 1] \to S$ by $W_K(u_1, u_2, t) = K(u_1, u_2, u_3, t, 1 - t, 0)$. Then W_K is a continuous strict convex structure on S. Moreover, for any $u_1, u_2, u_3 \in S$ and $t, \alpha \in [0, 1]$, we have*

$$W_K(W_K(u_1, u_2, \alpha), u_3, t) = K(u_l, u_2, u_3, \alpha t, t(1 - \alpha), 1 - t).$$

Definition 3.1.22. [*9, Definition 7*] *Consider a convex metric space (S, m, W). A* convex hull *of the set $A (A \subset S)$ is the intersection of all convex sets in S containing A, denoted by $conv(A)$. It is obvious that if A is a convex set, then*

$$\tilde{W}^n(A) = \tilde{W}(\tilde{W}(\tilde{W}(A) ...)) \subset A \ for \ any \ n \in \mathbb{N}.$$

If we set

$$A_n = \tilde{W}^n(A), (A \subset S),$$

then the sequence $\{A_n\}_{n \in \mathbb{N}}$ will be increasing and $\limsup A_n$ *exists, and*

$$\limsup A_n = \liminf A_n = \lim A_n = \bigcup_{n=1}^{\infty} A_n.$$

Proposition 3.1.23. [9, *Proposition 1*] *Let (S, m, W) be a convex metric space. Then*

$$conv(A) = \lim A_n = \bigcup_{n=1}^{\infty} A_n, (A \subset S)$$

Proof. If $u, v \in \bigcup_{n=1}^{\infty} A_n$, then there exists a positive integer n_0 such that $u, v \in A_{n_0}$. So, for $t \in [0, 1]$,

$$W(u, v, t) \in A_{n_0+1} \subset \bigcup_{n=1}^{\infty} A_n.$$

Thus, $\bigcup_{n=1}^{\infty} A_n$ is convex contains A. Further, for any convex set C containing A,

$$\tilde{W}^n \subset C \textit{ for every } n \in \mathbb{N}$$

i.e.

$$\bigcup_{n=1}^{\infty} A_n \subset C$$

So,

$$\bigcup_{n=1}^{\infty} A_n = conv(A).$$

□

Proposition 3.1.24. [9] *For any subset A of a convex metric space (S, m, W)*

$$\delta(convA) = \delta(A).$$

Proof. Since $A \subset convA$ then $\delta(A) \leq \delta(convA)$. Let $u, v \in convA$. If $u, v \in A$, then it is obvious that $m(u, v) \leq \delta(A)$. If we let one of them, for instance u, be in $convA$ and let the other one be in A. Since $u \in convA$, there exists $n_0 \in \mathbb{N}$ such that $u \in \tilde{W}^{n_0}(A)$. But, it means that there exist $u_1, u_2 \in \tilde{W}^{n_0-1}(A), t_1 \in [0, 1]$, so that $u = W(u_1, u_2, t_1)$ and then:

$$m(u, v) = m(W(u_1, u_2, t_1).v) \leq t_1 m(u_1, v) + (1 - t)m(u_2, v).$$

By induction it can be seen that there exists the subset

$$\{u_i\}_{i \in I} \subset A \textit{ where } I \textit{ is a finite set}$$

and

$$\{\alpha_i\}_{i \in I}, \alpha_i \geq 0, \sum_{i \in I} \alpha_i = 1,$$

such that

$$m(u, v) \leq \sum_{i \in I} \alpha_i m(\tilde{u}_i, v).$$

Since

$$m(\tilde{u}_i, v) \leq \delta(A) \; for \; i \in I,$$

we can prove that

$$m(u, v) \leq \delta(A).$$

Similarly, it can be seen that the above is valid even in the case for $v \in convA$. □

Definition 3.1.25. [1, *Definition 14*] *Suppose K is a subset of a complete convex metric space (S, m, W) and $\{f_\alpha : \alpha \in K\}$ be family of maps from $[0,1]$ into K, having property that for each $\alpha \in K$ we have $f_\alpha(1) = \alpha$. Such a family is said to be*

(a) contractive provided there exist a function $\varphi : (0,1) \to (0,1)$ such that for all $\alpha, \beta \in K$ and $t \in (0,1)$ we have $m(f_\alpha(t), f_\beta(t)) \leq \varphi(t) m(\alpha, \beta)$.

(b) jointly continuous if $t \to t_0$ in $[0,1]$ and $\alpha \to \alpha_0$ in K imply $f_\alpha(t) \to f_{\alpha_0}(t_0)$.

Definition 3.1.26. [18] *A convex metric space (S, m, W) is said to be* uniformly convex *if for any $\epsilon > 0$, there exists $\alpha = \alpha(\epsilon)$ such that, for all $r > 0$ and $u, v, z \in S$ with $m(z, u) \leq r$, $m(z, v) \leq r$ and $m(u, v) \leq r\epsilon$,*

$$m(z, W(u, v, \frac{1}{2})) \leq r(1 - \alpha) < r.$$

Example 3.1.3. Let H be a Hilbert space and let S be a nonempty closed subset of $\{u \in H : \| u \| = 1\}$ such that if $u, v \in S$ and $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$, then $\frac{(\alpha u + \beta v)}{\|\alpha u + \beta v\|} \in S$ and $\delta(S) = \frac{\sqrt{2}}{2}$. Let $m(u, v) = \cos^{-1}\{(u, v)\}$ for every $u, v \in S$, where $(\cdot, \cdot)$ is the inner product of H. When we define a convex structure W for (S, m) properly, it is easily seen that (S, m) becomes a complete and uniformly convex metric space.

Remark 3.1.27. [18] *Uniformly convex Banach spaces are uniformly convex metric spaces.*

Theorem 3.1.28. [18, *Theorem 1*] *Suppose (S, m, W) is a complete and uniformly convex metric space. Then S has the property (C).*

Lemma 3.1.29. [18] *Let (S, m, W) be a complete and uniformly convex metric space. Let K be a nonempty closed convex subset of S. If $\mathcal{F}$ is a filter on S which contains at least a bounded subset of S, then there exists a unique point $u_0 \in K$ such that*

$$r(u_0, \mathcal{F}) = \inf_{u \in K} r(u, \mathcal{F}).$$

Proof. Let $r = \inf_{u \in K} r(u, \mathcal{F})$ and define

$$K_n = \{z \in K : r(z, \mathcal{F}) \leq r + \frac{1}{n}\}$$

for every positive integer n. Then it is obvious that K_n is nonempty, closed and convex. Further, K_n is bounded. In fact, let $z, w \in K_n$. Then there exists $A \in \mathcal{F}$ such that

$$\sup_{v \in A} m(z, v) < r + \frac{2}{n} \quad \text{and} \quad \sup_{v \in A} m(w, v) < r + \frac{2}{n}.$$

So, we have

$$m(z,w) \leq \sup_{v\in A} m(z,v) + \sup_{v\in A} m(w,v) < 2(r + \frac{2}{n}).$$

Since $\{K_n\}$ is a bounded decreasing sequence of nonempty closed convex subsets of K, we have

$$\bigcap_{n=1}^{\infty} K_n \neq 0$$

Further, we prove that $\bigcap_{n=1}^{\infty} K_n$ consists of one point. Let $u, v \in \bigcap_{n=1}^{\infty} K_n$ If $r = 0$, then $m(u,v) < \frac{4}{n}$ for every positive integer n. Hence $u = v$. In the case of $r > 0$. suppose $u \neq v$. Then, for a fixed positive number b, there exists a positive number ε such that

$$m(u,v) \geq (r+a)\varepsilon$$

for every $a \in [0,b]$. We can also choose $a_0 \in (0,b)$ such that

$$(r + a_0)(1 - \alpha(\varepsilon)) < r.$$

Then there exists $A \in \mathcal{F}$ such that

$$\sup_{z\in A} m(u,z) < r + a_0 \quad \text{and} \quad \sup_{z\in A} m(v,z) < r + a_0.$$

Since S is uniformly convex, we have

$$m(z, W(u,v,\frac{1}{2})) \leq (r + a_0)(1 - \alpha(\varepsilon)) < r$$

for every $z \in A$. This implies

$$\sup_{z\in A} m(z, W(u,v,\frac{1}{2})) \leq (r + a_0)(1 - \alpha(\varepsilon)) < r$$

and hence $r(W(u,v,\frac{1}{2}), \mathcal{F}) < r$. This is a contradiction, because $W(u,v,\frac{1}{2}) \in K$. Therefore we have $u = v$. □

Proposition 3.1.30. [20] *Suppose that K is a compact convex subset of a Banach space and S be the set of all nonexpansive maps of K into S. Then for each pair of elements A and B of S, define a metric m by $m(A,B) = \sup\{\,|Au - Bu|| : u \in K\}$. S is a metric space with u. Define a map W that maps $S \times S \times [0,1]$ into S by $W(A,B.t)(u) = tAu + (1-t)Bu$ for $u \in K$ and $t \in [0,1]$.*

The previous proposition leads us to the following lemma:

Lemma 3.1.31. [20] *Suppose that K is a compact convex subset of a Banach space and S be the set of all nonexpansive maps of K into S. Then S is a compact convex metric space with respect to metric m and the convex structure W.*

Proof. For $A, B, C \in S$ and $t \in [0, 1]$,

$$\begin{aligned} m(A, W(B, C, t)) &= sup\{||Au - W(B, C, t)u|| : u \in K\} \\ &\leq \sup\{t||Au - Bu|| + (1 - t)||Au - Cu|| : u \in K\} \\ &= tm(A, B) + (1 - t)m(A, C) \end{aligned}$$

and hence W is a convex structure in S. We will show that the set S with metric m is compact. If $\varepsilon > 0$ is given, a subset E of K is called an $\varepsilon - net$ if E is finite and $K = \bigcup_{a \in E} S(a, \varepsilon)$. Let $\{U_n\}$ be a sequence of nonexpansive mappings of S. Then we show that there exists a subsequence $\{U_k\}$ of $\{U_n\}$ such that U_k converges to a point of S. Let N_n be a $\frac{1}{n} - net$ of K and $N = \bigcup\{N_n : n = 1, 2, \cdots\}$. Then it is obvious that there exists a subsequence $\{U_k\}$ of $\{U_n\}$ such that $U_k u$ for every $u \in N$ converges to a point of K. We show that $U_k z$ converges to a point for every z of K. Let z be any point of K and ε be any given positive number, then there exists $u \in N$ such that $||z - u|| \leq \frac{\varepsilon}{3}$. Now, since $U_k u$ converges, there exists a positive integer k_0 such that

$$\begin{aligned} ||U_{k_1} z - U_{k_2} z|| &\leq ||U_{k_1} z - U_{k_1} u|| + ||U_{k_1} u - U_{k_2} u|| + ||U_{k_2} u - U_{k_2} z|| \\ &\leq ||z - u|| + |U_{k_1} u - U_{k_2} u|| + ||u - z|| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

if $k1, k2 > k0$. Therefore $U_k z$ converges to a point of K. Let us define $U_z = \lim U_k z$ for every z of K. Then it is obvious that U is a nonexpansive mapping. We will show that the convergence is uniform. Let ε be any positive number and choose n_0 such that $\frac{1}{n_0} \leq \varepsilon < \varepsilon$, then N_{n_0} contains a point u such that $||u - z|| \leq \frac{\varepsilon}{3}$. Now k_0 exists such that $||U_{k_1} u - U_{k_2} u|| \leq \frac{\varepsilon}{3}$ for all $u \in N_{n_0}$ when $k_1, k_2 > k_0$. Thus k_0 is independent of z and $||U_{k_1} z - U_{k_2} z|| \leq \varepsilon$ when $k_1, k_2 > k_0$. This shows uniformity of the convergence. Therefore, there exists a subsequence $\{U_k\}$ of $\{U_n\}$ such that U_k converges. This completes the proof.

□

3.2. Fixed point theorems in convex structures

In this section, we are investigating fixed point theorems for nonexpansive mappings. We present previous results studied by Takahashi [20], Bula [7] and some other authors. Fixed point theory is a fascinating subject, with an enormous number of applications in various fields of mathematics.

Definition 3.2.1. [*15*] *Let* (S, m) *be a metric space. If* $T : S \to S$ *is a map, then* $u \in S$ *is a* fixed point *of* T *if* $T(u) = u$.
Moreover we denote the fixed point set of T *by* $Fix(T)$*. that is* $Fix(T) = \{u \in S : T(u) = u\}$.

Definition 3.2.2. [*7, Definition 2.3*] *A self-mapping* f *of a subset* K *of a normed linear space is said to be* quasi-nonexpansive *provided* f *has at least one fixed point in* K*, and if* $p \in K$ *is any fixed point of* f *then*

$$\| f(u) - p \| \leq \| u - p \|$$

for all $u \in K$.

Remark 3.2.3. [*1*] *The class of nonexpansive mappings is strictly contained in the class of quasi-nonexpansive mappings.*

We present the following example to support the previous Remark:

Example 3.2.1. Define the self map T on Euclidean space $\mathbf{R}$ with the usual norm as $T(u) = u\sin(\frac{1}{u})$ when $u \neq 0$ and $T(0) = 0$. Obviously 0 is the only fixed point of T and T is quasi-nonexpansive. Take $u = \frac{2}{\pi}$, $v = \frac{2}{3\pi}$, then,

$$| T(u) - T(v) | = \frac{8}{3\pi} > \frac{4}{3\pi} = | u - v |,$$

gives that T is not nonexpansive.

Definition 3.2.4. [7] *A self-mapping* f *of a subset* K *of a normed linear space is said to be asymptotically nonexpansive if for each* $u, v \in K$ *we have*

$$\| f^i(u) - f^i(v) \| \leq k_i \| u - v \|,$$

where $(k_i)_{i \in \mathbb{N}}$ *is a sequence of real numbers such that* $\lim_{i \to \infty} k_i = 1$ *(it is assumed that* $k_i \geq 1$ *and* $k_i \geq k_{i+1}$, $i = 1, 2, \cdots$*).*

Definition 3.2.5. [*20*] *Consider a compact convex metric space* (K, m, W)*. Then a family* $\mathcal{F}$ *of nonexpansive mappings* T *of* K *into itself is said to have* invariant property *in* K *if for any compact convex subset* $E \subset K$ *such that* $TE \subset E$ *for each* $T \in \mathcal{F}$*, there exists a compact subset* $M \subset E$ *such that* $TM = M$ *for each* $T \in \mathcal{F}$.

De Marr [8] showed that a commutative family of nonexpansive mappings of K into itself has invariant property in K. The following theorem asserts that this is true even if one considers a left amenable semigroup of nonexpansive mappings.

Theorem 3.2.6. [20] *Let K be a compact convex metric space. If $\mathcal{F}$ is a left amenable semigroup of nonexpansive mappings T of K into K. then the family $\mathcal{F}$ has invariant property in K.*

Proof. Let E be a compact convex subset of K such that E is invariant under each $T \in \mathcal{F}$. By using Zorn's lemma, we can find a minimal nonempty compact set $M \subset E$ such that M is invariant under each $T \in \mathcal{F}$. Let $C(M)$ be the space of bounded continuous real valued functions on M and $C(M)^*$ be the dual space of $C(M)$ and

$$K[C(M)] = \{L \in C(M)^* : L(e) = \| L \| = 1\}$$

where e denotes the constant 1 function on M. Since the semigroup of restrictions of mappings T to M is left amenable, there exists an element $L \in K[C(M)]$ such that $L(U_T f) = L(f)$ for all $f \in C(M)$ and $T \in \mathcal{F}$ where $U_T f$ denotes an element in $C(M)$ such that $(U_T f)(u) = f(Tu)$ for each $u \in M$. The Riesz's theorem states that to the element L, there corresponds a unique probability measure m on M such that

$$L(f) = \int_M f\, dm$$

for each $f \in C(M)$ and m is invariant under each $T \in \mathcal{F}$. Therefore it is obvious that $m(TM) = m(T^{-1}TM) = m(M) = 1$. Let $F \subset M$ be the support of the measure m. Then F, TF and $T^{-1}T$ are closed and contained in M. Since $1 = m(F) = m(T^{-1}T)$, it is obvious that F is contained in $T^{-1}F$ for each $T \in \mathcal{F}$. Therefore F is invariant under each $T \in \mathcal{F}$. This implies $TM = M$ for each T in $\mathcal{F}$. □

Theorem 3.2.7. [20, Theorem 1] *Suppose that (S, m, W) has Property (C). Let K be a nonempty bounded closed convex subset of S with normal structure. If T is a nonexpansive mapping of K into itself, then T has a fixed point in K.*

Proof. Let Φ be a family of all nonempty closed and convex subsets of K, each of which is mapped into itself by T. By Property (C) and Zorn's lemma, Φ has a minimal element E. We show that E consists of a single point. Let $u \in E_c$. Then $m(Tu, Tv) \leq m(u, v) \leq R_u(E)$ for all $v \in E$, and hence $T(E)$ is contained in the spherical ball $\bar{S}(T(u), R(E))$. Since $T(E \cap \bar{S}) \subset E \cap \bar{S}$, the minimality of E implies $E \subset \bar{S}$. Hence $R_{T(u)}(E) \leq R(E)$. Since $R(E) \leq R_u(E)$ for all $u \in E$, $R_{T(u)}(E) = R(E)$. Hence $T_{(u)} \in E_c$ and $T(E_c) \subset E_c$. We then have, $E_c \in \Phi$. If $z, w \in E_c$, then $m(z, w) \leq R_z(E) = R(E)$. Hence, by normal structure, $\delta(E_c) \leq R(E) < \delta(E)$. Since this contradicts the minimality of E, $\delta(E) = 0$ and E consists of a single point. □

Theorem 3.2.8. [9, Theorem 1] *Let (S, m) be a metric space with continuous convex structure and let K be a closed convex bounded subset of (S, m) with normal structure and property (C).*

If $A : K \to K$ is a continuous mapping such that for $u, v \in K$,

$$m(Au, Av) \leq \max\{m(u,v), m(u,Au), m(v,Av), m(u,Av), m(v,Au), \\ m(u,A^2u), m(v,A^2v), m(Au,A^2u), m(Av,A^2v)\}$$

then A has a fixed point.

Theorem 3.2.9. [20, Theorem 2] *Consider a strictly convex metric space (S, m, W) with Property (C). Let K be a nonempty bounded closed convex subset of S with normal structure. If $\mathcal{F}$ is a commuting family of nonexpansive mappings of K into itself, then the family contains a common fixed point in K.*

Proof. Suppose that T is a nonexpansive mapping in a strictly convex metric space, the set F of fixed points of T is a nonempty closed convex set. In fact, as $W(u,v,t) \in K$ for $u, v \in F$ and $t \in [0,1]$,

$$\begin{aligned} m(Tu, Tv) &\leq m(Tu, T(W(u,v,t)) + m(T(W(u,v,t), Tv) \\ &\leq m(u, W(u,v,t) + m(W(u,v,t), v) \\ &= m(u,v) \end{aligned}$$

and hence, by strict convexity of the space, $T(W(u,v,t)) = W(u,v,t)$. This implies that F is convex. Let F_α be the fixed point sets of $T_\alpha \in \mathcal{F}$. If $z \in F_\alpha$, then for any α',

$$T_\alpha T_{\alpha'} z = T_{\alpha'} T_\alpha z = T_{\alpha'} z$$

i.e., $T_{\alpha'} z$ lies in F_α and each $T_{\alpha'}$ maps F_α into itself. Suppose $\alpha_1, \alpha_2, \cdots, \alpha_n$, is a finite sequence and consider T_{α_n} as a nonexpansive mapping of $F_{\alpha_1} \cap F_{\alpha_2} \cap \ldots \cap F_{\alpha_n}$ into itself, it follows that $\cap_{k=1}^n F_{\alpha_k} \neq \emptyset$. By property (C), the family $\{F_\alpha\}$ has a nonempty intersection, but this consists of the common fixed point. □

In general case, the set of fixed points of nonexpansive mappings is not necessary convex. However, we will prove the following Theorem by assuming compactness.

Theorem 3.2.10. [20, Theorem 3] *Consider a compact convex metric space (S, m, W). Let $\mathcal{F}$ be a family of nonexpansive mappings with invariant property in S, then the family $\mathcal{F}$ contains a common fixed point.*

Proof. By using Zorn's lemma, we can find a minimal nonempty convex compact set $E \subset K$ such that E is an invariant under each $T \in \mathcal{F}$. If E consists of a single point, then Theorem is proved. By hypothesis, there exists a compact subset M of E such that $M = \{T(u) : u \in M\}$ for each $T \in \mathcal{F}$. If M contains more than one point, then there exists an element z in the least convex set containing M such that

$$\rho = \sup\{m(z,u) : u \in M\} < \delta(M),$$

where $\delta(M)$ is the diameter of M. We define

$$E_0 = \cap_{u \in M}\{v \in E : m(u,v) \leq \rho\},$$

then E_0 is the nonempty closed convex proper subset of E invariant under each $T \in \mathcal{F}$. This is a contradiction to the minimality of E. $\square$

Theorem 3.2.11. *[1, Theorem 17] Let F be a compact subset of a complete convex metric space (S, m, W). Suppose there exists a contractive, jointly continuous family of maps associated with F, then any nonexpansive mapping T of F into itself has a fixed point in F.*

Proof. For each $n = 1, 2, 3, \cdots$ let $k_n = \frac{n}{n+1} \in [0,1]$. Define a mapping T_n from F into itself by $T_n(u) = f_{T(u)}(k_n)$ for all $u \in F$. Since $T(F) \subset F$ and $k_n < 1$, each T_n is well defined map from F into itself. Moreover, for each n and all $u, v \in F$, we have

$$\begin{aligned} m(T_n(u), T_n(v)) &= m(f_{T(u)}(k_n), f_{T(v)}(k_n)) \\ &\leq \phi(k_n) m(T(u), T(v)) \\ &\leq \phi(k_n) m(u,v) \end{aligned}$$

So for each n, T_n is a contraction mapping on F. Each T_n has unique fixed point $u_n \in F$. Now compactness of F gives a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $u_{n_j} \to u_0$ in F. Since $T_{n_j}(u_{n_j}) = u_{n_j}$, we have $T_{n_j}(u_{n_j}) \to u_0$. By the continuity of T, $T(u_{n_j}) \to T(u_0)$, also by the joint continuity of the family, we have

$$T_{n_j}(u_{n_j}) = f_{T(u_{n_j})}(k_{n_j}) \to f_{T(u_0)}(1) = T(u_0).$$

It gives that u_0 is the fixed point of T. $\square$

Lemma 3.2.12. *[7, Lemma 4.1] Let K be convex and closed subset of strictly convex metric space (S, m, W). If $f : K \to K$ is a quasi-nonexpansive map then the set of all fixed points of f ($Fix\ f$) is closed and convex.*

Proof. Since f is quasi-nonexpansive then $Fix\ f \neq \emptyset$ and f is a continuous map in all fixed points. We assume, that $Fix\ f$ is not closed. Then there exists u that belongs to a boundary of $Fix\ f$ and does not belong to $Fix\ f$. Since K is closed then $u \in K$. Since u does not belong to $Fix\ f$ then $f(u) \neq u$. We define $r = \frac{1}{3}m(f(u), u) > 0$. Then there exists $v \in Fix\ f$ such that $m(u,v) \leq r$. Since f is quasi-nonexpansive then $m(f(u), v) \leq m(u,v) \leq r$, and we have

$$3r = m(f(u), u) \leq m(f(u), v) + m(v, u) \leq 2r.$$

This contradiction shows that assumption of $Fix\ f$ un-closedness is false. Now we prove that $Fix\ f$ is convex. We choose two points u and v ($u \neq v$) in $Fix\ f$. Let $t \in [0,1]$, we find the corresponding $z \in K$ such that

$$m(u,z) = tm(u,v) \quad \text{and} \quad m(z,v) = (1-t)m(u,v),$$

which is unique by strictly convexity of (S, m, W). We want to prove that z belongs to $Fix\ f$. Since f is quasi-nonexpansive then

$$m(f(z), u) \leq m(z, u) \text{ and } m(f(z), v) \leq m(z, v).$$

Therefore

$$\begin{aligned} m(u, v) &\leq m(u, f(z)) + m(f(z), v) \\ &\leq m(z, u) + m(z, v) \\ &= tm(u, v) + (1-t)m(u, v) \\ &= m(u, v). \end{aligned}$$

It follows that

$$m(u, f(z)) = m(z, u) = tm(u, v), \quad m(f(z), v) = m(z, v) = m(z, v) = (1-t)m(u, v).$$

By strictly convexity of (S, m, W) we have $z = f(z)$ and $z \in Fix\ f$. i.e., $Fix\ f$ is convex. □

Theorem 3.2.13. [2, Theorem 19] *Let F be a closed subset of a convex complete metric space (S, m, W) and $T : F \to S$ be a quasi-nonexpansive mapping. Suppose there exists a point $u_0 \in F$ with $u_n = T^n(u_0) \in F$, where $n \in \mathbb{N}$. Then the sequence $\{u_n\}$ converges to the fixed point of T if and only if* $\lim_{n\to\infty}(u_n, Fix(T)) = 0$.

Proof. Since $m(u_n, Fix(T)) = \sup_{u \in Fix(T)} m(u_n, u)$ then $\lim_{n\to\infty}(u_n, Fix(T)) = 0$, immediately follows by the convergence of sequence $\{u_n\}$ to the fixed point of T.
For the converse let $\varepsilon > 0$, then there exists n_0 such that for all $n \geq n_0$, we have $m(u_n, Fix(T)) < \frac{\varepsilon}{2}$. Take the point p in $Fix(T)$ and $m, n \geq n_0$, we have

$$\begin{aligned} m(u_n, u_m) &\leq m(u_n, p) + m(p, u_m) \\ &= m(T^n(u_0), p) + m(p, T^m(u_0)) \\ &\leq 2m(T^{n_0}(u_0), p) \\ &\leq 2m(u_{n_0}, p) \\ &\leq 2m(u_0, Fix(T)), \end{aligned}$$

$\{u_n\}$ is a Cauchy sequence in F. So there exists a point z in F such that $u_n \to z$. Hence $m(z, Fix(T)) = 0$, $Fix(T)$ is closed due to quasi-nonexpansiveness of T, we have $z \in Fix(T)$. □

Theorem 3.2.14. [4, Theorem 2.1] *Let C be a nonempty closed convex subset of a convex complete metric space (S, m, W) and T be a self-mapping of C. If there exist a, b, c, e, f and k such that*

$$\frac{b + e - |f|(1-t) - |c|t}{1-t} \leq k < \frac{a + b + c + e + f - |c|t - |f|(1-t)}{1-t}$$

$$am(u, Tu) + bm(v, Tv) + cm(Tu, Tv) + em(u, Tv) + fm(v, Tu) \leq km(u, v)$$

for all $u, v \in C$, then T has at least one fixed point.

Proof. Fix $t \in (0,1)$. Suppose $u_0 \in C$ is arbitrary. We define a sequence $\{u_n\}_{n=1}$ in the following way:

$$u_n = W(u_{n-1}, T(u_{n-1}, t)), \quad n = 1, 2, 3, \cdots.$$

As C is convex, $u_n \in C$ for all $n \in \mathbb{N}$. We have

$$m(u_{n+1}, u_n) = (1-t)m(u_n, Tu_n),$$

$$\begin{aligned} m(u_n, Tu_{n-1}) &= tm(u_{n-1}, Tu_{n-1}) \\ &= \frac{t}{1-t}m(u_n, u_{n-1}). \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{1-t}m(u_{n+1}, u_n) &= m(u_n, Tu_n) \\ &\leq m(u_n, Tu_{n-1}) + m(Tu_{n-1}, Tu_n) \end{aligned}$$

and

$$\frac{c}{1-t}m(u_{n+1}, u_n) - \frac{|c|t}{1-t}m(u_n, u_{n-1}) \leq cm(Tu_{n-1}, Tu_n).$$

By triangle inequality we have

$$\begin{aligned} \frac{1}{1-t}m(u_{n+1}, u_n) &= m(u_n, Tu_n) \\ &\leq m(u_n, u_{n-1}) + m(u_{n-1}, Tu_n) \end{aligned}$$

and

$$\frac{f}{1-t}m(u_{n+1}, u_n) - |f|m(u_n, u_{n-1}) \leq fm(Tu_{n-1}, Tu_n)$$

for all $n \in \mathbb{N}$. Now, by substituting u with u_n and v with u_{n-1} in

$$am(u, Tu) + bm(v, Tv) + cm(Tu, Tv) + em(u, Tv) + fm(v, Tu) \leq km(u, v)$$

we get

$$am(u_n, Tu_n) + bm(u_{n-1}, Tu_{n-1}) + cm(Tu_n, Tu_{n-1}) + em(u_n, Tu_{n-1}) + fm(u_{n-1}, Tu_{n-1}) \leq km(u_n, u_{n-1})$$

we obtain

$$\Big(\frac{a+c+f}{1-t}\Big)m(u_{n+1}, u_n) + \Big(\frac{b-|c|t+et}{1-t} - |f|\Big)m(u_{n-1}, u_n) \leq km(u_n, u_{n-1}).$$

Thus

$$m(u_n, u_{n+1}) \leq \Big(\frac{k(1-t)+|f|(1-t)-b-e+|c|t}{a+c+f}\Big)m(u_n, u_{n-1})$$

for all $n \in \mathbb{N}$. $\frac{k(1-t)+|f|(1-t)-b-e+|c|t}{a+c+f} \in [0,1)$ and hence, $\{u_n\} \subseteq C$. Therefore, $\{u_n\}$ it is a Cauchy sequence. Since C is a closed subset of a complete space, so $\lim_{n\to\infty} u_n = u^*$ for some $u^* \in C$. We now have

$$\frac{1}{1-t}m(u_{n+1}, u_n) = m(u_n, Tu_n) \leq m(u_n, u^*) + m(u^*, Tu_n)$$

we obtain $\lim_{n\to\infty} m(u_n, Tu_n) = 0$ and by

$$m(u^*, Tu_n) \leq m(u^*, u_n) + m(u_n, Tu_n)$$

we get $\lim_{n\to\infty} Tu_n = u^*$.
Now, by substituting u with u^* and v with u_n in

$$am(u, Tu) + bm(v, Tv) + cm(Tu, Tv) + em(u, Tv) + fm(v, Tu) \leq km(u, v),$$

we obtain

$$am(u^*, Tu^*) + bm(u_n, Tu_n) + cm(Tu^*, Tu_n) + em(u^*, Tu_n) + fm(u_n, Tu^*) \leq km(u^*, u_n).$$

So

$$(a + c + f)m(u^*, Tu^*) \leq 0.$$

But $a + c + f \geq 0$ thus $Tu^* = u^*$. □

4

Convex T_0-quasi-metric space

In this chapter we investigate a concept of convexity structure in the sense of Takahashi [20] on T_0-quasi-metric spaces. We generalize further well-known results about convexity structure in metric setting to quasi-metric point of view. Recently Kunzi and Yilzid [13] studied convexity structures in T_0-quasi-metric spaces. They proved that numerous important results about convexity structures in metric spaces can be generalized to the quasi-metric setting.

4.1. Convex structure in quasi-pseudometric spaces

In this section, we generalize the notion of convexity structures in metric spaces studied in the previous chapter into quasi-metric spaces.

Definition 4.1.1. *[13] Let (S,q) be a quasi-pseudometric space. A mapping $W : S \times S \times [0,1] \to S$ is said to be a* convex structure *on S if for all $u, v \in S$ and $t \in [0,1]$,*

$$q(u, W(u,v,t)) \leq tq(z,u) + (1-t)q(u,v),$$

and

$$q(W(u,z,t),u) \leq tq(u,z) + (1-t)q(v,z)$$

whenever $z \in S$.
Then (S,q) equipped with a convex structure is said to be a convex quasi-metric space *denoted by (S,q,W).*

Note that the last condition on the definition above is just the first condition formulated for the dual T_0-quasi-metric. So by definition if $W(u,v,t)$ is a convex structure for (S,q), then it is also a convex structure for (S,q^{-1}).

Example 4.1.1. Let $\mathbb{R}$ be equipped with the standard T_0-quasi-metric space $q(u,v) = u \dot{-} v = \max\{0, u-v\}$, whenever $u, v \in \mathbb{R}$. If we define $W(u,v,t) = tu + (1-t)v$ whenever $u, v \in \mathbb{R}$ and

$t \in [0,1]$, then $(\mathbb{R}, q, W)$ is a convex quasi-metric space.

Indeed, let $z, u, v \in \mathbb{R}$ and $t \in [0,1]$, we have

$$\begin{aligned} q(z, W(u,v,t)) &= \max\{0, z - (tu + (1-t)v)\} \\ &= \max\{0, z + tz - tz - tu - (1-t)v)\} \end{aligned}$$

which implies that

$$q(u, W(u,v,t)) \leq \max\{0, t(z-u)\} + \max\{0, (1-t)(z-v)\}.$$

Moreover

$$q(z, W(u,v,t)) \leq tq(z,u) + (1-t)q(z,v),$$

similarly

$$q(W(u,v,t), z) \leq tq(u,z) + (1-t)q(v,z).$$

Proposition 4.1.2. [13] *Suppose that (S, q, W) is a convex quasi-pseudometric space, then (S, q^{-1}, W) is a convex quasi-pseudometric space too.*

Proposition 4.1.3. [13] *Suppose that (S, q, W) is a convex T_0-quasi-metric space, then (S, q^s, W) is a convex metric space.*

Proof. Suppose that (S, q, W) is a T_0-quasi-metric space. Then for all $z, u, v \in S$ and $t \in [0,1]$, we have

$$q(z, W(u,v,t)) \leq tq(z,u) + (1-t)q(z,v)$$

and

$$q(W(u,v,t), z) \leq tq(u,z) + (1-t)q(v,z)$$

Since

$$tq(z,u) \leq tq^s(z,u) \quad \text{and} \quad (1-t)q(z,v) \leq (1-t)q^s(z,v),$$

then we have

$$tq(z,u) + (1-t)q(z,v) \leq tq^s(z,u) + (1-t)q^s(z,v).$$

We have

$$tq(u,z) + (1-t)q(v,z) \leq tq^s(z,u) + (1-t)q^s(z,v).$$

We obtain

$$q^s(z, W(u,v,t)) \leq \max\{tq(z,u) + (1-t)q(z,v),\ tq(u,z) + (1-t)q(v,z)\}.$$

Therefore, we conclude that

$$q^s(z, W(u,v,t)) \leq tq^s(z,u) + (1-t)q^s(z,v).$$

Thus (S, q^s, W) is a convex metric space. □

In the following Remark we prove that if W is a convex structure on the symmetrize metric (S, q^s) of a T_0-quasi-metric space (S, q), it does not imply that W is the convex structure on the T_0-quasi-metric space (S, q).

Remark 4.1.4. (*compare [13, Example 6]*) *Note that the converse of the previous Proposition is not true.*

Proof. If we equip $S = [0,1] \times [-\frac{1}{4}, \frac{1}{4}]$ with the T_0-quasi-metric induced by the asymmetric norm $\| u | = \max\{\| u_1 |, \| u_2 |\}$ whenever $u = (u_1, u_2) \in \mathbb{R}^2$. Note that the symmetrized norm of $\| \cdot |$ is denoted by $\| \cdot \|$. Let us consider $u_1 = (0,0)$, $v_1 = (1,0) \in S$. Let a convex structure $T(u, v, t) = tu + (1-t)v$ whenever $u, v \in S$ and $t \in [0,1]$. At the point $(u_1, v_1, \frac{1}{2}) \in S \times S \times [0,1]$, we define

$$W(u, v, t) = \begin{cases} (\frac{1}{2}, -\frac{1}{4}) & \text{if } (u, v, t) = (u_1, v_1, \frac{1}{2}) \\ S(u, v, t) & \text{if } (u, v, t) \neq (u_1, v_1, \frac{1}{2}). \end{cases}$$

Obviously W is a convex structure on (S, q^s). We have to check only that W is a convex structure on (S, q^s) at the point $(u_1, v_1, \frac{1}{2})$. Let $z = (z_1, z_2) \in S$, we have $z_1 \in [0,1]$ and thus $\frac{1}{2} = \frac{1}{2}(\| z_1 - 0 \| + \| z_1 - 1 \|) \leq \frac{1}{2} \| z - u_1 \| + \frac{1}{2} \| z - v_1 \|$. Moreover we have $\| z - W(u_1, v_1, \frac{1}{2}) \| \leq \frac{1}{2}$. Hence W is a convex structure on (S, q^s).
Furthermore, W is not a convex structure on (S, q) since at $z = (\frac{1}{2}, \frac{1}{4})$, we have $\frac{1}{2} \| z - u_1 | + \frac{1}{2} \| z - v_1 | = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{4} = \frac{3}{8}$, but $\| z - W(u_1, v_1, \frac{1}{2}) | = \frac{1}{2}$. Hence $\| z - W(u_1, v_1, \frac{1}{2}) | > \frac{1}{2} \| z - u_1 | + \frac{1}{2} \| z - v_1 |$. Therefore W is not a convex structure on (S, q).

□

Proposition 4.1.5. [*13, Remark 4*] *Suppose that (S, q, W) is a convex T_0-quasi-metric space. Then $W^{-1}(u, v, t) = W(v, u, 1-t)$ whenever $u, v \in S$ and $t \in [0,1]$ is a convex structure on a T_0-quasi-metric space (S, q).*

Proof. We have

$$\begin{aligned} q(u, W^{-1}(u, v, t)) &= q(z, W(v, u, 1-t)) \\ &\leq (1-t)q(z, v) + tq(z, u) \\ &= tq(z, u) + (1-t)q(z, v), \end{aligned}$$

similarly

$$\begin{aligned} q(W^{-1}(u, v, t), z) &= q(W(v, u, 1-t), z) \\ &\leq (1-t)q(v, z) + tq(u, z) \\ &= tq(u, z) + (1-t)q(v, z) \end{aligned}$$

whenever $z, u, v \in S$ and $t \in [0,1]$. Hence $W^{-1}(u, v, t)$ is a convex structure on a T_0-quasi-metric space (S, q). □

Definition 4.1.6. [*13, Definition 2*] *A convex structure $W(u,v,t)$ on a T_0-quasi-metric space (S,q) will be called* synchronized *if $W^{-1}(u,v,t) = W(u,v,t)$ whenever $u,v \in S$ and $t \in [0,1]$.*

Remark 4.1.7. [*13, Remark 5*] *Let (S,q) be a convex T_0-quasi-metric. Then $W(u,u,t) = u$ whenever $u \in S$ and $t \in [0,1]$. Furthermore we have $W(v,u,0) = u$ and $W(v,u,1) = v$ whenever $u,v \in S$.*

Proof. The proof essentially relies on the T_0-property. Indeed for any $t \in [0,1]$,

$$q(u, W(u,u,t)) \leq tq(u,u) + (1-t)q(u,u) = 0.$$

Similarly

$$q(W(u,u,t), u) \leq tq(u,u) + (1-t)q(u,u) = 0,$$

so $W(u,u,t) = u$ by the T_0-property of (S,q). Also

$$q(u(v,u,0)) \leq 0q(u,v) + (1-0)q(u,u) = 0.$$

Similarly

$$q(W(v,u,0), u) \leq 0q(v,u) + (1-0)q(u,u) = 0.$$

Thus $u = W(v,u,0)$ by the T_0-property.

$$q(v, W(v,u,1)) \leq 1q(v,v) + (1-1)q(v,u) = 0.$$

Furthermore

$$q(W(v,u,1), v) \leq 1q(v,v) + (1-1)q(u,v) = 0.$$

Thus by the T_0-property, we have $v = W(v,u,1)$. □

Proposition 4.1.8. [*13, Proposition 1*] *Consider a convex T_0-quasi-metric space (X,q,W). For any $u,v \in S$ and $t \in [0,1]$ we have*

$$q(u, W(u,v,t)) = (1-t)q(u,v) \quad \textit{and} \quad q(W(u,v,t), v) = tq(u,v).$$

Furthermore

$$q(v, W(u,v,t)) = tq(v,u) \quad \textit{and} \quad q(W(u,v,t), u) = (1-t)q(v,u).$$

Proof. We obtain

$$\begin{aligned} q(u,v) &\leq q(u, W(u,v,t)) + q(W(u,v,t), v) \\ &\leq tq(u,u) + (1-t)q(u,v) + tq(u,v) + (1-t)q(v,v) \\ &= (1-t)q(u,v) + tq(u,v) \\ &= q(u,v) \end{aligned}$$

whenever $u, v \in S$ and $t \in [0,1]$. In particular

$$q(u, W(u, v, t)) \leq (1-t)q(u, v)$$

and

$$q(W(u, v, t), v) \leq tq(u, v).$$

Therefore

$$(1-t)q(u, v) = q(u, W(u, v, t)) \text{ and } q(W(u, v, t), v) = tq(u, v)$$

whenever $u, v \in S$ and $t \in [0,1]$. Similarly

$$\begin{aligned} q(v, u) &\leq q(v, W(u, v, t)) + q(W(u, v, t), u) \\ &\leq tq(v, u) + (1-t)q(v, v) + tq(u, u) + (1-t)q(v, u) \\ &= q(v, u) \end{aligned}$$

whenever $u, v \in S$ and $t \in [0,1]$. Therefore

$$q(v, W(u, v, t)) = tq(v, u) \text{ and } q(W(u, v, t), u) = (1-t)q(v, u).$$

□

Proposition 4.1.9. *(compare [Proposition 3.1.7]) Suppose (S, q, W) is a convex T_0-quasi-metric space, then $W(u, v, t) \in \langle u, v\rangle_q = \{z \in S : q(u, v) = q(u, z) + q(z, v)\}$ whenever $u, v \in S$.*

Proof. (compare [proposition 3.1.7]) □

Remark 4.1.10. *[13, Remark 6] Let (S, q) be a T_0-quasi-metric space and $W(u, v, t)$ and $W'(u, v, t)$ be two convexity structures on (S, q). Suppose that $u, v \in S$ and $t', t \in [0,1]$ with $t' \leq t$. Then*

$$\begin{aligned} q(u, v) &\leq q(u, W(u, v, t)) + q(W(u, v, t), W(u, v, t')) + q(W(u, v, t'), v) \\ &\leq (1-t)q(u, v) + q(W(u, v, t), W(u, v, t')) + t'q(u, v). \end{aligned}$$

Thus $(t - t')q(u, v) \leq q(W(u, v, t), W(u, v, t'))$.
A dual argument yields that if $u, v \in S$ and $t', t \in [0,1]$ with $t' \geq t$, then

$$(t' - t)q(v, u) \leq q(W(u, v, t), W(u, v, t')).$$

Proposition 4.1.11. *[Proposition 2] Let W be a convex structure on a T_0-quasi-metric space (S, q). Then for each $u \in S$ and $t \in [0,1]$, W is continuous at (u, u, t) where S carries the topology $\tau(q)$ or $\tau(q^{-1})$.*

Proof. Let $((u_n, v_n, t))$ be a sequence converging to (u, u, t) we equip S with the topology $\tau(q)$ on S. It does not matter which topology we take, we want to show that $(W(u_n, v_n, t_n))$ converges to $W(u, u, t)$, which indeed equals u. We then have $q(u, W(u_n, v_n, t_n)) \leq t_n q(u, u_n) + (1 - t_n)q(u, v_n)$ whenever $n \in \mathbb{N}$, which proves the statement. □

Definition 4.1.12. [*13, Example 8*] *Consider a convex quasi-pseudometric space (X, q, W). For any $u, v \in S$, the set $S[u, v] = \{W(u, v, t) : t \in [0, 1]\}$ is called a* quasi-metric segment *with endpoints u, v.*

Remark 4.1.13. [*13*] *Suppose (S, q, W) is a convex T_0-quasi-metric space, then for any $u, v \in S$, with $u \neq v$ the quasi-metric interval $\langle u, v\rangle_q$ contains $S[u, v]$. If $u = v$ then the quasi-metric interval which is a singleton coincides with the quasi-metric segment.*

We define the uniqueness of a convex structure as follows:

Definition 4.1.14. [*13, Definition 3*] *Consider a convex T_0-quasi-metric space (S, q, W). We say that W is a* unique convex structure *on (S, q) if for any $w \in S$ such that there exists $(u, v, t) \in S \times S \times [0, 1]$ with*

$$q(z, w) \leq tq(z, u) + (1 - t)q(z, v)$$

and

$$q(w, z) \leq tq(u, z) + (1 - t)q(v, z)$$

whenever $z \in S$. then $w = W(u, v, t)$.

Remark 4.1.15. [*13*] *The uniqueness of a W convex structure on T_0-quasi-metric space (S, q) means really that W is a unique convex structure on (S, q). Since for any T_0-quasi-metric space (S, q) if W is a unique convex structure on (S, q^s), then W is unique convex structure on (S, q). The uniqueness of the convex structure W on a T_0-quasi-metric space has connections with the concept of strict convexity.*

Lemma 4.1.16. [*13, Lemma 1*] *Let W be the unique convex structure on T_0-quasi-metric space (S, q). Then*

$$W(W(u, v, t), v, t') = W(u, v, tt')$$

whenever $u, v \in S$ and $t, t' \in [0, 1]$

Proof. Let $u, v, z \in S$ and $t, t' \in [0, 1]$. Then

$$\begin{aligned} q(z, W(W(u, v, t'), v, t)) &\leq tq(z, W(u, v, t')) + (1 - t)q(z, v) \\ &\leq t[t'q(z, u) + (1 - t')q(z, v)] + (1 - t)q(z, v) \\ &\leq tt'q(z, u) + (1 - tt')q(z, v). \end{aligned}$$

We obtain

$$\begin{aligned} q^{-1}(z, W(W(u, v, t'), v, t)) &\leq tq^{-1}(z, W(u, v, t')) + (1 - t)q^{-1}(z, v) \\ &\leq t[t'q^{-1}(z, u) + (1 - t')q^{-1}(z, v)] + (1 - t)q^{-1}(z, v) \\ &\leq tt'q^{-1}(z, u) + (1 - tt')q^{-1}(z, v). \end{aligned}$$

So by uniqueness of W, we obtain $W(W(u, v, t), v, t') = W(u, v, tt')$. ∎

Proposition 4.1.17. [13] *Suppose W is the unique convex structure on a T_0-quasi-metric space (S,q), then the map $\psi : (S[u,v],q) \to ([0,q(u,v)], u_{q(u,v)q(v,u)})$ defined by $\psi(W(u,v,t)) = tq(u,v)$ whenever $u,v \in S$ with $u \neq v$ and $t \in [0,1]$ is an isometry embedding of $S[u,v]$ into $[0,q(u,v)]$.*

Proof. For any $u,v \in S$ and $t \in [0,1]$, we have

$$q(W(u,v,t),u) = (1-t)q(v,u) \text{ and } q(v,W(u,v,t)) = tq(v,u)$$

and

$$q(u,W(u,v,t)) = (1-t)q(u,v) \text{ and } q(W(u,v,t),v) = tq(u,v).$$

To prove that ψ is an isometry, we need to prove that for any $t,t' \in [0,1]$,

$$w_{q(u,v)q(v,u)}(\psi(t),\psi(t')) = q(W(u,v,t),W(u,v,t'))$$

whenever $u,v \in S$.
We have two cases to prove since the case where $t = t'$ is obvious.
Case 1. If $t < t'$, then we have

$$q(W(u,v,t),W(u,v,t')) = q(W(u,v,t'\frac{t}{t'}),W(u,v,t')).$$

It follows

$$q(W(u,v,t),W(u,v,t')) = q(W(W(u,v,t'),v,\frac{t}{t'}),W(u,v,t')).$$

We obtain

$$q(W(u,v,t),W(u,v,t')) = (1-\frac{t}{t'})q(v,(W(u,v,t')) = (1-\frac{t}{t'})t'q(v,u).$$

So

$$q(W(u,v,t),W(u,v,t')) = (t'-t)q(v,u) = w_{q(u,v)q(v,u)}(\psi(t),\psi(t')).$$

Case 2. If $t > t'$, then

$$q(W(u,v,t),W(u,v,t')) = q(W(u,v,t),W(u,v,t\frac{t'}{t})).$$

By similar arguments as **Case 1** we have

$$q(W(u,v,t),W(u,v,t')) = (t'-t)q(v,u) = w_{q(u,v)q(v,u)}(\psi(t),\psi(t')).$$

□

Example 4.1.2. Equip $\mathbb{R}^2$ with the asymmetric norm $\| u \| = \max\{\| u_1 \|, \| u_2 \|\}$ whenever $u = (u_1,u_2) \in \mathbb{R}^2$ and associated T_0-quasi-metric q. Consider the points $u = (0,0)$ and $v = (2,0)$ in $\mathbb{R}^2$. Then $q(u,v) = 0$ and $q(v,u) = 2$. Hence in this case we obtain the following system of equations at u,v : $q(u,W(u,v,t)) = 0$, $q(W(u,v,t),v) = 0$. $q(v,W(u,v,t)) = 2t$ and $q(W(u,v,t),u) = 2(1-t)$

with $t \in [0,1]$.
The system has a unique solution, as we show next: Indeed $q(u, W(u,v,t)) = 0$ implies that $(W(u,v,t))_1 \geq 0$ and $(W(u,v,t))_2 \geq 0$. Similarly since $q(W(u,v,t),v) = 0$, we have $(W(u,v,t))_1 \leq 2$ and $(W(u,v,t))_2 = 0$. Thus $(W(u,v,t))_2 = 0$ and $(W(u,v,t))_1 = 2-2t$ whenever $t \in [0,1]$. Hence $W(u,v,t)$ with $t \in [0,1]$ is unique, as stated.
Consider now the symmetrized metric space $(\mathbb{R}^2, q^s)$ with the same points u and v. In this case the system of equations now consists of two equations, namely $q^s(v, W(u,v,t)) = 2t$ and $q^s(W(u,v,t).u) = 2(1-t)$. This system does not have a unique solution, besides the standard solution given by $\bar{S}(u,v,t) = (2-2t, 0)$ with $t \in [0,1]$ another solution is determined by $W(u,v,t) = (2-2t, 2t)$ if $t \leq \frac{1}{2}$ and $W(u,v,t) = (2-2t, 2-2t)$ if $t > \frac{1}{2}$.

Definition 4.1.18. *[13, Definition 4] A convex structure W in a T_0-quasi-metric space (S,q) has* property (S) *if*

$$q(W(u,v,t), W(u',v',t)) \leq tq(u,u') + (1-t)q(v,v')$$

whenever $u, v, u', v' \in S$ and $t \in [0,1]$.

Remark 4.1.19. *[13] Suppose W is a convex structure on a T_0-quasi-metric space (S,q). Then W satisfies* property *S' if*

$$q(W(u.v,t), W(u,z),t)) \leq (1-t)q(v,z)$$

for all u, v, z in S and t in $[0,1]$.

Proof. For all u, v, z in S and t in $[0,1]$ we have

$$\begin{aligned}
q(W(u,v,t), W(u,z),t)) &\leq tq(W(u,v.t),u) + (1-t)q(W(u,v,t),z) \\
&\leq t[tq(u,u) + (1-t)q(v,u)] + (1-t)[tq(u,z) + (1-t)q(v,z)] \\
&\leq (1-t)[t(q(v,u) + q(u,z)) + (1-t)q(v,z)] \\
&\leq (1-t)[tq(v,z) + (1-t)q(v,z)] \quad \text{by using triangle inequality} \\
&\leq (1-t)q(v,z).
\end{aligned}$$

□

4.2. Convex subsets in T_0-quasi-metric spaces

In this section we present and give an example of convex subsets in T_0-quasi-metric spaces.

For a convex T_0-quasi-metric space (S, q, W), we denote by $\mathcal{CB}_0(S)$ the collection of bounded convex elements of $\mathcal{P}_0(S)$ and denote by $\mathcal{DCB}_0(S)$ the set of all nonempty doubly closed convex bounded subsets of S.

Definition 4.2.1. *(compare Definition 3.1.3) Consider a convex quasi-pseudometric space (S, q, W) and let C be a subset of S. We say that C is convex if $W(u, v, t) \in C$ whenever $u, v \in C$ and $t \in [0, 1]$.*

For any convex quasi-pseudometric space (S, q, W), S is convex and any convex subset C of (S, q) is naturally a convex quasi-pseudometric.

Example 4.2.1. The subset [0,1] of the set $\mathbb{R}$ is convex since whenever $u, v \in [0, 1]$, we have $W(u, v, t) = tu - (1 - t)v \in [0, 1]$ with $t \in [0, 1]$.

Lemma 4.2.2. *(compare Proposition 3.1.6) Suppose (S, q, W) is a convex quasi-pseudometric space, then whenever $u, v \in S$ and $r, s \geq 0$, the closed balls $C_q(u, r)$, $C_{q^{-1}}(u, s)$ and open balls $B_q(u, r)$, $B_{q^{-1}}(u, s)$ in S are convex subsets of S. Moreover $C_q(u, r) \cap C_{q^{-1}}(v, s)$ and $B_q(u, r) \cap B_{q^{-1}}(v, s)$ are convex subsets of S.*

Proof. Let $v, z \in C_q(u, r) = \{v \in S : q(u, v) \leq r\}$, we want to show that $W(v, z, t) \in C_q(u, r)$ whenever $t \in [0, 1]$. Since (S, q, W) is a convex quasi-pseudometric space, then we obtain

$$\begin{aligned} q(u, W(v, z, t)) &\leq q(u, v) + (1 - t)q(u, z) \\ &\leq tr + (1 - t)r \\ &= tr + r - tr \\ &= r. \end{aligned}$$

Therefore $W(v, z, t) \in C_q(u, r)$. Similarly, $C_{q^{-1}}(u, s)$ is also a convex subset of S.

For the open ball $B_q(u, r) = \{v \in S : q(u, v) < r\}$ we obtain

$$\begin{aligned} q(u, W(v, z, t)) &\leq q(u, v) + (1 - t)q(u, z) \\ &< tr + (1 - t)r \\ &= tr + r - tr \\ &= r. \end{aligned}$$

Therefore $W(v,z,t) \in B_q(u,r)$. Similarly, $B_{q^{-1}}(u,r)$ is also a convex subset of S.

Let $v, z \in C_q(u,r) \cap C_{q^{-1}}(v,s)$ and let $t \in [0,1]$. It follows that

$$\begin{aligned} q(u, W(v,z,t)) &\leq q(u,v) + (1-t)q(u,z) \\ &\leq tr + (1-t)r \\ &= r. \end{aligned}$$

and

$$\begin{aligned} q(W(v,z,t),u) &\leq q(v,u) + (1-t)q(z,u) \\ &\leq ts + (1-t)s \\ &= s. \end{aligned}$$

Since W is convex structure on S. Hence $W(v,z,t) \in C_q(u,r) \cap C_{q^{-1}}(v,s)$. Similarly, $W(v,z,t) \in B_q(u,r) \cap B_{q^{-1}}(v,s)$.

□

Proposition 4.2.3. *(compare Proposition 3.1.5) Consider a convex quasipseudometric space (S,q,W) and $(C_i)_{i\in I}$ a family of convex subsets of S. Then the intersection $\cap_{i\in I}C_i$ is also a convex subset of S.*

Proof. Let $C = \cap_{i\in I}C_i$, to prove convexity of $\cap_{i\in I}C_i$ we have to show that $W(u,v,t) \in C$ for every $u,v \in S$ and $t \in [0,1]$. Since $(C_i)_{i\in I}$ is a family of convex subsets of (S,q,W) it follows that $W(u,v,t) \in C_i$ for every $i \in I$, i.e. $W(u,v,t) \in C$, which means that $\cap_{i\in I}C_i$ is also a convex subset of (S,q,W). □

Lemma 4.2.4. [13, Lemma 3] *Suppose that (S,q) is a T_0-quasi-metric space with a convex structure W that satisfies property (S). If $A \in \mathcal{CB}_0(S)$, then its double closure $cl_\tau(q)A \cap cl_\tau(q^{-1})A$ also belongs to $\mathcal{CB}_0(S)$.*

Proof. Suppose that $u,v \in cl_\tau(q)A \cap cl_\tau(q^{-1})A$. Then for any $\epsilon > 0$ we can find $a, a' \in A$ such that $q(u,a) \vee q(a',v) < \epsilon$ and thus by the triangle inequality we see that

$$q(u,v) \leq q(u,a) + q(a,a') + q(a',v) = q(a,a') + 2\epsilon \leq \delta(A) + 2\epsilon.$$

Thus the double closure of A and A have the same diameter. It follows that the double closure of A is bounded. The double closure of a convex subset A is also convex. Fix $t \in [0,1]$. For any $a,b \in cl_\tau(q)A \cap cl_\tau(q^{-1})A$ there are sequences (a_n) and (b_n) in A such that $q(a,a_n) \to 0$ and $q(b,b_n) \to 0$. Furthermore we have

$$q(W(a,b,t), W(a_n,b_n,t)) \leq tq(a,a_n) + (1-t)q(b,b_n),$$

which implies that the sequence $(W(a_n, b_n, t))$ converges to $W(a, b, t)$ with respect to the topology $\tau(q)$. Hence $W(a, b, t) \in cl_\tau(q)A$, since $W(a_n, b_n, t) \in A$ whenever $n \in \mathbb{N}$ by convexity of A. Thus $cl_\tau(q)A$ is convex. Similarly $cl_\tau(q^{-1})A$ is convex. Hence the double closure of A as the intersection of these two convex sets is convex, too. □

Definition 4.2.5. [9] *Consider a convex quasi-pseudo-metric space (S, q, W) and let $A \subset S$. Then the map $\widetilde{W} : \mathcal{P}(S) \to \mathcal{P}$ defined by*

$$\widetilde{W}(A) = \{W(u, v, t) : u, v \in A \quad and \quad t \in [0, 1]\}$$

satisfies the following properties:
(a) $A \subset \widetilde{W}(A)$ whenever $A \in \mathcal{P}(S)$,
(b) $A \subset B$ implies $\widetilde{W}(A) \subset \widetilde{W}(B)$, whenever $A \subset B \in \mathcal{P}(S)$,
(c) $\widetilde{W}(A \cap B) \subset \widetilde{W}(A) \cap \widetilde{W}(B)$, whenever $A, B \in \mathcal{P}(S)$.

Remark 4.2.6. *If A is subset of a convex quasi-pseudometric space (S, q, W). Then A is convex if and only if $\widetilde{W}(A) \subset A$.*

Remark 4.2.7. (*compare [9, Definition 2.0.23]*) *If A is convex subset of a convex quasi-pseudometric space (S, q, W), then*

$$\widetilde{W}^n(A) = \widetilde{W}(\widetilde{W}((\widetilde{W}(A) \cdots))) \subset A$$

whenever $n \in \mathbb{N}$.
If (S, q, W) is a convex quasi-pseudometric space, we set

$$A_n = \widetilde{W}^n(A) \quad for\ any \quad A \subset S.$$

The sequence $(A_n)_{n \in \mathbb{N}}$ is monotonic increasing. Therefore,

$$\limsup A_n = \bigcup_{n=1}^{\infty} A_n.$$

5

Fixed points in convex T_0-quasi-metric spaces

In this chapter, we present some results on fixed point theorems in convex quasi-metric spaces. We extend the well-known results of Takahashi [20] to the framework of quasi-pseudometric spaces.

In [20] the concept of normal structure has been introduced for convex metric spaces. The following definition extends this concept in the context of convex quasi-metric spaces.

Definition 4.0.1. *(*compare *[16, Definition 3.2])* A convex T_0-quasi metric space (S, q, W) is said to have a *normal structure* if for each doubly closed convex bounded subset A of S, $r(A) < diam(A)$ and $diam(A) > 0$.

The following definition can be compared to *property (C)* in the sense of Takahashi (20 [p.144]).

Definition 4.0.2. A convex T_0-quasi-metric space (S, q, W) is said to have *property (H)* if any decreasing family $\{D_i\}_{i\in I}$ of nonempty doubly closed convex bounded subsets of S such that $D_j \subset D_i$ with $i \leq j$, has nonempty intersection.

Remark 4.0.3. If a convex quasi-pseudometric space (S, q, W) satisfies the property (H), then (S, q^s, W) satisfies the property (C) in the sense of Takahashi [20]. Indeed, if a subset A of S is doubly closed then A is $\tau(q^s)$-closed. Obviously, if A is q-bounded, then A is q^s-bounded too. Therefore, any nonempty family of doubly closed convex bounded subsets of S is a nonempty family of $\tau(q^s)$-closed convex and q^s-bounded.

Remark 4.0.4. If (S, q, W) is a convex T_0-quasi-metric space which has the property (H) and $A \subseteq S$, then the Chebychev center $C(A) \in \mathcal{DCB}_0(S)$.

Proof. Let $u \in S$ and $n \in \mathbb{N}$. Consider $A_n(u)$ a subset of A defined by

$$A_n(u) = \left\{ \ v \in A : q(u, v) \leq r(A) + \tfrac{1}{n} \text{ and } \ q(v, u) \leq r(A) + \tfrac{1}{n} \ \right\}.$$

Then

$$\begin{aligned}
C_n &= \bigcap_{u\in S} A_n(u) \\
&= \bigcap_{u\in S} \Big[C_q\Big(u, r(A)+\frac{1}{n}\Big)\cap C_{q^{-1}}\Big(u, r(A)+\frac{1}{n}\Big)\Big] \\
&= \bigcap_{u\in S} C_{q^s}\Big(u, r(A)+\frac{1}{n}\Big).
\end{aligned}$$

Let $z, t \in C_n$. Then for any $u \in S$, we have

$$q^s(u,t) \le r(A) + \frac{1}{n}$$

and

$$q^s(u,z) \le r(A) + \frac{1}{n}.$$

It follows that

$$q(t,z) \le q(t,u) + q(u,z) \le q^s(u,t) + q^s(u,z) \le 2r(A) + \frac{2}{n}.$$

So C_n is bounded whenever $n \in \mathbb{N}$.
We have to prove that C_n is doubly closed and convex whenever $n \in \mathbb{N}$.
Indeed we have $C_n \subseteq cl_\tau(q)C_n \cap cl_\tau(q^{-1})C_n$.
Let $v \in cl_\tau(q)C_n \cap cl_\tau(q^{-1})C_n$. Then we have

$$u \in B_q\Big(v, r(A)+\frac{1}{n}\Big)\cap C_n$$

and

$$u' \in B_{q^{-1}}\Big(v, r(A)+\frac{1}{n}\Big)\cap C_n.$$

Since $u \in C_n$ and $q(v,u) < r(A) + \frac{1}{n}$ it follows that

$$v \in B_q\Big(u, r(A)+\frac{1}{n}\Big) \subseteq C_q\Big(u, r(A)+\frac{1}{n}\Big).$$

Thus

$$v \in \bigcap_{u\in S} C_q\Big(u, r(A)+\frac{1}{n}\Big).$$

By similar arguments we have

$$v \in \bigcap_{u\in S} C_{q^{-1}}\Big(u, r(A)+\frac{1}{n}\Big).$$

Therefore

$$v \in \bigcap_{u\in S} C_q\Big(u, r(A)+\frac{1}{n}\Big)\cap C_{q^{-1}}\Big(u, r(A)+\frac{1}{n}\Big).$$

Hence

$$v \in \bigcap_{u\in S} C_{q^s}\Big(u, r(A)+\frac{1}{n}\Big) = C_n.$$

One sees that $\{C_n\}_{n\in\mathbb{N}}$ is a sequence of subset of S such that $C_{n+1} \subseteq C_n$ and C_n is convex as intersection of convex subsets of S. So $C_n \in \mathcal{DCB}_0(S)$ whenever $n \in \mathbb{N}$.
Furthermore, since (S, q, W) has property (H), it follows that $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$.
Observe that

$$C(A) = \bigcap_{n=1}^{\infty} C_n$$

Therefore $C(A)$ is a doubly closed convex bounded subset of S.

□

The following theorem can be compared to Theorem 2.2.5 in the sense of Takahashi [20].

Theorem 4.0.5. Let (S, q, W) be a convex T_0-quasi metric space and K be a nonempty doubly closed convex bounded subset of S with the normal structure. If $T : (K, q) \to (K, q)$ is a nonexpansive map, then T has fixed point.

Proof. We consider a set Γ defined by

$$\Gamma = \{D \in \mathcal{DCB}_0(S)\} \text{ such that } T : (D, q) \to (D, q) \text{ nonexpansive map } \}.$$

It follows that $\Gamma \neq \emptyset$ since $K \in \Gamma$. We partially ordered Γ by $A \leq B$ if and only if $A \subset B$ whenever $A, B \in \Gamma$. If $\mathcal{C} = \{C_\alpha\}_{\alpha\in T}$ is chain in Γ, then $\bigcap_{\alpha\in T} C_\alpha \in \Gamma$. Moreover whenever $\alpha \in \Gamma$, $\bigcap_{\alpha\in T} C_\alpha \subset C_\alpha$, hence Γ is bounded below by $\bigcap_{\alpha\in T} C_\alpha$. By Zorn's lemma it follows that Γ has a minimal element. Let A be the minimal element of Γ. Then $A \neq \emptyset$ and $T : (A, q) \to (A, q)$ is a nonexpansive map.
We want to show that A consists of a single point. Therefore that single point will be the fixed fixed point.
Indeed, let

$$u \in C(A) = \{u \in A : r_u(A) = r(A)\},$$

then by nonexpansiveness of T, we have

$$q(T(u), T(v)) \leq q(u, v) \leq r(A)$$

and

$$q(T(v), T(u)) \leq q(v, u) \leq r(A)$$

whenever $v \in A$.
It implies that $T(v) \in C_{q^s}(T(u), r(A))$, thus $T(A) \subset C_{q^s}(T(u), r(A))$. Furthermore, for $u \in A$,

$$r_{T(u)}(A) \leq r_u(A) = r(A)$$

and $r(A) \leq r_{T(u)}(A)$ since $T(u) \in A$. We have $r_{T(u)}(A)$ since $T(u) \in A$. We have $r_{T(u)}(A) = r(A)$. So $T(u) \in C(A)$ whenever $u \in A$. Hence $T : C(A) \to C(A)$ is a nonexpansive map, then we have that $C(A) \in \mathcal{DCB}_0(S)$. Furthermore $C(A) \in \Gamma$. We claim that $diam(A) = 0$. Suppose that $diam(A) > 0$, then since S has normal structure, it implies that $r(A) < diam(A)$. Let $z, w \in C(A)$, then

$$q(z, w) \leq r_z(A)_q \leq r_z(A) = r(A)$$

and

$$q(w, z) \leq r_z(A)_{q^{-1}} \leq r_z(A) = r(A).$$

Hence

$$diam(C(A)) = \sup\{q(z, w) : z, w \in C(A)\} \leq r(A) < diam(A)\}.$$

It follows that $C(A)$ is a proper subset of A and it contains the minimality of A, hence $diam(A) = 0$. □

Theorem 4.0.6. (*compare Theorem 3.2.7*) Let W be the unique convex structure on a T_0-quasi metric space (S, q) with the property (H). If K is a nonempty doubly closed convex bounded subset of S with the normal structure, then any commuting family $\{T_i : i = 1, \cdots, n\}$ of nonexpansive self-maps on (K, q) has a nonempty common fixed point set ($i.e.\ \cap_{i=1}^{n} Fix(T_i) \neq \emptyset$).

Proof. If W is the unique convex structure on (S, q), then $W = W|_K$ is a unique convex structure on (K, q). Suppose $T : (K, q) \to (K, q)$ is a nonexpansive map. Then there exists $u \in K$ such that $T(u) = u$. Hence $Fix(T) = \{u \in K : T(u) = u\} \neq \emptyset$ and is doubly closed and bounded. For any $u, v \in Fix(T)$, we have $W(u, v, t) \in K$ since K is a convex subset of S. Furthermore, let $z \in K$ then

$$\begin{aligned} q(T(z), T(W(u, v, t))) &\leq q(z, W(u, v, t)) \\ &\leq tq(z, u) + (1-t)q(z, v) \end{aligned}$$

and

$$\begin{aligned} q(T(W(u, v, t), T(z)) &\leq q(W(u, v, t), z) \\ &\leq tq(u, z) + (1-t)q(v, z). \end{aligned}$$

So by uniqueness of W we have $T(W(u, v, t)) = W(u, v, t)$ whenever $u, v \in Fix(T)$. Hence $W(u, v, t) \in Fix(T)$ and therefore $Fix(T)$ is a convex subset of S.
Consider T_1 and T_2 to be nonexpansive self-maps on (K, q) such that $T_2(T_1) = T_1(T_2)$. We have to show that $T_2(Fix(T_1)) \subset Fix(T_1)$. Let $u \in Fix(T_1)$, then $T_1(u) = u$ and $T_2(u) = T_2(T_1(u)) = T_1(T_2(u))$. Hence $T_2(u) \in Fix(T_1)$.
Furthermore, we have $T_2 : Fix(T_1) \to Fix(T_1)$ has a fixed point since $Fix(T_1)$ is a nonempty doubly closed convex bounded subset of (K, q). We say v is a fixed point of T_2 then v is also a fixed point of T_1. Then by induction on the family $\{T_i : i = 1, \cdots, n\}$ of nonexpansive self-maps on (K, q), the set of common fixed point $\bigcap_{i=1}^{n} Fix(T_i) \neq \emptyset$. □

6

Conclusion

In this MSc dissertation, we have successfully generalized the results of Takahashi [20] on convexity in metric spaces and nonexpansive mappings to the framework of quasi-metric spaces. In this last chapter of our work, we present a summary of our results.

In the first part of our investigations, we generalized convexity structures in the sense of Takahashi into quasi-metric settings. We discussed properties of convexity structures in quasi-metric spaces which we used in the second part of our investigations. In the second part we presented fixed point theorems in convex T_0-quasi-metric spaces for nonexpansive mappings. Our results generalized previous results of Takahashi [20], Shimizu and Takahashi [19] and many other authors.

Our conclusion leads us to some open problems encountered throughout the present investigations. We hope to study these problems in future work.

Problem 1. *How can one establish the relationship between W-convexity and the usual notion of (linear) convexity?*
Problem 2. *How can one extend the concept of convexity structures in quasi-pseudometric spaces to the ultra-quasi-pseudometric spaces?*
Problem 3. *How can one study fixed point theorems in the sense of Takahashi in ultra-quasi-pseudometric spaces?*

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