A new distribution function estimator based on a nonparametric transformation of the data with applications.

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Abstract

The purpose of this study is to investigate the properties of a bias reduction kernel estimator of a distribution function and to compare it with existing estimation techniques in the bootstrap. The procedure which is to be investigated, was proposed by Swanepoel and van Graan (2003). Monte Carlo simulation studies were performed to compare this procedure with existing procedures in the bootstrap methodology. The simulations involved constructing 90% and 95% two-sided percentile confidence intervals and upper bounds for the mean. The simulation study provided estimates for the coverage probabilities and expected lengths of the intervals. Findings and conclusions of these simulations are reported.

Uittreksel

Die doel van hierdie studie is om die eienskappe van 'n sydigheids-verlaging kernberamer van 'n verdelingsfunksie te bestudeer, en om dit te vergelyk met bestaande skoenlusmetodes. Die metode onder beskouing is deur Swanepoel en van Graan (2003) voorgestel. Monte Carlo simulasie-studies is gedoen deur hierdie metode te gebruik en te vergelyk met bestaande skoenlusmetodes. Die metode is aangewend om 90% en 95% tweekantige vertrouensintervalle en bogrense vir die gemiddelde te bereken. Die simulasie-studies verskaf berainings vir die oordekkingswaarskynlikhede en verwagte lengtes van die intervalle. Bevindinge en gevolgtrekkings van die studies word bespreek.

Summary

The purpose of this study is to investigate the properties of a bias reduction kernel estimator of a distribution function and to compare it with existing estimation techniques in the bootstrap. The procedure which is to be investigated, was proposed by Swanepoel and van Graan (2003). Monte Carlo simulation studies were performed to compare this procedure with existing procedures in the bootstrap methodology. The simulations involved constructing 90% and 95% two-sided percentile confidence intervals and upper bounds for the mean. The simulation studies provided estimates for the coverage probabilities and expected lengths of the intervals.

Chapter 1 gives a broad overview of the non-parametric (classical) bootstrap procedure with applications. Chapter 2 describes the smoothed bootstrap procedure and how to implement this. Chapter 3 introduces the new bias reduction bootstrap method. Chapter 4 describes the methodology of the Monte Carlo studies used to compare the different methods.

In chapter 1 we explore the classical bootstrap procedure. It explains various statistical inference methodologies, like estimation of population parameters, construction of confidence intervals, estimation of the bias, implementation of regression models, and the modified bootstrap procedure.

Chapter 2 explains the mechanics of the smoothed bootstrap procedure. It explores the analytic calculation of the asymptotic optimal choice for the bandwidth parameter h, and provides some examples of calculating h under various distributions. It further explores a method to sample from the smoothed distribution function to reduce the bias of the resample.

Chapter 3 introduces the new bias reduction smoothed bootstrap. It gives a method to calculate the asymptotic optimal choice for h under this method for certain examples, and provides an algorithm on how to implement this method.

Chapter 4 describes the Monte Carlo simulations used to compare the different procedures. It describes the algorithm and inputs used in the study, as well as the outputs (which can be found in Appendix A). It also provides the findings and conclusions derived. Appendix B contains the source code of the program to perform the Monte Carlo simulations.

Opsomming

Die doel van hierdie studie is om die eienskappe van 'n sydigheids-verlagings kernberamer van 'n verdelingsfunksie te bestudeer, en om dit te vergelyk met bestaande skoenlusmetodes. Die metode onder beskouing is deur Swanepoel en van Graan (2003) voorgestel. Monte Carlo simulasie-studies is gedoen deur hierdie metode te gebruik en te vergelyk met bestaande skoenlusmetodes. Die metode is aangewend om 90% en 95% tweekantige vertrouensintervalle en bogrense vir die gemiddelde te bereken. Die simulasie-studies verskaf beramings vir die oordekkingswaarskynlikhede en verwagte lengtes van die intervalle.

Hoofstuk 1 gee 'n oorsig van die nie-parametriese (klassieke) skoenlusmetode met toepassings. Hoofstuk 2 beskryf die gladde skoenlusmetode, asook die praktiese toepassing hiervan. Hoofstuk 3 beskryf die nuwe sydigheids-verlagings skoenlusmetode. Hoofstuk 4 beskryf die Monte Carlo simulasies wat gebruik is om die verskillende metodes te vergelyk.

In hoofstuk 1 word die klassieke skoenlusmetode ondersoek. Dit verduidelik verskeie statistiese inferensie metodologieë, soos beraming van populasieparameters, konstruering van vertrouensintervalle, beraming van sydigheid, toepassing van regressiemodelle asook die gewysigde skoenlusmetode.

Hoofstuk 2 beskryf die werking van die gladde skoenlusmetode. Dit ondersoek die analitiese berekening van die asimptoties optimale keuse van die bandwydte parameter h, en dit verskaf 'n aantal voorbeelde hoe om h te bereken vir verskeie verdelings. Verder ondersoek dit 'n metode om steekproefneming te doen uit die gladde verdelingsfunksie ten einde sydigheid van die hersteekproewe te verminder.

Hoofstuk 3 beskryf die nuwe sydigheids-verlaging gladde skoenlusmetode. Dit bespreek 'n metode om die asimptoties optimale keuse van h vir hierdie nuwe metode vir sekere voorbeelde te bereken, en dit verskaf 'n algoritme om die prosedure te implementeer.

Hoofstuk 4 beskryf die Monte Carlo studies wat die verskillende metodes vergelyk. Dit beskryf die algoritme en invoer wat gebruik is in die studie, asook die uitvoer (wat in Appendiks A verkry kan word). Verder verskaf dit bevindinge en konklusies wat bereik is. Appendiks B bevat die bronkode van die program waarmee die Monte Carlo studies gedoen is.

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Notation

Symbol	Description
$\mathbf{X}_n$	Sample of size n of independent, identically distributed random variables from unknown distribution function F.
$\mathbf{X}_n^*$	Bootstrap sample of size n.
F	Unknown distribution function of some random variable.
F_n	Empirical distribution function.
θ	Population parameter.
$T_n(\mathbf{X}_n;F)$ or $\hat{\theta}$	Random variable based on observations $\mathbf{X}_n$ and unknown distribution function F.
$T_n(\mathbf{X}_n^*;F_n)$ or $\hat{\theta}^*$	Random variable based on observations $\mathbf{X}_n^*$ and Empirical distribution function . F_n.
$P(A)$	Probability of event A under F.
$P^*(A)$	Probability of event A under F_n.
$Var(X)$	Variance of random variable X under F.
$Var^*(X^*)$	Variance of random variable X^* under F_n.
$E(X)$	Expected value of random variable X under F.
$E^*(X^*)$	Expected value of random variable X^* under F_n.
n	Sample size.
M	Number of Monte Carlo iterations.
B	Number of bootstrap iterations.
h	Bandwidth parameter of smoothed bootstrap.
[x]	Integer part of some value x.

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Chapter 1

The Classical Bootstrap

1.1 Introduction

The term bootstrap comes from the phrase "to pull oneself up by one's bootstraps". The phrase was coined by Rudolph Erich Raspe in "The Surprising Adventures of Baron Münchausen", written in 1786. In one of these fantastic stories, the Baron fell to the bottom of a deep lake. Just when it looked as if all was lost, he thought to pull himself up by his bootstraps.

The modern bootstrap procedure is not far removed from this. The methodology involves resampling from the original sample of independent observations. The success of the bootstrap methodology is highly dependent on the choice of estimate for the unknown population distribution. The method requires a lot of processing power, and it is only since the advent of modern computers that the method has taken root. Today a lot of research has gone into this method of analysing data, and with more powerful computers at hand, simulation has become easier, more flexible and straightforward.

In this chapter we will introduce the classical bootstrap procedure to the reader as described by Efron and Tibshirani (1993).

1.2 The Classical Bootstrap Procedure

The success of the bootstrap methodology in Statistics is highly dependent on the choice of estimate for the unknown population distribution. In the classical bootstrap methodology, the empirical distribution function is used as an estimate of the population distribution. Let $\mathbf{X_n} = (X_1, X_2, \dots, X_n)$ be an independent, identically distributed sample of data with unknown distribution function F and density function f. Then the empirical distribution function is defined by:

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \le x),$$

where I(A) is the indicator function of the event A.

This estimate puts equal probabilities n^{-1} at each sample value. Furthermore, $nF_n(x)$ is a binomial random variable (n trials, probability F(x) of success), therefore

$$E(F_n(x)) = F(x),$$

and

$$Var(F_n(x)) = [F(x)(1-F(x))]/n.$$

Now, let $T_n(\mathbf{X_n};F)$ be some specified random variable that we are interested in. In the classical bootstrap, we will estimate the sampling distribution of $T_n(\mathbf{X_n};F)$ under F with the bootstrap distribution of $T_n(\mathbf{X_n^*};F_n)$ under F_n. In this context, $\mathbf{X_n^*} = (X_1^*, X_2^*, ..., X_n^*)$, is a random sample (independent, with replacement) of size n from F_n.

The estimate $T_n(\mathbf{X_n^*};F_n)$ depends on how well F_n approximates F. The Glivenko-Cantelli theorem shows that F can be approximated by F_n in a uniform manner for large sample sizes, or

$$\lim_{n\to\infty} \sup_{-\infty<x<\infty} |F_n(x) - F(x)| = 0 \text{ almost surely.}$$

It can also be shown that the rate of this convergence is $O(n^{-1/2}(\log\log n)^{1/2})$ a.s. (Jacod and Protter: 2000).

The bootstrap distribution can be calculated by Taylor series expansion, direct theoretical calculation (which is not always possible), or with Monte Carlo approximation. This is a computerised method, and the algorithm is as follows:

Generate repeated, independent realisations of $\mathbf{X_n^*}$ by taking random samples with replacement of size n from F_n, and do this B times. Then we have B bootstrap samples $\mathbf{X_n^*}(1) = (X_{11}^*, X_{12}^*, ..., X_{1n}^*)$ to $\mathbf{X_n^*}(B) = (X_{B1}^*, X_{B2}^*, ..., X_{Bn}^*)$. For each of these B samples, calculate $T_n(\mathbf{X_n^*};F_n)$. The distribution of $T_n(\mathbf{X_n^*};F_n)$ is then estimated by the empirical distribution of $T_n(\mathbf{X_n^*}(1);F_n)$ to $T_n(\mathbf{X_n^*}(B);F_n)$. By increasing the size of B, we can increase the accuracy of this estimate.

1.3 The Bootstrap Estimate of Standard Deviation

From the discussion above we have seen how we can approximate the bootstrap distribution of $T_n(\mathbf{X}_n^*;F_n)$ with the empirical distribution of the $T_n(\mathbf{X}_n^*(i);F_n)$'s for i=1,...,B. This is then an estimate of the real population distribution of $T_n(\mathbf{X}_n;F)$.

Suppose we have observed a random sample of data $\mathbf{x}=(x_1,x_2,...,x_n)$ from distribution F. Then the sample estimate of θ is $\hat{\theta}=\hat{\theta}(x_1,x_2,...,x_n)$ and the standard deviation is

$$\sigma(F)=[Var_F(\hat{\theta})]^{1/2}.$$

Because F is unknown, $\sigma(F)$ is unknown. An approach to find the bootstrap estimate of the standard deviation is as follows:

1. Construct F_n by putting mass n^{-1} at each point in $\mathbf{x}$.
2. Draw a random sample of size n, $X_1^*,X_2^*,...,X_n^*$ from F_n with replacement, and calculate $\hat{\theta}^*(1)=\hat{\theta}(X_1^*,X_2^*,...,X_n^*)$.
3. Independently repeat step 2 B times to obtain replications $\hat{\theta}^*(1),\hat{\theta}^*(2),...,\hat{\theta}^*(B)$.
4. The bootstrap estimate of the standard deviation is:

$$\begin{aligned}\hat{\sigma}^* &= [Var^*(\hat{\theta}^*)]^{1/2}\\ &= [E^*(\hat{\theta}^*-E^*(\hat{\theta}^*))^2]^{1/2}\\ \mathrm{B}&\left[\frac{1}{B-1}\sum_{b=1}^{B}\{\hat{\theta}^*(b)-\hat{\theta}^*(\cdot)\}^2\right]^{1/2},\end{aligned}$$

where $\hat{\theta}^*(\cdot)=\frac{1}{B}\sum_{b=1}^{B}\hat{\theta}^*(b)$.

If $B\to\infty$, then $\hat{\sigma}^*$ converges to $\sigma(F_n)$. Var^* is the variance under the bootstrap sample $X_1^*,X_2^*,...,X_n^*$. In most cases B between 50 and 200 is sufficient to estimate standard deviations (Efron and Tibshirani: 1993). For other bootstrap estimates, a larger value of B is required.

1.4 Bootstrap Estimation of the Bias

An approach to estimate the bias of an estimator for a population parameter θ is as follows: suppose we have observed a random sample of data $\mathbf{x} = (x_1, x_2, ..., x_n)$ from a distribution F. Then the sample estimate of θ is $\hat{\theta} = \hat{\theta}(x_1, x_2, ..., x_n)$. The bias of $\hat{\theta}$ is defined by:

$$b(F) = E(\hat{\theta} \mid F) - \theta .$$

An estimate of b(F) is then:

$$b(F_n) = E(\hat{\theta} \mid F_n) - \hat{\theta} = E^*(\hat{\theta}^*) - \hat{\theta} .$$

This can be approximated by

$$\frac{1}{B}\sum_{b=1}^{B}\hat{\theta}^*(b) - \hat{\theta} .$$

1.5 Bootstrap Applied to Regression Models

We will now illustrate how the bootstrap methodology can be applied to more complicated data structures, like regression models.

Let $\mathbf{X} = (X_1, X_2, ..., X_n)$ with $X_i = g(\boldsymbol{\beta}, t_i) + \varepsilon_i$ and i=1,2,...,n. Here $\boldsymbol{\beta}$ is a $k \times 1$ vector of unknown parameters we wish to estimate, t_i is a $k \times 1$ deterministic vector and g is a known function. The ε_i's are independent, identically distributed with distribution function F and $E(\varepsilon_i) = 0$.

Having observed $X_1 = x_1, X_2 = x_2, ..., X_n = x_n$, we now wish to find an estimate for $\boldsymbol{\beta}$. We will employ the method of least squares, developed by Legendre and Gauss, which minimizes the residual squared error:

$$\hat{\boldsymbol{\beta}} = \arg\min_{\boldsymbol{\beta}} \sum_{i=1}^{n}[x_i - g(\boldsymbol{\beta}, t_i)]^2 .$$

We can employ the bootstrap as follows to find an estimate of the sampling distribution of $\hat{\boldsymbol{\beta}}$:

1. Construct F_n by putting mass n^{-1} on each of the centred residuals:

$$\hat{\varepsilon}_i = x_i - g(\hat{\boldsymbol{\beta}}, t_i) - \frac{1}{n}\sum_{j=1}^{n}\{x_j - g(\hat{\boldsymbol{\beta}}, t_j)\}, \text{ for i=1,...,n}.$$

2. The bootstrap sample is then

$$X_i^* = g(\hat{\boldsymbol{\beta}}, t_i) + \varepsilon_i^*,$$

where $\varepsilon_1^*, \varepsilon_2^*, ..., \varepsilon_n^*$ are independent, identically distributed random variables from F_n.

3. Calculate:

$$\hat{\boldsymbol{\beta}}^* = \arg\min_{\boldsymbol{\beta}} \sum_{i=1}^{n}[x_i^* - g(\boldsymbol{\beta}, t_i)]^2,$$

where $x_1^*, x_2^*, ..., x_n^*$ the realisations of $X_1^*, X_2^*, ..., X_n^*$.

4. Independently repeat steps 2 and 3 B times to obtain bootstrap replications $\hat{\boldsymbol{\beta}}^*(1), \hat{\boldsymbol{\beta}}^*(2), ..., \hat{\boldsymbol{\beta}}^*(B)$. We then estimate the sampling distribution of $\hat{\boldsymbol{\beta}}$ with the bootstrap distribution of $\hat{\boldsymbol{\beta}}^*$.

1.6 Bootstrap Confidence Intervals

We will now introduce to the reader bootstrap methodologies to construct confidence intervals for population parameters. We will also use confidence intervals in a later chapter to illustrate differences between the various bootstrap methodologies.

A confidence interval is a tool to assess the uncertainty of parameter estimators. The estimated standard deviation $\hat{\sigma}$ of an estimator $\hat{\theta}$ of an unknown parameter θ is crucial in constructing such intervals, because the standard deviation gives us some idea of the reliability or precision of the estimator. A confidence interval will be defined by limits $\hat{\theta}_{\alpha_1}$ and $\hat{\theta}_{1-\alpha_2}$ such that $P(\theta < \hat{\theta}_{\alpha_1}) = \alpha_1$ and $P(\theta < \hat{\theta}_{1-\alpha_2}) = 1 - \alpha_2$.

The coverage of the interval $[\hat{\theta}_{\alpha_1}, \hat{\theta}_{1-\alpha_2}]$ is $1-(\alpha_1+\alpha_2)$. Typically we will choose equal error probabilities on the two tails, i.e. $\alpha_1 = \alpha_2 = \alpha$. For the interval $[\hat{\theta}_{\alpha}, \hat{\theta}_{1-\alpha}]$, we then have coverage $1-2\alpha$. The one-sided confidence bound $(-\infty, \hat{\theta}_{1-\alpha}]$ has coverage $1-\alpha$.

The standard confidence interval with coverage probability ($1-2\alpha$) for a parameter θ is

$$[\hat{\theta} - z(1-\alpha)\hat{\sigma}, \hat{\theta} - z(\alpha)\hat{\sigma}],$$

where $z(1-\alpha) = \Phi^{-1}(1-\alpha)$ is the $100(1-\alpha)$ percentile point of the standard normal distribution. For this confidence interval, we assumed that

$$(\hat{\theta} - \theta)/\hat{\sigma} \sim N(0,1).$$

This is only approximately true, however, so intervals of this type usually do not have very good coverage, and we will show how to employ the bootstrap to create intervals with better coverage.

The Bootstrap-t

Let $a_\alpha = a_\alpha(F)$ and $a_{1-\alpha} = a_{1-\alpha}(F)$ be constants that satisfy:

$$P(a_\alpha \le (\hat{\theta} - \theta)/\hat{\sigma} \le a_{1-\alpha}) \approx 1-2\alpha,$$

where $\hat{\theta}$ is the estimator of the unknown parameter θ and $\hat{\sigma}$ is the estimated standard deviation of $\hat{\theta}$. If $a_\alpha(F)$ and $a_{1-\alpha}(F)$ were known, then a $(1-2\alpha)$-confidence interval for θ would be

$$[\hat{\theta} - a_{1-\alpha}(F)\hat{\sigma}, \hat{\theta} - a_\alpha(F)\hat{\sigma}].$$

All we need to do is find an estimate for a, and we can do this by plugging in the bootstrap estimate for F, namely F_n. Then we have the following approximate $(1-2\alpha)$-confidence interval for θ:

$$[\hat{\theta} - a_{1-\alpha}(F_n)\hat{\sigma}, \hat{\theta} - a_\alpha(F_n)\hat{\sigma}].$$

The bootstrap estimates $a_\alpha(F_n)$ and $a_{1-\alpha}(F_n)$ are defined by

$$P^*(a_\alpha(F_n) \le (\hat{\theta}^* - \hat{\theta}) / \hat{\sigma}^* \le a_{1-\alpha}(F_n)) \approx 1-2\alpha ,$$

where $\hat{\theta}^*$ and $\hat{\sigma}^*$ are the estimates $\hat{\theta}$ and $\hat{\sigma}$ based on the bootstrap sample $X_1^*, X_2^*, ..., X_n^*$ from F_n. P^* is the probability calculation under the bootstrap distribution of $X_1^*, X_2^*, ..., X_n^*$, with F_n given. The following Monte Carlo algorithm can be used to find an estimate of $a_\alpha(F_n)$ and $a_{1-\alpha}(F_n)$:

1. Construct F_n by putting mass n^{-1} on each point in $\mathbf{x}$, the observed sample.

2. Draw a random sample of size n, $X_1^*, X_2^*, ..., X_n^*$ with replacement from F_n, and calculate $\hat{\theta}^*(1)$, $\hat{\sigma}^*(1)$ and $T_n^*(1) = (\hat{\theta}^*(1) - \hat{\theta}) / \hat{\sigma}^*(1)$. $\hat{\sigma}^*(1)$ is the estimated standard error of the first bootstrap sample. This means that we will need to do an additional bootstrap within the bootstrap to get an estimator for the standard deviation.

3. Independently repeat step 2 B times to obtain replications $T_n^*(1), T_n^*(2), ..., T_n^*(B)$.

4. Arrange these replications in ascending order (order statistics), denoted by $Z^*_{(1)} \le Z^*_{(2)} \le ... \le Z^*_{(n)}$.

5. The bootstrap estimate of $a_\alpha(F_n)$ is then $Z^*_{([(B+1)\alpha])}$ and for $a_{1-\alpha}(F_n)$ it is $Z^*_{([(B+1)(1-\alpha)])}$, with [y] denoting the largest integer less than or equal to y. The $(1-2\alpha)$ bootstrap interval is then:

$$[\hat{\theta} - Z^*_{([(B+1)(1-\alpha)])}\hat{\sigma}, \hat{\theta} - Z^*_{([(B+1)\alpha])}\hat{\sigma}],$$

and a $(1-\alpha)$ confidence bound for θ is $(-\infty, \hat{\theta} - Z^*_{([(B+1)\alpha])}\hat{\sigma}]$.

The Percentile Method

Let $\hat{G}$ be the cumulative distribution function of $\hat{\theta}^* = \hat{\theta}(X_1^*, X_2^*, ..., X_n^*)$. That is

$$\hat{G}(t) = P^*(\hat{\theta}^* < t).$$

The $(1-2\alpha)$ percentile interval is defined by the α and the $1-\alpha$ percentiles of $\hat{G}$, it is

$$[\hat{G}^{-1}(\alpha), \hat{G}^{-1}(1-\alpha)].$$

Per definition $\hat{G}^{-1}(\alpha) = \hat{\theta}^{*(\alpha)}$ where $\hat{\theta}^{*(\alpha)}$ is the $100\cdot\alpha$th percentile of the bootstrap distribution. Then the above interval can be written as

$$[\hat{\theta}^{*(\alpha)}, \hat{\theta}^{*(1-\alpha)}].$$

In practice, we can generate these intervals as follows: generate B independent bootstrap samples and compute the bootstrap replications $\hat{\theta}^*(1), \hat{\theta}^*(2), ..., \hat{\theta}^*(B)$. Take the order statistics of these replications, then the $100\cdot\alpha$th percentile of the bootstrap distribution is $\hat{\theta}^*_{([B\alpha])}$ and the $100\cdot(1-\alpha)$th percentile is $\hat{\theta}^*_{([B(1-\alpha)])}$. The percentile interval is then $[\hat{\theta}^*_{([B\alpha])}, \hat{\theta}^*_{([B(1-\alpha)])}]$, and the $(1-\alpha)$ confidence bound for θ is $(-\infty, \hat{\theta}^*_{([B(1-\alpha)])}]$.

Further improvements can be made on these intervals by taking into account bias and skewness, and making appropriate adjustments for these factors. Also, a larger value for B needs to be chosen, usually greater than 1000, for these procedures to work well.

1.7 The Modified Bootstrap.

Consider $T_n(\mathbf{X_n}; F)$, a random variable dependent on the unknown distribution F. The bootstrap method so far discussed gives an approximation for the sampling distribution of $T_n(\mathbf{X_n}; F)$ under F with the bootstrap distribution of $T_n(\mathbf{X_n^*}; F_n)$ under F_n, where $\mathbf{X_n^*} = (X_1^*, X_2^*, ..., X_n^*)$ denotes a random sample of size n from F_n, i.e.

$$P(T_n(\mathbf{X_n}; F) \in \mathfrak{B}) \approx P^*(T_n(\mathbf{X_n^*}; F_n) \in \mathfrak{B}),$$

for any Borel set $\mathfrak{B}$. Singh (1981), and Bickel and Freedman (1981) showed that this approximation is asymptotically correct when $n \to \infty$ for a large number of situations. However, they also provided examples where the bootstrap seems to fail. Swanepoel (1986) showed how these cases could be rectified by the modified bootstrap. What it boils down to is replacing $T_n(\mathbf{X}_n^*; F_n)$ with $T_m(\mathbf{X}_m^*; F_n)$ for a suitable value of m, the bootstrap sample size.

Chapter 2

The Smoothed Bootstrap

2.1 Introduction

Up to now, we have used the discrete empirical distribution function F_n as an estimator for the unknown population distribution F. Now we will consider a smoothed estimate for F. Let k (henceforth known as a kernel function) be a known density function where we assume k is symmetric around 0, i.e. $k(-x) = k(x)$.

The assumption that k is a density function implies that $k \geq 0$, $\int_{-\infty}^{+\infty} k(x)dx = 1$ and the fact that k is symmetric around 0 further implies that $\int_{-\infty}^{+\infty} xk(x)dx = 0$.

In the literature there exists an estimator for the population density function f based on the kernel function, namely

$$\hat{f}_{n,h}(x) = \frac{1}{nh}\sum_{i=1}^{n} k(\frac{x - X_i}{h}),$$

where h is a sequence of smoothing parameters, or bandwidth parameters, for which we require $h \to 0$ and $nh \to \infty$ as $n \to \infty$.

The estimator for the population distribution function F is then defined as follows:

$$\hat{F}_{n,h}(x) = \int_{-\infty}^{x} \hat{f}_{n,h}(t)dt = \frac{1}{n}\sum_{i=1}^{n} K(\frac{x - X_i}{h}), \tag{2.1.1}$$

where K is the distribution function corresponding to k, or $K(x) = \int_{-\infty}^{x} k(t)dt$.

Silverman (1979) has shown that the choice of K isn't that important, so it can be chosen as any known continuous distribution function, for example, the standard normal distribution Φ. The choice of h is more critical and this will be further investigated in this chapter.

2.2 Asymptotic Optimal choice for h

We will now investigate a method of deriving an optimal value for h, similar to that of Azzalini (1981). Under certain conditions placed on F and K as $n \to \infty$, the following holds:

$$E\{K(\frac{x-X}{h})\} = \int_{-\infty}^{+\infty} K(\frac{x-y}{h})dF(y).$$

With partial integration this can be written as:

$$\begin{aligned} E\{K(\frac{x-X}{h})\} &= \int_{-\infty}^{+\infty} K(\frac{x-y}{h})dF(y) \\ &= K(\frac{x-y}{h})F(y)\mid_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} F(y)dK(\frac{x-y}{h}) \\ &= \frac{1}{h}\int_{-\infty}^{+\infty} F(y)k(\frac{x-y}{h})dy. \end{aligned}$$

Substituting $\frac{x-y}{h} = z$ and using Taylor series expansion we get:

$$\begin{aligned} E\{K(\frac{x-X}{h})\} &= \frac{1}{h}\int_{-\infty}^{+\infty} F(y)k(\frac{x-y}{h})dy \\ &= \int_{-\infty}^{+\infty} F(x-hz)k(z)dz \\ &= \int_{-\infty}^{+\infty} [F(x) - hzf(x) + \tfrac{1}{2}h^2z^2 f'(x) + h^3R_1(x,z)]k(z)dz. \end{aligned}$$

Completing the integration, we get:

$$\begin{aligned} E\{K(\frac{x-X}{h})\} &= \int_{-\infty}^{+\infty} [F(x) - hzf(x) + \tfrac{1}{2}h^2z^2 f'(x) + h^3R_1(x,z)]k(z)dz \\ &= F(x)\int_{-\infty}^{+\infty} k(z)dz - hf(x)\int_{-\infty}^{+\infty} zk(z)dz + \tfrac{1}{2}h^2 f'(x)\int_{-\infty}^{+\infty} z^2k(z)dz + h^3\int_{-\infty}^{+\infty} R_1(x,z)k(z)dz \\ &= F(x) + \tfrac{1}{2}f'(x)h^2\mu_2(k) + O(h^3), \end{aligned}$$

where we made use of the fact that $\int_{-\infty}^{+\infty} zk(z)dz = 0$, and where $\mu_2(k) = \int_{-\infty}^{+\infty} z^2k(z)dz$, the variance of K. We can now find $E(\hat{F}_{n,h}(x))$, which is simply:

$$E\{\hat{F}_{n,h}(x)\} = F(x) + A_1(x)h^2 + O(h^3), \tag{2.2.1}$$

where $A_1(x) = \frac{1}{2} f'(x)\mu_2(k)$. From this we see that the bias of $\hat{F}_{n,h}$ is $O(h^2)$, which approaches 0 as h approaches 0.

By a similar approach as above, we can find $E((\frac{x-X}{h})^2)$:

$$\begin{aligned}
E((\frac{x-X}{h})^2) &= \int_{-\infty}^{+\infty} K(\frac{x-y}{h})^2 dF(y) \\
&= [K(\frac{x-y}{h})^2 F(y) |_{-\infty}^{+\infty} + \frac{2}{h} \int_{-\infty}^{+\infty} F(y) K(\frac{x-y}{h}) k(\frac{x-y}{h}) dy] \\
&= \frac{2}{h} \int_{-\infty}^{+\infty} F(y) K(\frac{x-y}{h}) k(\frac{x-y}{h}) dy.
\end{aligned}$$

By similar substitution for z as $\frac{x-y}{h} = z$ and using Taylor series expansion we get:

$$\begin{aligned}
E((\frac{x-X}{h})^2) &= 2\int_{-\infty}^{+\infty} F(x-hz) K(z) k(z) dz \\
&= 2\int_{-\infty}^{+\infty} K(z)k(z)[F(x) - hzf(x) + \tfrac{1}{2}h^2 z^2 f'(x) + h^3 R_1(x,z)] dz \\
&= 2F(x)\int_{-\infty}^{+\infty} K(z)k(z)dz - 2hf(x)\int_{-\infty}^{+\infty} zK(z)k(z)dz + O(h^2) \\
&= F(x)K(z)^2 |_{-\infty}^{+\infty} - 2hf(x)\int_{-\infty}^{+\infty} zK(z)k(z)dz + O(h^2) \\
&= F(x) - 2hf(x)\int_{-\infty}^{+\infty} zK(z)k(z)dz + O(h^2) \\
&= F(x) - 2hf(x)\beta(k) + O(h^2),
\end{aligned}$$

where $\beta(k) = \int_{-\infty}^{+\infty} zk(z)K(z)dz$. We can now find an expression for $Var\{\hat{F}_{n,h}(x)\}$, namely:

$$\begin{aligned}
Var\{\hat{F}_{n,h}(x)\} &= Var\{\frac{1}{n}\sum_{i=1}^{n} K(\frac{x-X_i}{h})\} \\
&= \frac{1}{n^2}\sum_{i=1}^{n} Var\{K(\frac{x-X_i}{h})\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n^2}\sum_{i=1}^{n}[E\{K(\frac{x-X_i}{h})\}^2 - \{E(K(\frac{x-X_i}{h}))\}^2] \\
&= \frac{1}{n}[E\{K(\frac{x-X}{h})\}^2 - \{E(K(\frac{x-X}{h}))\}^2] \\
&= \frac{1}{n}[F(x) - 2hf(x)\beta(k) + O(h^2) - \{F(x) + \tfrac{1}{2}f'(x)\mu_2(k)h^2 + O(h^3)\}^2] \\
&= \frac{1}{n}F(x) - \frac{1}{n}F(x)^2 - 2\beta(k)f(x)\frac{h}{n} + O(\frac{h^2}{n}),
\end{aligned}
$$

which can be written as:

$$
Var\{\hat{F}_{n,h}(x)\} = \frac{F(x)(1-F(x))}{n} - A_2(x)\frac{h}{n} + O(\frac{h^2}{n}), \tag{2.2.2}
$$

where $A_2(x) = 2\beta(k)f(x) > 0$. From this we see the variance of $\hat{F}_{n,h}$ is asymptotically smaller than the variance of the empirical distribution function F_n.

The mean squared error (MSE) of $\hat{F}_{n,h}$ is a pointwise measure of how well $\hat{F}_{n,h}$ estimates F and is defined by

$$
MSE(\hat{F}_{n,h}) = E(\hat{F}_{n,h}(x) - F(x))^2.
$$

This is equivalent to:

$$
\begin{aligned}
MSE(\hat{F}_{n,h}) &= Var(\hat{F}_{n,h}(x)) + [E(\hat{F}_{n,h}(x)) - F(x)]^2 \\
&= Var(\hat{F}_{n,h}(x)) + Bias(\hat{F}_{n,h}(x))^2.
\end{aligned}
$$

From this and from (2.2.1) and (2.2.2), we can find the asymptotic mean squared error (AMSE) of $\hat{F}_{n,h}$:

$$
AMSE(\hat{F}_{n,h}) = \frac{F(x)(1-F(x))}{n} - A_2(x)\frac{h}{n} + A_1^2(x)h^4. \tag{2.2.3}
$$

A global measure of accuracy is the mean integrated squared error, which is defined by

$$MISE(\hat{F}_{n,h}, w) = \int_{-\infty}^{+\infty} E\{(\hat{F}_{n,h}(x) - F(x))^2\} w(x) dF(x),$$

where w(x) is a weight function. For the purpose of this dissertation, we will consider the case where $w(x) = (F'(x))^2$ (Swanepoel and van Graan: 2003).

From (2.2.3) and substitution of $w(x) = (F'(x))^2$, we can find the asymptotic mean integrated squared error, which is:

$$\begin{aligned} AMISE(\hat{F}_{n,h}) &= \int_{-\infty}^{+\infty} AMSE\{\hat{F}_{n,h}(x)\}\{F'(x)\}^2 dF(x) \\ &= \int_{-\infty}^{+\infty} \{\frac{F(x)(1-F(x))}{n} - A_2(x)\frac{h}{n} + A_1^2(x)h^4\} F'(x)^2 dF(x). \end{aligned} \tag{2.2.4}$$

We can now find an asymptotic optimal choice of h by an argument similar to that of Epanechikov (1967), by finding the h that minimizes the AMISE. This is done as follows:

$$\begin{aligned} AMISE(\hat{F}_{n,h}) &= \int_{-\infty}^{+\infty} \{\frac{F(x)(1-F(x))}{n} - A_2(x)\frac{h}{n} + A_1(x)^2 h^4\} F'(x)^2 dF(x) \\ &= \frac{1}{n}\int_{-\infty}^{+\infty} F(x)(1-F(x))F'(x)^2 dF(x) - \frac{h}{n}\int_{-\infty}^{+\infty} A_2(x)F'(x)^2 dF(x) \\ &\quad + h^4 \int_{-\infty}^{+\infty} A_1(x)^2 F'(x)^2 dF(x). \end{aligned}$$

Differentiation with respect to h gives us:

$$\begin{aligned} &\frac{d}{dh}[\frac{1}{n}\int_{-\infty}^{+\infty} F(x)(1-F(x))F'(x)^2 dF(x) - \frac{h}{n}\int_{-\infty}^{+\infty} A_2(x)F'(x)^2 dF(x) \\ &+ h^4 \int_{-\infty}^{+\infty} A_1(x)^2 F'(x)^2 dF(x)] \\ &= -\frac{1}{n}\int_{-\infty}^{+\infty} A_2(x)F'(x)^2 dF(x) + 4h^3 \int_{-\infty}^{+\infty} A_1(x)^2 F'(x)^2 dF(x), \end{aligned}$$

and setting this equal to 0 we find:

$$-\frac{1}{n}\int_{-\infty}^{+\infty} A_2(x)F'(x)^2\,dF(x)+4h_0^{\,3}\int_{-\infty}^{+\infty} A_1(x)^2F'(x)^2\,dF(x)=0$$

$$\Rightarrow h_0=\left[\frac{1}{4n}\frac{\int_{-\infty}^{+\infty} A_2(x)F'(x)^2\,dF(x)}{\int_{-\infty}^{+\infty} A_1(x)^2F'(x)^2\,dF(x)}\right]^{\frac{1}{3}}.$$

This can be written as:

$$h_0=\left\{\frac{1}{4n}\frac{\int_{-\infty}^{+\infty} A_2(x)f^3(x)\,dx}{\int_{-\infty}^{+\infty} A_1(x)^2f^3(x)\,dx}\right\}^{\frac{1}{3}}.$$

Substitution of this value for h in (2.2.4) gives us:

$$\begin{aligned}AMISE(\hat{F}_{n,h_0})&=\frac{1}{n}\int_{-\infty}^{+\infty} F(x)(1-F(x))F'(x)^2\,dF(x)-\frac{h_0}{n}\int_{-\infty}^{+\infty} A_2(x)F'(x)^2\,dF(x)\\&\quad+h_0^{\,4}\int_{-\infty}^{+\infty} A_1(x)^2F'(x)^2\,dF(x)\\&=\frac{1}{n}\int_{-\infty}^{+\infty} F(x)(1-F(x))f(x)^3\,dx-\frac{1}{n^{4/3}}\frac{\left((\frac{1}{4})^{\frac{1}{3}}\frac{3}{4}\right)\left(\int_{-\infty}^{+\infty} A_2(x)f^3(x)\,dx\right)^{\frac{4}{3}}}{\left(\int_{-\infty}^{+\infty} A_1(x)^2f^3(x)\,dx\right)^{\frac{1}{3}}},\end{aligned}$$

which can be written in the form:

$$AMISE(\hat{F}_{n,h_0})=\frac{B_1}{n}-\frac{B_2}{n^{4/3}},\ \text{with } \mathrm{B}_1>0,\ \mathrm{B}_2>0.$$

2.3 Examples of the Asymptotic Optimal choice for h

We will now illustrate the optimal choice for h with some examples. We will consider examples where k and f are normally distributed as well as where k has a uniform distribution.

Normal Kernel and Normal Density

We will now consider the case where both the kernel and density functions are normally distributed, i.e.

$$\begin{aligned} f(x) &= \frac{1}{\sigma}\phi\left(\frac{x}{\sigma}\right) \\ &= \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} \text{ for } -\infty < x < +\infty, \end{aligned}$$

and

$$k(z) = \phi(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2} \text{ for } -\infty < z < +\infty.$$

In this case,

$$\begin{aligned} \beta(k) &= \int_{-\infty}^{+\infty} zk(z)K(z)dz \\ &= -\int_{-\infty}^{+\infty} k'(z)K(z)dz \\ &= -K(z)k(z)|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} k(z)^2 dz \\ &= \int_{-\infty}^{+\infty} k(z)^2 dz \\ &= \frac{1}{2}\frac{1}{\sqrt{\pi}}. \end{aligned}$$

Also:

$$\begin{aligned} \mu_2(k) &= \int_{-\infty}^{+\infty} z^2 k(z)dz \\ &= 1, \end{aligned}$$

and

$$\begin{aligned} A_2(x) &= 2\beta(k)f(x) \\ &= 2\frac{1}{2}\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} \\ &= \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}, \end{aligned}$$

and

$$\begin{aligned} A_1(x) &= \frac{1}{2}f'(x)\mu_2(k) \\ &= -\frac{x}{2\sigma^2}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}. \end{aligned}$$

It now follows that:

$$
\begin{aligned}
h_0 &= \left\{ \frac{1}{4n} \frac{\int_{-\infty}^{+\infty} A_2(x) F'(x)^2 \, dF(x)}{\int_{-\infty}^{+\infty} A_1(x)^2 F'(x)^2 \, dF(x)} \right\}^{\frac{1}{3}} \\
&= \left\{ \frac{1}{4n} \frac{\int_{-\infty}^{+\infty} \left(\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} \right) \left(\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} \right)^3 dx}{\int_{-\infty}^{+\infty} \left(-\frac{1}{2} x \frac{1}{\sqrt{2\pi}\sigma^3} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} \right)^2 \left(\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} \right)^3 dx} \right\}^{\frac{1}{3}} \\
&= \left\{ \frac{1}{n} \sqrt{2}\sigma^5 \frac{\int_{-\infty}^{+\infty} e^{-2\left(\frac{x}{\sigma}\right)^2} dx}{\int_{-\infty}^{+\infty} x^2 e^{-\frac{5}{2}\left(\frac{x}{\sigma}\right)^2} dx} \right\}^{\frac{1}{3}} \\
&= \left\{ \frac{1}{n} \sqrt{2}\sigma^5 \frac{\sigma \frac{1}{2} \sqrt{2\pi}}{\sigma^3 \frac{\sqrt{10}}{25} \sqrt{\pi}} \right\}^{\frac{1}{3}} \\
&= \left\{ \frac{1}{n} \sigma^3 \frac{25}{\sqrt{10}} \right\}^{\frac{1}{3}} \\
&= 1.992 \sigma n^{-1/3}.
\end{aligned}
$$

We can find an approximate value for h_0 by approximating σ with some estimate $\hat{\sigma}$, i.e., $\hat{h}_0 = 1.992 \hat{\sigma} n^{-1/3}$.

Uniform Kernel and Normal Density

Now let us consider the case where the kernel has a uniform distribution and the density function is normally distributed, i.e.

$$
\begin{aligned}
f(x) &= \frac{1}{\sigma} \phi\left(\frac{x}{\sigma}\right) \\
&= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} \quad \text{for } -\infty < x < +\infty,
\end{aligned}
$$

and

$$
k(z) = \frac{1}{2\sqrt{3}} \quad \text{for } |z| \le \sqrt{3}.
$$

In this case,

$$\begin{aligned}\beta(k) &= \int_{-\infty}^{+\infty} zk(z)K(z)dz \\ &= \int_{-\sqrt{3}}^{+\sqrt{3}} \frac{z(z+\sqrt{3})}{(2\sqrt{3})^2} dz \\ &= \frac{1}{2\sqrt{3}}.\end{aligned}$$

Also:

$$\begin{aligned}\mu_2(k) &= \int_{-\infty}^{+\infty} z^2 k(z)dz \\ &= \int_{-\sqrt{3}}^{+\sqrt{3}} \frac{z^2}{2\sqrt{3}} dz \\ &= 1,\end{aligned}$$

and

$$\begin{aligned}A_2(x) &= 2\beta(k)f(x) \\ &= \frac{1}{\sqrt{6\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2},\end{aligned}$$

and

$$\begin{aligned}A_1(x) &= \frac{1}{2} f'(x)\mu_2(k) \\ &= -\frac{x}{2\sigma^2} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}.\end{aligned}$$

It now follows that:

$$\begin{aligned}h_0 &= \left\{ \frac{1}{4n} \frac{\int_{-\infty}^{+\infty} A_2(x) F'(x)^2 dF(x)}{\int_{-\infty}^{+\infty} A_1(x)^2 F'(x)^2 dF(x)} \right\}^{\frac{1}{3}} \\ &= \left\{ \frac{1}{4n} \frac{\int_{-\infty}^{+\infty} \left(\frac{1}{\sqrt{6\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} \right) \left(\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} \right)^3 dx}{\int_{-\infty}^{+\infty} \left(-\frac{x}{2} \frac{1}{\sqrt{2\pi}\sigma^3} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} \right)^2 \left(\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} \right)^3 dx} \right\}^{\frac{1}{3}}\end{aligned}$$

$$
\begin{aligned}
&= \left\{ \frac{1}{n} \frac{2\sqrt{\pi}}{\sqrt{6}} \sigma^5 \frac{\int_{-\infty}^{+\infty} e^{-2\left(\frac{x}{\sigma}\right)^2} dx}{\int_{-\infty}^{+\infty} x^2 e^{-\frac{5}{2}\left(\frac{x}{\sigma}\right)^2} dx} \right\}^{\frac{1}{3}} \\
&= \left\{ \frac{1}{n} \frac{2\sqrt{\pi}}{\sqrt{6}} \sigma^5 \frac{\sigma \frac{1}{2}\sqrt{2\pi}}{\sigma^3 \frac{\sqrt{10}}{25}\sqrt{\pi}} \right\}^{\frac{1}{3}} \\
&= \left\{ \frac{1}{n} \sigma^3 \frac{25\sqrt{\pi}}{\sqrt{30}} \right\}^{\frac{1}{3}} \\
&= 2.0075\sigma n^{-1/3}.
\end{aligned}
$$

As previously indicated, we can find an estimate for h_0 by estimating σ with $\hat{\sigma}$, i.e., $\hat{h}_0 = 2.0075\hat{\sigma} n^{-1/3}$.

2.4 Smoothed Bootstrap Methodology

An algorithm to construct a bootstrap sample from $\hat{F}_{n,h}$ is as follows:

1. Generate independent random variables $Y_1^*, Y_2^*, ..., Y_n^*$ from F_n (the empirical distribution function of the data).
2. Independently generate independent random variables $Z_1, Z_2, ..., Z_n$ from K (the kernel distribution function).
3. Let $X_i^* = Y_i^* + hZ_i$, be the bootstrap sample from $\hat{F}_{n,h}$. The implementation of the smoothed bootstrap follows exactly as that of the classical bootstrap, with these new X_i^*'s used in the process.

Generating confidence intervals, calculating standard deviations, bias and regression can still be done as explained in chapter 1, with the above method in mind. We are in effect just replacing F_n with $\hat{F}_{n,h}$.

Something to keep in mind is that the variance of X_i^* is not unbiased. If we use the fact that $Var(Z_i)=1$, $E(Z_i)=0$ and that Z_i and Y_i^* are independent, we can calculate the variance of X_i^* as follows:

$$\begin{aligned}
Var^*(X_i^*) &= Var^*(Y_i^* + hZ_i) \\
&= Var^*(Y_i^*) + Var^*(hZ_i) \\
&= E^*(Y_i^{*2}) - E^*(Y_i^*)^2 + h^2 \\
&= \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2 + h^2 \\
&= \frac{1}{n}\sum_{i=1}^{n} (X_i - \bar{X}_n)^2 + h^2 \\
&= s_n^{\ 2} + h^2.
\end{aligned}$$

An alternative for step 3 is to incorporate a variance-stabilizing factor, i.e.,

$$X_i^* = \bar{Y}_n^* + \left(1 + \frac{h^2}{s_n^{\ 2}}\right)^{-\frac{1}{2}} (Y_i^* - \bar{Y}_n^* + hZ_i).$$

Chapter 3

The Bias Reduction Method

3.1 Introduction

In this chapter we will investigate a new bias reduction method for nonparametric distribution function estimation as developed by Swanepoel and van Graan (2003). The same assumptions we made in chapter 2 still hold here, i.e. the kernel function k is symmetric around 0, is a density function and K is the distribution function corresponding to k. We will look at some of the asymptotic properties of this new estimator of the distribution function, and will also look at some examples analogous to those in chapter 2.

The new nonparametric distribution function estimator is defined as

$$\tilde{F}_{n,h}(x)=\frac{1}{n}\sum_{i=1}^{n}K\left(\frac{\hat{F}_{n,h}(x)-\hat{F}_{n,h}(X_i)}{h}\right),$$

where $\hat{F}_{n,h}(x)$ is the usual kernel distribution function estimator as defined in (2.1.1) and h is the bandwidth or smoothing parameter.

3.2 Asymptotic Optimal choice for h

Swanepoel and van Graan (2003) have shown that under certain conditions on F and K the following holds:

$$E(\tilde{F}_{n,h})=F(x)+C_1(x)h^4+O(h^5),$$

where $C_1(x)=\frac{1}{4}\mu_2^{\,2}(k)\left(\frac{f'(x)f''(x)}{f^3(x)}-\frac{f'''(x)}{f^2(x)}\right)$ and $\mu_2(k)=\int_{-\infty}^{+\infty}z^2k(z)dz$, the variance of K. This is an improvement on the kernel distribution function estimator of chapter 2, as it yields a smaller bias ($O(h^4)$ compared to $O(h^2)$).

Furthermore,

$$Var(\tilde{F}_{n,h}) = \frac{F(x)(1-F(x))}{n} - C_2\frac{h}{n} + O(h^2/n),$$

where $C_2 = 2\beta(k) > 0$ and $\beta(k) = \int_{-\infty}^{+\infty} zk(z)K(z)dz$.

The mean squared error of $\tilde{F}_{n,h}$ is

$$\begin{aligned} MSE(\tilde{F}_{n,h}) &= E(\tilde{F}_{n,h}(x) - F(x))^2 \\ &= Var(\tilde{F}_{n,h}(x)) + [E(\tilde{F}_{n,h}) - F(x)]^2 \\ &= Var(\tilde{F}_{n,h}) + Bias(\tilde{F}_{n,h})^2 \\ &= \frac{F(x)(1-F(x))}{n} - C_2\frac{h}{n} + O(h^2/n) + \left(C_1(x)h^4 + O(h^5)\right)^2. \end{aligned}$$

From this we can find the asymptotic mean squared error of $\tilde{F}_{n,h}$, which is:

$$\text{AMSE}(\tilde{F}_{n,h}) = \frac{F(x)(1-F(x))}{n} - C_2\frac{h}{n} + C_1^2(x)h^8.$$

The asymptotic mean integrated squared error of $\tilde{F}_{n,h}$ is

$$\begin{aligned} AMISE(\tilde{F}_{n,h}) &= \int_{-\infty}^{+\infty} AMSE\{\tilde{F}_{n,h}(x)\}\{F'(x)\}^2 dF(x) \\ &= \int_{-\infty}^{+\infty} \{\frac{F(x)(1-F(x))}{n} - C_2\frac{h}{n} + C_1^2(x)h^8\}F'(x)^2 dF(x) \\ &= \frac{1}{n}\int_{-\infty}^{+\infty} F(x)(1-F(x))F'(x)^2 dF(x) - 2\beta(k)\frac{h}{n}\int_{-\infty}^{+\infty} F'(x)^2 dF(x) \\ &+ \frac{h^8}{16}\mu_2^4(k)\int_{-\infty}^{+\infty}\left\{\left(\frac{f'(x)f''(x)}{f^3(x)} - \frac{f'''(x)}{f^2(x)}\right)\right\}^2 F'(x)^2 dF(x). \end{aligned}$$

Differentiation with respect to h gives us:

$$\frac{d}{dh}\Big[\frac{1}{n}\int_{-\infty}^{+\infty}F(x)(1-F(x))F'(x)^2dF(x)-2\beta(k)\frac{h}{n}\int_{-\infty}^{+\infty}F'(x)^2dF(x)$$
$$+\frac{h^8}{16}\mu_2^4(k)\int_{-\infty}^{+\infty}\left\{\left(\frac{f'(x)f''(x)}{f^3(x)}-\frac{f'''(x)}{f^2(x)}\right)\right\}^2F'(x)^2dF(x)\Big]$$
$$=-\frac{2\beta(k)}{n}\int_{-\infty}^{+\infty}F'(x)^2dF(x)+\frac{h^7}{2}\mu_2^4(k)\int_{-\infty}^{+\infty}\left\{\left(\frac{f'(x)f''(x)}{f^3(x)}-\frac{f'''(x)}{f^2(x)}\right)\right\}^2F'(x)^2dF(x),$$

and setting this equal to 0 we find:

$$-\frac{2\beta(k)}{n}\int_{-\infty}^{+\infty}F'(x)^2dF(x)+\frac{h^7}{2}\mu_2^4(k)\int_{-\infty}^{+\infty}\left\{\left(\frac{f'(x)f''(x)}{f^3(x)}-\frac{f'''(x)}{f^2(x)}\right)\right\}^2F'(x)^2dF(x)=0$$
$$\Rightarrow h_0^*=\left[\frac{1}{8n}\frac{C_2\int_{-\infty}^{+\infty}F'(x)^2dF(x)}{\int_{-\infty}^{+\infty}C_1^2(x)F'(x)^2dF(x)}\right]^{\frac{1}{7}}.$$

This can be written as

$$h_0^*=\left\{\frac{1}{8n}\frac{C_2\int_{-\infty}^{+\infty}f^3(x)dx}{\int_{-\infty}^{+\infty}C_1^2(x)f^3(x)dx}\right\}^{\frac{1}{7}}.$$

Substitution of this value for h in the AMISE of $\tilde{F}_{n,h}$ yields:

$$AMISE(\tilde{F}_{n,h_0^*})=\int_{-\infty}^{+\infty}\{\frac{F(x)(1-F(x))}{n}-C_2\frac{h_0^*}{n}+C_1^2(x)h_0^{*8}\}F'(x)^2dF(x)$$
$$=\int_{-\infty}^{+\infty}\frac{F(x)(1-F(x))}{n}f^3(x)dx-C_2\frac{h_0^*}{n}\int_{-\infty}^{+\infty}f^3(x)dx$$
$$+h_0^{*8}\int_{-\infty}^{+\infty}C_1^2(x)f^3(x)dx$$

$$= \int_{-\infty}^{+\infty} \frac{F(x)(1-F(x))}{n} f^3(x)dx - \frac{C_2}{n}\left\{\frac{1}{8n}\frac{C_2\int_{-\infty}^{+\infty} f^3(x)dx}{\int_{-\infty}^{+\infty} C_1^2(x) f^3(x)dx}\right\}^{\frac{1}{7}} \int_{-\infty}^{+\infty} f^3(x)dx$$

$$+\left\{\frac{1}{8n}\frac{C_2\int_{-\infty}^{+\infty} f^3(x)dx}{\int_{-\infty}^{+\infty} C_1^2(x) f^3(x)dx}\right\}^{\frac{8}{7}} \int_{-\infty}^{+\infty} C_1^2(x) f^3(x)dx$$

$$= \int_{-\infty}^{+\infty} \frac{F(x)(1-F(x))}{n} f^3(x)dx - \frac{\left\{\frac{C_2}{n}\int_{-\infty}^{+\infty} f^3(x)dx\right\}^{\frac{8}{7}}}{\left\{8\int_{-\infty}^{+\infty} C_1^2(x) f^3(x)dx\right\}^{\frac{1}{7}}}$$

$$+\frac{1}{8}\frac{\left\{\frac{C_2}{n}\int_{-\infty}^{+\infty} f^3(x)dx\right\}^{\frac{8}{7}}}{\left\{8\int_{-\infty}^{+\infty} C_1^2(x) f^3(x)dx\right\}^{\frac{1}{7}}}$$

$$= \int_{-\infty}^{+\infty} \frac{F(x)(1-F(x))}{n} f^3(x)dx - \frac{1}{n^{8/7}}\frac{\frac{7}{8}\left\{C_2\int_{-\infty}^{+\infty} f^3(x)dx\right\}^{\frac{8}{7}}}{\left\{8\int_{-\infty}^{+\infty} C_1^2(x) f^3(x)dx\right\}^{\frac{1}{7}}},$$

which can be written in the form:

$$AMISE(\tilde{F}_{n,h_0^*}) = \frac{B_1}{n} - \frac{D}{n^{8/7}}, \text{ with } \mathrm{B}_1 > 0,\ \mathrm{D} > 0.$$

As we have shown in chapter 2, the AMISE of $\hat{F}_{n,h}$ is:

$$AMISE(\hat{F}_{n,h_0}) = \frac{B_1}{n} - \frac{B_2}{n^{4/3}}, \text{ with } \mathrm{B}_1 > 0,\ \mathrm{B}_2 > 0.$$

The AMISE of $\tilde{F}_{n,h}$ is smaller than the AMISE of $\hat{F}_{n,h}$ as $n \to \infty$. It is sufficient to show that $\frac{B_2}{n^{4/3}} < \frac{D}{n^{8/7}}$. The inequality reduces to

$$n^{4/21} > \frac{B_2}{D},$$

which is always true as $n \to \infty$ because D and B_2 is always greater than 0.

3.3 Examples of the Asymptotic Optimal choice for h

We will now illustrate the optimal choice for h with some examples. We will consider examples where k and f are normally distributed as well as where k has a uniform distribution.

Normal Kernel and Normal Density

We will now consider the case where both the kernel and density functions are normally distributed, i.e.

$$\begin{aligned} f(x) &= \frac{1}{\sigma}\phi\left(\frac{x}{\sigma}\right) \\ &= \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} \text{ for } -\infty < x < +\infty, \end{aligned}$$

and

$$k(z) = \phi(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2} \text{ for } -\infty < z < +\infty.$$

In this case,

$$\begin{aligned} \beta(k) &= \int_{-\infty}^{+\infty} zk(z)K(z)dz \\ &= -\int_{-\infty}^{+\infty} k'(z)K(z)dz \\ &= -K(z)k(z)\big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} k(z)^2 dz \\ &= \int_{-\infty}^{+\infty} k(z)^2 dz \\ &= \frac{1}{2}\frac{1}{\sqrt{\pi}}. \end{aligned}$$

Also: $\mu_2(k) = \int_{-\infty}^{+\infty} z^2 k(z)dz = 1$; therefore: $C_2(x) = 2\beta(k) = \frac{1}{\sqrt{\pi}}$, and

$$C_1(x)=\frac{1}{4}\mu_2^{\,2}(k)\left(\frac{f'(x)f''(x)}{f^3(x)}-\frac{f'''(x)}{f^2(x)}\right)$$

$$=\frac{1}{4}\frac{\frac{-x}{\sqrt{2\pi}\sigma^3}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}\left(\frac{x^2}{\sqrt{2\pi}\sigma^5}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}-\frac{1}{\sqrt{2\pi}\sigma^3}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}\right)}{\left(\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}\right)^3}$$

$$-\frac{1}{4}\frac{\left(\frac{3x}{\sqrt{2\pi}\sigma^5}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}-\frac{x^3}{\sqrt{2\pi}\sigma^7}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}\right)}{\left(\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}\right)^2}$$

$$=\frac{1}{4}\frac{\left(\frac{-x^3}{2\pi\sigma^8}e^{-\left(\frac{x}{\sigma}\right)^2}+\frac{x}{2\pi\sigma^6}e^{-\left(\frac{x}{\sigma}\right)^2}\right)}{\left(\frac{1}{(2\pi)^{3/2}\sigma^3}e^{-\frac{3}{2}\left(\frac{x}{\sigma}\right)^2}\right)}$$

$$-\frac{1}{4}\frac{\left(\frac{3x}{\sqrt{2\pi}\sigma^5}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}-\frac{x^3}{\sqrt{2\pi}\sigma^7}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}\right)}{\left(\frac{1}{2\pi\sigma^2}e^{-\left(\frac{x}{\sigma}\right)^2}\right)}$$

$$=\frac{1}{4}\left(\frac{\frac{-2xe^{-2\left(\frac{x}{\sigma}\right)^2}}{(2\pi)^2\sigma^8}}{\left(\frac{1}{(2\pi)^{5/2}\sigma^5}e^{-\frac{5}{2}\left(\frac{x}{\sigma}\right)^2}\right)}\right)$$

$$=-\frac{\sqrt{\pi}}{\sqrt{2}}\frac{xe^{\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}}{\sigma^3}.$$

It now follows that:

$$h_0^*=\left\{\frac{1}{8n}\frac{C_2\int_{-\infty}^{+\infty}f^3(x)dx}{\int_{-\infty}^{+\infty}C_1^{\,2}(x)f^3(x)dx}\right\}^{\frac{1}{7}}$$

$$
=\left\{\frac{1}{8n}\frac{\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\left(\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}\right)^3 dx}{\int_{-\infty}^{+\infty}\left(-\frac{\sqrt{\pi}}{\sqrt{2}}\frac{xe^{\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}}{\sigma^3}\right)^2\left(\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}\right)^3 dx}\right\}^{\frac{1}{7}}
$$

$$
=\left\{\frac{1}{8n}\frac{\frac{1}{\pi^2 2^{3/2}\sigma^3}\int_{-\infty}^{+\infty}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}dx}{\frac{1}{\sqrt{\pi}2^{5/2}\sigma^9}\int_{-\infty}^{+\infty}x^2e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}dx}\right\}^{\frac{1}{7}}
$$

$$
=\left\{\frac{1}{12n\pi^{3/2}}\sqrt{3}\sigma^4\right\}^{\frac{1}{7}}
$$

$$
=0.5934\sigma^{4/7}n^{-1/7}.
$$

An estimate for h_0^* is then $\hat{h}_0^* = 0.5934\hat{\sigma}^{4/7}n^{-1/7}$ where we approximate σ with some estimator $\hat{\sigma}$.

Uniform Kernel and Normal Density

Now let us consider the case where the kernel has a uniform distribution and the density function is normally distributed, i.e.,

$$
\begin{aligned}
f(x) &= \frac{1}{\sigma}\phi\left(\frac{x}{\sigma}\right) \\
&= \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} \quad \text{for } -\infty < x < +\infty,
\end{aligned}
$$

and

$$
k(z) = \frac{1}{2\sqrt{3}} \text{ for } |z| \le \sqrt{3}.
$$

In this case, $\beta(k) = \int_{-\infty}^{+\infty} zk(z)K(z)dz = \frac{1}{2\sqrt{3}}$, and $\mu_2(k) = \int_{-\infty}^{+\infty} z^2k(z)dz = 1$, therefore $C_2(x) = 2\beta(k) = \frac{1}{\sqrt{3}}$ and

$$C_1(x)=\frac{1}{4}\mu_2^2(k)\left(\frac{f'(x)f''(x)}{f^3(x)}-\frac{f'''(x)}{f^2(x)}\right)$$

$$=\frac{1}{4}\frac{\frac{-x}{\sqrt{2\pi}\sigma^3}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}\left(\frac{x^2}{\sqrt{2\pi}\sigma^5}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}-\frac{1}{\sqrt{2\pi}\sigma^3}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}\right)}{\left(\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}\right)^3}$$

$$-\frac{1}{4}\frac{\left(\frac{3x}{\sqrt{2\pi}\sigma^5}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}-\frac{x^3}{\sqrt{2\pi}\sigma^7}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}\right)}{\left(\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}\right)^2}$$

$$=\frac{1}{4}\frac{\left(\frac{-x^3}{2\pi\sigma^8}e^{-\left(\frac{x}{\sigma}\right)^2}+\frac{x}{2\pi\sigma^6}e^{-\left(\frac{x}{\sigma}\right)^2}\right)}{\left(\frac{1}{(2\pi)^{3/2}\sigma^3}e^{-\frac{3}{2}\left(\frac{x}{\sigma}\right)^2}\right)}$$

$$-\frac{1}{4}\frac{\left(\frac{3x}{\sqrt{2\pi}\sigma^5}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}-\frac{x^3}{\sqrt{2\pi}\sigma^7}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}\right)}{\left(\frac{1}{2\pi\sigma^2}e^{-\left(\frac{x}{\sigma}\right)^2}\right)}$$

$$=\frac{1}{4}\left(\frac{\frac{-2xe^{-2\left(\frac{x}{\sigma}\right)^2}}{(2\pi)^2\sigma^8}}{\left(\frac{1}{(2\pi)^{5/2}\sigma^5}e^{-\frac{5}{2}\left(\frac{x}{\sigma}\right)^2}\right)}\right)$$

$$=-\frac{\sqrt{\pi}}{\sqrt{2}}\frac{xe^{\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}}{\sigma^3}.$$

It now follows that:

$$h_0^*=\left\{\frac{1}{8n}\frac{C_2\int_{-\infty}^{+\infty}f^3(x)dx}{\int_{-\infty}^{+\infty}C_1^2(x)f^3(x)dx}\right\}^{\frac{1}{7}}$$

$$
= \left\{ \frac{1}{8n} \frac{\frac{1}{\sqrt{3}} \int_{-\infty}^{+\infty} \left(\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} \right)^3 dx}{\int_{-\infty}^{+\infty} \left(-\frac{\sqrt{\pi}}{\sqrt{2}} \frac{x e^{\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}}{\sigma^3} \right)^2 \left(\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} \right)^3 dx} \right\}^{\frac{1}{7}}
$$

$$
= \left\{ \frac{1}{8n} \frac{\frac{1}{\sqrt{3}(2\pi)^{3/2}\sigma^3} \int_{-\infty}^{+\infty} e^{-\frac{3}{2}\left(\frac{x}{\sigma}\right)^2} dx}{\frac{1}{\sqrt{\pi}\, 2^{5/2}\sigma^9} \int_{-\infty}^{+\infty} x^2 e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} dx} \right\}^{\frac{1}{7}}
$$

$$
= \left\{ \frac{1}{12n\pi} \sigma^4 \right\}^{\frac{1}{7}}
$$

$$
= 0.5954\sigma^{4/7} n^{-1/7}.
$$

An estimate for h_0^* is therefore $\hat{h}_0^* = 0.5954\hat{\sigma}^{4/7} n^{-1/7}$.

3.4 Bootstrap Methodology

An algorithm to construct a bootstrap sample from $\tilde{F}_{n,h}$ is as follows:

1. Generate independent random variables $\tilde{Y}_1^*, \tilde{Y}_2^*, \ldots, \tilde{Y}_n^*$ from the empirical distribution function of $\hat{F}_{n,h}(X_1), \hat{F}_{n,h}(X_2), \ldots, \hat{F}_{n,h}(X_n)$.

2. Independently generate independent random variables $Z_1, Z_2, \ldots, Z_n$ from K (the kernel distribution function).

3. Let $\tilde{X}_i^* = \hat{F}_{n,h}{}^{-1}(\tilde{Y}_i^* + hZ_i)$, be the bootstrap sample from $\tilde{F}_{n,h}$. The rest of the bootstrap follows exactly as previously, with these new $\tilde{X}_i^*$'s used in the process.

Chapter 4

Monte Carlo Simulation

4.1 Introduction

In this chapter we will present the results of the Monte Carlo studies on the coverage probabilities and expected lengths of two-sided confidence intervals and one-sided upper bounds (using the percentile method, see paragraph 1.6) for the mean of the normal, log-normal, contaminated normal and the logistic distributions.

The distributions used are defined as follows:

1) Normal distribution: $N(\mu,\sigma^2)$, with mean μ and variance σ^2.

2) Log-normal distribution with underlying normal distribution $N(\mu,\sigma^2)$. The mean is $e^{(\mu+\sigma^2/2)}$ and the variance is $e^{(2\mu+2\sigma^2)}-e^{(2\mu+\sigma^2)}$.

3) Contaminated normal distribution: $(1-\varepsilon)N(\mu_1,\sigma_1^2)+\varepsilon N(\mu_2,\sigma_2^2)$ where $\varepsilon=0.2$. The mean is μ and the variance is $(1-\varepsilon)\sigma_1^2+\varepsilon\sigma_2^2+\varepsilon(1-\varepsilon)(\mu_1-\mu_2)^2$. We chose $\mu_1=\mu_2=0$ and $\sigma_1^2=1$, $\sigma_2^2=\sigma^2$.

4) Logistic distribution: $Logi(\mu,\sigma)$. Mean is μ and variance is $\dfrac{\pi^2\sigma^2}{3}$.

In all these cases we chose $\mu=0$, and we varied $\sigma=0.5$, 1, 2 and 3. We constructed the confidence intervals and upper bounds on values of the sample size (n) of 20, 40, 60, 80,100 and 150. We used M=2000 (Monte Carlo iterations), B=1000 (bootstrap iterations) and used $1-2\alpha=0.95$ and $1-2\alpha=0.9$. The kernel distribution function we used is uniformly distributed between $-\sqrt{3}$ and $\sqrt{3}$.

4.2 Monte Carlo Simulation Procedure

We will now discuss the methodology used to generate the results of the tables found in Appendix A. The source code of the Fortran program can be found in Appendix B. For every

Monte Carlo iteration, we generated an independent sample of size n from the current distribution. This is done as follows:

1) For the normal distribution, we generated independent random variables $Z_1, Z_2, \ldots, Z_n$ from the standard normal distribution, and then applied the scale and location parameters as follows:

$$X_i = Z_i\sigma + \mu, \text{ where } Z_i \sim N(0,1), \text{ i=1,...,n}.$$

2) For the log-normal distribution, we generated independent random variables e^{X_i}, i=1,...,n where X_i is $N(\mu, \sigma^2)$ distributed.

3) For the contaminated normal distribution, we first generate an uniformly distributed random number between 0 and 1. If the number generated is less than $(1-\varepsilon)$, we generate a random variable from $N(\mu,1)$ as in 1) above, otherwise we generate a random variable from $N(\mu,\sigma^2)$.

4) For the logistic distribution, we generate independent random variables $U_1, U_2, \ldots, U_n$ from the uniform [0,1] distribution, and then set

$$X_i = \mu + \sigma \ln\left(\frac{U_i}{1-U_i}\right), \text{ i=1,...,n}.$$

X_i is then $Logi(\mu,\sigma)$ distributed, for each i=1,...,n.

For the classical bootstrap procedure, we generate a random sample $Y_1^*, Y_2^*, \ldots, Y_n^*$, with replacement from the generated Monte Carlo sample $X_1, \ldots, X_n$. This is then our bootstrap sample.

The next step is to calculate the data dependent bandwidth parameters h for the two smoothed procedures. From paragraphs 2.3 and 3.3, we use the following expressions:

1) For the normal smoothed bootstrap: $\hat{h}_0 = 2.0075\hat{\sigma}n^{-1/3}$.

2) For the new transformed smoothed bootstrap: $\hat{h}_0^* = 0.5954\hat{\sigma}^{4/7}n^{-1/7}$.

$\hat{\sigma}$ is calculated from the Monte Carlo sample as follows:

$$\hat{\sigma} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}\ .$$

For the normal smoothed procedure, we apply smoothing to $Y_1^*, Y_2^*, ..., Y_n^*$ as follows (see paragraph 2.4):

$$X_i^* = Y_i^* + \hat{h}_0 Z_i\,,$$

where $Z_1, Z_2, ..., Z_n$ is a random sample from K (the kernel distribution function). We then have our bootstrap sample $X_1^*, X_2^*, ..., X_n^*$.

For the new transformed smoothed bootstrap procedure, we first need to construct $\hat{F}_{n,h}(X_1), \hat{F}_{n,h}(X_2), ..., \hat{F}_{n,h}(X_n)$, where for i=1,...,n,

$$\hat{F}_{n,h}(X_i) = \frac{1}{n}\sum_{j=1}^{n} K\left(\frac{X_i - X_j}{\hat{h}_0^*}\right),$$

and K is the kernel distribution function. We then order this from small to large to obtain $\hat{F}_{n,h}(X_{(1)}), \hat{F}_{n,h}(X_{(2)}), ..., \hat{F}_{n,h}(X_{(n)})$, where $X_{(1)}, ..., X_{(n)}$ are the order statistics of $X_1, ..., X_n$.

For each bootstrap iteration, we generate a random sample with replacement from $\hat{F}_{n,h}(X_1), \hat{F}_{n,h}(X_2), ..., \hat{F}_{n,h}(X_n)$, and call it $\tilde{Y}_1^*, \tilde{Y}_2^*, ..., \tilde{Y}_n^*$. We independently generate independent random variables $Z_1, Z_2, ..., Z_n$ from K.

The bootstrap sample $\tilde{X}_1^*, \tilde{X}_2^*, ..., \tilde{X}_n^*$ is then

$$\tilde{X}_i^* = \hat{F}_{n,h}^{-1}(\tilde{Y}_i^* + \hat{h}_0^* Z_i)\,.$$

To find the inverse of $\hat{F}_{n,h}(\bullet)$, we need to perform an interpolation on the curve of $\hat{F}_{n,h}(X_{(1)}), \hat{F}_{n,h}(X_{(2)}), ..., \hat{F}_{n,h}(X_{(n)})$ against $X_{(1)}, X_{(2)}, ..., X_{(n)}$. The first method we used was to create a cubic spline with the "not-a-knot" condition (i.e., the third derivative of the curve is continuous at the second and next to last nodes). Further improvement on this might be necessary, as the success of the bootstrap method is very sensitive to the interpolation method. If the sample size n is too small, the method might also fail. This will be apparent from the results in the tables in Appendix A.

The second method we used was to approximate the inverse with a Taylor series expansion. This is done as follows:

$$X_i^* = Y_i^* + \frac{\hat{h}_0^* Z_i}{\hat{f}_{\hat{h}_0^*}(Y_i^*)}\varepsilon,$$

where $Z_1, Z_2, \ldots, Z_n$ is a random sample from K and

$$\hat{f}_{\hat{h}_0^*}(Y_i^*) = \frac{1}{n\hat{h}_0^*}\sum_{j=1}^{n} k\left(\frac{Y_i^* - X_j}{\hat{h}_0^*}\right),$$

the kernel density function estimate. We have chosen $\varepsilon = \frac{1}{5}$ as an error reduction factor in the Taylor series expansion.

The rest of the bootstrap procedure is now the same for all three methods. For each bootstrap sample $X_1^*, X_2^*, \ldots, X_n^*$, we calculate the value of the relevant statistic (in our case, the mean). This is simply $\overline{X}^* = \frac{1}{n}\sum_{i=1}^{n} X_i^*$. We then have a vector of bootstrap replicates $\overline{X}_1^*, \overline{X}_2^*, \ldots, \overline{X}_B^*$. Calculate the order statistics of the bootstrap replicates, say $\overline{X}_{(1)}^*, \overline{X}_{(2)}^*, \ldots, \overline{X}_{(B)}^*$. The two-sided $1-2\alpha$ confidence interval is:

$$[\overline{X}_{([B\alpha])}^*, \overline{X}_{([B(1-\alpha)])}^*].$$

The one-sided $1-2\alpha$ confidence upper bound is:

$$(-\infty, \overline{X}_{([B(1-2\alpha)])}^*].$$

We used indicator variables P_i to calculate the coverage of the intervals. If the population mean lies in the interval, let $P_i = 1$, else $P_i = 0$ for i=1,...,M, where M is the number of Monte Carlo iterations. The estimated coverage is then

$$\hat{P}_M = \frac{1}{M}\sum_{i=1}^{M} P_i .$$

The standard error of the coverage is calculated as

$$\sqrt{\frac{\hat{P}_M(1-\hat{P}_M)}{M}} .$$

The length of the two-sided confidence interval for the i-th Monte Carlo trial is simply:

$$L_i = \bar{X}^{*}_{([B(1-\alpha)])} - \bar{X}^{*}_{([B\alpha])}, \text{ for i=1,...,M}.$$

The estimated average length is then:

$$\bar{L}_M = \frac{1}{M}\sum_{i=1}^{M} L_i ,$$

and the standard error of the average length is:

$$SE_L = \sqrt{\frac{\frac{1}{M}\sum_{i=1}^{M}(L_i - \bar{L}_M)^2}{M}} .$$

4.3 Conclusions

In Appendix A we present the results of the Monte Carlo simulations in tabular format. Estimates of coverage probabilities and expected lengths of the two-sided confidence intervals are displayed in Tables 1-48 (for $1-2\alpha = 0.95$) and Tables 73-120 (for $1-2\alpha = 0.90$). Furthermore, Monte Carlo estimates of the coverage probabilities of the one-sided upper bounds are presented in Tables 49-72 (for $1-2\alpha = 0.95$) and Tables 121-144 (for $1-2\alpha = 0.90$).

In the case of the two-sided interval (when $1-2\alpha = 0.95$) for the normal distribution, we see that the new transformed smoothed method provides better coverage than the normal smoothed procedure, except where the sample sizes and standard deviations are small ($n = 20$ and $\sigma \le 2$, $n = 40$, 60 and $\sigma \le 1$ and $n = 80$, 100 and $\sigma = 0.5$,). It provides better coverage than the classical method in all cases. The same holds in the case of the upper bound.

For the two-sided interval (where $1-2\alpha = 0.90$) for the normal distribution, the new method provides better coverage, except where $n = 20$ and $\sigma \le 2$, $n = 40$ and $\sigma \le 1$ and in the cases where $n = 60$, 80, 100 and $\sigma = 0.5$. The same holds in the case of the upper bound.

In the case of the log-normal distribution, the new transformed smoothed method provided better coverage than the normal smoothed and classical methods for the upper bound and two-sided cases (for $1-2\alpha=0.95$ and $1-2\alpha=0.90$), except where $n=20$ and $\sigma\leq 1$ and where $n=40$ and $\sigma=0.5$.

For the contaminated normal distribution, the new transformed smoothed method again failed for small sample sizes and standard deviations. It performed better than the other two methods except where $n=20$ and where $n=40$ and $\sigma\leq 1$. This holds in both the upper bound and two-sided interval cases, for 95% and 90% prescribed confidence levels.

Comparisons in the case of the logistic distribution reveal that the new transformed smoothed method again outperforms the other two methods, except where $n=20$ and $\sigma\leq 1$ and where $n=40$ and $\sigma=0.5$.

The main conclusion from the Monte Carlo experiments is that for small values of n, the new transformed smoothed method does not perform as well as the normal smoothed method. The converse is true for moderate and large sample sizes. This has been noted previously, and can be attributed to the fact that the former method requires an inverse interpolation, which might not be as accurate for small values of the sample size n. We also found that for large n, the transformed method produced intervals and upper bounds that are in many cases too conservative. This can be circumvented by choosing an ε not fixed, as we have done in the Monte Carlo studies, but rather as a suitable function of the sample size n, say ε_n, such that $\varepsilon_n \to 1$ as $n\to\infty$. Deriving an effective data-based choice of ε_n should also be a challenging future research project.

Appendix A

Two-sided 95% coverage for μ

Table 1

Two-sided Coverage: Normal Distribution, n=20, $1-2\alpha = 0.95$						
σ	Classical	Classical Standard Error	Smooth	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9285	0.0058	0.9800	0.0031	0.9400	0.0053
1	0.9320	0.0056	0.9755	0.0035	0.9560	0.0046
2	0.9335	0.0056	0.9755	0.0035	0.9680	0.0039
3	0.9360	0.0055	0.9805	0.0031	0.9810	0.0031

Table 2

Two-sided Length: Normal Distribution, n=20, $1-2\alpha = 0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.4242	0.0015	0.5366	0.0019	0.4520	0.0018
1	0.8481	0.0032	1.0738	0.0040	0.9574	0.0041
2	1.6934	0.0061	2.1440	0.0077	2.0788	0.0091
3	2.5446	0.0095	3.2220	0.0121	3.3463	0.0158

Table 3

Two-sided Coverage: Normal Distribution, n=40, $1-2\alpha = 0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9330	0.0056	0.9695	0.0038	0.9525	0.0048
1	0.9415	0.0052	0.9750	0.0035	0.9685	0.0039
2	0.9330	0.0056	0.9720	0.0037	0.9790	0.0032
3	0.9400	0.0053	0.9670	0.0040	0.9810	0.0031

Table 4

Two-sided Length: Normal Distribution, n=40, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.3072	0.0008	0.3592	0.0009	0.3305	0.0010
1	0.6106	0.0016	0.7146	0.0018	0.7025	0.0022
2	1.2223	0.0031	1.4307	0.0036	1.5553	0.0055
3	1.8374	0.0050	2.1491	0.0058	2.5335	0.0098

Table 5

Two-sided Coverage: Normal Distribution, n=60, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9380	0.0054	0.9645	0.0041	0.9505	0.0049
1	0.9465	0.0050	0.9765	0.0034	0.9740	0.0036
2	0.9455	0.0051	0.9675	0.0040	0.9825	0.0029
3	0.9475	0.0050	0.9725	0.0037	0.9905	0.0022

Table 6

Two-sided Length: Normal Distribution, n=60, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.2517	0.0005	0.2849	0.0006	0.2734	0.0007
1	0.5049	0.0011	0.5707	0.0012	0.5916	0.0018
2	1.0060	0.0022	1.1372	0.0025	1.3147	0.0045
3	1.5100	0.0033	1.7071	0.0037	2.1426	0.0076

Table 7

Two-sided Coverage: Normal Distribution, n=80, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9435	0.0052	0.9630	0.0042	0.9605	0.0044
1	0.9470	0.0050	0.9660	0.0041	0.9745	0.0035
2	0.9375	0.0054	0.9550	0.0046	0.9795	0.0032
3	0.9460	0.0051	0.9650	0.0041	0.9905	0.0022

Table 8

Two-sided Length: Normal Distribution, n=80, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.2193	0.0004	0.2428	0.0005	0.2389	0.0006
1	0.4372	0.0008	0.4842	0.0009	0.5203	0.0015
2	0.8741	0.0017	0.9693	0.0019	1.1707	0.0038
3	1.3118	0.0024	1.4548	0.0027	1.9176	0.0063

Table 9

Two-sided Coverage: Normal Distribution, n=100, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9430	0.0052	0.9610	0.0043	0.9595	0.0044
1	0.9485	0.0049	0.9660	0.0041	0.9765	0.0034
2	0.9475	0.0050	0.9680	0.0039	0.9920	0.0020
3	0.9445	0.0051	0.9670	0.0040	0.9955	0.0015

Table 10

Two-sided Length: Normal Distribution, n=100, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.1962	0.0003	0.2147	0.0004	0.2162	0.0005
1	0.3915	0.0007	0.4284	0.0007	0.4700	0.0013
2	0.7840	0.0013	0.8572	0.0015	1.0713	0.0035
3	1.1782	0.0019	1.2891	0.0021	1.7554	0.0057

Table 11

Two-sided Coverage: Normal Distribution, n=150, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9525	0.0048	0.9640	0.0042	0.9695	0.0038
1	0.9480	0.0050	0.9600	0.0044	0.9755	0.0035
2	0.9550	0.0046	0.9705	0.0038	0.9870	0.0025
3	0.9515	0.0048	0.9630	0.0042	0.9970	0.0012

Table 12

Two-sided Length: Normal Distribution, n=150, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.1607	0.0002	0.1722	0.0003	0.1793	0.0004
1	0.3211	0.0005	0.3445	0.0005	0.3958	0.0012
2	0.6428	0.0010	0.6890	0.0010	0.9066	0.0031
3	0.9636	0.0014	1.0338	0.0015	1.5038	0.0051

Table 13

Two-sided Coverage: Log-normal Distribution, n=20, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9120	0.0063	0.9600	0.0044	0.9290	0.0057
1	0.8610	0.0077	0.9065	0.0065	0.8965	0.0068
2	0.6305	0.0108	0.6705	0.0105	0.7090	0.0102
3	0.3500	0.0107	0.3850	0.0109	0.4510	0.0111

Table 14

Two-sided Length: Log-normal Distribution, n=20, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.4972	0.0032	0.6316	0.0041	0.5424	0.0039
1	1.5402	0.0182	1.9880	0.0247	1.8914	0.0256
2	14.6833	0.5586	20.3958	0.8494	25.2938	1.1270
3	217.8147	21.5898	331.3334	35.1265	616.4154	74.7553

Table 15

Two-sided Coverage: Log-normal Distribution, n=40, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9345	0.0055	0.9640	0.0042	0.9580	0.0045
1	0.8980	0.0068	0.9325	0.0056	0.9435	0.0052
2	0.7120	0.0101	0.7420	0.0098	0.8275	0.0084
3	0.3935	0.0109	0.4155	0.0110	0.5705	0.0111

Table 16

Two-sided Length: Log-normal Distribution, n=40, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.3637	0.0017	0.4259	0.0020	0.4072	0.0023
1	1.2061	0.0118	1.4269	0.0146	1.6148	0.0192
2	12.9422	0.3497	16.2218	0.4795	26.7313	0.8775
3	207.6734	18.6773	282.8651	26.9831	720.4447	75.1334

Table 17

Two-sided Coverage: Log-normal Distribution, n=60, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9390	0.0054	0.9635	0.0042	0.9610	0.0043
1	0.9160	0.0062	0.9370	0.0054	0.9580	0.0045
2	0.7290	0.0099	0.7525	0.0096	0.8715	0.0075
3	0.4065	0.0110	0.4285	0.0111	0.6435	0.0107

Table 18

Two-sided Length: Log-normal Distribution, n=60, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.2994	0.0011	0.3390	0.0013	0.3431	0.0016
1	1.0179	0.0086	1.1596	0.0101	1.4530	0.0151
2	11.5156	0.3023	13.7556	0.3937	26.7785	0.8213
3	244.3635	29.3957	321.5783	41.3379	1042.9519	141.3462

Table 19

Two-sided Coverage: Log-normal Distribution, n=80, $1-2\alpha = 0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9375	0.0054	0.9610	0.0043	0.9670	0.0040
1	0.9185	0.0061	0.9415	0.0052	0.9705	0.0038
2	0.7425	0.0098	0.7630	0.0095	0.9050	0.0066
3	0.4555	0.0111	0.4715	0.0112	0.7055	0.0102

Table 20

Two-sided Length: Log-normal Distribution, n=80, $1-2\alpha = 0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.2622	0.0009	0.2911	0.0010	0.3079	0.0015
1	0.8825	0.0071	0.9850	0.0083	1.3281	0.0137
2	10.6621	0.2548	12.3791	0.3182	27.1479	0.7501
3	295.3065	64.8079	389.5131	94.3219	1467.9720	378.7585

Table 21

Two-sided Coverage: Log-normal Distribution, n=100, $1-2\alpha = 0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9365	0.0055	0.9600	0.0044	0.9690	0.0039
1	0.9270	0.0058	0.9405	0.0053	0.9820	0.0030
2	0.7705	0.0094	0.7890	0.0091	0.9235	0.0059
3	0.4670	0.0112	0.4845	0.0112	0.7455	0.0097

Table 22

Two-sided Length: Log-normal Distribution, n=100, $1-2\alpha = 0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.2345	0.0007	0.2570	0.0008	0.2807	0.0012
1	0.7914	0.0052	0.8683	0.0058	1.2408	0.0105
2	10.4802	0.3102	12.0380	0.3978	29.7107	1.0585
3	190.4992	16.8263	235.0997	22.5609	989.5258	126.3512

Table 23

Two-sided Coverage: Log-normal Distribution, n=150, $1-2\alpha = 0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9400	0.0053	0.9580	0.0045	0.9770	0.0034
1	0.9190	0.0061	0.9335	0.0056	0.9860	0.0026
2	0.7950	0.0090	0.8120	0.0087	0.9565	0.0046
3	0.5105	0.0112	0.5240	0.0112	0.8655	0.0076

Table 24

Two-sided Length: Log-normal Distribution, n=150, $1-2\alpha = 0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.1927	0.0005	0.2068	0.0005	0.2399	0.0010
1	0.6578	0.0040	0.7070	0.0044	1.1338	0.0095
2	9.0354	0.2363	10.0032	0.2829	29.6126	0.9837
3	208.7134	25.5801	250.4671	33.8734	1284.8367	204.8160

Table 25

Two-sided Coverage: Contaminated Normal Distribution, n=20, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9155	0.0062	0.9630	0.0042	0.9360	0.0055
1	0.9250	0.0059	0.9680	0.0039	0.9430	0.0052
2	0.9280	0.0058	0.9770	0.0034	0.9545	0.0047
3	0.9210	0.0060	0.9800	0.0031	0.9545	0.0047

Table 26

Two-sided Length: Contaminated Normal Distribution, n=20, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.7798	0.0031	0.9866	0.0040	0.8828	0.0041
1	0.8504	0.0032	1.0766	0.0040	0.9568	0.0040
2	1.0651	0.0050	1.3481	0.0064	1.2632	0.0073
3	1.3357	0.0082	1.6920	0.0105	1.6636	0.0124

Table 27

Two-sided Coverage: Contaminated Normal Distribution, n=40, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9305	0.0057	0.9660	0.0041	0.9595	0.0044
1	0.9390	0.0054	0.9750	0.0035	0.9695	0.0038
2	0.9300	0.0057	0.9665	0.0040	0.9655	0.0041
3	0.9340	0.0056	0.9700	0.0038	0.9795	0.0032

Table 28

Two-sided Length: Contaminated Normal Distribution, n=40, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.5637	0.0017	0.6588	0.0019	0.6511	0.0024
1	0.6135	0.0016	0.7174	0.0019	0.7052	0.0023
2	0.7741	0.0026	0.9061	0.0031	0.9707	0.0045
3	0.9741	0.0044	1.1402	0.0052	1.3257	0.0081

Table 29

Two-sided Coverage: Contaminated Normal Distribution, n=60, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9400	0.0053	0.9645	0.0041	0.9715	0.0037
1	0.9370	0.0054	0.9685	0.0039	0.9720	0.0037
2	0.9375	0.0054	0.9640	0.0042	0.9785	0.0032
3	0.9450	0.0051	0.9665	0.0040	0.9890	0.0023

Table 30

Two-sided Length: Contaminated Normal Distribution, n=60, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.4626	0.0011	0.5234	0.0012	0.5476	0.0017
1	0.5027	0.0011	0.5690	0.0012	0.5885	0.0017
2	0.6380	0.0018	0.7208	0.0020	0.8357	0.0036
3	0.8034	0.0030	0.9076	0.0034	1.1728	0.0063

Table 31

Two-sided Coverage: Contaminated Normal Distribution, n=80, $1-2\alpha = 0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9520	0.0048	0.9685	0.0039	0.9735	0.0036
1	0.9520	0.0048	0.9680	0.0039	0.9760	0.0034
2	0.9410	0.0053	0.9575	0.0045	0.9825	0.0029
3	0.9360	0.0055	0.9645	0.0041	0.9915	0.0021

Table 32

Two-sided Length: Contaminated Normal Distribution, n=80, $1-2\alpha = 0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.4025	0.0008	0.4462	0.0009	0.4842	0.0015
1	0.4380	0.0008	0.4857	0.0009	0.5192	0.0015
2	0.5541	0.0014	0.6146	0.0015	0.7529	0.0031
3	0.7044	0.0023	0.7812	0.0025	1.0936	0.0054

Table 33

Two-sided Coverage: Contaminated Normal Distribution, n=100, $1-2\alpha = 0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9410	0.0053	0.9610	0.0043	0.9780	0.0033
1	0.9515	0.0048	0.9680	0.0039	0.9775	0.0033
2	0.9475	0.0050	0.9645	0.0041	0.9840	0.0028
3	0.9415	0.0052	0.9620	0.0043	0.9950	0.0016

Table 34

Two-sided Length: Contaminated Normal Distribution, n=100, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.3614	0.0007	0.3952	0.0008	0.4386	0.0013
1	0.3918	0.0007	0.4287	0.0007	0.4720	0.0014
2	0.4955	0.0011	0.5425	0.0012	0.6942	0.0028
3	0.6268	0.0019	0.6858	0.0021	1.0094	0.0047

Table 35

Two-sided Coverage: Contaminated Normal Distribution, n=150, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9515	0.0048	0.9660	0.0041	0.9840	0.0028
1	0.9445	0.0051	0.9640	0.0042	0.9800	0.0031
2	0.9505	0.0049	0.9620	0.0043	0.9900	0.0022
3	0.9470	0.0050	0.9655	0.0041	0.9985	0.0009

Table 36

Two-sided Length: Contaminated Normal Distribution, n=150, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.2954	0.0005	0.3166	0.0005	0.3689	0.0011
1	0.3207	0.0005	0.3439	0.0005	0.3961	0.0012
2	0.4059	0.0007	0.4354	0.0008	0.6055	0.0022
3	0.5157	0.0013	0.5522	0.0014	0.8988	0.0038

Table 37

Two-sided Coverage: Logistic Distribution, n=20, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9300	0.0057	0.9765	0.0034	0.9525	0.0048
1	0.9355	0.0055	0.9750	0.0035	0.9630	0.0042
2	0.9320	0.0056	0.9790	0.0032	0.9775	0.0033
3	0.9245	0.0059	0.9775	0.0033	0.9840	0.0028

Table 38

Two-sided Length: Logistic Distribution, n=20, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.7688	0.0034	0.9732	0.0043	0.8769	0.0045
1	1.5239	0.0068	1.9292	0.0087	1.8827	0.0102
2	3.0797	0.0141	3.8979	0.0179	4.2526	0.0252
3	4.5930	0.0205	5.8168	0.0260	6.8669	0.0395

Table 39

Two-sided Coverage: Logistic Distribution, n=40, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9405	0.0053	0.9720	0.0037	0.9665	0.0040
1	0.9410	0.0053	0.9765	0.0034	0.9825	0.0029
2	0.9375	0.0054	0.9710	0.0038	0.9890	0.0023
3	0.9425	0.0052	0.9735	0.0036	0.9950	0.0016

Table 40

Two-sided Length: Logistic Distribution, n=40, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.5544	0.0018	0.6479	0.0021	0.6566	0.0028
1	1.1069	0.0036	1.2952	0.0042	1.4464	0.0065
2	2.2161	0.0070	2.5920	0.0082	3.3003	0.0152
3	3.3340	0.0107	3.8966	0.0126	5.4354	0.0262

Table 41

Two-sided Coverage: Logistic Distribution, n=60, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9530	0.0047	0.9755	0.0035	0.9785	0.0032
1	0.9455	0.0051	0.9710	0.0038	0.9920	0.0020
2	0.9415	0.0052	0.9630	0.0042	0.9940	0.0017
3	0.9525	0.0048	0.9730	0.0036	0.9985	0.0009

Table 42

Two-sided Length: Logistic Distribution, n=60, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.4568	0.0012	0.5165	0.0014	0.5578	0.0022
1	0.9134	0.0024	1.0327	0.0027	1.2502	0.0053
2	1.8150	0.0049	2.0493	0.0056	2.8449	0.0125
3	2.7401	0.0071	3.0957	0.0081	4.7762	0.0204

Table 43

Two-sided Coverage: Logistic Distribution, n=80, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9445	0.0051	0.9640	0.0042	0.9740	0.0036
1	0.9450	0.0051	0.9630	0.0042	0.9860	0.0026
2	0.9315	0.0056	0.9575	0.0045	0.9940	0.0017
3	0.9540	0.0047	0.9705	0.0038	0.9975	0.0011

Table 44

Two-sided Length: Logistic Distribution, n=80, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.3980	0.0010	0.4408	0.0011	0.4997	0.0019
1	0.7920	0.0019	0.8786	0.0021	1.1212	0.0044
2	1.5848	0.0037	1.7552	0.0042	2.6075	0.0109
3	2.3784	0.0055	2.6363	0.0060	4.3245	0.0173

Table 45

Two-sided Coverage: Logistic Distribution, n=100, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9390	0.0054	0.9590	0.0044	0.9770	0.0034
1	0.9385	0.0054	0.9610	0.0043	0.9890	0.0023
2	0.9555	0.0046	0.9675	0.0040	0.9970	0.0012
3	0.9430	0.0052	0.9615	0.0043	0.9995	0.0005

Table 46

Two-sided Length: Logistic Distribution, n=100, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.3539	0.0007	0.3875	0.0008	0.4522	0.0016
1	0.7121	0.0015	0.7783	0.0016	1.0361	0.0040
2	1.4193	0.0031	1.5521	0.0034	2.4286	0.0100
3	2.1216	0.0045	2.3219	0.0049	4.0109	0.0161

Table 47

Two-sided Coverage: Logistic Distribution, n=150, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9395	0.0053	0.9580	0.0045	0.9850	0.0027
1	0.9445	0.0051	0.9550	0.0046	0.9940	0.0017
2	0.9485	0.0049	0.9610	0.0043	0.9975	0.0011
3	0.9435	0.0052	0.9575	0.0045	0.9995	0.0005

Table 48

Two-sided Length: Logistic Distribution, n=150, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.2916	0.0005	0.3128	0.0005	0.3914	0.0013
1	0.5821	0.0010	0.6239	0.0011	0.9026	0.0034
2	1.1586	0.0021	1.2418	0.0022	2.1128	0.0080
3	1.7468	0.0031	1.8726	0.0034	3.5862	0.0140

Upper bound 95% coverage for μ

Table 49

Upper Bound Coverage: Normal Distribution, n=20, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9340	0.0056	0.9715	0.0037	0.9420	0.0052
1	0.9440	0.0051	0.9760	0.0034	0.9590	0.0044
2	0.9440	0.0051	0.9785	0.0032	0.9695	0.0038
3	0.9385	0.0054	0.9730	0.0036	0.9730	0.0036

Table 50

Upper Bound Coverage: Normal Distribution, n=40, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9460	0.0051	0.9635	0.0042	0.9550	0.0046
1	0.9450	0.0051	0.9690	0.0039	0.9610	0.0043
2	0.9475	0.0050	0.9660	0.0041	0.9735	0.0036
3	0.9485	0.0049	0.9700	0.0038	0.9840	0.0028

Table 51

Upper Bound Coverage: Normal Distribution, n=60, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9400	0.0053	0.9600	0.0044	0.9505	0.0049
1	0.9535	0.0047	0.9705	0.0038	0.9715	0.0037
2	0.9475	0.0050	0.9665	0.0040	0.9750	0.0035
3	0.9440	0.0051	0.9635	0.0042	0.9855	0.0027

Table 52

Upper Bound Coverage: Normal Distribution, n=80, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9475	0.0050	0.9620	0.0043	0.9575	0.0045
1	0.9450	0.0051	0.9635	0.0042	0.9690	0.0039
2	0.9535	0.0047	0.9655	0.0041	0.9835	0.0028
3	0.9500	0.0049	0.9660	0.0041	0.9920	0.0020

Table 53

Upper Bound Coverage: Normal Distribution, n=100, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9365	0.0055	0.9565	0.0046	0.9570	0.0045
1	0.9470	0.0050	0.9620	0.0043	0.9715	0.0037
2	0.9515	0.0048	0.9650	0.0041	0.9855	0.0027
3	0.9510	0.0048	0.9610	0.0043	0.9880	0.0024

Table 54

Upper Bound Coverage: Normal Distribution, n=150, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9505	0.0049	0.9635	0.0042	0.9675	0.0040
1	0.9530	0.0047	0.9610	0.0043	0.9765	0.0034
2	0.9520	0.0048	0.9670	0.0040	0.9850	0.0027
3	0.9485	0.0049	0.9675	0.0040	0.9940	0.0017

Table 55

Upper Bound Coverage: Log-normal Distribution, n=20, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9015	0.0067	0.9415	0.0052	0.9170	0.0062
1	0.8290	0.0084	0.8750	0.0074	0.8645	0.0077
2	0.5925	0.0110	0.6310	0.0108	0.6660	0.0105
3	0.3270	0.0105	0.3505	0.0107	0.4215	0.0110

Table 56

Upper Bound Coverage: Log-normal Distribution, n=40, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9175	0.0062	0.9480	0.0050	0.9330	0.0056
1	0.8785	0.0073	0.9015	0.0067	0.9175	0.0062
2	0.6705	0.0105	0.6965	0.0103	0.7885	0.0091
3	0.3705	0.0108	0.3825	0.0109	0.5225	0.0112

Table 57

Upper Bound Coverage: Log-normal Distribution, n=60, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9245	0.0059	0.9445	0.0051	0.9450	0.0051
1	0.8910	0.0070	0.9100	0.0064	0.9370	0.0054
2	0.6805	0.0104	0.7075	0.0102	0.8365	0.0083
3	0.3725	0.0108	0.3850	0.0109	0.6005	0.0110

Table 58

Upper Bound Coverage: Log-normal Distribution, n=80, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9270	0.0058	0.9435	0.0052	0.9490	0.0049
1	0.8880	0.0071	0.9100	0.0064	0.9500	0.0049
2	0.6945	0.0103	0.7155	0.0101	0.8685	0.0076
3	0.4170	0.0110	0.4300	0.0111	0.6540	0.0106

Table 59

Upper Bound Coverage: Log-normal Distribution, n=100, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9355	0.0055	0.9505	0.0049	0.9630	0.0042
1	0.9015	0.0067	0.9150	0.0062	0.9630	0.0042
2	0.7165	0.0101	0.7370	0.0098	0.8930	0.0069
3	0.4315	0.0111	0.4430	0.0111	0.7090	0.0102

Table 60

Upper Bound Coverage: Log-normal Distribution, n=150, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9270	0.0058	0.9430	0.0052	0.9595	0.0044
1	0.8950	0.0069	0.9065	0.0065	0.9700	0.0038
2	0.7440	0.0098	0.7595	0.0096	0.9365	0.0055
3	0.4700	0.0112	0.4780	0.0112	0.8240	0.0085

Table 61

Upper Bound Coverage: Contaminated Normal Distribution, n=20, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9330	0.0056	0.9670	0.0040	0.9490	0.0049
1	0.9295	0.0057	0.9665	0.0040	0.9470	0.0050
2	0.9350	0.0055	0.9735	0.0036	0.9575	0.0045
3	0.9305	0.0057	0.9745	0.0035	0.9620	0.0043

Table 62

Upper Bound Coverage: Contaminated Normal Distribution, n=40, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9400	0.0053	0.9625	0.0042	0.9585	0.0045
1	0.9430	0.0052	0.9675	0.0040	0.9620	0.0043
2	0.9495	0.0049	0.9685	0.0039	0.9695	0.0038
3	0.9325	0.0056	0.9650	0.0041	0.9730	0.0036

Table 63

Upper Bound Coverage: Contaminated Normal Distribution, n=60, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9420	0.0052	0.9610	0.0043	0.9640	0.0042
1	0.9390	0.0054	0.9610	0.0043	0.9610	0.0043
2	0.9415	0.0052	0.9635	0.0042	0.9750	0.0035
3	0.9550	0.0046	0.9720	0.0037	0.9870	0.0025

Table 64

Upper Bound Coverage: Contaminated Normal Distribution, n=80, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9495	0.0049	0.9705	0.0038	0.9745	0.0035
1	0.9535	0.0047	0.9645	0.0041	0.9725	0.0037
2	0.9445	0.0051	0.9600	0.0044	0.9750	0.0035
3	0.9380	0.0054	0.9585	0.0045	0.9890	0.0023

Table 65

Upper Bound Coverage: Contaminated Normal Distribution, n=100, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9440	0.0051	0.9575	0.0045	0.9715	0.0037
1	0.9485	0.0049	0.9625	0.0042	0.9715	0.0037
2	0.9485	0.0049	0.9650	0.0041	0.9825	0.0029
3	0.9445	0.0051	0.9575	0.0045	0.9880	0.0024

Table 66

Upper Bound Coverage: Contaminated Normal Distribution, n=150, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9590	0.0044	0.9655	0.0041	0.9755	0.0035
1	0.9440	0.0051	0.9510	0.0048	0.9705	0.0038
2	0.9505	0.0049	0.9590	0.0044	0.9895	0.0023
3	0.9455	0.0051	0.9615	0.0043	0.9955	0.0015

Table 67

Upper Bound Coverage: Logistic Distribution, n=20, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9285	0.0058	0.9735	0.0036	0.9520	0.0048
1	0.9360	0.0055	0.9745	0.0035	0.9640	0.0042
2	0.9405	0.0053	0.9735	0.0036	0.9735	0.0036
3	0.9335	0.0056	0.9695	0.0038	0.9810	0.0031

Table 68

Upper Bound Coverage: Logistic Distribution, n=40, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9460	0.0051	0.9710	0.0038	0.9655	0.0041
1	0.9430	0.0052	0.9680	0.0039	0.9715	0.0037
2	0.9415	0.0052	0.9635	0.0042	0.9860	0.0026
3	0.9460	0.0051	0.9645	0.0041	0.9890	0.0023

Table 69

Upper Bound Coverage: Logistic Distribution, n=60, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9555	0.0046	0.9715	0.0037	0.9775	0.0033
1	0.9385	0.0054	0.9625	0.0042	0.9845	0.0028
2	0.9495	0.0049	0.9690	0.0039	0.9915	0.0021
3	0.9510	0.0048	0.9730	0.0036	0.9965	0.0013

Table 70

Upper Bound Coverage: Logistic Distribution, n=80, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9465	0.0050	0.9615	0.0043	0.9710	0.0038
1	0.9570	0.0045	0.9710	0.0038	0.9845	0.0028
2	0.9420	0.0052	0.9600	0.0044	0.9900	0.0022
3	0.9590	0.0044	0.9705	0.0038	0.9955	0.0015

Table 71

Upper Bound Coverage: Logistic Distribution, n=100, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9455	0.0051	0.9575	0.0045	0.9705	0.0038
1	0.9450	0.0051	0.9605	0.0044	0.9825	0.0029
2	0.9505	0.0049	0.9665	0.0040	0.9930	0.0019
3	0.9510	0.0048	0.9605	0.0044	0.9970	0.0012

Table 72

Upper Bound Coverage: Logistic Distribution, n=150, $1-2\alpha=0.95$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9505	0.0049	0.9590	0.0044	0.9800	0.0031
1	0.9470	0.0050	0.9595	0.0044	0.9885	0.0024
2	0.9485	0.0049	0.9575	0.0045	0.9945	0.0017
3	0.9510	0.0048	0.9590	0.0044	0.9975	0.0011

Two-sided 90% coverage for μ

Table 73

Two-sided Coverage: Normal Distribution, n=20, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8695	0.0075	0.9350	0.0055	0.8895	0.0070
1	0.8660	0.0076	0.9425	0.0052	0.9030	0.0066
2	0.8835	0.0072	0.9445	0.0051	0.9345	0.0055
3	0.8745	0.0074	0.9455	0.0051	0.9480	0.0050

Table 74

Two-sided Length: Normal Distribution, n=20, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.3558	0.0013	0.4506	0.0017	0.3780	0.0015
1	0.7124	0.0027	0.9019	0.0034	0.7993	0.0035
2	1.4224	0.0054	1.7990	0.0068	1.7396	0.0080
3	2.1326	0.0077	2.6968	0.0098	2.7775	0.0126

Table 75

Two-sided Coverage: Normal Distribution, n=40, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8905	0.0070	0.9450	0.0051	0.9150	0.0062
1	0.8780	0.0073	0.9345	0.0055	0.9245	0.0059
2	0.8820	0.0072	0.9325	0.0056	0.9470	0.0050
3	0.8875	0.0071	0.9340	0.0056	0.9600	0.0044

Table 76

Two-sided Length: Normal Distribution, n=40, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.2563	0.0007	0.2996	0.0008	0.2748	0.0008
1	0.5128	0.0013	0.5999	0.0015	0.5882	0.0019
2	1.0266	0.0026	1.2011	0.0031	1.2987	0.0046
3	1.5341	0.0040	1.7927	0.0047	2.0853	0.0078

Table 77

Two-sided Coverage: Normal Distribution, n=60, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9080	0.0065	0.9395	0.0053	0.9245	0.0059
1	0.8925	0.0069	0.9350	0.0055	0.9380	0.0054
2	0.9040	0.0066	0.9410	0.0053	0.9655	0.0041
3	0.8950	0.0069	0.9355	0.0055	0.9745	0.0035

Table 78

Two-sided Length: Normal Distribution, n=60, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.2104	0.0004	0.2376	0.0005	0.2277	0.0006
1	0.4198	0.0009	0.4747	0.0010	0.4896	0.0015
2	0.8471	0.0018	0.9576	0.0020	1.0947	0.0035
3	1.2632	0.0029	1.4283	0.0032	1.7770	0.0064

Table 79

Two-sided Coverage: Normal Distribution, n=80, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8880	0.0071	0.9190	0.0061	0.9095	0.0064
1	0.9065	0.0065	0.9345	0.0055	0.9505	0.0049
2	0.8955	0.0068	0.9280	0.0058	0.9605	0.0044
3	0.8860	0.0071	0.9225	0.0060	0.9745	0.0035

Table 80

Two-sided Length: Normal Distribution, n=80, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.1834	0.0004	0.2034	0.0004	0.2002	0.0005
1	0.3663	0.0007	0.4062	0.0008	0.4320	0.0012
2	0.7347	0.0014	0.8145	0.0016	0.9729	0.0032
3	1.0978	0.0021	1.2169	0.0023	1.5856	0.0052

Table 81

Two-sided Coverage: Normal Distribution, n=100, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8865	0.0071	0.9190	0.0061	0.9145	0.0063
1	0.8975	0.0068	0.9310	0.0057	0.9475	0.0050
2	0.8985	0.0068	0.9225	0.0060	0.9620	0.0043
3	0.8980	0.0068	0.9230	0.0060	0.9765	0.0034

Table 82

Two-sided Length: Normal Distribution, n=100, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.1644	0.0003	0.1797	0.0003	0.1808	0.0004
1	0.3281	0.0006	0.3591	0.0006	0.3926	0.0011
2	0.6568	0.0011	0.7184	0.0012	0.8887	0.0028
3	0.9845	0.0017	1.0759	0.0019	1.4470	0.0048

Table 83

Two-sided Coverage: Normal Distribution, n=150, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8995	0.0067	0.9210	0.0060	0.9295	0.0057
1	0.8980	0.0068	0.9170	0.0062	0.9485	0.0049
2	0.8920	0.0069	0.9160	0.0062	0.9705	0.0038
3	0.8925	0.0069	0.9160	0.0062	0.9830	0.0029

Table 84

Two-sided Length: Normal Distribution, n=150, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.1345	0.0002	0.1443	0.0002	0.1504	0.0004
1	0.2690	0.0004	0.2886	0.0004	0.3294	0.0009
2	0.5369	0.0008	0.5756	0.0008	0.7487	0.0024
3	0.8060	0.0011	0.8646	0.0012	1.2448	0.0042

Table 85

Two-sided Coverage: Log-normal Distribution, n=20, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8520	0.0079	0.9230	0.0060	0.8755	0.0074
1	0.7860	0.0092	0.8540	0.0079	0.8350	0.0083
2	0.5690	0.0111	0.6160	0.0109	0.6565	0.0106
3	0.3080	0.0103	0.3365	0.0106	0.4155	0.0110

Table 86

Two-sided Length: Log-normal Distribution, n=20, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.4160	0.0027	0.5275	0.0034	0.4518	0.0032
1	1.3198	0.0182	1.6839	0.0237	1.6021	0.0243
2	13.2121	0.5119	17.1528	0.6775	21.0750	0.9086
3	207.5252	22.6327	275.5011	30.6208	511.8604	71.6855

Table 87

Two-sided Coverage: Log-normal Distribution, n=40, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8835	0.0072	0.9280	0.0058	0.9080	0.0065
1	0.8520	0.0079	0.8880	0.0071	0.9070	0.0065
2	0.6230	0.0108	0.6555	0.0106	0.7550	0.0096
3	0.3670	0.0108	0.3830	0.0109	0.5320	0.0112

Table 88

Two-sided Length: Log-normal Distribution, n=40, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.3034	0.0014	0.3551	0.0016	0.3394	0.0018
1	0.9949	0.0102	1.1706	0.0122	1.3210	0.0157
2	12.4854	0.8075	15.0257	1.0143	24.6584	1.6947
3	216.7975	23.3330	266.0590	29.1457	668.7862	80.9417

Table 89

Two-sided Coverage: Log-normal Distribution, n=60, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8835	0.0072	0.9220	0.0060	0.9175	0.0062
1	0.8480	0.0080	0.8825	0.0072	0.9145	0.0063
2	0.6865	0.0104	0.7125	0.0101	0.8415	0.0082
3	0.4075	0.0110	0.4210	0.0110	0.6250	0.0108

Table 90

Two-sided Length: Log-normal Distribution, n=60, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.2511	0.0010	0.2839	0.0011	0.2870	0.0014
1	0.8441	0.0077	0.9582	0.0088	1.1974	0.0129
2	10.1558	0.2832	11.7158	0.3355	22.5335	0.6991
3	221.0598	31.6486	260.8043	37.5633	830.3741	133.4588

Table 91

Two-sided Coverage: Log-normal Distribution, n=80, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8840	0.0072	0.9160	0.0062	0.9240	0.0059
1	0.8525	0.0079	0.8890	0.0070	0.9380	0.0054
2	0.6920	0.0103	0.7145	0.0101	0.8765	0.0074
3	0.4280	0.0111	0.4410	0.0111	0.6645	0.0106

Table 92

Two-sided Length: Log-normal Distribution, n=80, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.2185	0.0007	0.2422	0.0008	0.2554	0.0012
1	0.7387	0.0057	0.8218	0.0064	1.1033	0.0106
2	9.2499	0.3007	10.4365	0.3511	22.9574	0.8412
3	266.8684	77.0159	310.9986	92.0718	1244.1809	480.1048

Table 93

Two-sided Coverage: Log-normal Distribution, n=100, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8880	0.0071	0.9190	0.0061	0.9315	0.0056
1	0.8625	0.0077	0.8930	0.0069	0.9535	0.0047
2	0.7030	0.0102	0.7300	0.0099	0.9005	0.0067
3	0.4330	0.0111	0.4470	0.0111	0.7190	0.0101

Table 94

Two-sided Length: Log-normal Distribution, n=100, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.1969	0.0006	0.2154	0.0007	0.2343	0.0011
1	0.6699	0.0052	0.7340	0.0057	1.0438	0.0101
2	9.0583	0.3065	10.0899	0.3532	24.1391	0.8763
3	196.8239	32.2191	221.4939	36.0551	877.2134	157.8483

Table 95

Two-sided Coverage: Log-normal Distribution, n=150, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8940	0.0069	0.9160	0.0062	0.9430	0.0052
1	0.8725	0.0075	0.8965	0.0068	0.9675	0.0040
2	0.7310	0.0099	0.7485	0.0097	0.9390	0.0054
3	0.4580	0.0111	0.4705	0.0112	0.8105	0.0088

Table 96

Two-sided Length: Log-normal Distribution, n=150, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.1619	0.0004	0.1737	0.0004	0.2003	0.0008
1	0.5544	0.0036	0.5957	0.0039	0.9484	0.0083
2	7.5257	0.1802	8.1540	0.1996	23.3445	0.5971
3	153.1990	8.5383	168.5520	9.5330	816.4093	52.3134

Table 97

Two-sided Coverage: Contaminated Normal Distribution, n=20, $1-2\alpha = 0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8610	0.0077	0.9430	0.0052	0.8940	0.0069
1	0.8780	0.0073	0.9455	0.0051	0.9120	0.0063
2	0.8710	0.0075	0.9455	0.0051	0.9135	0.0063
3	0.8605	0.0077	0.9410	0.0053	0.9135	0.0063

Table 98

Two-sided Length: Contaminated Normal Distribution, n=20, $1-2\alpha = 0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.6489	0.0026	0.8223	0.0033	0.7312	0.0034
1	0.7113	0.0027	0.8993	0.0034	0.7972	0.0034
2	0.9000	0.0043	1.1391	0.0054	1.0627	0.0061
3	1.1164	0.0069	1.4142	0.0088	1.3785	0.0103

Table 99

Two-sided Coverage: Contaminated Normal Distribution, n=40, $1-2\alpha = 0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8950	0.0069	0.9385	0.0054	0.9320	0.0056
1	0.8810	0.0072	0.9360	0.0055	0.9250	0.0059
2	0.8760	0.0074	0.9275	0.0058	0.9295	0.0057
3	0.8885	0.0070	0.9330	0.0056	0.9505	0.0049

Table 100

Two-sided Length: Contaminated Normal Distribution, n=40, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.4711	0.0013	0.5510	0.0016	0.5435	0.0019
1	0.5124	0.0014	0.5992	0.0016	0.5862	0.0019
2	0.6452	0.0022	0.7552	0.0026	0.7998	0.0038
3	0.8177	0.0037	0.9575	0.0043	1.1070	0.0067

Table 101

Two-sided Coverage: Contaminated Normal Distribution, n=60, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8900	0.0070	0.9290	0.0057	0.9310	0.0057
1	0.8790	0.0073	0.9200	0.0061	0.9290	0.0057
2	0.8930	0.0069	0.9325	0.0056	0.9555	0.0046
3	0.8840	0.0072	0.9270	0.0058	0.9640	0.0042

Table 102

Two-sided Length: Contaminated Normal Distribution, n=60, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.3898	0.0009	0.4405	0.0010	0.4597	0.0015
1	0.4211	0.0009	0.4765	0.0011	0.4904	0.0015
2	0.5305	0.0015	0.5999	0.0017	0.6892	0.0029
3	0.6753	0.0026	0.7632	0.0029	0.9756	0.0052

Table 103

Two-sided Coverage: Contaminated Normal Distribution, n=80, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8905	0.0070	0.9205	0.0060	0.9350	0.0055
1	0.8910	0.0070	0.9195	0.0061	0.9370	0.0054
2	0.8910	0.0070	0.9265	0.0058	0.9615	0.0043
3	0.8870	0.0071	0.9260	0.0059	0.9800	0.0031

Table 104

Two-sided Length: Contaminated Normal Distribution, n=80, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.3379	0.0007	0.3744	0.0008	0.4032	0.0012
1	0.3657	0.0007	0.4056	0.0008	0.4319	0.0012
2	0.4634	0.0012	0.5142	0.0013	0.6247	0.0026
3	0.5847	0.0019	0.6484	0.0021	0.8928	0.0044

Table 105

Two-sided Coverage: Contaminated Normal Distribution, n=100, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9015	0.0067	0.9280	0.0058	0.9450	0.0051
1	0.8895	0.0070	0.9225	0.0060	0.9450	0.0051
2	0.9000	0.0067	0.9280	0.0058	0.9710	0.0038
3	0.8920	0.0069	0.9160	0.0062	0.9810	0.0031

Table 106

Two-sided Length: Contaminated Normal Distribution, n=100, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.3030	0.0006	0.3312	0.0006	0.3666	0.0011
1	0.3288	0.0006	0.3598	0.0006	0.3917	0.0011
2	0.4147	0.0009	0.4541	0.0010	0.5766	0.0023
3	0.5245	0.0016	0.5740	0.0017	0.8400	0.0040

Table 107

Two-sided Coverage: Contaminated Normal Distribution, n=150, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8980	0.0068	0.9225	0.0060	0.9535	0.0047
1	0.8905	0.0070	0.9125	0.0063	0.9405	0.0053
2	0.9065	0.0065	0.9245	0.0059	0.9770	0.0034
3	0.8960	0.0068	0.9175	0.0062	0.9915	0.0021

Table 108

Two-sided Length: Contaminated Normal Distribution, n=150, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.2478	0.0004	0.2657	0.0004	0.3091	0.0009
1	0.2682	0.0004	0.2876	0.0004	0.3300	0.0009
2	0.3398	0.0006	0.3645	0.0007	0.5015	0.0018
3	0.4321	0.0010	0.4639	0.0011	0.7499	0.0031

Table 109

Two-sided Coverage: Logistic Distribution, n=20, $1-2\alpha = 0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8880	0.0071	0.9460	0.0051	0.9150	0.0062
1	0.8730	0.0074	0.9450	0.0051	0.9250	0.0059
2	0.8695	0.0075	0.9475	0.0050	0.9535	0.0047
3	0.8720	0.0075	0.9395	0.0053	0.9570	0.0045

Table 110

Two-sided Length: Logistic Distribution, n=20, $1-2\alpha = 0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.6414	0.0029	0.8112	0.0036	0.7258	0.0038
1	1.2879	0.0058	1.6300	0.0073	1.5817	0.0086
2	2.5736	0.0116	3.2543	0.0147	3.5223	0.0203
3	3.8465	0.0172	4.8615	0.0216	5.6906	0.0329

Table 111

Two-sided Coverage: Logistic Distribution, n=40, $1-2\alpha = 0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8870	0.0071	0.9330	0.0056	0.9300	0.0057
1	0.8810	0.0072	0.9285	0.0058	0.9450	0.0051
2	0.8785	0.0073	0.9330	0.0056	0.9720	0.0037
3	0.8915	0.0070	0.9340	0.0056	0.9805	0.0031

Table 112

Two-sided Length: Logistic Distribution, n=40, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.4614	0.0015	0.5396	0.0017	0.5419	0.0022
1	0.9226	0.0030	1.0785	0.0035	1.1971	0.0055
2	1.8511	0.0060	2.1649	0.0071	2.7418	0.0131
3	2.7737	0.0091	3.2457	0.0107	4.4905	0.0220

Table 113

Two-sided Coverage: Logistic Distribution, n=60, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9040	0.0066	0.9335	0.0056	0.9470	0.0050
1	0.8950	0.0069	0.9275	0.0058	0.9605	0.0044
2	0.8920	0.0069	0.9270	0.0058	0.9790	0.0032
3	0.9015	0.0067	0.9285	0.0058	0.9885	0.0024

Table 114

Two-sided Length: Logistic Distribution, n=60, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.3819	0.0010	0.4316	0.0011	0.4623	0.0017
1	0.7624	0.0020	0.8621	0.0023	1.0272	0.0041
2	1.5233	0.0041	1.7213	0.0046	2.3818	0.0103
3	2.2858	0.0061	2.5828	0.0069	3.9158	0.0170

Table 115

Two-sided Coverage: Logistic Distribution, n=80, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8930	0.0069	0.9240	0.0059	0.9440	0.0051
1	0.8960	0.0068	0.9310	0.0057	0.9730	0.0036
2	0.8835	0.0072	0.9145	0.0063	0.9820	0.0030
3	0.8825	0.0072	0.9185	0.0061	0.9935	0.0018

Table 116

Two-sided Length: Logistic Distribution, n=80, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.3303	0.0007	0.3662	0.0008	0.4107	0.0015
1	0.6618	0.0015	0.7348	0.0017	0.9287	0.0036
2	1.3234	0.0031	1.4682	0.0034	2.1523	0.0089
3	1.9891	0.0046	2.2054	0.0051	3.6007	0.0148

Table 117

Two-sided Coverage: Logistic Distribution, n=100, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8935	0.0069	0.9190	0.0061	0.9550	0.0046
1	0.8930	0.0069	0.9150	0.0062	0.9775	0.0033
2	0.8920	0.0069	0.9200	0.0061	0.9870	0.0025
3	0.8915	0.0070	0.9255	0.0059	0.9955	0.0015

Table 118

Two-sided Length: Logistic Distribution, n=100, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.2968	0.0006	0.3248	0.0007	0.3771	0.0014
1	0.5947	0.0012	0.6501	0.0014	0.8600	0.0033
2	1.1871	0.0024	1.2987	0.0027	2.0037	0.0077
3	1.7788	0.0037	1.9467	0.0041	3.3406	0.0131

Table 119

Two-sided Coverage: Logistic Distribution, n=150, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8810	0.0072	0.9015	0.0067	0.9545	0.0047
1	0.9000	0.0067	0.9190	0.0061	0.9785	0.0032
2	0.8900	0.0070	0.9135	0.0063	0.9905	0.0022
3	0.8950	0.0069	0.9215	0.0060	0.9985	0.0009

Table 120

Two-sided Length: Logistic Distribution, n=150, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.2445	0.0004	0.2620	0.0005	0.3258	0.0011
1	0.4879	0.0009	0.5226	0.0009	0.7464	0.0027
2	0.9750	0.0017	1.0445	0.0018	1.7740	0.0067
3	1.4641	0.0026	1.5701	0.0028	2.9899	0.0117

Upper bound 90% coverage for μ

Table 121

Upper Bound Coverage: Normal Distribution, n=20, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8710	0.0075	0.9255	0.0059	0.8875	0.0071
1	0.8815	0.0072	0.9305	0.0057	0.9110	0.0064
2	0.8905	0.0070	0.9425	0.0052	0.9285	0.0058
3	0.8885	0.0070	0.9330	0.0056	0.9345	0.0055

Table 122

Upper Bound Coverage: Normal Distribution, n=40, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9025	0.0066	0.9290	0.0057	0.9175	0.0062
1	0.8885	0.0070	0.9180	0.0061	0.9160	0.0062
2	0.9010	0.0067	0.9235	0.0059	0.9330	0.0056
3	0.8940	0.0069	0.9325	0.0056	0.9500	0.0049

Table 123

Upper Bound Coverage: Normal Distribution, n=60, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9005	0.0067	0.9315	0.0056	0.9225	0.0060
1	0.8985	0.0068	0.9230	0.0060	0.9275	0.0058
2	0.9065	0.0065	0.9265	0.0058	0.9485	0.0049
3	0.8955	0.0068	0.9185	0.0061	0.9590	0.0044

Table 124

Upper Bound Coverage: Normal Distribution, n=80, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9045	0.0066	0.9245	0.0059	0.9210	0.0060
1	0.9055	0.0065	0.9250	0.0059	0.9365	0.0055
2	0.9025	0.0066	0.9235	0.0059	0.9485	0.0049
3	0.8840	0.0072	0.9140	0.0063	0.9600	0.0044

Table 125

Upper Bound Coverage: Normal Distribution, n=100, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8995	0.0067	0.9165	0.0062	0.9160	0.0062
1	0.8945	0.0069	0.9140	0.0063	0.9330	0.0056
2	0.9045	0.0066	0.9230	0.0060	0.9555	0.0046
3	0.9070	0.0065	0.9245	0.0059	0.9605	0.0044

Table 126

Upper Bound Coverage: Normal Distribution, n=150, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8955	0.0068	0.9130	0.0063	0.9220	0.0060
1	0.8970	0.0068	0.9150	0.0062	0.9405	0.0053
2	0.9070	0.0065	0.9165	0.0062	0.9570	0.0045
3	0.8995	0.0067	0.9120	0.0063	0.9730	0.0036

Table 127

Upper Bound Coverage: Log-normal Distribution, n=20, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8430	0.0081	0.8895	0.0070	0.8540	0.0079
1	0.7485	0.0097	0.7985	0.0090	0.7805	0.0093
2	0.5080	0.0112	0.5500	0.0111	0.5935	0.0110
3	0.2660	0.0099	0.2935	0.0102	0.3625	0.0107

Table 128

Upper Bound Coverage: Log-normal Distribution, n=40, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8615	0.0077	0.9020	0.0066	0.8820	0.0072
1	0.7970	0.0090	0.8440	0.0081	0.8590	0.0078
2	0.5590	0.0111	0.5930	0.0110	0.6865	0.0104
3	0.3135	0.0104	0.3385	0.0106	0.4715	0.0112

Table 129

Upper Bound Coverage: Log-normal Distribution, n=60, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8755	0.0074	0.8990	0.0067	0.8995	0.0067
1	0.8130	0.0087	0.8380	0.0082	0.8745	0.0074
2	0.6280	0.0108	0.6525	0.0106	0.7770	0.0093
3	0.3580	0.0107	0.3820	0.0109	0.5615	0.0111

Table 130

Upper Bound Coverage: Log-normal Distribution, n=80, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8690	0.0075	0.8940	0.0069	0.9040	0.0066
1	0.8210	0.0086	0.8445	0.0081	0.8950	0.0069
2	0.6270	0.0108	0.6500	0.0107	0.8165	0.0087
3	0.3800	0.0109	0.3960	0.0109	0.6085	0.0109

Table 131

Upper Bound Coverage: Log-normal Distribution, n=100, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8725	0.0075	0.8980	0.0068	0.9095	0.0064
1	0.8290	0.0084	0.8485	0.0080	0.9080	0.0065
2	0.6470	0.0107	0.6660	0.0105	0.8465	0.0081
3	0.3795	0.0109	0.3935	0.0109	0.6605	0.0106

Table 132

Upper Bound Coverage: Log-normal Distribution, n=150, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8850	0.0071	0.8975	0.0068	0.9230	0.0060
1	0.8425	0.0081	0.8575	0.0078	0.9345	0.0055
2	0.6625	0.0106	0.6790	0.0104	0.9025	0.0066
3	0.3945	0.0109	0.4095	0.0110	0.7475	0.0097

Table 133

Upper Bound Coverage: Contaminated Normal Distribution, n=20, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8775	0.0073	0.9235	0.0059	0.8960	0.0068
1	0.8915	0.0070	0.9275	0.0058	0.9100	0.0064
2	0.8870	0.0071	0.9340	0.0056	0.9185	0.0061
3	0.8705	0.0075	0.9280	0.0058	0.9075	0.0065

Table 134

Upper Bound Coverage: Contaminated Normal Distribution, n=40, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9090	0.0064	0.9370	0.0054	0.9315	0.0056
1	0.8870	0.0071	0.9230	0.0060	0.9175	0.0062
2	0.8830	0.0072	0.9170	0.0062	0.9250	0.0059
3	0.9015	0.0067	0.9330	0.0056	0.9455	0.0051

Table 135

Upper Bound Coverage: Contaminated Normal Distribution, n=60, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8975	0.0068	0.9270	0.0058	0.9290	0.0057
1	0.8965	0.0068	0.9205	0.0060	0.9220	0.0060
2	0.9060	0.0065	0.9305	0.0057	0.9450	0.0051
3	0.8950	0.0069	0.9250	0.0059	0.9605	0.0044

Table 136

Upper Bound Coverage: Contaminated Normal Distribution, n=80, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9010	0.0067	0.9255	0.0059	0.9325	0.0056
1	0.8950	0.0069	0.9185	0.0061	0.9295	0.0057
2	0.9040	0.0066	0.9285	0.0058	0.9510	0.0048
3	0.8990	0.0067	0.9220	0.0060	0.9655	0.0041

Table 137

Upper Bound Coverage: Contaminated Normal Distribution, n=100, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9060	0.0065	0.9220	0.0060	0.9380	0.0054
1	0.8805	0.0073	0.9060	0.0065	0.9230	0.0060
2	0.9025	0.0066	0.9170	0.0062	0.9555	0.0046
3	0.8945	0.0069	0.9205	0.0060	0.9740	0.0036

Table 138

Upper Bound Coverage: Contaminated Normal Distribution, n=150, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9075	0.0065	0.9235	0.0059	0.9410	0.0053
1	0.8920	0.0069	0.9070	0.0065	0.9310	0.0057
2	0.9025	0.0066	0.9240	0.0059	0.9665	0.0040
3	0.8965	0.0068	0.9105	0.0064	0.9815	0.0030

Table 139

Upper Bound Coverage: Logistic Distribution, n=20, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8885	0.0070	0.9415	0.0052	0.9145	0.0063
1	0.8935	0.0069	0.9365	0.0055	0.9250	0.0059
2	0.8815	0.0072	0.9295	0.0057	0.9415	0.0052
3	0.8825	0.0072	0.9345	0.0055	0.9540	0.0047

Table 140

Upper Bound Coverage: Logistic Distribution, n=40, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8920	0.0069	0.9250	0.0059	0.9210	0.0060
1	0.8950	0.0069	0.9265	0.0058	0.9380	0.0054
2	0.8845	0.0071	0.9145	0.0063	0.9545	0.0047
3	0.9005	0.0067	0.9375	0.0054	0.9745	0.0035

Table 141

Upper Bound Coverage: Logistic Distribution, n=60, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8885	0.0070	0.9205	0.0060	0.9305	0.0057
1	0.8960	0.0068	0.9260	0.0059	0.9495	0.0049
2	0.8965	0.0068	0.9245	0.0059	0.9670	0.0040
3	0.8950	0.0069	0.9220	0.0060	0.9790	0.0032

Table 142

Upper Bound Coverage: Logistic Distribution, n=80, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9075	0.0065	0.9265	0.0058	0.9430	0.0052
1	0.8945	0.0069	0.9175	0.0062	0.9550	0.0046
2	0.8890	0.0070	0.9115	0.0064	0.9675	0.0040
3	0.8960	0.0068	0.9175	0.0062	0.9865	0.0026

Table 143

Upper Bound Coverage: Logistic Distribution, n=100, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.9015	0.0067	0.9210	0.0060	0.9445	0.0051
1	0.9005	0.0067	0.9210	0.0060	0.9615	0.0043
2	0.8940	0.0069	0.9190	0.0061	0.9765	0.0034
3	0.8985	0.0068	0.9215	0.0060	0.9900	0.0022

Table 144

Upper Bound Coverage: Logistic Distribution, n=150, $1-2\alpha=0.90$						
σ	Classical	Classical Standard Error	Smoothed	Smoothed Standard Error	Transformed Smoothed	Transformed Smoothed Standard Error
0.5	0.8905	0.0070	0.9055	0.0065	0.9400	0.0053
1	0.9040	0.0066	0.9225	0.0060	0.9710	0.0038
2	0.9030	0.0066	0.9125	0.0063	0.9880	0.0024
3	0.8925	0.0069	0.9095	0.0064	0.9915	0.0021

Appendix B

Source Code

```
program montecarlo

real*8 mu,sigma(4),alfa(2),epsilon,c,c_star
integer*4 MC_rep,B_rep,ss(6),stat,distr,B_rep2,count_ss,count_sigma,alfa_count
character(80) filename

data sigma/0.5d0,1.0d0,2.0d0,3.0d0/
data alfa/0.025d0,0.05d0/
data ss/20,40,60,80,100,150/

filename='ouput.txt'
open(1,FILE=filename,status="unknown")
close(1)

!Mean value
mu=0

!monte carlo repetitions
MC_REP=2000

!bootstrap repitions
B_REP=999

B_rep2=99

!Only for contaminated normal distribution
epsilon=0.2d0

!Smooth bootstrap constant
c=2. 2.00748098

!Transformed Smooth bootstrap constant
c_star=0.595401384d0

!coverage
do alfa_count=1,2

        !statistic to do inference on
        !do stat=0,1              !0 for mean, 1 for variance
        stat=0

        !distribution: 0 Normal, 1 log normal, 2 contaminated normal,3 logistic
        do distr=0,3

                !sample size
                do count_ss=1,6
```

```
                        !standard deviation parameter
                        do count_sigma=1,4

                            call
bootstrap(filename,mu,sigma(count_sigma),alfa(alfa_count),MC_rep,B_Rep,B_rep2,ss(count_ss
),stat,distr,epsilon,c,c_star)

                        enddo !sigma

                enddo !sample size

        enddo !distribution

        !enddo !stat

enddo !alfa

close(1)

end program




subroutine
bootstrap(filename,mu,sigma,alfa,MC_rep,B_Rep,B_rep2,ss,stat,distr,epsilon,c,c_star)

use msimsl
implicit none

!initialise variables
character(80) filename

integer*4 b,B_REP,MC_REP,MC,KK,stat,distr,k
integer*4 ss,B_rep2,i
integer*4 IR(ss)

real*8 ds(ss) ,t, epsilon,get_theta,f_hat_matrix(ss)
real*8 boot_sample(ss),alfa
real*8 mu,sigma,tmu,h1,h2,U(ss),Zsm(ss),F_smooth(ss),c,c_star
real*8 s,mean
real*8 p(3), p3(3), pSE(3),pSEupper(3), L(3), Lse(3)
real*8 total(3), total3(3)

real*8, allocatable::replicate_class(:)
real*8, allocatable::replicate_smooth(:)
real*8, allocatable::replicate_newsmooth(:)
real*8, allocatable::replicate2(:)
real*8, allocatable::Length(:,:)
real*8, allocatable::upperbound(:,:)
real*8, allocatable::LK(:,:)
real*8, allocatable::RK(:,:)
```

```
allocate (replicate_class(B_rep))
allocate (replicate_smooth(B_rep))
allocate (replicate_newsmooth(B_rep))
allocate (replicate2(B_rep2))
allocate (Length(MC_REP,3))
allocate (upperbound(MC_REP,3))
allocate (LK(MC_REP,3))
allocate (RK(MC_REP,3))

!open file for output
open(1,FILE=filename,status="old")
write(1,*) ''

!set random seed number
call rnset(0)

!Calculate population parameter according to distribution and set parameters
tmu=get_theta(mu,sigma,epsilon,distr,stat)

!set initial count values (used to calculate coverage)
total=0.0d0
total3=0.0d0

!NR=ss

do mc=1,MC_REP !Monte Carlo repetitions

        KK=ss

        !Data - random sample form distribution
        call generate_sample(ss,mu,sigma,epsilon,distr,ds)

        !calculate sample mean, variance as well as smoothing paramaters
        mean=sum(ds)/ss
        s=(sum((ds-mean)**2.0d0)/(ss-1.0d0))**0.50d0

        !For normal smoothing method
        h1= c * s * (ss**( -1.0d0/3.0d0))

        !For Transformed Smoothing method
        h2= c_star * (s**(4.0d0/7.0d0)) * (ss**( -1.0d0/7.0d0))

        !For First Method of Transformed Smoothing procedure, construct EDF of F hat
        !do i=1,ss

        !       tot=0

        !       do j=1,ss
        !               tot=tot+K_dist((ds(i)-ds(j))/h2)
        !       end do !j
```

```
!       F_hat(i)=(tot/real(ss))

!       if (F_hat(i) .lt. 0.0d0) then

!               F_hat(i)=0.0d0

!       elseif (F_hat(i) .gt. 1.0d0) then

!               F_hat(i)=1.0d0

!       endif

!       IPERM(i)=i

!end do !i

!Sorts the EDF of F hat, and puts it in F.Returns permutation in IPERM
!call DSVRGP (ss, F_hat, F, IPERM)

!initialises X to be the same permuation as F of the sorted data
!X=ds(IPERM)

do b=1,B_rep  !Bootstrap sampling

        call RNUND(ss,KK,IR)  !Generate random permutation
        boot_sample=ds(IR)


        !-----------------------------------------------------------------------
        !classical bootstrap
        !------------------
        F_smooth=boot_sample
        replicate_class(b)=t(ss,F_smooth,stat,0.0d0)


        !-----------------------------------------------------------------------
        !smooth bootstrap
        !---------------
        call dRNUN (ss, U)
        Zsm=-(3.0d0**0.5d0) + (2.0d0*(3.0d0**0.5d0))* U
        F_smooth=boot_sample+h1*Zsm
        !F_smooth=sum(boot_sample)/real(ss)+(boot_sample-
        sum(boot_sample)/real(ss)+h*Zsm)*(1+(h/s)**2.0d0)**(-0.5d0)
        replicate_smooth(b)=t(ss,F_smooth,stat,h1)


        !-----------------------------------------------------------------------
        !First Method for Transformed Smooth bootstrap
        !-------------------

        !X tilde a random sample from edf of F hat
        !X_tilde=F(IR)
```

```
        !Generate probabilities
        !p_tilde=X_tilde +h2*Zsm

        !Set all probabilties greater than 1 to 1 and less than 0 to 0.
        !This is the old method for Transformed Smooth Boostrap
        !do i=1,ss

        !       if (p_tilde(i) .le. 0.0d0) then
                        !add or subtract small amount (dependant on i)
                        !to ensure all p_tilde values are strictly increasing
        !               p_tilde(i)=0.0d0 + i/1000000000000000.0d0
        !       elseif (p_tilde(i) .ge. 1.0d0) then
        !               p_tilde(i)=1.0d0 - i/1000000000000000.0d0
        !       endif

        !end do

        !F_smooth is the inverse of the EDF of F hat in the point p_tilde
        !Cubic Spline interpolation
        !call DCSIEZ (ss, F, X, ss, p_tilde, F_smooth)

        !replicate_newsmooth(b)=t(ss,F_smooth,stat,0.0d0)



        !------------------------------------------------------------------------
        !Second Method for Transformed Smooth bootstrap
        !(replace inverse with Taylor series expansion)
        !------------------

        call f_hat_star(boot_sample,ds,h2,ss,f_hat_matrix)
        F_smooth=boot_sample+h2*Zsm / (5.0d0*f_hat_matrix)
        replicate_newsmooth(b)=t(ss,F_smooth,stat,0.0d0)

end do !Bootstrap repetitions

k=floor((B_Rep)*alfa)


!Sort bootstrap replicates
call dSVRGN (B_Rep, replicate_class, replicate_class)
call dSVRGN (B_Rep, replicate_smooth, replicate_smooth)
call dSVRGN (B_Rep, replicate_newsmooth, replicate_newsmooth)

!Get two sided and upper bounds:
!Classical
RK(mc,1)=replicate_class(B_Rep-k)
LK(mc,1)=replicate_class(k)
upperbound(mc,1)=replicate_class(B_Rep-2.0d0*k)

!Smooth
```

```
        RK(mc,2)=replicate_smooth(B_Rep-k)
        LK(mc,2)=replicate_smooth(k)
        upperbound(mc,2)=replicate_smooth(B_Rep-2.0d0*k)

        !Transformed Smooth
        RK(mc,3)=replicate_newsmooth(B_Rep-k)
        LK(mc,3)=replicate_newsmooth(k)
        upperbound(mc,3)=replicate_newsmooth(B_Rep-2.0d0*k)

        !Adds values to total variables if interval covers parameter value
        do i=1,3

                IF ((tmu<=RK(mc,i)) .AND. (tmu>=LK(mc,i))) total(i) = total(i)+1.0d0
                IF (tmu>upperbound(mc,i)) total3(i)=total3(i)+1
                Length(mc,i)=(RK(mc,i)-LK(mc,i))

        enddo

end do !Monte Carlo

L=0

!Get average length of intervals
L=real(sum(Length,DIM=1))/real(MC_rep)

!Get standard error of average lengths:
Lse=0
do i=1,3
        do mc=1,MC_rep
                Lse(i)=Lse(i)+(Length(mc,i)-L(i))**2.0d0
        enddo

        Lse(i)=(real(Lse(i))/real(MC_rep)/real(MC_rep))**0.5d0

enddo

!Calculate coverage of intervals for different methods:
do i=1,3

        p(i)=real(total(i))/real(MC_rep)
        pSE(i)= (p(i)*(1.0d0-p(i))/MC_rep)**0.5d0
        p3(i)=real(total3(i))/real(MC_rep)
        pSEupper(i)=(p3(i)*(1.0d0-p3(i))/MC_rep)**0.5d0

end do

!Output results to file:
write(1,999)'','mu ', mu
write(1,999)'','sigma ',sigma
write(1,999)'','alfa ',alfa
write(1,888)'','n ',ss
write(1,888)'','stat ',stat
```

```
write(1,888)'','distr ',distr

write(1,666)'','Classical ','Smooth ','Transformed Smooth'
write(1,777)'Coverage two sided ',p(1),p(2),p(3)
write(1,777)'Standard error two sided ',pSE(1),pSE(2),pSE(3)
write(1,777)'Avg length ',L(1),L(2),L(3)
write(1,777)'Standard error avg length ',Lse(1),Lse(2),Lse(3)
write(1,777)'Coverage upper ',1-p3(1),1-p3(2),1-p3(3)
write(1,777)'Standard error upper ',pSEupper(1),pSEupper(2),pSEupper(3)

666 Format(A30,A30,A30,A30)
777 Format(A30,F30.10,F30.10,F30.10)
888 Format(A60,A30,I30)

555 Format(I30,I30)

999 Format(A60,A30,F30.10,I30)

deallocate (replicate_class)
deallocate (replicate_smooth)
deallocate (replicate_newsmooth)
deallocate (replicate2)
deallocate (Length)
deallocate (upperbound)
deallocate (LK)
deallocate (RK)

end subroutine



subroutine f_hat_star(Yi_star,ds,h2,ss,f_hat_matrix)

!kernel density function estimator
implicit none

integer*4 ss,i,j
real*8 ds(ss),Yi_star(ss),h2,f_hat_matrix(ss),k(ss)

do j=1,ss

k=0

do i=1,ss

        if ( ((Yi_star(j)-ds(i))/h2 .ge. -3.0d0**0.5d0) .AND. ((Yi_star(j)-ds(i))/h2 .le.
3.0d0**0.5d0)) then

                !k is the kernel density
                k(i)=1.0d0/(2.0d0*3.0d0**0.5d0)
```

```
else

        k(i)=0.0d0

    endif

enddo !i

f_hat_matrix(j)=sum(k)/(real(ss)*h2)

end do !j

end subroutine



function K_dist(x)

!Kernel distribution function
implicit none

real*8 x,K_dist

!We use k(t)=1/(2*sqrt(3))
if (x .lt. -3.0d0**0.5d0) then

    K_dist=0

elseif (x .gt. 3.0d0**0.5d0) then

    K_dist=1

else

    K_dist=(x+3.0d0**0.5d0)/(2.0d0*(3.0d0**0.5d0))

endif

end function



function get_theta(mu,sigma,epsilon,distr,stat)

!Calculate population parameters
implicit none

real*8 mu,sigma,epsilon,get_theta,pi
integer*4 distr,stat
```

```
!Mean
if (stat .eq. 0)  then

        !Normal
        if (distr .eq. 0) then
                get_theta=mu

        !Log Normal
        elseif (distr .eq. 1) then
                get_theta=dexp(mu+((sigma**2.0d0)/2.0d0))

        !Contaminated Normal
        elseif (distr .eq. 2) then
                get_theta=mu

        !Standard Logistic
        elseif (distr .eq. 3) then
                get_theta=mu

        endif

!Variance
else if (stat .eq. 1)  then

        !Normal
        if (distr .eq. 0) then
                get_theta=sigma*sigma

        !Log Normal
        elseif (distr .eq. 1) then
                get_theta=dexp(2.0d0*mu+2.0d0*sigma**2.0d0)-
dexp(2.0d0*mu+sigma**2.0d0)

        !Contaminated Normal
        elseif (distr .eq. 2) then
                get_theta=(1.0d0-epsilon)+epsilon*sigma**2.0d0

        !Standard Logistic
        elseif (distr .eq. 3) then
                pi = 4.0d0 * datan(1.0d0)
                get_theta=(sigma**2.0d0)*((pi)**2.0d0)/3.0d0

        endif

endif

end function
```

```
subroutine generate_sample(NR,mu,sigma,epsilon,distr,ds)

!generate Monte Carlo samples
implicit none

integer*4 distr,NR     ,i
real*8 mu,sigma,epsilon,ds(NR),Z(NR),W(NR),U(NR)


!Normal
if (distr .eq. 0) then

        call dRNNOR (NR, Z)
        ds=Z*sigma+mu

!Log Normal
elseif (distr .eq. 1) then

        call dRNLNL (NR, mu, sigma, Z)
        ds=Z

!Contaminated Normal
elseif (distr .eq. 2) then

        call dRNUN (NR, U)
        call dRNNOR (NR,W)

        do i=1, NR

                if(U(i)<=1.0d0-epsilon) then
                        Z(i)=mu+W(i)
                elseif (U(i)>1.0d0-epsilon) then
                        Z(i)=mu + sigma*W(i)
                endif

        end do

        ds=Z

!Logistic distribution
elseif (distr .eq. 3) then

        call dRNUN (NR,Z)
        ds=mu+sigma*dlog(Z/(1-Z))

endif

end subroutine
```

```
function t(yn,ysample,stat,h)

!Calculate sample estimates for population parameters
implicit none

integer*4 yn,stat
real*8 ysample(yn),t,avg,h

!mean
if (stat .eq. 0) then

        t=sum(ysample)/yn

elseif (stat .eq. 1) then

!variance
avg=sum(ysample)/yn
t=sum((ysample-avg)**2.0d0)/(yn-1.0d0) -(h**2.0d0)
if (t .lt. 0.0d0) then
      t=0.0000001d0
endif

endif

end function
```

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