

A model for the development and transformation of teachers' mathematical content knowledge

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Abstract

In this paper we offer a theoretical discussion on the mathematical content knowledge development of secondary school mathematics teachers. We discuss relevant studies from literature and use different views and conceptions of school mathematics to propose a model to develop or transform the mathematical content knowledge of secondary school mathematics teachers in such a way that they are able to reconsider their beliefs about the nature of school mathematics. In the model we view mathematics teaching as a kind of problem solving where mathematical content knowledge has to be unpacked and communicated after teachers have reflected on the components of the model, namely different kinds of mathematical knowledge (procedural and conceptual), cognitive processes (problem solving, reasoning, communication, connections and representations) involved in doing mathematics, and working in different problem contexts.

Key words: mathematical content knowledge, pedagogical content knowledge, teacher education, secondary school mathematics, mathematics education

Introduction

Researchers and other role players involved in mathematics teacher education agree that teachers need sound content knowledge in order to be good teachers of mathematics (e.g. Ma, 1999; Mewborn, 2003), because “one cannot teach what one does not know” (Fennema & Franke, 1992, p. 147). There is the underlying idea, supported by many mathematicians, that “whoever knows mathematics, knows how to teach mathematics” (Furinghetti, 2000, p. 44). The latter idea encourages the belief that the teaching of mathematics entails the transfer of knowledge from the teacher (as expert) to the student (as clean slate) through a process of transmission. The implication for teacher training is that teachers should learn more mathematics, that the higher the level of mathematics a teacher knows the better a teacher he or she becomes (Vistro-Yu, 2005). However, research on mathematics education reveals that effective teaching depends more on the quality of teachers' knowledge, i.e. how their knowledge is constructed (Hill & Ball, 2004). While a mathematician knows very high levels of mathematics, the teaching of mathematics implies taking complex subject matter relevant to school mathematics and translating it into representations that learners can understand (Fennema & Franke, 1992). This translation of mathematics into understandable representations is one of the critical aspects that distinguish a mathematics teacher from a career mathematician.

Educational researchers argue that teachers need to develop pedagogical content knowledge (PCK), a special kind of teacher knowledge that links subject matter and pedagogy (e.g. Brodie, 2001; Davis & Simmt, 2006; Park & Oliver, 2008; Shulman, 1986). Traditionally, pre-service

courses for mathematics teachers consisted of separate pure mathematics and pedagogic courses. However, integration of content knowledge of mathematics (MCK) with knowledge of teaching mathematics does not happen naturally. Research indicates that teachers tend to teach the way they have been taught mathematics and not as pedagogical courses have prepared them to do (Furinghetti, 2000; Nieuwoudt, 1998). It seems that the development of PCK is closely related to a change in beliefs about the nature of mathematics. Research on the development of PCK in teacher education programs currently foregrounds teachers' classroom practices (e.g. Ball & Bass, 2000; Brodie, 2001; Davis & Simmt, 2006; Kinach, 2002, Schoenfeld & Kirkpatrick, 2008). Most of the research and material on the development of PCK focuses on mathematics teaching in the intermediate and senior phase (middle school) using innovative teaching strategies or real-life contexts e.g. modelling fractions with physical models or explaining integers using real-life examples.

We are involved in a professional development programme which attempts to improve the mathematical content knowledge (MCK) of under-qualified senior secondary school teachers. This programme is situated in a Faculty of Natural Sciences; therefore we are compelled to approach our research on PCK by focussing on the nature and structure of MCK. These teachers have usually been exposed to higher levels of undergraduate mathematics in their initial training in order to prepare them for teaching in the senior secondary school. Higher levels of mathematics tend to be more formalist in nature and therefore less conceptual and/or transferable to real-life problem contexts (e.g. Furinghetti, 2000; Hiebert & Lefevre, 1986; Kinach, 2002). In a study by Naidoo and Parker (2005) on the beliefs of a group of grade 9 teachers they found that most of these teachers regarded formal mathematics e.g. factorisation of algebraic expressions as "proper" or valid mathematics in contrast to the solving of mathematical problems in a real-life context. Indications are that it will be difficult to change the beliefs of secondary school mathematics teachers and to transform their MCK into appropriate PCK.

In this article we unpack the MCK of teachers by reviewing related literature on the relationship between MCK and PCK and by reflecting on our experiences while working with teachers in a professional development program. Based on our findings we propose a model for the development of the MCK of secondary school mathematics teachers and give examples from senior secondary school mathematics to illustrate the model.

The relationship between MCK and PCK

In an attempt to bridge the gap between knowledge of a subject and pedagogical method, researchers began to differentiate between different kinds of teachers' knowledge. Lee Shulman (1986) made a valuable contribution to this field by distinguishing among three categories of teacher knowledge, namely subject matter content knowledge (SMK), pedagogical content knowledge (PCK) and curricular knowledge (CK). In mathematics education, the subject matter knowledge is mathematical content knowledge (MCK), and PCK is a particular form of MCK which represents the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented and adapted to the diverse interests and abilities of learners, and presented for instruction (Shulman, 1987). Ball and Bass (2000) defined PCK as a special form of knowledge that combines mathematical knowledge with knowledge of learners, learning and pedagogy.

MCK is the first knowledge base for teaching (Fennema & Franke, 1992), as it is the main factor in recognizing and seizing teachable moments in the mathematics classroom (Kahan,

Cooper & Bethea, 2003). According to Piccolo (2008, p. 48) “all [mathematical] pedagogical competencies are based upon having a deep, vast and thorough understanding of mathematical content”. Kahan et al. (2003, p. 231) suggest that “a mathematically strong teacher is flexible enough to ask impromptu questions and to address unexpected statements or conjectures that arise in the classroom”, but give evidence that supports Monk’s (1994, p. 142) claim “that a good grasp of one’s subject area is a necessary but not sufficient condition for effective teaching”. Teachers enter teacher education programs with MCK that can be described as “in-school acquired knowledge” (Fennema & Franke, 1992, p. 160). During these programs, their mathematical competence starts to connect to their primary concern about teaching and learning of school mathematics and they start to develop PCK (Ma, 1999). However, no amount of PCK can prepare a teacher fully for practice. Mathematics teaching can be seen as a particular kind of problem solving (Ball & Bass, 2000) in the sense that a significant proportion of teaching is unpredictable, and some components of teachers’ knowledge evolve through teaching. Teaching in a mathematics classroom is the structure that defines the components of knowledge and beliefs that come into play. Within this context, teachers’ knowledge of content interacts with their knowledge of pedagogy and learners’ cognition, and combines with beliefs to create a unique knowledge system that drives classroom behaviour (Fennema & Franke, 1992).

The transformation of teachers’ mathematical knowledge is not a “one-direction process” from MCK to PCK, but occurs as a result of a “dialectical conversation between prospective teachers’ conceptions of mathematics and appropriate pedagogy” (Kinach, 2002, pp. 54-55). Reflective practices are therefore an important part of the development and activation of PCK. Peterson (1988) argues for cognitional knowledge for classroom teaching and learning. This includes teachers’ self-awareness of their own cognitive processes and being able to reflect on their own thoughts and actions as they attempt to act on their knowledge in their classroom teaching. MCK that is isolated from learners’ cognition of mathematics and from teachers’ meta-cognition does not appear to be valuable in teaching mathematics (Peterson, 1988). Unless a teacher can understand his/her own thinking in mathematics, MCK will not be useful in structuring the classroom so that learners can learn mathematics with understanding.

We acknowledge that the acquisition of PCK is a complex and ongoing process that evolves through teaching practice. Changes in mathematical PCK occur as a result of changes in teachers’ knowledge of mathematical content (MCK), and vice versa. However, in our research context, our point of departure regarding the relationship between MCK and PCK is that MCK is the driving force behind PCK, and that underdeveloped MCK will make it difficult to attain appropriate PCK.

The influence of beliefs on MCK

Teachers’ beliefs about the nature of mathematics are closely related to how they have learned mathematics and how their own mathematical knowledge is structured, as teachers frequently treat their beliefs as knowledge (Thompson, 1992). Teachers’ views of the nature and essence of mathematics influence the way in which the teaching and learning of mathematics take place. Traditionally mathematics was viewed as an invariable (“fixed”) and static body of knowledge consisting of a logical and meaningful network of inter-related truths which consist mainly of facts, rules and algorithms (Ernest, 1989). The assumption was that one can gradually and in neat chunks unfold and discover this body of knowledge, and consequentially, that the mathematics teacher is then able to transfer these chunks of knowledge to the learner. During the latter stages

of the 20th century a shift in focus has occurred from knowing mathematics to doing mathematics. Mathematics is not seen as a finished product, but as a science of systematically regulated patterns, logical order, and a process of coming to know (e.g. Ernest, 1989; Goldin, 2002; Schoenfeld, 1992). Mathematics is regarded as a human endeavour in which people of all ages construct concepts, discover relationships, invent methods, execute algorithms, communicate, and solve problems posed by their own real worlds (Cangelosi, 2003). This definition does not ignore the fact that mathematical knowledge has a hierarchical network and logical structure of inter-related ideas, relations and procedures, but the focus has shifted from the transmission of procedural knowledge to the actions taken to build up this logical integrated structure of mathematics in an individual's mind.

Research on teacher education contains numerous examples of a discrepancy between the aims of teacher education programs and teachers' knowledge and beliefs. Cohen (1990) described "The case of Mrs Oublier" (p. 339), a teacher in the USA who was convinced that she has introduced new and innovative ideas into her teaching practices, but on observation failed to do so. In a South African study on the effect of beliefs on teaching practices, researchers tested participating teachers' beliefs about the nature of mathematics and found that it tended to be innovative and correlated with a constructivist paradigm as encouraged by their courses on pedagogy, but in their classrooms traditional approaches to teaching prevailed (Webb & Webb, 2004). Another study on the identities of teachers in grade 9 classes reported how teachers justified their teaching of formal, pure mathematical knowledge and skills on the basis of the demands of the high-stakes matric examination (Naidoo & Parker, 2005). These teachers experienced cognitive dissonance and superficially complied with official expectations of them.

Various studies reported on attempts to change behaviour through pedagogy or "methods" courses by challenging prospective teachers' beliefs in instructional situations. Kinach (2002) proposed cognitive strategies that can be implemented in method courses to help transform prospective teachers' instructional explanations from an instrumental to a relational view of mathematics. McDiarmid (1990) reported on a field experiment where teacher education students were deliberately brought face-to-face with their assumptions through encounters with negative numbers, third-graders, and an unconventional teacher. Although the students appeared to reconsider their beliefs, such changes appeared to be superficial and short-lived. Indications are that in a relatively short span of time, a transformation can be observed in an individual's understanding of a particular piece of mathematics, but transformations of the body of mathematics will take considerably longer (Davis & Simmt, 2006).

A renewed focus on MCK

Quality of MCK

Recently, Ball, Thames and Phelps (2008) revisited their ideas on PCK and MCK, stating that teaching may require a specialised form of pure content knowledge that is not mixed with knowledge of students or pedagogy. In an editorial of the *Journal of Research in Mathematics Education* Kathleen Heid (2010) asks the question "Where is the Math (in Mathematics Education Research)?" (p. 102). As early as 1986 Shulman postulated that to think properly about content knowledge requires going beyond knowledge of the facts or concepts of a domain to an understanding of the structures of the subject matter. In mathematics education, the

organization of MCK, which implies connections among mathematical ideas such as concepts, procedures and problem-solving processes (Fennema & Franke, 1992), may be more important than high levels of undergraduate mathematics. Hill and Ball (2004) posit that, at least in mathematics, the way in which teachers' knowledge is constructed may matter more than how comprehensive their knowledge base is. In other words, teaching proficiency might not relate so much to performance on standard tests of mathematics achievement as it does to whether a teacher's knowledge is procedural or conceptual, connected to big ideas or isolated into small bits, compressed or conceptually unpacked (Ball, 1990; Ma, 1999). Ball and Wilson (1990) advocate that educational mathematics courses need to give teachers opportunities to unpack their understanding. Ball, Bass and Hill (as quoted by Adler, 2005:4) suggest that unpacking may be one of the essential and distinctive features of "knowing mathematics for teaching". Teachers need to be able to "decompress" the mathematical knowledge that they have learned previously, and which may have become automated (Ball & Bass, 2000). Similar to an example of the difference of assessment in compressed and decompressed settings given by Adler (2005), we compiled the following example for a Grade 12 topic:

Q1: Determine the following: $\lim_{x \rightarrow 1} x^2 = \dots$. (compressed)

Q2: Explain how you would determine $\lim_{x \rightarrow 1} x^2$ using an appropriate sketch. (decompressed)

We conclude that mathematics teachers need to be able to view earlier mathematical content from an advanced perspective; to decompress and unpack their understanding of the mathematical topics they need to teach; to clear their mathematical knowledge of misconceptions and to reflect on their practices.

Structure of MCK

Mathematics educators have found it useful to distinguish between two types of mathematical knowledge, namely procedural knowledge and conceptual knowledge (Hiebert & Lefevre, 1986). Procedural knowledge of mathematics is knowledge of the conventions, rules and procedures that one uses in carrying out routine mathematical tasks, and also the symbolism that is used to represent mathematics (Van de Walle, 2004). Conceptual knowledge is characterized as knowledge where relationships are created by recognizing similar core features in pieces of information that are seemingly different (Cangelosi, 2003). For Hiebert and Lefevre (1986), conceptual knowledge is characterized most clearly as knowledge that is rich in relationships. However, the classification of mathematical content into categories is not such a clear-cut process. Mathematical content that has been developed as conceptual knowledge can become simple procedural knowledge at a later stage (Hiebert & Carpenter, 1992). Another aspect is that the distinction between procedural and conceptual knowledge does not give an adequate impression of the interrelatedness of these two kinds of knowledge. With procedural knowledge one may have an intuitive feel for mathematics, but not be able to solve a real-life problem, or, alternatively, answers may be generated without understanding. Relevant conceptual knowledge assists in identifying the underlying structures and conceptual features of the problem context. Procedural knowledge that is supported by conceptual knowledge, leads to procedures that will be remembered better and used more effectively (Hiebert & Lefevre, 1986).

Procedural knowledge is usually bound to the initial context in which it has been learned. The factoring of an expression consisting of the difference between two squares is initially learned in an algebraic context, for example $x^2 - 1 = (x + 1)(x - 1)$, but a similar expression in a trigonometric context, for example $\sin^2 x - 1$, may not necessarily be identified as the difference between two squares. Conceptual knowledge releases the procedure from the surface context in which it has been learnt and encourages its use in other structurally similar problems. The level of abstractness increases as knowledge becomes freed from a specific problem context (Hiebert & Lefevre, 1986). This leads to reflective level relationships, which requires a process of stepping back and reflecting on the information being connected. These relationships are constructed at a higher, more abstract level than the pieces of information they connect, and are therefore, less tied to specific context. For example, a student may recognize the need to use the double-angle formula to expand $\tan 2x$, but may fail to make the abstraction that in general $f(ax) \neq af(x)$ in a subsequent problem when asked to expand $\tan \frac{1}{2}x$.

Understanding mathematics

There is a distinction between knowing mathematics as isolated packages of knowledge, and understanding mathematics. Understanding can be defined as a measure of the quality and quantity of connections that an idea has with existing ideas, or that new knowledge has with prior knowledge (Fennema & Franke, 1992; Van de Walle, 2004). The more connections that are made in a network of ideas, the better the understanding. In order to have a pedagogically powerful representation for a topic, a teacher should first have a comprehensive understanding of it that is deep, broad and thorough (Ma, 1999). Breadth of understanding is the capacity to connect a topic with topics of similar or less conceptual power. Depth of understanding is the capacity to connect a topic with those of greater conceptual power and to understand how mathematical ideas grow conceptually (Schoenfeld & Kirkpatrick, 2008). Thoroughness is the capacity to connect all topics - it "glues" knowledge of mathematics into a coherent whole (Ma, 1999:124).

Skemp (2005) distinguished between two competing views of mathematical understanding, namely an instrumental and a relational understanding. These two different conceptions of what constitutes mathematical understanding account for sharp differences in instructional practices, goals and emphases in the mathematics classroom. An instrumental understanding gives insight into the "what and how" and refers to the possession of a set of "fixed plans" for performing mathematical tasks. A feature of these plans is that they prescribe a step-by-step procedure to be followed in performing a given task, with each step determining the next. Within its own context, an instrumental understanding is easier to obtain and provides quick and easy ways to get answers to problems. For some teachers, the ability to use procedures and get answers is what they mean by understanding mathematics. However, an instrumental understanding of mathematics is not adequate to prepare teachers to teach mathematics according to reformed curricula (Kinach, 2002). They need to develop a relational understanding of mathematics that also includes the "why" of mathematics. This reflective practice helps a person to know how to carry out an algorithm and to know why it makes sense mathematically. For example, many school textbooks for grade 10 contain lists of expressions that have to be factorised, but the motivation behind these exercises is not communicated clearly to learners; i.e. that one can use a property of zero to solve an equation consisting of two factors are equal to zero. A relational understanding of mathematics is characterized by the possession of conceptual structures that enable the possessor to construct several plans for performing a given task. For example, the x -coordinate of the turning point of a parabola can be determined using the algebraic formula

$x = \frac{-b}{2a}$, or by differentiation or by using the symmetry property of the graph of a parabola. A teacher with an instrumental understanding will enforce one of these methods as being the “correct” method, instead of encouraging students to explore their own strategies.

The need for development and transformation of MCK

Traditional methods of teaching mathematics stress content of mathematical theory rather than the motivations and thoughts that underlie this content. Hence, the subject-matter understanding with which pre-service teachers enter teacher education courses is not the conceptual understanding that they need to develop in their future students. According to Kinach (2002) it is well documented that the procedural mathematical knowledge that pre-service teachers obtain in university mathematics courses is not adequate to teach reform-mathematics curricula. In our own study on teachers’ MCK with teachers in our program (Plotz, 2007) we found that these teachers’ available mathematical content knowledge states are mainly procedural and/or formalist in nature. Results from an investigation into current teacher education practices in South Africa conducted by Adler and Davis (2006) revealed that mathematics courses designed specifically for teachers tend to be dominated by compression or abbreviation of mathematical ideas. We conclude that many practising teachers do not have a well-structured and integrated knowledge base. On the other hand, those teachers who do possess adequate levels of mathematical knowledge might not necessarily have an adequate conceptual knowledge base of school mathematics. We believe that in order to effect change in teachers’ beliefs about the nature of mathematics and their teaching practices, the way that they themselves experience mathematics also needs to change.

If it is implausible to expect learners to understand mathematics simply by being told, why is it any less implausible to expect teachers to learn a “new” mathematics simply by being told? (Cohen, 1990, p. 343).

Development and transformation of MCK

Cognitive processes of mathematics

In order to develop a comprehensive understanding of mathematics one needs to focus on the cognitive processes through which students of mathematics acquire and use mathematical knowledge (Mamona-Downs & Downs, 2002). These processes direct the methods of doing all mathematics and, therefore, should be seen as integral components of all mathematics learning and teaching (Van de Walle, 2004). Problem solving is an integral part of learning and doing mathematics. It is essential to understand that problem solving is much more than finding answers to word problems and exercises. It also entails engaging in a task for which the solution method is not known in advance.

Problem solving involves both deductive and inductive reasoning. Deductive reasoning is used to formulate proofs, whereas concept construction and relationships discovery resulting in conjectures require the use of inductive reasoning (Cangelosi, 2003). For example, when introducing sequences at school level, one could first expose learners to different sequences, allowing them to recognize patterns and make conjectures about possible generalities, before they are subjected to the deduction of the recursive algorithms for sequences. When confronted with a new situation or problem context, one reasons deductively by deciding how, if at all, a

previously learned generality, such as a concept, relationship or algorithm, is relevant to that situation. At the same time, one has to reason inductively to build mental representations of the problem and to connect the representations to appropriate procedures (Hiebert & Lefevre, 1986). Connections, within and among mathematical concepts and ideas, lead to greater flexibility in problem solving (Dossey, McCrone, Giordano & Weir, 2002). Students, who have difficulty translating a concept from one representation to another, will have difficulty solving problems in different contexts (Lesh, Post & Behr, 1987).

Representations can become the vehicle through which connections are made (Goldin, 2002), while representations without appropriate connections will limit a teacher's use of representations. A student's ability to develop and interpret various representations increases the ability to do and understand mathematics. When students gain access to mathematical representations and the ideas they represent, they have a set of tools that significantly expands their capacity to think mathematically. Representations extend a person's understanding of a concept, and shed light on an idea not fully understood in another form (Dossey et al., 2002). Some students develop a stronger understanding when they see drawings or graphs; others prefer algebraic or symbolic representations. For example, limit laws can be used to determine the value of $\lim_{x \rightarrow 2} \frac{x^2 - 2x}{x - 2}$, but a better understanding of the limit concept can be obtained by observing a graphical or table representation of the function. In order to facilitate constructivist learning of concepts, a teacher must have knowledge of different representations of a concept which will allow the teacher to choose appropriate representations in the practice of teaching.

A schematic model for the development of the MCK of teachers

In figure 1 we describe a schematic model for the development or transformation of the MCK of teachers in the senior secondary school. The model provides a base for the restructuring of views on MCK in order to assist teachers and teacher educators to develop their new or change their current and available mathematical content knowledge states into a knowledge structure that is more conducive to teaching.

The driving force behind the model is reflective mathematics teaching and learning. As early as 1904, Dewey (as quoted by Mewborn, 1999) argued that teachers who lack an inquiring mind will not be able to bridge the gap between knowing and doing. However, one can only reflect on what one knows, and a teacher who knows only facts, algorithms and rules, can only reflect on these. What teachers do in every step of a procedure and why they do it should involve continuously unpacking, decompressing, and then connecting mathematical knowledge to be able to use it in different appropriate problem contexts. Reflecting on appropriate components of mathematical knowledge is of no use to a teacher if it does not lead to an ability to unpack mathematical problems and to communicate aspects of the problem to learners. These two features (unpacking and communicating) represent the cognitive skills by which a teacher disseminates knowledge in the act of teaching. These processes are linked by reflective practices to the integrated components of the model. For example, if a learner solves the equation $\frac{2(x-1)}{5} = 0$ by rewriting it in the form $\frac{2(x-1)}{5} = \frac{0}{5}$, the teacher should be able to unpack this procedure, connect it with the deductive rule that one can substitute an expression in an equation with an equivalent expression, and be able to communicate this reasoning in words.

Related literature mostly categorizes mathematical knowledge into procedural or conceptual knowledge. However, this rigorous classification is not sufficient to describe teachers' MCK, therefore we place representations, connections, reasoning and transfer to different problem

contexts on the same level as procedural and conceptual understanding to emphasize that these cognitive processes become knowledge *per se*. According to our model, teachers’ knowledge is “broader” than just having procedural knowledge, in the sense that they have knowledge and understanding of procedures, concepts, representations and reasoning, and can use this knowledge and connect it to other ideas from other mathematical contexts. With an integrated knowledge structure, which combines MCK and the processes of mathematics, teachers will be able to reflect on previously learned mathematics that might have become automated, and be able to unpack and communicate mathematics with understanding.

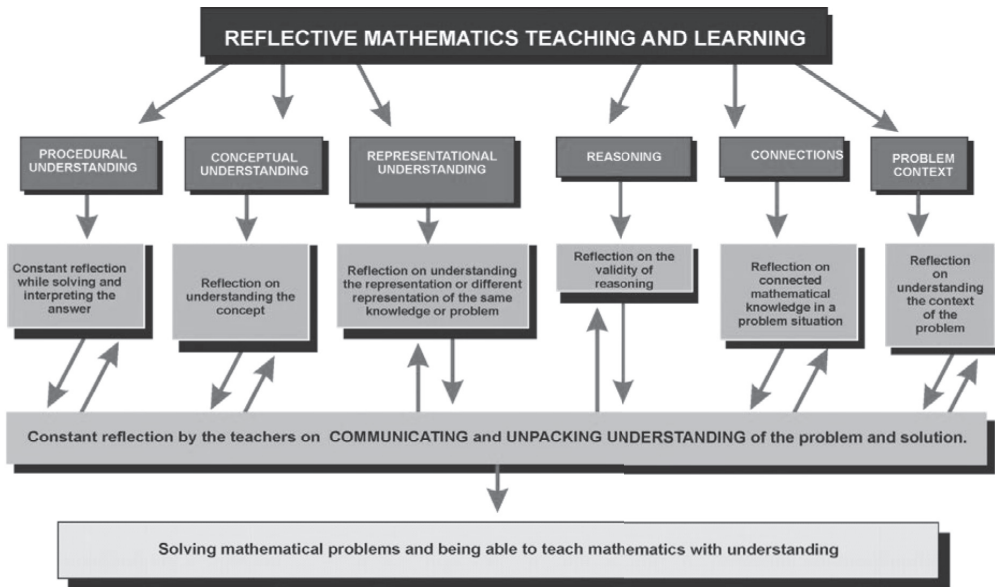


Figure 1: A model for restructuring the mathematical content knowledge of teachers

Application of model

The question is how does one “teach” someone to engage in reflective practices? The key to the development of a reflective frame of mind is connections. However, the building of connections is a higher order thinking skill and it cannot be taught explicitly, which makes it problematic to compile a definitive curriculum based on this model. The potential danger is that the information required to make the connections explicitly can be internalized as one more piece of isolated knowledge rather than supporting the construction of useful connections (Hiebert & Carpenter, 1992). The value of the model lies in its ability to expand one’s understanding of a specific piece of mathematics. For example, when working with the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, a teacher with a knowledge structure consisting mainly of procedural knowledge will regard it as an algorithm to solve quadratic equations which cannot be factorised easily. A teacher who engages with this formula using the model, will develop a connected understanding of this formula in terms of its conceptual origin (derived using completion of squares to solve the general formula $ax^2 + bx + c = 0$), its connections with other mathematical ideas (understanding that the square

root in the formula limits the potential values of the solution), and its link to other representations, such as the x – intercepts of the graph of the function $f(x) = ax^2 + bx + c$, and other problem contexts, such as a distance versus time graph.

Let us consider the introduction of trigonometry at school level as another example of the application of the model. Knowledge of trigonometry at tertiary level is usually compressed to knowing how to substitute one trigonometric expression with a more appropriate expression when solving differential equations. By engaging with the model a teacher will try to decompress this knowledge by searching for a deeper conceptual understanding of trigonometry and to use different representations (e.g. graphs and diagrams) and different mathematical contexts (right-angled triangle or functions) to introduce these concepts. With these different conceptualizations of the formula a teacher will be better able to seize teachable moments in the act of teaching. In our program we work with teachers with insufficient knowledge of the school mathematics they are supposed to teach. We use the model to structure their mathematics courses. For example, when introducing trigonometry, we reflect on the conceptual progression in trigonometry and start with the underlying concepts of ratio and proportion and similar triangles to define the trigonometric ratios. Then we connect these definitions with the concept of a function and use different contexts, such as triangles, the Cartesian plane, a circle with radius r and the unit circle to develop their conceptual understanding. Along the way, we seize teachable moments based on the model. For example, in the learning process we use inductive reasoning by allowing teachers to search for patterns, formulate conjectures, and then prove these conjectures using deductive reasoning.

Conclusions

An important discourse in mathematics teacher education is the amount and kind of MCK that teachers need in order to teach mathematics with understanding. We acknowledge the undeniable role that both MCK and PCK play in the training of mathematics teachers. The model that we propose for the pure MCK of teachers is an attempt to assist teachers and teacher educators in bridging the gap between knowing mathematics and teaching mathematics with and for understanding. The focus in the model shifts from what teachers know mathematically to how they understand mathematics. We believe that this transformation is not acquired incidentally, but that teachers have to reflect on their compressed mathematical knowledge, and learn new mathematical knowledge using the model.

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