A study of equal-width and Zakharov-Kuznetsov-Burgers equations

K Plaatjie

orcid.org 0000-0003-0400-0801

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Supervisor: Prof CM Khalique

Co-supervisor: Dr T Motsepa

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Student number: 25451308

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by

KARABO PLAATJIE (25451308)

Dissertation submitted for the degree of Master of Science in Applied Mathematics in the Department of Mathematical Sciences in the Faculty of Natural and Agricultural Sciences,

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Supervisor: Professor C M Khalique

Co-Supervisor: Dr T Motsepa

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Declaration

I KARABO PLAATJIE, student number 25451308, declare that this dissertation for the degree of Master of Science in Applied Mathematics at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other University, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

Signed: ....

MR. KARABO PLAATJIE

Date: ....

This dissertation has been submitted with my approval as a University supervisor and would certify that the requirements for the applicable Master of Science degree rules and regulations have been fulfilled.

Signed:...

PROF C.M. KHALIQUE

Date: ....

Signed:...

DR T. MOTSEPA

Date: ....

Declaration of Publications

Details of contribution to publications that form part of this thesis.

Chapter 3

K. Plaatjie, C.M. Khalique and T. Motsepa, On optimal system, exact solutions and conservation laws of the equal-width equation, To be submitted for publication.

Chapter 4

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Chapter 5

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Dedication

To my family

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Abstract

In this work we study some nonlinear partial differential equations which evolve in mathematical physics and other fields of science. We start by considering a second-order nonlinear Burgers equation as an illustration of what will be done in this dissertation. We then study the main equations of this dissertation. These equations are the equal-width equation and Zakharov-Kuznetsov-Burgers equations. Lie group analysis is used to construct exact solutions of these equations. Furthermore, conservation laws are derived for the underlying equations using the multiplier method.

Introduction

Nonlinear partial differential equations (NLPDEs) model many natural phenomena. We study these equations in order to obtain solutions that will enable us to better understand the real world. Extensive research on NLPDEs has been made and many researchers continue to seek new methods of solutions for these kind of equations.

To date several methods for computing exact solutions of NLPDEs have been developed since there is no general theory to find their exact solutions. These methods include the Jacobi elliptic function expansion method [1], the homogeneous balance method [2], the Kudryashov method [3], the ansatz method [4], the inverse scattering transform method [5], the Bäcklund transformation [6], the Darboux transformation [7], the Hirota bilinear method [8], the $(G'/G)-$expansion method [9] and the Lie symmetry method [10–15], just to mention a few.

In the late 19th century, the Norwegian mathematician Marius Sophus Lie (1844-1899), developed a powerful symmetry-based technique for solving differential equations known today as Lie group analysis. This method made it possible to obtain exact solutions of differential equations. A robust amount of research based on Lie's work has been published by various researchers [10–15].

In 1918, a German mathematician Emmy Noether (1882-1935) presented a procedure for deriving conservation laws for systems of differential equations that admit variational principle and this procedure is referred to as Noether's theorem [16]. For a given differential equation system to admit a variational principle, it should have a Lagrangian.

Conservation laws play a vital role in the study of differential equations. They describe physical conserved quantities, e.g., mass, energy, momentum, charge and other constants of motion. They are also important for the investigation of integrability and uniqueness of solutions. See, for example [16–24] and references therein.

The outline of this dissertation is as follows:

In Chapter one, we present important preliminaries regarding Lie's theory, Kudryashov's method, extended Jacobi elliptic function method, Noether's theorem and the multiplier approach.

In Chapter two, we present an illustration of what this research project will be about. We consider a second-order Burgers equation and use Lie's theory to obtain its group-invariant solutions. Thereafter, we employ the multiplier method to derive its conservation laws.

In Chapter three, Lie point symmetries for an equal-width equation are computed. We then calculate an optimal system of one-dimensional subalgebras and use it to determine an optimal system of group-invariant solutions. Travelling wave solutions are obtained by applying Kudryashov's and extended Jacobi elliptic function expansion methods. Finally, conservation laws are derived using the multiplier approach and Noether's theorem.

In Chapter four, Kudryashov's method is used to obtain exact solutions of the ZK-Burgers equation. Conservation laws of this equation are derived using the multiplier method.

In Chapter five, we study the ZK-Burgers equation with power law nonlinearity by obtaining its exact solutions using Kudryashov's method. The multiplier method is then employed to derive its conservation laws.

In Chapter six, a summary of results is provided and future work proposed.

Bibliography is given at the end of this dissertation.

Chapter 1

Preliminaries

In this chapter we give a synopsis of pertinent concepts that are essential to this dissertation. These include the algorithm to determine the Lie point symmetries of PDEs, Noether's theorem, the variational derivative approach and some methods for obtaining exact solutions of PDEs.

1.1 Introduction

In the late 19th century, Marius Sophus Lie, an eminent mathematician from Norway, developed a revolutionary symmetry-based method for solving differential equations. This method today is known as Lie group analysis and gives a systematic way to obtain exact solutions of differential equations. Recently several books on Lie group analysis have been published [10–14]. The definitions and results presented in this chapter are taken from the books mentioned above.

1.2 One-parameter groups

Suppose $x = (x^1, \cdots, x^n)$ is the independent variable with coordinates x^i and $u = (u^1, \cdots, u^m)$ is the dependent variable with coordinates u^α (n and m finite). We consider the following change of the variables x and u:

$$T_a : \bar{x}^i = f^i(x,u,a),\ \ \bar{u}^\alpha = \phi^\alpha(x,u,a), \tag{1.1}$$

where a is a real parameter which continuously takes values from a neighbourhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$, and f^i and ϕ^α are differentiable functions.

Definition 1.1 A continuous one-parameter (local) Lie group of transformations in the space of variables x and u is a set G of transformations (1.1) which satisfies the following properties:

(i) If T_a, $T_b \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then $T_b\,T_a = T_c \in G,\ c = \phi(a,b) \in \mathcal{D}$ (Closure)

(ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0\,T_a = T_a\,T_0 = T_a$ (Identity)

(iii) There exists $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that $T_a\,T_{a^{-1}} = T_{a^{-1}}\,T_a = T_0$ (Inverse)

We note that from (i) the associativity property is satisfied. The group property (i) can be written as

$$\begin{aligned} \bar{\bar{x}}^i &\equiv \ f^i(\bar{x},\bar{u},b) = f^i(x,u,\phi(a,b)), \\ \bar{\bar{u}}^\alpha &\equiv \ \phi^\alpha(\bar{x},\bar{u},b) = \phi^\alpha(x,u,\phi(a,b)) \end{aligned} \tag{1.2}$$

and the function ϕ is called the group composition law. A group parameter a is called canonical if the group composition law is additive, i.e. $\phi(a,b) = a + b$.

Theorem 1.1 For any composition law $\phi(a,b)$, there exists the canonical parameter $\tilde{a}$ defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)},$$

where

$$w(s) = \left.\frac{\partial\,\phi(s,b)}{\partial b}\right|_{b=0}.$$

1.3 Prolongation of group transformations and their generators

The derivatives of u with respect to x are defined as

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j D_i(u_i), \cdots , \tag{1.3}$$

where the operator of total differentiation is defined by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots , \quad i = 1, ..., n \tag{1.4}$$

The collection of all first derivatives u_i^α is denoted by $u_{(1)}$, i.e.,

$$u_{(1)} = \{u_i^\alpha\} \quad \alpha = 1, ..., m, \quad i = 1, ..., n.$$

Similarly

$$u_{(2)} = \{u_{ij}^\alpha\} \quad \alpha = 1, ..., m, \quad i, j = 1, ..., n$$

and $u_{(3)} = \{u_{ijk}^\alpha\}$ and likewise $u_{(4)}$ etc. Since $u_{ij}^\alpha = u_{ji}^\alpha$, $u_{(2)}$ contains only u_{ij}^α for $i \leq j$. In the same manner $u_{(3)}$ has only terms for $i \leq j \leq k$.

In group analysis all variables $x, u, u_{(1)}, \cdots$ are considered functionally independent variables connected only by the differential relations (1.3). Therefore the u_s^α are called differential variables.

1.3.1 Prolonged or extended groups

If $z = (x, u)$, one-parameter group of transformations G is

$$\begin{aligned} \bar{x}^i = f^i(x, u, a), \quad f^i|_{a=0} = x^i, \\ \bar{u}^\alpha \phi^\alpha(x, u, a), \quad \phi^\alpha|_{a=0} = u^\alpha. \end{aligned} \tag{1.5}$$

According to the Lie's theory, finding the symmetry group G is equivalent to the determination of the corresponding infinitesimal transformations:

$$\bar{x}^i \approx x^i + a\,\xi^i(x, u), \quad \bar{u}^\alpha \approx u^\alpha + a\,\eta^\alpha(x, u) \tag{1.6}$$

obtained from (1.1) by expanding the functions f^i and ϕ^α into Taylor series in a about $a = 0$ and also taking into account the initial conditions

$$f^i\big|_{a=0} = x^i, \quad \phi^\alpha\big|_{a=0} = u^\alpha.$$

Consequently, we have

$$\xi^i(x,u) = \frac{\partial f^i}{\partial a}\bigg|_{a=0}, \quad \eta^\alpha(x,u) = \frac{\partial \phi^\alpha}{\partial a}\bigg|_{a=0}. \tag{1.7}$$

We now introduce the symbol of the infinitesimal transformations by writing (1.6) as

$$\bar{x}^i \approx (1 + a\,X)x, \quad \bar{u}^\alpha \approx (1 + a\,X)u,$$

where the differential operator

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} \tag{1.8}$$

is known as the infinitesimal operator or generator of the group G.

We now show how the derivatives are transformed.

The D_i transforms as

$$D_i = D_i(f^j)\bar{D}_j, \tag{1.9}$$

where $\bar{D}_j$ is the total differentiations in transformed variables $\bar{x}^i$. So

$$\bar{u}_i^\alpha = \bar{D}_j(u^\alpha), \qquad \bar{u}_{ij}^\alpha = \bar{D}_j(\bar{u}_i^\alpha) = \bar{D}_i(\bar{u}_j^\alpha), \cdots.$$

Let us now apply (1.9) and (1.5)

$$\begin{aligned} D_i(\phi^\alpha) &= D_i(f^j)\bar{D}_j(\bar{u}^\alpha) \\ &= D_i(f^j)\bar{u}_j^\alpha. \end{aligned} \tag{1.10}$$

Thus

$$\left(\frac{\partial f^j}{\partial x^i} + u_i^\beta\frac{\partial f^j}{\partial u^\beta}\right)\bar{u}_j^\alpha = \frac{\partial \phi^\alpha}{\partial x^i} + u_i^\beta\frac{\partial \phi^\alpha}{\partial u^\beta}. \tag{1.11}$$

The quantities $\bar{u}_j^\alpha$ can be represented as functions of $x, u, u_{(i)}$, for small a, i.e., (1.11) is locally invertible:

$$\bar{u}_i^\alpha = \psi_i^\alpha(x, u, u_{(1)}, a), \quad \psi^\alpha|_{a=0} = u_i^\alpha. \tag{1.12}$$

The transformations in $(x, u, u_{(1)})$ space given by (1.5) and (1.12) form a one-parameter group called the first prolongation or just extension of the group G and denoted by $G^{[1]}$.

We let

$$\bar{u}_i^\alpha \approx u_i^\alpha + a\zeta_i^\alpha \tag{1.13}$$

be the infinitesimal transformation of the first derivatives so that the infinitesimal transformation of the group $G^{[1]}$ is (1.6) and (1.13). Higher-order prolongations of G, viz., $G^{[2]}$, $G^{[3]}$ can be obtained by derivatives of (1.10).

1.3.2 Prolonged generators

Using (1.10) together with (1.6) and (1.13) we obtain

$$\begin{aligned} D_i(f^j)(\bar{u}_j^\alpha) &= D_i(\phi^\alpha) \\ D_i(x^j + a\xi^j)(u_j^\alpha + a\zeta_j^\alpha) &= D_i(u^\alpha + a\eta^\alpha) \\ u_i^\alpha + a\zeta_i^\alpha + au_j^\alpha D_i\xi^j &= u_i^\alpha + aD_i\eta^\alpha \\ \zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \quad \text{(sum on } j). \end{aligned} \tag{1.14}$$

This is called the first prolongation formula. Similarly, one can obtain the second prolongation

$$\zeta_{ij}^\alpha = D_j(\eta_i^\alpha) - u_{ik}^\alpha D_j(\xi^k), \quad \text{(sum on } k). \tag{1.15}$$

By induction (recursively)

$$\zeta_{i_1,i_2,\ldots,i_p}^\alpha = D_{i_p}(\zeta_{i_1,i_2,\ldots,i_{p-1}}^\alpha) - u_{i_1,i_2,\ldots,i_{p-1}\,j}^\alpha D_{i_p}(\xi^j), \quad \text{(sum on } j). \tag{1.16}$$

The first and higher prolongations of the group G form a group denoted by $G^{[1]}, \cdots, G^{[p]}$. The corresponding prolonged generators are

$$\begin{aligned} X^{[1]} &= X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} \quad \text{(sum on } i, \alpha), \\ &\vdots \\ X^{[p]} &= X^{[p-1]} + \zeta_{i_1,\ldots,i_p}^\alpha \frac{\partial}{\partial u_{i_1,\ldots,i_p}^\alpha} \quad p \geq 1, \end{aligned} \tag{1.17}$$

where

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}. \tag{1.18}$$

1.4 Group admitted by a PDE

Consider a pth-order PDE, namely

$$E(x, u, u_{(1)}, ..., u_{(p)}) = 0. \tag{1.19}$$

Definition 1.2 The vector field

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} \tag{1.20}$$

is a Lie point symmetry of the pth-order PDE (1.19), if

$$X^{[p]} E\big|_{E=0} = 0, \tag{1.21}$$

where the symbol $|_{E=0}$ means evaluated on the equation $E = 0$.

Definition 1.3 An equation (1.21) that determines all the infinitesimal symmetries of (1.19) is called the determining equation.

Definition 1.4 A one-parameter group G of continuous transformations (1.1) is called a symmetry group of equation (1.19) if (1.19) is invariant (has the same form) in the new variables $\bar{x}$ and $\bar{u}$, i.e.,

$$E(\bar{x}, \bar{u}, \bar{u}_{(1)}, \cdots, \bar{u}_{(p)}) = 0, \tag{1.22}$$

where the function E is the same as in equation (1.19).

1.5 Invariant functions

Definition 1.5 A function $F(x, u)$ is called an invariant of the group of transformation (1.1) if

$$F(\bar{x}, \bar{u}) \equiv F(f^i(x, u, a), \phi^\alpha(x, u, a)) = F(x, u), \tag{1.23}$$

identically in x, u and a.

Theorem 1.2 A necessary and sufficient condition for a function $F(x, u)$ to be an invariant is that

$$X\,F \equiv \xi^i(x, u)\frac{\partial F}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial F}{\partial u^\alpha} = 0\,. \tag{1.24}$$

It follows from the above theorem that every one-parameter group of point transformations (1.1) has $n-1$ functionally independent invariants. One can take, as basic invariants the left-hand side $n-1$ first integrals

$$J_1(x, u) = c_1, \cdots, J_{n-1}(x, u) = c_{n-1}$$

of the characteristic equations

$$\frac{dx^1}{\xi^1(x, u)} = \cdots = \frac{dx^n}{\xi^n(x, u)} = \frac{du^1}{\eta^1(x, u)} = \cdots = \frac{du^m}{\eta^n(x, u)}.$$

Theorem 1.3 If the infinitesimal transformation (1.6) or its symbol X is given, then the corresponding one-parameter group G is obtained by solving the Lie equations

$$\frac{d\bar{x}^i}{da} = \xi^i(\bar{x}, \bar{u}), \quad \frac{d\bar{u}^\alpha}{da} = \eta^\alpha(\bar{x}, \bar{u}) \tag{1.25}$$

subject to the initial conditions

$$\bar{x}^i\big|_{a=0} = x, \quad \bar{u}^\alpha\big|_{a=0} = u\,.$$

1.6 Lie algebra

Let us consider two operators X_1 and X_2 defined by

$$X_1 = \xi^i_1(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha_1(x, u)\frac{\partial}{\partial u^\alpha}$$

and

$$X_2 = \xi_2^i(x,u)\frac{\partial}{\partial x^i} + \eta_2^\alpha(x,u)\frac{\partial}{\partial u^\alpha}.$$

Definition 1.6 The commutator of X_1 and X_2, written as $[X_1, X_2]$, is defined by $[X_1, X_2] = X_1(X_2) - X_2(X_1)$.

Definition 1.7 A Lie algebra is a vector space L (over the field of real numbers) of operators $X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u}$ with the following property. If the operators

$$X_1 = \xi_1^i(x,u)\frac{\partial}{\partial x^i} + \eta_1^\alpha(x,u)\frac{\partial}{\partial u}, \quad X_2 = \xi_2^i(x,u)\frac{\partial}{\partial x^i} + \eta_2^\alpha(x,u)\frac{\partial}{\partial u}$$

are any elements of L, then their commutator

$$[X_1, X_2] = X_1(X_2) - X_2(X_1)$$

is also an element of L. It follows that the commutator is

1. Bilinear: for any $X, Y, Z \in L$ and $a, b \in \mathbb{R}$,

$$[aX + bY, Z] = a[X, Z] + b[Y, Z], \ [X, aY + bZ] = a[X, Y] + b[X, Z];$$

2. Skew-symmetric: for any $X, Y \in L$,

$$[X, Y] = -[Y, X];$$

3. and satisfies the Jacobi identity: for any $X, Y, Z \in L$,

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

1.7 Solution methods for differential equations

In this section we present a few methods for finding exact solutions of differential equations.

1.7.1 Kudryashov's method

This method was introduced by Kudryashov in his paper [3] and is used for finding exact solutions of NLPDEs.

Consider the NLPDE

$$E_1(t, x, u, u_t, u_x, u_{tt}, u_{xx}, \cdots) = 0. \tag{1.26}$$

The algorithm of Kudryashov method is given below:

Step 1. The substitution $u(x, t) = U(z),\ z = kx + \omega t$, where k and ω are constants, reduces equation (1.26) to the ordinary differential equation

$$E_2(U, \omega U', kU', \omega^2 U'', k^2 U'', \cdots) = 0. \tag{1.27}$$

Step 2. Suppose that the exact solution of equation (1.27) can be expressed in the form

$$U(z) = \sum_{n=0}^{N} a_n Q^n(z), \tag{1.28}$$

where the coefficients a_n $(n = 0, 1, 2, \cdots, N)$ are constants to be determined, such that $a_N \neq 0$, and $Q(z)$ is the solution of the first-order nonlinear ODE

$$Q'(z) = Q^2(z) - Q(z). \tag{1.29}$$

Equation (1.29) has the solution

$$Q(z) = \frac{1}{1 + e^z}. \tag{1.30}$$

Step 3. We substitute the value for $U(z)$ into equation (1.27) and use equation (1.29) to obtain an equation involving powers of Q.

Step 4. Equating different powers of Q to zero, we obtain the system of algebraic equations

$$P_n(a_N, a_{N-1}, \cdots, a_0, k, \omega, \cdots) = 0, \quad (n = 0, \cdots, N). \tag{1.31}$$

Step 5. The solution of the system of algebraic equations gives the values of coefficients $a_0, a_1, \cdots, a_{N-1}, a_N$ and relations for parameters of equation (1.27). As a result, we obtain exact solutions of equation (1.27) in the form (1.28).

1.7.2 The extended Jacobi elliptic function expansion method

We briefly outline the extended Jacobi elliptic function expansion method, another algorithm for determining the exact solutions of differential equations. There is extensive literature, some of which dates back several decades, covering different aspects of Jacobi elliptic functions [25–27] such as their derivation, interrelationships and applications. Several researchers [28–30] have recently employed the properties of some of these elliptic functions to determine exact solutions of differential equations.

The procedure for implementing the extended Jacobi elliptic function expansion method is as follows:

Firstly we transform the NLPDE (1.26) to a nonlinear ordinary differential equation (ODE) by making use of the substitution

$$u(t,x) = U(z), \quad z = x - ct. \tag{1.32}$$

Using the above substitutions, (1.26) is transformed into the nonlinear ODE

$$E(U, -cU', U', c^2U'', -cU'', U'', \cdots) = 0. \tag{1.33}$$

Secondly, we assume that our solutions can be expressed in the form

$$U(z) = \sum_{i=-M}^{M} A_i H(z)^i, \tag{1.34}$$

where M is a positive integer obtained by the balancing procedure and

$$H(z) = \mathrm{cn}(z|\omega), \tag{1.35}$$

the cosine-amplitude function, is a solution to the first-order ODE [26, 28, 30]

$$H'(z) = -\sqrt{(1 - H^2(z))(1 - \omega + \omega H^2(z))}, \tag{1.36}$$

and the sine amplitude function

$$H(z) = \mathrm{sn}(z|w) \tag{1.37}$$

is a solution to the first-order ODE

$$H'(z) = \sqrt{(1 - H^2(z))(1 - \omega H^2(z)}. \tag{1.38}$$

The third step of our procedure entails substituting (1.34) subject to (1.36) or (1.38) into (1.33) to obtain a polynomial in powers of $H(z)$. Separating coefficients with respect to like powers of $H(z)$ yields an algebraic system of equations. These can be solved to obtain the values of $A_i,\ i = 0, \pm 1, \pm 2, \cdots \pm M$.

1.8 Conservation laws

Consider a kth-order system of PDEs of n independent variables $x = (x^1, x^2, \cdots, x^n)$ and m dependent variables $u = (u^1, u^2, \cdots, u^m)$, which are defined as

$$E_\alpha(x, u, u_{(1)}, u_{(2)} \cdots, u_{(k)}) = 0, \quad \alpha = 1, \cdots, m, \tag{1.39}$$

where $u_{(i)}$ denotes the collection of all i-th-order partial derivatives of u. The n-tuple vector which is given by $T = (T^1, T^2, \ldots, T^n)$, $T^j \epsilon\, \mathcal{A}$, $j = 1, \ldots, n$, ($\mathcal{A}$ is the space of differential functions) is a conserved vector of (1.39) if T^i satisfies

$$D_i T^i|_{(1.39)} = 0. \tag{1.40}$$

1.8.1 The multiplier approach

This approach has been applied by many researchers. See for example [31–35]. A local conservation law of a given differential system arises from a linear combination formed by local multipliers with each differential equation in the system, where the multipliers Λ_α are functions of the dependent and independent variables as well as of a finite number of derivatives with respect to the dependent variables of the system of differential equations.

A multiplier $\Lambda_\alpha(x, u, u_1, \ldots)$ has the property that [12, 35]

$$\Lambda_\alpha E_\alpha = D_i T^i \tag{1.41}$$

holds identically. The determining equation for the multiplier Λ_α is

$$\frac{\delta(\Lambda_\alpha E_\alpha)}{\delta u^\alpha} = 0, \tag{1.42}$$

where $\delta/\delta u^\alpha$ is the Euler-Lagrange operator defined as

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s=1}^{\infty}(-1)^s D_{j_1}\dots D_{j_s}\frac{\partial}{\partial u^\alpha_{ij_1\cdots j_s}}, \alpha = 1,\dots,m. \tag{1.43}$$

Once the multipliers have been obtained from (1.42), we can determine our conserved vectors by invoking equation (1.41) as illustrated in [35].

1.8.2 Noether's theorem

Consider the system (1.39) that admits the infinitesimal generator

$$X = \quad \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha}. \tag{1.44}$$

Suppose this system is variational, i.e., there exist a Lagrangian $\mathcal{L}$ for it such that

$$\left.\frac{\delta\mathcal{L}}{\delta u^\alpha}\right|_{(1.39)} = 0, \tag{1.45}$$

where $\delta/\delta u^\alpha$ is the Euler-Lagrange operator given by (1.43). Then the Noether point symmetries of the system (1.39) are determined by solving the equation

$$X(\mathcal{L}) + \mathcal{L}D_i(\xi^i) = D_i(B^i), \tag{1.46}$$

where B^i are the gauge terms and D_i are the total derivatives. Once the Noether symmetries of the system (1.39) have been obtained then conservation laws corresponding to each Noether symmetry may be obtained by explicitly using the formula [36]

$$T^k = \xi^i\mathcal{L} + \left(\eta - u_{x^j}\xi^j\right)\left\{\frac{\partial\mathcal{L}}{\partial u_{x^k}} - \sum_{l=k}^{n} D_{x^i}\left(\frac{\partial\mathcal{L}}{\partial u_{x^jx^k}}\right)\right\} + \sum_{l=k}^{n}\left\{\zeta_l - u_{x^lx^j}\xi^j\right\}\frac{\partial\mathcal{L}}{\partial u_{x^kx^l}} - B^i. \tag{1.47}$$

1.9 Conclusions

In this Chapter a brief introduction to Lie symmetry methods was presented. Some solution methods for finding exact solutions of NLPDEs were discussed. A synopsis of two methods for deriving conservation laws were recalled. The material deliberated upon in this Chapter will be utilised throughout this dissertation.

Chapter 2

Solutions and conservation laws for the Burgers equation: an illustrative example

In this Chapter we study the second-order Burgers equation. First we compute group-invariant solutions of the equation. Thereafter conservation laws of this equation will be derived by making use of the multiplier method.

2.1 Introduction

Burgers' equation [37], named after Johannes Martinus Burgers (1895-1981), is considered to be a fundamental partial differential equation which appears in many areas of applied mathematics, such as fluid mechanics, nonlinear acoustics, gas dynamics and traffic flow. In this chapter we study the Burgers equation

$$u_t - uu_x - u_{xx} = 0, \tag{2.1}$$

which describes the motion of weak nonlinear waves in gases when dissipative effects are sufficiently small to be considered in the first approximation only. When dissipation tends to zero, this equation gives an adequate description of waves in a non-viscous medium. Here u is the dependent variable whereas t and x are independent variables.

2.2 Solutions of Burgers equation (2.1)

2.2.1 Lie point symmetries of (2.1)

We start first by computing Lie point symmetries of the Burgers equation (2.1). Equation (2.1) admits the one-parameter Lie group of transformations with infinitesimal generator

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{2.2}$$

if and only if

$$X^{[2]}(u_t - uu_x - u_{xx})\big|_{(2.1)} = 0. \tag{2.3}$$

Using the definition of $X^{[2]}$ from Chapter 1 we get

$$\left(\tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \zeta_1\frac{\partial}{\partial u_t} + \zeta_2\frac{\partial}{\partial u_x} + \zeta_{22}\frac{\partial}{\partial u_{xx}}\right)(u_t - uu_x - u_{xx})\,|_{u_t = uu_x + u_{xx}} = 0,$$

which gives

$$-\eta u_x + \zeta_1 - u\zeta_2 - \zeta_{22}|_{u_{xx} = u_t - uu_x} = 0, \tag{2.4}$$

where ζ_1 and ζ_2 are defined by (1.14) and ζ_{22} is given by (1.15). Substituting the values of ζ_1, ζ_2 and ζ_{22} in (2.4) we obtain the following determining equation:

$$\begin{aligned}
&\eta_t - \eta u_x + (\eta_u - \tau_t)u_t - \xi_t u_x - \tau_u u_t^2 - \xi_u u_t u_x - u\eta_x - (\eta_u - \xi_x)uu_x + \tau_x uu_t + \xi_u uu_x^2 \\
&+ \tau_u uu_x u_t - \eta_{xx} - (2\eta_{xu} - \xi_{xx})u_x + \tau_{xx}u_t + 2\tau_{xu}u_x u_t - (\eta_{uu} - 2\xi_{xu})u_x^2 + \xi_{uu}u_x^3 \\
&+ \tau_{uu}u_t u_x^2 - (\eta_u - 2\xi_x)u_{xx} + 3\xi_u u_x u_{xx} + \tau_u u_t u_{xx} + (2u_{xt}\tau_x + \tau_u u_x)|_{u_{xx} = u_t - uu_x} = 0.
\end{aligned}$$

Now replacing u_{xx} by $u_t - uu_x$ in the above equation we obtain

$$\eta_t - \eta u_x + (\eta_u - \tau_t)u_t - \xi u_x - \tau_u u_t^2 - \xi_u u_t u_x - u\eta_x - (\eta_u - \xi_x)uu_x + \tau_x uu_t + \xi_x uu_x^2$$

$$+ \tau_u u u_t u_x - \eta_{xx} - (2\eta_{xu} - \xi_{xx})u_x + \tau_{xx}u_t - (\eta_{uu} - 2\xi_{xu})u_x^2 + 2\tau_{xu}u_x u_t + \xi_{uu}u_x^3$$
$$+ \tau_{uu}u_t u_x^2 - \eta_u(u_t - uu_x) + 2(u_t - uu_x)\xi_x + 3u_x(u_t - uu_x)\xi_u + u_t(u_t - uu_x)\tau_u$$
$$+ 2u_{xt}\tau_x + \tau_u u_x = 0.$$

Since the functions τ, ξ and η depend only on t, x and u and are independent of the derivatives of u, we can then split the above equation on the derivatives of u and obtain

$$\tau_x = 0, \tag{2.5}$$
$$\tau_u = 0, \tag{2.6}$$
$$\xi_u = 0, \tag{2.7}$$
$$\eta_{uu} = 0, \tag{2.8}$$
$$2\xi_x - \tau_t = 0, \tag{2.9}$$
$$\xi_{xx} - 2\eta_{xu} - \xi_x u - \xi_t - \eta = 0, \tag{2.10}$$
$$\eta_t - \eta_x u - \eta_{xx} = 0. \tag{2.11}$$

Equations (2.5) and (2.6) imply that

$$\tau = B(t), \tag{2.12}$$

where $B(t)$ is an arbitrary function of t. From (2.7) we get

$$\xi = A(t, x), \tag{2.13}$$

where $A(t, x)$ is an arbitrary function of t and x. Integrating equation (2.8) we get

$$\eta = D(t, x)u + E(t, x),$$

where $D(t, x)$ and $E(t, x)$ are arbitrary functions of t and x. Substituting the values of ξ and η into equation (2.10), we obtain

$$A_{xx} - Du - E - A_t - A_x u - 2D_x = 0. \tag{2.14}$$

Splitting equation (2.14) on powers u yields

$$u \;\; : \;\; D + A_x = 0, \tag{2.15}$$

$$u^0 \quad : \quad A_{xx} - E - A_t - 2D_x = 0. \tag{2.16}$$

Now substituting the value of η into equation (2.11) we get

$$D_t u + E_t - D_x u^2 - E_x u - D_{xx} u - E_{xx} = 0.$$

Splitting the above equation on powers of u yields

$$u^2 \quad : \quad D_x = 0, \tag{2.17}$$

$$u \quad : \quad D_t - E_x - D_{xx} = 0, \tag{2.18}$$

$$u^0 \quad : \quad E_t - E_{xx} = 0. \tag{2.19}$$

From (2.17) we have $D = D(t)$. Thus from equation (2.15) we have $A_x = -D(t)$, which on integration gives

$$A = -D(t)x + F(t), \tag{2.20}$$

where $F(t)$ is an arbitrary function of t. From equations (2.16) and (2.18), we obtain respectively

$$A_t = -E \quad \text{and} \quad E_x = D'(t). \tag{2.21}$$

Now integrating the second equation with respect to x we obtain

$$E = D'(t)x + G(t), \tag{2.22}$$

where $G(t)$ is an arbitrary function of t. Substituting the above value of E into equation (2.19) we get

$$D''(t)x + G'(t) = 0. \tag{2.23}$$

Splitting equation (2.23) on x we have

$$x \quad : \quad D''(t) = 0, \tag{2.24}$$

$$x^0 \quad : \quad G'(t) = 0. \tag{2.25}$$

Therefore

$$D(t) = C_1 t + C_2 \tag{2.26}$$

$$\text{and} \quad G(t) = C_3, \tag{2.27}$$

where C_1, C_2 and C_3 are arbitrary constants, and so from equation (2.22) we have

$$E = C_1 x + C_3. \tag{2.28}$$

The first equation of (2.21) simplifies to

$$A_t = -C_1 x - C_3. \tag{2.29}$$

Now substituting equations (2.20) and (2.26) into (2.9), we have

$$B_t = -2C_1 t - 2C_2.$$

Integrating the above equation with respect to t we get

$$B = -C_1 t^2 - 2C_2 t + C_4, \tag{2.30}$$

where C_4 is an arbitrary constant. Hence $\tau = -C_1 t^2 - 2C_2 t + C_4$. Using equation (2.20) we have

$$A_t = -D'(t)x + F'(t). \tag{2.31}$$

This means that

$$A_t = -C_1 x + F'(t). \tag{2.32}$$

Finally equations (2.29) and (2.32) imply that

$$-C_1 x - C_3 = -C_1 x + F'(t), \tag{2.33}$$

which gives $F'(t) = -C_3$. Integrating yields

$$F(t) = -C_3 t + C_5, \tag{2.34}$$

where C_5 is an arbitrary constant. Thus we obtain

$$\begin{aligned} \tau &= -C_1 t^2 - 2C_2 t + C_4, \\ \xi &= -C_1 t x - C_2 x - C_3 t + C_5, \end{aligned}$$

$$\eta = C_1(tu + x) + C_2 u + C_3.$$

Hence the Lie point symmetries of the Burgers equation (2.1) are

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t},\\
X_2 &= \frac{\partial}{\partial x},\\
X_3 &= t\frac{\partial}{\partial x} - \frac{\partial}{\partial u},\\
X_4 &= 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u},\\
X_5 &= t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} - (x+tu)\frac{\partial}{\partial u}.
\end{aligned}$$

2.2.2 Commutator table for the symmetries of (2.1)

We now calculate the commutation relations for all the symmetry generators obtained above. Firstly we compute $[X_2, X_5]$. By the definition of Lie bracket we have

$$\begin{aligned}
[X_2, X_5] =&\ X_2X_5 - X_5X_2\\
=&\ \frac{\partial}{\partial x}\left(t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} - (x+tu)\frac{\partial}{\partial u}\right) - \left(t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} - (x+tu)\frac{\partial}{\partial u}\right)\frac{\partial}{\partial x}\\
=&\ t\frac{\partial}{\partial x} - \frac{\partial}{\partial u}\\
=&\ X_3.
\end{aligned}$$

Likewise, one can compute all the remaining commutation relations using the above procedure. The table below shows the commutation relations in table form.

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	0	X_2	$2X_1$	X_4
X_2	0	0	0	X_2	X_3
X_3	$-X_2$	0	0	$-X_3$	0
X_4	$-2X_1$	$-X_2$	X_3	0	X_5
X_5	$-X_4$	$-X_3$	0	$-X_5$	0

2.2.3 One-parameter groups of (2.1)

We now employ the Lie equations

$$\frac{d\bar{t}}{da} = \tau(\bar{t},\bar{x},\bar{u}), \quad \bar{t}|_{a=0} = t,$$
$$\frac{d\bar{x}}{da} = \xi(\bar{t},\bar{x},\bar{u}), \quad \bar{x}|_{a=0} = x,$$
$$\frac{d\bar{u}}{da} = \eta(\bar{t},\bar{x},\bar{u}), \quad \bar{u}|_{a=0} = u,$$

to compute the one-parameter group of transformations. For each X_i, let T_{a_i} be the corresponding group. We first compute one-parameter group corresponding to infinitesimal generator X_4, namely

$$X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}.$$

Using Lie equations, we have

$$\frac{d\bar{t}}{da} = 2\bar{t},\ \bar{t}|_{a=0} = t, \quad \frac{d\bar{x}}{da} = \bar{x},\ \bar{x}|_{a=0} = x, \quad \frac{d\bar{u}}{da} = -u,\ \bar{u}|_{a=0} = u.$$

Solving the above equations we obtain

$$\bar{t} = te^{2a_4} \quad \bar{x} = xe^{a_4}, \quad \bar{u} = ue^{-a_4}.$$

Thus the one-parameter group T_{a_4} corresponding to the operator X_4 is given by

$$T_{a_4} : (\bar{t},\bar{x},\bar{u}) \longrightarrow (te^{2a_4}, xe^{a_4}, ue^{-a_4}).$$

If we continue in the same manner as above, we get the following one-parameter groups for the remaining operators:

$$T_{a_1} :\ (\bar{x},\bar{t},\bar{u}) \longrightarrow (t + a_1, x, u),$$
$$T_{a_2} :\ (\bar{t},\bar{x},\bar{u}) \longrightarrow (t, x + a_2, u),$$
$$T_{a_3} :\ (\bar{t},\bar{x},\bar{u}) \longrightarrow (t, x + a_3 t, u + a_3),$$
$$T_{a_5} :\ (\bar{t},\bar{x},\bar{u}) \longrightarrow \left(\frac{t}{1 - a_5 t}, \frac{x}{1 - a_5 t}, \frac{u}{1 - a_5 t}\right).$$

2.2.4 Constructing group-invariant solutions of (2.1)

Given a Lie point symmetry

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{2.35}$$

of the Burgers equation (2.1), the group-invariant solutions under the one-parameter group generated by X are obtained as follows: We calculate two linearly independent invariants

$$J_1 = \phi(t,x), \quad J_2 = \psi(t,x)$$

by solving the first-order quasi-linear PDE

$$X\,J \equiv \tau(t,x,u)\frac{\partial J}{\partial x} + \xi(t,x,u)\frac{\partial J}{\partial x} + \eta(t,x,u)\frac{\partial J}{\partial u} = 0\,.$$

Then we write one invariant as a fuction of the other, e.g.,

$$J_2 = f(J_1) \tag{2.36}$$

and solve (2.36) for u. Finally, the expression of u is substituted into equation (2.1) and this yields an ODE for the unknown function f. This procedure reduces the number of independent variables by one.

We now use the above method to construct group-invariant solutions of the Burgers equation (2.1). Since the Burgers equation has five symmetries this means we are going to have five cases.

Case 1. We firstly consider the symmetry operator

$$X_1 = \frac{\partial}{\partial t}. \tag{2.37}$$

The characteristic equations assosiated with the operator (2.37) are

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{0},$$

which gives two invariants

$$J_1 = x \text{ and } J_2 = u.$$

Thus the group-invariant solution is given by $J_2 = f(J_1)$. This implies that $u = f(x)$, where f is an arbitrary function. Substituting this value of u into (2.1) we obtain the second-order ordinary differential equation (ODE)

$$f''(x) + f(x)f'(x) = 0,$$

whose solution is given by

$$f(x) = A \coth\left(\frac{1}{2}Ax + B\right),$$

where A and B are arbitrary constants of integration. Thus the group-invariant solution of (2.1) under X_1 is

$$u(t, x) = A \coth\left(\frac{1}{2}Ax + B\right).$$

Case 2. Next we consider the symmetry operator

$$X_2 = \frac{\partial}{\partial x}$$

and obtain a group-invariant solution under X_2. The characteristic equations associated with this operator are

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{0}.$$

Thus we get the following two invariants:

$$J_1 = t, \text{ and } J_2 = u.$$

Hence the group-invariant solution can be written as $J_2 = \psi(J_1)$, that is $u = \psi(t)$, where ψ is an arbitrary function. By differentiating u with respect to x and t, we get

$$u_t = \psi'(t), \quad u_x = 0, \quad u_{xx} = 0.$$

Substituting these expressions into (2.1) we obtain the first-oder ODE

$$\psi'(t) = 0,$$

which on integration gives $\psi(t) = C_1$, where C_1 is an arbitrary constant. Thus the group-invariant solution for (2.1) under X_2 is $u(t, x) = C_1$.

Case 3. We now consider the Lie point symmetry

$$X_3 = t\frac{\partial}{\partial x} - \frac{\partial}{\partial u}. \tag{2.38}$$

The characteristic equations for (2.38) are

$$\frac{dt}{0} = \frac{dx}{t} = \frac{du}{-1},$$

which yield the two invariants $J_1 = t$ and $J_2 = u + x/t$. Consequently, the group-invariant solution of (2.1) under X_3 is $J_2 = \phi(J_1)$, where ϕ is an arbitrary function. This implies that

$$u(t, x) = \phi(t) - \frac{x}{t}. \tag{2.39}$$

Differentiating (2.39) with respect to x and t we get

$$u_t = \phi' + \frac{x}{t^2}, \quad u_x = -\frac{1}{t} \quad \text{and} \quad u_{xx} = 0. \tag{2.40}$$

Subtituting (2.40) into (2.1) yields a first-order ODE

$$\phi' + \frac{1}{t}\phi = 0,$$

which is a first-order variables separable ordinary differential equation whose solution is

$$\phi(t) = \frac{C}{t},$$

where C is an arbitrary constant. Hence the group-invariant solution of (2.1) under X_3 is given by

$$u(t, x) = \frac{C - x}{t}. \tag{2.41}$$

Case 4. We consider the symmetry operator

$$X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}.$$

The assosiated characteristic equations to X_4 are

$$\frac{dt}{2t} = \frac{dx}{x} = \frac{du}{-u}.$$

The equations

$$\frac{dt}{2t} = \frac{dx}{x} \quad \text{and} \quad \frac{dt}{2t} = \frac{du}{-u}, \tag{2.42}$$

respectively, yield the two invariants $J_1 = x/\sqrt{t}$ and $J_2 = \sqrt{t}u$. Hence the group-invariant solution is given by $J_2 = f(J_1)$, where f is an arbitrary function. This implies

$$u(t,x) = \frac{1}{\sqrt{t}} f(\lambda), \quad \lambda = \frac{x}{\sqrt{t}}. \tag{2.43}$$

Substituting this value of u into (2.1) we obtain

$$f'' + ff' + \frac{1}{2}(\lambda f' + f) = 0. \tag{2.44}$$

Equations (2.44) can also be written as

$$(f')' + \frac{1}{2}(f^2)' + \frac{1}{2}(\lambda f)' = 0 \tag{2.45}$$

and integrating it once gives

$$f' + \frac{1}{2}f^2 + \frac{1}{2}\lambda f = C, \tag{2.46}$$

where C is an arbitrary constant of integration. Here if $C = 0$, (2.46) reduces to a Bernoulli's equation for f and when solved we obtain

$$f(\lambda) = \frac{2}{\sqrt{\pi}} \left[\frac{e^{\frac{-\lambda^2}{4}}}{A + erf\left(\frac{\lambda}{2}\right)} \right], \tag{2.47}$$

where A is an arbitrary constant and

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2} ds \tag{2.48}$$

is the error function. Thus the group-invariant solution for (2.1) under X_4 is

$$u(t,x) = \frac{2}{\sqrt{\pi t}} \left(\frac{e^{\frac{-x^2}{4t}}}{A + erf\left(\frac{x}{2\sqrt{t}}\right)} \right). \tag{2.49}$$

Case 5. Finally we consider the symmetry operator

$$X_5 = t^2\frac{\partial}{\partial t} + xt\frac{\partial}{\partial x} - (x+tu)\frac{\partial}{\partial u},$$

whose characteristic equations are

$$\frac{dt}{t^2} = \frac{dx}{xt} = \frac{du}{-(x+tu)}.$$

From

$$\frac{dt}{t^2} = \frac{dx}{xt}$$

we get the first invariant as $J_1 = x/t$. Now from

$$\frac{dx}{t^2} = \frac{du}{-(x+tu)},$$

we obtain the second invariant as $J_2 = x + ut$. Hence the group-invariant solution is given by $J_2 = f(J_1)$, where f is an arbitrary function. This implies

$$u(t,x) = \frac{1}{t}f\left(\frac{x}{t}\right) - \frac{x}{t}.$$

Substituting the above value of u into (2.1) gives

$$f'' + ff' = 0. \tag{2.50}$$

Solving (2.50) we get

$$f\left(\frac{x}{t}\right) = A\coth\left\{\frac{Ax}{2t} + B\right\},$$

where A and B are arbitrary constants. Thus the group-invariant solutions of (2.1) under X_5 is

$$u(t,x) = \frac{A}{t}\coth\left\{\frac{Ax}{2t} + B\right\} - \frac{x}{t}.$$

2.3 Conservation laws of the Burgers equation

In this section we employ the multiplier method to derive the conservation laws of the Burgers equation (2.1). From (1.43), we get Euler-Lagrange operator as

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_x^2\frac{\partial}{\partial u_{xx}}\cdots. \tag{2.51}$$

We look for the zeroth-order multiplier $\Lambda = \Lambda(t, x, u)$. The determining equation for the multiplier is given by

$$\frac{\delta}{\delta u}\left[\Lambda(t, x, u)\left\{u_t - uu_x - u_{xx}\right\}\right] = 0. \tag{2.52}$$

Expanding equation (2.52) gives

$$\Lambda_u(u_t - uu_x - u_{xx}) - D_t(\Lambda) - \Lambda u_x + D_x(u\Lambda) - D_x^2(\Lambda) = 0. \tag{2.53}$$

Applying the total derivatives (1.4) to equation (2.53) gives

$$\Lambda_u - \Lambda_u u_{xx} - \Lambda_t + \Lambda_x u - \Lambda_{xx} - 2\Lambda_{xu}u_x - \Lambda_{uu}u_x^2 = 0.$$

Splitting the above equation on the derivatives of u we obtain

$$u_{xx} \;:\; \Lambda_u = 0, \tag{2.54}$$

$$u_x^2 \;:\; \Lambda_{uu} = 0, \tag{2.55}$$

$$u_x \;:\; \Lambda_{xu} = 0, \tag{2.56}$$

$$\text{rest} \;:\; u\Lambda_x - \Lambda_t - \Lambda_{xx} + \Lambda_u = 0. \tag{2.57}$$

Equations (2.55) and (2.56) are already satisfied by equation (2.54), thus integrating (2.54) gives

$$\Lambda = A(t, x), \tag{2.58}$$

where $A(t, x)$ is an arbitrary function of t and x. Now substituting this value of Λ into equation (2.57) we get

$$A_x u - A_t - A_{xx} = 0.$$

Since A is independent of u therefore we can split the above equation on powers of u and get

$$u \;:\; A_x = 0, \tag{2.59}$$

$$u^0 \;:\; A_t + A_{xx} = 0. \tag{2.60}$$

Equation (2.59) implies that $A = A(t)$. Thus substituting this value of A into equation (2.60) we get

$$A'(t) = 0. \tag{2.61}$$

Integrating equation (2.61) gives $A(t) = C_1$, where C_1 is an arbitrary constant of integration. Thus our multiplier is given by $\Lambda = C_1$. A multiplier Λ for Burgers equation (2.1) has the property that

$$\Lambda(u_t - uu_x - u_{xx}) = D_t T^t + D_x T^x, \tag{2.62}$$

where $T^t = T^t(t, x, u, u_x)$ and $T^x = T^x(t, x, u, u_x)$. Here since our multiplier $\Lambda = C_1$, we take $C_1 = 1$ and solve equation (2.62). Expanding equation (2.62) we have

$$u_t - uu_x - u_{xx} = T^t_t + T^t_u u_t + T^t_{u_x} u_{tx} + T^x_x + T^x_u u_x + T^x_{u_x} u_{xx}.$$

Splitting the above equation on second derivatives of u we obtain

$$u_{tx} \ : \ T^t_{u_x} = 0, \tag{2.63}$$

$$u_{xx} \ : \ T^x_{u_x} = -1, \tag{2.64}$$

$$\text{Rest} \ : \ u_t - uu_x = T^t_t + T^t_u u_t + T^x_x + T^x_u u_x. \tag{2.65}$$

Equation (2.63) implies that

$$T^t = A(t, x, u),$$

where A is an arbitrary function of its arguments. From equation (2.64) we get

$$T^x = -u_x + B(t, x, u),$$

where B is an arbitrary function of its arguments. Substituting the above values of T^t and T^x into equation (2.65) we get

$$u_t - uu_x = A_t + A_u u_t + B_x + B_u u_x \tag{2.66}$$

Splitting equation (2.66) on derivatives of u yieds

$$u_t : A_u = 1, \tag{2.67}$$

$$u_x : B_u = -u, \tag{2.68}$$

$$\text{rest} : A_t + B_x = 0. \tag{2.69}$$

Equation (2.67) implies that

$$A = u + C(t, x),$$

where C is an arbitrary function of t and x. Equation (2.68) gives

$$B = -\frac{1}{2}u^2 + D(t, x),$$

where D is an arbitrary function of its arguments. Substituting the values of A and B into equation (2.69) gives $C_t + D_x = 0$, thus we take C and D as zero because they contribute to the trivial part of the conservation law. Thus the conservation law of the Burgers equation (2.1) is given by

$$\begin{aligned} T^t &= u, \\ T^x &= -u_x - \frac{1}{2}u^2. \end{aligned}$$

2.4 Concluding remarks

In this chapter we presented an illustration of what this research project will be about by constructing Lie point symmetries of the second-order Burgers equation (2.1). We then used the obtained symmetries to find group-invariant solutions. Finally conservation laws of this equation were derived using the multiplier approach.

Chapter 3

Solutions and conservation laws of the equal-width equation

In this chapter we study a third-order equal-width equation by first computing its Lie point symmetries and then performing symmetry reductions. Exact solutions are obtained using Kudryashov's and extended Jacobi elliptic expansion methods. Furthermore conservation laws for this equation will be derived using both the multiplier and Noether's methods.

3.1 Introduction

An equal-width equation is given by

$$u_t + 2\alpha u u_x - \beta u_{txx} = 0,\ \ \alpha \neq 0,\ \ \beta \neq 0, \tag{3.1}$$

where α is the nonlinearity parameter and β is the dispersion parameter. This equation was first introduced by Morrison et al. [38] and is used as a model equation that describes nonlinear dispersive waves, e.g., waves generated in a shallow water channel. Several methods have been used to derive solutions of this equation, for instance the authors in [39] used a Petrov-Galerkin method using quadratic B-spline finite element. In [41] the authors used least-squares technique to construct numerical solutions of this equation.

3.2 Solutions of the equal-width equation (3.1)

In this section we present Lie point symmetries, optimal systems and symmetry reductions of equation (3.1). Moreover we obtain travelling wave solution of (3.1) by employing extended Jacobi elliptic expansion and Kudryashov's method.

3.2.1 Lie point symmetries of (3.1)

Here we construct Lie point symmetries for the equal-width equation (3.1). The vector field (1.20) for this equation (3.1) is written as

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}. \tag{3.2}$$

We recall that (3.2) is a Lie point symmetry of (3.1) if

$$\mathrm{X}^{[3]}F|_{F=0} = 0, \tag{3.3}$$

where

$$F \equiv u_t + 2\alpha u u_x - \beta u_{txx} = 0.$$

Here $\mathrm{X}^{[3]}$ is the third prolongation [12] of (3.2) defined by

$$\mathrm{X}^{[3]} = X + \zeta_1\frac{\partial}{\partial u_t} + \zeta_2\frac{\partial}{\partial u_x} + \zeta_{122}\frac{\partial}{\partial u_{txx}} \tag{3.4}$$

and ζ_1, ζ_2 and ζ_{122} are determined as follows:

$$\begin{aligned}
\zeta_1 &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi),\\
\zeta_2 &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi),\\
\zeta_{12} &= D_x(\zeta_1) - u_{tt} D_x(\tau) - u_{tx} D_x(\xi),\\
\zeta_{122} &= D_x(\zeta_{12}) - u_{ttx} D_x(\tau) - u_{txx} D_x(\xi),
\end{aligned}$$

where the total derivatives D_t and D_x are given by

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + \cdots ,$$
$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_t} + u_{xx}\frac{\partial}{\partial u_x} + \cdots .$$

From equation (3.3) we obtain

$$\left[\tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \zeta_1\frac{\partial}{\partial u_t} + \zeta_2\frac{\partial}{\partial u_x} + \zeta_{122}\frac{\partial}{\partial u_{txx}}\right]\left(u_t + 2\alpha u u_x - \beta u_{txx}\right)\Big|_{(3.1)} = 0,$$

which on expansion gives

$$\zeta_1 + 2\alpha u\zeta_2 + 2\alpha u_x\eta - \beta\zeta_{122}\Big|_{(3.1)} = 0.$$

Subtitution of values of ζ_1, ζ_2 and ζ_{122} in the above equation gives

$$\begin{aligned}
&\eta_t - u_t\eta_u - u_t\tau_u - u_t^2\tau_u - u_x\xi_t - u_tu_x\xi_u + 2\alpha\eta_x + 2\alpha u_x\eta_u - 2\alpha u_t\tau_x - 2\alpha u_tu_x\tau_u \\
&- 2\alpha u_x\xi_x - 2\alpha u_x^2\xi_u + 2\alpha u_x\eta - \beta\left[\eta_{txx} - 4u_{tx}u_tu_x\tau_{uu} - 3u_{xx}u_tu_x\xi_{uu} - u_xu_{txx}\tau_u\right. \\
&-3u_xu_{xx}\xi_{tu} - 3u_{tx}u_x^2\xi_{uu} - 2u_{tx}u_x\tau_{tu} - 3u_{tx}u_{xx}\xi_u - 2u_tu_{txx}\tau_u - u_{tt}u_{xx}\tau_u \\
&-4u_{tx}u_t\tau_{xu} - u_{ttx}u_x\tau_u - 2u_tu_{xx}\xi_{xu} - u_tu_{xx}\tau_{tu} + u_tu_{xx}\eta_{uu} + u_tu_x\eta_{xuu} + u_tu_x^2\eta_{uuu} \\
&+u_x^2\eta_{tuu} - u_{tx}\xi_{xx} - u_{txx}\tau_x - u_{xxx}\xi_x - 2u_{tx}\tau_{tx} - 2u_{tx}^2\tau_uu_{xxx}\xi_t - 2u_{xx}\xi_{tx} - u_{ttx}\tau_x \\
&+2u_{tx}\eta_{xu} - u_{txx}\tau_t + u_{txx}\eta_u + u_{xx}\eta_{tu} - u_{tt}\tau_{xx} - u_t\tau_{txx} - u_t^2\tau_{xxu} - u_x\xi_{txx} - u_{txx}\xi_x \\
&-2u_x^2\xi_{txu} - u_tu_x^3\xi_{uuu} - u_t^2u_{xx}\tau_{uu} - 4u_xu_{tx}\xi_{xu} - u_tu_{xxx}\xi_u - 2u_xu_{txx}\xi_u + 2u_xu_{tx}\eta_{uu} \\
&-2u_{tt}u_x\tau_{xu} - u_{tt}u_x^2\tau_{uu} - u_xu_{xxx}\xi_u - 2u_tu_x^2\xi_{xuu} - u_tu_x\xi_{xxu} + u_t\eta_{xxu} + 2u_x\eta_{txu} \\
&\left.-u_x^3\xi_{tuu} - u_t^2u_x^2\tau_{uuu} - 2u_t^2u_x\tau_{xuu} - u_tu_x^2\tau_{tuu} - 2u_tu_x\tau_{txu}\right]|_{(3.1)} = 0.
\end{aligned}$$

Replacing u_{txx} by $\frac{1}{\beta}(u_t + 2\alpha uu_x)$ in the above equation yields

$$\begin{aligned}
&\eta_t - u_t\eta_u - u_t\tau_u - u_t^2\tau_u - u_x\xi_t - u_tu_x\xi_u + 2\alpha u_x\eta_u - 2\alpha u_t\tau_x - 2\alpha u_tu_x\tau_u - 2\alpha u_x\xi_x \\
&- 2\alpha u_x^2\xi_u + 2\alpha u_x\eta + 2\alpha\eta_x - \beta\eta_{txx} + 4\beta u_{tx}u_tu_x\tau_{uu} + 3\beta u_{xx}u_tu_x\xi_{uu} + \beta u_xu_{txx}\tau_u \\
&+ 3\beta u_xu_{xx}\xi_{tu} + 3\beta u_{tx}u_x^2\xi_{uu} + 2\beta u_{tx}u_x\tau_{tu} + 3\beta u_{tx}u_{xx}\xi_u + 2\beta u_tu_{txx}\tau_u + \beta u_{tt}u_{xx}\tau_u \\
&+ 4\beta u_{tx}u_t\tau_{xu} + \beta u_{ttx}u_x\tau_u + 2\beta u_tu_{xx}\xi_{xu} + \beta u_tu_{xx}\tau_{tu} - \beta u_tu_{xx}\eta_{uu} - \beta u_tu_x\eta_{xuu} \\
&- \beta u_tu_x^2\eta_{uuu} - \beta u_x^2\eta_{tuu} + \beta u_{tx}\xi_{xx} + \beta u_{txx}\tau_x + \beta u_{xxx}\xi_x + 2\beta u_{tx}\tau_{tx} + 2\beta u_{tx}^2\tau_uu_{xxx}\xi_t \\
&+ 2\beta u_{xx}\xi_{tx} + \beta u_{ttx}\tau_x - 2\beta u_{tx}\eta_{xu} + \beta u_{txx}\tau_t - \beta u_{txx}\eta_u - \beta u_{xx}\eta_{tu} + \beta u_{tt}\tau_{xx} + \beta u_t\tau_{txx}
\end{aligned}$$

$$+\beta u_t^2\tau_{xxu} + \beta u_x\xi_{txx} + \beta u_{txx}\xi_x + 2\beta u_x^2\xi_{txu} + \beta u_t u_x^3\xi_{uuu} + \beta u_t^2 u_{xx}\tau_{uu} + 4\beta u_x u_{tx}\xi_{xu}$$

$$+ u_t u_{xxx}\xi_u + 2\beta u_x u_{txx}\xi_u - 2\beta u_x u_{tx}\eta_{uu} + 2\beta u_{tt}u_x\tau_{xu} + \beta u_{tt}u_x^2\tau_{uu} + \beta u_x u_{xxx}\xi_u$$

$$+ 2\beta u_t u_x^2\xi_{xuu} + \beta u_t u_x\xi_{xxu} - \beta u_t\eta_{xxu} - 2\beta u_x\eta_{txu} + \beta u_x^3\xi_{tuu} + \beta u_t^2 u_x^2\tau_{uuu}$$

$$+ 2\beta u_t^2 u_x\tau_{xuu} + \beta u_t u_x^2\tau_{tuu} + 2\beta u_t u_x\tau_{txu} = 0.$$

Splitting the above equation on derivatives of u gives the following over determined system of linear partial differential equations:

$$\tau_x = 0, \tag{3.5}$$

$$\tau_u = 0, \tag{3.6}$$

$$\xi_u = 0, \tag{3.7}$$

$$\eta_{uu} = 0, \tag{3.8}$$

$$\xi_x - \xi_t = 0, \tag{3.9}$$

$$\xi_x - \beta\eta_{xxu} = 0, \tag{3.10}$$

$$2\xi_{tx} - \eta_{tu} = 0, \tag{3.11}$$

$$\xi_{xx} - 2\eta_{xu} = 0, \tag{3.12}$$

$$2\alpha\eta + 2\alpha u\tau_t + \beta\xi_{xxt} - \xi_t - 2\beta\eta_{txu} = 0, \tag{3.13}$$

$$\eta_t - \beta\eta_{txx} + 2\alpha u\eta_x = 0. \tag{3.14}$$

Equations (3.5) and (3.6) imply that

$$\tau = A(t), \tag{3.15}$$

where $A(t)$ is an arbitrary function of t. Integrating equation (3.7) gives

$$\xi = B(t,x), \tag{3.16}$$

where $B(t,x)$ is an arbitrary function of t and x. Solving equation (3.8) yields

$$\eta = C(t,x)u + D(t,x),$$

where $C(t,x)$ and $D(t,x)$ are arbitrary functions of t and x. Subtituting the above values of τ, ξ and η into equation (3.13) we get

$$2\alpha C(t,x)u + 2\alpha D(t,x) + 2\alpha A'(t)u + \beta B_{txx}(t,x) - B_t(t,x) - 2\beta C_{tx}(t,x) = 0. \tag{3.17}$$

Splitting equation (3.17) on u gives

$$u \quad : \quad C(t,x) + A'(t) = 0, \tag{3.18}$$

$$u^0 \quad : \quad 2\alpha D + \beta B_{txx} - B_t - 2\beta C_{tx} = 0. \tag{3.19}$$

From equation (3.18) we see that $C(t,x) = -A'(t)$ and this means that

$$\eta = -A'(t)u + D(t,x).$$

Subtituting the values of τ and η into equation (3.10) we get

$$B_x = 0,$$

which implies that $B = B(t)$, hence

$$\xi = B(t). \tag{3.20}$$

From (3.9) we get $B'(t) = 0$. This gives $B(t) = C_1$, where C_1 is an arbitrary constant of integration. Thus

$$\xi = C_1.$$

Equation (3.19) now simplifies to

$$D(t,x) = 0.$$

Hence

$$\eta = -A'(t)u.$$

Equation (3.11) gives

$$A''(t) = 0,$$

which on integration yields

$$A(t) = C_2 t + C_3,$$

where C_2 and C_3 are arbitrary constants of integration. Thus

$$\tau = C_2 t + C_3,$$

$$\xi = C_1,$$
$$\eta = -C_2 u.$$

It can easily be verified that equations (3.12) and (3.14) are safisfied by the above values of τ, ξ, and η. Hence equation (3.1) has the following three Lie point symmetry generators:

$$X_1 = \frac{\partial}{\partial t}$$
$$X_2 = \frac{\partial}{\partial x}$$
$$X_3 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}.$$

We note that the first two symmetries represent translations in t and x respectively, whereas the third decribes a scaling symmetry.

3.2.2 Optimal system of (3.1)

In this section we construct an optimal system of one-dimensional subalgebras. The table of commutators of the Lie point symmetries for (3.1) and the adjoint representations of the symmetry group of (3.1) on its Lie algebra are presented in Table 1 and Table 2, respectively. Consequently, Table 1 and Table 2 are used to compute an optimal system of one-dimensional subalgebras for equation (3.1).

Table 1. Lie brackets for equation (3.1)

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	X_1
X_2	0	0	0
X_3	$-X_1$	0	0

The entries of the adjoint representation are computed as follows:

$$\mathrm{Ad}(\exp(\varepsilon X_i))X_j = X_j - \varepsilon[X_i, X_j] + \frac{\varepsilon^2}{2}[X_i, [X_i, X_j]] - \cdots .$$

Letting $i = 1$ and $j = 1$ by definition

$$\mathrm{Ad}(\exp(\varepsilon X_1))X_1 \equiv X_1 - \varepsilon[X_1, X_1] + \frac{\varepsilon^2}{2}[X_1, [X_1, X_1]] - \cdots = X_1,$$

this implies that $\mathrm{Ad}(\exp(\varepsilon X_2))X_2 = X_2,\ \mathrm{Ad}(\exp(\varepsilon X_3))X_3 = X_3$. Computing $\mathrm{Ad}(\exp(\varepsilon X_1))X_3$, we obtain

$$\begin{aligned}\mathrm{Ad}(\exp(\varepsilon X_1))X_3 &= X_3 - \varepsilon[X_1, X_3] + \frac{\varepsilon^2}{2}[X_1, [X_1, X_3]] - \cdots \\ &= X_3 - \varepsilon X_1 + \frac{\varepsilon^2}{2}(0) - \cdots, \\ &= X_3 - \varepsilon X_1.\end{aligned}$$

Similarly

$$\begin{aligned}\mathrm{Ad}(\exp(\varepsilon X_3))X_1 &= X_1 - \varepsilon[X_3, X_1] + \frac{\varepsilon^2}{2}[X_3, [X_3, X_1]] - \cdots \\ &= X_1 + \varepsilon X_1 + \frac{\varepsilon^2}{2}[X_3, -X_1] - \cdots, \\ &= X_1 + \varepsilon X_1 - \frac{\varepsilon^2}{2}X_1 - \cdots \\ &= e^{\varepsilon}X_1.\end{aligned}$$

Table 2. Adjoint representation of subalgebras

Ad	X_1	X_2	X_3
X_1	X_1	X_2	$-\varepsilon X_1 + X_3$
X_2	X_1	X_2	X_3
X_3	$e^{\varepsilon}X_1$	X_2	X_3

Thus following [12] and utilising Tables 1 and 2 we can obtain an optimal system of one-dimensional subalgebras, which is given by $\{X_2, X_3 + aX_2, X_1 + cX_2\}$, where c and a are arbitrary constants.

3.2.3 Symmetry reductions and solutions

We now utilise the optimal system of one-dimensional subalgebras obtained above in the previous subsection and find symmetry reductions and group-invariant solutions for equation (3.1).

Case 1. We start with the operator X_2. This operator has the invariants

$$J_1 = t, \;\; J_2 = u.$$

Thus its group-invariant solution is given by $u = \phi(t)$. Subtituting this value into equation (3.1) yields $\phi'(t) = 0$, whose solution is $\phi(t) = C_1$, where C_1 is an arbitrary constant of integration. Hence the group-invariant solution corresponding to this operator is given by

$$u(t, x) = C_1.$$

Case 2. We consider the operator $X_3 + aX_2$. This operator has the characteristic equations

$$\frac{dt}{t} = \frac{dx}{a} = \frac{du}{-u}.$$

The above characteristic equations gives the invariants $J_1 = x - a\ln(t)$ and $J_2 = tu$. This has the group-invariant solution

$$u = \frac{1}{t} F(x - a\ln(t)). \tag{3.21}$$

Substituting the above into (3.1) yields

$$a\beta F'''(z) + \beta F''(z) + 2\alpha F(z)F'(z) - aF'(z) - F(z) = 0, \quad \text{where} \quad z = x - a\ln(t).$$

Case 3. For the operator $X_1 + cX_2$ of the optimal system, we get two invariants

$$\xi = x - ct \;\text{ and }\; U = u, \tag{3.22}$$

which give the group-invariant solution $U = U(\xi)$. Using ξ as our new independent variable, equation (3.1) is transformed into the nonlinear ordinary differential equation (ODE)

$$c\beta U'''(\xi) + 2\alpha U(\xi)U'(\xi) - cU'(\xi) = 0. \tag{3.23}$$

W now solve equation (3.23) by using two methods; namely Kudryashov's and the extended Jacobi elliptic function expansion method.

Solution of (3.23) using Kudryashov's method

Here we note that (3.22) represent the travelling wave variables, thus we solve equation (3.23) using Kudryashov's technique which has been fully outlined in Chapter one. To utilize Kudryashov's method we begin by assuming the solution of equation (3.23) to be of the form

$$U(z) = \sum_{i=0}^{M} B_i \psi^i(z), \tag{3.24}$$

where $\psi(z)$ satisfies the Riccati equation

$$\psi'(z) = \psi^2(z) - \psi(z), \tag{3.25}$$

whose solution is

$$\psi(z) = \frac{1}{1+e^z}. \tag{3.26}$$

From the balancing we get $M = 2$, thus the solution (3.24) can be written as

$$U(z) = B_0 + B_1\psi(z) + B_2\psi^2(z). \tag{3.27}$$

Substituting the value of $U(z)$ from equation (3.27) into equation (3.23) and using (3.25) we obtain the following equation in $\psi(z)$:

$$\begin{aligned}
&2\alpha B_0B_1\psi^2(z) - 2\alpha B_0B_1\psi(z) + 4\alpha B_0B_2\psi^3(z) - 4\alpha B_0B_2\psi^2(z) + 2\alpha B_1^2\psi^3(z)\\
&+ 6\alpha B_1B_2\psi^4(z) - 6\alpha B_1B_2\psi^3(z) + 4\alpha B_2^2\psi^5(z) - 2\alpha B_2^2\psi^4(z) - cB_1\psi^2(z)\\
&+ cB_1\psi(z) - 2cB_2\psi^3(z) + 2cB_2\psi^2(z) + 6\beta cB_1\psi^4(z) - 12\beta cB_1\psi^3(z)\\
&+ 7\beta cB_1\psi^2(z) - \beta cB_1\psi(z) + 24\beta cB_2\psi^5(z) - 54\beta cB_2\psi^4(z)\\
&+ 38\beta cB_2\psi^3(z) - 8\beta cB_1\psi^2(z) = 0.
\end{aligned} \tag{3.28}$$

Splitting equation (3.28) on powers of ψ gives

$$\psi^5(z):\ 4\alpha B_2^2 + 24\beta cB_2 = 0, \tag{3.29}$$

$$\psi^4(z):\ 6\alpha B_1B_2 - 4\alpha B_2^2 + 6\beta cB_1 - 54\beta cB_2 = 0, \tag{3.30}$$

$$\psi^3(z):\ 4\alpha B_0B_2 - 2cB_2 + 2\alpha B_1^2 - 6\alpha B_1B_2 - 12\ \beta cB_1 + 38\beta cB_2 = 0, \tag{3.31}$$

$$\psi^2(z):\ 2cB_2 - cB_1 + 2\alpha B_0B_1 - 4\alpha B_0B_2 - 2\alpha B_1^2 + 7\beta cB_1 - 8\beta cB_2 = 0, \tag{3.32}$$

$$\psi(z): \ cB_1 - 2\alpha B_0 B_1 - \beta c B_1 = 0. \tag{3.33}$$

Solving equation (3.29) we get

$$B_2 = -\frac{6\beta c}{\alpha}.$$

Assuming $B_1 \neq 0$, equation (3.33) implies that

$$B_0 = \frac{c}{2\alpha} - \frac{\beta c}{2\alpha}.$$

Substituting the above obtained value of B_2 into equation (3.30) and solving for B_1 we get

$$B_1 = \frac{6\beta c}{\alpha}.$$

Equations (3.31) and (3.32) are satisfied by the above values of B_0, B_1 and B_2. Thus the solution is obtained as

$$u(t,x) = \frac{c}{2\alpha} - \frac{\beta c}{2\alpha} + \frac{6\beta c}{\alpha(1+\exp{(x-ct)})} - \frac{6\beta c}{\alpha(1+\exp{(x-ct)})^2}.$$

Solutions of (3.23) using extended Jacobi elliptic function expansion method

We now use the extended Jacobi elliptic function expansion method [40] to obtain exact solutions of (3.23).

We assume that the solutions of the third-order NLODE (3.23) can be expressed in the form

$$U(\xi) = \sum_{i=-M}^{M} A_i H(\xi)^i, \tag{3.34}$$

where M is a positive integer obtained by the balancing procedure. Here $H(\xi)$ satisfies the first-order ODE

$$H'(\xi) = -\sqrt{(1-H^2(\xi))(1-\omega+\omega H^2(\xi))} \tag{3.35}$$

or

$$H'(\xi) = \sqrt{(1-H^2(\xi))(1-\omega H^2(\xi))}. \tag{3.36}$$

We recall that

$$H(\xi) = \text{cn}(\xi|\omega), \tag{3.37}$$

the Jacobi cosine-amplitude function, is a solution to (3.35), whereas the Jacobi sine-amplitude function

$$H(\xi) = \text{sn}(\xi|w) \tag{3.38}$$

is a solution to (3.36). Here ω is a parameter such that $0 \le \omega \le 1$ [26, 27].

We note that when $\omega \to 1$, then $\text{cn}(\xi|\omega) \to \text{sech}(\xi)$ and $\text{sn}(\xi|\omega) \to \tanh(\xi)$. Also, when $\omega \to 0$, then $\text{cn}(\xi|\omega) \to \cos(\xi)$ and $\text{sn}(\xi|\omega) \to \sin(\xi)$.

Cnoidal wave solutions

Considering the NLODE (3.23), the balancing procedure yields $M = 2$, thus (3.34) is

$$U(\xi) = A_{-2}H^{-2}(\xi) + A_{-1}H^{-1}(\xi) + A_0 + A_1 H(\xi) + A_2 H^2(\xi). \tag{3.39}$$

We now substitute the value of U from (3.39) into (3.23) and utilise (3.35) to obtain

$$\begin{aligned}
&4\,H(\xi)^{10}\alpha\,{A_2}^2 - 48\,\beta\,c\omega\,A_{-2} - 4\,\alpha\,\omega\,{A_{-2}}^2 + 2\,H(\xi)^8\alpha\,{A_1}^2 - 4\,H(\xi)^8\alpha\,{A_2}^2\\
&- 2\,H(\xi)^8 cA_2 - H(\xi)^7 cA_1 - 2\,H(\xi)^6\alpha\,{A_1}^2 + 2\,H(\xi)^6 cA_2 + H(\xi)^5 cA_{-1} + H(\xi)^5 cA_1\\
&- 2\,H(\xi)^4\alpha\,{A_{-1}}^2 + 2\,H(\xi)^4 cA_{-2} - H(\xi)^3 cA_{-1} - 4\,H(\xi)^2\alpha\,{A_{-2}}^2 + 2\,H(\xi)^2\alpha\,{A_{-1}}^2\\
&- 2\,H(\xi)^2 cA_{-2} + 8\,H(\xi)^4\alpha\,\omega\,A_{-2}A_0 + 12\,H(\xi)^3\alpha\,\omega\,A_{-2}A_{-1} - 2\,H(\xi)^3\alpha\,\omega\,A_{-2}A_1\\
&- 2\,H(\xi)^3\alpha\,\omega\,A_{-1}A_0 - 4\,H(\xi)^2\alpha\,\omega\,A_{-2}A_0 + 6\,H(\xi)\,\beta\,c\omega^2 A_{-1} - 6\,H(\xi)\,\alpha\,\omega\,A_{-2}A_{-1}\\
&- 12\,H(\xi)\,\beta\,c\omega\,A_{-1} + 6\,H(\xi)^{11}\alpha\,\omega\,A_1A_2 + 4\,H(\xi)^{10}\alpha\,\omega\,A_0A_2 + 2\,H(\xi)^9\alpha\,\omega\,A_{-1}A_2\\
&+ 2\,H(\xi)^9\alpha\,\omega\,A_0A_1 - 12\,H(\xi)^9\alpha\,\omega\,A_1A_2 - 8\,H(\xi)^8\alpha\,\omega\,A_0A_2 - 2\,H(\xi)^7\alpha\,\omega\,A_{-2}A_1\\
&- 2\,H(\xi)^7\alpha\,\omega\,A_{-1}A_0 - 4\,H(\xi)^7\alpha\,\omega\,A_{-1}A_2 - 4\,H(\xi)^7\alpha\,\omega\,A_0A_1 + 6\,H(\xi)^7\alpha\,\omega\,A_1A_2\\
&- 4\,H(\xi)^6\alpha\,\omega\,A_{-2}A_0 + 4\,H(\xi)^6\alpha\,\omega\,A_0A_2 - 6\,H(\xi)^5\alpha\,\omega\,A_{-2}A_{-1} + 4\,H(\xi)^5\alpha\,\omega\,A_{-2}A_1\\
&+ 4\,H(\xi)^5\alpha\,\omega\,A_{-1}A_0 + 2\,H(\xi)^5\alpha\,\omega\,A_{-1}A_2 + 2\,H(\xi)^5\alpha\,\omega\,A_0A_1 + 24\,\beta\,cA_{-2}\\
&+ 2\,H(\xi)^2 c\omega\,A_{-2} + 6\,H(\xi)\,\alpha\,A_{-2}A_{-1} + 6\,H(\xi)\,\beta\,cA_{-1} + 24\,\beta\,c\omega^2 A_{-2} + 4\,H(\xi)^{12}\alpha\,\omega\,{A_2}^2
\end{aligned}$$

$$+ 2\,H(\xi)^{10}\alpha\,\omega\,{A_1}^2 - 8\,H(\xi)^{10}\alpha\,\omega\,{A_2}^2 - 2\,H(\xi)^{10}c\omega\,A_2 + 6\,H(\xi)^9\alpha\,A_1A_2 - H(\xi)^9c\omega\,A_1$$

$$- 4\,H(\xi)^8\alpha\,\omega\,{A_1}^2 + 4\,H(\xi)^8\alpha\,\omega\,{A_2}^2 + 4\,H(\xi)^8\alpha\,A_0A_2 + 4\,H(\xi)^8c\omega\,A_2 + 2\,H(\xi)^7\alpha\,A_{-1}A_2$$

$$+ 2\,H(\xi)^7\alpha\,A_0A_1 - 6\,H(\xi)^7\alpha\,A_1A_2 + H(\xi)^7c\omega\,A_{-1} + 2\,H(\xi)^7c\omega\,A_1 - 2\,H(\xi)^6\alpha\,\omega\,{A_{-1}}^2$$

$$+ 2\,H(\xi)^6\alpha\,\omega\,{A_1}^2 - 4\,H(\xi)^6\alpha\,A_0A_2 + 2\,H(\xi)^6c\omega\,A_{-2} - 2\,H(\xi)^6c\omega\,A_2 - 2\,\left(H\left(\xi\right)\right)^5\alpha\,A_{-2}A_1$$

$$- 2\,H(\xi)^5\alpha\,A_{-1}A_0 - 2\,H(\xi)^5\alpha\,A_{-1}A_2 - 2\,H(\xi)^5\alpha\,A_0A_1 - 2\,H(\xi)^5c\omega\,A_{-1} - H(\xi)^5c\omega\,A_1$$

$$- 4\,H(\xi)^4\alpha\,\omega\,{A_{-2}}^2 + 4\,H(\xi)^4\alpha\,\omega\,{A_{-1}}^2 - 4\,H(\xi)^4\alpha\,A_{-2}A_0 - 4\,H(\xi)^4c\omega\,A_{-2}$$

$$+ 2\,H(\xi)^3\alpha\,A_{-2}A_1 + 2\,H(\xi)^3\alpha\,A_{-1}A_0 + H(\xi)^3c\omega\,A_{-1} + 8\,H(\xi)^2\alpha\,\omega\,{A_{-2}}^2$$

$$- 2\,H(\xi)^2\alpha\,\omega\,{A_{-1}}^2 + 4\,H(\xi)^2\alpha\,A_{-2}A_0 + 4\,\alpha\,{A_{-2}}^2 - 8\,\left(H\left(\xi\right)\right)^8\beta\,cA_2 - H(\xi)^7\beta\,cA_1$$

$$+ H(\xi)^5\beta\,cA_{-1} + H(\xi)^5\beta\,cA_1 + 8\,H(\xi)^4\beta\,cA_{-2} - 7\,H(\xi)^3\beta\,cA_{-1} - 32\,H(\xi)^2\beta\,cA_{-2}$$

$$- 7\,H(\xi)^9\beta\,c\omega\,A_1 - 56\,H(\xi)^8\beta\,c\omega^2A_2 + 56\,H(\xi)^8\beta\,c\omega\,A_2 - 2\,\left(H\left(\xi\right)\right)^7\beta\,c\omega^2A_{-1}$$

$$- 10\,H(\xi)^7\beta\,c\omega^2A_1 + H(\xi)^7\beta\,c\omega\,A_{-1} + 10\,H(\xi)^7\beta\,c\omega\,A_1 - 16\,H(\xi)^6\beta\,c\omega^2A_{-2}$$

$$+ 16\,H(\xi)^6\beta\,c\omega^2A_2 + 8\,H(\xi)^6\beta\,c\omega\,A_{-2} - 24\,H(\xi)^6\beta\,c\omega\,A_2 + 10\,H(\xi)^5\beta\,c\omega^2A_{-1}$$

$$+ 2\,H(\xi)^5\beta\,c\omega^2A_1 - 10\,H(\xi)^5\beta\,c\omega\,A_{-1} - 3\,H(\xi)^5\beta\,c\omega\,A_1 + 56\,H(\xi)^4\beta\,c\omega^2A_{-2}$$

$$- 56\,H(\xi)^4\beta\,c\omega\,A_{-2} - 14\,H(\xi)^3\beta\,c\omega^2A_{-1} + 21\,H(\xi)^3\beta\,c\omega\,A_{-1} - 64\,H(\xi)^2\beta\,c\omega^2A_{-2}$$

$$+ 96\,H(\xi)^2\beta\,c\omega\,A_{-2} - 24\,H(\xi)^{12}\beta\,c\omega^2A_2 - 6\,H(\xi)^{11}\beta\,c\omega^2A_1 + 64\,H(\xi)^{10}\beta\,c\omega^2A_2$$

$$- 32\,H(\xi)^{10}\beta\,c\omega\,A_2 + 14\,H(\xi)^9\beta\,c\omega^2A_1 + 8\,H(\xi)^6\beta\,cA_2 - 6\,H(\xi)^3\alpha\,A_{-2}A_{-1} = 0.$$

The above equation can be separated on like powers of $H(\xi)$ to obtain the overdetermined system of thirteen algebraic equations

$$\alpha\,\omega\,{A_2}^2 - 6\,\beta\,c\omega^2A_2 = 0,$$

$$\alpha\,\omega\,A_1A_2 - \beta\,c\omega^2A_1 = 0,$$

$$6\,\beta\,c\omega^2A_{-2} - \alpha\,\omega\,{A_{-2}}^2 - 12\,\beta\,c\omega\,A_{-2} + \alpha\,{A_{-2}}^2 + 6\,\beta\,cA_{-2} = 0,$$

$$\beta\,c\omega^2A_{-1} - \alpha\,\omega\,A_{-2}A_{-1} - 2\,\beta\,c\omega\,A_{-1} + \alpha\,A_{-2}A_{-1} + \beta\,cA_{-1} = 0,$$

$$32\,\beta\,c\omega^2A_2 + 2\,\alpha\,\omega\,A_0A_2 + \alpha\,\omega\,{A_1}^2 - 4\,\alpha\,\omega\,{A_2}^2 - 16\,\beta\,c\omega\,A_2 + 2\,\alpha\,{A_2}^2 - c\omega\,A_2 = 0,$$

$$14\,\beta\,c\omega^2A_1 + 2\,\alpha\,\omega\,A_{-1}A_2 + 2\,\alpha\,\omega\,A_0A_1 - 12\,\alpha\,\omega\,A_1A_2 - 7\,\beta\,c\omega\,A_1 + 6\,\alpha\,A_1A_2$$

$$- c\omega\,A_1 = 0,$$

$$28\,\beta\,c\omega^2A_{-2} - 2\,\alpha\,\omega\,{A_{-2}}^2 + 4\,\alpha\,\omega\,A_{-2}A_0 + 2\,\alpha\,\omega\,{A_{-1}}^2 - 28\,\beta\,c\omega\,A_{-2} - 2\,\alpha\,A_{-2}A_0$$

$$- \alpha A_{-1}{}^2 + 4\beta cA_{-2} - 2c\omega A_{-2} + cA_{-2} = 0,$$

$$4\alpha\omega A_{-2}{}^2 - 32\beta c\omega^2 A_{-2} - 2\alpha\omega A_{-2}A_0 - \alpha\omega A_{-1}{}^2 + 48\beta c\omega A_{-2} - 2\alpha A_{-2}{}^2$$

$$+ 2\alpha A_{-2}A_0 + \alpha A_{-1}{}^2 - 16\beta cA_{-2} + c\omega A_{-2} - cA_{-2} = 0,$$

$$\alpha A_1{}^2 - 28\beta c\omega^2 A_2 - 4\alpha\omega A_0A_2 - 2\alpha\omega A_1{}^2 + 2\alpha\omega A_2{}^2 + 28\beta c\omega A_2 + 2\alpha A_0A_2$$

$$- 2\alpha A_2{}^2 - 4\beta cA_2 + 2c\omega A_2 - cA_2 = 0,$$

$$21\beta c\omega A_{-1} - 14\beta c\omega^2 A_{-1} + 12\alpha\omega A_{-2}A_{-1} - 2\alpha\omega A_{-2}A_1 - 2\alpha\omega A_{-1}A_0$$

$$- 6\alpha A_{-2}A_{-1} + 2\alpha A_{-2}A_1 + 2\alpha A_{-1}A_0 - 7\beta cA_{-1} + c\omega A_{-1} - cA_{-1} = 0,$$

$$\alpha\omega A_1{}^2 - 8\beta c\omega^2 A_{-2} + 8\beta c\omega^2 A_2 - 2\alpha\omega A_{-2}A_0 - \alpha\omega A_{-1}{}^2 + 2\alpha\omega A_0A_2$$

$$+ 4\beta c\omega A_{-2} - 12\beta c\omega A_2 - 2\alpha A_0A_2 - \alpha A_1{}^2 + 4\beta cA_2 + c\omega A_{-2} - c\omega A_2 + cA_2 = 0,$$

$$\beta c\omega A_{-1} - 2\beta c\omega^2 A_{-1} - 10\beta c\omega^2 A_1 - 2\alpha\omega A_{-2}A_1 - 2\alpha\omega A_{-1}A_0 - 4\alpha\omega A_{-1}A_2$$

$$- 4\alpha\omega A_0A_1 + 6\alpha\omega A_1A_2 + 10\beta c\omega A_1 + 2\alpha A_{-1}A_2 + 2\alpha A_0A_1 - 6\alpha A_1A_2 - \beta cA_1$$

$$+ c\omega A_{-1} + 2c\omega A_1 - cA_1 = 0,$$

$$10\beta c\omega^2 A_{-1} + 2\beta c\omega^2 A_1 - 6\alpha\omega A_{-2}A_{-1} + 4\alpha\omega A_{-2}A_1 + 4\alpha\omega A_{-1}A_0 + 2\alpha\omega A_{-1}A_2$$

$$+ 2\alpha\omega A_0A_1 - 10\beta c\omega A_{-1} - 3\beta c\omega A_1 - 2\alpha A_{-2}A_1 - 2\alpha A_{-1}A_0 - 2\alpha A_{-1}A_2$$

$$- 2\alpha A_0A_1 + \beta cA_{-1} + \beta cA_1 - 2c\omega A_{-1} - c\omega A_1 + cA_{-1} + cA_1 = 0.$$

Solving the above system we get two cases, namely

Case 1

$$A_{-2} = 0, \quad A_{-1} = 0, \quad A_0 = -\frac{c\,(8\beta\omega - 4\beta - 1)}{2\alpha}, \; A_1 = 0, A_2 = \frac{6\beta c\omega}{\alpha}. \tag{3.40}$$

Case 2

$$A_{-2} = \frac{6\beta c\,(\omega - 1)}{\alpha}, \quad A_{-1} = 0, \quad A_0 = -\frac{c\,(8\beta\omega - 4\beta - 1)}{2\alpha},$$
$$A_1 = 0, \quad A_2 = \frac{6\beta c\omega}{\alpha}. \tag{3.41}$$

Consequently we obtain two cnoidal wave solutions of the equal-width equation (3.1) as

$$u(t,x) = \frac{c\,(1 + 4\beta - 8\beta\omega)}{2\alpha} + \frac{6\beta c\omega}{\alpha}\mathrm{cn}^2(\xi\,|\omega)$$

and

$$u(t,x) = \frac{6\beta c\,(\omega - 1)}{\alpha}\mathrm{nc}^2(\xi\,|\omega) - \frac{c\,(8\beta\omega - 4\beta - 1)}{2\alpha} + \frac{6\beta c\omega}{\alpha}\mathrm{cn}^2(\xi\,|\omega)$$

with $0 \leq \omega \leq 1$.

Snoidal wave solutions

In this subsection we obtain snoidal wave solutions for the equation (3.1). We recall that the balancing procedure yields $M = 2$, thus substituting the value of U from (3.39) into (3.23) and making use of (3.36) we obtain the determining equation

$$8\,H(\xi)^8\beta\,cA_2 + H\,(\xi)^7\,\beta\,cA_1 - 8\,H\,(\xi)^6\,\beta\,cA_2 - H\,(\xi)^5\,\beta\,cA_{-1} - H(\xi)^5\beta\,cA_1$$
$$-\,8\,H(\xi)^4\beta\,cA_{-2} + 7\,H(\xi)^3\beta\,cA_{-1} + 32\,H(\xi)^2\beta\,cA_{-2} + 4\,H(\xi)^2\alpha\,\omega\,A_{-2}{}^2$$
$$-\,4\,H(\xi)^2\alpha\,A_{-2}A_0 - 6\,H\,(\xi)\,\alpha\,A_{-2}A_{-1} - 6\,H\,(\xi)\,\beta\,cA_{-1} + 4\,H(\xi)^{12}\alpha\,\omega\,A_2{}^2$$
$$+\,2\,H\,(\xi)^{10}\,\alpha\,\omega\,A_1{}^2 - 4\,H\,(\xi)^{10}\,\alpha\,\omega\,A_2{}^2 - 2\,H\,(\xi)^{10}\,c\omega\,A_2 - 6\,H\,(\xi)^9\,\alpha\,A_1A_2$$
$$-\,H\,(\xi)^9\,c\omega\,A_1 - 2\,H\,(\xi)^8\,\alpha\,\omega\,A_1{}^2 - 4\,H\,(\xi)^8\,\alpha\,A_0A_2 + 2\,H\,(\xi)^8\,c\omega\,A_2$$
$$-\,2\,H\,(\xi)^7\,\alpha\,A_{-1}A_2 - 2\,H\,(\xi)^7\,\alpha\,A_0A_1 + 6\,H\,(\xi)^7\,\alpha\,A_1A_2 + H\,(\xi)^7\,c\omega\,A_{-1}$$
$$+\,H\,(\xi)^7\,c\omega\,A_1 - 2\,H\,(\xi)^6\,\alpha\,\omega\,A_{-1}{}^2 + 4\,H\,(\xi)^6\,\alpha\,A_0A_2 + 2\,H\,(\xi)^6\,c\omega\,A_{-2}$$
$$+\,2\,H\,(\xi)^5\,\alpha\,A_{-2}A_1 + 2\,H\,(\xi)^5\,\alpha\,A_{-1}A_0 + 2\,H\,(\xi)^5\,\alpha\,A_{-1}A_2 + 2\,H\,(\xi)^5\,\alpha\,A_0A_1$$
$$-\,H\,(\xi)^5\,c\omega\,A_{-1} - 4\,H\,(\xi)^4\,\alpha\,\omega\,A_{-2}{}^2 + 2\,H\,(\xi)^4\,\alpha\,\omega\,A_{-1}{}^2 + 4\,H(\xi)^4\alpha\,A_{-2}A_0$$
$$-\,2\,H(\xi)^4c\omega\,A_{-2} + 6\,H(\xi)^3\alpha\,A_{-2}A_{-1} - 2\,H(\xi)^3\alpha\,A_{-2}A_1 - 2\,H(\xi)^3\alpha\,A_{-1}A_0 - 4\,\alpha\,A_{-2}{}^2$$
$$-\,4\,H(\xi)^{10}\alpha\,A_2{}^2 2\,H(\xi)^8\alpha\,A_1{}^2 + 4\,H(\xi)^8\alpha\,A_2{}^2 + 2\,H(\xi)^8cA_2 + H(\xi)^7cA_1$$
$$+\,2\,H\,(\xi)^6\,\alpha\,A_1{}^2 2\,H\,(\xi)^6\,cA_2 - H\,(\xi)^5\,cA_{-1} - H(\xi)^5cA_1 + 2\,H(\xi)^4\alpha\,A_{-1}{}^2$$
$$-\,2\,H(\xi)^4cA_{-2} + H(\xi)^3cA_{-1} + 4\,H\,(\xi)^2\,\alpha\,A_{-2}{}^2 - 2\,H\,(\xi)^2\,\alpha\,A_{-1}{}^2 + 2\,H\,(\xi)^2\,cA_{-2}$$
$$+\,6\,H\,(\xi)^{11}\,\alpha\,\omega\,A_1A_2 + 4\,H\,(\xi)^{10}\,\alpha\,\omega\,A_0A_2 + 2\,H(\xi)^9\alpha\,\omega\,A_{-1}A_2 + 2\,H(\xi)^9\alpha\,\omega\,A_0A_1$$
$$-\,6\,H(\xi)^9\alpha\,\omega\,A_1A_2 - 4\,H(\xi)^8\alpha\,\omega\,A_0A_2 - 2\,H(\xi)^7\alpha\,\omega\,A_{-2}A_1 - 2\,H\,(\xi)^7\,\alpha\,\omega\,A_{-1}A_0$$
$$-\,2\,H\,(\xi)^7\,\alpha\,\omega\,A_{-1}A_2 - 2\,H\,(\xi)^7\,\alpha\,\omega\,A_0A_1 - 4\,H(\xi)^6\alpha\,\omega\,A_{-2}A_0 - 6\,H(\xi)^5\alpha\,\omega\,A_{-2}A_{-1}$$
$$+\,2\,H(\xi)^5\alpha\,\omega\,A_{-2}A_1 + 2\,H(\xi)^5\alpha\,\omega\,A_{-1}A_0 + 4\,H(\xi)^4\alpha\,\omega\,A_{-2}A_0 + 6\,H\,(\xi)^3\,\alpha\,\omega\,A_{-2}A_{-1}$$
$$-\,24\,\beta\,cA_{-2} + H(\xi)^7\beta\,c\omega^2A_1 + H(\xi)^7\beta\,c\omega\,A_{-1} + 8\,H(\xi)^7\beta\,c\omega\,A_1 + 8\,H(\xi)^6\beta\,c\omega^2A_{-2}$$
$$+\,8\,H(\xi)^6\beta\,c\omega\,A_{-2} - 8\,H(\xi)^6\beta\,c\omega\,A_2 - H(\xi)^5\beta\,c\omega^2A_{-1} - 8\,H(\xi)^5\beta\,c\omega\,A_{-1}$$
$$-\,8\,H(\xi)^4\beta\,c\omega^2A_{-2} - 40\,H(\xi)^4\beta\,c\omega\,A_{-2} + 7\,H(\xi)^3\beta\,c\omega\,A_{-1} + 32\,H(\xi)^2\beta\,c\omega\,A_{-2}$$
$$+\,24\,H(\xi)^{12}\beta\,c\omega^2A_2 + 6\,H(\xi)^{11}\beta\,c\omega^2A_1 - 32\,H(\xi)^{10}\beta\,c\omega^2A_2 - 32\,H(\xi)^{10}\beta\,c\omega\,A_2$$

$$- 7\,H(\xi)^9\beta\,c\omega^2 A_1 - 7\,H(\xi)^9\beta\,c\omega\,A_1 + 8\,H(\xi)^8\beta\,c\omega^2 A_2 + 40\,H\,(\xi)^8\,\beta\,c\omega\,A_2$$

$$+ H\,(\xi)^7\,\beta\,c\omega^2 A_{-1} - H(\xi)^5\beta\,c\omega\,A_1 = 0,$$

which splits into thirteen algebraic equations

$$\alpha {A_{-2}}^2 + 6\beta c A_{-2} = 0,$$

$$6\beta c\omega^2 A_2 + \alpha\omega {A_2}^2 = 0,$$

$$\alpha A_{-2}A_{-1} + \beta c A_{-1} = 0,$$

$$\beta c\omega^2 A_1 + \alpha\omega A_1 A_2 = 0,$$

$$2\alpha\omega {A_{-2}}^2 + 16\beta c\omega A_{-2} + 2\alpha {A_{-2}}^2 - 2\alpha A_{-2}A_0 - \alpha {A_{-1}}^2$$

$$+ 16\beta c A_{-2} + cA_{-2} = 0,$$

$$2\alpha\omega A_0 A_2 - 16\beta c\omega^2 A_2 + \alpha\omega {A_1}^2 - 2\alpha\omega {A_2}^2 - 16\beta c\omega A_2 - 2\alpha {A_2}^2 - c\omega A_2 = 0,$$

$$6\alpha\omega A_{-2}A_{-1} + 7\beta c\omega A_{-1} + 6\alpha A_{-2}A_{-1} - 2\alpha A_{-2}A_1 - 2\alpha A_{-1}A_0 + 7\beta c A_{-1} + cA_{-1} = 0,$$

$$2\alpha\omega A_{-1}A_2 - 7\beta c\omega^2 A_1 + 2\alpha\omega A_0 A_1 - 6\alpha\omega A_1 A_2 - 7\beta c\omega A_1 - 6\alpha A_1 A_2 - c\omega A_1 = 0,$$

$$2\alpha\omega A_{-2}A_0 - 4\beta c\omega^2 A_{-2} - 2\alpha\omega {A_{-2}}^2 + \alpha\omega {A_{-1}}^2 - 20\beta c\omega A_{-2} + 2\alpha A_{-2}A_0 + \alpha {A_{-1}}^2$$

$$- 4\beta c A_{-2} - c\omega A_{-2} - cA_{-2} = 0,$$

$$4\beta c\omega^2 A_2 - 2\alpha\omega A_0 A_2 - \alpha\omega {A_1}^2 + 20\beta c\omega A_2 - 2\alpha A_0 A_2 - \alpha {A_1}^2 + 2\alpha {A_2}^2 + 4\beta c A_2$$

$$+ c\omega A_2 + cA_2 = 0,$$

$$4\beta c\omega^2 A_{-2} - \alpha\omega A_{-2}A_0 - \alpha\omega {A_{-1}}^2 + 2\beta c\omega A_{-2} - 2\beta c\omega A_2 + \alpha A_0 A_2$$

$$+ \alpha {A_1}^2 - 2\beta c A_2 + c\omega A_{-2} - cA_2 = 0,$$

$$2\alpha\omega A_{-2}A_1 - \beta c\omega^2 A_{-1} - 6\alpha\omega A_{-2}A_{-1} + 2\alpha\omega A_{-1}A_0 - 8\beta c\omega A_{-1} - \beta c\omega A_1 + 2\alpha A_{-2}A_1$$

$$+ 2\alpha A_{-1}A_0 + 2\alpha A_{-1}A_2 + 2\alpha A_0 A_1 - \beta c A_{-1} - \beta c A_1 - c\omega A_{-1} - cA_{-1} - cA_1 = 0,$$

$$\beta c\omega^2 A_{-1} + \beta c\omega^2 A_1 - 2\alpha\omega A_{-2}A_1 - 2\alpha\omega A_{-1}A_0 - 2\alpha\omega A_{-1}A_2 - 2\alpha\omega A_0 A_1 + \beta c\omega A_{-1}$$

$$+ 8\beta c\omega A_1 - 2\alpha A_{-1}A_2 - 2\alpha A_0 A_1 + 6\alpha A_1 A_2 + \beta c A_1 + c\omega A_{-1} + c\omega A_1 + cA_1 = 0.$$

Solving the above equations for A_{-2}, A_{-1}, A_0, A_1 and A_2 yields the following two cases:

Case1

$$A_{-2} = 0,\ \ A_{-1} = 0,\ \ A_0 = \frac{c}{2\alpha}\left(4\,\beta\,\omega + 4\,\beta + 1\right),\ \ A_1 = 0,\ \ A_2 = -\frac{6\beta\,c\omega}{\alpha}. \quad (3.42)$$

Case 2

$$A_{-2} = -\frac{6\beta\, c}{\alpha}, \;\; A_{-1} = 0, \;\; A_0 = \frac{c}{2\alpha}\left(4\,\beta\,\omega + 4\,\beta + 1\right), \;\; A_1 = 0, \;\; A_2 = -\frac{6\beta\, c\omega}{\alpha}. \tag{3.43}$$

Thus we obtain two snoidal wave solutions of the equal-width equation (3.1) as

$$u(t,x) = \frac{c\left\{1 + 4\,\beta + 4\,\beta\,\omega(1 - 3\,\mathrm{sn}^2\left(\xi\,|\omega\,\right))\right\}}{2\alpha}, \tag{3.44}$$

and

$$u(t,x) = -\frac{6\beta\, c}{\alpha}\mathrm{ns}^2\left(\xi\,|\omega\,\right) + \frac{c\left(4\,\beta\,\omega + 4\,\beta + 1\right)}{2\alpha} - \frac{6\beta\, c\omega}{\alpha}\mathrm{sn}^2\left(\xi\,|\omega\,\right) \tag{3.45}$$

with $0 \leq \omega \leq 1$.

3.3 Conservation laws of (3.1)

In this section we construct conservation laws of the equal-width equation (3.1) by using two different approaches, namely, the multiplier method and Noether's approach.

3.3.1 Conservation laws of (3.1) using the multiplier approach

We apply the algorithm described in section 1.8.1 to seek zeroth-order multiplier $Q = Q(t,x,u)$. The determining equation for the multiplier is given by

$$\frac{\delta}{\delta u}\left[Q(t,x,u)\left\{u_t + 2\alpha u u_x - \beta u_{txx}\right\}\right] = 0, \tag{3.46}$$

where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} - D_t D_x^2\frac{\partial}{\partial u_{txx}}\cdots$$

is the Euler-Lagrange operator. Equation (3.46) yields

$$u_t Q_u + 2\alpha u u_x Q_u - \beta u_{txx} Q_u + 2\alpha u_x Q - D_t(Q) - D_x(2\alpha u Q) - D_x^2 D_t(-\beta Q) = 0.$$

Applying the total derivatives D_t and D_x on the above equation gives

$$\beta Q_{txx} - Q_t - 2\alpha u Q_x + 2\beta u_x Q_{txu} + \beta u_x^2 Q_{tuu} + \beta u_{xx} Q_{tu} + \beta u_t Q_{xxu} + 2\beta u_t u_x Q_{xuu}$$
$$+ 2\beta u_{tx} Q_{xu} - 2 + \beta u_t u_x^2 Q_{uu} + \beta u_t u_{xx} Q_{uu} + 2\beta u_x u_{tx} Q_{uu} = 0.$$

Splitting the above equation on derivatives of u yields the following simplified determining equations:

$$Q_{tu} = 0, \tag{3.47}$$

$$Q_{xu} = 0, \tag{3.48}$$

$$Q_{uu} = 0, \tag{3.49}$$

$$Q_t + 2\alpha u Q_x - \beta Q_{txx} = 0. \tag{3.50}$$

Integrating equation (3.49) gives

$$Q = A(t, x)u + B(t, x),$$

where $A(t, x)$ and $B(t, x)$ are arbitrary functions of t and x. Substituting this value of Q into (3.47) we obtain

$$A_t = 0,$$

which implies that $A = A(x)$. Thus

$$Q = A(x)u + B(t, x).$$

Using equation (3.48) we obtain

$$A'(x) = 0. \tag{3.51}$$

Integrating equation (3.51) gives $A(x) = C_1$, where C_1 is an arbitrary constant of integration. Thus

$$Q = C_1 u + B(t, x).$$

Substituting the value of Q into equation (3.50) we have

$$B_t + 2\alpha u B_x - \beta B_{txx} = 0.$$

Since B is independent of u, the above equation can split on u to give

$$u : B_x = 0, \tag{3.52}$$

$$u^0 : B_t - \beta B_{txx} = 0. \tag{3.53}$$

Equation (3.52) implies that

$$B = B(t). \tag{3.54}$$

Substituting equation (3.54) into equation (3.53) gives $B'(t) = 0$ and integrating this gives

$$B(t) = C_2, \tag{3.55}$$

where C_2 is an arbitrary constant of integration. Thus the multiplier is given by

$$Q = C_1 u + C_2,$$

which is a linear combination of two nontrivial conservation law multipliers $Q_1 = 1$ and $Q_2 = u$. We now recall and apply the property of a multiplier given in (1.41), that is,

$$Q(t, x, u) \{u_t + 2\alpha u u_x - \beta u_{txx}\} = D_t T^t + D_x T^x, \tag{3.56}$$

where $T^t = T^t(t, x, u, u_x)$ and $T^x = T^x(t, x, u, u_x, u_{tx})$.

Case 1. For the first multiplier $Q_1 = 1$, we expand (3.56) and get

$$u_t + 2\alpha u u_x - \beta u_{txx} = T^t_t + T^t_u u_t + T^t_{u_x} u_{tx} + T^x_x + T^x_u u_x + T^x_{u_x} u_{xx} + T^x_{u_{tx}} u_{txx}.$$

Splitting the above equation on third derivatives of u yields

$$u_{txx} : T^x_{u_{tx}} = -\beta, \tag{3.57}$$

$$\text{Rest} : u_t + 2\alpha u u_x = T^t_t + T^t_u u_t + T^t_{u_x} u_{tx} + T^x_x + T^x_u u_x + T^x_{u_x} u_{xx}. \tag{3.58}$$

Equation (3.57) implies that

$$T^x = -\beta u_{tx} + A(t, x, u, u_x),$$

where A is an arbitrary function of its arguments. Substituting this value of T^x into equation (3.58) we get

$$u_t + 2\alpha u u_x = T^t_t + T^t_u u_x t + T^t_{u_x} u_{tx} + A_x + A_u u_x + A_{u_x} u_{xx},$$

The above equation is split on second derivatives of u, to give

$$u_{tx} : T^t_{u_x} = 0, \tag{3.59}$$

$$u_{xx} : A_{u_x} = 0, \tag{3.60}$$

$$\text{Rest} : u_t + 2\alpha u u_x = T^t_t + T^t_u u_t + A_x + A_u u_x. \tag{3.61}$$

Equations (3.60) implies that

$$A = A(t, x, u).$$

From equation (3.59) we get

$$T^t = B(t, x, u),$$

where B is an arbitrary function of t, x and u. Substituting the value of T^t into (3.61) gives

$$u_t + 2\alpha u u_x = B_t + B_u u_t + A_x + A_u u_x. \tag{3.62}$$

Splitting equation (3.62) on the derivatives of u yields

$$u_t : B_u = 1, \tag{3.63}$$

$$u_x : A_u = 2\alpha u, \tag{3.64}$$

$$\text{Rest} : A_x + B_t = 0. \tag{3.65}$$

Integrating equation (3.63) we obtain

$$B = u + C(t, x),$$

where C is an arbitrary function of t and x. Equation (3.64) gives

$$A = \alpha u^2 + D(t, x), \tag{3.66}$$

where D is an arbitrary function of t and x. Substituting the values A and B into equation (3.65) we get $D_x + C_t = 0$. Since C and D contribute to the trivial part of the conservation law, we therefore take $C = D = 0$. Thus the conservation law corresponding to this multiplier is given by

$$T^t = u,$$
$$T^x = \alpha u^2 - \beta u_{tx}.$$

Case 2. Now we consider the multiplier $Q_2 = u$ and construct the corresponding conservation law. For this multiplier equation (3.56) gives

$$u\left\{u_t + 2\alpha u u_x - \beta u_{txx}\right\} = D_t T^t + D_x T^x,$$

where $T^t = T^t(t, x, u, u_x)$ and $T^x = T^x(t, x, u, u_x, u_{tx})$.

Expanding the above equation we get

$$uu_t + 2\alpha u^2 u_x - \beta u u_{txx} = T^t_t + T^t_u u_t + T^t_{u_x} u_{tx} + T^x_x + T^x_u u_x + T^x_{u_x} u_{xx} + T^x_{u_{tx}} u_{txx}. \tag{3.67}$$

Splitting equation (3.67) on third derivatives of u gives

$$u_{txx} : T^x_{u_{tx}} = -\beta u, \tag{3.68}$$
$$\text{Rest} : uu_t + 2\alpha u^2 u_x = T^t_t + T^t_u u_t + T^t_{u_x} u_{tx} + T^x_x + T^x_u u_x + T^x_{u_x} u_{xx}. \tag{3.69}$$

Integrating equation (3.68) yields

$$T^x = -\beta u u_{tx} + A(t, x, u, u_x),$$

where A is an arbitrary function of its arguments. Substituting the above value of T^x into equation (3.69) we have

$$uu_t + 2\alpha u^2 u_x = T^t_t + T^t_u u_t + T^t_{u_x} u_{tx} + A_x + A_u u_x - \beta u_x u_{tx} + A_{u_x} u_{xx}.$$

Splitting the above equation on second derivatives of u gives

$$u_{tx} : T^t_{u_x} = \beta u_x, \tag{3.70}$$

$$u_{xx} : A_{u_x} = 0, \tag{3.71}$$

$$\text{Rest} : uu_t + 2\alpha u^2 u_x = T^t_t + T^t_u u_t + A_x + A_u u_x. \tag{3.72}$$

Equation (3.71) implies that $A = A(t, x, u)$. From (3.70) we get

$$T^t = \frac{1}{2}\beta u_x^2 + B(t, x, u),$$

where B is an arbitrary function of its arguments. Substituting the above value of T^t into equation (3.72) gives

$$uu_t + 2\alpha u^2 u_x = B_t + B_u u_t + A_x + A_u u_x.$$

Splitting the above equation on derivatives of u we get

$$u_t : B_u = u, \tag{3.73}$$

$$u_x : A_u = 2\alpha u^2, \tag{3.74}$$

$$\text{Rest} : B_t + A_x = 0. \tag{3.75}$$

Equation (3.73) gives

$$B = \frac{1}{2}u^2 + C(t, x), \tag{3.76}$$

where C is an arbitrary function of t and x. Integrating equation (3.74) yields

$$A = \frac{2}{3}\alpha u^3 + D(t, x),$$

where D is an arbitrary function of t and x. Substituting the values of A and B into equation (3.75) gives $C_t + D_x = 0$. We take note that C and D contribute to the trivial part of the conservation law thus we take them to be zero. Hence the conservation law corresponding to this multiplier is given by

$$\begin{aligned} T^t &= \frac{1}{2}\beta u_x^2 + \frac{1}{2}u^2, \\ T^x &= \frac{2}{3}\alpha u^3 - \beta u u_{tx}. \end{aligned}$$

3.3.2 Conservation Laws of (3.1) using Noether's approach

In this subsection we utilize Noether's approach to derive consevation laws for the equal-width equation (3.1). This equation is of third order and as such does not have a Lagrangian. To make it variational we increase its order by letting $u = v_x$. Thus we have

$$v_{tx} + 2\alpha v_x v_{xx} - \beta v_{txxx} = 0. \tag{3.77}$$

It can be verified that equation (3.77) has a second-order Lagrangian $\mathcal{L}$ given by

$$\mathcal{L} = -\frac{1}{2}v_t v_x - \frac{1}{3}\alpha v_x^3 - \frac{1}{2}\beta v_{tx} v_{xx},$$

as

$$\frac{\delta \mathcal{L}}{\delta v} = 0$$

on the equation. Here the Euler-Lagrange operator is given by

$$\frac{\delta}{\delta v} = \frac{\partial}{\partial v} - D_t \frac{\partial}{\partial v_t} - D_x \frac{\partial}{\partial v_x} + D_t D_x \frac{\partial}{\partial v_{tx}} + D_x^2 \frac{\partial}{\partial v_{xx}} + \cdots .$$

The determining equation for Noether point symmetries is given by

$$X^{[2]}(\mathcal{L}) + \mathcal{L}\left[D_t(\tau) + D_x(\xi)\right] = D_t(B^1) + D_x(B^2), \tag{3.78}$$

where $B^1 = B^1(t,x,v)$ and $B^2 = B^2(t,x,v)$ are gauge terms and $X^{[2]}$ is the second prolongation of the infinitesimal generator

$$X = \tau(t,x,v)\frac{\partial}{\partial t} + \xi(t,x,v)\frac{\partial}{\partial x} + \eta(t,x,v)\frac{\partial}{\partial v}.$$

Hence expanding equation (3.78) with the Lagrangian $\mathcal{L}$ gives

$$-\frac{1}{2}v_x\left\{\eta_t + v_t\eta_v - v_t\tau_t - v_t^2\tau_v - v_x\xi_t - v_t v_x\xi_v\right\} - \frac{1}{2}v_t\left\{\eta_x + v_x\eta_v - v_t\tau_x - v_t v_x\tau_v\right.$$
$$\left. - v_x\xi_x - v_x^2\xi_v\right\} - \alpha v_x^2\left\{\eta_x + v_x\eta_v - v_t\tau_x - v_t v_x\tau_v - v_x\xi_x - v_x^2\xi_v\right\} - \frac{\beta}{2}v_{xx}\left\{\eta_{tx}\right.$$
$$+ v_x\eta_{tv} + v_t\eta_{xv} + v_{xt}\eta_v + v_t v_x\eta_{vv} - v_{xt}\tau_{tv} - v_{xt}\xi_{xv} - v_t^2\tau_{xv} - 2v_t v_{xt}\tau_v - v_x v_{tt}\tau_v$$
$$\left. - v_t^2 v_x\tau_{vv} - v_x\xi_{xt} - v_{xx}\xi_t - v_x^2\xi_{tv} - 2v_x v_{xt}\xi_v - v_t v_{xx}\xi_v - v_t v_x^2\xi_{vv}\right\} - \frac{1}{2}\beta v_{xt}\left\{\eta_{xx}\right.$$

$$+2v_x\eta_{xv} + v_{xx}\eta_v + v_x^2\eta_{vv} - 2v_{xx}\xi_x - v_x\xi_{xx} - 2v_x^2\xi_{xv} - 3v_xv_{xx}\xi_v - v_x^3\xi_{vv} - 2v_{xt}\tau_x$$
$$-v_t\tau_{xx} - 2v_tv_x\tau_{xv} - v_tv_{xx}\tau_v - 2v_xv_{xt}\tau_v - v_tv_x^2\tau_{vv}\} - B_t^1 - v_tB_v^1 - B_x^2 - v_xB_v^2 = 0.$$

Splitting the above equation on derivatives of v yields

$$\tau_t = 0, \tag{3.79}$$
$$\tau_x = 0, \tag{3.80}$$
$$\tau_v = 0, \tag{3.81}$$
$$\xi_x = 0, \tag{3.82}$$
$$\xi_t = 0, \tag{3.83}$$
$$\xi_v = 0, \tag{3.84}$$
$$\eta_x = 0, \tag{3.85}$$
$$\eta_v = 0, \tag{3.86}$$
$$\eta_x + 2B_v^1 = 0, \tag{3.87}$$
$$\eta_t + 2B_v^2 = 0, \tag{3.88}$$
$$B_x^1 + B_t^2 = 0. \tag{3.89}$$

Equations (3.79)-(3.81) imply that $\tau = C_1$, where C_1 is an arbitrary constant of integration. Similarly equations (3.82) to (3.84) imply that $\xi = C_2$, where C_2 is an arbitrary constant of integration. From equations (3.85) and (3.86) we get

$$\eta = f(t), \tag{3.90}$$

where $f(t)$ is an arbitrary function of t. Thus equation (3.88) results in

$$B^2 = -\frac{1}{2}f'(t)v + E(t,x), \tag{3.91}$$

where $E(t,x)$ is an arbitrary function of t and x. From (3.87) we get $B^1 = G(t,x)$, where $G(t,x)$ is an arbitrary function of t and x. Equation (3.89) gives $E_t + G_x = 0$, thus we take E and G to be zero as they contribute to the trivial part of the conserved vector. Thus Noether point symmetries and their gauge functions are given by

$$X_1 = \frac{\partial}{\partial t}, \quad B^1 = 0, \quad B^2 = 0,$$

$$X_2 = \frac{\partial}{\partial x}, \quad B^1 = 0, \quad B^2 = 0,$$
$$X_f = f(t)\frac{\partial}{\partial v}, \quad B^1 = 0, \quad B^2 = -\frac{1}{2}vf'(t).$$

We now construct the conservation laws corresponding to each Noether point symmetry using [36]

$$\begin{aligned}
T^t = &\ \mathcal{L}\tau + (\eta - u_t\tau - u_x\xi)\frac{\partial \mathcal{L}}{\partial u_t} + (\zeta_2 - u_{tx}\tau - u_{xx}\xi)\frac{\partial \mathcal{L}}{u_{tx}} - B^1, \\
T^x = &\ \mathcal{L}\xi + (\eta - u_t\tau - u_x\xi)\left(\frac{\partial \mathcal{L}}{\partial u_x} - D_t\left(\frac{\partial \mathcal{L}}{\partial u_{tx}}\right) - D_x\left(\frac{\partial \mathcal{L}}{\partial u_{xx}}\right)\right) \\
&+ (\zeta_2 - u_{tx}\tau - u_{xx}\xi)\frac{\partial \mathcal{L}}{u_{xx}} - B^2.
\end{aligned}$$

Case 1. The first Noether symmetry $X_1 = \partial/\partial t$ has the conserved vector

$$T_1^t = -\frac{1}{3}\alpha v_x^3,$$
$$T_1^x = \frac{1}{2}v_t^2 - \alpha v_t v_x^2 - \beta v_t v_{txx} + \frac{1}{2}v_{tx}^2.$$

Case 2. The second operator $X_2 = \partial/\partial x$ yields the consevation law

$$T_2^t = \frac{1}{2}v_x^2 + \frac{1}{2}\beta v_{xx}^2,$$
$$T_2^x = \frac{2}{3}\alpha v_x^3 - \beta v_x v_{txx}.$$

Case 3. Finally for the operator $X_f = f(t)\partial/\partial v$, we get the conserved vector

$$T_f^t = -\frac{1}{2}f(t)v_x,$$
$$T_f^x = \beta f(t)v_{txx} - \frac{1}{2}f(t)v_t - \alpha f(t)v_x^2 + \frac{1}{2}f'(t)u.$$

Reverting back to our original variable u, we have

$$T_1^t = -\frac{1}{3}\alpha u^3,$$
$$T_1^x = \frac{1}{2}\left(\int u_t dx\right)^2 + \alpha u^2\int u_t dx - \beta u_{tx}\int u_t dx + \frac{1}{2}\beta u_t^2;$$

$$T_2^t = \frac{1}{2}u^2 + \frac{1}{2}\beta u_x^2,$$
$$T_2^x = \frac{2}{3}\alpha u^3 - \beta u u_{tx};$$

$$
\begin{aligned}
T^t_f &= \; -\frac{1}{2}f(t)u,\\
T^x_f &= \beta f(t)\, u_{tx} - \frac{1}{2}f(t)\int u_t dx - \alpha f(t)u^2 + \frac{1}{2}f'(t)\int u dx.
\end{aligned}
$$

3.4 Concluding remarks

In this chapter we presented the Lie point symmetries, symmetry reductions and exact solutions of the equal-width equation (3.1) using Kudryashov's method and Jacobi elliptic function expansion method. We then derived the conservation laws for this equation by using the multiplier approach which produced two multipliers and consequently, two conservation laws. Finally, we employed the celebrated Noether's approach to construct conservation laws for the underlying equation.

Chapter 4

Solutions and conservation laws of Zakharov-Kuznetsov-Burgers equation

In this Chapter we present exact solutions of the Zakharov-Kuznetsov-Burgers equation using Kudryashov's method and derive conservation laws by employing the multiplier method.

4.1 Introduction

The (2+1)-dimensional Zakharov-Kuznetsov-Burgers equation is of the form [42]

$$u_t + u_x + uu_x - u_{xx} + u_{xxx} + u_{xyy} = 0. \tag{4.1}$$

This equation is a two dimensional analog of Korteweg-de Vries-Burgers equation which includes dissipation due to viscosity of a medium and dispersion. Several works have been done on this equation because of its applications in mechanics and physics. See, for example, [43].

4.2 Exact solutions of (4.1)

In this section Kudryashov's method is employed to get exact solutions of the Zakharov-Kuznetsov-Burgers equation (4.1). We use the travelling wave variable, namely, $z = \alpha x + \beta y - ct$ to reduce the $(2+1)-$dimensional Zakhorov-Kuznetsov-Burgers equation (4.1) to an ordinary differential equation. Thus we let

$$u(t, x, y) = U(z), \quad \text{where} \quad z = \alpha x + \beta y - ct. \tag{4.2}$$

The above substitution reduces equation (4.1) to a third-order nonlinear ordinary differential equation

$$\alpha(\beta^2 + \alpha^2)U''' - \alpha^2 U'' + \alpha U'U + (\alpha - c)U' = 0. \tag{4.3}$$

To utilize Kudryashov's method we firstly assume the solution of (4.3) to be of the form

$$U(z) = \sum_{i=0}^{M} B_i \psi^i(z), \tag{4.4}$$

where $\psi(z)$ satisfies the Riccati equation

$$\psi'(z) = \psi^2(z) - \psi(z), \tag{4.5}$$

whose solution is

$$\psi(z) = \frac{1}{1 + e^z}. \tag{4.6}$$

In this case the balancing yields $M = 2$, hence the solution (4.4) can be written as

$$U(z) = B_0 + B_1\psi(z) + B_2\psi^2(z). \tag{4.7}$$

Substituting the value of U from equation (4.7) into (4.3) and using (4.5) we obtain the following equation in $\psi(z)$:

$$\alpha B_0 B_1 \psi^2(z) - \alpha B_0 B_1 \psi(z) + 2\alpha B_0 B_2 \psi^3(z) - 2\alpha B_0 B_2 \psi^2(z) + \alpha B_1^2 \psi^3(z) - \alpha B_1^2 \psi^2(z)$$
$$+ 3\alpha B_1 B_2 \psi^4(z) - 3\alpha B_1 B_2 \psi^3(z) + 2\alpha B_2^2 \psi^5(z) - 2\alpha B_2^2 \psi^4(z) - cB_1\psi^2(z) + cB_1\psi(z)$$
$$- 2cB_2\psi^3(z) + 2cB_2\psi^2(z) + \alpha B_1\psi^2(z) - \alpha B_1\psi(z) + 2\alpha B_2\psi^3(z) - 2\alpha B_2\psi^2(z)$$
$$- 2\alpha^2 B_1\psi^3(z) + 3\alpha^2 B_1\psi^2(z) - \alpha B_2\psi^2(z) - \alpha^2 B_1\psi(z) - 6\alpha^2 B_2\psi^4(z)$$

$$+ 10\alpha^2 B_2\psi^3(z) - 4\alpha^2 B_2\psi^2(z) + 6\alpha^3 B_1\psi^4(z) - 12\alpha^3 B_1\psi^3(z) + 7\alpha^3 B_1\psi^2(z)$$
$$- \alpha^3 B_1\psi(z) + 24\alpha^3 B_2\psi^5(z) - 54\alpha^3 B_2\psi^4(z) + 38\alpha^3 B_2\psi^3(z) - 8\alpha^3 B_2\psi^2(z)$$
$$+ 6\alpha\beta^2 B_1\psi^4(z) - 12\alpha\beta^2 B_1\psi^3(z) + 7\alpha\beta^2 B_1\psi^2(z) - \alpha\beta^2 B_1\psi(z)$$
$$+ \alpha\beta^2 B_2\psi^5(z) - 54\alpha\beta^2 B_2\psi^4(z) + 38\alpha\beta^2 B_2\psi^3(z) - 8\alpha\beta^2 B_2\psi^2(z) = 0. \qquad (4.8)$$

Equating the coefficients of like powers of $\psi(z)$ in equation (4.8) we obtain the following five algebraic equations in terms of B_0, B_1 and B_2:

$$cB_1 - \alpha^3 B_1 - \alpha^2 B_1 - \alpha\beta^2 B_1 - \alpha B_0 B_1 - \alpha B_1 = 0,$$
$$7\alpha^3 B_1 - 8\alpha^3 B_2 + 3\alpha^2 B_1 - 4\alpha^2 B_2 + 7\alpha\beta^2 B_1 - 8\alpha\beta^2 B_2 - \alpha B_1^2 + \alpha B_0 B_1$$
$$+ \alpha B_1 - 2\alpha B_0 B_2 - 2\alpha B_2 - B_1 c + 2B_2 c = 0,$$
$$38\alpha^3 B_1 2 - 12\alpha^3 B_1 - 2\alpha^2 B_1 + 10\alpha^2 B_2 - 12\alpha\beta^2 B_1 + 38\alpha\beta^2 B_2 + \alpha B_1^2$$
$$+ 2\alpha B_0 B_2 - 3\alpha B_1 B_2 + 2\alpha B_2 - 2cB_2 = 0,$$
$$6\alpha^3 B_1 - 54\alpha^3 B_2 - 6\alpha^2 B_2 + 6\alpha\beta^2 B_1 - 54\alpha\beta^2 B_2 - 2\alpha B_2^2 + 3\alpha B_1 B_2 = 0,$$
$$24\alpha^3 B_2 + 24\alpha\beta^2 B_2 + 2\alpha B_2^2 = 0.$$

Solving the above system with the aid of Maple one gets

$$\beta = \frac{\sqrt{(1-5\alpha)\alpha}}{\sqrt{5}}, \quad B_0 = \frac{c}{\alpha} - \frac{6\alpha}{5} - 1, \quad B_1 = \frac{24\alpha}{5}, \quad B_2 = -\frac{12\alpha}{5}.$$

Thus the solution of (4.1) in the original variables is given by

$$u(t,x,y) = \frac{c}{\alpha} - \frac{6\alpha}{5} - 1 + \frac{24\alpha}{5(1+\exp(-ct+\alpha x+\frac{1}{5}\sqrt{5\alpha-25\alpha}\,y))} - \frac{12\alpha}{5(1+\exp(-ct+\alpha x+\frac{1}{5}\sqrt{5\alpha-25\alpha}\,y))^2}.$$

4.3 Conservation laws of (4.1)

In this section we construct conservation laws for the Zakharov-Kuznetsov-Burgers equation (4.1) using the multiplier approach. For this equation, the Euler-Lagrange operator is given by

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} - D_y\frac{\partial}{\partial u_y} + D_x^2\frac{\partial}{\partial u_{xx}} - D_x^3\frac{\partial}{\partial u_{xxx}} - D_y D_x^2\frac{\partial}{\partial u_{yxx}}, \quad (4.9)$$

where D_t, D_x and D_y are the total derivatives given by

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + u_{ty} \frac{\partial}{\partial u_y} \cdots , \\
D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x} + u_{xy} \frac{\partial}{\partial u_y} \cdots , \\
D_y &= \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{ty} \frac{\partial}{\partial u_t} + u_{xy} \frac{\partial}{\partial u_x} + u_{yy} \frac{\partial}{\partial u_y} \cdots .
\end{aligned} \tag{4.10}$$

We consider zeroth-order multipliers, i.e., $\Lambda = \Lambda(t, x, y, u)$ and proceed to derive the conservation laws of (4.1) using the multiplier approach. Recall that the zeroth-order multiplier Λ of the Zakharov-Kuznetsov-Burgers equation (4.1) is obtained from the determining equation

$$\frac{\delta}{\delta u} \left[\Lambda(t, x, y, u) \left(u_t + u_x + u u_x - u_{xx} + u_{xxx} + u_{xyy} \right) \right] = 0.$$

Expanding the above equation yields

$$\begin{aligned}
&\Lambda_u (u_t + u_x + u u_x - u_{xx} + u_{xxx} + u_{xyy}) + u_x \Lambda - D_t(\Lambda) - D_x(\Lambda) - D_x(u\Lambda) + D_x^2(-\Lambda) \\
&- D_x^3(\Lambda) - D_x D_y^2(\Lambda) = 0.
\end{aligned}$$

Using total derivatives (4.10) to expand the above equation we get

$$\begin{aligned}
&2u_{xx}\Lambda_u + \Lambda_t + \Lambda_{xx} + 2u_x\Lambda_{xu} + u_x^2\Lambda_{uu} + \Lambda_{xxx} + 3u_x\Lambda_{xxu} + 3u_x^2\Lambda_{xuu} + 3u_{xx}\Lambda_{xu} \\
&+ 3u_{xx}u_x\Lambda_{uu} + u_x^3\Lambda_{uuu} + u\Lambda_x + \Lambda_x + \Lambda_{xyy} + 2u_y\Lambda_{xyu} + u_y^2\Lambda_{xuu} + u_{yy}\Lambda_{xu} + u_x\Lambda_{yyu} \\
&+ 2u_xu_y\Lambda_{yuu} + 2u_{xy}\Lambda_{yu} + u_xu_y^2\Lambda_{uuu} + 2u_{xy}u_y\Lambda_{uu} + u_xu_{yy}\Lambda_{uu} = 0.
\end{aligned}$$

Splitting the above equation on the derivatives of u yields the following system of overdetermined equations:

$$\begin{aligned}
&2\Lambda_u + 3\Lambda_{xu} = 0, \quad \Lambda_{uuu} = 0, \quad \Lambda_{xuu} = 0, \quad 2\Lambda_{xyu}, \quad \Lambda_{xu} = 0, \quad \Lambda_{yu} = 0, \quad \Lambda_{yu} = 0 \\
&\Lambda_{uu} = 0, \quad \Lambda_{yuu} = 0, \quad 3\Lambda_{xxu} + 2\Lambda_{xu} + \Lambda_{yyu} = 0, \quad \Lambda_t + \Lambda_{xx} + u\Lambda_x + \Lambda_{xyy} = 0.
\end{aligned}$$

Solving the above system gives the multiplier of equation (4.1) as

$$\Lambda(t, x, y, u) = f(y), \tag{4.11}$$

where $f(y)$ is an arbitrary function of y.

A multiplier (4.11) of the Zakharov-Kuznetsov-Burgers equation (4.1) has the property that

$$\Lambda(u_t + u_x + uu_x - u_{xx} + u_{xxx} + u_{xyy}) = D_t(T^t) + D_x(T^x) + D_y(T^y), \tag{4.12}$$

where

$$\begin{aligned} T^t &= T^t(t, x, y, u, u_x), \\ T^x &= T^x(t, x, y, u, u_x, u_{xx}, u_{yy}), \\ T^y &= T^y(t, x, y, u, u_x). \end{aligned}$$

Expansion of equation (4.12) with the multiplier $\Lambda(t, x, y, u) = f(y)$ gives

$$\begin{aligned} &f(y)(u_t + u_x + uu_x - u_{xx} + u_{xxx} + u_{xyy}) - T^t_t - T^t_u u_t - T^t_{u_x} u_{tx} - T^x_x - T^x_u u_x \\ &- T^x_{u_x} u_{xx} - T^x_{u_{xx}} u_{xxx} - T^x_{u_{yy}} u_{xyy} - T^y_y - T^y_u u_y - T^y_{u_x} u_{xy} = 0. \end{aligned} \tag{4.13}$$

Splitting equation (4.13) on third derivatives of u yields

$$u_{xxx} : T^x_{u_{xx}} - f(y) = 0, \tag{4.14}$$

$$u_{xyy} : T^x_{u_{yy}} - f(y) = 0, \tag{4.15}$$

$$\begin{aligned} \text{Rest} : \ & f(y)u_t + f(y)u_x + f(y)uu_x - f(y)u_{xx} - T^t_t - T^t_u u_t - T^t_{u_x} u_{tx} - T^x_x \\ & - T^x_u u_x - T^x_{u_x} u_{xx} - T^y_y - T^y_u u_y - T^y_{u_x} u_{xy} = 0. \end{aligned} \tag{4.16}$$

Equation (4.14) gives

$$T^x = f(y)u_{xx} + A(t, x, y, u, u_x, u_{yy}),$$

where A is an arbitrary function of its arguments. Substituting the above value of T^x into equation (4.15) we get $A_{u_{yy}} = f(y)$ and integrating this gives

$$A = f(y)u_{yy} + B(t, x, y, u, u_x),$$

where B is an arbitrary function of its arguments. Thus

$$T^x = f(y)u_{xx} + f(y)u_{yy} + B(t, x, y, u, u_x).$$

Substituting the above value of T^x into equation (4.16) we have

$$\begin{aligned} &f(y)u_t + f(y)u_x + f(y)uu_x - f(y)u_{xx} - T^t_t - T^t_u u_t - T^t_{u_x} u_{tx} - B_x \\ &- B_u u_x - B_{u_x} u_{xx} - T^y_y - T^y_u u_y - T^y_{u_x} u_{xy} = 0. \end{aligned} \tag{4.17}$$

Splitting equation (4.17) on the second derivatives of u we get

$$u_{tx} : T^t_{u_x} = 0, \tag{4.18}$$

$$u_{xx} : B_{u_x} + f(y) = 0, \tag{4.19}$$

$$u_{xy} : T^y_{u_x} = 0, \tag{4.20}$$

$$\text{Rest} : f(y)u_t + f(y)u_x + f(y)uu_x - T^t_t - T^t_u u_t - B_x - B_u u_x - T^y_y - T^y_u u_y = 0. \tag{4.21}$$

Equation (4.18) implies that

$$T^t = C(t, x, y, u),$$

where C is an arbitrary function of its arguments. Similarly equation (4.20) implies that

$$T^y = D(t, x, y, u),$$

where D is an arbitrary function of its arguments. Equation (4.19) gives

$$B = -f(y)u_x + F(t, x, y, u), \tag{4.22}$$

where F is an arbitrary function of its arguments. Thus

$$T^x = f(y)u_{xx} + f(y)u_{yy} - f(y)u_x + F(t, x, y, u).$$

Substituting the above values of T^t, B and T^y into equation (4.21) gives

$$f(y)u_t + f(y)u_x + f(y)uu_x - C_t - C_u u_t - F_x - F_u u_x - D_y - D_u u_y = 0.$$

Splitting the above equation on the derivatives of u we obtain

$$u_t : C_u = f(t), \tag{4.23}$$

$$u_x : F_u = f(y) + f(y)u, \tag{4.24}$$

$$u_y : D_u = 0, \tag{4.25}$$

$$\text{Rest} : C_t + F_x + D_y = 0. \tag{4.26}$$

From equation (4.23) we get

$$C = f(y)u + J(t, x, y), \tag{4.27}$$

where J is an abitrary function of t, x and y. Integrating equation (4.24) gives

$$F = f(y)u + \frac{1}{2} f(y)u^2 + H(t, x, y), \tag{4.28}$$

where H is an abitrary function of t, x and y. Equation (4.25) implies that

$$D = D(t, x, y). \tag{4.29}$$

Substituting the values of C, F and D into equation (4.26) gives $J_t + H_x + D_y = 0$. We may set J, H and D to zero as they contribute to the trivial part of the conservation law. Therefore conservation law for the Zakharov-Kuznetsov-Burgers equation (4.1) is given by

$$\begin{aligned}
T^t &= f(y)u, \\
T^x &= f(y)u_{xx} + f(y)u_{yy} - f(y)u_x + f(y)u + \frac{1}{2} f(y)u^2, \\
T^y &= 0.
\end{aligned}$$

4.4 Concluding remarks

In this chapter we studied the Zakharov-Kuznetsov-Burgers equation (4.1). The travelling wave variable was used to transform (4.1) into a nonlinear ODE. Thereafter Kudryashov's method was invoked to obtain exact solutions of the nonlinear ODE. We then obtained travelling wave solutions of (4.1). Finally the multiplier method was employed to derive conservation laws of this equation.

Chapter 5

Solutions and conservation laws of a ZK-Burgers equation with power law nonlinearity

In this Chapter we study the Zakharov-Kuznetsov-Burgers equation with power law nonlinearity. We first transform this equation into an ordinary differential equation using the travelling wave variable. Kudryashov's method is then employed to obtain exact solutions. Finally conservation laws are derived using the multiplier method.

5.1 Introduction

A Zakharov-Kuznetsov-Burgers equation was derived from the ion continuity equation, ion momentum equation with kinematic viscosity among ionic fluid, electrons, and positrons having kappa distribution together with the Poisson equation in the form [44]

$$u_t + Auu_x + Bu_{xxx} + Cu_{xyy} + Cu_{zz} - D\left(u_{xx} + u_{yy} + u_{zz}\right) = 0.$$

Here A, B, C and D are positive constant quantities which involve physical quantities such as plasma frequecy, magnetic field kinetic viscosity, etc., with the independent variables t, x, y and z representing spatial and temporal variables. In [45] another Zakharov-Kuznetsov-Burgers equation was derived in dense plasmas using the small amplitude perturbation expansion method and is given by

$$u_t + auu_x + bu_{xxx} + cu_{xyy} - du_{xx} - eu_{yy} = 0. \tag{5.1}$$

By employing the tanh expansion method travelling wave solutions of this equation were obtained. In this work we study equation (5.1) by replacing the term u with u^n where n is a nonzero constant and taking the constants $a, b, c, d,$ and e to be equal to one. Thus, we consider the Zakharov-Kuznetsov-Burgers (ZKB) equation with power law nonlinearity of the form

$$u_t + u^n u_x - u_{xx} - u_{yy} + u_{xxx} + u_{xyy} = 0. \tag{5.2}$$

We obtain the travelling wave solutions of (5.2) using Kudryashov's method and derive conservation laws for it using the multiplier method.

5.2 Travelling wave solution of (5.2)

For the travelling wave solution we start by letting

$$u(t, x, y) = U(z), \quad z = \alpha x + \beta y - ct. \tag{5.3}$$

The above transforms ZKB equation (5.2) into a third-order nonlinear ordinary differential equation

$$\alpha\omega U''' - \omega U'' + \alpha U^n U' - cU' = 0,$$

where $\omega = \alpha^2 + \beta^2$. Letting $U(z) = F(z)^{1/n}$, the above ODE becomes

$$\begin{aligned} &n^2\alpha F^3 F' - cn^2 F^2 F' - n\omega F F'^2 - n^2\omega F^2 F'' + n^2\omega F F'^2 + \alpha\omega F'^3 + 3\alpha\omega n F F' F'' \\ &- 3n\alpha\omega F'^3 + n^2\alpha\omega F^2 F''' - 3n^2 F\alpha\omega F' F'' + 2n^2\alpha\omega F'^3 = 0. \end{aligned} \tag{5.4}$$

Solutions of (5.4) using Kudryashov's method

To utilize Kudryashov's method we firstly assume the solution of (5.4) to be of the form

$$F(z) = \sum_{i=0}^{M} B_i \psi^i(z), \tag{5.5}$$

where $\psi(z)$ satisfies the Riccati equation

$$\psi'(z) = \psi^2(z) - \psi(z), \tag{5.6}$$

whose solution is

$$\psi(z) = \frac{1}{1+e^z}. \tag{5.7}$$

In this case the balancing procedure gives $M = 2$, thus the solution (5.5) can be written as

$$F(z) = B_0 + B_1\psi(z) + B_2\psi^2(z). \tag{5.8}$$

Substituting the value of $F(z)$ from equation (5.8) into the equation (5.4) and using (5.6) we obtain the following equation in $\psi(z)$:

$$n^2\alpha B_1^4\psi(z)^5 - n^2\alpha B_1^4\psi(z)^4 + 2n^2\alpha B_2^4\psi(z)^9 - 2n^2\alpha B_2^4\psi(z)^8 - n^2cB_1^3\psi(z)^4 + n^2cB_1^3\psi(z)^3$$
$$- 2n^2cB_2^3\psi(z)^7 + 2n^2cB_2^3\psi(z)^6 + 9n^2\alpha B_0^2B_1\psi(z)^4B_2 - 9n^2\alpha B_0^2B_1\psi(z)^3B_2$$
$$+ 12n^2\alpha B_0B_1^2\psi(z)^5B_2 - 12n^2\alpha B_0B_1^2\psi(z)^4B_2 + 15n^2\alpha B_0B_1\psi(z)^6B_2^2$$
$$- 15n^2\alpha B_0B_1\psi(z)^5B_2^2 - 6n^2cB_0B_1\psi(z)^4B_2 + 6n^2cB_0B_1\psi(z)^3B_2 - 4n\omega B_0B_1\psi(z)^3B_2$$
$$+ 8n\omega B_0B_1\psi(z)^4B_2 - 4n\omega B_0B_1\psi(z)^5B_2 - 6n^2\omega B_0B_1\psi(z)^3B_2 + 18n^2\omega B_0B_1\psi(z)^4B_2$$
$$- 12n^2\omega B_0B_1\psi(z)^5B_2 + 36n\alpha\omega B_0B_2^2\psi(z)^7 + 18n\alpha\omega B_1^2\psi(z)^7B_2 + 30n\alpha\omega B_1\psi(z)^8B_2^2$$
$$+ 6n\alpha\omega B_0B_1^2\psi(z)^5 - 15n\alpha\omega B_0B_1^2\psi(z)^4 - 96n\alpha\omega B_0B_2^2\psi(z)^6 - 39n\alpha\omega B_1^2\psi(z)^6B_2$$
$$- 66n\alpha\omega B_1\psi(z)^7B_2^2 + 12n\alpha\omega B_0B_1^2\psi(z)^3 + 84n\alpha\omega B_0B_2^2\psi(z)^5 + 24n\alpha\omega B_1^2\psi(z)^5B_2$$
$$+ 42n\alpha\omega B_1\psi(z)^6B_2^2 - 3n\alpha\omega B_0B_1^2\psi(z)^2 - 24n\alpha\omega B_0B_2^2\psi(z)^4 - 3n\alpha\omega B_1^2\psi(z)^4B_2$$
$$- 6n\alpha\omega B_1\psi(z)^5B_2^2 + 6n^2\alpha\omega B_0^2B_1\psi(z)^4 + 24n^2\alpha\omega B_0^2B_2\psi(z)^5 + 12n^2\alpha\omega B_0B_2^2\psi(z)^7$$
$$+ 12n^2\alpha\omega B_1^2\psi(z)^7B_2 + 12n^2\alpha\omega B_1\psi(z)^8B_2^2 + 6n^2\alpha\omega B_0B_1^2\psi(z)^5 - 9n^2\alpha\omega B_0B_1^2\psi(z)^4$$
$$- 12n^2\alpha\omega B_0B_2^2\psi(z)^6 - 21n^2\alpha\omega B_1^2\psi(z)^6B_2 - 18n^2\alpha\omega B_1\psi(z)^7B_2^2 - 12n^2\alpha\omega B_0^2B_1\psi(z)^3$$
$$- 54n^2\alpha\omega B_0^2B_2\psi(z)^4 + 7n^2\alpha\omega B_0^2B_1\psi(z)^2 + 38n^2\alpha\omega B_0^2B_2\psi(z)^3 + 2n^2\alpha\omega B_0B_1^2\psi(z)^3$$

$$- 8n^2\alpha\omega B_0B_2^2\psi(z)^5 + 10n^2\alpha\omega B_1^2\psi(z)^5B_2 + 5n^2\alpha\omega B_1\psi(z)^6B_2^2 - n^2\alpha\omega B_0^2B_1\psi(z)$$

$$- 8n^2\alpha\omega B_0^2B_2\psi(z)^2 + n^2\alpha\omega B_0B_1^2\psi(z)^2 + 8n^2\alpha\omega B_0B_2^2\psi(z)^4 - n^2\alpha\omega B_1^2\psi(z)^4B_2$$

$$+ n^2\alpha\omega B_1\psi(z)^5B_2^2 + 2n\omega B_1^3\psi(z)^4 + 8n\omega B_2^3\psi(z)^7 - n\omega B_1^3\psi(z)^5 - 4n\omega B_2^3\psi(z)^8$$

$$- n\omega B_1^3\psi(z)^3 - 4n\omega B_2^3\psi(z)^6 + n^2\omega B_1^3\psi(z)^4 + 2n^2\omega B_2^3\psi(z)^7 - n^2\omega B_1^3\psi(z)^5$$

$$- 2n^2\omega B_2^3\psi(z)^8 + \alpha\omega B_1^3\psi(z)^6 + 8\alpha\omega B_2^3\psi(z)^9 + 3\alpha\omega B_1^3\psi(z)^4 + 24\alpha\omega B_2^3\psi(z)^7$$

$$- 3\alpha\omega B_1^3\psi(z)^5 - 24\alpha\omega B_2^3\psi(z)^8 - \alpha\omega B_1^3\psi(z)^3 - 8\alpha\omega B_2^3\psi(z)^6 + 6\alpha\omega B_1^2\psi(z)^7B_2$$

$$+ 12\alpha\omega B_1\psi(z)^8B_2^2 - 18\alpha\omega B_1^2\psi(z)^6B_2 - 36\alpha\omega B_1\psi(z)^7B_2^2 + 18\alpha\omega B_1^2\psi(z)^5B_2$$

$$+ 36\alpha\omega B_1\psi(z)^6B_2^2 - 6\alpha\omega B_1^2\psi(z)^4B_2 - 12\alpha\omega B_1\psi(z)^5B_2^2 + 3n\alpha\omega B_1^3\psi(z)^6$$

$$+ 12n\alpha\omega B_2^3\psi(z)^9 + 3n\alpha\omega B_1^3\psi(z)^4 + 12n\alpha\omega B_2^3\psi(z)^7 - 6n\alpha\omega B_1^3\psi(z)^5 - 24n\alpha\omega B_2^3\psi(z)^8$$

$$+ 2n^2\alpha\omega B_1^3\psi(z)^6 + 4n^2\alpha\omega B_2^3\psi(z)^9 + n^2\alpha\omega B_1^3\psi(z)^4 + 2n^2\alpha\omega B_2^3\psi(z)^7$$

$$- 3n^2\alpha\omega B_1^3\psi(z)^5 - 6n^2\alpha\omega B_2^3\psi(z)^8 + 2n^2cB_0^2B_2\psi(z)^2 - 2n^2cB_0B_1^2\psi(z)^3$$

$$+ 2n^2cB_0B_1^2\psi(z)^2 - 4n^2cB_0B_2^2\psi(z)^5 + 4n^2cB_0B_2^2\psi(z)^4 - 4n^2cB_1^2\psi(z)^5B_2$$

$$+ 4n^2cB_1^2\psi(z)^4B_2 - 5n^2cB_1\psi(z)^6B_2^2 + 5n^2cB_1\psi(z)^5B_2^2 - n\omega B_0B_1^2\psi(z)^4$$

$$- 5n\omega B_1^2\psi(z)^6B_2 - 8n\omega B_1\psi(z)^7B_2^2 + 2n\omega B_0B_1^2\psi(z)^3 + 8n\omega B_0B_2^2\psi(z)^5$$

$$+ 16n\omega B_1\psi(z)^6B_2^2 - n\omega B_0B_1^2\psi(z)^2 - 4n\omega B_0B_2^2\psi(z)^4 - 5n\omega B_1^2\psi(z)^4B_2$$

$$- 3n^2\omega B_0B_1^2\psi(z)^4 - 8n^2\omega B_0B_2^2\psi(z)^6 - 5n^2\omega B_1^2\psi(z)^6B_2 - 6n^2\omega B_1\psi(z)^7B_2^2$$

$$- 6n^2\omega B_0^2B_2\psi(z)^4 + 3n^2\omega B_0^2B_1\psi(z)^2 + 10n^2\omega B_0^2B_2\psi(z)^3 + 4n^2\omega B_0B_1^2\psi(z)^3$$

$$+ 12n^2\omega B_0B_2^2\psi(z)^5 + 6n^2\omega B_1^2\psi(z)^5B_2 + 7n^2\omega B_1\psi(z)^6B_2^2 - n^2\omega B_0^2B_1\psi(z)$$

$$- n^2\omega B_0B_1^2\psi(z)^2 - 4n^2\omega B_0B_2^2\psi(z)^4 - n^2\omega B_1^2\psi(z)^4B_2 - n^2\omega B_1\psi(z)^5B_2^2$$

$$- n^2\alpha B_0^3B_1\psi(z) + 2n^2\alpha B_0^3B_2\psi(z)^3 - 2n^2\alpha B_0^3B_2\psi(z)^2 + 3n^2\alpha B_0^2B_1^2\psi(z)^3$$

$$+ 6n^2\alpha B_0^2B_2^2\psi(z)^5 - 6n^2\alpha B_0^2B_2^2\psi(z)^4 + 3n^2\alpha B_0B_1^3\psi(z)^4 - 3n^2\alpha B_0B_1^3\psi(z)^3$$

$$- 6n^2\alpha B_0B_2^3\psi(z)^6 + 5n^2\alpha B_1^3\psi(z)^6B_2 - 5n^2\alpha B_1^3\psi(z)^5B_2 + 9n^2\alpha B_1^2\psi(z)^7B_2^2$$

$$- 9n^2\alpha B_1^2\psi(z)^6B_2^2 + 7n^2\alpha B_1\psi(z)^8B_2^3 - 7n^2\alpha B_1\psi(z)^7B_2^3 - n^2cB_0^2B_1\psi(z)^2$$

$$+ n^2cB_0^2B_1\psi(z) - 2n^2cB_0^2B_2\psi(z)^3 - 18n\alpha\omega B_0B_1\psi(z)^3B_2 + 66n\alpha\omega B_0B_1\psi(z)^4B_2$$

$$- 78n\alpha\omega B_0B_1\psi(z)^5B_2 + 30 * n\alpha\omega B_0B_1\psi(z)^6B_2 + 24n^2\alpha\omega B_0B_1\psi(z)^4B_2$$

$$- 54n^2\alpha\omega B_0B_1\psi(z)^5B_2 + 30n^2\alpha\omega B_0B_1\psi(z)^6B_2 + 6n^2\alpha B_0B_2^3\psi(z)^7 - 3n^2\alpha B_0^2B_1^2\psi(z)^2$$

$$+ n^2\alpha B_0^3B_1\psi(z)^2 - 4n^2\omega B_0^2B_2\psi(z)^2 - 2n^2\omega B_0^2B_1\psi(z)^3 - 8n\omega B_1\psi(z)^5B_2^2$$

$$+10n\omega B_1^2\psi(z)^5B_2-4n\omega B_0B_2^2\psi(z)^6=0.$$

Equating the coefficients of like powers of $\psi(z)$ in the above equation we obtain the following nine algebraic equations in terms of B_0, B_1 and B_2:

$$4\alpha n^2\omega B_2^3+2\alpha n^2B_2^4+12\alpha n\omega B_2^3+8\alpha\omega B_2^3=0,$$

$$-\alpha n^2\omega B_0^2B_1-\alpha n^2B_0^3B_1+cn^2B_0^2B_1-n^2\omega B_0^2B_1=0,$$

$$12\alpha n^2\omega B_1B_2^2-6\alpha n^2\omega B_2^3+7\alpha n^2B_1B_2^3-2\alpha n^2B_2^4+30\alpha n\omega B_1B_2^2-24\alpha n\omega B_2^3-2n^2\omega B_2^3$$
$$+12\alpha\omega B_1B_2^2-24\alpha\omega B_2^3-4n\omega B_2^3=0$$

$$7\alpha n^2\omega B_0^2B_1-8\alpha n^2\omega B_0^2B_2+\alpha n^2\omega B_0B_1^2+\alpha n^2B_0^3B_1-2\alpha n^2B_0^3B_2-3\alpha n^2B_0^2B_1^2$$
$$-3\alpha n\omega B_0B_1^2-cn^2B_0^2B_1+2cn^2B_0^2B_2+2cn^2B_0B_1^2+3n^2\omega B_0^2B_1-4n^2\omega B_0^2B_2$$
$$-n^2\omega B_0B_1^2-n\omega B_0B_1^2=0$$

$$12\alpha n^2\omega B_0B_2^2+12\alpha n^2\omega B_1^2B_2-18\alpha n^2\omega B_1B_2^2+2\alpha n^2\omega B_2^3+6\alpha n^2B_0B_2^3+9\alpha n^2B_1^2B_2^2$$
$$-7\alpha n^2B_1B_2^3+36\alpha n\omega B_0B_2^2+18\alpha n\omega B_1^2B_2-66\alpha n\omega B_1B_2^2+12\alpha n\omega B_2^3-2cn^2B_2^3$$
$$-6n^2\omega B_1B_2^2+2n^2\omega B_2^3+6\alpha\omega B_1^2B_2-36\alpha\omega B_1B_2^2+24\alpha\omega B_2^3-8n\omega B_1B_2^2$$
$$+8n\omega B_2^3=0$$

$$-12\alpha n^2\omega B_0^2B_1+38\alpha n^2\omega B_0^2B_2+2\alpha n^2\omega B_0B_1^2+2\alpha n^2B_0^3B_2+3\alpha n^2B_0^2B_1^2$$
$$-9\alpha n^2B_0^2B_1B_2-3\alpha n^2B_0B_1^3+12\alpha n\omega B_0B_1^2-18\alpha n\omega B_0B_1B_2-2cn^2B_0^2B_2-2cn^2B_0B_1^2$$
$$+6cn^2B_0B_1B_2+cn^2B_1^3-2n^2\omega B_0^2B_1+10n^2\omega B_0^2B_2+4n^2\omega B_0B_1^2-6n^2\omega B_0B_1B_2$$
$$-\alpha\omega B_1^3+2n\omega B_0B_1^2-4n\omega B_0B_1B_2-n\omega B_1^3=0$$

$$30\alpha n^2\omega B_0B_1B_2-12\alpha n^2\omega B_0B_2^2+2\alpha n^2\omega B_1^3-21\alpha n^2\omega B_1^2B_2+5\alpha n^2\omega B_1B_2^2$$
$$+15\alpha n^2B_0B_1B_2^2-6\alpha n^2B_0B_2^3+5\alpha n^2B_1^3B_2-9\alpha n^2B_1^2B_2^2+30\alpha n\omega B_0B_1B_2$$
$$-96\alpha n\omega B_0B_2^2+3\alpha n\omega B_1^3-39\alpha n\omega B_1^2B_2+42\alpha n\omega B_1B_2^2-5cn^2B_1B_2^2+2cn^2B_2^3$$
$$-8n^2\omega B_0B_2^2-5n^2\omega B_1^2B_2+7n^2\omega B_1B_2^2+\alpha\omega B_1^3-18\alpha\omega B_1^2B_2+36\alpha\omega B_1B_2^2-8\alpha\ \omega B_2^3$$
$$-4n\omega B_0B_2^2-5n\omega B_1^2B_2+16n\omega B_1B_2^2-4n\omega B_2^3=0,$$

$$24\alpha n^2\omega B_0^2B_2+6\alpha n^2\omega B_0B_1^2-54\alpha n^2\omega B_0B_1B_2-8\alpha n^2\omega B_0B_2^2-3\alpha n^2\omega B_1^3+10\alpha n^2\omega B_1^2B_2$$
$$+\alpha n^2\omega B_1B_2^2+6\alpha n^2B_0^2B_2^2+12\alpha n^2B_0B_1^2B_2-15\alpha n^2B_0B_1B_2^2+\alpha n^2B_1^4-5\alpha n^2B_1^3B_2$$
$$+6\alpha n\omega B_0B_1^2-78\alpha n\omega B_0B_1B_2+84\alpha n\omega B_0B_2^2-6\alpha n\omega B_1^3+24\alpha n\omega B_1^2B_2-6\alpha n\omega B_1B_2^2$$

$$- 4cn^2B_0B_2^2 - 4cn^2B_1^2B_2 + 5cn^2B_1B_2^2 - 12n^2\omega B_0B_1B_2 + 12n^2\omega B_0B_2^2 - n^2\omega B_1^3$$
$$+ 6n^2\omega B_1^2B_2 - n^2\omega B_1B_2^2 - 3\alpha\omega B_1^3 + 18\alpha\omega B_1^2B_2 - 12\alpha\omega B_1B_2^2 - 4n\omega B_0B_1B_2 + 8n\omega B_0B_2^2$$
$$- n\omega B_1^3 + 10n\omega B_1^2B_2 - 8n\omega B_1B_2^2 = 0,$$
$$6\alpha n^2\omega B_0^2B_1 - 54\alpha n^2\omega B_0^2B_2 - 9\alpha n^2\omega B_0B_1^2 + 24\alpha n^2\omega B_0B_1B_2 + 8\alpha n^2\omega B_0B_2^2$$
$$+ \alpha n^2\omega B_1^3 - \alpha n^2\omega B_1^2B_2 + 9\alpha n^2B_0^2B_1B_2 - 6\alpha n^2B_0^2B_2^2 + 3\alpha n^2B_0B_1^3 - 12\alpha n^2B_0B_1^2B_2$$
$$- \alpha n^2B_1^4 - 15\alpha n\omega B_0B_1^2 + 66\alpha n\omega B_0B_1B_2 - 24\alpha n\omega B_0B_2^2 + 3\alpha n\omega B_1^3 - 3\alpha n\omega B_1^2B_2$$
$$- 6cn^2B_0B_1B_2 + 4cn^2B_0B_2^2 - cn^2B_1^3 + 4cn^2B_1^2B_2 - 6n^2\omega B_0^2B_2 - 3n^2\omega B_0B_1^2$$
$$+ 18n^2\omega B_0B_1B_2 - 4n^2\omega B_0B_2^2 + n^2\omega B_1^3 - n^2\omega B_1^2B_2 + 3\alpha\omega B_1^3 - 6\alpha\omega B_1^2B_2 - n\omega B_0B_1^2$$
$$+ 8n\omega B_0B_1B_2 - 4n\omega B_0B_2^2 + 2n\omega B_1^3 - 5n\omega B_1^2B_2 = 0.$$

Solving the above equations for B_0, B_1 and B_2 with the aid of Maple yields the following two cases:

Case1

$$\begin{aligned} &\alpha = \frac{n}{n+4}, \quad c = -\frac{2\omega(n+2)}{n(n+4)}, \quad B_0 = -\frac{2\omega(n^2+3n+2)}{n^2}, \\ &B_1 = \frac{4\omega(n^2+3n+2)}{n^2}, \quad B_2 = -\frac{2\omega(n^2+3n+2)}{n^2}. \end{aligned} \tag{5.9}$$

Case 2

$$\begin{aligned} &\alpha = -\frac{n}{n+4}, \quad c = \frac{2\omega(n+2)}{n(n+4)}, \quad B_0 = 0, B_1 = 0, \\ &B_2 = -\frac{2\omega(n^2+3n+2)}{n^2}. \end{aligned} \tag{5.10}$$

Thus for the first case we obtain the solution of the ZKB equation (5.2) as

$$u(t,x,y) = \left\{ B_0 + \frac{B_1}{1+e^z} + \frac{B_2}{(1+e^z)^2} \right\}^n,$$

where the values of B_0, B_1, B_2 are given by (5.9) and $z = \alpha x + \beta y - ct$. For the second case the solution of the ZKB equation (5.2) is

$$u(t,x,y) = \left\{ \frac{B_2}{(1+e^z)^2} \right\}^n,$$

where the value of B_2 is given by (5.10) and $z = \alpha x + \beta y - ct$.

5.3 Conservation laws of (5.2)

In this section conservation laws of the ZKB equation (5.2) are derived using the multiplier method. Here we seek the zeroth-order multipliers $Q = Q(t, x, y, u)$. Recall that a multiplier $Q(t, x, y, u)$ for the ZKB (5.2) is obtained from the determining equation

$$\frac{\delta}{\delta u}\left[Q(t,x,y)(u_t + u^n u_x - u_{xx} - u_{yy} + u_{xxx} + u_{xyy})\right] = 0. \tag{5.11}$$

Expanding equation (5.11) yields

$$\begin{aligned} &Q_u(u_t + u^n u_x + u_{xxx} + u_{xyy} - u_{xx} - u_{yy}) - D_t(Q) + nu^{n-1}u_x Q - D_x(u^n Q) \\ &- D_x^3(Q) - D_x D_y^2(Q) + D_x^2(-Q) + D_y^2(-Q) = 0. \end{aligned} \tag{5.12}$$

Applying total derivatives D_t, D_x and D_y on the above equation we obtain

$$\begin{aligned} &2u_{xx}Q_u + 2u_{yy}Q_u - u^n Q_x + Q_{xxx} + 3u_x Q_{xxu} + 3u_x^2 Q_{xuu} + 3u_{xx}Q_{xu} + u_x^3 Q_{uuu} \\ &+ 3u_x u_{xx} Q_{uu} + Q_{xyy} + u_x Q_{yyu} + 2u_y Q_{xyu} + 2u_x u_y Q_{yuu} + 2u_{xy} Q_{yu} + u_y^2 Q_{xuu} \\ &+ u_x u_y^2 Q_{uuu} + 2u_y u_{xy} Q_{uu} + u_{yy} Q_{xu} + u_x u_{yy} Q_{uu} + Q_{xx} + 2u_x Q_{xu} + u_x^2 Q_{uu} \\ &+ Q_{yy} + 2u_y Q_{yu} + u_y^2 Q_{uu} + Q_t = 0. \end{aligned} \tag{5.13}$$

Splitting the above equation on derivatives of u we get

$$2Q_u + 3Q_{xu} = 0, \tag{5.14}$$

$$2Q_u + Q_{ux} = 0, \tag{5.15}$$

$$Q_{yu} = 0, \tag{5.16}$$

$$Q_{uu} = 0, \tag{5.17}$$

$$Q_t + u^n Q_x + Q_{xx} + Q_{yy} + Q_{xxx} + Q_{xyy} = 0. \tag{5.18}$$

From equations (5.14) and (5.15) we obtain

$$Q_u = 0,$$

which on integration gives

$$Q = A(t, x, y),$$

where A is an arbitrary function of t, x and y. Equations (5.16) and (5.17) are satisfied by the above value of Q. Substituting the value of Q into equation (5.18) we obtain

$$A_t + u^n A_x + A_{xx} + A_{yy} + A_{xxx} + A_{xyy} = 0. \tag{5.19}$$

Splitting equation (5.19) on powers of u yields

$$u^n : A_x = 0, \tag{5.20}$$

$$\text{Rest} : A_t + A_{xx} + A_{yy} + A_{xxx} + A_{xyy} = 0. \tag{5.21}$$

Equation (5.20) implies $A = A(t, y)$, thus equation (5.21) becomes

$$A_t + A_{yy} = 0. \tag{5.22}$$

To solve equation (5.22), we assume that

$$A(t, y) = T(t)Y(y).$$

Substituting the above value of A into equation (5.22) gives

$$Y''T + T'Y = 0. \tag{5.23}$$

Hence

$$\frac{Y''}{Y} = -\frac{T'}{T} = -a, \quad \text{(say)} \tag{5.24}$$

where a is an arbitrary constant. Thus from equation (5.24) we have

$$T' - aT = 0 \tag{5.25}$$

and

$$Y'' + aY = 0. \tag{5.26}$$

Solving equation (5.25), we obtain

$$T(t) = Ce^{at}, \tag{5.27}$$

where C is a constant of integration. Solving equation (5.26) yields

$$Y(y) = A_1 \sin \sqrt{a}y + B_1 \cos \sqrt{a}y, \tag{5.28}$$

where A_1 and B_1 are arbitrary constants. Thus

$$A(t,y) = Ce^{at}\left(A_1 \sin\sqrt{a}y + B_1 \cos\sqrt{a}y\right).$$

Therefore the zeroth-order multiplier for the ZKB equation (5.2) is given by

$$Q = e^{at}\left(C_1 \sin\sqrt{a}y + C_2 \cos\sqrt{a}y\right),$$

where C_1 and C_2 are arbitrary constants. Now that the multipliers of ZKB equation have been obtained, then each multiplier is used to derive the corresponding conservation law.

Case1. Here the first multiplier $Q_1 = e^{at}\sin\sqrt{a}y$ is used to derive a conservation law. The above multiplier of the ZKB equation (5.2) has the property that

$$e^{at}\sin\sqrt{a}y\,(u_t + u^n u_x - u_{xx} - u_{yy} + u_{xxx} + u_{xyy}) = D_t(T^t) + D_x(T^x) + D_y(T^y), \tag{5.29}$$

where

$$\begin{aligned} T^t &= T^t(t,x,y,u,u_x),\\ T^x &= T^x(t,x,y,u,u_x,u_{xx},u_{yy}),\\ T^y &= T^y(t,x,y,u,u_x,u_y). \end{aligned} \tag{5.30}$$

Expanding equation (5.29) we have

$$\begin{aligned} &e^{at}\sin\sqrt{a}y\,(u_t + u^n u_x - u_{xx} - u_{yy} + u_{xxx} + u_{xyy}) - T^t_t - T^t_u u_t - T^t_{u_x} u_{tx} - T^x_x\\ &- T^x_u u_x - T^x_{u_x} u_{xx} - T^x_{u_{xx}} u_{xxx} - T^x_{u_{yy}} u_{xyy} - T^y_y - T^y_u u_y - T^y_{u_x} u_{xy} - T^y_{u_y} u_{yy} = 0. \end{aligned}$$

Splitting the above on the third derivatives of u yields

$$u_{xxx} : T^x_{u_{xx}} - e^{at}\sin\sqrt{a}y = 0, \tag{5.31}$$

$$u_{xyy} : T^x_{u_{yy}} - e^{at}\sin\sqrt{a}y = 0, \tag{5.32}$$

$$\begin{aligned} \text{Rest} : &\, e^{at}\sin\sqrt{a}y\,(u_t + u^n u_x - u_{xx} - u_{yy}) - T^t_t - T^t_u u_t - T^t_{u_x} u_{tx} - T^x_x\\ &- T^x_u u_x - T^x_{u_x} u_{xx} - T^y_y + T^y_u u_y - T^y_{u_x} u_{xy} - T^y_{u_y} u_{yy} = 0. \end{aligned} \tag{5.33}$$

Equations (5.31) and (5.32) imply that

$$T^x = u_{xx}e^{at}\sin\sqrt{a}y + u_{yy}e^{at}\sin\sqrt{a}y + A(t,x,y,u,u_x),$$

where A is an arbitrary function of its arguments. Substituting the above value of T^x into equation (5.33) we get

$$e^{at}\sin\sqrt{a}y\,(u_t+u^nu_x-u_{xx}-u_{yy})-T^t_t-T^t_uu_t-T^t_{u_x}u_{tx}-A_x-A_uu_x$$
$$-A_{u_x}u_{xx}-T^y_y-T^y_uu_y-T^y_{u_x}u_{xy}-T^y_{u_y}u_{yy}=0.$$

Splitting the above equation on second derivatives of u gives

$$u_{tx}:T^t_{u_x}=0, \tag{5.34}$$
$$u_{xy}:T^y_{u_x}=0, \tag{5.35}$$
$$u_{xx}:A_{u_x}+e^{at}\sin\sqrt{a}y=0, \tag{5.36}$$
$$u_{yy}:T^y_{u_y}+e^{at}\sin\sqrt{a}y=0, \tag{5.37}$$
$$\text{Rest}:e^{at}\sin\sqrt{a}y\,(u_t+u^nu_x)-T^t_t-T^t_uu_t-A_x-A_uu_x-T^y_y-T^y_uu_y=0. \tag{5.38}$$

Equation (5.34) yields

$$T^t=B(t,x,y,u), \tag{5.39}$$

where B is an arbitrary function of its arguments. Equations (5.35) and (5.37) imply that

$$T^y=-u_ye^{at}\sin\sqrt{a}y+C(t,x,y,u),$$

where C is an arbitrary function of its arguments. Integrating equation (5.36) gives

$$A=-u_xe^{at}\sin\sqrt{a}y+D(t,x,y,u),$$

where D is an arbitrary function of its arguments. Substituting the values of T^t, T^y and A into equation (5.38) we have

$$e^{at}\sin\sqrt{a}y\,(u_t+u^nu_x)-B_t-B_uu_t-D_x-D_uu_x+u_y\sqrt{a}\,e^{at}\cos\sqrt{a}y-C_y-C_uu_y=0.$$

Splitting the above equation on the derivatives of u, we obtain

$$u_t:B_u-e^{at}\sin\sqrt{a}y=0, \tag{5.40}$$
$$u_x:D_u-u^ne^{at}\sin\sqrt{a}y=0, \tag{5.41}$$

$$u_y : C_u - \sqrt{a}\, e^{at} \cos \sqrt{a} y = 0, \tag{5.42}$$

$$\text{Rest} : B_t + D_x + C_y = 0. \tag{5.43}$$

Equations (5.40) gives

$$B = ue^{at} \sin \sqrt{a} y + E(t, x, y),$$

where E is an arbitrary function of t, x and y. Integrating equation (5.41) we get

$$D = \frac{u^{n+1}}{n+1} e^{at} \sin \sqrt{a} y + F(t, x, y) \quad \text{n} \neq -1,$$

where F is an arbitrary function of t, x and y. From equation (5.42) we get

$$C = \sqrt{a} u e^{at} \cos \sqrt{a} y + G(t, x, y),$$

where G is an arbitrary of t, x and y. Substituting this values of B, C and D into equation (5.43) we have

$$E_t + F_x + G_y = 0. \tag{5.44}$$

Since E, F and G are contributing to the trivial part of the conservation law we then set them to be zero. Thus the conservation law corresponding to the multiplier Q_1 for the ZKB equation is given by

$$\begin{aligned}
T^t &= ue^{at} \sin \sqrt{a} y, \\
T^x &= u_{xx} e^{at} \sin \sqrt{a} y + u_{yy} e^{at} \sin \sqrt{a} y - u_x e^{at} \sin \sqrt{a} y + \frac{u^{n+1}}{n+1} e^{at} \sin \sqrt{a} y \quad \text{n} \neq -1, \\
T^y &= \sqrt{a} u e^{at} \cos \sqrt{a} y - u_y e^{at} \sin \sqrt{a} y.
\end{aligned}$$

Case2. Now we use the multiplier $Q_2 = e^{at} \cos \sqrt{a} y$ to derive the conservation law for the ZKB equation. For this multiplier we have

$$e^{at} \cos \sqrt{a} y \,(u_t + u^n u_x - u_{xx} - u_{yy} + u_{xxx} + u_{xyy}) = D_t(T^t) + D_x(T^x) + D_y(T^y),$$

where

$$T^t = T^t(t, x, y, u, u_x),$$

$$T^x = T^x(t, x, y, u, u_x, u_{xx}, u_{yy}),$$

$$T^y = T^y(t, x, y, u, u_x, u_y).$$

Expanding the above equation we get

$$e^{at}\cos\sqrt{a}y\,(u_t + u^n u_x + u_{xxx} + u_{xyy} - u_{xx} - u_{yy}) - T^t_t - T^t_u u_t - T^t_{u_x} u_{tx} - T^x_x - T^x_u u_x$$
$$- T^x_{u_x} u_{xx} - T^x_{u_{xx}} u_{xxx} - T^x_{u_{yy}} u_{xyy} - T^y_y - T^y_u u_y - T^y_{u_x} u_{xy} - T^y_{u_y} u_{yy} = 0.$$

Splitting the above on the third derivatives of u yields

$$u_{xxx} : T^x_{u_{xx}} - e^{at}\cos\sqrt{a}y = 0, \tag{5.45}$$

$$u_{xyy} : T^x_{u_{yy}} - e^{at}\cos\sqrt{a}y = 0, \tag{5.46}$$

$$\text{Rest} : e^{at}\cos\sqrt{a}y\,(u_t + u^n u_x - u_{xx} - u_{yy}) - T^t_t - T^t_u u_t - T^t_{u_x} u_{tx} - T^x_x$$
$$- T^x_u u_x - T^x_{u_x} u_{xx} - T^y_y - T^y_u u_y - T^y_{u_x} u_{xy} - T^y_{u_y} u_{yy} = 0. \tag{5.47}$$

Equations (5.45) and (5.46) imply that

$$T^x = u_{xx} e^{at}\cos\sqrt{a}y + u_{yy} e^{at}\cos\sqrt{a}y + A(t, x, y, u, u_x),$$

where A is an arbitrary function of its arguments. Substituting this value of T^x into equation (5.47) we have

$$e^{at}\cos\sqrt{a}y\,(u_t + u^n u_x - u_{xx} - u_{yy}) - T^t_t - T^t_u u_t - T^t_{u_x} u_{tx} - A_x - A_u u_x$$
$$- A_{u_x} u_{xx} - T^y_y - T^y_u u_y - T^y_{u_x} u_{xy} - T^y_{u_y} u_{yy} = 0.$$

Splitting the above equation on the second derivatives of u gives

$$u_{tx} : T^t_{u_x} = 0, \tag{5.48}$$

$$u_{xy} : T^y_{u_x} = 0, \tag{5.49}$$

$$u_{xx} : A_{u_x} + e^{at}\cos\sqrt{a}y = 0, \tag{5.50}$$

$$u_{yy} : T^y_{u_y} + e^{at}\cos\sqrt{a}y = 0, \tag{5.51}$$

$$\text{Rest} : e^{at}\cos\sqrt{a}y\,(u_t + u^n u_x) - T^t_t - T^t_u u_t - A_x - A_u u_x - T^y_y - T^y_u u_y = 0. \tag{5.52}$$

Equation (5.48) gives

$$T^t = B(t, x, y, u),$$

where B is an arbitrary function of its arguments. Equations (5.49) and (5.51) imply that

$$T^y = -u_y\, e^{at} \cos \sqrt{a} y + C(t, x, y, u),$$

where C is an arbitrary function of its arguments. From equation (5.50) we get

$$A = -u_x\, e^{at} \cos \sqrt{a} y + D(t, x, y, u),$$

where D is an arbitrary function of its arguments. Substituting the above values of T^t, T^y and A into equation (5.52) yields

$$e^{at} \cos \sqrt{a} y\, (u_t + u^n u_x) - B_t - B_u u_t - D_x - D_u u_x - u_y \sqrt{a}\, e^{at} \sin \sqrt{a} y - C_y - C_u u_y = 0.$$

Splitting the above equation on derivatives of u yields

$$u_t : B_u - e^{at} \cos \sqrt{a} y = 0, \tag{5.53}$$

$$u_x : D_u - u^n e^{at} \cos \sqrt{a} y = 0, \tag{5.54}$$

$$u_y : C_u + \sqrt{a}\, e^{at} \sin \sqrt{a} y = 0, \tag{5.55}$$

$$\text{Rest} : B_t + D_x + C_y = 0. \tag{5.56}$$

Equations (5.53) gives

$$B = u e^{at} \cos \sqrt{a} y + E(t, x, y),$$

where E is an arbitrary function of t, x and y. Integrating equation (5.54) we get

$$D = \frac{u^{n+1}}{n+1} e^{at} \cos \sqrt{a} y + F(t, x, y) \quad \text{n} \neq -1,$$

where F is an arbitrary function of t, x and y. From equation (5.55) we obtain

$$C = -\sqrt{a} u e^{at} \sin \sqrt{a} y + G(t, x, y),$$

where G is an arbitrary function of t, x and y. Substituting these values of B, C and D into equation (5.56) we have

$$E_t + F_x + G_y = 0. \tag{5.57}$$

The arbitrary functions E, F and G are set to be zero as they contribute to the trivial part of the conservation law. Thus the conservation law for the ZKB equation corresponding to the multiplier Q_2 is given by

$$
\begin{aligned}
T^t &= ue^{at}\cos\sqrt{a}y,\\
T^x &= u_{xx}e^{at}\cos\sqrt{a}y + u_{yy}e^{at}\cos\sqrt{a}y - u_x e^{at}\cos\sqrt{a}y + \frac{u^{n+1}}{n+1}e^{at}\cos\sqrt{a}y \quad \text{n} \neq -1,\\
T^y &= -\sqrt{a}ue^{at}\sin\sqrt{a}y - u_y e^{at}\cos\sqrt{a}y.
\end{aligned}
$$

5.4 Concluding remarks

In this Chapter the Zakharov-Kuznetzov-Burgers equation with power law nonlinearity (5.2) was studied. We started the study by using the travelling wave variable to transform this equation into an ODE. Kudryashov's method was then used to find exact solutions of this equation. Furthermore, conservation laws of (5.2) were derived using the multiplier approach.

Chapter 6

Concluding remarks and future work

Most real world problems are modelled by nonlinear partial differential equations. It is therefore crucial to study such equations by finding their exact solutions and conservation laws. In this dissertation we studied four nonlinear partial differential equations, namely, the Burgers equation, the equal-width equation and two different versions of Zakharov-Kuznetsov-Burgers equations.

In Chapter one we presented relevant literature which was used in this dissertation. Various methods for finding the exact solutions of partial differential equations were described and two methods for deriving conservation laws were also discussed.

In Chapter two we studied the Burgers equation. We computed all Lie point symmetries of this equation and then determined its group-invariant solutions under each of its symmetries. The multiplier method was employed to derive conservation laws of this equation.

In the third Chapter we studied the first main problem of this dissertation, i.e., an equal-width equation. Lie point symmetries, one dimensional optimal system of subalgebras and exact solutions of this equation were obtained using Kudryashov's method and the extended Jacobi elliptic function expansion method. Moreover conservation laws of this equation were derived by invoking the multiplier method and Noether's theorem.

In Chapters four and five we studied two different versions of Zakharov-Kuznetsov-Burgers equation. Exact solutions were obtained for both equations using Kudryashov's method. Conservation laws were derived using multiplier method for both the equations.

In future work, we plan to study the equal-width equation with power law nonlinearity. We also intend to explore double reduction using conservation laws on all the equations studied in this dissertation.

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