Price estimation of basket credit default swaps using numerical and quasi-analytical methods

NO Umeorah

orcid.org 0000-0002-0307-5011

Thesis accepted in fulfilment of the requirements for the degree *Doctor of Philosophy in Science with Risk Analysis* at the North-West University

Promoter: Prof HP Mashele

Co-promoter: Prof M Ehrhardt

Graduation July 2020

27658457

Dedication

To everyone who love what is right!!!

"Trust in Jehovah with all your heart, and do not rely on your own understanding. In all your ways take notice of him, and he will make your paths straight."

Proverbs 3 vs 5-6.

Declaration

I, Umeorah Nneka Ozioma, hereby declare that this thesis title: “Price estimation of basket credit default swaps using numerical and quasi-analytical methods”, submitted in fulfilment of the requirements for the degree Philosophiae Doctor (PhD) in Science with Risk Analysis is my work and has not previously been submitted to any other institution in whole or in part. Written consent from authors had been obtained for publications where co-authors have been involved.

Umeorah Nneka Ozioma

16-06-2020 Potchefstroom

Date Place

Preface

Format of thesis

The format of this thesis is in accordance with the academic rules of the North-West University (approved on November 22nd, 2013), where rule **A.5.4.2.7** states: "Where a candidate is permitted to submit a thesis in the form of a published research article or articles, or as an unpublished manuscript or manuscripts in article format and more than one such article or manuscript is used, the thesis must still be presented as a unit, supplemented with an inclusive problem statement, a focused literature analysis and integration and with a synoptic conclusion, and the guidelines of the journal concerned must also be included."

Rule **A.5.4.2.8** states: "Where any research article or manuscript and/or internationally examined patent is used for the purpose of a thesis in article format to which other authors and/or inventors than the candidate contributed, the candidate must obtain a written statement from each co-author and/or co-inventor in which it is stated that such co-author and/or coinventor grants permission that the research article or manuscript and/or patent may be used for the stated purpose and in which it is further indicated what each co-author's and/or coinventor's share in the relevant research article or manuscript and/or patent was."

Rule **A.5.4.2.9** states: "Where co-authors or co-inventors as referred to in **A.5.4.2.8** above were involved, the candidate must mention that fact in the preface and must include the statement of each co-author or co-inventor in the thesis immediately following the preface."

Format of numbering and referencing

It should be noted that the formatting, referencing style, numbering of tables and figures, and general outline of the manuscripts were adapted to ensure uniformity throughout the thesis. The format of manuscripts which have been submitted and/or published adhere to the author guidelines as stipulated by the editor of each journal, and may appear in a different format to what is presented in this thesis. The headings and original technical content of the manuscripts were not modified from the submitted and/or published versions, and only minor spelling and typographical errors were corrected. The bibliography (reference list) was included at the end of the overall thesis.

Authorship/ Letter of Consent

To whom it may concern,

All the research works explained in this thesis are mine. I was absolutely involved in the whole aspect of this work, including the methodologies, problem-solving, computational algorithm, data analysis and the manuscript write-ups (principal author), though I benefited from the helpful remarks of my supervisors Prof. Phillip Mashele and Prof. Matthias Ehrhardt.

The listed co-authors hereby give their consent that **Umeorah Nneka Ozioma** may submit the following manuscripts as part of her thesis entitled: **Price estimation of basket credit default swaps using numerical and quasi-analytical methods**, for the degree Philosophiae Doctor in Risk Analysis, at the North-West University:

Umeorah, N., Ehrhardt, M. & Mashele, P. (2020). Valuation of basket credit default swaps under stochastic default intensity models. *Advances in Applied Mathematics and Mechanics*, **12**(4), 1-26, DOI: 10.4208/aamm.OA-2019-0141.

Umeorah, N., Mashele, P., & Ehrhardt, M. (2020). Elliptical and Archimedean copula models: an application to the price estimation of portfolio credit derivatives. **Accepted and to appear in Risk Journals (Submission ID: MS 2025)**.

Umeorah, N., Mashele, P., & Ehrhardt, M. (2020). Pricing basket default swaps using quasi-analytic techniques. Submitted to **Decisions in Economics and Finance (Manuscript ID: DEAF-D-20-00080)**.

Note: This letter of consent complies with rules **A.5.4.2.8** and **A.5.4.2.9** of the academic rules, as stipulated by the North-West University.

Mashele Phillip

16-06-2020 Potchefstroom

Date Place

Ehrhardt Matthias

16-06-2020 Wuppertal

Date Place

List of Publications

Journal articles

1. **Umeorah, N.**, Mashele, P., & Ehrhardt, M. (2020). Pricing basket default swaps using quasi-analytic techniques. Submitted to **Decisions in Economics and Finance (Manuscript ID DEAF-D-20-00080)**.

2. **Umeorah, N.**, Mashele, P., & Ehrhardt, M. (2020). Elliptical and Archimedean copula models: an application to the price estimation of portfolio credit derivatives. **Accepted and to appear in Risk Journals (Submission number MS 2025)**.

3. **Umeorah, N.**, Ehrhardt, M. & Mashele, P. (2020). Valuation of basket credit default swaps under stochastic default intensity models. *Advances in Applied Mathematics and Mechanics*, **12**(4), 1-26, DOI: 10.4208/aamm.OA-2019-0141.

4. **Umeorah N.** & Mashele, P. (2019). 'A Crank-Nicolson finite difference approach on the numerical estimation of rebate barrier option prices', *Cogent Economics & Finance*, **7**(1). DOI: 10.1080/23322039.2019.1598835

5. **Umeorah N.** & Mashele, P. (2018). 'A Comparative Study on Barrier Option Pricing using Antithetic and Quasi Monte-Carlo Simulations'. *Journal of Mathematics and Statistics*, **14**, 94–106. DOI: 10.3844/jmssp.2018.94.106.

Presentations

1. **Umeorah N.O. (presenter)**, Mashele, P. & Ehrhardt, M. (2019). 'Pricing basket credit default swaps using numerical and quasi-analytical methods'. **Oral Presentation** at the African Workshop on Mathematical Optimisation, Department of Mathematical Sciences, University of Zululand. November 11–15, 2019.

2. **Umeorah N.O. (presenter)**, Ehrhardt, M. & Mashele, P. (2019). 'Valuation of basket credit default swaps under stochastic default intensity models'. **Oral Presentation** at the Vienna Congress on Mathematical Finance/ Educational Workshop, Vienna University of Economics and Business, Austria. September 9–13, 2019.

3. **Umeorah N.O. (presenter)** & Mashele, P. (2018). ‘Interest rate modelling with application to swaption pricing’. **Oral Presentation** at the Fakultat fur Mathematik und Naturwissenschaften, Applied Mathematics, Bergische Universität Wuppertal, Germany, August 8, 2018. During the short-term research visit (April to Sept 2018), sponsored by DAAD.

4. **Umeorah N.O. (presenter)** & Mashele, P. (2018). ‘Estimation of zero rebate knock-out barrier options using antithetic Monte-Carlo simulations’. **Oral Presentation** at the 2018 International Women in Science Without Borders (WISWB)– Indaba, University of Johannesburg, South Africa. March 21–23, 2018. Available at https://researchspace.csir.co.za/dspace/handle/10204/10251.

5. **Umeorah N.O. (presenter)** & Mashele, P. (2017). ‘Application of finite difference methods in barrier option pricing’. **Oral Presentation** at the 60th Annual Congress of the South African Mathematical Society, North-West University, Potchefstroom, South Africa. November 20–22, 2017.

6. **Umeorah N.O. (presenter)** & Mashele, P. (2017). ‘A Crank-Nicolson finite difference approach on the numerical approximation of rebate barrier option prices’. **Oral Presentation** at the 6th International Conference on Mathematics in Finance held at Skukuza, Kruger National Park, South Africa. August 6–12, 2017.

7. Dankwa, E., Andam, P.S., **Umeorah, N.O** & Maboulou, A.P.B. (2017). ‘Analysis of data sets using dimensionality reduction and clustering techniques’. **Group Presentation** at the Atlantic Association for Research in the Mathematical Sciences (AARMS) Summer School in Mathematical Finance and Actuarial Sciences at Prince Edward Island University, Charlottetown, Canada. July 3-28, 2017.

Acknowledgements

First and foremost, the author wish to acknowledge the finger of the almighty Jehovah God in this research work; for his love, mercy and grace endure forever.

This research work is based on the joint financial supports from the Deutscher Akademischer Austauschdienst (DAAD), in collaboration with the African Institute for Mathematical Sciences (AIMS), South Africa and the North-West University (NWU), Potchefstroom, South Africa. Any opinion, finding or conclusion or recommendation expressed in this work is exclusively that of the author(s).

I equally wish to use this medium to thank my supervisor Prof. Phillip Mashele for his immense contributions, motivation and profound support in the course of this research journey. Your constructive criticisms and excellent guidance equally made this work possible. To my co-supervisor, Prof. Matthias Ehrhardt, it was a unique privilege for you to serve as my academic host during my short-term research visit at the Bergische Universität, Wuppertal, Germany. Ever since, you have been kind in assisting and co-supervising my work; many thanks for that. I am highly grateful and opportune to have both of you as my research mentors.

I also wish to acknowledge Prof. Riaan De Jongh at the Centre for Business Mathematics and Informatics (BMI), NWU, I want to thank you for your assistance and financial support throughout the duration of this research. Also, to the staff of BMI for all your kind support. I also acknowledge the National Research Foundation (Ref no: RA180111301913) for the research fund made available to me for international travel (Vienna conference).

To my mum and my siblings: Stanley, Chima, Chiamaka, Ifeoma, Nnamdi; You guys are my awesome family and simply the best, thank you all for your prayers and encouragement. To my twin cousin Gift; and to my friends: Dr. R.C Uwaoma, Assumpta, Jordan, Amanda; and others who in one or the other contributed to the success of this research. I am grateful to you all.

Executive Summary

The introduction of the credit derivatives in finance has facilitated the concept of credit risk transfer, together with its analysis and management. The reason is obviously due to the fact that these derivative instruments can be pre-defined and tailor-made to conform to the needs of its investors, thereby expediting the hedging and diversification of credit risk. This research focuses and gives a broader overview of the multi-name credit derivatives, which have received less attention as compared to the single-name credit derivatives. The basket credit default swaps, as multi-name derivatives have appealing features to the financial investors owing to their substantial leveraging benefits, as well as their less expensive nature. Hence, the overall theme of this thesis is to price the basket credit default swaps (BCDS) using numerical and quasi-analytical techniques. These methods are targeted towards estimating the joint default probability distributions or the joint dependent defaults which characterizes the basket pricing concepts. This investigation was channelled into three major phases, as evident in the main three-part chapters presented in this work.

Phase one of this thesis focused on pricing the BCDS using the stochastic default intensity models. Here, we modelled the hazard rate or the intensity default process using the one-factor Vasicek and the Cox-Ingersoll-Ross (CIR) models. Next, we approximated the joint survival probability distribution functions which describes the intensity models under the risk-neutral pricing measure, for both the homogeneous and the heterogeneous portfolios. We next utilized the Monte-Carlo method, under the Gaussian copula model to numerically approximate the default time distribution function. The nth-to-default basket credit swaps, in which the spreads depend on the nth default time, were successfully priced using the above methodologies, as well as the consideration of the effects of different swap parameters to various nth-to-default swaps. Furthermore, this phase equally considered the estimation of the survival probabilities, the swap spreads and the equal-weighted portfolio values defined within the context of the Vasicek and the CIR model. Results obtained showed that the CIR is more applicable in modelling the hazard rate process, and this, in turn, resulted in more efficient pricing of the basket swaps, as compared to the Vasicek counterpart. Also, from the results, we observed that the nth-to-default swap prices behave differently with respect to varying in the intensity rate and the default correlation, as the rank of default protection increases. Hence, we can recommend that investors who wish to trade first-to-default swaps with highest swap premium should sell protections on entities with low correlations.

Phase two of the thesis estimated the valuation of the portfolio of credit derivatives (specifically the nth-to-default basket swap), as well as the comparative analysis on the effect of using the one-factor elliptical and the Archimedean copula models to swap pricing. Here, we used the Gaussian and the Student-t as elliptical models, as well as the Clayton, the Frank, and the Gumbel as Archimedean copulas, to model the corresponding default times. For the numerical computation, we employed the Monte-Carlo simulations as the benchmark of the estimation process. The break-even swap premium valuation was made viable through the estimation of the default times and the payment leg streams of the contingent claims. Finally, from our results, we investigated the choice of copula models from existing works of literature, and then made our appropriate choice of copulas for BCDS pricing based on their computation time. Furthermore, the corresponding numerical experiments which were presented clearly showed that the selection of the copula model hugely affects the quantitative risk analysis of the portfolio.

Finally, the quasi-analytic techniques were employed in Phase three of the thesis to the valuation of the BCDS premiums. Here, the main focus was on the one-factor copula model, which was applied to minimize the dimensionality issues resulting from the basket default swap pricing. The one-factor Gaussian, student-t and the Clayton copula models were used to estimate the conditional default probability. Furthermore, the conditional characteristic function for the corresponding portfolio loss distribution using the Fast Fourier transform was obtained, and then, we retrieved the unconditional characteristic function with the aid of the inverse fast Fourier transform using numerical integration. Hence, the quasi-analytical expressions for the computation of the premium payment leg, the default payment leg and then the nth-to-default swap formulas were derived by incorporating the concept of the Fourier transform, together with the distribution function of a counting process. From our findings, we observed that in the absence of the trending simulation method, a semi-analytic method which involves the applications of the discrete Fourier transform could be utilized to price the basket credit default swaps effectively.

Keywords:

Archimedean Copulas, Basket Default Swaps, Characteristics Function, CIR Model, Convolution, Copulas, Default Times, Discrete Fourier Transform, Elliptical Copulas, Fast Fourier Transform, Gaussian Copula, Hazard Rate, Joint Survival Probability Distribution, Monte-Carlo Simulations, Portfolio Credit Derivatives, Probability Distributions, Stochastic Intensity Modelling, Sensitivity Analysis, Vasicek Model.

List of Acronyms

AIC	-	Akaike Information Criterion
AL	-	Accrued Leg
BCDS	-	Basket Credit Default Swaps
BIC	-	Bayesian Information Criterion
CDF	-	Cumulative Density Function
CDO	-	Collateralized Debt Obligations
CDS	-	Credit Default Swaps
CEV	-	Constant Elasticity of Variance
CKLS	-	Chan, Karolyi, Longstaff and Sanders
CIR	-	Cox-Ingersoll-Ross
DFT	-	Discrete Fourier Transform
DIFT	-	Discrete Inverse Fourier Transform
DL	-	Default Leg
F2D	-	First-to-Default
FT	-	Fourier Transform
FFT	-	Fast Fourier Transform

Fo2D - Fourth-to-Default

IFFT - Inverse Fast Fourier Transform

IFT - Inverse Fourier Transform

ISDA - International Swap Derivatives Association

JSPD - Joint Survival Probability Distribution

MCS - Monte-Carlo Simulation

MLE - Maximum Likelihood Estimation

n2D - nth-to-Default

OAS - Option-Adjusted Spread

OTC - Over-the-Counter

PCD - Portfolio Credit Derivatives

PDF - Probability Density Function

PL - Premium Leg

PRNG - Pseudo-Random Number Generator

S2D - Second-to-Default

T2D - Third-to-Default

Contents

List of Figures

List of Tables

1. General Introduction

This introductory chapter is partitioned into six subsections: background information, problem statement, aim and objectives of the thesis, motivation, method of investigation, and the overview of the thesis.

1.1 Background Information

Without a doubt, the introduction of credit derivatives in the financial and industrial sectors have revolutionized the act of credit risk trade and its management. This is due to the fact that these derivative tools can be modified to suit the risk-return profile which is made specifically for its investors, and financial institutions as well can have their credit risk effectively hedged and diversified. In the credit derivative market, the participants or the major dealers and in the mid-2013 are given in Figure 1.1 below:

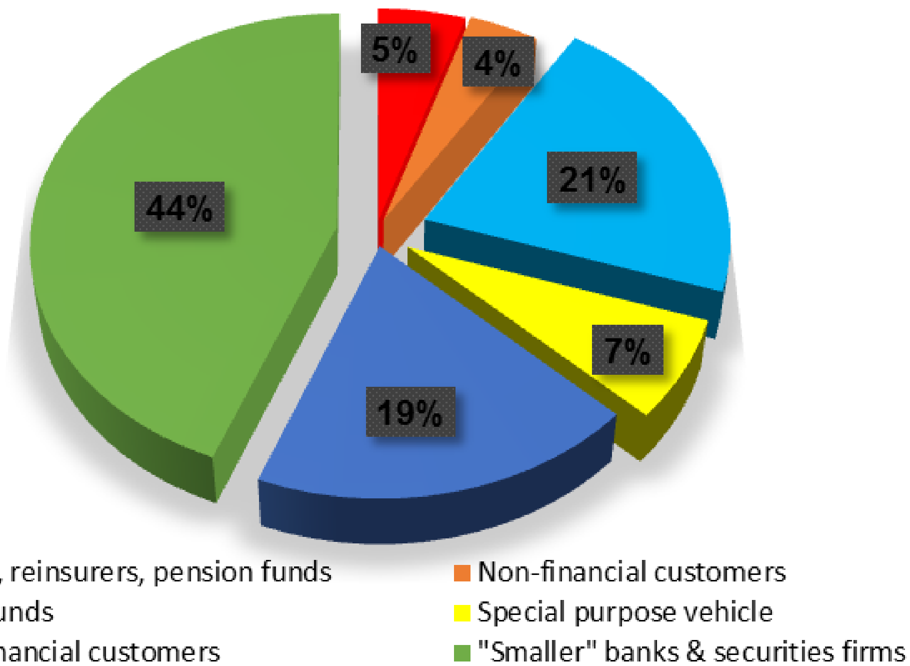


Figure 1.1: Participants of credit derivatives (*Source:* Bomfim (2015))

The banks are seen to be the most significant dealer in the market of credit derivatives, and as such, have tremendous trading activities. They provide liquidity by their willingness to assume the risk responsibility on their trading booklets, which they, in turn, seek to hedge. Hedge funds account for a little bigger than one-fifth of the total notional amounts, and they have grown significantly in terms of their trading activities in the market of credit derivatives measured over the past few years. In 2013, the notional amounts resulting from the contracts of the hedge funds with dealers increased tremendously from \$395 billion in 2007 to about \$1.1 trillion (Bomfim 2015). Furthermore, other financial customers, such as the mutual funds account for a little less than 20% of the outstanding notional, and their values were almost close to the hedge funds percentage in 2013. The non-financial customers, special purpose vehicle and the pension funds perform insignificant trading activities in the market, and often times, they are restricted with regards to the composition of assets which they are allowed to hold, and this mostly excludes credit derivatives.

Furthermore, with regards to the percentages of the market shares for the credit derivatives instrument, Figure 1.2 gives the statistics.

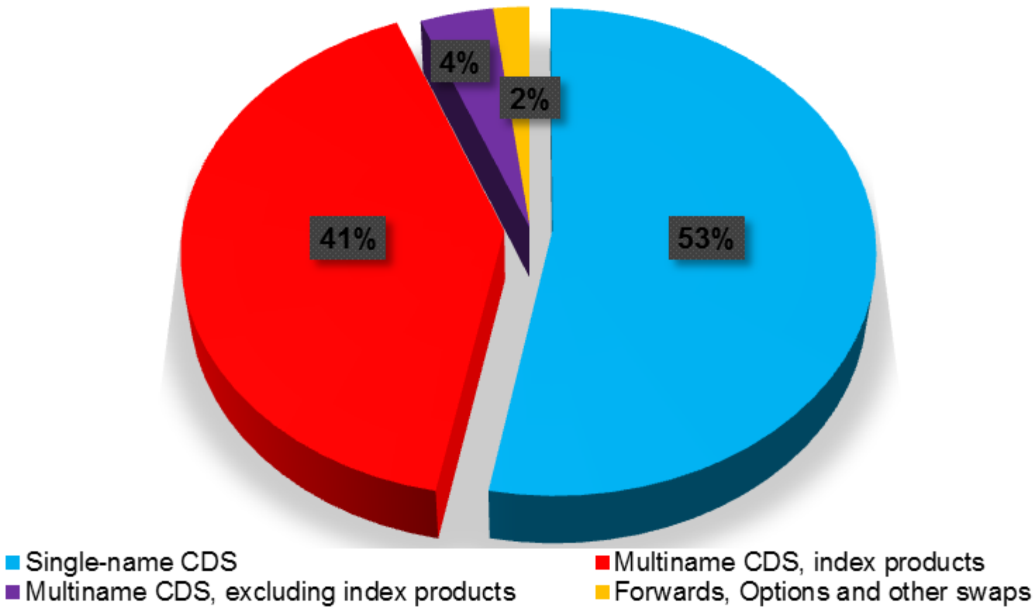


Figure 1.2: Percentage of market shares (*Source:* Bomfim (2015))

It is observed that the single-names derivative obviously constituted the vast majority. The index products, such as the full index trade, follow next with an increment from 24% market share in the mid-2010, to about 41% after 3 years, as conducted by the Bank for International Settlement in 2013. The third was the multi-name credit default swaps (CDS) without the index products, and this category contained the synthetic collateralized debt obligations (CDO), basket default swaps, mortgage-backed and asset-backed securities. The last components involve the forward contracts, options and other swaps, thereby making up roughly 2% of the whole market shares.

The simplest and the most crucial credit derivative, as stated above, refers to the CDS, which is a bilateral contract designed to alleviate the investor against potential losses in the event of a credit default. Schönbucher (2003) defined credit event with reference to repudiation, restructuring, bankruptcy, failure to pay, credit spread change, obligatory default and a downgrade by rating agencies.

Swaps

Swaps are derivative contracts whereby two counterparties agree to trade-off streams of cash flows or liabilities which are based on particular notional amount over a specified time period. There is always a risk of default in swaps, that is, a situation where one party fails to meet up with the obligatory requirements, and this can be regarded as counterparty risk. These financial instruments are generally not traded on exchanges but are over-the-counter derivatives, designed to suit the needs of the parties involved. One of the legs of the swap cash flow is fixed, and the other is floating. The floating leg changes and can be based on interest rates (interest rate swaps), currency (currency swaps), stock or equity index (equity swaps), default (credit default swaps) and commodity prices (commodity swaps).

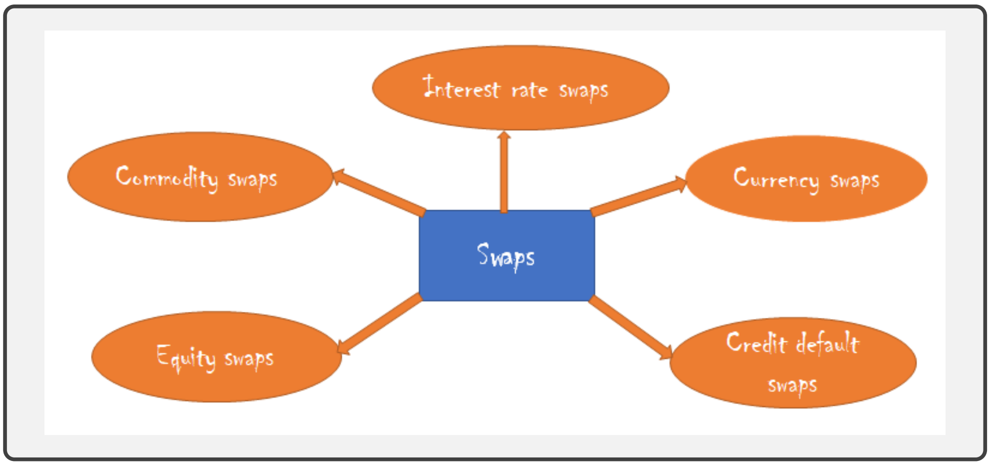


Figure 1.3: Introduction to swaps

Figure 1.3 shows the general classification of swaps. Interest rate swaps and currency swaps are closely related, the difference being that the former exchanges typically interest payments. In contrast, the latter is a foreign exchange agreement of cash amounts in two different currencies. According to the end-June 2010 report by von Kleist et al. (2010), the amount of the interest rate swaps dominate the other component swaps, with 367,541 for the interest rate swaps, 31,331 for the CDS, 18,890 for the currency swaps, 1,854 for the equity swaps, and 1,686 for the commodity swaps (all in billions of US \$). Often times, laws or investment policies prohibit certain portfolios from trading swaps; others are prohibited as a result of credit reasons. This is due to the fact that there is a high potential for credit risk since no clearing house is involved. Hence, the market of swap trading is presided over by the standard International Swap Derivatives Association (ISDA)[1] contract, which has to be endorsed by the counterparties prior to the swap trading. Adopting this systematic documentation for the CDS contracts by the ISDA has resulted in the significant liquidity, as well as rapid evolution pertinent in the CDS market.

The first swap agreements were developed and negotiated four decades ago, and now, swaps are among the most crucial and practical derivative instruments used in the debt capital trade. The vast majority of institutions, ranging from banks, corporates, mortgage banks, local authorities, etc., make effective use of swaps. As a result, the demand for swaps has grown exponentially and has gained more extensive acceptance in the debt capital markets. In fact, with regards to their applicability, Flavell (2010) explained that swap markets were designed to facilitate and raise cheap funds, as well as efficiently manage risk through the mechanism of comparative advantage. For the scheme of this research, we will limit our discussion to the credit default swaps, and more precisely, the basket credit default swaps.

Credit Default Swaps

The market of CDS contracts evolved originally from the customized agreements that existed between the banks and their respective clients, and as such, no exact date can be ascribed to these derivatives' origin. Hirsa and Neftci (2013) explained that commercial banks which are the largest provider and net buyers of CDS, use the contract to aid in the diversification of their portfolios, which often concentrate in certain geographical areas. One of the remarkable attraction of using the CDS is the fact that zero capital is required to validate the contract, and this makes it easy for investors to leverage their positions. Also, in the absence of the reference entity's cash bonds, one can still trade the CDS, and by buying the contract, one can take a 'short position' on the underlying credit risk. With regards to the identification of the

[1] https://www.isda.org/

CDS, Schönbucher (2003) explained that the following features have to be noted: reference entity, credit event (either downgrading, restructuring, failure to pay, etc.), notional CDS amount, start and the contract's maturity period, frequency of swap spread payment, as well as the default payment.

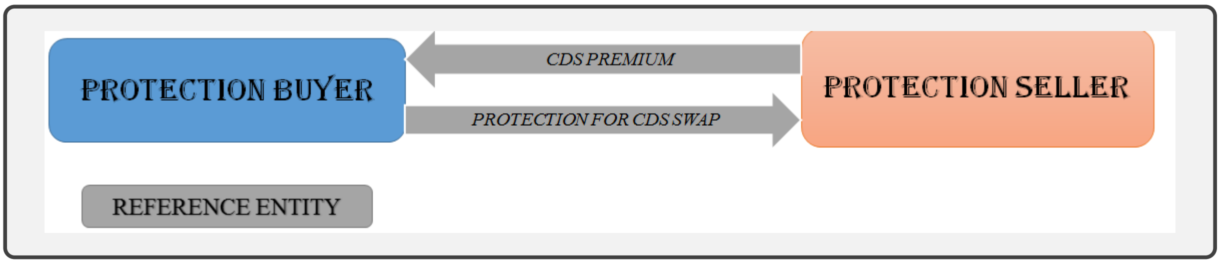


Figure 1.4: Credit default swaps

The CDS mechanism is normally undertaken by two counterparties A and B who are fully involved in this financial agreement, and it costs nothing to enter this swap contract. In this context, we refer the CDS contract as the single-entity CDS contract, since it is in connection with one reference entity, as shown in Figure 1.4. Party A, also known as the protection seller, agrees to make a default payment to party B, also known as the protection buyer if a credit event happens prior to the contract's expiration time. The payments collectively made by B to A make up the swap premium leg, whereas the contingent payment that A will make is known as the protection leg. The structure of this default payment replaces the loss which an ordinary lender would experience should the reference entity defaults. In the event of zero default till maturity, party A makes no payment; however, the counterparty B makes regular payments at intervals or an upfront lump-sum for default protection. Furthermore, suppose the accrued premium is included in the contract, and if the credit event happens in between the two payment dates, party B will have to make a payment that is normally a small portion of the subsequent payment which has accrued till the default time.

The payments in the premium legs of the CDS, referred to as spreads, are often quoted in basis points (1 bp = 0.01%) per year of the notional amount of the contract, and they are mostly settled quarter-annually. The CDS quarter payments usually are based on the 'actual/360 basis', meaning that there is an assumption of 360 days per annum and 'actual' refers to the actual number of days which exists in a quarter. These spread values are not quoted on any interest rate or even a risk-free bond; rather, they are based on the contract's notional value. Hirsa and Neftci (2013) further explained that the CDS spread works like a premium which is defined on a put option because the buyer will have to deliver the face value of the defaulted bond, or still get the difference between the par value of the bond and its recovery value when a credit event happens. Concerning the settlement of CDS contracts, these can be done via cash payment or physical delivery. For the former, A pays to B a cash amount which is equal to the difference existing between the underlying asset's value and the corresponding recovery value. If the settlement is done in a physical sense, counterparty B supplies A with the reference obligation, thereby getting an exchange for a face value payment. Figure 1.5 can equally be used to describe the payment streams of the CDS contract; the only difference being that CDS being considered here refers to the one-reference-entity agreement.

Mathematically, suppose that the expiration of the CDS contract is at time T, such that the payment schedule is $(T_1, T_2, \cdots, T_N = T)$, and let Δ_i be the day-count fractions or the frequency of swap premium payment. Also, let β be the swap premium; $f(0,t)$ be the risk-neutral discount factor at time t; τ be the entity's default time; R, the recovery rate and A, the notional amount of the entity.

Then the present value of the protection leg V_{pro} can be given as (Elouerkhaoui 2017):

$$V_{pro} = \mathbb{E}\left[A\int_0^T f(0,t)(1-R)\mathrm{d}\lambda_t\right],$$

where $\lambda_T = \mathbb{I}_{\{\tau \leq T\}}$ refers to the default indicator.

The present value describing the premium leg V_{pre} can also be written as (Elouerkhaoui 2017):

$$V_{pre} = \mathbb{E}\left[A\sum_{i=1}^{N} f(0,T_i)(1-\lambda_{T_i})\beta\Delta_i\right].$$

Thus, we solve for $\beta : V_{pro} = V_{pre}$, that is, we equate the two streams of payment legs to obtain the value of the CDS premium.

If the accrued premium is being considered, then this can be obtained by solving for the probability of default at each time period which exists between two different premium dates. The effect of this accrued premium on the swap spread is significantly small, even though the values are not negligible. Hence, the present value of the accrual premium leg V_{apre} can be approximated using the following expression shown in O'Kane and Turnbull (2003):

$$V_{apre} = \mathbb{E}\left[A\sum_{i=1}^{N} f(0,T_i)(\lambda_{T_i-1}-\lambda_{T_i})\frac{\beta}{2}\Delta_i\right].$$

Hence, we solve for the swap spread $\beta : V_{pre} + V_{apre} = V_{pro}$.

Basket Credit Default Swaps

BCDS are financial contracts whereby the protection seller agrees to compensate the protection buyer if the basket of reference entities experiences a specified credit event. Focardi and Fabozzi (2004) classified the BCDS as subordinate BCDS, the senior BCDS and the nth-to-default (n2D). In the subordinate BCDS, a maximum payout is allocated for each of the defaulted entity, as well as a maximum cumulative payout which is distributed out over the contract's tenor for the entities in the given portfolio. In a senior BCDS, each reference entity has a maximum payout, and this payout is triggered when a certain threshold is attained. For the n2D basket swaps, there is usually a payoff by the seller in case the nth reference entity defaults, and for the first $n-1$ entities' default, no payment will be made.

Figure 1.5 shows a typical n2D basket swap, and it will be the benchmark of this thesis. For $n = 1$, we have the first-to-default (F2D) basket swap, and this is triggered if only one of the entities defaults. For $n = 2$, we have the second-to-default (S2D) basket swap which comes into effect when the second entity defaults. Thus, the contract continues in existence even if there is just one default amongst the reference entities, and ceases to exists if there is s second default. There are yet other classes of the basket default which are not highly liquid, that is, the n-out-of-m-to-default, which payoff when there exist a first n defaults in an m-entity portfolio. Then, the all-to-default, whereby the whole reference entities in the basket defaults before the contract will be void.

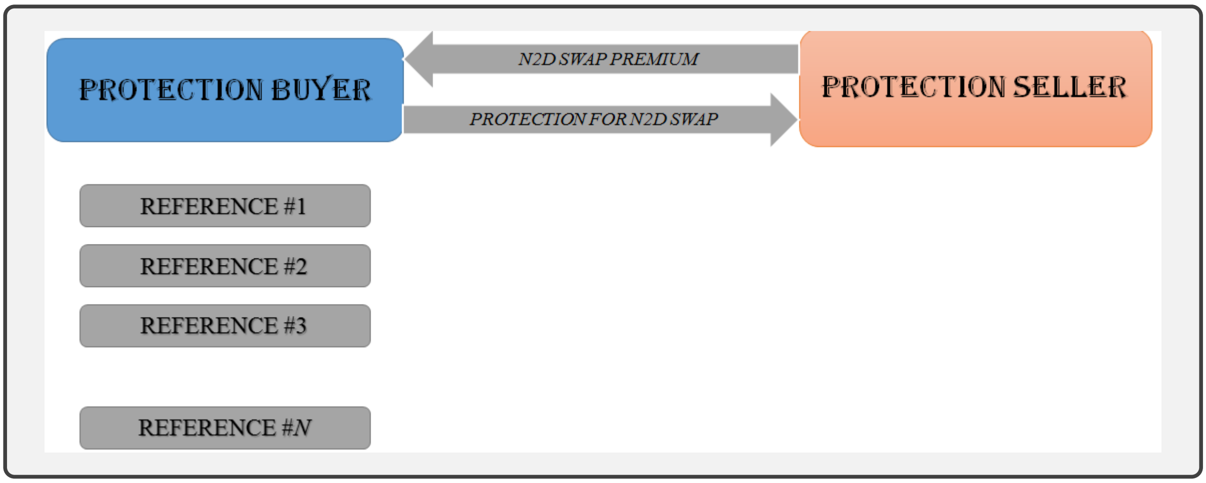


Figure 1.5: Basket credit default swaps

The number of obligors or reference entities in the basket or portfolio (typically between 5 and 10), the default correlation amongst the entities, the credit quality of the entities, as well as their corresponding recover rates hugely determine the payment that the buyer will make. Amongst these parameters, the default correlation assumes a considerable role in the BCDS pricing, and that made some investors to categorize these financial instruments as default correlation products (Bomfim 2015). If the default correlation is higher in a given basket, then there will be a greater likelihood that default on one entity would result to a simultaneous default on another entity, and this can equally result to default correlation risk. The key associated to the modelling of the BCDS involves the estimation of the joint probability distribution function describing the default times τ_i, which in turn can affect the correlation degree of default risk in the portfolio. Duffie and Singleton (2012) explained that to model the correlated default in a basket, one can use the following methods in the estimation: copulas, CreditMetrics, correlated default intensities which follow the doubly stochastic models, and finally, the intensity-based models having joint default events. Thus, this thesis mostly utilized the copula models in the estimation of the joint probabilities of default times.

Although the single-name CDS are hugely traded on the credit derivatives market, the multi-name CDS, which includes the basket CDS (BCDS) or the CDS index have equally received much attention over the past few years. O'Kane (2011) explained that the emergence of default baskets began in the late 1900s, and they initially dominated other correlated credit derivative products, though the synthetic CDO instruments have replaced this position. In spite of this, default basket products continue to appeal to credit investors owing to the attractive features these contracts possess, and one of those advantages is described in the subsequent example below.

Example 1.1. *Assume that there seven entities in a given basket and let the notional amount of each reference entities is \$5 million. Also, let the basket premium be quoted at 100 bp per annum, with zero recovery rates, then the buyer will make an annual payment of*

$$\frac{100}{100^2} \times \$5 \; million = \$50000\,,$$

or equivalently, \$12,500 for quarter-annual payment.

From Example 1.1 above, the total worth of the portfolio is \$35 million, but \$5 million was taken into perspective in the computation of the buyer's payment. This is essentially because the seller only has to cover for the first default (F2D swaps), and the worth for each default only is quoted at \$5 million. The buyer thus pays less to cover this portfolio, instead of purchasing individual protections which are obtainable in the single-name CDS contracts. On the other end, the protection seller will be opportune to experience limited downside risk, as well as offer substantial leverage on its credit exposure (Bomfim 2015).

Mathematically, assume that there are N basket CDS contracts which mature at time T, such that, the payment schedule is $(T_1, T_2, \cdots, T_N = T)$, and let Δ_i be the day-count fractions or the frequency of swap premium payment. Also, let $f(0,t)$ be the risk-free discount factor at time t; β be the swap premium; R_i, the recovery rates of each entity; A_i, the notional amounts of each entity; τ_i, the default times where $i = 1, 2, \cdots, N$; and $\lambda_T^i = \mathbb{I}_{\{\tau_i \leq T\}}$, the indicator functions of each of the reference entities. Furthermore, define the nth ordered statistic of default times, such that $\tau^1 < \tau^2 < \cdots < \tau^N$, where τ^1 refers to the time when the F2D credit event occurred (Choe and Jang 2011a), and let the default indicator functions of the nth ordered default times be $\lambda_T^{\tau^n} = \mathbb{I}_{\{\tau^n \leq T\}}$, where $1 \leq n \leq N$.

Then the present value of the protection leg V_{pro}^n can be written as follows (Elouerkhaoui 2017):

$$V_{pro}^n = \sum_{i=1}^{N} \mathbb{E}\left[A_i \int_0^T f(0,t)(1 - R_i)(1 - \lambda_t^{\tau^n}) \mathrm{d}\lambda_t^i\right].$$

The present value of the premium leg V_{pre}^n can also be written as (Elouerkhaoui 2017):

$$V_{pre}^n = \sum_{i=1}^{N} \mathbb{E}\left[A_i \sum_{j=1}^{N} f(0,T_j)(1 - \lambda_{T_j}^{\tau^n})\beta\Delta_j\right].$$

Thus, we solve for $\beta : V_{pro}^n = V_{pre}^n$, that is, we equate the two streams of payment legs to obtain the value of the basket CDS premium.

1.2 Problem Statement

The values for credit derivatives are hugely influenced by certain underlying assets or entities which are credit-sensitive. They are used by market participants as tools, not for transferring legal ownership, but for shifting credit risks from one party to the other. Weistroffer (2009) further gave a quick review on the theoretical underpinnings surrounding the usage of the CDS, thereby explained that the CDS is a useful tool for risk management, a trading instrument, a means to measure credit risk, an efficient risk allocation, and as a tool used by banks to manage their loan portfolios. These appealing concepts have made the market of credit derivatives to be increasingly flexible, thereby resulting in their rapid utilization by investors and their enormous growth over the past years (Choe and Jang 2011a). When compared with the other over-the-counter (OTC)[2] credit derivatives traded in the markets, von Kleist et al. (2010) stated that the CDS (on a single entity) accounted for about 99% of positions amongst other credit derivatives. This end-June 2010 survey was in contrast to the 88% of positions during the end-June 2007 report, and this accelerated growth reflects an immense interest in the use of these instruments.

On the other hand, the multi-name credit derivatives[3], with much emphasis on the basket CDS, have equally attracted some attention over the last few years (O'Kane 2011). These multi-name derivatives contracts are advantageous because they offer the investors some appealing opportunities of leveraging the spread premium, and of using one single contract to hedge a portfolio of contingent claims, such as bonds or loans. Thus, the need for obtaining a series of contracts for single securities has been averted, and also, the improvement of the relative risk-return account as compared to other equivalent credit investment tools will be made viable. These credit derivatives are connected with multiple defaults which trigger some remedial payments from the protection seller down to the protection buyer, in the case of credit events; else, these contracts remain valid till the expiration date upon which they expire. The premium paid by the protection buyer in a basket CDS contract is duly affected by several factors such as the number of entities or obligors in a given portfolio, default correlations amongst entities, the credit quality of each portfolio component, as well as their expected recovery rates. With regards to default correlations, they are fully evident in the joint probability of the default times, and as such, measures the tendency of entities in a portfolio to default together. Since defaults are rare occurrences, empirical studies which focus on default correlations seem difficult. However, models such as copula models, conditional independence models and the contagion model have been developed and widely investigated in works of literature.

One of the major issues pertaining to the pricing of the BCDS involves the computation of the joint survival or the joint default probability distributions, and also the joint dependent defaults. In fact, Bomfim (2015) explained that modelling defaults is quite problematic due to the fact that they are regarded as exogenous events which occur at random times. As a result, some researchers have employed different techniques to handle these default probabilities and

[2]This refers to an off-exchange trade that is done exclusively between two counterparties, and often times without keen supervision.

[3]These include basket CDS, the CDOs, the CDS indices, the CDO-squared tranches, leveraged super-senior (LSS) tranche, etc.

dependent defaults evident in the valuation of credit derivatives. For instance, Cifuentes and O'Connor (1996) in Moody's Investors Service introduced the binomial expansion method to model independent default; Andersen et al. (2003) used the recursion techniques; Zhou (2001) employed the idea of the first-passage-time model for multiple default modelling. Hull and White (2004) introduced the concept of 'probability bucketing' numerical approach; Laurent and Gregory (2005) and Gregory and Laurent (2003) employed a semi-analytic scheme for the computation of the default distributions.

Furthermore, when the contagion effects are introduced in the process, that is, when the correlation or even the recovery rates becomes loss dependent, then Bush et al. (2011) explained that the loss distribution functions would become more skewed, thereby resulting to more flexibility of data alignment. Hence, incorporating the infectious default or the default contagion model, Haworth and Reisinger (2007), Davis and Lo (2001) and Herbertsson (2008) were able to model dependent and correlated defaults under different specified approaches. Additionally, Lin et al. (2011) focused on the primary-subsidiary heterogeneous case which is defined in a framework of interacting intensity; Duffie and Garleanu (2001) applied the idea of correlated default intensities but with default times which are conditionally independent, to compute the default distributions. Whereas, others equally incorporated the copula model to aid in the estimation of multiple default risks associated in the portfolio (Aas (2004); Ackerer and Vatter (2017); Embrechts et al. (2001); Embrechts et al. (2002) and Frey and McNeil (2003)).

With regards to the numerical implementation of the basket CDS, several simulation techniques and semi-analytical schemes have been developed. Muroi (2006) applied the concept of asymptotic expansion approach to obtain the closed-form approximation equations for the pricing of contingent claims such as the basket CDS and the swaptions equivalent. In this approach, the interest rate and the default hazard rates were first modelled via the Gaussian, the Cox-Ingersoll-Ross (CIR) and the Constant Elasticity of Variance (CEV) models, and the corresponding values were compared to the Monte-Carlo simulated values. The results obtained showed that the asymptotic expansion approach is more applicable and suitable for the price valuations of various credit derivatives. Furthermore, Bastide et al. (2008) employed the Stein numerical approach, which was introduced by El Karoui, Jiao and Kurtz[4], to price the basket CDS and the CDO tranches. They also compared their results to the values obtained by the use of the probability generating function approach, the Monte-Carlo simulations, the recursive method which was put forward by Hull and White (2004), and

[4]This work later appeared both in El Karoui and Jiao (2009) and Jiao et al. (2008).

discovered that the Stein method outperforms other methods in terms of its accuracy and its speed. Other numerical approaches to basket CDS can be found in Zheng (2006), Schröter and Heider (2013a), Schröter and Heider (2013b), Fadel (2010) and Chen and Glasserman (2008).

Generally, the valuation of the BCDS involves the computation of the default legs, accrued premium legs and the premium legs, which in turn, are used to obtain the fair swap prices. The joint probability of default is the crucial element utilized in the estimation of these payment streams, and the thesis gives a broader viewpoint on this issue. Hence, this thesis will tackle the BCDS valuation problem in two folds, as shown in the flowchart in Figure 1.6. First, through the numerical approach which involves the implementation of the Monte-Carlo simulation method and second, the quasi-analytical scheme which will be carried out via the implementation of the discrete Fourier transform. In the first two phases of the thesis, we will only focus on the pricing of the BCDS using the Monte-Carlo simulation method and the copula models. In contrast, the last part of the thesis will focus on the application of the discrete Fourier transform and the copula model in the estimation of the swap premiums.

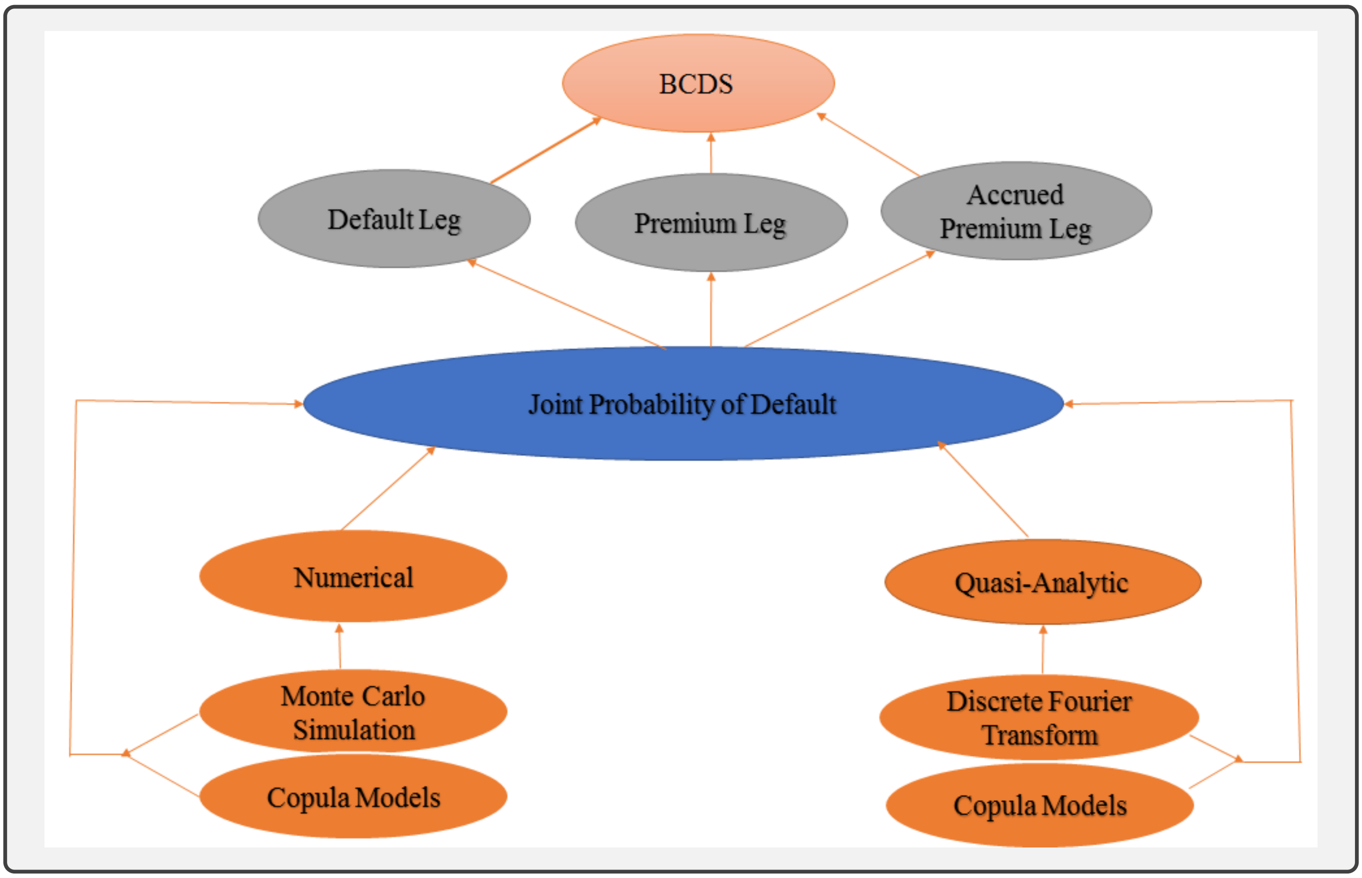


Figure 1.6: A flow chart of the problem statement

1.3 Aim and objectives of the study

The main aim of this thesis is to obtain the valuation of the basket credit default swaps (nth-to-default) prices using numerical and quasi-analytical methods.

The specific objectives of this research include:

- Incorporating the combination of the stochastic default intensity models (Vasicek and the CIR, for instance) and the Monte-Carlo simulations in the valuation of hazard rates and default times, thereby valuing the n2D basket swaps.
- Incorporating the combination of the elliptical and the Archimedean copula models, together with the Monte-Carlo simulations in the estimation of the default times, thereby valuing the n2D basket swaps.
- Applying the combination of the discrete Fourier transform and copula models (as a quasi-analytic scheme) in the approximation of the joint probability of defaults, thereby valuing the n2D basket swaps.

1.4 Motivation

Notwithstanding the financial crisis which existed in 2008, the importance of valuing credit default swaps and their basket equivalence remains unmatchable, as investors, commercial and financial practitioners continuously aim at transferring or redistributing credit risk using these portfolio credit derivatives while ensuring liquidity in the financial markets. Thus, the emergence of such market has attracted traders such as speculators, hedgers, arbitrageurs, and it has given them a chain of innovative investment strategies. The impact of this research work is to obtain the fair prices of credit default swaps (CDS), of the basket type, using copula models. The following are seen to be the motivation, as well as the potential impact of this thesis:

To the theoretical framework, this work will contribute scientifically to research and academia. To our knowledge, this research topic has received little or no attention in the Financial Mathematics and Quantitative Finance sector and thus, we seek to make an immense contribution to the existing knowledge. Therefore, the knowledge will be fully evident in the scientific communication of our research in terms of scientific presentations and articles publishing.

This thesis, together with the mathematical theory and the computational techniques, will give financial researchers a broader perspective in the proper understanding of credit derivatives, as multiple sources of risks are inherent in the modelling of these derivatives. The knowledge and the incorporation of copulas to default time modelling can help in the efficient analysis and management of credit risks, thereby resulting in profitable portfolio decisions in the financial markets. The quasi-analytic techniques employed in the estimation of the likelihood of default amongst a portfolio of entities will provide a more accurate and efficient method on how credit derivatives can be valued, without the random simulation of basket default times.

1.5 Method of Investigation

The method of the investigation which will be conducted in this thesis includes:

- **Mathematical Modelling and Analysis**: To model the joint survival probability distribution of the homogenized portfolio under both the Vasicek and the CIR model. To model the hazard rate process using stochastic default intensity models. To model the distribution functions of the default time using the Monte-Carlo simulations, under the one-factor elliptical and the Archimedean copula models. To conduct some sensitivity studies of the n2D basket swaps with respect to the time of expiration T, the hazard rates λ and the concordance parameters (such as ρ- for the elliptical copulas, and θ- for the Archimedean copulas).

- **Data Analysis**: To employ the concept of the Maximum Likelihood Estimation (MLE) in the estimation of the default intensity parameters for each of the entities, using the option-adjusted spread (OAS) bid values of corporate entities, and it will be applicable to the Vasicek model. To employ the approximate optimal estimating function approach which describes the CIR model in estimating the parameters of the CIR model. To conduct some empirical and statistical analysis of 5-year monthly CDS spread data of some randomly selected firms which range from the higher-rated to the lower-rated firms.

- **Pricing**: To obtain the values of the hazard rates, swap spreads as well as the corresponding equal-weighted portfolio values under the heterogeneous portfolios. To obtain the values of the default legs, accrued premium legs, the premium legs and the swap prices, under both the elliptical and the Archimedean copulas, via the Monte-Carlo simulations. To estimate the prices of the default legs, the premium legs and the fair swap spreads using a combination of the copula models and the discrete Fourier transform.

- **Numerical Implementation**: All the numerical experiments in this thesis, which will be conducted via the Monte-Carlo simulations, the Fast Fourier transform and the Inverse Fourier transform will be implemented in the `jupyter notebook`, version `python 3.6.1`.

1.6 Thesis Overview

Chapter 1: This chapter will focus on the general introduction of the thesis. This prelude will include an extensive background statement, the problem statement, the principal aim and specific objectives of the study. The motivation for the research and the method of investigating the research problem will also be projected. Finally, the chapter will conclude by providing a specific outline or the scope of the overall thesis.

Chapter 2: This chapter will explain more about the thesis' conceptual overview, and this will involve introducing the mathematical methods, such as the Monte-Carlo simulations and the Fourier transform techniques. These mathematical approaches will serve as the framework methods utilized in the thesis. Further details on the financial and the statistical concepts, inclusive of the CDS, the CDS of the basket type, elliptical and the Archimedean copula modelling will also be discussed.

Chapter 3: This chapter will provide an investigative study on the concept of basket default swap pricing. Here, we will model the hazard rate or the intensity default process using the one-factor Vasicek and CIR models. The survival probability distribution of the underlying entities will be obtained, and the Monte-Carlo simulations will be employed to model the default time, hence, leading to the valuation of the n2D basket swaps. The pricing done in this chapter will equally be applicable to both the homogeneous and the heterogeneous basket. Note: This chapter will be entirely linked to the article titled, 'Valuation of basket credit default swaps under stochastic default intensity models', which has been accepted to be published by *Advances in Applied Mathematics and Mechanics.*

Chapter 4: An overview of the copula models, which are essential quantitative methods, will be discussed in this chapter. This chapter will conduct a comparative analysis of the effect of pricing portfolio credit derivatives using various copula models, including both the elliptical models and the Archimedean models. They will be incorporated to model the corresponding default times, using the Monte-Carlo simulations as the benchmark in the estimation of the corresponding prices of the swap premiums. This chapter will be entirely linked to the article titled, 'Elliptical and Archimedean copula models: an application to the price estimation of portfolio credit derivatives', which has been published by *Risk Journal*.

Chapter 5: In order to overcome the issues of huge simulation number encountered via the simulation method and to accelerate the convergence analysis, this chapter will introduce the discrete Fourier transform as an alternative to the computation of the semi-explicit premiums of the basket default swaps. This numerical method will be utilized to reduce the computational complexity, as well as the difficulties of the dimensionality problems encountered during the approximation procedures. This chapter will be entirely linked to the article titled, 'Pricing basket default swaps using quasi-analytic techniques', which has been submitted to the *Decisions in Economics and Finance* for publication.

Chapter 6: This chapter will highlight and discuss the main significance of the thesis, thereby providing the conclusion and some appropriate recommendations of the study. Future research will also be suggested.

2. Mathematical Methods

This chapter will explain in detail the numerical and statistical methodologies employed in this thesis.

2.1 Numerical Techniques

With regards to the numerical approaches, the Monte-Carlo simulations (MCS) and the Fourier transform (FT) techniques are the main mathematical methods applied in this research. The following subsections give the overview.

2.1.1 Techniques of the Monte-Carlo Simulations

MCS is a class of numerical algorithm that depend on statistical analysis and continuous random sampling in order to output their results. They provide a more flexible method of approximating the average μ of random variables X with explicitly unknown distribution. For instance, in finance, the variable X could be the discounted payoff of a contingent claim, whose values depend on the subsequent performance of the entities or assets which follow a stochastic model. As such, the μ assumes the fair option price. The key idea to this technique is that the mean of random samples, $X_1, X_2, \cdots, X_K$, tends to μ as K goes to infinity, and the strong law of large numbers (Theorem 2.1) ensures that this convergence occurs almost surely.

Theorem 2.1. *(Strong Law of Large Numbers, (Kroese et al. 2013)).*

Define the integrable sequence $[x_k]_{K\in\mathbb{N}}$ as independent identically distributed random variables on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mu = \mathbb{E}[x_K]$ for all $\eta \in \Omega$, then

$$\frac{1}{K}\sum_{k=1}^{K} x_k(\eta) \xrightarrow{a.s} \mu, \quad K \to \infty\,.$$

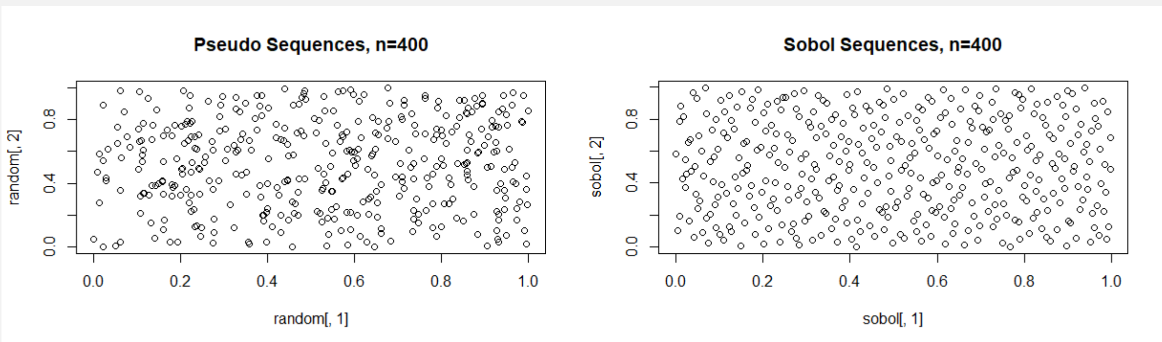
In finance and according to Jäckel (2002), the most frequent implementation of the MCS is obtainable during the computation of the expected value of a given function $f(\mathbf{z})$ which is defined in a particular probability density function $\psi(\mathbf{z})$, over $\mathbf{z} \in \mathbb{R}^p$,

$$\mathbb{E}_{\psi(\mathbf{z})}[f(\mathbf{z})] = \int f(\mathbf{z})\psi(\mathbf{z})\mathrm{d}\mathbf{z}^p \,. \tag{2.1.1}$$

Normally, the standard MCS suffers from two primary sources of estimation errors which span from the $\frac{1}{\sqrt{K}}$ factor and the standard error σ resulting from the output. If the variance, as well as the standard deviation, become smaller, then there will be a high likelihood of obtaining a more accurate result. Some variance reduction, such as antithetic variates, importance sampling, control variate and the stratification techniques have been designed to reduce the error estimate of the MCS method. Following the techniques of the random numbers generation, the MCS method can be classified into the standard MCS and the Quasi MCS. The essence of using both the standard and the quasi MCS involves approximating the integral of a given function f defined over a p-dimensional unit interval, as the mean of f evaluated at discrete points $x_1, x_2, \cdots, x_K$, as

$$\int_{[0,1]^p} f(\mathbf{z})\mathrm{d}\mathbf{z} = \frac{1}{K}\sum_{k=1}^{K} f(x_k)\,,$$

where $\mathbf{z}$ represents the uniform random variables which are independently distributed on $[0,1]^p$. The standard MCS are defined based on the generation of their random points, and these come from the pseudo-random sequences, whereas the quasi MCS uses the low discrepancy sequences such as the Sobol, Koborov, Faure and Halton sequences to choose these random points deterministically and systematically. Figure 2.1 shows the nature of the 400 random points for $p = 2$.



(a) Standard MCS (b) Quasi MCS

Figure 2.1: Random numbers simulation

The pseudo-random sequences are generated by Pseudo-Random Number Generator (PRNG), and a good example is the linear congruential method (Kroese et al. 2013). They have a slower convergence rate $\mathcal{O}\left(\frac{1}{\sqrt{K}}\right)$ in terms of the K number of nodes, with probabilistic error bound of (Glasserman 2013):

$$\left|\frac{1}{K}\sum_{k=1}^{K} f(x_k) - \int_{[0,1]^p} f(z)\mathrm{d}\mathbf{z}\right| \leq Z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{K}}\,.$$

Definition 2.2. *(Discrepancy, (Glasserman 2013)).*

Let $\mathcal{R}$ be a collection of Lebesgue measurable subsets R defined on $[0,1)^p$, the discrepancy of the points $\{x_1,\cdots,x_K\}$ relative to $\mathcal{R}$ is defined as:

$$D(x_1,\cdots,x_K;\mathcal{R}) = \underset{R\in\mathcal{R}}{Sup}\left|\frac{\#(x_k\in R)}{K} - vol(R)\right|\,, \tag{2.1.2}$$

where $\#(x_k \in R)$ refers to the number of points x_k which are contained in set R; and $vol(R)$ refers to the measure or the volume of R.

Thus, the discrepancy level measures the supremum over all the errors encountered during the integration of the indicator function of the set R using x_k points. The quasi-random points, on the other hand, have a higher degree of uniformity which are mostly reflected in their discrepancy levels, as compared to the pseudorandom points. They have faster convergence rate $\mathcal{O}\left(\frac{(\log K)^p}{K}\right)$ in terms of the K number of nodes, and the deterministic error bound is given as (Glasserman 2013):

$$\left|\frac{1}{K}\sum_{k=1}^{K} f(x_k) - \int_{[0,1]^p} f(z)\mathrm{d}\mathbf{z}\right| \leq V(f)\times D^*(x_k)\,,$$

where D^* refers to the star discrepancy. Given Definition 2.2, the star discrepancy also denoted by $D^*(x_1,\cdots,x_K;\mathcal{R})$ exists if $\mathcal{R}$ consists of the collection of every rectangular subsets in $[0,1)^p$ which has the form $R=\prod_{i=1}^{p}[0,u_i)$, where $0\leq u_i\leq 1$ for $1\leq i\leq p$. Also, the term $V(f)$ is a constant function which depends only on f. Thus, these discrepancy measures are essential in the convergence rate analysis of multi-dimensional integration.

In the pricing of BCDS, we employed the MCS to numerically simulate mostly the default times via different distribution functions, which in turn were used to compute the default and the survival probabilities. These simulations were used in the thesis in connection with the copula models. For instance, we generated uniformly distributed random variables, and then random numbers from the normal distribution (Gaussian copula) gamma distribution (Clayton copula), exponential distribution (Gumbel copula), chi-square distribution (student-t copula) and the logarithmic distribution (Frank copula). Finally, we utilized these probability distributions to estimate the swap payment legs and the break-even swap prices.

2.1.2 Fourier Transform Techniques

The FT approach is mode efficient, reliable, and it presents many benefits compared to the MCS method. It is very fast in its computation, avoids the convergence problems which are typically associated with the MCS approach, and it is easily implementable in certain computing software such as Python, Microsoft Excel, etc. One motive of the FT techniques emanated from studying Fourier series, as FT aims at extending Fourier series limit to approach infinity. The periodic functions associated with the Fourier series can be decomposed into sine and cosine functions or in its discrete exponential forms, whereas the FT is mostly for non-periodic functions. The Fourier transform (FT) technique has remained an increasingly common and significant tool in finance and economics. According to O'Kane (2011), the idea of the FT approach spanned from mapping a specific problem into another space which renders the problem analytically tractable. Next, the problem will be solved in that space, after which the solution will be transformed back to the original space.

2.1.2.1 Discrete Fourier transform

The discrete Fourier transform (DFT) converts one function into another referred to as the frequency domain representation, and it is an ideal technique for processing information which are stored in the computers. They are mostly applied in the field of digital signal processing and other related fields to perform convolution operations, to solve PDEs, as well as in time-frequency analysis and signal spectral analysis (Cherubini et al. 2010). The techniques of the DFT focus on evenly distributed points which are defined in a circle, and mathematically, these evenly spaced points are mostly described by complex numbers. Černỳ (2004) gives a more geometrical representation of the concept of DFT.

Definition 2.3. *(DFT, Debuysscher et al. (2003)).*

Let $z_k = \mathrm{e}^{\frac{2\pi i}{K}}$ *be a* K*th root of unity;* $\{x_k\} := x_0, x_1, \cdots, x_{K-1}$ *and* $\{X_m\} := X_0, X_1, \cdots, X_{K-1}$

be sequences of K complex numbers. The DFT of $\{x_k\}$ is $\{X_m\}$ such that

$$X_m = x_0(z_k^m)^0 + x_1(z_k^m)^1 + \cdots + x_{K-1}(z_k^m)^{K-1} = \sum_{k=0}^{K-1} x_k z_k^{mk}\,.$$

The only differences between the DFT and the discrete inverse Fourier transform (DIFT) are their opposite exponent signs, together with the presence of the $\frac{1}{K}$ coefficient for the DIFT. Thus, this implies that the methodology for solving the DFT is equally applicable for the inverse counterpart, but only with a minor modification. From Definition 2.3, the vectors of x_k's are multiplied by a matrix that has (m, k)th element to be the constant z raised to the power of $m \times k$. This matrix multiplication results in the vector components of the X_m. Thus, in practice, estimating x_k from X_k, or vice versa with K data size requires K^2 multiplications which will be made of complex quantities, together with an addition of $K(K-1)$ complex sums. These numerical computations are generally independent and highly computationally intensive. Thus, the introduction of an efficient algorithm (Fast Fourier Transform) with only $K\log_2(K)$ computer operations made such computation easier (Fusai and Roncoroni 2007). The larger the data size or the Fourier resolution K, the higher the 'resolution' which means higher accuracy.

2.1.2.2 Fast Fourier transform

The Fast Fourier Transform (FFT) was originally developed by Cooley and Tukey (1965), as a fast and efficient computer algorithm used to estimate the values of the DFT of a given sequence. Generally, by performing binary decimation, it converts a sampled signal in space or time to the same signal which is sampled in frequency, and this technique is widely employed in science, engineering, music, information science and mathematics. This technique is referred to as 'divide-and-conquer', as it aims at recursive disintegration of large-sized DFT into many minute DFTs, together with series of $\mathcal{O}(K)$ multiplications by roots of unity which are complex (Cherubini et al. 2010). According to Press et al. (2002), the computing time for the implementation of the FFT is entirely considerable. For instance, if $K = 10^6$, the FFT method will take about 30 secs of the CPU time to output its results. This is in contrast to the standard microsecond cycle time computer which will take an estimated number of 2 weeks for its implementation, and the major drawback for using the FFT is that the data size K must be in powers of 2, that is, $K = 2^x$. The inverse fast Fourier transform (IFFT), on the other hand, is a fast implementation of estimating the DIFT values which describe a given sequence of points.

One of the earliest significant breakthroughs in derivative pricing using the FT techniques was done by Carr and Madan (1999), where they efficiently employed the FFT to obtain the price valuation of options. To efficiently price these derivatives, they assumed a known analytical value of the characteristic function which defines the risk-free density function, and this numerical concept was later applied by Fusai (2004) to Asian options. Hirsa and Neftci (2013) further explained that the probability distribution function of asset or entity prices can be recovered from their characteristic function via the method of the Fourier inversion method, and this concept is essentially crucial for many models whose characteristics function could be represented in their analytical forms or even semi-analytically. Abate and Whitt (1992) equally employed the FT approach to approximate the values of the probability mass functions and the cumulative distribution functions, by the numerical inversion of their characteristic functions, generating functions and Laplace functions. Whereas, Debuysscher et al. (2003) explained that the FT approach is applicable in determining the market value distribution, return distribution and the loss distribution of portfolios of entities in a specified time horizon.

In this thesis, we investigated the Fourier transform techniques in order to estimate the default distribution or the portfolio loss distribution over a given time period. This approach, in turn, was used in connection with the copula models to semi-analytically price the BCDS. Here, we employed the concept of O'Kane (2011) who used the FT, together with the conditional independence of basket entities, to calculate the conditional loss distribution function. Here, the characteristic function of the aggregate portfolio loss distribution, which is conditional on a common factor, is first defined. Next, we estimate the unconditional characteristic function for the total loss written in terms of the conditional characteristic function. This can be done via numerical integration over the market's common factor, and techniques such as Gaussian approximation (Shelton 2004); Gauss-Hermite quadrature (Press et al. 1992); trapezoidal rule, binomial and the adjusted binomial approximations (O'Kane 2011) can be applied to it. Finally, the FFT techniques are applied to invert these unconditional characteristics functions in order to output the final probability loss distribution function. These probability distributions are in turn used to obtain the payment streams of the BCDS premium values.

2.2 Statistical Techniques

In this section, we introduce and describe the statistical techniques employed in this thesis, and this is predominantly the copula models.

2.2.1 Copula Models

A copula function generally links the univariate distributions, thereby generates the corresponding multivariate distribution functions. The concept of copula makes the issue of modelling joint stochastic processes of a basket of entities less complicated. This is because the marginal distribution functions can be separately estimated from the corresponding joint distribution functions, instead of simultaneously obtaining the estimate of all the parameters of the distribution. Thus, in the analysis of credit risks using copula models, it will be easier to allocate each primary issuer to various marginal distributions given a copula. Hence, it will be possible to investigate how each default in a credit basket affects the behaviour of multiple defaults of many credit portfolio issuers.

Furthermore, with the marginal distributions, one can vary the structure of the correlations by selecting different copulas, or even the same copula model that has various parameter values. As such, it will be possible to numerically measure the impacts of default correlations in a credit risk portfolio. Copulas have become paramount tools in multivariate modelling, which are employed by diverse fields. For instance, actuarial science use the copula models to estimate dependent losses and credibility (Frees and Wang 2005); Finance use copulas in risk, as well as in portfolio management (Embrechts et al. 2001), asset return and allocation (Bouyé et al. 2000); Econometrics employ the copulas for modelling joint parametric distributions (Trivedi and Zimmer 2007); as well as in the field of civil engineering to estimate the joint probability flow at stream confluences (Kilgore and Thompson 2011).

Definition 2.4. *(Copula, Trivedi and Zimmer (2007)).*

A copula $C = C(v_1, v_2, \cdots, v_d)$ *is a multivariate probability distribution, possessing uniform marginals in a unit interval. Mathematically, it is defined by* $C : [0,1]^d \to [0,1]$*, having the following properties:*

- $C(v_1, v_2, \cdots, v_d) = 0$ *if* $v_j = 0$*, for any* $j \leq d$.
- $C(1, 1, \cdots, 1, v_j, 1, \cdots, 1) = v_j$*, for every* $j \leq d$ *and for all* $v_j \in [0,1]$*;*

- $C(v_1, v_2, \cdots, v_d)$ *is d-increasing, that is* $\forall M = M_1 \times M_2 \times \cdots \times M_d$ *where* $M_j = [a_j, b_j] \subset [0, 1]$ *for* $j = 1, 2, \cdots, d$*. Then we have*

$$V_C(M) = \int_M \mathrm{d}C(v_1, v_2, \cdots, v_d) = \sum_{\alpha \in M} sign\ (\alpha) C(\alpha) \geq 0\,,$$

where the summation is over all the vertices $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d)$ *of the hyper-rectangle* M*, that is,* $\alpha_j = a_j$ *or* $\alpha_j = b_j$ *and*

$$sign\ (\alpha) = \begin{cases} 1 & \text{if } \alpha_l = a_l \quad \text{for } l\text{-even number} \\ 1 & \text{if } \alpha_l = a_l \quad \text{for } -l\text{-odd number}. \end{cases}$$

Thus, we can assert that a d-copula is a d-dimensional function on $[0, 1]^d$ having standard marginals which are uniformly distributed. A very crucial statistical theory with regards to the copula models is the Sklar's theorem, and this theorem asserts the disintegration of any multivariate probability distribution into its copula components and their respective univariate margins.

Theorem 2.5. *(Sklar's Theorem, Rüschendorf (2013)).*

Let $\mathbf{F} = (F_1, F_2, \cdots, F_d)$ *be a d-dimensional probability distribution function having marginals* $F_1, F_2, \cdots, F_d$*. Then, there exists a d-copula function C, with uniform marginals such that for every* $\mathbf{v} \in \mathbb{R}^d$*,*

$$\mathbf{F}(v_1, v_2, \cdots, v_d) = C\big(F_1(v_1), F_2(v_2), \cdots, F_d(v_d)\big)\,. \tag{2.2.1}$$

Suppose $F_1, F_2, \cdots, F_d$ *are all continuous functions, then C is unique. Else, C is uniquely determined on Range* $F_1 \times$ *Range* $F_2 \times \cdots \times$ *Range* F_d*. Conversely, if C is a given copula and* $F_1, F_2, \cdots, F_d$ *are marginal distribution functions, then, the function* $\mathbf{F}$ *defined in equation (2.2.1) is a jointly distributed function with* $F_1, F_2, \cdots, F_d$ *margins.*

Copulas are generally classified into the elliptical and the Archimedean class. The former includes the Gaussian and the Student-t copula, whereas the Archimedean family are mostly defined by a parameter called 'generator function' which is convex and decreasing on $[0, 1]$ and whose pseudo-inverse exists. Some prominent examples include the Clayton copula, Joe copula, Gumbel copula, Frank copula and, others are the independence Copula and the Ali–Mikhail–Haq copula models. See Section 4.3.1 for more illustrative pictures on some of the prominent examples of these copula model.

2.2.1.1 Gaussian copula

This is a symmetric copula and easily implementable for simulation purposes. It is not suitable for modelling tail dependency events, and can easily be generalized to dimension higher than 2. Let Φ^{-1} be the inverse of Φ, which is the standard normal distribution function and let Σ be a symmetric correlation matrix which is positive definite. Thus, according to Cherubini et al. (2004), the CDF of the d-dimensional Gaussian copula, with $\mathbf{v} = (v_1, v_2, \cdots, v_d)$ is:

$$C^{Ga}(\mathbf{v}; \Sigma) = \Phi^d_\Sigma \left(\Phi^{-1}(v_1), \Phi^{-1}(v_2), \cdots, \Phi^{-1}(v_d)\right) .$$

Correspondingly, the density function is given as:

$$\tilde{C}^{Ga}(\mathbf{v}; \Sigma) = \frac{1}{\sqrt{|\Sigma|}} \exp\left(\frac{-1}{2}\mathbf{r}^T(\Sigma^{-1} - I)\mathbf{r}\right) ,$$

where $I \in \mathbb{R}^{d\times d}$ is an identity matrix, $|\Sigma| \in \mathbb{R}^{d\times d}$ is the determinant of the Σ which triggers the dependency and varies between -1 and 1, $\mathbf{r} = (r_1, r_2, \cdots, r_d)^T$ with the normal scores $r_i = \Phi^{-1}(v_i)$ for $i = 1, 2, \cdots, d$.

2.2.1.2 Student-t copula

Let t_β^{-1} be the inverse of t_β, which is the univariate Student-t distribution function with β degrees of freedom, and let Σ be a symmetric correlation matrix which is positive definite. Thus, according to Cherubini et al. (2004), the CDF of the d-dimensional Student-t copula, with $\mathbf{v} = v_1, v_2, \cdots, v_d$ is:

$$C^S(\mathbf{v}; \Sigma) = t_\beta^{d,\Sigma} \left(t_\beta^{-1}(v_1), t_\beta^{-1}(v_2), \cdots, t_\beta^{-1}(v_d)\right) .$$

Correspondingly, the density function is given as:

$$\tilde{C}^S(\mathbf{v}; \Sigma) = \frac{\Gamma\left(\frac{\beta+d}{2}\right)}{\sqrt{|\Sigma|}\Gamma\left(\frac{\beta}{2}\right)} \left(\frac{\Gamma\left(\frac{\beta}{2}\right)}{\Gamma\left(\frac{\beta+1}{2}\right)}\right)^d \frac{\lambda^{\frac{-\beta+d}{2}}}{\eta^{\frac{-\beta+1}{2}}} ,$$

where

$$\lambda = \left(1 + \frac{1}{\beta}\mathbf{r}^T\Sigma^{-1}\mathbf{r}\right), \quad \eta = \prod_{i=1}^{d}\left(1 + \frac{r_i^2}{\beta}\right) ,$$

$|\Sigma| \in \mathbb{R}^{d\times d}$ is the determinant of the Σ which triggers the dependency and varies between -1 and 1, and $\mathbf{r} = (r_1, r_2, \cdots, r_d)^T$ with the normal scores $r_i = t_\beta^{-1}(v_i)$ for $i = 1, 2, \cdots, d$. As $\beta \to \infty$, the student-t distribution function tends to the normal distribution function, and equally, the copula correspondents. The student-t copula has non-zero tail structure as a result of extreme clustered values, and their tails which are heavier compared to the normal distribution show the same characteristics like the Pareto laws (See Beirlant et al. (2004)).

2.2.1.3 Clayton copula

This is an asymmetric copula whose weight is more concentrated on the left tail. According to Gunawan et al. (2017), the CDF of a d-dimensional Clayton copula, with $\theta > 0$ and $\mathbf{v} = v_1, v_2, \cdots, v_d$, is given by:

$$C^C(\mathbf{v};\theta) = \left[\sum_{j=1}^{d} v_j^{-\theta} - d + 1\right]^{\frac{-1}{\theta}} .$$

Correspondingly, the density function is given as:

$$\tilde{C}^C(\mathbf{v};\theta) = \prod_{\lambda=0}^{d-1}(\theta\lambda + 1) \times \left(\prod_{l=1}^{d} v_l\right)^{-(1+\theta)} \times \left[\sum_{j=1}^{d} v_j^{-\theta} - d + 1\right]^{-\left(d+\frac{1}{\theta}\right)} .$$

2.2.1.4 Gumbel copula

This is an asymmetric copula whose weight is more concentrated on the right tail. It is also referred to as logistic or extreme value copula. According to Gunawan et al. (2017), the CDF of a d-dimensional Gumbel copula, with $\theta \in [1, \infty)$ and $\mathbf{v} = v_1, v_2, \cdots, v_d$, is given by:

$$C^G(\mathbf{v};\theta) = \exp\left\{\left[\sum_{j=1}^{d}(-\log(v_j))^{\theta}\right]^{\frac{1}{\theta}}\right\} .$$

Correspondingly, the density function is given as:

$$\tilde{C}^G(\mathbf{v};\theta) = \theta^d C^G(\mathbf{v};\theta) \times \frac{\prod_{l=1}^{d}(-\log(v_l))^{\theta-1}}{\left(\sum_{l=1}^{d}(-\log(v_l))^{\theta}\right)^{d}\prod_{l=1}^{d} v_l} \times P^G_{d,\theta}\left(\left[\sum_{j=1}^{d}(-\log(v_i))^{\theta}\right]^{\frac{1}{\theta}}\right) ,$$

$$\text{where} \qquad P^G_{d,\theta}(z) = \sum_{k=1}^{d} s^G_{dk}(\theta) z^k \quad \text{and} \quad s^G_{dk}(\theta) = \frac{d!}{k!} \sum_{l=1}^{k} \binom{k}{l} \binom{\theta l}{d} (-1)^{d-l} \, .$$

2.2.1.5 Frank copula

This is a symmetric copula which is often used to model codependency. According to Hofert et al. (2012), the CDF of a d-dimensional Frank copula, with $\theta \in (-\infty, \infty) \setminus \{0\}$ and $\mathbf{v} = v_1, v_2, \cdots, v_d$, is given by:

$$C^F(\mathbf{v}; \theta) = \frac{-1}{\theta} \log \left\{ 1 + \frac{\prod_{i=1}^{d} \left(\mathrm{e}^{-\theta v_i} - 1 \right)}{(\mathrm{e}^{-\theta} - 1)^{d-1}} \right\} .$$

Correspondingly, the density function is given as:

$$\tilde{C}^F(\mathbf{v}; \theta) = \left(\frac{\theta}{1 - \mathrm{e}^{-\theta}} \right)^{d-1} \times \mathrm{Li}_{d-1} f^F(\mathbf{v}; \theta) \times \frac{\exp(-\theta \sum_{l=1}^{d} v_l)}{f^F(\mathbf{v}; \theta)} ,$$

where

$$f^F(\mathbf{v}; \theta) = (1 - \mathrm{e}^{-\theta})^{1-d} \prod_{i=1}^{d} \{1 - \mathrm{e}^{-\theta v_i}\} .$$

2.2.1.6 Joe copula

According to Joe (1997), the CDF of a d-dimensional Joe copula, with $\theta \in [1, \infty)$, $\mathbf{v} = v_1, v_2, \cdots, v_d$ and $\bar{v} = 1 - v$, is given by:

$$C^J(\mathbf{v}; \theta) = 1 - \left(\sum_{i=1}^{d} \bar{v}_i^\theta - \prod_{i=1}^{d} \bar{v}_i^\theta \right) .$$

Correspondingly, the density function is given as (Hofert et al. 2012):

$$\tilde{C}^J(\mathbf{v}; \theta) = \theta^{d-1} \frac{\prod_{l=1}^{d} (1 - \log(v_l))^{\theta - 1}}{[1 - f^J(\mathbf{v}; \theta)]^{1 - \frac{1}{\theta}}} \times P^J_{d,\theta} \left(\frac{f^J(\mathbf{v}; \theta)}{1 - f^J(\mathbf{v}; \theta)} \right) ,$$

where

$$f^J(\mathbf{v}; \theta) = \prod_{i=1}^{d} \{1 - (1 - v_i)^\theta\}, \quad P^J_{d,\theta}(z) = \sum_{k=0}^{d-1} s^J_{dk}(\theta) z^k, \quad s^J_{dk}(\theta) = S(d, k+1) \frac{\Gamma\left(k + 1 - \frac{1}{\theta}\right)}{\Gamma\left(1 - \frac{1}{\theta}\right)} ,$$

with $S(a, b)$ to be the Stirling partition numbers or the Stirling numbers of the second kind.

2.2.1.7 Ali–Mikhail–Haq copula

This class of copula was derived by employing the probability transform and the algebraic transform on the joint probability distribution functions of the Gumbel's bivariate distribution (Kumar 2010). It is the only copula present in the Archimedean family which measures both the positive and negative dependence. According to Hofert et al. (2012), the CDF of a d-dimensional Ali–Mikhail–Haq copula, with $\theta \in [-1, 1]$ and $\mathbf{v} = v_1, v_2, \cdots, v_d$, is given by:

$$C^A(\mathbf{v}; \theta) = \prod_{i=1}^{d} \frac{v_i}{1 - \theta(1 - v_i)} .$$

Correspondingly, the density function is given as:

$$\tilde{C}^A(\mathbf{v}; \theta) = \frac{(1-\theta)^{d+1}}{\theta} \times \frac{C^A(\mathbf{v}; \theta)}{\prod_{l=1}^{d} v_l^2} \times \mathrm{Li}_{-d}(\theta C^A(\mathbf{v}; \theta)) ,$$

where $\mathrm{Li}_p(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^p}$ refers to the polylogarithm of p-order and x-argument.

2.2.1.8 Independence copula

This copula model has no upper and lower tail dependency, and the CDF for this class of copula can be obtained by setting $\theta = 0$ from the CDF of the Ali–Mikhail–Haq copula. Thus, for $\mathbf{v} = v_1, v_2, \cdots, v_d$ and according to Nelsen (2007), the CDF of the d-dimensional independence or the product copula is given by:

$$C^I(\mathbf{v}; \theta) = \prod_{i=1}^{d} v_i .$$

Correspondingly, the density function of the independence copula is simply a constant. Thus, following the Sklar's theorem, if there exists independent random variables, then the copula describing them is an independence copula and vice versa (Schmidt 2007).

3. Valuation of Basket Credit Default Swaps under Stochastic Default Intensity Models

This chapter has been published as:

Umeorah, N., Ehrhardt, M. & Mashele, P. (2020). Valuation of basket credit default swaps under stochastic default intensity models. *Advances in Applied Mathematics and Mechanics*, **12**(4), 1-26, DOI: 10.4208/aamm.OA-2019-0141.

Abstract

Portfolio credit derivatives, including the basket credit default swaps, are designed to facilitate the transfer of credit risk amongst market participants. Investors consider them as cheap tools to hedge a portfolio of credits, instead of individual hedging of the credits. The prime aim of this work is to model the hazard rate process using stochastic default intensity models, as well as extend the results to the pricing of basket default swaps. We focused on the nth-to-default swaps whereby the spreads are dependent on the nth default time, and we estimated the joint survival probability distribution functions of the intensity models under the risk-neutral pricing measure, for both the homogeneous and the heterogeneous portfolio. This work further employed the Monte-Carlo method, under the one-factor Gaussian copula model to numerically approximate the distribution function of the default time, and thus, the numerical experiments for pricing the nth default swaps were made viable under the two portfolio types. Finally, we compared the effects of different swap parameters to various nth-to-default swaps.

AMS subject classifications: 91G20, 91G30, 91G40, 91G60, 91G70, 62P05, 65C05

Keywords

Portfolio credit derivatives, basket default swaps, Gaussian copula, Monte-Carlo simulations, stochastic intensity modelling, hazard rate, joint survival probability distribution, Vasicek model, CIR model.

3.1 Introduction

Basket default swaps (BDS) are financial contracts that payoff whenever there is a default or multiple defaults among a portfolio of entities or obligors. BDS are generally classified into first-to-default ($F2$D), nth-to-default ($n2$D), n-out-of-m-to-default and then, all-to-default. From the investor's point of view, BDS are still preferable because they limit the credit portfolio that an investor can easily monitor compared to the large portfolio obtainable in a typical synthetic collateralized debt obligation (CDO)[1](O'Kane 2011). The protection buyer finds the BDS attractive because the cost of purchasing protection on a portfolio of entities is less expensive compared to the purchase of individual protection. Considering the $n2$D, the protection seller enjoys a limited downside risk, and this stems from the fact that at most one default is expected for the seller to cover (Bomfim 2015). After the payments pending default and contract finalized, the loss incurred resulting from further default is borne only by the protection buyer. The recovery rate, the number of entities in the portfolio, the entity's credit ratings, as well as the default correlations are some of the major factors that affect the BDS spread.

Several research works have been carried out in the area of basket default swaps. Li (2000) first implemented the copula functions to model credit derivatives, and this was extended by several authors. For instance, Laurent and Gregory (2005) considered the valuation of BDS and the CDO. They obtained semi-analytic values for the above contingent claims under the assumption of independent default times conditioned on a low dimensional factor, and based their valuation on mean-variance mixture models and frailty models. Kijima and Muromachi (2000) priced BDS using a stochastic intensity-based model by assuming conditional independence. Using the joint survival probability, they further provided closed form values when the hazard rate process is defined within the extended Vasicek model.

Liang et al. (2011) provided the limitation of using the Vasicek model in modelling the intensity process. They explained using some numerical examples that the Vasicek model can be efficient when the portfolio has relatively few correlated risky assets and their valuations are extended to the prices of BDS, credit default swap (CDS) index and CDOs. Fathi and Nader (2007a) provided Monte-Carlo methods and semi-explicit expressions which improves the cost-effectiveness of the Monte-Carlo approach to price multi-named credit derivatives like the basket default swaps and the CDO tranches. They further calibrated the prices of BDS to the Japanese financial markets (Fathi and Nader 2007a). Iscoe and Kreinin

[1] Small scale portfolio size includes 5-10 credits, and large portfolio scale ranges from 100-150 credits.

(2006) proposed a recursive algorithm to value BDS based on a continuous-time model in the conditional independence framework. They employed the concept of the order statistics of the default times of the entities in a portfolio, and then applied it in the estimation of the $F2$D swaps and the second-to-default ($S2$D) swap contracts.

Regardless of whether the reduced form method or the structural approach in modelling the joint default events in portfolio credit derivatives, there is always the problem of the joint probability distribution of the default times, and many researchers have employed Copula models. Frey and McNeil (2003) explored the roles of the copula functions in latent variable models and used the modified Gaussian copula method to model the dependent defaults evident in a portfolio credit risk. Mashal and Naldi (2002) used the student t-copulas, which possess the non-trivial tail dependency structure, as well as the ability for more joint extreme events. They applied it to the price estimations of the multi-name instruments, like the $n2$D baskets and CDOs. Schönbucher and Schubert (2001) used the Archimedean copulas in general, and in particular, employed the Gumbel and Clayton copulas in the modelling of the default dependency structure in the intensity models. Li et al. (2015) focused on the use of the single-factor Gaussian-NIG-copula model in the simulation of the distribution functions, as well as to obtain the structure of the correlation existing between the assets and further applied the concepts to the valuation of BDS.

Choe and Jang (2011a) included the one-factor Gaussian copula model to derive the probability distribution function of the nth default time explicitly and thus, valued the $n2$D and n-out-of-m-to-default. Bluhm et al. (2002) defined the nth default time as an order statistic for $n \leq N$, where N refers the number of reference entities and thus, they obtained the distribution of the time $\tau^{n\mathrm{th}}$ of the nth default. The Joshi-Kainth algorithm (Joshi and Kainth 2004) is an innovative importance sampling technique also employed in modelling the default time of the process. Chen and Glasserman (2008) proposed a modification of the technique and ensured that there exist variance reduction even when the defaults does not seem to occur. Usually, in the pricing of $n2$D swaps, as the size of the entities increase, the pricing becomes computationally intense. Schröter and Heider (2013a) derived the default time distribution and simplified the valuation issues from an n-dimensional quadrature into a one-dimensional quadrature, thereby breaking its curse of dimensionality. Jouanin et al. (2002) incorporated the modelling of correlated default events in the intensity-based framework. They estimated the marginal probability distribution functions for each default and then, modelled the joint distribution with the aid of a copula function.

In this work, however, we modelled the intensity process using the Vasicek model and Cox-

Ingersoll-Ross (CIR) model, owing to their analytical tractability. The joint survival probability distributions (JSPD) for the homogeneous portfolio under both models and the JSPD for the heterogeneous portfolio under the Vasicek model are obtained. For a heterogeneous portfolio, we consider one which consists of five entities from the corporate sector (with different credit ratings), obtained the default intensity (hazard rates), estimated the parameters of the default intensity framework and solved for their JSPD. Furthermore, the survival probabilities and the swap spreads of each firm were equally estimated from both the Vasicek and the CIR model, and the portfolio value of the basket was obtained from the equal-weighted assumption. Next, we considered a homogenized portfolio and employed the Monte-Carlo method to simulate their default time distribution function and thus, obtained the prices for different categories of n2D.

The organization of the work is as follows: Section 3.1 introduces the topic and highlights some of the recent work done on the pricing of BDS. Section 3.2 discusses the model structure, outlines the density functions of both the Vasicek and the CIR model. It further introduces the model for intensity default and highlights the model for bond valuation under both Vasicek and the CIR processes. Section 3.3 focuses on the pricing of BDS and explains how the swap spread can be obtained. It further introduces the concept of default time modelling. Section 3.4 outputs results on the parameter estimation of the heterogeneous portfolios, hazard rates, swap spreads as well as the corresponding portfolio values. It also gives some numerical experiments on the BDS valuations defined on both homogeneous and heterogeneous baskets, as well as some sensitivity analysis on the swap spreads. Section 3.5 concludes our research study.

3.2 Model Structure

In modelling the intensity process, we consider the 'economically viable' property of mean-reverting and Ornstein–Uhlenbeck process. This process ensures that the hazard rate does not explode and hence tends to infinity. Consider the given stochastic differential equation (SDE)

$$\mathrm{d}\lambda(t) = \alpha\Big(\beta - \lambda(t)\Big)\mathrm{d}t + \sigma\Big(\lambda(t)\Big)^{\gamma}\mathrm{d}W(t)\,, \tag{3.2.1}$$

where α denotes the speed of reversion, β is the long term mean, σ is volatility and $W(t)$ denotes the standard Brownian motion. The parameters α, β, σ are all positive constants.

The drift term which is of the form $\alpha(\beta - \lambda(t))$ is linear and captures the full rate of mean reversion. The volatility term defined by $\sigma(\lambda(t))^\gamma$ is non-linear and the term γ parametrizes the extent to which the process $\lambda(t)$ depends on its level (Choi and Wirjanto 2007). The above SDE in equation (3.2.1) follows the model proposed by Chan, Karolyi, Longstaff and Sanders (CKLS) in which many one-factor and multi-factor stochastic interest rate models are nested together (Chan et al. 1992). They compared the models to obtain, which best fits the short-term interest rate process. From their findings, the best fit models possess the characteristics that their conditional volatility of interest rate changes fully depend on the interest rate level. For $\gamma = 0$, we have the Vasicek model, $\gamma = 1/2$ gives the CIR model, $\gamma = 1$ gives the Brennan-Schwartz model, etc. Let the given process $\lambda(t)$ which is being considered to be the hazard rate function or the default process. This function refers to the probability of default for a given time interval which is conditioned on no prior default. These defaults, also known as credit events, could be as a result of failure to pay, bankruptcy, downgrade, restructuring, etc (O'Kane 2011). In this work, our focus shall be on the Vasicek and CIR models.

Under the risk-neutral measure Q, the stochastic process λ defined by the Vasicek model follows a normal distribution with mean and variance given respectively as:

$$\begin{aligned} \mathbb{E}^Q\Big[\lambda(T)|\mathcal{F}(t)\Big] &= \lambda(t)\mathrm{e}^{-\alpha(T-t)} + \beta(1-\mathrm{e}^{-\alpha(T-t)})\,, \\ \mathrm{Var}^Q\Big[\lambda(T)\Big] &= \frac{\sigma^2}{2\alpha}(1-\mathrm{e}^{-2\alpha(T-t)})\,. \end{aligned}$$

For the CIR model, the process $\lambda(t)$ follows the non-central chi-squared distribution $\chi^2_{v,\eta}$ with v and η denoting the degree of freedom and the non-centrality parameter, respectively. Under the risk-neutral pricing measure, the probability density function is defined as (Brigo and Mercurio 2007):

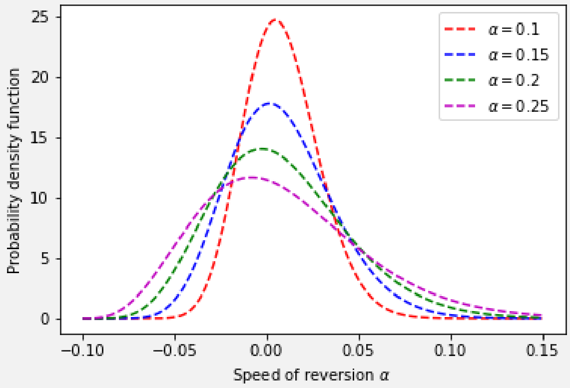
$$f\Big[\lambda(T)|\lambda(t)\Big] := c\cdot\chi^2_{v,\eta}\Big[c\lambda(t)\Big]\,,$$

where

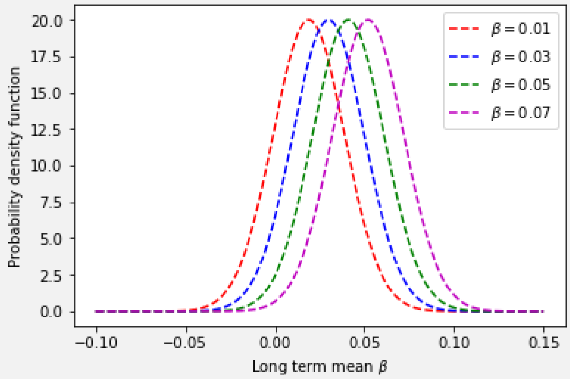
$$c = \frac{4\alpha}{\sigma^2(1-\mathrm{e}^{-\alpha(T-t)})}, \qquad v = \frac{4\alpha\beta}{\sigma^2}, \qquad \eta = c\lambda(T)\mathrm{e}^{-\alpha(T-t)}\,.$$

The following plots in Figure 3.1a till 3.1f depict the probability density functions (PDF) of both the Vasicek and the CIR models[2]. We plot the PDF with respect to various parameters of each of the model.

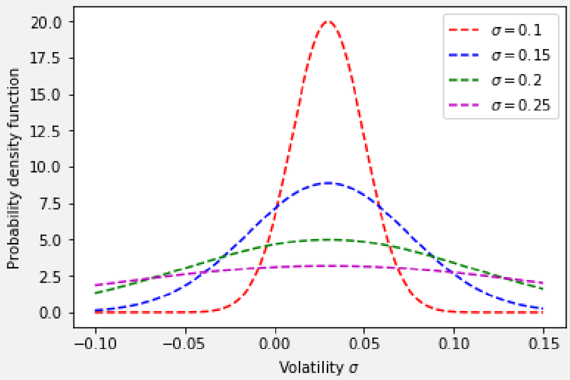
[2]See the link https://github.com/NnekaU/Codes/blob/master/BDS%20codes.ipynb for the `ipython notebook` codes of all the plots and tables obtained in this chapter.



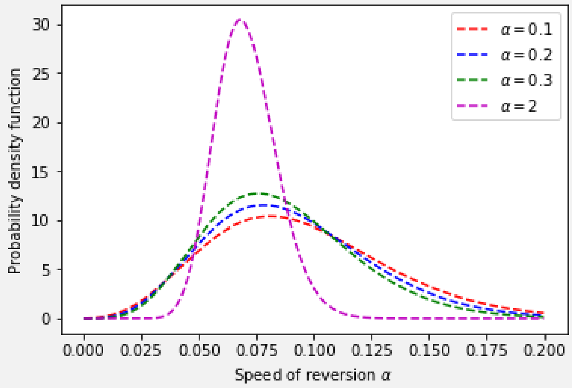
(a) PDF of Vasicek vs α



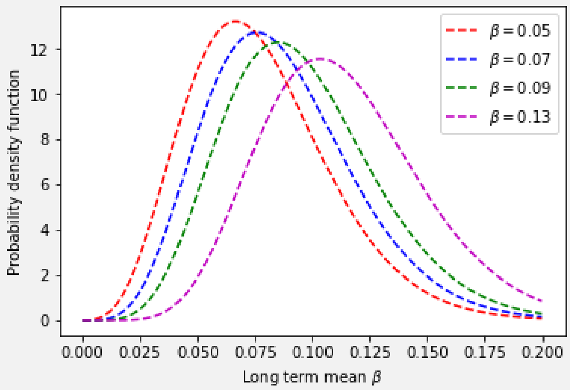
(b) PDF of Vasicek vs β



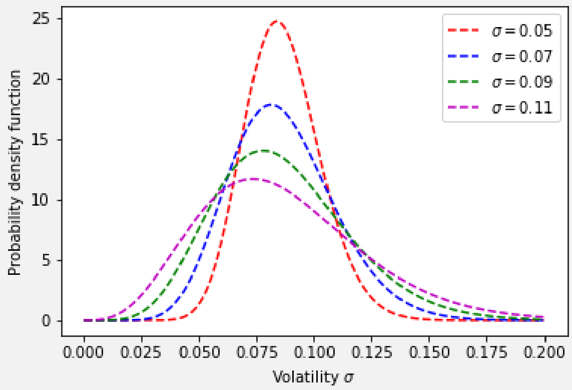
(c) PDF of Vasicek vs σ



(d) PDF of CIR vs α



(e) PDF of CIR vs β



(f) PDF of CIR vs σ

Figure 3.1: PDF of the Vasicek and the CIR model versus the parameters α, β and σ.

For both models, the speed reversion parameter α identifies the velocity at which the process revolves around the long term mean β. Increasing α reduces the variance, and the process attains to β faster. The size of the distribution is not affected when β is increased, though the mean of the distribution is affected. Larger volatility σ results to a more flatter surface and thus greater randomness of the process is observed. Additional observations from the CIR model are the presence of the non-negative values of the hazard rate function, resulting from $\sigma\sqrt{\lambda(t)}$, which is the standard deviation factor, as well as an adjustment in the shape and structure of the density function owing to the β increment.

3.2.1 Default intensity model describing a heterogeneous portfolio

CIR Model: Let the default process $\lambda_i(t)$ be modelled by the following SDE:

$$\mathrm{d}\lambda_i(t) = \alpha_i(\beta_i - \lambda_i(t))\mathrm{d}t + \sigma_i\sqrt{\lambda_i(t)}\mathrm{d}W_i(t)\,, \tag{3.2.2}$$

where $t \geq 0$, $\alpha_i, \beta_i, \sigma_i$ are all positive constants, with the condition $2\alpha_i\beta_i > \sigma_i^2$ and $i = 0, 1, \cdots, n$. The correlation of the standard Brownian motion $W_i(t)$ and $W_j(t)$, where $i, j = 0, 1, \cdots, n$, under the risk-neutral probability measure $\mathbb{P}$ reads:

$$\mathrm{d}W_i(t)\mathrm{d}W_j(t) = \begin{cases} \rho_{ij}\mathrm{d}t & \text{for } \ i \neq j \\ \mathrm{d}t & \text{for } \ i = j\,. \end{cases}$$

The stochastic process of the CIR model belongs to one of the many classes of SDE's which is not analytically solvable. However, explicit forms of its mean and variance are obtainable, and can be extracted from the corresponding integral form. Thus, the integral representation of the model in (3.2.2) using the Ito's lemma can be given as:

$$\lambda_i(t) = \lambda_i(0)\mathrm{e}^{-\alpha_i t} + \int_0^t \alpha_i\beta_i\mathrm{e}^{-\alpha_i(t-s)}\mathrm{d}s + \sigma_i\int_0^t \mathrm{e}^{-\alpha_i(t-s)}\sqrt{\lambda_i(s)}\mathrm{d}W_i(s)\,. \tag{3.2.3}$$

According to Brigo and Mercurio (2007), the first two moments associated with the CIR process in equation (3.2.2) can be obtained as:

$$\mathbb{E}^{\mathbb{P}}\Big[\lambda_i(t)\Big] = \lambda_i(0)\mathrm{e}^{-\alpha_i t} + \beta_i\left(1 - \mathrm{e}^{-\alpha_i t}\right)\,, \tag{3.2.4}$$

$$\mathrm{Var}^{\mathbb{P}}\Big[\lambda_i(t)\Big] = \frac{\sigma_i^2\lambda_i(0)}{\alpha_i}(\mathrm{e}^{-\alpha_i t} - \mathrm{e}^{-2a_i t}) + \frac{\sigma_i^2\beta_i}{2\alpha_i}(1 - \mathrm{e}^{-\alpha_i t})^2\,. \tag{3.2.5}$$

Let $\mathcal{A} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a given probability space, where Ω a sample space is a set consisting of all possible events which are defined in a given time horizon; $\mathcal{F}$ is the σ-field which represents the collection of all events; $\{\mathcal{F}_t\}_{t\geq 0}$ refers to filtration which carries information with specific times; and $\mathbb{P}$ is the risk-neutral probability measure since the no-arbitrage assumption is taken into consideration. Furthermore, $\mathcal{A}$ is equipped with filtration $\mathcal{F} = \mathbb{H} \vee \mathbb{F}$, where $\mathbb{H}$ contains the information which describes the evolutions of credit events and $\mathbb{F}$ refers to some reference filtration (Bielecki et al. 2007).

This work will first aim at obtaining the joint probability distribution of default in order to price basket derivatives under intensity process. Consider first a credit event of n values whose default times are denoted by τ_i, for $i = 1, 2, \cdots, n$, and also let the corresponding default intensity processes be denoted by $\lambda_i(t)$, where $i = 0, 1, \cdots, n$. For $\mathbb{P}$ being the appropriate probability measure of the pricing problem, let the marginal survival and marginal default probability function of each ith entity be defined respectively by:

$$S_i(t_i) = \mathbb{P}(\tau_i > t_i) \quad \text{and} \quad F_i(t_i) = \mathbb{P}(\tau_i \leq t_i)\,.$$

Also, let the joint survival probability distribution (JSPD) and the joint default probability distribution functions be defined respectively by:

$$\begin{aligned} S(t_0, t_1, \cdots, t_n) &= \mathbb{P}(\tau_0 > t_0, \tau_1 > t_1, \cdots, \tau_n > t_n) \quad \text{and} \\ F(t_0, t_1, \cdots, t_n) &= \mathbb{P}(\tau_0 \leq t_0, \tau_1 \leq t_1, \cdots, \tau_n \leq t_n)\,. \end{aligned}$$

Then Kijima (2000) defined the JSPD of default times under the intensity process by:

$$S(t_0, t_1, \cdots, t_n) = \mathbb{P}(\tau_0 > t_0, \tau_1 > t_1, \cdots, \tau_n > t_n) := \mathbb{E}\left[\exp\left(-\sum_{i=0}^{n} H_i(t_i)\right)\right], \qquad (3.2.6)$$

where $H_i(t) = \int_0^t \lambda_i(s)\mathrm{d}s$. Thus, we have

$$\begin{aligned} S(t_0, t_1, \cdots, t_n) = \mathbb{E}\Bigg[\exp\Bigg(-\sum_{i=0}^{n}\Bigg[\int_0^{t_i}\Bigg(\lambda_i(0)\mathrm{e}^{-\alpha_i t} + \int_0^t \alpha_i\beta_i \mathrm{e}^{-\alpha_i(t-s)}\mathrm{d}s + \\ \sigma_i \int_0^t \mathrm{e}^{-a_i(t-s)}\sqrt{\lambda_i(s)}\mathrm{d}W_i(s)\Bigg)\mathrm{d}t\Bigg]\Bigg)\Bigg] = \mathbb{E}\left[\exp\left(-\sum_{i=0}^{n} B_i(t_i) - \sum_{i=0}^{n}\sigma_i I_i(t_i)\right)\right], \end{aligned}$$

where

$$B_i(t_i) = \int_0^{t_i} \left(\lambda_i(0)\mathrm{e}^{-\alpha_i t} + \int_0^t \alpha_i \beta_i \mathrm{e}^{-\alpha_i(t-s)} \mathrm{d}s \right) \mathrm{d}t = \frac{\lambda_i(0)}{\alpha_i}(1-\mathrm{e}^{-\alpha_i t_i}) + \int_0^{t_i} \beta_i(1-\mathrm{e}^{-\alpha_i(t_i-s)})\mathrm{d}s \tag{3.2.7}$$

and

$$I_i(t_i) = \int_0^{t_i} \int_0^t \mathrm{e}^{-\alpha_i(t-s)} \sqrt{\lambda_i(s)} \mathrm{d}W_i(s) \mathrm{d}t = \frac{1}{\alpha_i} \int_0^{t_i} (1-\mathrm{e}^{-\alpha_i(t-s)}) \sqrt{\lambda_i(s)} \mathrm{d}W_i(s)\,.$$

We seek for the covariance.

$$\mathrm{Cov}(I_i(t_i), I_j(t_j)) = \mathbb{E}\Big[I_i(t_i) \cdot I_j(t_j)\Big] - \mathbb{E}\Big[I_i(t_i)\Big] \cdot \mathbb{E}\Big[I_j(t_j)\Big]$$

Denote the covariance for the *ith* and *jth* counterparties as $c_{ij}(t_i, t_j)$. Then

$$\begin{aligned}
c_{ij}(t_i,t_j) =& \mathbb{E}\left[\int_0^{t_i}\int_0^{t_j} \frac{1}{\alpha_i\alpha_j}(1-\mathrm{e}^{-\alpha_i(t_i-s)})(1-\mathrm{e}^{-\alpha_j(t_j-s)}) \cdot \sqrt{\lambda_i(s)}\mathrm{d}W_i(s)\sqrt{\lambda_j(s)}\mathrm{d}W_j(s)\right] \\
=& \frac{1}{\alpha_i\alpha_j}\int_0^{t_i}\int_0^{t_j} (1-\mathrm{e}^{-\alpha_i(t_i-s)})(1-\mathrm{e}^{-\alpha_j(t_j-s)}) \cdot \mathbb{E}\left[\sqrt{\lambda_i(s)\lambda_j(s)}\mathrm{d}W_i(s)\mathrm{d}W_j(s)\right] \\
=& \frac{\rho_{ij}}{\alpha_i\alpha_j}\int_0^{t_i\wedge t_j} (1-\mathrm{e}^{-\alpha_i(t_i-s)})(1-\mathrm{e}^{-\alpha_j(t_j-s)}) \cdot \mathbb{E}\left[\sqrt{\lambda_i(s)\lambda_j(s)}\right] \mathrm{d}s
\end{aligned} \tag{3.2.8}$$

from the correlation properties of standard Brownian motion. Also, define $t_i \wedge t_j$ as the minimum of t_i and t_j.

Denote the quantity $I = \sum_{i=0}^n \sigma_i I_i(t_i)$. We note that I follows a normal distribution

$$I \sim \mathcal{N}\left(0, \sum_{j=0}^n \sum_{i=0}^n \sigma_i\sigma_j c_{ij}(t_i,t_j)\right)\,.$$

That is,

$$\mathrm{Var}(I) = \mathrm{Var}\left[\sum_{i=0}^n \sigma_i I_i(t_i)\right] = \mathrm{Var}\left[\sum_{i=0}^n \sigma_i \frac{1}{\alpha_i}\int_0^{t_i}(1-\mathrm{e}^{-\alpha_i(t-s)})\sqrt{\lambda_i(s)}\mathrm{d}W_i(s)\right]$$

$$
\begin{aligned}
\mathrm{Var}(I) &= \mathbb{E}\left[\sum_{i=0}^{n}\sigma_i\frac{1}{\alpha_i}\int_0^{t_i}(1-\mathrm{e}^{-\alpha_i(t-s)})\sqrt{\lambda_i(s)}\mathrm{d}W_i(s)\cdot\sum_{j=0}^{n}\sigma_j\frac{1}{\alpha_j}\int_0^{t_j}(1-\mathrm{e}^{-\alpha_j(t-s)})\sqrt{\lambda_j(s)}\mathrm{d}W_j(s)\right]\\
&= \sum_{j=0}^{n}\sum_{i=0}^{n}\sigma_i\sigma_j.\frac{\rho_{ij}}{\alpha_i\alpha_j}\int_0^{t_i\wedge t_j}(1-\mathrm{e}^{-\alpha_i(t_i-s)})(1-\mathrm{e}^{-\alpha_j(t_j-s)})\cdot\mathbb{E}\left[\sqrt{\lambda_i(s)\lambda_j(s)}\right]\mathrm{d}s\\
&= \sum_{j=0}^{n}\sum_{i=0}^{n}\sigma_i\sigma_j c_{ij}(t_i,t_j)\,.
\end{aligned}
$$

From the moment generating function of I, we have that

$$\mathbb{E}\big[\mathrm{e}^{-I}\big] = \mathrm{e}^{\frac{1}{2}\mathrm{Var}(I)}\,. \tag{3.2.9}$$

Summing up equations (3.2.6) and (3.2.9), the JSPD function of the default times τ_i for the heterogeneous portfolio, under the CIR model is given as

$$S(t_0,\cdots,t_n) = \exp\left(-\sum_{i=0}^{n}B_i(t_i)+\frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{n}\sigma_i\sigma_j c_{ij}(t_i,t_j)\right)\,. \tag{3.2.10}$$

For the **Vasicek Model**, the same process is obtainable with the CIR model above, and the difference is the nature of the diffusion process which is of the form $\sigma_i\mathrm{d}W_i$. That is, the default process $\lambda_i(t)$ is modelled by the following SDE:

$$\mathrm{d}\lambda_i(t) = \alpha_i(\beta_i-\lambda_i(t))\mathrm{d}t+\sigma_i\mathrm{d}W_i(t)\,, \tag{3.2.11}$$

where $t\geq 0$, $\alpha_i,\beta_i,\sigma_i$ are all positive constants for $i=0,1,\cdots,n$, with β_i long term mean and long term variance of $\frac{\sigma_i^2}{2\alpha_i}$. The model is analytical tractable with the same or slightly differing expectation value as that of the CIR model, but with different variance structure. The JSPD function of the default times τ_i for the heterogeneous portfolio, under the Vasicek model is given by (Kijima 2000):

$$S(t_0,t_1,\cdots,t_n) = \exp\left(-\sum_{i=0}^{n}B_i(t_i)+\frac{1}{2}\sum_{i=0}^{n}\sum_{j=0}^{n}\sigma_i\sigma_j c_{ij}(t_i,t_j)\right)\,, \tag{3.2.12}$$

where $B_i(t_i)$ is the same as in equation (3.2.7) and c_{ij} is defined as

$$\begin{aligned}
c_{ij}(t_i,t_j) &= \mathbb{E}\left[\int_0^{t_i}\int_0^{t_j}\frac{1}{\alpha_i a_j}(1-\mathrm{e}^{-\alpha_i(t_i-s)})(1-\mathrm{e}^{-\alpha_j(t_j-s)})\cdot \mathrm{d}W_i(s)\mathrm{d}W_j(s)\right]\\
&= \frac{1}{\alpha_i\alpha_j}\int_0^{t_i}\int_0^{t_j}(1-\mathrm{e}^{-\alpha_i(t_i-s)})(1-\mathrm{e}^{-\alpha_j(t_j-s)})\cdot \mathbb{E}\left[\mathrm{d}W_i(s)\mathrm{d}W_j(s)\right]\\
&= \frac{\rho_{ij}}{\alpha_i\alpha_j}\int_0^{t_i\wedge t_j}(1-\mathrm{e}^{-\alpha_i(t_i-s)})(1-\mathrm{e}^{-\alpha_j(t_j-s)})\mathrm{d}s \qquad (3.2.13)\\
&= \frac{\rho_{ij}}{\alpha_i\alpha_j}\left[s-\frac{e^{-\alpha_i(t_i-s)}}{\alpha_i}-\frac{e^{-\alpha_j(t_j-s)}}{\alpha_j}+\frac{e^{-\alpha_i(t_i-s)-\alpha_j(t_j-s)}}{\alpha_i+\alpha_j}\right]_{s=0}^{s=t_i\wedge t_j}. \qquad (3.2.14)
\end{aligned}$$

3.2.2 Default intensity model describing a homogeneous portfolio

Consider homogenized risky assets in the basket portfolio, where the parameters of each entity are assumed to be the same, and the default intensities of all ith and jth firms are identical. This is in contrast to the heterogeneous portfolio explained in section 3.2.1, in which the intensity dynamics describing each entity are assumed to be different. Thus, we have that for $\alpha_i = \alpha_j = \alpha, \beta_i = \beta_j = \beta, \sigma_i = \sigma_j = \sigma, \rho_{ij} = \rho$ and $\lambda_i = \lambda_j = \lambda$ (where $1 \leq i, j, \leq N$), the JSPD function under this identical portfolio can be written as:

$$S(t_0,\cdots,t_n) = \exp\left(-B(t)+\frac{1}{2}\sigma^2 c(t,t)\right),$$

where $B(t)$ for both the Vasicek and CIR is

$$B(t) = \frac{\lambda(0)(1-e^{-at})}{a}+b\left(t-\frac{1}{a}(1-e^{-at})\right),$$

and $c(t,t)$ for the Vasicek is

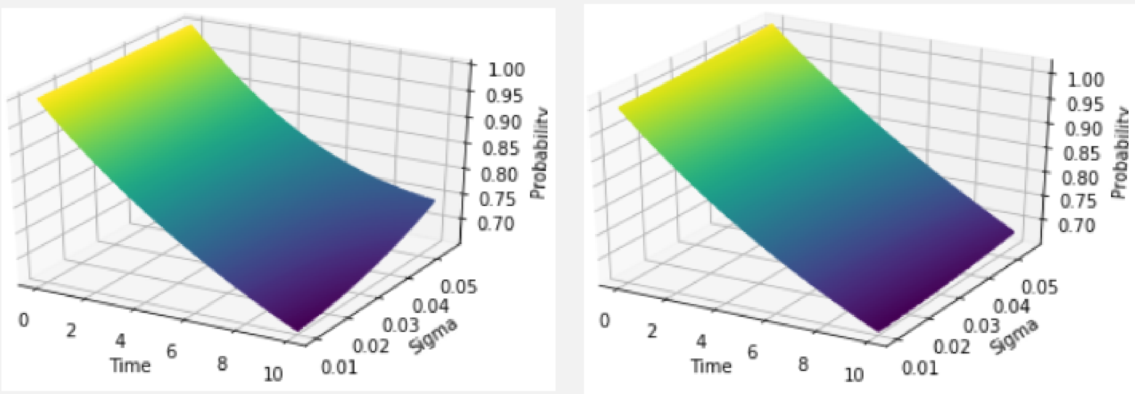
$$c(t,t) = \frac{\rho(2at-3+4e^{-at}-e^{-2at})}{2a^3},$$

and for the CIR model, we have

$$c(t,t) = \rho\left[\frac{e^{-2at}(b-2\lambda(0)+4e^{at}(b+a(b-\lambda(0))t)+e^{2at}(2\lambda(0)+b(2at-5))}{2a^3}\right].$$

The plots of the above homogenized process for both the Vasicek and the CIR are given below in Figure 5.2. The parameters considered are $\sigma = 0.05, \alpha = 0.1, \beta = 0.03, \lambda(0) =$

$0.05, T = 10, t = 0$ and $\rho = 0.5$. From the plots, we observed that the survival probabilities are declining with increase in time, which in turn, increases the probability of default.



(a) Vasicek Model (b) CIR Model

Figure 3.2: Joint survival probability distribution for homogenized portfolio

3.2.3 Bond Valuation

Let $C(\bar{\lambda}, t, T)$ be the value of a defaultable zero coupon bond having a face value of 1, with no recovery rate. Let $\bar{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_N)$, where λ_i is the default intensity or default hazard rate defined in (3.2.11). Define the pre-first-default bond value as $\hat{C}(\bar{\lambda}, t, T)$ and according to Liang et al. (2011), the price at time t of the bond can be written as:

$$C(\bar{\lambda}, t, T) := e^{-r(T-t)} P\{\tau_1 > T, \cdots, \tau_N > T | \mathcal{F}_t\},$$

following the Markov process of $\bar{\lambda}$. Here, P is the survival probability and $\mathcal{F}_t$ is the filtration available at the current time t. Next assume that the conditional probability that default occurs between an infinitesimal period $[t, t + \mathrm{d}t]$, provided that the survival was up to time t is $\lambda \mathrm{d}t$, and $\sum_{i=1}^{N} \lambda_i(t)\mathrm{d}t$, for the first-to-default case. Then, under the risk-neutral measure, $\mathbb{E}[\mathrm{d}\hat{C}] = r\hat{C}\mathrm{d}t$ and applying Itô's formula, the following pre-default pricing PDE for $\hat{C}(\bar{\lambda}, t, T)$ is satisfied (Bielecki et al. (2004), Liang et al. (2011)):

$$\frac{\partial \hat{C}}{\partial t} + \sum_{i=1}^{N} \alpha_i(\beta_i - \lambda_i)\frac{\partial \hat{C}}{\partial \lambda_i} + \frac{1}{2}\sum_{i,j=1}^{N} \frac{\partial^2 \hat{C}}{\partial \lambda_i \lambda_j}\left[\lambda_i^{2\gamma}\sigma_i\sigma_j\rho_{ij}\right] = \left(r + \sum_{i=1}^{N}\lambda_i\right)\hat{C}\,. \tag{3.2.15}$$

The affine term structure of the bond is given below:

$$\hat{C} = \exp\left\{A(t,r;T) - \sum_{i=1}^{N} B_i(t;T)\lambda_i\right\}, \tag{3.2.16}$$

The parameters for the **Vasicek model** are obtained from the solution of the ODE:

$$\begin{cases} \dfrac{\partial A(t,r;T)}{\partial t} - \displaystyle\sum_{i=1}^{N} \alpha_i\beta_i B_i(t,T) + \frac{1}{2}\sum_{i,j=1}^{N} B_i(t;T)B_j(t;T)\sigma_i\sigma_j\rho_{ij} - r = 0 \\ \dfrac{\partial B_i(t;T)}{\partial t} - \alpha_i B_i(t;T) + 1 = 0, \end{cases} \tag{3.2.17}$$

which is solved using the following conditions: $A(T,r;T) = 0$ and $B_i(T;T) = 0$.

The solution to equation (3.2.17) has the following parameters:

$$B_i(t;T) = \frac{1}{\alpha_i}(1 - \mathrm{e}^{-\alpha_i(T-t)})$$

and

$$A(t,r;T) = \frac{1}{2}\sum_{i,j=1}^{N} \frac{\rho_{ij}\sigma_i\sigma_j}{\alpha_i\alpha_j}\left[(T-t) - B_i(t;T) - B_j(t;T) + \frac{1-\mathrm{e}^{-(\alpha_i+\alpha_j)(T-t)}}{\alpha_i+\alpha_j}\right]$$
$$- \sum_{i=1}^{N} \beta_i[(T-t) - B_i(t;T)] - r(T-t)$$

Thus, the bond price is

$$\hat{C}(\bar{\lambda}, t, T) = \exp\left\{A(t;T) - \sum_{i=1}^{N} B_i(t;T)\lambda_i\right\}. \tag{3.2.18}$$

The parameters for the **CIR model** are obtained from the solution of the ODE:

$$\begin{cases} \dfrac{\partial A(t,r;T)}{\partial t} - \displaystyle\sum_{i=1}^{N} \alpha_i\beta_i B_i(t,T) - r = 0 \\ \dfrac{\partial B_i(t;T)}{\partial t} - \alpha_i B_i(t;T) - \displaystyle\frac{1}{2}\sum_{i,j=1}^{N} B_i(t;T)B_j(t;T)\sigma_i\sigma_j\rho_{ij} + 1 = 0, \end{cases} \tag{3.2.19}$$

which is solved using the following conditions: $A(T,r;T) = 0$ and $B_i(T;T) = 0$.

For homogeneous portfolio basket derivatives, the ODE reduces to

$$\begin{cases} \dfrac{\partial A(t,r;T)}{\partial t} - \alpha\beta B(t;T) - r = 0 \\ \dfrac{\partial B(t;T)}{\partial t} - \alpha B(t;T) - \dfrac{1}{2}B^2(t;T)\sigma^2\rho + 1 = 0, \end{cases} \tag{3.2.20}$$

Remark 3.1. *The second part of (3.2.20) takes a Ricatti form, with solution*

$$B(t;T) = \frac{2(\mathrm{e}^{h(T-t)} - 1)}{2h + (\alpha\beta + h)(\mathrm{e}^{h(T-t)} - 1)}, \tag{3.2.21}$$

where $h = \sqrt{(\alpha\beta)^2 + 2\sigma^2\rho}$.

Furthermore,

$$A(t,r;T) = A(T,r;T) - \int_t^T \partial_s A(s,r,T)\mathrm{d}s + r(T-t) - \alpha\beta\int_t^T B(s;T)\mathrm{d}s - r(T-t)$$

$$= \frac{-2(\alpha\beta)^2[\zeta(T-t) + \log 4 - \alpha\log \mathrm{e}^{h(T-t)}\zeta + 2\log\alpha\beta]}{\zeta(\alpha\beta - h)} - r(T-t),$$

with $\zeta = \alpha\beta + h$. Thus, we have the bond price $\hat{C}$ under the CIR model for the homogeneous basket portfolio with parameters $A(t,r;T)$ and $B(t;T)$.

3.2.3.1 Demerits on the use of the Vasicek model for intensity process

The JSPD of default time τ_i is given by

$$P = P(\tau_1 > T, \tau_2 > T, \cdots, \tau_N > T|\mathcal{F}_t) = \mathrm{e}^{r(T-t)}\hat{C}(\bar{\lambda}, t, T), \tag{3.2.22}$$

where $\hat{C}(\bar{\lambda}, t, T)$ is already defined in equation (3.2.18). Liang et al. (2011) defined the JSPD function, in connection with the number of reference entities N as:

$$P(N) = \exp\left\{\frac{\sigma^2\rho K N^2}{2\alpha^2} - \frac{\sigma^2(\rho-1)KN}{2\alpha^2} + ZN\right\}, \tag{3.2.23}$$

where

$$K = (T-t) - \frac{3}{2\alpha} + \frac{2e^{-\alpha(T-t)}}{\alpha} - \frac{e^{-2\alpha(T-t)}}{2\alpha} \qquad \text{and}$$

$$Z = \frac{(\beta - \lambda(t))(1 - e^{-\alpha(T-t)})}{\alpha} - \beta(T-t)\,.$$

To illustrate the demerits, consider a homogenized basket portfolio of $N = 40$ reference obligors, with other parameters[3] $\alpha = 0.3, \beta = 0.03, \rho = 0.5, \lambda(0) = 0.05, \sigma = 0.035, t = 0, T = 5$. The JSPD for the portfolio under the Vasicek is given below:

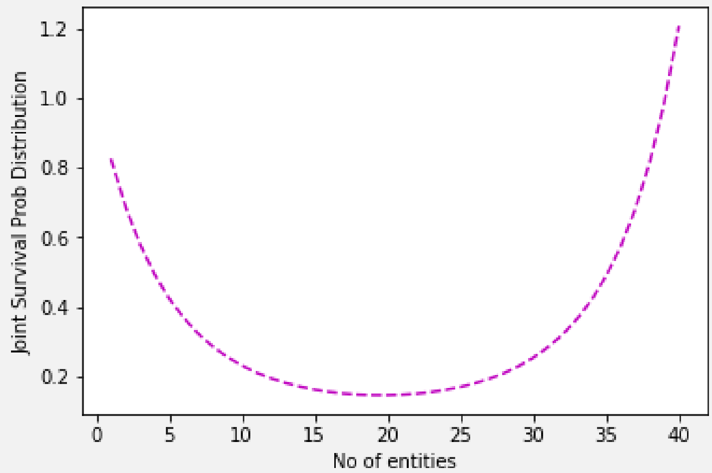


Figure 3.3: Survival distribution function with respect to reference entities.

From Figure 3.3, we observed that the survival curve decreased from $P(1) = 0.8255$ with an increase in the number of reference entities; attained minimum at $P(20) = 0.14652$, and then went up to $P(40) = 1.2068$. It is also seen that the probability $P(N) > 1$ after $N > 39$. The Vasicek model produced a negative default intensity, and as a result, the corresponding JSPD function exceeded 1 at some points. Such characteristics makes it non-feasible to completely model the intensity process using the Vasicek model. An exception is to place a restriction on the use of the model, such as limiting the number of risky assets in the given portfolio (Kijima (2000), Kijima (2000) and Liang et al. (2011)).

[3]These parameters are purely be assumption.

3.3 Pricing basket credit default swaps

Consider an n2D swap which is a bilateral swap contract whose payoff is dependent on the specified cumulative credit default event of a given portfolio of reference entities. The reference entities mentioned here could be bonds, loans, corporations, etc. The pricing of the above contingent claim needs the knowledge of the joint probability distribution functions of the default times. The default leg (DL) which is the payment by the protection seller in case of a credit event and premium leg (PL) which is paid by the protection buyer prior to default must be calculated. Under the risk-neutral probability measure, the fair swap spread is obtained as the ratio between the default payments and the premium payments.

Let the price of a bond with t maturity be

$$B(0,t) = \exp\left(-\int_0^t f(0,x)\mathrm{d}x\right),$$

where $f(0,x)$ is the instantaneous forward rate at initial time $t=0$. Let the discrete premium dates be $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$ and the frequency payment dates be $\delta_i = t_i - t_{i-1}$ (in units of years). Furthermore, denote S_n = swap spread; τ_n = the time of the nth default in the basket; n = the seniority of the basket (for instance, $n = 1$ means the first-to-default basket); $M = \sum_{i=1}^N M_i$ = total face value of the portfolio where M_i is the face value of the ith entity; $R = \sum_{i=1}^N R_i$ = constant recovery rate where R_i is the recovery rate of the ith entity; $\mathbb{I}_{\{\tau_n > t\}}$ = indicator function of the credit event; and T = maturity. If the nth asset defaults on or before maturity, the protection seller has to pay $M_i(1-R_i)$ to the protection buyer at time τ_n or at maturity. On the other hand, suppose no default occurs, the buyer continues to pay $S_n\delta$ at time t_i.

The present value for the **default leg** is given by

$$\mathrm{DL} = M(1-R)B(0,\tau_n)\mathbb{I}_{\{\tau_n \leq T\}}. \tag{3.3.1}$$

Taking its expectation value, we have

$$\begin{aligned}\mathbb{E}[DL] &= \mathbb{E}[M(1-R)B(0,\tau_n)\mathbb{I}_{\{\tau_n \leq T\}}] \\ &= M(1-R)\int_0^T B(0,\tau_n)\mathrm{d}F^n(t) \\ &= M(1-R)\int_0^T \exp\left(-\int_0^t f(0,x)\mathrm{d}x\right)\mathrm{d}F^n(t) \\ &= M(1-R)\left[B(0,T)F^n(T) + \int_0^T f(0,t)B(0,t)F^n(t)\mathrm{d}t\right]. \end{aligned} \tag{3.3.2}$$

The present value of the **premium leg** is given by

$$PL = S_n M \sum_{i=1}^{N} \delta B(0,t_i)\mathbb{I}_{\{\tau_n > t\}}. \tag{3.3.3}$$

Taking its expectation, we have

$$\mathbb{E}[PL] = \sum_{i=1}^{N} S_n M \delta B(0,t_i)[1 - F^n(t_i)]. \tag{3.3.4}$$

Under the risk-neutral measure, the value of the fair swap spread is solved by equating the expected value for the discounted premium leg (PL) with that of the default leg (DL) and solving for S_n. Theorem 3.2 and Theorem 3.3 give the values of the n2D swaps in the absence and the presence of accrued premium respectively.

Theorem 3.2. *(n2D Swaps, Fathi and Nader (2007b), Choe and Jang (2011a)).*

Assume that there is no accrued premium, and let $F^n(t) = \mathbb{P}(\tau_n \leq t)$ is the distribution function of the default time τ_n. Then, the risk neutral pricing measure for the annualized n2D swap is given below:

$$S_n = \frac{(1-R)\left[B(0,T)F^n(T) + \int_0^T f(0,t)B(0,t)F^n(t)\mathrm{d}t\right]}{\sum_{i=1}^{N} \delta B(0,t_i)[1 - F^n(t_i)]}.$$

In the presence of accrued premium, suppose a credit event occurs at the commencement of a premium period, then there will be no premium which must have accrued. Suppose the default occurred at the end of the time period, then a complete premium payment will have to be sorted out. But if however, any of the entities default within the two premium time period (t_{i-1}, t_i), then the protection buyer is under obligation to pay an extra amount of the accrued premium of $S_n\delta(t_{i-1}, \tau_i)$. The calculation follows the same basic convention used to pay the other premium.

Thus, the present value of the **Accrued leg** due to a default in the nth premium period is given by

$$AP = S_n M \sum_{i=1}^{N} \delta \left(\frac{\tau_n - t_{i-1}}{t_i - t_{i-1}} \right) B(0, \tau_n) \mathbb{I}_{\{t_{i-1} < \tau_n \leq t_i\}} \,. \tag{3.3.5}$$

Taking its expectation, we have

$$\mathbb{E}[AP] = S_n M \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} (q - t_{i-1}) \delta B(0, q) F_n \mathrm{d}q \,. \tag{3.3.6}$$

Theorem 3.3. *(n2D Swaps with Accrued Premium, (Fathi and Nader 2007b)).*

In the presence of accrued premium, and let $F^n(t) = \mathbb{P}(\tau_n \leq t)$ is the distribution function of the default time τ_n. Then, the risk neutral pricing measure for the annualized n2D swap is given as

$$S_n = \frac{(1 - R) \left[B(0, T) F^n(T) + \int_0^T f(0, t) B(0, t) F^n(t) \mathrm{d}t \right]}{\sum_{i=1}^{N} \delta B(0, t_i)[1 - F^n(t_i)] + \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} (q - t_{i-1}) B(0, q) F_n \mathrm{d}q} \,.$$

3.3.1 Modelling the default time

The expectation values in the pricing of an n2D basket swap are over the joint distribution of default times, and thus, it is vital to know the joint probability distribution of the default times of the entities in the given portfolio. This probability can be modelled via a copula function, which connects the marginal default probability to their joint probability. Market information relating to that entity can provide the corresponding marginal probability distribution of each of the entities (Chen and Glasserman 2008). This information is essential in the approximation of the instantaneous intensity rate λ_i for each entity i, and then we can define the probability distribution (Kijima and Muromachi 2000):

$$F_{\tau_i}(t_i) = \mathbb{P}(\tau_i \leq t_i) := \mathbb{E} \left[1 - \exp \left(- \int_0^{t_i} \lambda_i(u) \mathrm{d}u \right) \right] \,. \tag{3.3.7}$$

The corresponding probability density function $f_{\tau_i}(t_i)$, which can be obtained from the derivative of $F_{\tau_i}(t_i)$ exists provided that the expectation and the differentiation of the given functions are interchangeable. According to Bartle (1976), a sufficient condition which ensures the existence of this interchangeability is that: For $Q(t_i) = 1 - \exp\left(-\int_0^{t_i} \lambda_i(u)\mathrm{d}u\right)$ in equation (3.3.7), $Q(t_i)$ and $\frac{\partial}{\partial t_i}Q(t_i)$ are continuous in t_i, $\mathbb{E}[Q(t_i)]$ exists, and $\mathbb{E}\left[\frac{\partial}{\partial t_i}Q(t_i)\right]$ is uniformly convergent in t_i. Thus, we have that

$$f_{\tau_i}(t_i) := \frac{\partial}{\partial t_i}\mathbb{P}(\tau_i \leq t_i) := \mathbb{E}\left[\lambda_i(t)\exp\left(-\int_0^{t_i} \lambda_i(u)\mathrm{d}u\right)\right]. \tag{3.3.8}$$

Using the bond market data and CDS spread data, one can calibrate the distribution function $F_{\tau_i}(t)$ to the market data. Thus, in the analysis of the default time modelling, the joint distribution function is decomposed into univariate marginals and the dependency structure of default time (as described by the suitable copula), and they are made evident in the following theorem.

Theorem 3.4. *(Sklar's Theorem, Schönbucher (2003)).*

Denote $\tau_1, \tau_2, \cdots, \tau_N$ as random variables having marginal distribution functions F_{τ_i} and joint distribution function $F_{\tau_1,\tau_2,\cdots,\tau_N}$. Then, there exists an N-dimensional copula function $C : [0,1]^N \to [0,1]$, such that $\forall\ \tau_{\mathbf{i}} \in \mathbb{R}^N$, we have:

$$F_{\tau_1,\tau_2,\cdots,\tau_N}(\mathbf{t}) = \mathbb{P}(\tau_1 \leq t_1, \tau_2 \leq t_2, \cdots, \tau_N \leq t_N) = C(F_{\tau_1}(t_1), F_{\tau_2}(t_2), \cdots, F_{\tau_N}(t_N)).$$

If $F_{\tau_1}, F_{\tau_2} \cdots, F_{\tau_N}$ are continuous, then C is unique. Else, C is uniquely determined on $RanF_{\tau_1} \times RanF_{\tau_2} \times \cdots \times RanF_{\tau_N}$, where $RanF_{\tau_i}$ denotes the range of F_{τ_i}, for $i = 1, 2, \cdots, N$.

Thus, the copula function C is significant in the modelling of joint default time distribution, and it reflects the dependency amongst the default times in the pricing of BDS and CDO tranches.

In this work, we focus on implementing the **one-factor Gaussian copula** given by Laurent and Gregory (2005) and Choe and Jang (2011a). The Gaussian copula model is defined by $C(t_1, \cdots, t_n) = \Phi_\Sigma(\Phi^{-1}(t_1), \cdots, \Phi^{-1}(t_n))$, where Φ is the cumulative standard normal distribution function and Φ_Σ is the joint distribution function for a multivariate random normal vector, having a covariance matrix Σ and zero mean. Define the correlated standard

Gaussian random variables Y_i as:

$$Y_i = \rho_i Z + \sqrt{1-\rho_i^2}\epsilon_i = \Phi^{-1}\Big(F_i(\tau_i)\Big)\,,$$

where Z is a common factor (systematic risk) for all the entities and ϵ_i is the error term (or entity-specific risk). Both the random variables Z and ϵ_i are independent standard normally distributed, so that for $i \neq j$, $\mathrm{Cov}(Y_i, Y_j) = \rho$ and $\mathrm{Var}(Y_i) = 1$. We seek to obtain the joint probability distribution function for the correlated default times:

$$\mathbb{P}(\tau_1 \le t_1, \tau_2 \le t_2, \cdots, \tau_N \le t_N) = \mathbb{P}\Big(F_1^{-1}\big(\Phi(Y_1)\big) \le t_1, \cdots, F_N^{-1}\big(\Phi(Y_N) \le t_N\big)\Big)\,.$$

Taking expectations of both sides with respect to Z, we have

$$\begin{aligned}\mathbb{E}_Z\left[\mathbb{P}(\tau_1 \le t_1, \tau_2 \le t_2, \cdots, \tau_N \le t_N)|Z\right] &= \mathbb{E}_Z[\mathbb{P}(F_1^{-1}(\Phi(Y_1) \le t_1, \cdots, F_N^{-1}(\Phi(Y_N) \le t_N)|Z] \\ &= \mathbb{E}_Z[\mathbb{P}(F_1^{-1}(\Phi(Y_1) \le t_1|Z) \times \cdots \times \mathbb{P}(F_N^{-1}(\Phi(Y_N) \le t_N|Z)]\,.\end{aligned}$$

Since

$$\mathbb{P}(F_i^{-1}(\Phi(Y_i) \le t_i|Z) = \Phi\left(\frac{\Phi^{-1}(F_i(t_i)) - \rho_i Z}{\sqrt{1-\rho_i^2}}\right)\,,$$

and since the default times conditional on Z are independent, the joint probability of default times reads

$$\mathbb{P}(\tau_1 \le t_1, \tau_2 \le t_2, \cdots, \tau_N \le t_N) = \int_{-\infty}^{\infty} \prod_{i=1}^{N} \Phi\left(\frac{\Phi^{-1}(F_i(t_i)) - \rho_i z}{\sqrt{1-\rho_i^2}}\right)\psi(z)\mathrm{d}z\,.$$

On the other hand, the joint probability of the survival time is

$$\mathbb{P}(\tau_1 > t_1, \tau_2 > t_2, \cdots, \tau_N > t_N) = \int_{-\infty}^{\infty} \prod_{i=1}^{N} \Phi\left(\frac{\rho_i z - \Phi^{-1}(F_i(t_i))}{\sqrt{1-\rho_i^2}}\right)\psi(z)\mathrm{d}z\,,$$

where $\psi(z) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\frac{-z^2}{2}}$ denotes the standard normal density function of Z.

For the simulation of the default time, we employ the MCS method. In the numerical experiment for section 3.4, we used the default times to calculate the expectation of each leg by averaging over the discounted payoff for each the protection legs and premium. When the default boundary is a deterministic function, then each of the default time τ_i, for random variables Y_i can be defined as (Laurent and Gregory 2005):

$$\tau_i := \inf\left\{t \geq 0 : \int_0^t \lambda_i(q)\mathrm{d}q \geq -\ln(Y_i)\right\}.$$

Thus, by mapping the cumulative normal distribution between the Gaussian variable Y_i and the default time τ_i, the default time of the reference entity i can be simulated using

$$\tau_i = \frac{-\ln(1-\Phi(Y_i))}{\lambda_i}. \tag{3.3.9}$$

3.4 Parameter Analysis and Numerical Experiments

In this section, we investigate the concept of n2D basket swap pricing under both the homogeneous and the heterogeneous portfolios.

3.4.1 Heterogeneous Portfolio

This section will consider a portfolio of five corporate entities with different credit ratings given by Standard & Poor's as of 28-02-2017. The entities have the grades[4] AAA, AA, A, BBB and BB. For the data, we used the OAS spread bid of the entities from the periods of 30-10-2015 till 28-02-2017, monitored monthly with a dollar (\$) denomination.

Table 3.1: Entities with their credit ratings

Entities	Microsoft	Apple	Pepsi	General Motors	Free Port Inc.
Ratings	AAA	AA	A	BBB	BB

For the parameters of the default intensities for each of the entities, we employ the Maximum Likelihood Estimation (MLE) under the Vasicek model, using the historical bond spread data. The MLE maximizes the log-likelihood function which spans over the given parameter space, and this estimation process was conducted via the `excel` software. Let S be a random variable having $f(S;\theta)$, as the frequency function or the probability density function of S which is dependent on the parameters $\theta = \{\alpha, \beta, \sigma\}$. Here, we used the individual OAS

[4]Here, AA^+ is viewed as AA; AA^- as A^+; A^- as A; BBB^+ as BBB; BBB^- as BB^+; and BB^- as BB

spread bids as our sample $\{S_1, S_2, \cdots, S_n\}$ where n is the number of the spread values, and in which the log-likelihood function[5] could be described as:

$$\log L(\theta; S) = \log\left(\prod_{i=1}^{n} f(S_i;\theta)\right) = \sum_{i=1}^{n} \log\left(f(S_i;\theta)\right) .$$

Thus, the maximum likelihood estimator is defined below as:

$$\hat{\theta} = \arg\max_{\theta} \log L(\theta; S) = \arg\max_{\theta} \sum_{i=1}^{n} \log\left(f(S_i;\theta)\right) .$$

For the CIR model, we used the explicit estimator functions of the α, β and σ parameters of the historical N-observed market spreads (for $N = 5$ in this context), which are described by Bibby et al. (2010). These estimators were obtained from the construction of an approximate optimal estimating function which describes the CIR model.

Applying the MLE to the Vasicek model, we modelled the intensity process $\lambda_i(t)$ for each of the entity using equation (3.2.11), by setting the initial intensity process at time $t = 0$, to be $\lambda_i(0) = 0.02$, and the initial parameters $\alpha_i = 0.2, \beta_i = 0.02, \sigma_i = 0.002$ and swap maturity time, $T = 5$ years. Table 3.2 gives the calibrated parameters, the first and second moments[6] for each of the entities using the Vasicek model, together with the default intensity for each entity. In the CIR model, we estimated the intensity process $\lambda_i(t)$, as well as the corresponding parameters using equation (3.2.2), and we equally assumed a swap maturity of $t = 5$ years. The first and the second moments under the CIR model which were obtained from equation (3.2.4) and equation (3.2.5) respectively, were equally presented in Table 3.3.

[5]The solution of the log-likelihood function can be done either analytically or numerically. For convenience, we employed the MLE – Solver Add-Inn in the Microsoft Excel to obtain the estimated parameters.

[6]Note that $A\mathcal{E}^{-a} = A \times 10^{-a}$.

Table 3.2: Parameter estimation, first & second moments for N entities under the Vasicek model

Entities	i	DI	Parameters (α, β, σ)	$\mathbb{E}[\lambda_i(t)]$	Var$[\lambda_i(t)]$
AAA	1	42	(0.66196, 0.00263, 0.00064)	32.64413	$3.08972\mathcal{E}^{-3}$
AA	2	69	(0.95874, 0.00421, 0.00059)	43.40769	$1.81528\mathcal{E}^{-3}$
A	3	128	(0.38473, 0.00834, 0.00088)	100.43209	$9.84946\mathcal{E}^{-3}$
BBB	4	266	(0.26734, 0.01816, 0.00248)	186.43389	$1.07091\mathcal{E}^{-1}$
BB	5	1019	(0.25302, 0.06547, 0.01482)	526.37863	$3.99455\mathcal{E}^{0}$

Table 3.3: Parameter estimation, first & second moments of N entities under CIR model

Entities	i	DI	Parameters (α, β, σ)	$\mathbb{E}[\lambda_i(t)]$	Var$[\lambda_i(t)]$
AAA	1	42	(0.67433, 0.00235, 0.00044)	29.55976	$5.04947\mathcal{E}^{-6}$
AA	2	69	(0.43228, 0.00410, 0.00022)	59.31104	$4.07891\mathcal{E}^{-6}$
A	3	128	(0.71303, 0.00766, 0.00021)	80.09132	$2.57674\mathcal{E}^{-6}$
BBB	4	266	(1.22805, 0.01566, 0.00149)	156.69350	$1.42483\mathcal{E}^{-4}$
BB	5	1019	(1.42376, 0.05759, 0.01401)	575.59562	$3.96555\mathcal{E}^{-2}$

From Tables 3.2 and 3.3, we observed that firms with higher credit ratings have lesser default intensity (i.e., their conditional probabilities of no earlier default per year) compared to lower-rated entities. These highly rated firms have a strong capacity of meeting up to their financial obligations, and their corresponding low bond yield serves as a security and high repayment probability. The lower-rated firms, on the other hand, have higher bond spreads, and this follows from their vulnerability and a higher risk of default. We further observe that as the entity gradually transits to a lower credit rating, the expected values of their hazard rates increase, together with their probabilities of default[7]. Using a constant recovery rate $R_i = R = 40\%$, we obtain the average default intensity of each firm from the given historical data, using the formula $S_i(1-R)^{-1}$, where S_i is the spread of each entity. This recovery rate is a common assumption of market participants (Hull et al. 2005). The expectation values of the hazard rates differ slightly in both the Vasicek and the CIR model, whereas

[7]This probability can be obtained from CDS spreads, Merton's structural model, bond prices, or historical data.

huge discrepancies between their variances exist. The result is evident because in the CIR model, the volatility parameter contains an extra $\sqrt{\lambda_i(t)}$ term and this function reduces the effect of the volatility change. We observed that as a fixed change in the volatility affect the intensity rate of the CIR more than the Vasicek, thus, leading to a smaller variance, as shown in Table 3.3.

Furthermore, when the portfolio of the 5 entities are observed, the marginal survival distribution is smooth and strictly decreasing. These survival functions give the probability that the portfolio will attain at a specific time, and using equation (3.2.12), their JSPD values are shown in Table 3.4:

Table 3.4: JSPD of the entities in Table 3.1 with different maturity time

T	1	2	3	4	5	6	7	8	9
JSPD	0.91022	0.83490	0.76703	0.70416	0.64547	0.59071	0.53978	0.49261	0.44909

In Table 3.5, we output the values of the estimated hazard rates, survival probabilities, break-even swap spreads and the weighted-portfolio values of the given entities using both the Vasicek and the CIR model. We assumed that the portfolio is equal-weighted, that is, $V_p = \sum_{i=1}^{N} x_i S_i$, where V_p is the portfolio value, S_i is the spread of each ith entity, and $x_1 = x_2 = \cdots = x_N$ are the equal weights, with $N = 5$. For the hazard rates of each entity, we used the Euler-Maruyama scheme which discretizes the CIR and the Vasicek models given in equations (3.2.2) and (3.2.11) into the following equations respectively:

$$\lambda_i(t+\mathrm{d}t) = \lambda_i(t) + \alpha_i(\beta_i - \lambda_i(t))\Delta t + \sigma_i\sqrt{\Delta t|\lambda_i(t)|}\epsilon_i \quad \text{and} \tag{3.4.1}$$

$$\lambda_i(t+\mathrm{d}t) = \lambda_i(t) + \alpha_i(\beta_i - \lambda_i(t))\Delta t + \sigma_i\sqrt{\Delta t}\epsilon_i\,, \tag{3.4.2}$$

where $\epsilon_i \sim \mathcal{N}(0,1)$ and $i = 1, \cdots, 5$. In the estimation of the hazard rates $\lambda_i(t)$, we considered the zero initial hazard rate, time $T = 5$ years, $K = 1000$ time points such that $\Delta t = \frac{T}{K}$ time steps. The estimated parameters α_i, β_i and σ_i explicitly given in Table 3.2 (Vasicek model) and Table 3.3 (CIR model) were equally incorporated in equations (3.4.2) and (3.4.1) respectively.

Table 3.5: Spread and portfolio values using the Vasicek and the CIR intensity-based models

Model	Entity	Hazard Rates	Survival Probability	Swap Spreads	Portfolio Value (V_p)
Vasicek	AAA	0.00176	0.96198	47.09089	351.41831
	AA	0.00356	0.92683	92.79046	
	A	0.00616	0.88097	155.02713	
	BBB	0.01340	0.76706	325.54832	
	BB	0.05081	0.39897	1136.63477	
CIR	AAA	0.00168	0.96396	44.76380	333.00992
	AA	0.00243	0.94803	65.10054	
	A	0.00557	0.89.80	141.44050	
	BBB	0.01305	0.77250	316.85286	
	BB	0.04886	0.41219	1096.89188	

From Table 3.5, we observed that the hazard rates increased drastically as the higher-rated firms transcend to lower-rated firms, and these characteristics, in turn, led to larger spread values. The survival probabilities which measure the likelihood of the firms not to experience any prescribed credit event equally reduced as the firms get downgraded to lower-rated entities. For the computation of the default times, we utilized the one-factor Gaussian copula model together with the Monte-Carlo method. Random samples were drawn from the multivariate normal distribution with the incorporation of the entities' correlation matrix (Table 3.6), which was calculated using Kendall's tau correlation method. Finally, the default times were estimated using equation (3.3.9).

Table 3.6: Correlation coefficient of entities using Kendall's tau method

Correlation (ρ)	AAA	AA	A	BBB	BB
AAA	1.00000	0.42647	0.08824	-0.04412	-0.20588
AA	0.42647	1.00000	0.45588	0.32353	-0.07353
A	0.08824	0.45588	1.00000	0.66177	0.38235
BBB	-0.04412	0.32353	0.66177	1.00000	0.60294
BB	-0.20588	-0.07353	0.38235	0.60294	1.00000

Furthermore, from Table 3.5, it was shown that the combination of the higher hazard rates and lower survival probabilities gave rise to the larger probability of default, which in turn, resulted to larger spreads. Other parameters which were assumed in computing the swap spreads includes: $r = 0.05, R = 0.4$, frequency of payment $\delta = 0.5$, 500000 Monte-Carlo simulated points, $N = 5$ and $T = 5$. Generally, we estimated the hazard rates of the firms and priced the corresponding basket swaps using both the Vasicek and the CIR models. Results from the model comparison showed that the CIR model outperformed the Vasicek model in the pricing problem, even though the discrepancies are not significantly well-pronounced. This is because the values obtained from the CIR model are relatively close to the observed market spread depicted by "DI" in Tables 3.2 and 3.3 above. Graphically, the error estimates are shown in Figure 3.4 below:

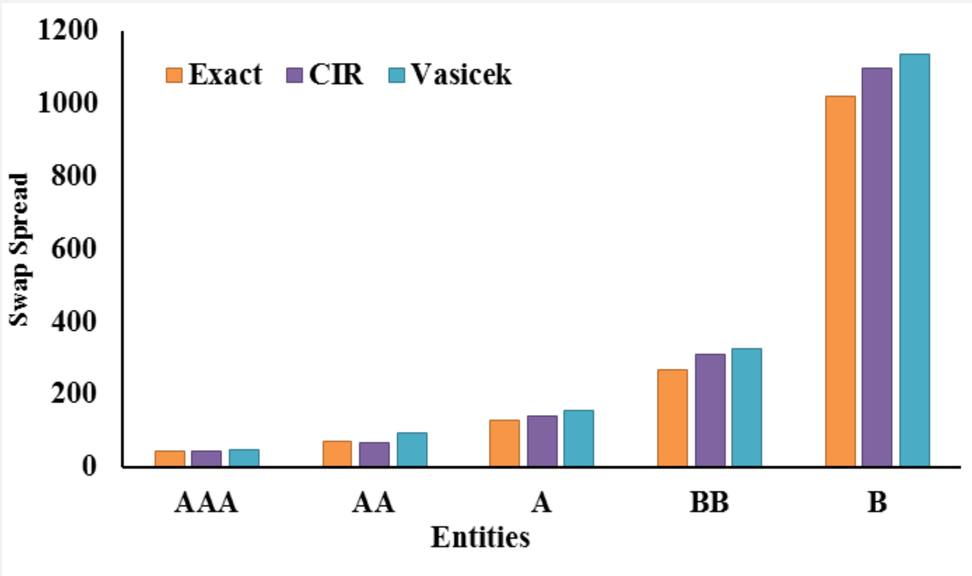


Figure 3.4: Swap spreads of entities modelled using stochastic hazard rates

3.4.2 Homogeneous Portfolio

Consider the valuation of the basket default swap for a homogeneous portfolio of $N = 10$ entities. They have the same notional value M, recovery rate $R = 0.5$. The maturity of the swap is $T = 5$ years, the payment frequency is $\delta = 0.25$ (quarterly payment), r=0.04, $\lambda = 0.07$ and $\rho = 0.45$. With 10000 number of simulations, we estimate the values for the default leg, premium leg and accrued premium leg for different nth basket or rank of the default protection, by using the formulas in equations (3.3.2), (3.3.4) and (3.3.6) respectively. The results are displayed in Table 3.7 below.

Table 3.7: Default leg, premium leg, and accrued premium values for increasing rank of the default protection

Rank	Default leg	Premium leg	Accrued premium leg
1	0.36126	2.14996	0.08695
2	0.27968	2.92183	0.07028
3	0.21665	3.40114	0.05442
4	0.16368	3.73715	0.04011
5	0.12217	3.96857	0.03058
6	0.08657	4.14992	0.02157
7	0.05925	4.27591	0.01476
8	0.03720	4.37210	0.00953
9	0.01951	4.44181	0.00496
10	0.00721	4.44864	0.00188

Table 3.7 shows that as the rank level ($n = 1$ for F2D, $n = 2$ for S2D, etc.) increases, the default leg and the accrued premium legs decrease, whereas, the premium leg increases. This result, in turn, leads to a decrease in the value of the n2D basket spread and the lower the rank level, the less risky. The accrued premium is a fraction of the premium which has accrued starting from the date the previous payment was made, till the time when a default occurs. It is the smallest in comparison with the other legs, and its effect can be ignored since it is very insignificant. Considering the same 10 entities-portfolio, the probability of having a fewer number of entities to default is more than having many entities to default. The protection seller experiences loss when there is more than one default. Thus, the payment from the default leg in case of a default reduces as the number of entities to default increases. Furthermore, since the probability of default decreases as the rank level increases, and since there is less likelihood for many entities to default in the portfolio, the protection buyer has to pay more premium at regular intervals.

Table 3.8 below considers the same parameters as in Table 3.7, but we vary and increase the hazard rate (λ) and the default correlation (ρ) for the valuations of the first to fourth-default swap. Here, we considered the case where the accrued premium was paid, and using the Theorem 3.3, we output the following values displayed in Table 3.8 below:

Table 3.8: Effects of default intensities and default correlations on basket default swap prices

F2D											
λ / ρ	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.99
0.01	487	453	416	367	324	289	258	216	172	129	66
0.02	1008	879	770	675	588	512	427	354	295	220	133
0.03	1517	1306	1115	973	831	729	620	530	409	310	193
0.04	2003	1738	1474	1273	1100	944	780	663	540	423	260
0.05	2519	2119	1781	1564	1312	1126	956	794	636	498	318
S2D											
λ / ρ	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.99
0.01	86	100	111	119	124	120	110	107	97	84	61
0.02	275	281	263	262	259	254	226	219	192	169	120
0.03	494	467	461	418	407	376	342	316	273	245	183
0.04	743	684	636	595	551	505	458	419	376	310	236
0.05	1002	897	822	762	683	615	564	521	467	387	298
T2D											
λ / ρ	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.99
0.01	11	22	36	45	54	61	70	71	75	64	62
0.02	59	92	108	127	139	143	148	145	148	139	117
0.03	154	194	217	224	226	241	233	224	216	200	174
0.04	277	310	326	321	325	333	325	310	295	267	229
0.05	421	433	439	440	434	425	401	396	378	368	264
Fo2D											
λ / ρ	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.99
0.01	1	5	10	20	28	33	42	45	52	55	51
0.02	9	27	42	57	79	84	96	107	110	117	106
0.03	39	66	95	115	132	148	153	169	175	178	156
0.04	88	127	163	181	199	216	225	231	237	228	208
0.05	164	204	235	259	281	285	289	292	294	288	266

Table 3.8 gives the valuations of the premium prices for the $F2$D, second-to-default ($S2$D), third-to-default ($T2$D), and fourth-to-default ($Fo2$D) values, together with the effects of the default correlations and hazard rate. The prices for the S2D and T2D swaps are generally lesser compared to the F2D swaps, and Fo2D swaps are the cheapest among the others because the payoff is paid when the entity defaults four times, notwithstanding prior defaults. The protection buyer is entitled to a net loss on the entity which has defaulted, and not on the cumulative loss. The hazard rate is a significant factor in the pricing of $n2$D swaps. A positive correlation exists between the hazard rate and the premium, with every other parameter kept constant, and this evident in the following relation $S = \lambda(1 - R)$, for a constant recovery rate R. For example, when $\lambda = 0.02$ for an F2D swap, the swap spread becomes 1008 basis point(bp), but the spread increases to 2519 bp when the hazard rate increases to 0.05. Furthermore, we observe that the intensity for the probability of default is entirely dependent on the hazard rate value. Thus, from the perspective of the protection seller, lesser spread has lower default probability and more likelihood to offset its debt.

A negative correlation exists between the BDS spread values (in this case, F2D) and the default correlation, as the spread decreases with an increase in the default correlation. Lower default correlation implies that fewer losses are more likely to occur, and this strengthens the effects of risk diversification, thereby lowering the BDS spreads[8]. The more entities stay correlated, the higher the chances of the portfolio to exhibit characteristics of a single-named entity. Furthermore, zero correlation implies no relationship among the portfolio entities and selling protection on non-correlated entities (for F2D swaps) is the same as selling individual credit protections; thus; this accounts for their large spread. Hence, investors seeking to achieve the highest swap premium should sell protection on a portfolio of entities with low correlation or zero correlation, like over a portfolio of many unrelated industries.

As we gradually move away from the valuation of the F2D swaps, we noticed that monotonicity is no longer applicable. The result is evident from Table 3.8 because for a fixed hazard rate and increasing default correlation, the spread values started to fluctuate; that is, increasing and decreasing at some points. For example: consider the S2D swap, where $\lambda \geq 0.03$ in Table 3.8, we notice a consistent decrease in the spread value as the default correlation increases. But the reverse is the case for lower hazard rate $\lambda < 0.03$. Also, we observe the non-linearity of the BDS spread for a fixed correlation in the valuation of the T2D and Fo2D swaps. The primary reason behind this phenomena was given explicitly by

[8]This might be different for a portfolio which consists of sufficiently large entities, or when higher-order swaps are considered.

Jabbour et al. (2009).

3.5 Conclusion

Basket default swaps are financial credit derivatives which are linked to an underlying basket of entities, bonds, loans or assets. The pricing is dependent on the joint distribution of the corresponding default times of the entities in question. We modelled the dependency structure of the default time, defined under the one-factor Gaussian copula method, with the Monte-Carlo method. Furthermore, this work focused on the valuation of n2D swaps, which are based on homogeneous and heterogeneous portfolios of entities. Under the stochastic intensity models, we modelled the hazard rate and estimated the model parameters of five corporate entities, having different credit ratings. We further obtained the JSPD of the same entities, and we observed that an increase in the maturity time T leads to a decrease in the survival probability. We also considered and estimated the survival probabilities, the swap spreads and the equal-weighted portfolio values defined within the frameworks of the Vasicek and the CIR model. Results obtained showed that the CIR is more applicable in modelling the hazard rate process, and this, in turn, led to more efficient pricing of the basket swaps, as compared to the Vasicek counterpart.

Hedging portfolio credit derivatives involve an appropriate calculation of the sensitivities of the swap value with respect to the associated parameters. Thus, this work conducted a sensitivity analysis for the prices of the n2D basket, in connection to the rank of the default protection, the intensity rate and the default correlation. We observed that the value of the n2D swaps behaves differently with regards to changes in the intensity rate and the default correlation, as the rank of default protection increases. Finally, we observed that investors that want to trade F2D swaps with highest swap premium should sell protections on entities with low correlations.

4. Elliptical and Archimedean copula models: An Application to the Price Estimation of Portfolio Credit Derivatives

This chapter has been submitted and accepted to be published by *Risk Journals* (submission ID: MS 2025) as

Umeorah N., Mashele P. & Ehrhardt M. (2020). Elliptical and Archimedean copula models: an application to the price estimation of portfolio credit derivatives.

Abstract

This work explores the impact of elliptical and Archimedean copula models on the valuation of the basket default swaps. To achieve this aim, we employ the Monte-Carlo simulation, in connection with the copula models, to estimate the default times, and to calculate the swap payment legs and the cumulative swap premium. The numerical experiments also reveal some sensitivity analysis on the impact of swap parameters on the fair prices of the nth-to-default swaps. Finally, from the results presented, an appropriate choice of the copula model can be made based on the computation time of the valuation process, and such choice hugely affects the quantitative risk analysis of the portfolio.

Keywords

Copulas, Elliptical Copulas, Archimedean Copulas, Basket Default Swaps, Monte-Carlo Simulations, Default Times, Sensitivity Analysis.

4.1 Introduction

Portfolio credit derivatives (PCD) are financial instruments whose payoffs are dependent on the default times evident in a given portfolio of defaultable or risky assets, and these instruments include the Collateralized Debt Obligations (CDOs) and the Basket Credit Default Swaps (BCDS). The payoffs are normally due to credit events, such as restructuring, downgrade, bankruptcy, or any situation which affects the creditworthiness of an entity, in a portfolio of several entities. Depending on the seniority or the rank of the default protection, BCDS can be classified into first-to-default (F2D), second-to-default (S2D), or generally, the nth-to-default (n2D) basket swaps. Investors that trade the BCDS encounter correlation risk, or the likelihood of simultaneous defaults in the basket of entities, and these credit securities help in the transfer of credit risks amongst the participants, as well as in hedging and speculation (Umeorah, Ehrhardt and Mashele 2019). Their prices are dependent on the default correlations of the entities in the portfolio. Furthermore, the valuation of these credit derivatives entails the knowledge of the joint distribution of the default times, which poses a problem during the modelling, but the introduction of copulas had made it feasible.

This research work is prompted by the interplay of various copula models and the default time modelling with regards to their tangible applications in PCD valuations. It is vital to have substantial knowledge of the effects of different copula modelling to the pricing of the BCDS. Hence, our objectives in this research include conducting a comparative study on the pricing of BCDS, investigating the impact of modelling the default time parameter using both elliptical and Archimedean copula methods, as well as recommending a proper choice of copula models in BCDS pricing. This study will model default times using the five copula models listed in the introductory part of this work. The key to the choice of these copula models lies on their familiarities, simplicities and their widespread usage in terms of modelling defaults and incorporating them to the pricing mechanics of BCDS.

Under the elliptical copulas, the Gaussian model was chosen because of its analytical tractability, as well as its easy calibration benefit. The Gaussian copula focuses more on the multivariate normal distribution in the analysis of credit risks, and during financial crises, it fails to take into consideration the default clustering. Most corporate entities defaults often happen in clusters, and one entity defaulting increases typically the likelihood of the other defaulting within a short time frame. Hence, our incorporation of other alternative copulas with the tail dependency structures, like the student t-copula. This property is essential for risk management because the student t-copula tends to the Gaussian as their degrees of freedom increases. The Frank, Clayton, and the Gumbel are prominent copula models in the Archimedean family, with the Clayton and Gumbel modelling extreme lower and upper dependences respectively. The Frank copula, on the other hand, are symmetric and can permit a maximum interval of dependency. Thus, these copulas were chosen for illustrative purposes.

To ensure clarity and simplicity in the comparative study, we will consider a homogeneous basket, and further assume that the pricing parameters such as the hazard rate, concordance measures (ρ and θ), time to expiration, etc., are all kept constant irrespective of the copula models employed. The main results of the study will feature some numerical experiments, in accordance with the same Kendall's tau (τ) concordance structure in the estimation of the n2D swaps prices. We will equally apply the concept of ordered statistic employed by Choe and Jang (2011a), to numerically compute the corresponding payment legs associated with the swap pricing. Some findings on the estimation of the n2D swaps prices based on the Monte-Carlo methods will be presented, together with some comparative sensitivity studies with regards to the default swap parameters. Finally, inference will be made on the appropriate choice of the copula model from their computation time.

The remaining structure of this study is as follows: Section 4.2 will present some related literature studies on the pricing techniques of basket credit derivatives. Section 4.3 describes the structure of the model, which is the copula model and highlights its applicability in the derivatives pricing. Section 4.4 introduces the concept of the valuation of BCDS, explains how default time can be modelled via different copula methods and outlines how the swap premium can be obtained. It further introduces the payment legs associated with swap spread, and then discussed the Monte-Carlo method, as the simulation technique employed in the research. Section 4.5 focuses on results obtained in the numerical experiments of the BCDS pricing, some sensitivity analysis on the default swap spreads and the tabulating of the CPU time for the whole pricing process. Section 4.6 concludes our research study.

4.2 Literature Study

In the analysis of financial, economic and insurance risks, the concept of quantifying dependence has been a significant issue, and this sparked the interest of developing copula models. With regards to their significance, Peng and Qi (2017) stated that the copula models and their corresponding survival copula models had been recommended in Basel III and Solvency II, for both the banking sectors and the insurance industries, respectively. Patton (2012) further explained that copula models had been employed in the field of risk management; to effectively value derivatives contracts such as the CDS and the CDOs; in portfolio decision problems which involves utility maximization; as well as in other financial and economic time series. To ensure efficient and effective risk management profile, Kolev et al. (2006) noted that the concept of extreme value techniques needs to be introduced, with much emphasis on the tail dependency copula models. The authors further explained that ignoring such dependency effects could possibly lead to the underestimation of suitable risk measures. Other intensive review on the copula models, their key characteristics and their implementations can be found in Venter (2002), Charpentier (2003) and Kolev et al. (2006).

Copulas have equally been introduced to mitigate the inconsistencies, inefficiencies and the computational costliness of the widespread structural models, as well as the reduced form models, which are typically employed in the valuation of multi-name PCD. The default correlation factor, which is a crucial parameter in the PCD modelling, has motivated market participants to model credit events using the factor copula techniques. This default correlation estimates the likelihood of two entities to experience a simultaneous default. In the financial and insurance sector, copula models have been fully utilized in ensuring efficient and flexible modelling of dependence structures. There are lots of copula models, but not all of them are applicable in modelling PCDs. As one significant drawback, Watts (2016) explained that for independent variables with a correlation of +1 or -1, the corresponding copula used to model the variables are inherently inappropriate for real-life applications.

An increased amount of research has been channelled to the valuation of financial credit derivatives, primarily the CDOs and the BCDS, in the field of copula models. The first implementation of copula functions in the modelling of financial credit derivatives was investigated by Li (2000), who used the market prices of CDS to model the default dependency structures. Practically, he focused on pricing the CDS and the $F2$D swaps using the Gaussian copula model. Fathi and Nader (2007b) further focused on the impact of dependency structure and the choice of simulation methodology to value the $n2$D swap and CDO. Mashal and Naldi (2002) considered the $n2$D and the CDOs pricing using a hybrid of the structural and the reduced form techniques as default models. They further studied the student-t copula, which is a class of elliptical copulas with larger tail dependency structure. In the presence of stochastic recovery rates, the student-t copula has been employed by Goegebeur et al. (2007) in the modelling of default time dependencies and thus, they applied the concept to the valuation of the synthetic CDOs. Hull and White (2004) and Andersen et al. (2003) both developed analytical methodologies which are based on copula functions, to value credit derivatives. The former incorporated a recursion method to value n2D swaps, and the latter introduced an improved recursion-based technique which derives the portfolio loss distribution.

Furthermore, Choe and Jang (2011b) proposed an importance sampling algorithm, in connection with the nested Gumbel copula and the exchangeable Archimedean copulas, to value the BDS. To achieve this, they established a multi-level dependence structure using nested copulas to model the credit risks of a portfolio which consists of sub-portfolios. Importance sampling techniques were also employed by Schröter and Heider (2013b) to quantify the credit model risk of a given portfolio, and they further presented an analytical formula to value the n2D swaps (Schröter and Heider 2013a). Burtschell et al. (2005) compared different copula functions such as, Gaussian, student-t, double-t, Clayton and Marshall-Olkin copulas, to model the structure of the joint default of default times which is based on the factor model. They compared the concept of semi-analytical pricing to the approximation techniques associated with large portfolio securities and subsequently estimated the premiums of the BCDS and CDOs. However, the appropriate choice of the copula model remains an unavoidable question in financial modelling, as Durrleman et al. (2000) presented few methodologies for the right choice within the family of the Archimedean copulas. In each family, they considered the parametric estimation, like the maximum likelihood, information matrix systems, dependence measures for the parametric families; together with the non-parametric estimation, which are all rooted on the empirical copula models, and they further proposed appropriate selection criteria for the optimal copula.

4.3 Model Structure

In this section, we introduce the Gaussian copula, the student-t copula, Archimedean (Clayton, Gumbel and Frank) copula, as well as describe their effects in analysing the tail dependency structures of random variables. The tail dependency is noteworthy because they are significant factors in measuring the probability of simultaneous extreme losses, specifying the quantity of joint dependency in the tails of their distribution functions, and ignoring them will underestimate the default risk premium (Naifar 2011).

4.3.1 Introduction to Copula Models

A copula function essentially joins the univariate distributions and thus, generates multivariate distribution functions (Trivedi and Zimmer 2007). A crucial mathematical theory in connection with the copula models is the Sklar's theorem, which essentially explains that any multivariate probability distribution functions can be disintegrated into the components of a copula and its univariate margins. Conversely, a combination of some given margins together with a corresponding copula can be integrated back into a multivariate distribution (Rüschendorf 2013). A consequence of this is that copulas enable multivariate distribution to be expressed in form of their marginal distributions, and it has contributed immensely in capturing correlation structures.

4.3.1.1 Elliptical copula

Definition 4.1. *(Elliptical copula, Elouerkhaoui (2017)).*

A d-variate family of copula is referred to as elliptical if it is written in the form

$$C(v_1, v_2, \ldots, v_d; \Sigma) = \Psi_\Sigma^d\big(\Psi^{-1}(v_1), \Psi^{-1}(v_2), \ldots, \Psi^{-1}(v_d)\big)\,,$$

where Ψ_Σ^d is a d-dimensional multivariate ζ distribution with Σ as the correlation matrix; and Ψ^{-1} denotes an inverse of the univariate ζ distribution. The distribution ζ corresponds to either standard normal distribution or the Student-t distribution (t_β), having β degrees of freedom (df). The elliptical copulas are also known as inversion-method copulas.

Gaussian copula: Figure 4.1 depicts 5000 random numbers which were simulated using the Gaussian copula model with 0.8 correlation. For any correlation parameter, the tail dependency is zero (Scherer and Mai 2017, p. 176), as can be observed in the zoomed part of the corresponding Figure 4.1.

By definition, the Gaussian copula is given as

$$C_\rho^\Sigma(\mathbf{v}) = \frac{1}{2\pi\sqrt{|\Sigma|}} \int_{-\infty}^{\Phi^{-1}(v_1)} \int_{-\infty}^{\Phi^{-1}(v_2)} \left(-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2|\Sigma|}\right) \mathrm{d}\mathbf{x}\,,$$

where $\mathbf{v} = (v_1, v_2)$, $\mathbf{x} = (x_1, x_2)$, $\rho \in (-1, 1)$ refers to the correlation parameter in the 2×2

matrix Σ, and then, Φ^{-1} refers to the inverse of a univariate standard normal distribution (Trivedi and Zimmer (2007), O'Kane (2011)).

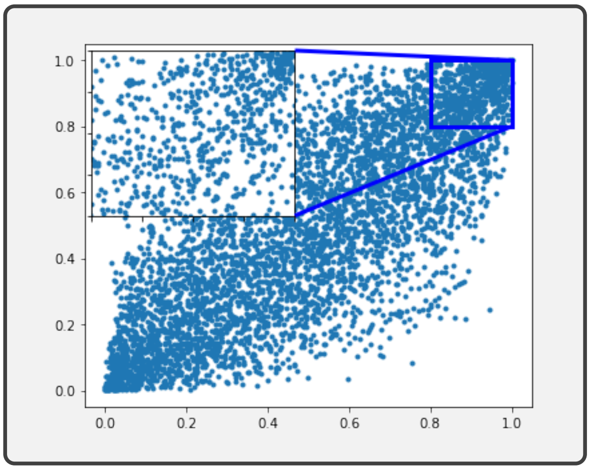


Figure 4.1: Gaussian random variables with tail structure

Student-t copula: Figure 4.2 depicts 5000 random numbers which were simulated using the Student-t copula model with 0.8 correlation and 1 degree of freedom. It equally outputs the structure of the tails at both ends. Thus, the upper and lower tail dependency are symmetric, given by $2t_{\beta+1}\left(-\sqrt{(\beta+1)\left(\frac{1-\rho}{1+\rho}\right)}\right)$, as can be seen in the corresponding figure below.

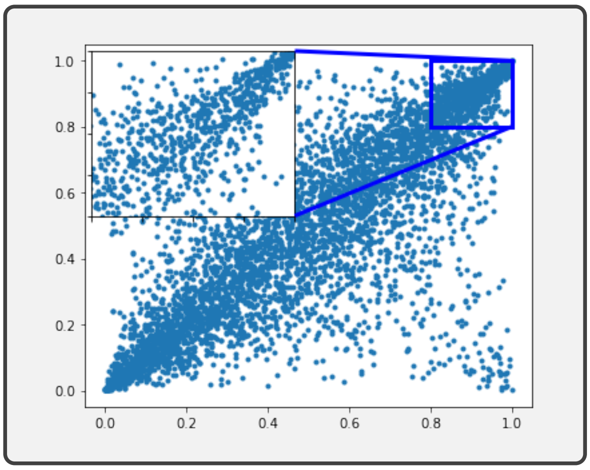


Figure 4.2: Student-t random variables with tail structure

By definition, the student-t copula is given as

$$C^{\Sigma}_{\rho,\beta}(\mathbf{v}) = \frac{1}{2\pi\sqrt{|\Sigma|}} \int_{-\infty}^{t_\beta^{-1}(v_1)} \int_{-\infty}^{t_\beta^{-1}(v_2)} \left(1 + \frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{\beta|\Sigma|}\right)^{-\frac{\beta+2}{2}} \mathrm{d}\mathbf{x}\,,$$

where $\mathbf{v} = (v_1, v_2)$, $\mathbf{x} = (x_1, x_2)$, $\rho \in (-1, 1)$ is the correlation parameter in the 2×2 matrix Σ, and t_β^{-1} is the inverse of a univariate t_β with β df (Trivedi and Zimmer (2007), O'Kane (2011)).

4.3.1.2 Archimedean copula

Definition 4.2. *(Archimedean copula, O'Kane (2011)).*

A d-variate copula is said to be in the Archimedean family if it is written in the form

$$C(u_1, u_2, \cdots, u_d) = \psi^{-1}\big(\psi(u_1) + \psi(u_2) + \cdots + \psi(u_d)\big)\,,$$

having $\psi(x)$ as the generator of the copula function, satisfying the following conditions: $\psi(0) = \infty$ and $\psi(1) = 0$; the inverse function $\psi^{-1}(x)$ corresponds to a probability, such that $\psi^{-1} : [0, \infty) \to [0, 1]$; and the function $\psi(x)$ is convex and decreasing, such that $x \in [0, \infty)$.

Clayton copula: Figure 4.3 depicts 5000 random numbers which were simulated using the Clayton copula model with $\theta = 2.8820$ dependence parameter. Clayton copula is known for its strong dependency at its lower tails, as can be seen in the zoomed part of the corresponding figure. Thus, the upper and the lower tail dependencies are $\lambda_U = 0$ and $\lambda_L = 2^{-\frac{1}{\theta}}$ respectively. The Clayton model is beneficial for modelling data points, which are strongly correlated at the lower values and weakly correlated at the upper values.

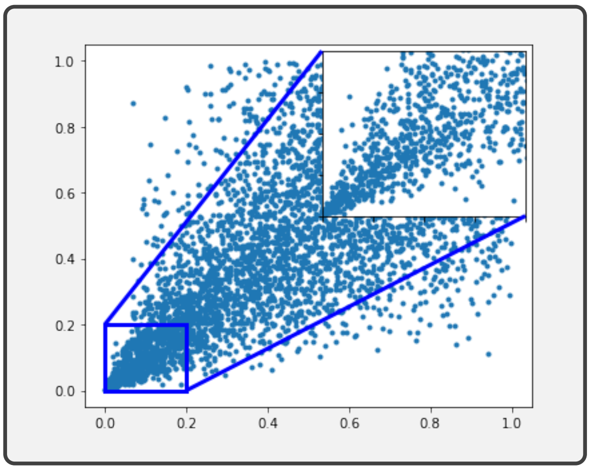


Figure 4.3: Clayton random variables with tail structure

Denote the generator function as $\psi(c) = \theta^{-1}(c^{-\theta} - 1)$, where $\theta \in [-1, \infty) \setminus \{0\}$, then the bivariate Clayton copula is defined as

$$C_\theta(v_1, v_2) = \max\{[v_1^{-\theta} + v_2^{-\theta} - 1]^{-\frac{1}{\theta}}, 0\}\,;$$

and for $\theta > 0$, we have

$$C_\theta(v_1, v_2) = [v_1^{-\theta} + v_2^{-\theta} - 1]^{-\frac{1}{\theta}}\,.$$

Gumbel copula: Figure 4.4 depicts 5000 random numbers which were simulated using the Gumbel copula model with $\theta = 2.4410$ dependence parameter. The figure also shows the strong correlation and dependency evident in the upper tails, which are apparent in the enlarged section of the graph. Thus, the upper tail dependency and lower tail dependency are given as $\lambda_U = 2 - 2^{\frac{1}{\theta}}$ and $\lambda_L = 0$ respectively. The Gumbel model is beneficial for modelling data points, which are strongly correlated at the upper values and weakly correlated at the lower values.

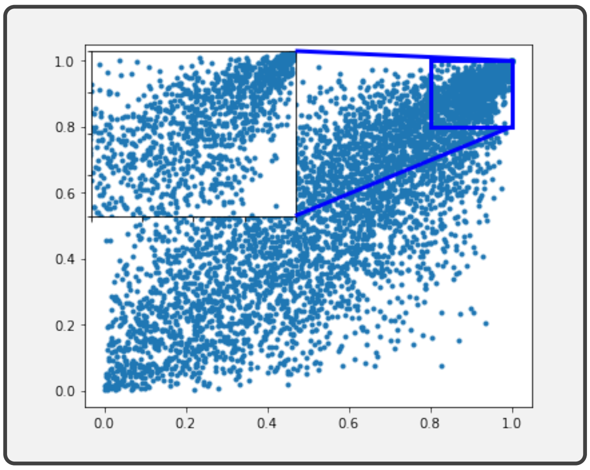


Figure 4.4: Gumbel random variables with tail structure

Denote the generator function as $\psi(c) = (-\log c)^{\theta}$, where $\theta \geq 1$, then the bivariate Gumbel copula is defined as

$$C_{\theta}(v_1, v_2) = \exp\{-[(-\log v_1)^{\theta} + (-\log v_2)^{\theta}]^{\frac{1}{\theta}}\}, \qquad \text{for } \theta > 0.$$

Frank copula: Figure 4.5 depicts 5000 random numbers which were simulated using the Frank copula model with $\theta = 7.6762$ dependence parameter. The figure also shows the structure of the tail dependency of these random variables modelled via the Frank copula, and this is much evident in the zoomed part of the figure. Thus, the upper and lower tail dependencies are zero, and hence, they are the only strict Archimedean copula with the so-called radial symmetry (Embrechts et al. 2001). For further details on Frank copula as radially symmetric we refer the reader to (Genest and Nešlehová 2014).

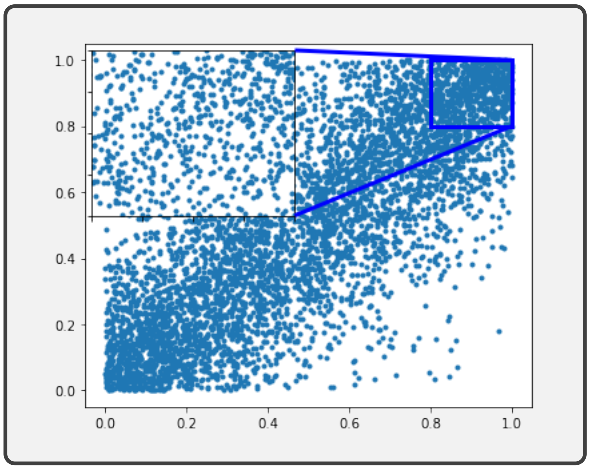


Figure 4.5: Frank random variables with tail structure

Denote the generator function as $\psi(c) = -\log[(e^{-\theta c} - 1)(e^{-\theta} - 1)^{-1}]$, where $\theta \in \mathbb{R} \setminus \{0\}$, then the bivariate Frank copula is given as

$$C_\theta(v_1, v_2) = -\frac{1}{\theta} \log\left(1 + \frac{(e^{-\theta v_1} - 1)(e^{-\theta v_2} - 1)}{(e^{-\theta} - 1)}\right).$$

4.3.1.3 Concordance structures

The measures of dependency, such as the Kendall's τ, the Spearman's rho, and the tail dependence coefficients are all bivariate concordance measures which are copula-based (Aas 2004). We employed the Kendall's τ concordance measure to describe the correlation among the portfolio entities, and this is because they are flexible for non-linear correlations. The Kendall's τ and their corresponding domains are given in Table 4.1, and the parameter ρ for both the Gaussian and the Student-t copulas are the correlation coefficients. The θ parameter in the models influences the dependency structures, as an increase in θ results to the increment of the dependence. Furthermore, it is possible to have the same concordance measure for different copula models, and subsequently different n2D basket prices. In all the numerical results of this study, we used the Kendall's τ concept to estimate the θ-parameters of the Archimedean copulas, which in turn, are employed in valuing the n2D swaps.

Table 4.1: Scherer and Mai (2017), Vose (2008): Concordance structure of different copula models

S/N	Copula type (elliptical)	$\tau = g(\rho)$	Domain $\tau \in \Omega$
1	Gaussian	$\frac{2}{\pi}\sin^{-1}(\rho)$	$[-1, 1]$
2	Student-t	$\frac{2}{\pi}\sin^{-1}(\rho)$	$[-1, 1]$
	Copula type (Archimedean)	$\tau = g(\theta)$	Domain $\tau \in \Omega$
3	Clayton	$\frac{\theta}{\theta+2}$	$(0, 1]$
4	Gumbel	$1 - \frac{1}{\theta}$	$[0, 1]$
5	Frank	$1 - \frac{4}{\theta}\left(1 - \frac{1}{\theta}\int_0^\theta \frac{q}{e^q - 1}dq\right)$	$[-1, 1] \setminus \{0\}$

Table 4.1, via the Kendall's τ can also be used to express the relation between the ρ-parameter

of the elliptical copula and the θ-parameter of the Archimedean copula models. Below, we provide the equivalent expressions:

$$\rho = \sin\left(\frac{\pi\theta}{2\theta+4}\right); \quad \rho = \sin\left(\frac{\pi(\theta-1)}{2\theta}\right); \quad \text{and} \;\; \rho = \sin\left(\frac{\pi}{2} - \frac{2\pi}{\theta^2}\int_0^\theta \frac{q}{\mathrm{e}^q - 1}\mathrm{d}q + \frac{2\pi}{\theta}\right),$$

for the Clayton, Gumbel and the Frank copula respectively. Having the ρ-parameter, one can solve for the parameters of the corresponding Archimedean copula. Thus, for $\rho = 0.35$, we have $\theta = 0.5895$ (Clayton), $\theta = 1.2947$ (Gumbel) and $\theta = 2.1393$ (Frank).

4.4 Pricing Basket Credit Default Swaps

This section will introduce and discuss the concept of the BCDS and default time modelling, as well as the Monte-Carlo methods used in the result section of this study.

4.4.1 Basket Credit Default Swaps

Basket default swap is a financial derivative contract existing between the protection buyer and the seller which promises a payoff in the case a credit default happens among the given portfolio of entities. This contractual agreement ensures the transfer of credit exposure of securities from one person to another. A periodic payment (spread) is made at specified regular intervals to the protection seller till a credit event occurs, or till the contract's expiration. Basket default swaps are generally classified into all-to-default, n2D, and n-out-of-m-to-default. This study will focus on n2D basket swaps. The intuition behind the construction of BCDS can be linked with the approach of redistributing the financial credit risk of a basket of CDS. The mechanism of both CDS and BCDS are essentially the same, but the difference lies in the trigger of the credit event. Consider for example the F2D swaps, a contingent payment made by the protection seller to the buyer, which is triggered only when a basket of underlying credit experiences a first credit event, and in this way, the buyer is only protected against the first default.

In the valuation of the spread for the BCDS, defined in a given homogeneous portfolio, the following notations will be used: The portfolio consists of N number of reference entities, with A as the notional value of the contract. Let the time to maturity of the contract be $T = t_k$, with current time $t_0 = 0$, and define the deterministic discount factor by $f(t) = \mathrm{e}^{-r(t-t_0)}$, with

r as the risk-free interest rate. $R = R_{(j)}$, for $j = 1, 2, \ldots, N$ is the homogenized recovery rate, and the rank of the swap or the seniority level is defined by n, such that upon nth default at time $\tau_{(n)}$ with $\tau_{(1)} < \cdots, < \tau_{(n)} < \cdots < \tau_{(N)}$, a default payment will be received by the protection buyer. Let DL be the default leg, which is the payment made by the protection seller in case of a default. This payment leg is calculated by the difference between the default payment (DP) and the accrued premium AP[1]. The frequency of the payment dates[2], payable in units of years is denoted by $\Delta = t_i - t_{i-1}$, where $i = 1, 2, \ldots, N$, and the time interval in which the nth default happened in the case of AP is given by $S = t_{i-1} < \tau_{(n)} \leq t_i$. Furthermore, let PL be the premium leg which involves the series of cash flows paid by the protection buyer until maturity or before maturity, in the case of a credit event, and finally define $\gamma_{(n)}$ as the fair price of the n2D swap.

Hence, according to Galiani (2003), the fair price under the risk-neutral pricing measure of the n2D swap is calculated by solving for $\gamma_{(n)}$ in the expression $\mathbb{E}[\text{DL}]=\mathbb{E}[\text{PL}]$.

The present value and the expected present value for the PL are given respectively by:

$$\text{PL} = \gamma_{(n)} A \sum_{i=1}^{N} \Delta f(t_i) \mathbf{1}_{\{\tau_{(n)} > t_i\}} \qquad \text{and} \qquad \mathbb{E}[\text{PL}] = \sum_{i=1}^{N} \gamma_{(n)} A \Delta f(t_i)[1 - F_{(n)}(t_i)]\,.$$

The present value and the expected present value for the DP are given respectively by:

$$\text{DP} = A \sum_{j=1}^{N} (1-R) f(\tau_{(n)}) \mathbf{1}_{\{\tau_{(n)} \leq T\}} \qquad \text{and} \qquad \mathbb{E}[\text{DP}] = A \sum_{j=1}^{N} (1-R) \int_0^{\top} f(t) F_{(n)}^{n^{th}=j}\, \mathrm{d}t\,.$$

The present value and the expected present value for the AP are given respectively by:

$$\text{AP} = \sum_{i=1}^{N} A\gamma_{(n)} \left(\tau_{(n)} - t_{i-1}\right) f(\tau_{(n)}) \mathbf{1}_{\{S\}} \quad \text{and} \quad \mathbb{E}[\text{AP}] = \sum_{i=1}^{N} A\gamma_{(n)} \int_{t_{i-1}}^{t_i} (a - t_{i-1}) f(a) F_{(n)}(\mathrm{d}a)\,.$$

Under the assumptions that credit events occur at discrete dates and the outstanding debts resulting from the DP are cleared promptly upon default, then Theorem 4.3 below gives the price of the n2D swap premium:

Theorem 4.3. *(Galiani (2003), Fathi and Nader (2007b), Choe and Jang (2011a)).*

[1]The accrued premium is payable from the last date the payment was made prior to default, till the time $\tau_{(n)}$ of the nth default.

[2]The frequency $\Delta = 1$ corresponds to an annual frequency, $\Delta = \frac{1}{2}$ for semi-annual frequency payment, etc, assuming we are considering a $\frac{30}{360}$ day count convention.

The risk neutral pricing measure for the fair price of the annualized n2D swap, in the presence of accrued premium, is given below:

$$\gamma_{(n)} = \frac{\sum_{j=1}^{N}(1-R)\int_0^{\top} f(t)F_{(n)}^{n^{th}=j}\mathrm{d}t}{\sum_{i=1}^{N}\Delta f(t_i)[1-F_{(n)}(t_i)] + \sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}(a-t_{i-1})f(a)F_{(n)}(\mathrm{d}a)},$$

where $F_{(n)}(t) = \mathbb{P}(\tau_{(n)} \leq t)$ *is the cumulative distribution function (CDF) of* $\tau_{(n)}$*, and* $F_{(n)}^{n^{th}=j}(t)$ *is the CDF of the* n*th basket of default times which is relative to the* j*th default.*

It is, however, pertinent to model the probabilities of these default times, and the next subsection will introduce the various concepts of copula models in the generation of the default times.

4.4.2 Default Time Modelling

In pricing portfolio derivatives, copulas are essential tools in modelling entities defaults, which in turn facilitates the BCDS valuation. Define $(\Omega, \mathcal{F}, \mathbb{P})$ denotes the given probability space, where Ω is the sample space of all feasible events defined in a horizon of finite time, $\mathcal{F}$ is the σ-algebra consisting of a set of events, and $\mathbb{P}$ is the probability measure, which under the no-arbitrage principle, becomes risk-neutral (Bielecki et al. 2007). Assume that τ_i, the default time, is the time at which each entity i experiences a credit event. Let $F_i(t)$ be the risk-neutral cumulative probability distribution that each i entity will default before a given time t, that is, $F_i(t) = \mathbb{P}(\tau_i \leq t)$, and let $S_i(t) = 1 - F_i(t) = \mathbb{P}(\tau_i > t)$ be the survival probability of the ith entity. According to Kijima (2000), the CDF of default times is given by:

$$F_i(t) := \mathbb{P}(\tau_i \leq t) = 1 - \mathrm{e}^{-\int_0^t \lambda_i(u)\mathrm{d}u},$$

where λ_i is the instantaneous intensity rate for each entity. Laurent and Gregory (2005) simulated the default time of each reference entities by using the following expression:

$$\tau_i = -\frac{1}{\lambda_i}\ln(1-X_i), \qquad (4.4.1)$$

where X_i arises from the corresponding copula model. We refer to Appendix A for the series of algorithm employed in modelling the default time. First, Li (2000) considered the case of the Gaussian copula, and he noted that the survival probability or the default probability can be employed to estimate the default times, provided that the copulas are symmetric

(Gaussian and student t). Furthermore, the default times for the **Gaussian copula model** are calculated as follows:

$$\tau_i = F_i^{-1}\big(\Phi(s_i)\big)\,,$$

where Φ is defined as the cumulative distribution function of a standard Gaussian random variable s_i, which in turn, is given by $s_i = \rho_i V + \sqrt{1-\rho_i^2}\eta_i$. The correlation term $\rho_i \in [-1, 1]$, and the independent random variables V and η_i are the single common factor and the error term respectively, which are obtained from the standard Gaussian distribution.

The corresponding default time for the **student-t copula model**, alternatively, follows a dual-factor copula model, and it is given by (Bastide et al. 2008)

$$\tau_i = F_i^{-1}\big(t_\beta(s_i)\big)\,,$$

where t_β is the CDF of a student random variables s_i, which in turn, is given by $s_i = \sqrt{V_2}(\rho_i V_1 + \sqrt{1-\rho_i^2}\eta_i)$, having β degrees of freedom. The factors V_1 and V_2 are independent random variables, together with the error term η_i. The variable V_1 follows a normal distribution and V_2 follows an inverse gamma distribution, with equal scale and shape parameters of $\beta/2$.

The default times modelled under the Archimedean copulas are quite different from those modelled under the elliptical copulas. The general form for obtaining their corresponding random variables X_i, is shown in the works of (Bluhm and Overbeck (2006), Hyrš and Schwarz (2015), Melchiori (2006)) as:

$$X_i = \psi^{-1}\Big(\frac{-\ln(k_i)}{M}\Big)\,, \tag{4.4.2}$$

where the uniform random variables k_i are independent and identically distributed, i.e. $k_i \sim \mathcal{U}([0,1])$, and the function $\psi^{-1}(.)$ refers to the inverse generator associated with each copula models. The random variable M is dependent on the copula model being considered. For instance, M is simulated from the gamma distribution if we are considering the Clayton copula model. Also, the variable M is derived from logarithmic series distribution and stable distribution for the Frank and the Gumbel copula respectively.

Consider the **Clayton copula model**, with $\theta > 0$ parameter, as example, the random variable X_i returns a uniformly distributed random variables. The positive random variable M is obtained from the standard gamma distribution having a unit scale parameter and $1/\theta$ shape parameter. In other words, $M \sim \Gamma(1/\theta, 1)$, and finally, the inverse generator $\psi^{-1}(c)$

for the Clayton model is calculated as

$$\psi^{-1}(c) = (1+c)^{-1/\theta}, \qquad \theta > 0\,.$$

For the **Frank copula model**, with parameter θ, the random variable M follows a Logarithmic series on $\mathbb{N}^+$ having a parametric value of $\alpha = 1 - \mathrm{e}^{-\theta} \in (0,1)$. In other words, $M \sim \text{Logarithmic}(1-\mathrm{e}^{-\theta})$. The corresponding PDF can be defined discretely as

$$\mathbb{P}(q) = \frac{-\alpha^q}{q\ln(1-\alpha)}, \qquad \text{for } q \in \mathbb{N}\,.$$

Furthermore, the inverse generator $\psi^{-1}(c)$ for the Frank model is calculated as

$$\psi^{-1}(c) = -\frac{1}{\theta}\log\big(1+\mathrm{e}^{-c}(\mathrm{e}^{-\theta}-1)\big), \qquad \theta \in \mathbb{R}\setminus\{0\}\,.$$

On the other hand, the random variable M for the **Gumbel copula model** follows an $\alpha -$ stable distribution, that is, $M \sim \text{Stable}(\alpha,\beta,\gamma,\delta)$. The distribution function is also referred to as the Lévy alpha-stable distribution, with parameters $\alpha \in (0,2]$, as the stability parameter; the skewness parameter $\beta \in [-1,1]$; the scale parameter $\gamma \in (0,\infty)$; and finally, $\delta \in \mathbb{R}$, as the location parameter. Specifically, the random variables follow

$$M \sim \text{Stable}\bigg(\frac{1}{\theta}, 1, \cos\left(\frac{\pi}{2\theta}\right)^{\theta}, 0\bigg)\,.$$

The inverse generator $\psi^{-1}(c)$ for the Gumbel model is calculated as

$$\psi^{-1}(c) = \mathrm{e}^{-c^{\frac{1}{\theta}}}, \qquad \theta \geq 1\,.$$

Generally, this study employed the MCS method in the generation of the random numbers and the default times, which correspond to different copula models. The present value of an n2D swap depends on the time in which the nth entity defaults, and so, obtaining the nth default time suffices to obtain the default times of all the entities in a given portfolio. We implemented the concept of order statistic (See Definition 4.4) to the unordered default times of the individual entities. The implementation was achieved by categorising the B-dimensional vector of the default times in their increasing sequence of order, and this sorting is in connection to their rank of default. For instance, for the F2D swap pricing, we selected the first coordinate. Using this ordered statistic, the corresponding values for the PL, DP, AP, and finally, the fair spreads of the n2D were all computed.

Definition 4.4. *(nth order statistic, Choe and Jang (2011a)).*

Denote $\tau_1, \tau_2, \ldots, \tau_n$ as independent and identically distributed (i.i.d.) random variables of sample size n from a continuous distribution. If $\tau_1, \tau_2, \ldots, \tau_n$ are rearranged in ascending order of magnitude, with the minimum $\tau_1 = \min_i(\tau_i)$ and the maximum $\tau_n = \max_i(\tau_i)$, then the nth order statistic is given as $\tau_1 < \tau_2 < \cdots < \tau_N$.

4.5 Numerical Experiments and Sensitivity Analysis

In this section, we explain the results obtained in the course of the numerical simulations, as well as some sensitivity studies on the research.

4.5.1 Results

This chapter used the `ipython notebook` in all the numerical computations related to the prices of the n2D swaps, as well as, with respect to their varied parameters. The full code can be assessed at https://github.com/NnekaU/Codes/blob/master/BDS%20pricing%20using%20Elliptical%20and%20Archimedean%20copulas.ipynb.

Let N be the number of entities; R, the recovery rate of entities; r, the deterministic risk-free interest rate; dt, the frequency payment dates; df, the degree of freedom; M, the Monte-Carlo simulation points; mm, the number of sub-time steps within each dt for integration; ρ, the correlation coefficient; θ_C, the theta parameter for the Clayton model; θ_F, the theta parameter for the Frank model; θ_G, the theta parameter for the Gumbel model; and λ, the intensity rate. For more consistent result, we used Kendall's $\boldsymbol{\tau}$ to get a rho-theta equivalent in terms of the model parameters.

We first generate 50000 data points of default times which correspond to the applicable models used in this work, and the results were obtained using the algorithms in Appendix A. We consider the other parameters: $\lambda = 0.04$, df $= 1$, $\rho = 0.7$, $\theta_C = 1.9497$, $\theta_F = 5.6212$, and $\theta_G = 1.9749$. Displayed in Table 4.2 are the outputs of the first 5 simulations, for the first-to-default BCDS consisting of 10 reference entities in each models. The results of the simulated default times for each of the five models are in the form of an (M, N) matrix.

Table 4.2: Simulation of default times corresponding from the copula models

		[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]	[,9]	[,10]
Gaussian	[1,]	23.10	7.96	16.34	9.92	4.63	21.21	16.32	33.61	3.96	17.25
	[2,]	31.92	16.16	11.03	27.22	21.47	24.88	26.95	42.79	17.74	79.85
	[3,]	74.10	25.75	8.49	12.13	14.82	33.24	15.22	9.80	20.46	22.05
	[4,]	85.77	56.25	46.85	48.93	101.04	32.29	33.28	89.99	56.65	36.64
	[5,]	38.37	41.06	95.49	38.03	82.34	28.16	53.00	72.06	23.36	95.13
		[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]	[,9]	[,10]
Student-t	[1,]	8.34	3.33	2.89	16.02	4.96	4.03	0.81	2.24	0.60	4.28
	[2,]	18.47	5.26	16.21	14.81	10.45	5.73	0.60	35.43	1.32	8.89
	[3,]	16.05	26.23	34.99	40.35	32.36	31.75	83.85	26.34	75.26	35.62
	[4,]	42.73	44.37	50.13	42.55	35.64	40.09	50.97	66.74	101.81	15.58
	[5,]	9.48	5.35	5.61	5.02	6.75	6.68	1.19	3.18	0.37	4.04
		[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]	[,9]	[,10]
Clayton	[1,]	15.85	11.48	18.98	14.65	6.37	16.47	12.44	43.91	18.75	22.73
	[2,]	86.01	40.70	64.92	48.02	118.37	35.74	80.50	80.37	60.87	49.09
	[3,]	6.83	0.64	6.32	0.95	1.26	2.97	2.76	1.58	3.34	0.89
	[4,]	0.23	1.66	0.36	1.64	0.55	0.80	1.79	0.88	0.55	2.59
	[5,]	4.33	6.77	12.43	23.87	4.41	5.49	3.42	6.85	6.74	6.59
		[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]	[,9]	[,10]
Gumbel	[1,]	55.87	39.76	30.02	53.88	54.85	48.49	39.11	50.29	41.25	41.15
	[2,]	23.50	9.18	34.41	4.43	11.62	12.57	4.26	22.16	16.55	8.53
	[3,]	8.86	6.01	12.80	7.53	1.32	9.68	1.66	27.56	10.66	12.50
	[4,]	15.04	17.19	6.18	17.29	9.30	11.66	11.12	8.93	26.62	16.78
	[5,]	26.09	29.51	44.83	38.42	28.11	65.95	66.55	45.22	39.59	54.65
		[,1]	[,2]	[,3]	[,4]	[,5]	[,6]	[,7]	[,8]	[,9]	[,10]
Frank	[1,]	1.54	15.53	21.24	1.94	0.82	9.36	1.61	1.79	2.31	0.89
	[2,]	16.39	26.54	14.63	2.54	11.37	13.60	7.04	14.84	5.25	18.51
	[3,]	7.75	4.81	5.93	30.32	7.59	9.21	30.80	9.25	14.21	16.49
	[4,]	37.90	66.18	8.12	10.91	10.48	23.09	11.02	20.73	11.02	8.62
	[5,]	14.92	21.95	12.56	32.10	20.22	22.75	19.91	66.28	9.97	25.32

One of the major objectives achieved in this study is to model default times using different copula models, and Table 4.2 above shows the discrepancies in the results obtained. Default times are first estimated, and then incorporated in the valuation of the n2D swaps.

The parameters in Table 4.3 below will be used to output the values of the premium legs, survival probabilities, as well as the actual premiums for different ranks of n2D swaps. For the recovery rate, we use the market standard value of 40%, the other parameters are purely by assumption, and the dependence parameters were chosen to harmonise with the Kendall's τ[3].

Table 4.3: Parameters for the valuation of n2D swap using the same Kendall's τ

N	T	R	r	dt	df	M	mm	λ	θ_C	θ_G	θ_F	ρ
10	5	40%	5%	0.5	4	50000	10	5%	0.5895	1.2947	2.1393	0.35

Table 4.4: Premium legs (PL) & Survival probabilities (SP) for n2D swap using different copulas

Rank	Gaussian		Student-t		Clayton		Gumbel		Frank	
	SP	PL	SP	PL	SP	PL	SP	PL	SP	PL
1	0.2698	2.1940	0.2653	2.1177	0.3575	2.6446	0.2884	2.1666	0.3422	2.2992
2	0.4792	3.0821	0.4887	3.0853	0.5307	3.2160	0.4479	2.9724	0.4733	2.9655
3	0.6361	3.5792	0.5843	3.4343	0.6551	3.5660	0.5897	3.5058	0.5858	3.4620
4	0.7548	3.8857	0.7501	3.8681	0.7434	3.7953	0.7280	3.8888	0.6985	3.8315
5	0.8445	4.0908	0.8240	4.0614	0.8127	3.9614	0.8453	4.1362	0.8177	4.1064
6	0.9026	4.2069	0.8943	4.2083	0.8707	4.0920	0.9253	4.2714	0.9088	4.2614
7	0.9446	4.2834	0.9481	4.2983	0.9114	4.1807	0.9726	4.3369	0.9674	4.3348
8	0.9715	4.3287	0.9791	4.3441	0.9471	4.2588	0.9923	4.3613	0.9925	4.3626
9	0.9888	4.3527	0.9946	4.3635	0.9737	4.3147	0.9981	4.3670	0.9987	4.3679
10	0.9968	4.3646	0.9978	4.3669	0.9905	4.3501	0.9998	4.3688	0.9999	4.3689

Table 4.4 shows the increasing sequence in terms of the ranks of the n2D swaps. Here, we used five copula models to output the premium legs and the survival probabilities of each of the n2D swaps. Notwithstanding the distinct values that all the models' outputs, they generally exhibit similar characteristics with regards to the observed values. With an increase

[3]The same is applicable to Table 4.6

in the ranks, the survival probabilities become negatively correlated with the premium legs. The survival probability is the likelihood of each entity or entities in a portfolio to survive a credit event before a given time t. Considering the portfolio of 10 entities, for instance, the probability of having one entity out of the whole portfolio to default is highly significant, and this, in turn, implies that the survival probability of other non-defaulting entities becomes very slim. Thus, as the rank increases, there is a higher chance for a few of the entities to survive, and hence, the protection buyer will be entitled to pay higher premiums. Finally, we output the n2D swap premiums using the parameters in Table 4.3, and the results will be displayed in Table 4.5 below:

Table 4.5: Premiums for n2D swap using different copula models

Rank	Gaussian copula	Student-t copula	Clayton copula	Gumbel copula	Frank copula
1	1649.8	1686.7	1251.1	1623.6	1417.0
2	885.5	866.2	762.9	977.9	944.6
3	532.2	638.2	507.5	611.8	630.0
4	327.8	336.5	354.4	359.6	405.2
5	195.6	222.7	247.8	189.4	224.0
6	118.4	123.9	165.7	87.5	106.3
7	65.7	60.7	111.0	31.4	37.0
8	33.1	23.9	65.1	8.7	8.4
9	13.0	6.1	31.9	2.1	1.5
10	3.7	2.4	11.4	0.2	0.0

In Table 4.5, we observe a sharp decline in the premiums of the n2D swaps using different copula models, as the rank of the n2D swaps increases. For the F2D swap price, the Clayton copula model outputs a significantly least value in comparison with other models. However, the Clayton values far exceed the others when the rank moves from fifth till tenth-to-default. In Burtschell et al. (2005), it can still be argued that for a specific value of θ, the outputs from the Clayton copula model are almost similar to the values of the Gaussian copula, but the reverse is the case when the Kendall's τ equivalent is being used. The degree of freedom (df) used in the student-t copula significantly affects the swap prices, but when there is an increase of the df, the student-t copula values tend to the Gaussian copula values. Consider for instance, when the df increased from 4 to 15, the student-t n2D swap premiums became 1650.7, 872.3, 534.9, 322.7, 191.5, 117.1, 66.3, 33.8, 15.1 and 3.7, as the rank equally increased.

Furthermore, we observe from Table 4.5, that the choice of the copula model has a considerable effect on the value of the n2D swap. The swap prices differ using different copula models, even if the modelling assumes the same concordance structure. We note that in our case, we used the same Kendall's τ equivalent. The θ parameters of the Archimedean copula are equivalent to the ρ value in the elliptical copula. Thus, in connection to this, Schröter and Heider (2013b) asserted that the choice of the copula function should be considered as part of the modelling since the function plays a critical role in the risk profile of the BDS, and they should be chosen wisely in connection to the regulatory requirements.

4.5.2 Sensitivity Studies

Sensitivity analysis plays a huge role in any model of derivative pricing, as they specify which trading strategy hedges the security effectively. Several model factors affect the riskiness of the portfolio and consider the F2D as an example. However, we note that the following analysis depends on the rank to default in the portfolio. That is, whether it is F2D, second-to-default (S2D), etc. The model factors include: concordance structure (ρ and θ) - the higher the concordance rate, the lower the probability of default; hazard rates - the lesser the rates, the smaller the likelihood of default; time to maturity - the longer the maturity time, the lesser the probability of default; recovery rate - higher rate of recovery results to lower default probability; and number of entities - the bigger the basket, the more the chances of default. In this work, we shall numerically consider the variations of the F2D till tenth-to-default swap premiums against their respective concordance structures, default rates and the times to expiration. Table 4.6 lists the parameters used in the results:

Table 4.6: Parameters for the sensitivity analysis of the n2D swap using same Kendall's τ

N	T	R	r	dt	df	M	mm	λ	θ_C	θ_F	θ_G	ρ
10	5	45%	3%	0.25	7	50000	10	3%	1.0000	3.3057	1.5000	0.50

The graphical description of the sensitivity analysis are depicted in Figures 4.6 and 4.7. Rows I and II of Figure 4.6 give results obtained using the Gaussian and the Student-t copula models respectively. Columns I, II and III respectively show different values of the n2D swaps against ρ, λ and T. Similarly, Rows I, II and III of Figure 4.7 are results obtained using the Clayton, Gumbel and Frank copulas respectively. The corresponding columns I, II and III show different swap values as plotted against θ, λ and T. The analysis are conducted with different default times arising from the corresponding models been used, and the explanations follow subsequently in Sections 4.5.2.1, 4.5.2.2, and 4.5.2.3.

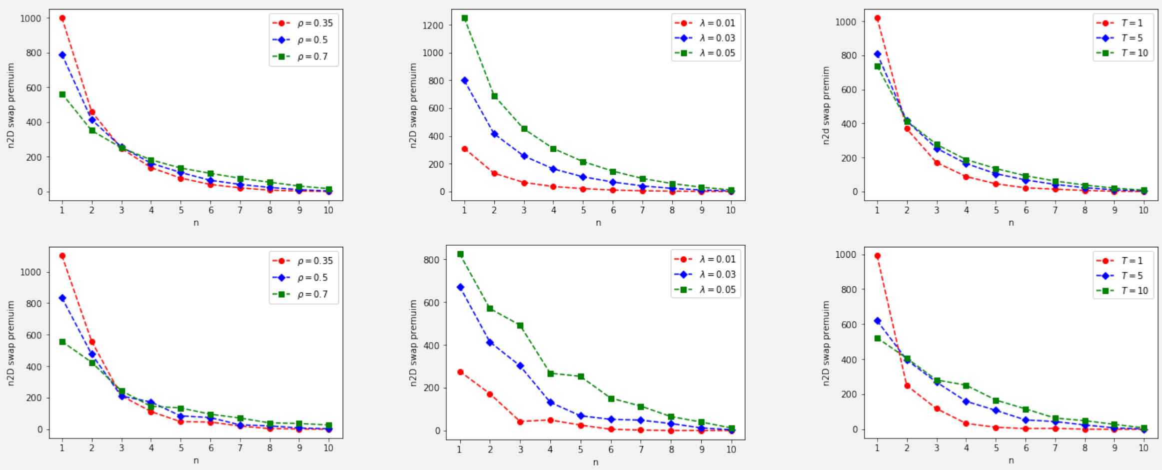


Figure 4.6: Elliptical copula: n2D swap prices versus the parameters ρ, λ and T respectively.

Generally, from Figure 4.6, the Student-t copula displayed fairly different results when compared with the Gaussian copula, and the equivalence of both results is dependent on the degree of freedom (df) value chosen for the Student-t copula. It is further observed that some form of non-monotonicity was seen in the case of the Student-t copula results, especially for higher-order basket form (where $n \geq 3$, with n been the rank of default protection). This behaviour is attributed to the fact that the Student-t takes into account extreme joint events, such that, when the dependence in the tail is increased, then the joint default probability will be simultaneously increased. For instance, when the df is increased from 7 to 20, we observed that the non-monotonic property of the Student-t results varnished, as the results became as close to that of the Gaussian copula as possible. Thus, the df plays a pivotal role in the monotone/non-monotone characteristics displayed by the results of the Student-t copula. Hence, this extra parameter, df, is crucial in modelling and in describing the extreme co-movements associated with financial variables in the valuation of default-sensitive securities.

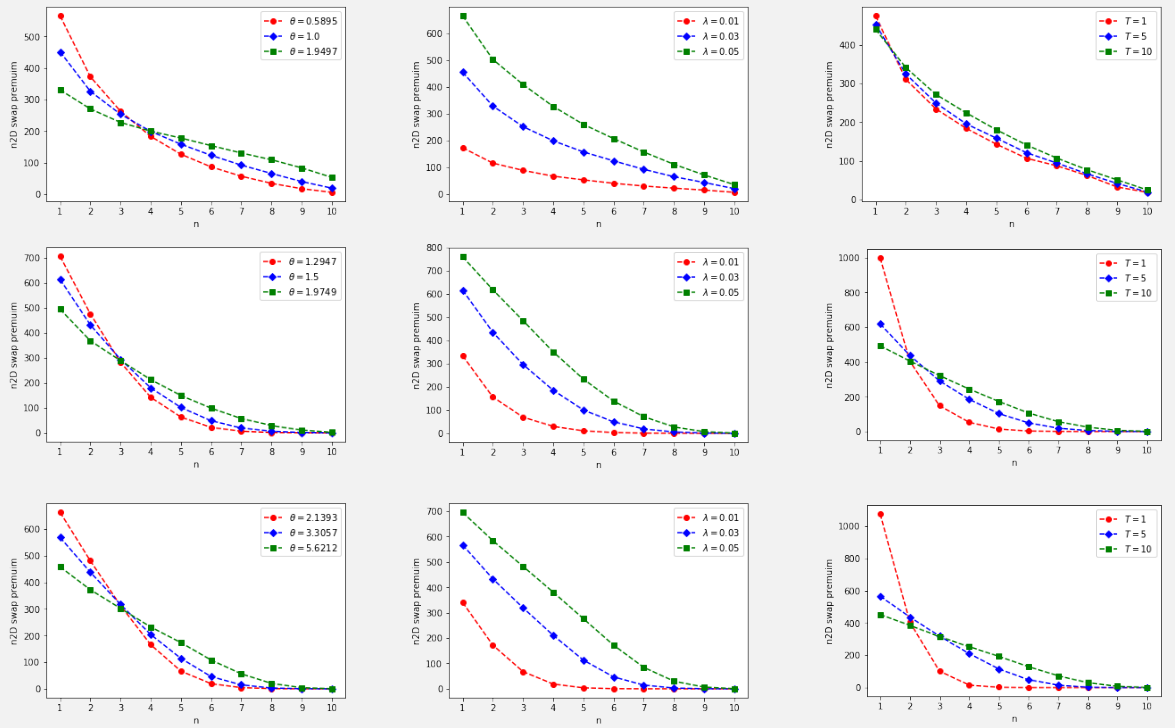


Figure 4.7: Archimedean copula: n2D swap prices versus the parameters θ, λ and T respectively.

From Figure 4.7, under the Archimedean family, it was observed that the Frank and the Gumbel copula displayed fairly similar results, with the Clayton slightly different. This behaviour could be as a result of the non-parametric dependence measure, such as the Kendall's τ, which was used throughout the research to ensure uniformity of results. Thus, different Kendall's τ leading up to different θ parameters could result in the n2D premium similarities amongst both families of copula models. For instance, Gaussian copula and Clayton copula gave rise to similar premium values under different Kendall's τ (See Burtschell, Gregory and Laurent (2005)).

4.5.2.1 Sensitivity Analysis (w.r.t. concordance parameters)

The concordance structure is one of the essential parameters which affect the sensitivity of the n2D swap prices. For the correlation factor, we observe that higher correlation results in decline in the swap spread, and this is evident for the F2D and the S2D spread. It could be said that if an investor takes a position of a short protection, then it will be a long correlation; which obviously implies that the investor will stand to gain from an equivalent increase in the probability of no default. Furthermore, as the rank of the default increases (as from the third-to-default (T2D)), we observe that an increase in the correlation factor subsequently led to an increase in the swap premium.

The θ-parameters of the Archimedean copulas measure the tail dependency, as both the parameters and the tail dependencies are positively correlated. When the default time is modelled using the Clayton copula, we observe that as from the fourth-to-default (Fo2D) upwards, the swap prices are increased owing to a consistent rise in the θ-parameters. Similar results are seen from the T2D swap spreads when the default times are modelled using both Frank and Gumbel copulas. However, as we increase the rank further on, the swap spreads coincide at the tenth-to-default, since we considered a portfolio of 10 entities. Hence, increasing the dependency structures in the tail of the distribution, that is θ, leads to a remarkable increase in the probability of joint defaults, and in turn, the default correlation.

In general, the concordance structures affect the tail dependencies of the models, and hence, investors seeking to gain from this derivative security must choose the basket swap type and the position strategically. For instance, the correlation sensitivity has resulted in speculating the direction of the correlation in the credit derivatives market. If the investor perceives that the default correlation of entities in a portfolio will rise, then the investor can sell F2D default protection with the sole aim of making a profit should default correlation eventually increase.

4.5.2.2 Sensitivity Analysis (w.r.t. hazard rate)

We observed similar trends in all the models, as default time values are estimated using various copula models. A steep decline in the rank of default is observed with respect to the n2D swap prices, and the swap prices are seen to coincide if the rank is increased further to 10. There is a consistent increase in the swap spread as the default intensities of all the copula models increase. Thus, this is intuitively right because the default intensities measure the loss expectation for a greater specified dependency structure, and this requires a higher premium (Jabbour et al. 2009). Furthermore, as the hazard rate increases, the chances of the entities to default become more likely, and this accounts to a more substantial premium. The behaviour of the hazard rate and the correlation coefficient are in sharp contrast because increasing both parameters result to a monotone decrease and monotone increase of the swap spreads respectively when the F2D swaps are considered.

In general, when the default intensity is high, the likelihood of joint defaults become exceedingly vast, resulting in larger spreads. On the other hand, at low default intensity or as the limit of default intensity tends to zero, the dependency structure becomes irrelevant, the spreads gradually decline, and no default will be experienced in the given portfolio.

4.5.2.3 Sensitivity Analysis (w.r.t. time to expiration)

We finally showed the dependence of the n2D swap spread on time to maturity of the basket. While the time to expiration of the contract increases, we observe a gradual reduction in the quarter-annual swap premium payments. This observation is evident for the F2D basket swap only, and the reverse becomes the case when the seniority of the default payment increases. Thus, for each of the models, the payments made by the protection seller upon default and the regular premium payments by the protection buyer are seen to increase simultaneously, with an increase in the contract's expiration time. This increment, in turn, lowers the cumulative swap value.

From the S2D swap and above, there will be more likelihood of joint defaults by the entities in the portfolio, provided the time horizon is extended further. The results obtained from the copula models equally showed some discrepancies, especially with regards to the swap prices when the rank of default is between 2 and 6. For a year n2D swap, the student-t copula, the Gumbel copula and the Frank copula exhibited a sharp decline in the swap premium prices, whereas, the Gaussian and the Clayton copula showed a steady decrease in the swap prices.

4.5.3 Choice of Copula Model

Despite the significant benefits of using copula models in the dependent default modelling, it, however, remains challenging to propose a suitable copula for a specific problem, and this is because of how the weights are allocated amongst the dependency structures. The selection of the best bit copula model can be made descriptively via the scatter plot diagrams of their dataset, as their unique shapes can be attributed to each of the copula functions. For instance, with regards to their tail structure, some copulas focus more on lower tails (Clayton), others on upper tails (Gumbel), others with zero tail dependency (Gaussian and Frank), and finally, others on both the upper and lower tails (student-t). For significant losses, in which high correlation exists in the right tail of the distribution, the extreme copula such as the Gumbel copula will be the best fit for the default time modelling. For negative dependence of the variables in the dataset, the Gaussian copula will be applicable. There does not exist any strong relationship between large losses in both the Gaussian copula and the Frank copula, even though the variable relationship in the Gaussian copula is slightly stronger than the Frank copula. The Clayton, on the other hand, depicts large concentration on its lower tails, and thus, is suitable for strongly correlated variables which exist there. Hence, the appropriate choice of the copula model can be made by examining the distribution functions of the associated dataset.

Furthermore, some statistical models rooted in information theory, such as the Akaike information criterion (AIC), and the Bayesian information criterion (BIC) have been developed to aid in the practical choice of the copula model (Grønneberg and Hjort 2014). These statistical procedures aim at employing the concept of maximum likelihood, or even the pseudo-likelihood estimation methods to fit individual copula models to a given data. Fang et al. (2014) equally performed some simulation tests to ensure that the AIC method was far more effective than the goodness-of-fit method, which seems to be the most common procedures. Other selection criteria for the copula models can be found in the works of Charpentier (2003), Durrleman et al. (2000), and Venter (2002).

From our study, however, we analyzed the choice of copula models from the computational time taken to output the results. Table 4.7 below shows the discrepancies:

Table 4.7: Computational time on different copula models (seconds)

Entities	Elliptical copulas		Archimedean copulas		
	Gaussian	Student-t	Clayton	Gumbel	Frank
1	0.3386	0.5288	3.4777	3.2934	6.3981
2	0.5954	0.9929	3.7357	3.6405	6.9158
3	0.7227	1.3367	3.8918	3.8392	6.9292
4	0.8355	1.6494	4.1035	3.9364	7.1784
5	1.0318	2.1319	4.1858	4.0372	7.2623
6	1.1885	2.3389	4.4469	4.2474	7.4078
7	1.2709	2.6831	4.5217	4.3804	7.5739
8	1.5100	2.8710	4.6123	4.4396	7.6673
9	1.6923	3.4578	4.6868	4.6347	7.7088
10	1.7782	3.6804	4.8522	4.7212	7.7412

In Table 4.7, we compared the effectiveness and the efficiency of the copula models by investigating the CPU time taken to output the final break-even swap spreads. The computer that was used to compute and analyze the results comprises of Windows 7 operating system and Intel(R) Core(TM)i5-3210M CPU @2.50GHz processor. Here, we used the same parameters found in Table 4.3 for the $F2D$ swap in different portfolio sizes, that is, N, and observed that the computation time for the swap spreads using all the copula models increased with an increase in the basket size. Furthermore, we classified the performances of the copula models based on their type, that is, whether elliptical or Archimedean, and this classification was as a result of how they are being generated. From our results in BCDS valuation, we found out that the student-t copula is much more computationally expensive as compared to the Gaussian copula, and as such, Gaussian copula is recommended in the class of these copula models. For the Archimedean class, the discrepancies of the CPU time between the Clayton copula and the Gumbel copula are not much. However, the Gumbel copula performs best in comparison to Frank and Clayton. The Frank copula was least recommended because of the significant computation time, and this could be as a result of the concept of its radial symmetry and non-bounded θ-dependence structure.

4.6 Conclusion

In this chapter, we considered a comprehensive analysis of the effect of valuing portfolio credit derivatives, primarily the $n2D$ basket swaps, using both elliptical and Archimedean copula models. We employed the Monte-Carlo simulation in connection to the swap premium prices using the Gaussian, student-t, Clayton, Gumbel, and the Frank copula models to estimate the default times first. Most credit default events are captured as tail events by nature; we further compared the results obtained using both non-tail distributions (Gaussian and Frank) and fat tail distributions (Gumbel, Clayton, and student-t). The study summed up analysing the inherent model risks, and this was achieved graphically by conducting some sensitivity analysis, with regards to the concordance structures, hazard rates and time to expiration of the contract. From the analysis of the copula models via their computation times, we recommend the Gumbel copula under the Archimedean class, as well as the Gaussian copula under the elliptical class in the pricing of $n2D$ basket swaps. Thus, having an in-depth knowledge of these studies can assist investors and financial participants in hedging and speculating the direction of the market.

Furthermore, to ensure efficient comparative statics, we introduced Kendall's τ concordance measures to model the dependency structures of the underlying entities, even though the corresponding results produced different swap values for the models used. The concordance measures of the portfolio should be utilized to determine the risk associated with the choice of a specific copula, as this ensures the dependency or the joint dependencies associated with the portfolio. To conclude, we can infer that the choice of copula used to model default time contributes immensely in the riskiness of a credit portfolio of multiple entities.

In terms of the theoretical implications, this research broadens the overview of copula modelling in connection to the pricing of BCDS premiums existing in the financial risk analysis sectors. The practical significance of this study lies in comparative statics on the estimation of n2D swap prices, using various copula models. The incorporation of default correlations of entities is beneficial for the financial institutions and enterprises to monitor and control their exposure to default risk arising from the trading of BCDS.

5. Pricing Basket Default Swaps Using Quasi-Analytic Techniques

This chapter has been submitted to *Decisions in Economics and Finance*, with Manuscript ID: DEAF-D-20-00080) as

Umeorah N., Mashele P. & Ehrhardt M. (2020). Pricing Basket Default Swaps Using Quasi-Analytic Techniques.

Abstract

This chapter is based on the concept of the one-factor copula model, which is applied to reduce the dimensionality problems associated with the basket default swap pricing. We employ the Gaussian, the student-t and the Clayton one-factor copula to estimate the conditional probability of default. Incorporating the Fourier transform together with the distribution function of a counting process, we derive the quasi-analytical expression for the computation of the swap payment legs. We compute the conditional characteristic function for the corresponding portfolio loss distribution using the Fast Fourier transform. Then, employ numerical integration with the aid of the Inverse Fast Fourier transform to retrieve the distribution function or the unconditional characteristic function. Our results show that in the absence of the trending simulation method, a semi-analytic method which involves the applications of the discrete Fourier transform can be utilized to price the basket credit default swaps.

Keywords

Discrete Fourier Transform, Fast Fourier Transform, Copulas, Convolution, Basket default swaps, Characteristics Function, Probability Distributions.

5.1 Introduction

Originally, the credit default swaps (CDS) market was created to equip financial institutions, such as banks, with the avenue of allocating, reducing, diversifying their credit risk exposure beyond customer base, as well as freeing up regulatory capital efficiently and effectively. Nowadays, CDS trading has become an indispensable tool which propels the market of credit derivatives, and as such, its market has been described as one of the most tremendous innovations in the financial markets over the past two decades. The banks, with their tremendous trading activities, are seen to be the largest dealer in the market of credit derivatives. They provide liquidity by their willingness to assume the risk responsibility on their trading booklets, which they, in turn, seek to hedge. The CDS have been existing as early as the 1990s, and its usage climaxed in the early 2000s prior to the 2007-2008 financial crises. Despite the huge difference between its gross notional amount mostly before the crises, the CDS trade continues to assume a crucial avenue and platform for trading credit risk. CDS are contracts that allow the seller to make a compensation payment to the buyer in the event of a default or any other stipulated credit events in the contract such as bankruptcy, payment failure, restructuring, obligation default, repudiation or obligation acceleration.

Rule (2001) defined a CDS as "'state-dependent' but 'outcome-independent'" contract, implying that CDS fully depends on whether or not credit events have occurred and not necessarily on whether the buyer suffers loss or not. The common reference entities which exist in a CDS market ranges from single firms to a portfolio of firms, or even to a large collection of firms which comprises an index. The CDS focus on a single entity's credit risk, and basket default swaps, on the other hand, are bilateral financial contracts that payoff upon default or multiple defaults amongst a portfolio of entities. These multi-name derivatives contracts are advantageous because they offer the investors some appealing opportunities of leveraging the spread premium, and of using one single contract to hedge a portfolio of contingent claims, such as bonds or loans. Thus, the need for obtaining a series of contracts for single securities has been averted, and also, the improvement of the relative risk-return account as compared to other equivalent credit investment tools will be made viable. They are aimed at transferring credit risk of multiple reference entities and are generally classified into first-to-default ($F2\text{D}$), $n2\text{D}$, n-out-of-m-to-default and finally, all-to-default.

Generally, valuing the nth-to-default ($n2\text{D}$) basket swaps involves computing the probability of default associated with the default times. This approach can be done in two folds: First, the Monte-Carlo simulations of random scenarios from the corresponding joint probability distribution functions. Our recent paper focused on this concept, and we were able to model the default times using copula models such as Clayton, Gumbel, Frank, Gaussian and the student-t copulas, in connection with the Monte-Carlo simulation techniques (Umeorah, Mashele and Ehrhardt 2019). These simulated values were in turn used in the valuation of the n2D basket swaps. The next fold is the quasi-analytic techniques, which is the main highlight of this article. As an objective of this research, we will use the concept of the discrete Fourier transform (DFT) to obtain the probability distribution of default times analytically. Thus, we will utilize the one-factor model, in conjunction with the concept of DFT to obtain semi-analytic computation, and in turn, directly apply to value the swaps. Most of the research works which deal with semi-tractable methods of pricing PCDs have been channelled to the CDO tranches and fewer attentions for the n2D basket swaps (Laurent and Gregory 2005). Thus, the contributions presented in this chapter are as follows:

- Empirical and statistical analysis of CDS spread data of some selected highly rated firms.
- Estimating the dependent defaults without simulations and pricing BCDS under the Gaussian and Clayton copula via the FFT and DFT.

The structure of this chapter is as follows: Section 5.1 introduces the topic, and in section 5.2, we offer some literature studies on the financial applications of the Fourier transform, as well as in the valuation of basket default swaps. Section 5.3 explains the idea of pricing BDS and the concept of dependency on the use of copula models. Section 5.4 investigates the notion of the discrete and fast Fourier transform and then links the applications of the discrete Fourier transform (DFT) to the evaluation of the premium legs and the default legs of the BDS valuation. Section 5.5 discusses the numerical results obtained in the course of the valuation, whereas section 5.6 concludes the study.

5.2 Literature Study

In the field of quantitative risk analysis, insurance and financial engineering, one of the significant challenges lies in how dependent default events can be modelled and equally analyzed over a specified period of time. Factor models are often used to describe correlated and dependent defaults, arising from some common systematic factors, which exist among a set of financial entities. However, when the dependence is conditioned on these latent factors, the corresponding defaults are fully independent. The factor model, in conjunction to the copula function, is fully utilized in the analytical formulation of the joint distribution function of the default times. This is because the copula function allows for statistical inference to be decoupled into inference from their dependence and inference from their marginals owing to the scanty information which exists on dependence in the area of credit risk (Cossin et al. 2010).

The copula models have been extensively used in finance, especially in contingent claims pricing owing to their indispensable applications. For instance, Li (2000) focused on the issue relating to default correlation and introduced the copula function to model the joint distribution functions of the survival times. He equally proved the equivalence of the typical normal CreditMetrics methods to the use of the copula function and then illustrated numerically with the pricing of CDS, as well as the first-to-default contingent claims. Using the US market as a case study, Das and Geng (2003) were able to utilize the copula models in the simulation of correlated default risk and to assess the corresponding joint distribution of default times. Schönbucher and Schubert (2001) incorporated the dynamic default dependency in the context of default risk models which are intensity-based in order to obtain the survival probabilities and thus, the price values of credit spreads. These concepts were connected to the copula models such as the Gumbel, the Clayton and the Gaussian models.

Under interacting default intensities, the pricing and hedging of the portfolio credit derivatives (PCDs), such as the basket credit default swaps (BCDS) and the collateralized debt obligations (CDOs) were also considered by Frey and Backhaus (2008). Here, they explicitly modelled the default contagion ,which is evident among interacting portfolios and thus, they analyzed the corresponding models through the Markov processes. Ackerer and Vatter (2017) modelled joint dependent defaults, joint dependent losses with the aid of factor copula models, and thus fitted the models to the valuation of credit index tranches. To estimate the loss distribution functions of these contingent claims, they computed the individual losses, discretely fitted on some finite grids and then calibrated some specific models to the tranche prices (Ackerer and Vatter 2017). The concept of correlations and copula dependence modelling can be found further in the works of (Burtschell et al. (2005), Embrechts et al. (2001), Embrechts et al. (2002)).

On the other hand, the Monte-Carlo simulations have attracted much attention in the PCD valuations because they are easily implemented to model some dependent and correlated default risks, as well as in the estimation of the expected loss. However, several analytical approximation techniques like the Parametric approximations, Fast Fourier transform

(FFT), the recursion method and "probability bucketing"(Hull and White 2004) have been introduced as alternatives to this random simulations because the former thrives best when the tail risk is being considered (Merino and Nyfeler 2002). The Fourier transform method (FTM) has been described as a numerical technique that offered high computational speed and accuracy in estimating the loss distribution functions of a specified PCDs over a certain period of time. The DFT and its natural algorithm, the FFT, are essential tools employed in the computation of the characteristic functions which are utilized in the pricing of such PCDs.

Fusai and Roncoroni (2007) conducted a comparative study on the BCDS pricing using the Monte-Carlo methods and two semi-analytic methods, that is, the Hull-White recursive algorithm and the FFT. They found out that the Monte-Carlo method requires an exceptionally high number of simulations to ensure convergence, and the simulation method is equally not feasible when modelling a basket of more than 100 entities. For a small heterogeneous set of portfolios, the probability generating function and the FFT were equally utilized by Gregory and Laurent to compute the distribution function of the cumulative defaults (Gregory and Laurent (2003), Laurent and Gregory (2005)). Whereas, Zheng (2006) employed a two-hybrid algorithm in the pricing of BCDS and CDOs by combining an analytical approximation of the loss distribution function and Monte-Carlo simulation, with more reference on large heterogeneous portfolios.

More numerical approaches are seen in the works of Bastide et al. (2008) where they conducted a comparative analysis on the pricing of basket default swaps by the application of the Stein method in connection to the factor copula models. They further incorporated the Monte-Carlo techniques, probability generating function method and the recursive-based method proposed by (Hull and White 2004), to the valuation of the basket default swaps and the CDO tranches. Bielecki et al. (2007) equally estimated the dependent defaults and credit migrations under the Markovian model which describes the financial market, and thus were able to price the BCDS, together with credits/loans portfolios. Debuysscher et al. (2003) employed the FTM to estimate the default or the loss distribution function by first modelling the risk exposure and the default correlation of the entities. Furthermore, the authors used a factor copula to obtain the Fourier transform of the basket's aggregate loss distribution, and then, inverting the corresponding portfolio Fourier transform to obtain the needed distribution function (Debuysscher et al. 2003). Merino and Nyfeler (2002) focused on the numerical computation of portfolio losses experienced by credit portfolios like the CDOs. Here, they used the Poisson approximation techniques, the FFT, and the numerical integrations of the quasi MCS methods to decrease the computational complexity, which is paramount in price estimations.

5.3 Basket Default Swaps

This section focuses on the concept of BDS pricing, with particular attention on the n2D swaps because they are more liquid when compared to the other BDS types. BDS offers considerable advantages to credit investors, even though their liquidity cannot be compared to the highly-traded synthetic CDOs. The former permits the financial practitioners to select, manage and easily monitor smaller credit portfolios (say about 5-10) than the broader portfolio of about 100-150 which are evident in synthetic CDOs. In this context, we shall consider a homogeneous loss portfolio for ease on convenience since they are the framework of the BDS pricing. Such portfolio implies that each entity incurs a similar amount of loss in the event of a credit default, even though they might possess different values for their correlations and their default swaps spread curves.

5.3.1 Pricing Basket Default Swap

Consider a portfolio with N reference entities or obligors, having A, as the face value of the contract. Let T be the maturity time of the contract, $B(0, t_i)$ be the discount factor supposing t_i maturity and $f(t)$ the instantaneous forward rate at time t. Define n as the seniority of the basket, such that for $n = 1$, we have the F2D, and let the portfolio pay a spread of β at dates $\{t_1, \cdots, t_N\}$. These periodic payments made by the protection buyer to the protection seller are known as the Premium Legs. The swap premium is paid at a frequency of $\Delta_i = t_i - t_{i-1}$, which could be annually for $\Delta = 1$, quarter-annual for $\Delta = \frac{1}{4}$. On the other hand, suppose a default occurs before the maturity time T, the seller will pay $A(1 - R)$ to the buyer at maturity, or at the default time τ^n. Thus, the risk-neutral pricing measure for obtaining the value of the spread, having zero accrued premium, is solved by equating the expected values of the premium leg and the default leg. The theorem below gives the fair spread value:

Theorem 5.1. *(n2D swap spread, Fusai and Roncoroni (2007)).*

Let $F^n(t)$ be the probability distribution function of τ^n, the risk-neutral pricing for the an-

nualized equilibrium spread of n2D swap is given below as:

$$\beta = \frac{(1-R)\left[B(0,T)F^n(T) + \int_0^T f(0,t)B(0,t)F^n(t)\mathrm{d}t\right]}{\sum_{i=1}^{N} \Delta B(0,t_i)[1-F^n(t_i)]}\,,$$

where $f(0,t) = r$ if the instantaneous forward rate equals a constant interest rate.

For proof, see (Fusai and Roncoroni 2007). □

Furthermore, in the event of an accrued premium, the buyer pays off an amount of $\beta\Delta(t_{i-1} - t_i)$ to the seller, and this payment stream is made viable when a credit event occurs between two premium time, that is, between t_{i-1} and t_i.

5.3.2 Dependence Structure via Copula Models

In this subsection, we focus on the one-factor Gaussian, Clayton and the Student-t Copula models to estimate the joint probability of default based on the assumption of conditionally independent default times.

Gaussian copula:

Let the dynamics of an entity value i, for $i = 1, \cdots, N$ be denoted by

$$Y_i = \rho_i Z + \sqrt{1-\rho_i^2}\epsilon_i\,, \tag{5.3.1}$$

where $\rho_i \in [0,1]$ and $\mathrm{Cov}(\epsilon_i, \epsilon_j) = \rho_i\rho_j$. Here, ϵ_i denote the idiosyncratic factors and Z the latent factor, with both Z and ϵ_i from a standard normal distribution. Let $\tau_1, \cdots, \tau_N$ denote the default times, such that $\tau_i = F_i^{-1}(Y_i)$ and let $F_1(t_1), \cdots, F_N(t_N) = \mathbb{P}(\tau_1 \leq t_1), \cdots, \mathbb{P}(\tau_N \leq t_N)$ be defined as the set of marginal distributions of default times. Then, we can have the function $F_i(t_i) = \mathbb{P}(Y_i \leq u_i)$ and $F(t_1, \cdots, t_N) = \mathbb{P}(Y_1 \leq u_1, \cdots, Y_N \leq u_N)$, where the parameter u_i refers to certain threshold at which the creditworthiness of the entity i falls below when the i-th asset defaults. Suppose the value of Z is known, then the (risk-neutral) joint conditional probability, $\mathbb{P}(Y_i \leq u_i|Z)$ that the ith entity defaults before the given time t can be computed as:

$$\mathbb{P}(Y_i \leq u_i|Z) = \mathbb{P}\left(\epsilon_i < \frac{\Phi^{-1}(F_i(t_i)) - \rho_i Z}{\sqrt{1-\rho_i^2}} \middle| Z\right), \tag{5.3.2}$$

and since ϵ_i are standard normally distributed, the joint conditional probability of default is finally given below as:

$$p_{t_i}^{i|Z} = \Phi\left(\frac{\Phi^{-1}(F_i(t_i)) - \rho_i Z}{\sqrt{1-\rho_i^2}}\right). \tag{5.3.3}$$

Integrating out the dependency structure evident on the conditional variable Z gives the unconditional probability of default, which can be denoted by $F(t_1, \cdots, t_N)$. Thus, the joint probability distribution is given below as:

$$F(t_1, \cdots, t_N) = \int_{-\infty}^{\infty} \left[\prod_{i=1}^{N} \Phi\left(\frac{\Phi^{-1}(F_i(t_i)) - \rho_i z}{\sqrt{1-\rho_i^2}}\right)\right] \phi(z)\mathrm{d}z\,, \tag{5.3.4}$$

where $\phi(z) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\frac{-z^2}{2}}$ is the standard normal density function.

Clayton copula:

Suppose the single factor Z follows a Gamma distribution with $\frac{1}{\theta}$ parameter, for $\theta > 0$. The conditional default probability can be defined as follows:

$$\mathbb{P}(Y_i \leq u_i|Z) = \exp\left[-z\psi^{-1}(F_i(t_i))\right] \quad \text{for all } t_1, \cdots, t_N\,, \tag{5.3.5}$$

where ψ defined as the generator function describing the Clayton copula with $\psi^{-1}(k) = k^{-\theta} - 1$ and $\psi(k) = (1+k)^{\frac{-1}{\theta}}$. Integrating out the dependency structure over the Gamma distributed random variable z yields the unconditional probability of default, and thus, the joint probability distribution is given below as (Laurent and Gregory 2005):

$$F(t_1, \cdots, t_N) = \int_{0}^{\infty} \left[\prod_{i=1}^{N} \exp\left(-z\psi^{-1}(F_i(t_i))\right)\right] \phi(z)\mathrm{d}z\,, \tag{5.3.6}$$

where $\phi(z) = \frac{1}{\Gamma(\frac{1}{\theta})}\mathrm{e}^{-z}z^{\frac{(1-\theta)}{\theta}}$ is the PDF of the Gamma function.

Student-t copula:

The student-t copula has a non-zero structure of tail dependence, and thus, it is more efficient in capturing the extreme joint events which might likely occur, especially in equity data (O'Kane 2011). Let R denote a χ^2 distribution random variable having η as the degree of freedom. Also, let Z and ϵ_i be standard normally distributed random variables such that R, Z and ϵ_i are all independent. Then, we can define the latent variable to be given by:

$$Y_i = \sqrt{\frac{\eta}{R}}\left(\rho_i Z + \sqrt{1-\rho_i^2}\epsilon_i\right) . \tag{5.3.7}$$

Furthermore, let $u_i = t_\eta^{-1}(F_i(t_i))$ be the same threshold in which the default occurs before time t. The joint conditional probability of default $p_{t_i}^{i|Z,R}$ of the ith entity prior to time t_i is given below as

$$\mathbb{P}(Y_i \le u_i|Z,R) = \mathbb{P}\left(\epsilon_i \le \frac{u_i\sqrt{\frac{R}{\eta}} - \rho_i Z}{\sqrt{1-\rho_i^2}}\middle| Z,R\right) .$$

Thus, we have that

$$p_{t_i}^{i|Z,R} = \Phi\left(\frac{\sqrt{\frac{R}{\eta}}t_\eta^{-1}(F_i(t_i)) - \rho_i Z}{\sqrt{1-\rho_i^2}}\right) . \tag{5.3.8}$$

Obtaining the unconditional probability of default using the student-t copula involves a double integration of the conditional loss distribution over two distributions, that is, the Gaussian and the Chi-square[1]. Thus, the joint probability distribution can be written as:

$$F(t_1,\cdots,t_N) = \int_0^\infty\int_{-\infty}^\infty\left[\prod_{i=1}^N \Phi\left(\frac{\sqrt{\frac{r}{\eta}}t_\eta^{-1}(F_i(t_i)) - \rho_i Z}{\sqrt{1-\rho_i^2}}\right)\right]\phi(z)\phi_\eta(r)\mathrm{d}z\mathrm{d}r\,, \tag{5.3.9}$$

where $\phi_\eta(r) = \frac{r^{\frac{\eta}{2}-1}e^{\frac{-r}{2}}}{2^{\frac{\eta}{2}}\Gamma\left(\frac{\eta}{2}\right)}$ is the PDF of the χ^2 distribution (O'Kane 2011).

This double integral is quite computationally intensive and this possess a notable disadvantage of implementing the student-t copula over the Gaussian copula. Thus, pricing problems and risk management of large baskets are quite impractical when the student-t copula is being utilized.

[1]The χ^2 distribution function is only valid for $r \ge 0$, having first and second moments as $\mathbb{E}[r] = \eta$ and $\mathbb{E}[r^2] = 2\eta$

5.4 Discrete and Fast Fourier Transform

The Fourier transform approach or the spectral method is another technique, apart from the usual recursion method, employed in the generation of the conditional and the unconditional loss distribution functions. According to Parodi (2014), the FTM works by mapping the original distribution function into another space in which the convolution operation becomes more analytically tractable, and then transferred back to its original space after the problem has been solved. In a nutshell, the FTM approach is entirely dependent on the successful implementation of the FFT.

Definition 5.2. *(Characteristic Function, Hirsa and Neftci (2013)).*

Let X be a continuous random variable, the characteristic function Ψ of X referred to as the expected value e^{itX} where $i = \sqrt{-1}$ and $t \in \mathbb{R}$ (the argument of Ψ), is defined as

$$\Psi_X(t) = \mathbb{E}\left[\mathrm{e}^{itX}\right] = \int_{-\infty}^{\infty} \mathrm{e}^{itX} f_X(x)\mathrm{d}x\,,$$

where $f_X(x)$ is the probability density function of X. □

Note that if X has a probability density of f_X, the characteristic function is equivalent to its Fourier transform having a sign reversal in its complex exponential.

Definition 5.3. *(Convolution, Fusai and Roncoroni (2007)).*

Let f and g be measurable functions, the convolution of both f and g is defined as

$$(f \otimes g)(x) \triangleq \int_{-\infty}^{\infty} f(y)g(x-y)\mathrm{d}y \equiv \int_{-\infty}^{\infty} f(x-y)g(y)\mathrm{d}y\,.$$

Definition 5.4. *(Fourier and Inverse Fourier Transform, Hirsa and Neftci (2013)):*

Define X as a continuous random variable and let f be an integrable function such that $\hat{f} : \mathbb{R} \to \mathbb{C}$ and $i = \sqrt{-1}$, the FT and the IFT are defined respectively as

$$\hat{f}(t) = \int_{-\infty}^{\infty} \mathrm{e}^{itX} f(x)\mathrm{d}x \qquad \textit{and} \qquad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-itk} \hat{f}(k)\mathrm{d}k\,.$$

Definition 5.5. *(Discrete Fourier and inverse discrete Fourier Transform, Debuysscher et al. (2003)).*

Let f be a discrete function such that $f : \{x_n\} \to \{X_k\}$, then the DFT and the IDFT are defined respectively as:

$$X_k = \sum_{n=0}^{N-1} x_n \mathrm{e}^{\frac{-2\pi ink}{N}} \qquad \textit{and} \qquad x_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k \mathrm{e}^{\frac{2\pi ink}{N}},$$

where $n, k = 0, 1, \cdots, N-1$. □

The Fast Fourier transform, however, is the algorithm for executing the DFT of a sequence of variables. The FFT lowers the computational complexity of the DFT from $\mathcal{O}(N^2)$ to $\mathcal{O}(N \log_2 N)$, and this is so noticeable when computing a vast dataset. With regards to the immense difference between the two order of complexity, Press et al. (2002) stated that for an $N = 10^6$ points, it is roughly the difference between 30 seconds (CPU time) and about two weeks (CPU time) which are measured on a microsecond cycle time computer.

5.4.1 Portfolio Loss Distribution

For simplicity, we link this subsection to the Gaussian copula defined in a one-factor model, and the corresponding estimated default probabilities can be used if another copula is being used. Let $Z(t)$ be the cumulative portfolio loss at a specified time t with a discrete distribution, and M_i the notional on each of the entity $i = 1, \cdots, N$. The conditional probability of default $p_t^{j|x}$ for the jth name up to time t, which we have conditioned on a specified Gaussian random variable $X = x$ follows from equation (5.3.3). For the Clayton and the student-t copula models, the conditional probabilities are given in equations (5.3.5) and (5.3.8) respectively. Define the indicator function of the default time counting process for each ith entity as $Q_i(t) = \mathbb{I}_{\{\tau_i \le t\}}$, and let R_i be the ith recovery rate, then

$$Z(t) = \sum_{i=1}^{N} (1 - R_i) M_i Q_i(t)\,.$$

For the valuation of multi-name PCDs like the CDOs and n2D basket swaps, it is pertinent to either simulate the default times or perform an estimate for the loss distribution over a time range $[0, T]$. To calculate the distribution function of $Z(t)$, we first need to estimate the characteristic function $\Psi_{Z(t)}(s)$ of the portfolio loss, and a natural choice to this would be the implementation of the FFT.

Furthermore, suppose the σ-algebras $\sigma(R)$ and $\sigma(X, \tau_1, \cdots, \tau_N)$ are independent functions, we calculate the characteristic function of the aggregate loss as follows:

$$\Psi_{Z(t)}(s) = \mathbb{E}[\mathrm{e}^{isZ(t)}] .$$

Proposition 5.6. *Let $(\mathcal{F}_t)_{t\geq 0}$ be a given filtration. If Y is a stochastic random variable and $s < t$, then the law of iterated expectation states that:*

$$\mathbb{E}[\mathbb{E}[Y|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}[Y|\mathcal{F}_s] .$$

For proof, see (Björk 2009). □

Conditioning on the common factor X, and from the properties of iterated expectations in Proposition 5.6, we have that

$$\Psi_{Z(t)}(s) = \mathbb{E}[\mathbb{E}[\mathrm{e}^{isZ(t)}|X = x]] = \mathbb{E}\left[\mathbb{E}\left[\prod_{j=1}^{N} \exp\{is(1-R_j)M_j\mathbb{I}_{\{\tau_i \leq t\}}\}\middle| X = x\right]\right] .$$

Since τ_j is conditionally independent on X and from the assumption of independent R_j recovery rates[2], the inner expected value can be written as

$$\begin{aligned}
=&\mathbb{E}\left[\prod_{j=1}^{N} \exp\{is(1-R)\mathbb{I}_{\{\tau_j \leq t\}}\}\middle| X = x\right] = \prod_{j=1}^{N}\mathbb{E}\left[\exp\{is(1-R)\mathbb{I}_{\{\tau_j \leq t\}}\}\middle| X = x\right] \\
=&\prod_{j=1}^{N}\left[(1-p_t^{j|X}) + p_t^{j|X}\mathrm{e}^{is(1-R)}\right] = \prod_{j=1}^{N}\left[1 + p_t^{j|X}\left(\mathrm{e}^{is(1-R)} - 1\right)\right] .
\end{aligned}$$

The unconditional FT can thus be computed by integrating out the common factor over the Gaussian density as shown below:

$$\Psi_{Z(t)}(s) = \mathbb{E}\left[\prod_{j=1}^{N}\left[1 + p_t^{j|X}\left(\mathrm{e}^{is(1-R)} - 1\right)\right]\right] = \int_{-\infty}^{\infty}\prod_{j=1}^{N}\left[1 + p_t^{j|X}\left(\mathrm{e}^{is(1-R)} - 1\right)\right]\phi(x)\mathrm{d}x .$$

Finally, the actual portfolio loss distribution describing $Z(t)$ can be obtained by inverting the characteristic function with the use of a more suitable Fourier transform function, such as the inverse form of the FFT [3], and this is more efficient when non-homogeneous portfolios

[2] The rate R_j is considered constant here and further assumption of a constant unit notional, i.e., $\sum M_i = 1$
[3] From `jupyter notebook`, `ifft` imported from `scipy.fftpack` package

are being considered (Press et al. 2002).

5.4.2 Derivation of Main Theorem

This section will give the main highlight of this research work, which involves the derivation of the basket swap premium legs, thereby presenting the quasi-analytic expression for the computation of the swap pricing premium.

Theorem 5.7. *(Main Theorem).*

The fair value for the n2D swap, with zero accrued premium, is given as:

$$\beta = \frac{(1-R)\left[B(0,t_N)\sum_{j=n}^{N}\mathbb{P}(Z(t)=j) + \sum_{i=1}^{N} f(0,t_i)B(0,t_i)\left(\sum_{j=n}^{N}\mathbb{P}(Z(t_i)=j)\right)\right]}{\sum_{i=1}^{N}\Delta_i B(0,t_i)\sum_{j=0}^{n-1}\mathbb{P}(Z(t)=j)}\,.$$

Proof:

Let $\lambda_i(t)$, also known as hazard rates or the default intensity, be the intensity of the Poisson process. Then, the marginal density function F_i of default times τ_i is given below as

$$F_i(t) := \mathbb{P}(\tau_i < t) := 1 - \exp\left(-\int_0^t \lambda_i(s)\mathrm{d}s\right), \qquad \text{where } \; i = 1, \cdots, N.$$

For τ^n been the time of the nth default, the survival and the default probability distribution functions for the nth default are given respectively as:

$$S^n(t) := \mathbb{P}(\tau^n > t) = \mathbb{P}(Z(t) < n) = \sum_{j=0}^{n-1}\mathbb{P}(Z(t)=j) \tag{5.4.1}$$

$$F^n(t) := \mathbb{P}(\tau^n \le t) = \mathbb{P}(Z(t) \ge n) = \sum_{j=n}^{N}\mathbb{P}(Z(t)=j)\,. \tag{5.4.2}$$

The value for the premium leg (PL) for the n2D basket swap can be defined as:

$$PL = A\sum_{i=1}^{N}\beta\Delta_i B(0,t_i)\mathbb{I}_{\{\tau^n > t_i\}}\,.$$

Taking the expectation values, we have

$$PL = A\beta \sum_{i=1}^{N} \Delta_i B(0,t_i)\mathbb{E}[\mathbb{I}_{\{\tau^n > t_i\}}] = A\beta \sum_{i=1}^{N} \Delta_i B(0,t_i)\mathbb{P}[Z(t_i) < n]\,,$$

where $\mathbb{P}[Z(t_i) < n]$ is referred to as the probability of having lower than n defaults.

Thus, the final discounted expected value for the **PL** is given below as

$$PL = A\beta \sum_{i=1}^{N} \Delta_i B(0,t_i) \sum_{j=0}^{n-1} \mathbb{P}(Z(t) = j)\,. \tag{5.4.3}$$

Suppose there exists one or multiple credit events before the expiration time T, then the swap seller will be obligated to offset the difference of the nominal value and the recovery rate value for the defaulted reference entity. The default leg (DL) value can be expressed as:

$$DL = A(1-R)B(0,\tau^n)\mathbb{I}_{\{\tau^n \leq T\}}\,.$$

Taking the expectation values, we have

$$DL = A(1-R)B(0,\tau^n)\mathbb{E}[\mathbb{I}_{\{\tau^n \leq T\}}] = A(1-R)\int_0^T B(0,t)\mathrm{d}F^n(t)\,.$$

Integration by parts yields

$$DL = A(1-R)\left[F^n(T)B(0,T) + \int_0^T f(0,t)B(0,t)F^n(t)\mathrm{d}t\right]\,.$$

Next, we convert the functions to discrete functions by substituting the distribution functions $F^n(t)$ with the probabilities of the counting processes as shown in equation (5.4.2). Thus, we have

$$DL = A(1-R)\left[B(0,t_N)\mathbb{P}(Z(T) \geq n) + \sum_{i=1}^{N} f(0,t_i)B(0,t_i)\mathbb{P}(Z(t_i) \geq n)\right]\,.$$

Hence, the value of the **DL** reduces to

$$DL = A(1-R)\left[B(0,t_N)\sum_{j=n}^{N}\mathbb{P}(Z(t) = j) + \sum_{i=1}^{N} f(0,t_i)B(0,t_i)\left(\sum_{j=n}^{N}\mathbb{P}(Z(t_i) = j)\right)\right]\,. \tag{5.4.4}$$

Equations (5.4.3) and (5.4.4) give the discrete forms of the premium legs and the default legs respectively. Hence, the fair spread value (in the absence of the accrued premium) for the n2D swap is such that $\beta \implies \mathbb{E}[PL] = \mathbb{E}[DL] = 0$, so that we have

$$\beta = \frac{(1-R)\left[B(0,t_N)\sum_{j=n}^{N}\mathbb{P}(Z(t)=j) + \sum_{i=1}^{N} f(0,t_i)B(0,t_i)\left(\sum_{j=n}^{N}\mathbb{P}(Z(t_i)=j)\right)\right]}{\sum_{i=1}^{N}\Delta_i B(0,t_i)\sum_{j=0}^{n-1}\mathbb{P}(Z(t)=j)}\,. \tag{5.4.5}$$

□

Equation (5.4.5) gives the equilibrium price (quasi-analytic expression) of the fair n2D swap spread, and the calculation is straightforward following from the estimation of the probabilities $\mathbb{P}(Z(t)=j)$. Our approach in this technique first uses the convolution concept to output vectors of probabilities which are conditioned on the given X as a common latent variable[4]. Next, we import the `fft` function from the `scipy.stats` library in `jupyter notebook` to obtain the product of the probability vectors, and the corresponding values are finally inverted back using the `ifft` function. Sub-section 5.4.3 further describes the methodology of obtaining the distribution function using the FT method.

5.4.3 Probability of having n Minimum Defaults

Let $Z_i(T)$ denote the indicator function that an ith entity defaults before maturity T, that is, $Z_i(T) = \mathbb{I}_{\{\tau_i \le T\}}$, and let $q_i = (1-p_i) = \mathbb{P}(\tau_i \le T)$ be the corresponding probability. If the number of defaults existing before T or the counting process in connection with the number of default is denoted by $L(T) = \sum_{i=1}^{N} Z_i(t)$, then the probability of having a minimum of n defaults can be written as

$$\mathbb{P}(Z(T) \ge n) = \sum_{k=n}^{N}\mathbb{P}(Z(T)=k)\,.$$

The distribution function of $Z(T)$ can be viewed as a convolution of the distribution functions of the $Z_i(T)$, which can be solved via the Fourier approach(Fusai and Roncoroni 2007). Let

[4]For now; we only consider the Gaussian copula. This approach is equally obtainable when the distribution functions are changed to reflect the copula model being considered.

the probability vectors of Bernoulli random variables $Z_i(T)$ be defined as follows:

$$\gamma_{ij} = \begin{cases} 1 - q_i & \text{if } j = 0 \\ q_i & \text{if } j = 1 \\ 0 & \text{otherwise}\,, \end{cases}$$

as well as its DFT as

$$\eta_{ij} = \sum_{k=0}^{M-1} \gamma_{ij} \exp\left(\frac{2\pi\sqrt{-1}kj}{M}\right) \qquad \text{for } \; j = 1, \cdots, M\,.$$

Then the corresponding $\mathbb{P}(Z(T) = j)$ is obtained by taking the inverse of the DFT of the product of the FFT values η_{ij}. That is

$$\mathbb{P}(Z(T) = j) = \frac{1}{M} \sum_{k=0}^{M-1} \left[\exp\left(\frac{-2\pi\sqrt{-1}kj}{M}\right) \prod_{i=1}^{N} \gamma_{ij} \right]\,.$$

After this step, it suffices to calculate the unconditional loss distribution and this can be done by integrating the results over the Gaussian factor distribution. Finally, this value is fixed into the n2D basket credit analytics to solve for the premium legs, the default legs and then the corresponding break-even spread.

5.5 Results and Discussions

This section focuses on the numerical experiments obtained in the course of the BDS valuation. First, we consider some empirical and statistical analysis of some selected CDS spread data and then secondly, we employed the concept of the Gaussian and the Clayton copula, in conjunction with the DFT to obtain the fair spread values.

5.5.1 Data Analysis

Our dataset consists of a 5-year monthly CDS spread quotes emanating from the following ten companies: Xerox, Coca Cola, Boeing, IBM, Johnson & Johnson, Oracle, Pepsi, McDonald, Walmart and AT & T, and these shall be denoted by Xe, Co, Bo, IB, JJ, Or, Pe, Mc, Wa, and AT respectively. We conduct some empirical and statistical analysis based on these spread values, with values in the units of basis points (bp). The sample comprises of 132 monthly observations from January 31, 2008, till December 31, 2018, which spans across different credit ratings (from AAA to BB rating) and tailored around the Corporate Sector. In Figure 5.1 below, we made a plot of the corresponding time series, and each of the lines in this plot represents the spread of different entities.

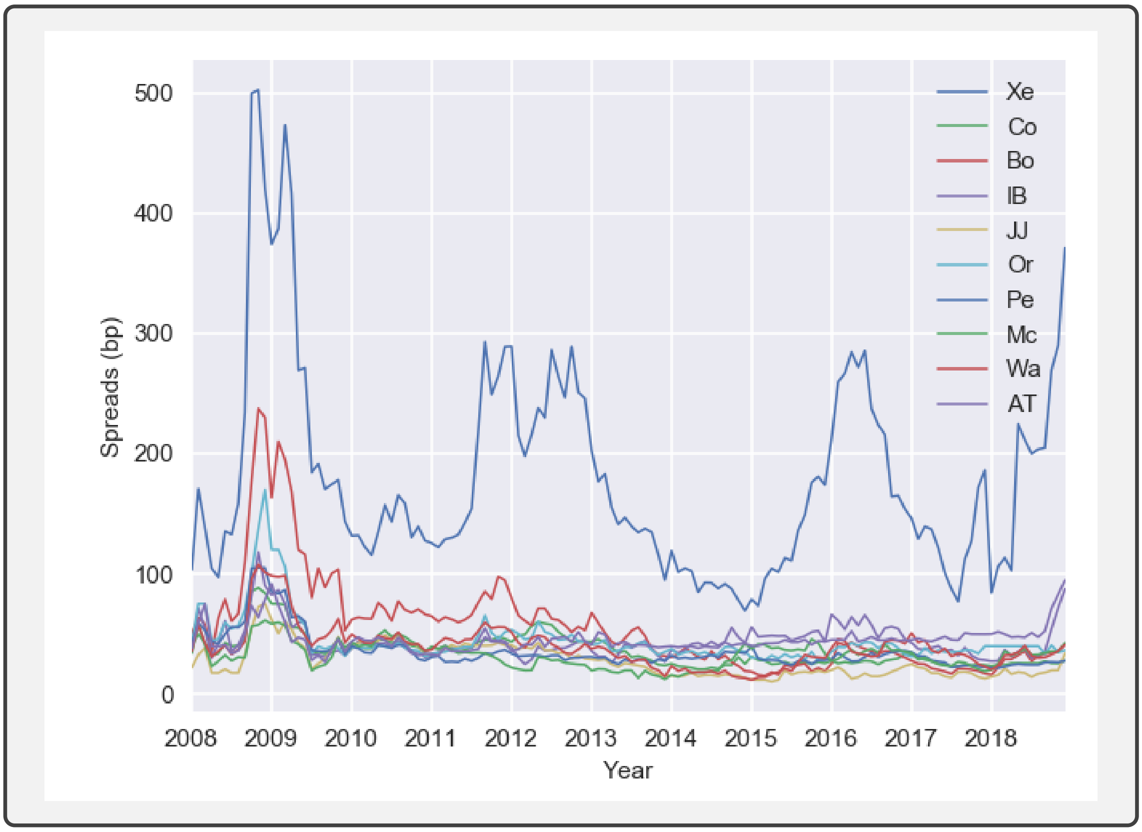
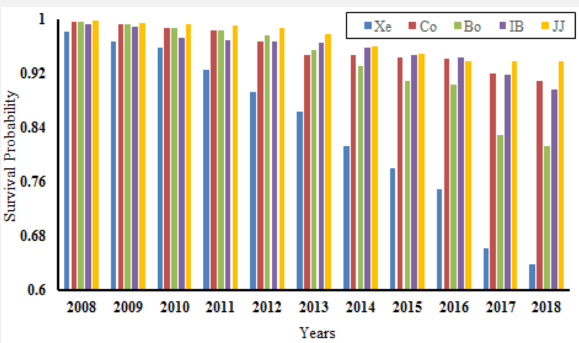


Figure 5.1: Time series for the 5-Year monthly CDS spreads for ten entities

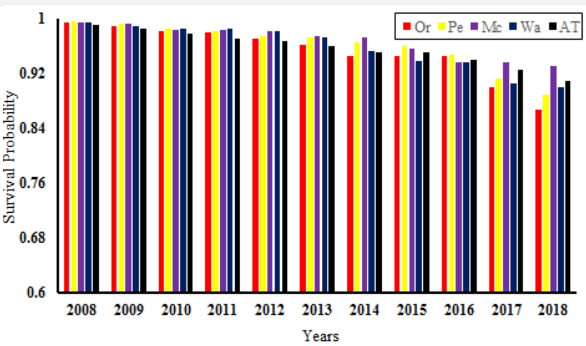
For each of the monthly CDS spread spanning through 11 year time period, we output the yearly observances by calculating the average combined quotes in each year. Based on this, we obtain the mean, the standard deviation, the skewness and the kurtosis for each obligor or reference entity, and this is shown in Table 5.1 below. Consider the lower rate firm, the Xerox Holdings Corporation for instance; it is observed that both its mean and the corresponding standard deviation spreads are significantly higher when compared to other less-risky firms in this chapter. Throughout the time period for this CDS swap analysis, the Xerox firm achieved high swap spread in October, November, December 2008 and then, March 2009, with values 499 bp, 503 bp, 420 bp and 473 bp, respectively. This increment in spread values might be the after-effect of the global financial crises of 2007-2008, which rendered many entities downgraded as a result of some perceived credit risks. Furthermore, the firm was able to recuperate during the last eight months of 2014 till the first quarter of 2015, where it became accountable for smaller spread values with the lowest in December 2014 at 69 bp.

As for the skewness and the kurtosis, the values are generally small for all the reference entities. Skewness ascertains the symmetric structure of the distribution about its mean. From Table 5.1, we observed that fewer reference entities like the Xerox, Coca-Cola, Boeing and AT & T had a majority of their dataset to be positively skewed, whereas the other reference entities had many of their data skewed to the left. Thus, negatively skewed datasets are mostly associated with lower credit spreads, and this was in line with the explanations reported by Cremers et al. (2008). Kurtosis, on the other hand, measures how light-tailed or heavy-tailed a distribution is in connection to the normal distribution. As can be observed from Table 5.1, we noticed that the majority of the entities dataset had lighter tails in comparison to their normal distribution tails, especially the Pepsi Corporation, which assumed the highest. Whereas, the distribution functions for firms such as Oracle and McDonald had much heavier tails.

Under the risk-neutral pricing measure $\mathcal{Q}$, consider a model of exponential default which have intensity λ. The CDS spread is directly proportional to $(1 - R)$, with λ as constant, and we further assume a standard recovery rate $R = 40\%$. The survival probability of the set of data frames can be obtained using the expression $\exp(-s * t/(1 - R))$. In Figure 5.2 below, we made plots for the probability of survival of the first five and last five entities, respectively.



(a) Survival Probabilities of First 5 Entities



(b) Survival Probabilities of Last 5 Entities

Figure 5.2: Probability of Survival

Table 5.1: Summary statistics defining the 5-year monthly CDS spreads for ten entities

Year	Entities (Mean ($\bar{X}$) and Standard Deviation (SDV))																			
	Xe		Co		Bo		IB		JJ		Or		Pe		Mc		Wa		AT	
	$\bar{X}$	SDV	$\bar{X}$	SDV	$\bar{X}$	SDV	$\bar{X}$	SDV	$\bar{X}$	SDV	$\bar{X}$	SDV	$\bar{X}$	SDV	$\bar{X}$	SDV	$\bar{X}$	SDV	$\bar{X}$	SDV
2008	224.8	156.1	25.4	1.2	30.5	6.7	42.0	19.1	18.7	5.2	38.5	2.4	24.5	1.9	33.0	5.8	32.0	5.9	57.6	16.1
2009	269.5	114.6	28.8	3.2	21.0	3.3	35.6	4.4	17.5	3.7	35.1	2.5	26.3	4.2	25.5	4.2	35.0	8.1	46.8	2.5
2010	138.1	15.3	27.4	2.3	35.9	5.2	56.8	6.7	17.6	3.4	39.5	3.3	31.8	3.1	34.6	3.8	38.8	5.8	45.6	2.3
2011	187.6	69.6	26.3	1.8	19.8	3.5	48.5	3.3	15.3	3.6	30.4	2.3	28.2	3.9	38.6	3.3	21.9	7.6	42.6	1.7
2012	247.4	30.5	23.3	2.8	30.0	5.3	41.8	5.6	15.2	1.1	35.2	2.4	31.8	2.2	22.5	8.5	17.8	2.9	39.9	1.4
2013	146.9	29.4	33.9	7.4	47.1	10.3	34.7	5.9	23.2	4.6	38.8	3.9	28.1	1.8	17.5	2.9	29.0	7.7	42.1	4.1
2014	92.5	13.1	50.2	6.5	62.2	8.4	36.5	5.9	35.7	4.1	48.6	4.9	31.2	1.4	24.8	3.8	41.7	5.8	43.3	4.7
2015	124.4	37.9	40.9	4.1	72.5	13.7	40.9	5.9	39.1	3.1	43.1	9.5	30.6	3.5	34.0	4.4	48.4	7.0	38.2	4.8
2016	228.4	47.7	40.6	3.1	67.4	5.5	38.9	4.2	42.6	4.4	38.1	2.8	36.4	4.5	44.0	5.3	44.5	4.2	41.7	5.4
2017	127.9	31.9	50.1	16.9	125.2	48.3	51.3	21.2	41.6	13.3	63.0	33.9	55.5	21.2	39.8	14.0	63.6	22.7	47.0	19.3
2018	198.5	86.5	51.9	21.0	102.4	71.5	59.6	26.3	34.8	21.4	77.5	39.8	64.5	25.1	39.7	14.2	55.0	28.9	52.2	16.8

	Entities (Kurtosis (Kurt) and Skewness (SKW))																			
	Xe		Co		Bo		IB		JJ		Or		Pe		Mc		Wa		AT	
	SKW	Kurt	SKW	Kurt	SKW	Kurt	SKW	Kurt	SKW	Kurt	SKW	Kurt	SKW	Kurt	SKW	Kurt	SKW	Kurt	SKW	Kurt
2008	1.19	-0.30	1.14	-0.47	1.24	0.03	1.06	0.68	1.12	-0.09	1.51	1.66	1.08	-0.56	0.55	-1.59	1.17	-0.43	0.51	-1.52
2009	0.63	-1.20	0.54	-1.28	0.51	-0.69	0.57	-1.30	0.01	-1.01	0.94	-0.74	0.42	-1.54	0.04	-1.20	0.80	-1.04	1.31	1.39
2010	0.54	-0.60	-0.07	-1.09	0.55	-0.81	-0.78	-0.16	-0.85	1.39	-1.29	3.33	-1.03	-0.01	0.05	-0.76	-0.30	0.31	-0.83	-0.78
2011	0.56	-1.68	0.33	-1.49	0.89	-0.68	-0.86	1.61	-1.32	0.86	1.19	1.18	0.35	-1.16	-0.81	0.93	0.05	-1.04	0.59	-0.69
2012	0.06	-0.98	0.31	-1.59	0.77	0.06	-0.82	-0.06	-0.14	-0.63	1.26	1.75	-0.77	1.33	-0.01	-1.25	0.14	-1.00	-0.08	-0.65
2013	0.31	0.28	0.21	-1.14	0.26	-0.34	0.04	-1.04	-0.60	-0.49	-0.14	-1.09	-0.29	-0.59	-0.84	-0.61	-0.45	-0.43	1.71	2.00
2014	0.27	0.66	1.61	3.43	-0.63	0.35	1.07	2.23	0.12	2.11	0.49	0.80	0.28	-1.62	1.36	0.96	-0.03	-0.11	0.96	0.31
2015	0.35	-1.28	1.44	2.39	0.18	-1.40	0.67	0.63	-0.25	-1.99	0.70	-0.21	2.40	7.41	0.72	0.40	0.13	-1.70	-0.13	-1.70
2016	-0.41	-1.24	1.54	3.37	-0.57	0.27	-0.27	-0.51	0.21	-1.08	-0.88	0.20	-0.28	-1.32	-0.58	0.66	-0.18	-1.95	2.52	7.05
2017	0.22	-0.13	1.28	0.90	0.72	-0.10	-0.45	-0.68	0.54	-0.33	0.19	-0.06	0.87	-0.44	0.45	0.41	0.41	-0.10	0.39	-1.63
2018	0.43	-0.18	0.23	0.57	-0.82	1.03	1.69	2.00	2.46	7.21	-1.55	0.95	-0.12	-0.78	-1.04	1.94	-0.51	0.55	1.63	1.47

As time increased, the probability of survival declined as observed in the plots above. Generally, the firms considered in this research are highly-rated entities by the credit agencies, and as such, the probability that each of them survives still remained high. However, this was in contrast to the Xerox firm, denoted by Xe, as a sharp descent was seen in its tendency to survive as time goes beyond 2018.

Next, we output the correlation matrix for the reference entity pairs on our dataset using Kendall's τ, a rank correlation measure which is based on the concept of concordance and the discordance measures. For $B = (m - \tilde{m})(n - \tilde{n})$ and according to Agresti (2010), Kendall's τ is given by

$$\tau = \mathbb{P}(B > 0) - \mathbb{P}(B < 0)\,,$$

where the pairs $(m, \tilde{m})$ and $(n, \tilde{n})$ following from the joint distribution function $F(m, n)$ are concordant supposing $m > \tilde{m}$ and $n > \tilde{n}$. Furthermore, for each pairs (m_i, n_i) and (m_j, n_j), let $m_{ij} = \operatorname{sgn}(m_i - m_j)$ and $n_{ij} = \operatorname{sgn}(n_i - n_j)$, then the Kendall's τ is[5]

$$\tau = \frac{2}{N(N-1)} \sum_{\substack{i=1 \\ i \neq j}}^{N} \sum_{i<j}^{N} m_{ij} n_{ij}\,.$$

We thus implemented `data.corr(method ='kendall')` in ipython notebook, where data is the file, in order to output the correlations with respect to any two data sets. Using Kendall's τ, we presented the summary of the Kendall's τ for our reference entities in Table 5.2 below:

[5]The sign or the signum function, also referred to as $\operatorname{sgn}(m)$ is defined as

$$\operatorname{sgn}(m) = \frac{\mathrm{d}}{\mathrm{d}m}|m|, \quad m \neq 0 \qquad \text{OR} \qquad \operatorname{sgn}(m) = \begin{cases} 1 & \text{if } m > 0 \\ -1 & \text{if } m < 1 \\ 0 & \text{if } m = 0\,. \end{cases}$$

Table 5.2: Kendall's τ correlation matrix for pairs of entities

Kendall's Tau (τ)	Xe	Co	Bo	IB	JJ	Or	Pe	Mc	Wa	AT
Xe	1.0000	0.4237	0.3986	0.2284	0.3680	0.4428	0.1793	0.1522	0.4645	0.2738
Co	0.4237	1.0000	0.6193	0.1047	0.6395	0.4772	0.4641	0.2490	0.5742	0.0407
Bo	0.3986	0.6193	1.0000	0.0862	0.6430	0.4867	0.4756	0.1760	0.6026	-0.0703
IB	0.2284	0.1047	0.0862	1.0000	0.0780	0.1411	0.2625	0.3698	0.1466	0.2466
JJ	0.3680	0.6395	0.6430	0.0780	1.0000	0.3569	0.3764	0.2766	0.6460	0.0322
Or	0.4428	0.4772	0.4867	0.1411	0.3568	1.0000	0.3322	0.0708	0.4147	0.2160
Pe	0.1793	0.4641	0.4756	0.2625	0.3764	0.3322	1.0000	0.2530	0.4046	-0.0466
Mc	0.1522	0.2490	0.1760	0.3698	0.2766	0.0708	0.2530	1.0000	0.3135	0.1788
Wa	0.4645	0.5742	0.6026	0.1466	0.6460	0.41470	0.4046	0.3135	1.0000	0.1257
AT	0.2738	0.0407	-0.0703	0.2466	0.0322	0.2160	-0.0466	0.1788	0.1257	1.0000

As can be observed, there are strong positive correlations which exist between firms Co, Bo, JJ and Wa. There also exist weak negative correlations between firms Bo, Pe versus AT, whereas the remaining are mostly weak positive correlations.

5.5.2 Swap Premiums using DFT

Gaussian Copula and DFT: In the computation of the n2D basket swap, we first compute the convolution of the corresponding vectors using the DFT techniques, the unconditional probabilities of default, the probabilities of having at least n defaults, and finally, the estimated swap prices. The following parameters $\lambda = 0.01, \rho = 0.35, N = 10, T = 1, \cdots, 5$ are utilized for Tables 5.3, 5.4 and Figure 5.3. Also, additional parameters of $R = 0.4, \Delta t = 1$ (that is, the frequency of the swap premium payment and $\Delta t = 1$ implies annual payment), $r = 0.06$ for Table 5.5, and the following results were obtained:

Table 5.3: Gaussian - Convolution of probabilities with varying time

$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
$5.3660\mathcal{E}^{-2}$	$5.3122\mathcal{E}^{-2}$	$5.2460\mathcal{E}^{-2}$	$5.1704\mathcal{E}^{-2}$	$5.0871\mathcal{E}^{-2}$
$3.2922\mathcal{E}^{-4}$	$8.6175\mathcal{E}^{-4}$	$1.5108\mathcal{E}^{-3}$	$2.2430\mathcal{E}^{-3}$	$3.0368\mathcal{E}^{-3}$
$9.0895\mathcal{E}^{-7}$	$6.2907\mathcal{E}^{-6}$	$1.9580\mathcal{E}^{-5}$	$4.3789\mathcal{E}^{-5}$	$8.1580\mathcal{E}^{-5}$
$1.4897\mathcal{E}^{-9}$	$2.7212\mathcal{E}^{-8}$	$1.5037\mathcal{E}^{-7}$	$5.0658\mathcal{E}^{-7}$	$1.2987\mathcal{E}^{-6}$
$1.5967\mathcal{E}^{-12}$	$7.7251\mathcal{E}^{-11}$	$2.6191\mathcal{E}^{-10}$	$3.8459\mathcal{E}^{-9}$	$1.3567\mathcal{E}^{-8}$
$1.1866\mathcal{E}^{-15}$	$1.5038\mathcal{E}^{-13}$	$2.6191\mathcal{E}^{-12}$	$2.0021\mathcal{E}^{-11}$	$9.7188\mathcal{E}^{-11}$
$1.1686\mathcal{E}^{-17}$	$2.0605\mathcal{E}^{-16}$	$6.2899\mathcal{E}^{-15}$	$7.2382\mathcal{E}^{-14}$	$4.8348\mathcal{E}^{-13}$
$-6.2582\mathcal{E}^{-18}$	$-6.4455\mathcal{E}^{-18}$	$1.0328\mathcal{E}^{-17}$	$1.8424\mathcal{E}^{-16}$	$1.6484\mathcal{E}^{-15}$
$-1.8498\mathcal{E}^{-18}$	$7.8333\mathcal{E}^{-19}$	$3.0567\mathcal{E}^{-18}$	$-3.0652\mathcal{E}^{-18}$	$9.5703\mathcal{E}^{-18}$
$-8.7018\mathcal{E}^{-18}$	$-6.0027\mathcal{E}^{-19}$	$8.2505\mathcal{E}^{-18}$	$-7.3395\mathcal{E}^{-18}$	$-4.6489\mathcal{E}^{-18}$
$5.3130\mathcal{E}^{-18}$	$-3.8145\mathcal{E}^{-18}$	$-5.1768\mathcal{E}^{-18}$	$9.8087\mathcal{E}^{-18}$	$8.1739\mathcal{E}^{-19}$

Table 5.3 outputs the probability vectors of obtaining α-defaults, such that $\alpha = 0, 1, \cdots, N$, which were conditioned on the latent variable X. Note that $A\mathcal{E}^{-a} = A \times 10^{-a}$. The FT approach played a unique role in this result outputs, as both the FFT and the IFFT were employed in the solution of the DFT problem set. Generally, on the positive probability scale, we observed from the table that the conditional probability vectors reduced as N increased, and this monotone decrease was altered when the negative scale was reached. For instance, the conditional probability of one default is normally larger when compared to having more than one defaults. Concerning time, the conditional probability vectors decreased with an increase in time if no default was considered, whereas, for defaults greater than 0 and up to 7, the probability values increased steadily with respect to time. For defaults greater than 7, we observed a haphazard movement of the probability vectors which could be as a result of the Gaussian model and from the fact that the latent variable was drawn from the whole real line $\mathbb{R}$.

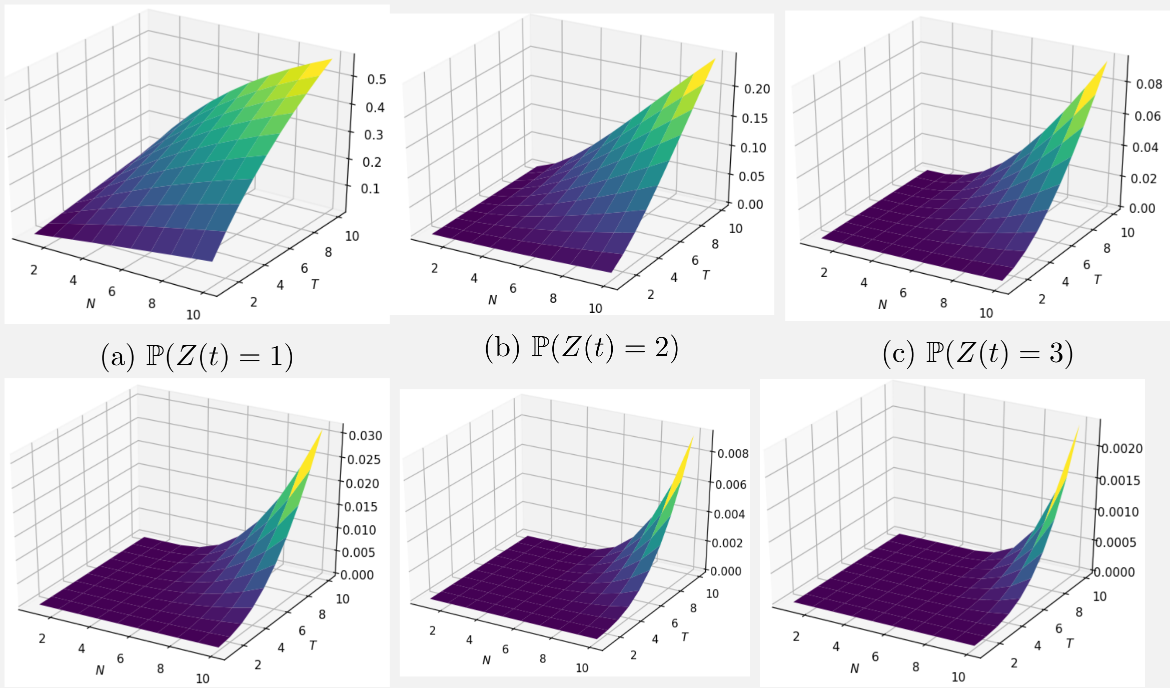
Next, we seek to compute the corresponding unconditional probabilities of default or the marginal probabilities, and these are obtained by numerical integration over the factors distribution, thereby integrating out the dependency property on the conditional variable. In this instance, we truncate the domain of the normal distribution to be $[-7, 7]$, and the following values give the probabilities:

Table 5.4: Gaussian - Unconditional probabilities of defaults with varying time

$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
$9.0940\mathcal{E}^{-1}$	$8.3114\mathcal{E}^{-1}$	$7.6194\mathcal{E}^{-1}$	$7.0009\mathcal{E}^{-1}$	$6.4445\mathcal{E}^{-1}$
$8.2542\mathcal{E}^{-2}$	$1.4388\mathcal{E}^{-1}$	$1.9103\mathcal{E}^{-1}$	$2.2750\mathcal{E}^{-1}$	$2.5560\mathcal{E}^{-1}$
$7.3046\mathcal{E}^{-3}$	$2.1390\mathcal{E}^{-2}$	$3.8378\mathcal{E}^{-2}$	$5.6549\mathcal{E}^{-2}$	$7.4922\mathcal{E}^{-2}$
$6.8320\mathcal{E}^{-4}$	$3.0885\mathcal{E}^{-3}$	$7.1520\mathcal{E}^{-3}$	$1.2642\mathcal{E}^{-2}$	$1.9307\mathcal{E}^{-2}$
$6.6161\mathcal{E}^{-5}$	$4.3551\mathcal{E}^{-4}$	$1.2609\mathcal{E}^{-3}$	$2.6173\mathcal{E}^{-3}$	$4.5344\mathcal{E}^{-3}$
$6.3904\mathcal{E}^{-6}$	$5.8752\mathcal{E}^{-5}$	$2.0788\mathcal{E}^{-4}$	$4.9902\mathcal{E}^{-4}$	$9.6970\mathcal{E}^{-4}$
$5.8942\mathcal{E}^{-7}$	$7.3413\mathcal{E}^{-6}$	$3.1224\mathcal{E}^{-5}$	$8.5716\mathcal{E}^{-5}$	$1.8529\mathcal{E}^{-4}$
$4.9250\mathcal{E}^{-8}$	$8.1241\mathcal{E}^{-7}$	$4.1028\mathcal{E}^{-6}$	$1.2776\mathcal{E}^{-5}$	$3.0535\mathcal{E}^{-5}$
$3.4646\mathcal{E}^{-9}$	$7.4428\mathcal{E}^{-8}$	$4.4232\mathcal{E}^{-7}$	$1.5530\mathcal{E}^{-6}$	$4.0864\mathcal{E}^{-6}$
$1.8130\mathcal{E}^{-10}$	$5.0100\mathcal{E}^{-9}$	$3.4811\mathcal{E}^{-8}$	$1.3724\mathcal{E}^{-7}$	$3.9641\mathcal{E}^{-7}$
$5.2713\mathcal{E}^{-12}$	$1.8581\mathcal{E}^{-10}$	$1.5029\mathcal{E}^{-9}$	$6.6358\mathcal{E}^{-9}$	$2.1003\mathcal{E}^{-8}$

Table 5.4 outputs the unconditional probability of having n defaults using numerical integration, and we used the `integrate.quad` function found in the `scipy` library to execute the integral. The probabilities became more uniform upon the removal of the dependency structure. For zero defaults, the unconditional probabilities decreased with an increase in time and increased with an increase in time when the defaults were bigger than zero. Also, a negative correlation between the number of defaults and the probabilities was observed, notwithstanding the time value.

Furthermore, we applied the concept of the counting process to estimate the probabilities of having less than n entities defaulting at a specified time t_i in a given portfolio. This is denoted by $\mathbb{P}(Z(t_i) < n)$ and the following Figure 5.3 showed the probabilities when the $n = 1, 2, 3, 4, 5, 6$ out of 10 entities. We observed that the probability of having a minimum of n defaults was generally greater than that of the $n + 1$ defaults. This was due to the fact that it took more time for $n + 1$ reference entities to default in comparison to n entities defaulting. The default probability equally increased with an increase in time since there was a greater likelihood of credit events amongst the entities in a given basket.



(a) $\mathbb{P}(Z(t) = 1)$ (b) $\mathbb{P}(Z(t) = 2)$ (c) $\mathbb{P}(Z(t) = 3)$

(d) $\mathbb{P}(Z(t) = 4)$ (e) $\mathbb{P}(Z(t) = 5)$ (f) $\mathbb{P}(Z(t) = 6)$

Figure 5.3: Gaussian - Probability of having n defaults against time T and N entities

Having computed the above steps, that is, the conditional probabilities of default with respect to the Gaussian density function, the unconditional probabilities of default, and the probabilities of having at least n entities defaulting, it suffices to incorporate all these in the valuation of the n2D basket swaps.

Table 5.5: Comparison of n2D BDS prices using FT under the one-factor Gaussian copula

Rank	n2D swap premium via Fourier Transform						
	$\lambda = 0.01$	$\lambda = 0.015$	$\lambda = 0.02$	$\lambda = 0.025$	$\lambda = 0.03$	$\lambda = 0.035$	$\lambda = 0.04$
1	553.1536	826.3604	1105.7815	1393.6095	1691.2408	1999.6699	2319.6655
2	164.6480	285.4171	415.1819	550.7922	690.7409	834.2651	980.9814
3	54.3331	112.7715	183.2751	262.0463	346.6463	435.4700	527.4395
4	17.0634	42.9693	79.2484	123.9775	175.4164	232.1254	292.9536
5	4.8727	14.9355	31.5276	54.5161	83.3601	117.3604	155.7923
6	1.2279	4.5737	11.0928	21.3393	35.5340	53.6560	75.5247
7	0.2632	1.1874	3.3139	7.1201	12.9823	21.1591	31.7966
8	0.0452	0.2466	0.7921	1.9056	3.8195	6.7525	10.8946
9	0.0056	0.0367	0.1357	0.3663	0.8079	1.5590	2.7135
10	0.0004	0.0030	0.0126	0.0383	0.0937	0.1972	0.3719

In Table 5.5, we output the fair spread values under the Gaussian copula model solved via the DFT approach, and we varied both the default intensity and the seniority of the portfolio. The hazard rates or the intensity defaults are indispensable variables in the valuation of n2D basket swap, as they measure the conditional probabilities of having no earlier default in any given year. The hazard rate increase led to a corresponding decrease in the probability of survival, and when the survival probability was significantly low, the chances of the portfolio having no default became extremely low. Hence this behaviour accounted for the presence of larger spread values and thus, both the spreads and the hazard rates were directly proportional in confirmation with the results of Jabbour et al. (2009). The rank, on the other hand, measures the seniority of the defaulting entity, as rank $= 1$ means the first entity to default in a portfolio of $N = 10$ entities. Thus, as the rank increased, the likelihood of experiencing a default became slimmer and thus giving rise to smaller swap spread.

Clayton Copula and DFT: We compared results using the Clayton copula in connection with the DFT techniques. This copula model arises from an asymmetric Archimedean family, with much dependence clustered in its negative tail than in their positive tail, and very useful for modelling correlated defaults. We employ the same techniques as used in the Gaussian copula model, but we now use the main probability distribution (Gamma Distribution) which describes the Clayton copula model. Contrary to the Gaussian correlation structure, here, we use the Clayton copula parameter $\theta \in (0, \infty)$ which measures the dependency level between the given variables. Using a parameter of $\theta = 0.193$, we output the probability vectors defined under the Clayton copula and the following results are given in Table 5.6:

Table 5.6: Clayton - Convolution of probabilities with varying time

$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
$6.4801\mathcal{E}^{-3}$	$3.8768\mathcal{E}^{-3}$	$2.4990\mathcal{E}^{-3}$	$1.6756\mathcal{E}^{-3}$	$1.1661\mathcal{E}^{-3}$
$3.8987\mathcal{E}^{-3}$	$4.4991\mathcal{E}^{-3}$	$4.1476\mathcal{E}^{-3}$	$3.5809\mathcal{E}^{-3}$	$3.0145\mathcal{E}^{-3}$
$1.0555\mathcal{E}^{-3}$	$2.3496\mathcal{E}^{-3}$	$3.1090\mathcal{E}^{-3}$	$3.4437\mathcal{E}^{-3}$	$3.5068\mathcal{E}^{-3}$
$1.6935\mathcal{E}^{-4}$	$7.2712\mathcal{E}^{-4}$	$1.3810\mathcal{E}^{-3}$	$1.9625\mathcal{E}^{-3}$	$2.4175\mathcal{E}^{-3}$
$1.7830\mathcal{E}^{-5}$	$1.4767\mathcal{E}^{-4}$	$4.0257\mathcal{E}^{-4}$	$7.3396\mathcal{E}^{-4}$	$1.0937\mathcal{E}^{-3}$
$1.2873\mathcal{E}^{-6}$	$2.0565\mathcal{E}^{-5}$	$8.0469\mathcal{E}^{-5}$	$1.8822\mathcal{E}^{-4}$	$3.3927\mathcal{E}^{-4}$
$6.4541\mathcal{E}^{-8}$	$1.9888\mathcal{E}^{-6}$	$1.1170\mathcal{E}^{-5}$	$3.3520\mathcal{E}^{-5}$	$7.3088\mathcal{E}^{-5}$
$2.2189\mathcal{E}^{-9}$	$1.3189\mathcal{E}^{-7}$	$1.0632\mathcal{E}^{-6}$	$4.0935\mathcal{E}^{-6}$	$1.0797\mathcal{E}^{-5}$
$5.0062\mathcal{E}^{-11}$	$5.7398\mathcal{E}^{-9}$	$6.6414\mathcal{E}^{-8}$	$3.2805\mathcal{E}^{-7}$	$1.0467\mathcal{E}^{-6}$
$6.6932\mathcal{E}^{-13}$	$1.4803\mathcal{E}^{-10}$	$2.4584\mathcal{E}^{-9}$	$1.5579\mathcal{E}^{-8}$	$6.0128\mathcal{E}^{-8}$
$4.0270\mathcal{E}^{-15}$	$1.7179\mathcal{E}^{-12}$	$4.0951\mathcal{E}^{-11}$	$3.3294\mathcal{E}^{-10}$	$1.5544\mathcal{E}^{-9}$

From Table 5.6, when there was no default, we observed a steady decrease in the values of the conditional probability vectors across time. This monotone decrease was altered when the default increased to one, and for 2 defaults up till 10, the probability vectors increased monotonically. These altered movements were as a result of the random component in the Gamma distribution function of the Clayton copula model. Additionally, the convolution of vectors reduced at time $= 1$ for all the number of defaults. From time $= 2$ till 5, we observed that the probability vectors increased at $n = 1$, and then reduced gradually when the number of defaults became greater than one.

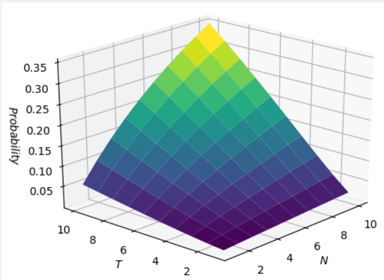
Next, the corresponding unconditional probabilities of default or the marginal probabilities were obtained by numerical integration over the factors distribution (Gamma distribution), thereby integrating out the dependency property on the conditional variable. From equation (5.3.6), the joint probability distribution function which served as our unconditional probability of default in this instance was obtained by integrating over the positive real line, that is, $\mathbb{R}_{>0} = (0, \infty)$, and the following values give the probabilities:

Table 5.7: Clayton - Unconditional probabilities of defaults with varying time

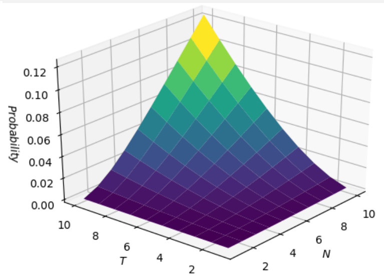
$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
$9.7835\mathcal{E}^{-1}$	$9.4475\mathcal{E}^{-1}$	$9.0761\mathcal{E}^{-1}$	$8.6921\mathcal{E}^{-1}$	$8.3062\mathcal{E}^{-1}$
$1.9830\mathcal{E}^{-2}$	$4.8995\mathcal{E}^{-2}$	$7.6774\mathcal{E}^{-2}$	$1.0440\mathcal{E}^{-1}$	$1.3019\mathcal{E}^{-1}$
$1.6222\mathcal{E}^{-3}$	$6.1101\mathcal{E}^{-3}$	$1.2613\mathcal{E}^{-2}$	$2.0569\mathcal{E}^{-2}$	$2.9577\mathcal{E}^{-2}$
$1.7854\mathcal{E}^{-4}$	$9.6272\mathcal{E}^{-4}$	$2.4317\mathcal{E}^{-3}$	$4.5705\mathcal{E}^{-3}$	$7.3400\mathcal{E}^{-3}$
$2.0347\mathcal{E}^{-5}$	$1.5406\mathcal{E}^{-4}$	$4.6942\mathcal{E}^{-4}$	$1.0050\mathcal{E}^{-3}$	$1.7839\mathcal{E}^{-3}$
$2.1372\mathcal{E}^{-6}$	$2.2716\mathcal{E}^{-5}$	$8.3362\mathcal{E}^{-5}$	$2.0285\mathcal{E}^{-4}$	$3.9715\mathcal{E}^{-4}$
$1.9194\mathcal{E}^{-7}$	$2.8810\mathcal{E}^{-6}$	$1.2777\mathcal{E}^{-5}$	$3.5414\mathcal{E}^{-5}$	$7.6567\mathcal{E}^{-5}$
$1.3785\mathcal{E}^{-8}$	$2.9456\mathcal{E}^{-7}$	$1.5869\mathcal{E}^{-6}$	$5.0285\mathcal{E}^{-6}$	$1.2040\mathcal{E}^{-5}$
$7.3106\mathcal{E}^{-10}$	$2.2416\mathcal{E}^{-8}$	$1.4751\mathcal{E}^{-7}$	$5.3669\mathcal{E}^{-7}$	$1.4281\mathcal{E}^{-7}$
$2.5210\mathcal{E}^{-11}$	$1.1175\mathcal{E}^{-9}$	$9.0305\mathcal{E}^{-9}$	$3.7882\mathcal{E}^{-8}$	$1.1241\mathcal{E}^{-7}$
$4.2145\mathcal{E}^{-13}$	$2.7192\mathcal{E}^{-11}$	$2.7113\mathcal{E}^{-10}$	$1.3164\mathcal{E}^{-9}$	$4.3706\mathcal{E}^{-9}$

In Table 5.7, we output the unconditional probability of defaults. The observed values decreased with an increase in time when there was no default but increased steadily when defaults occurred in the basket of the entities. Also, the values obtained for the Clayton copula were generally lesser when compared to the Gaussian copula, but only with the exception of the zero default values.

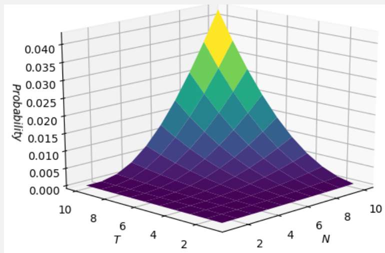
Next, we apply the concept of counting process in connection with the Clayton copula, to compute the probabilities of having less than n entities defaulting at a specified time t_i in a given portfolio. This is denoted by $\mathbb{P}(Z(t_i) < n)$ and the following figures give the probabilities when the $n = 1, 2, 3, 4, 5, 6$ out of 10 entities.



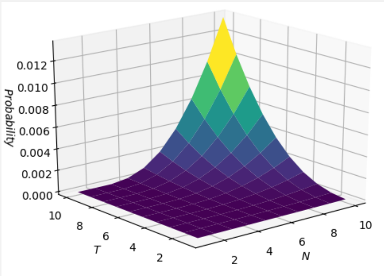
(a) $\mathbb{P}(Z(t) = 1)$



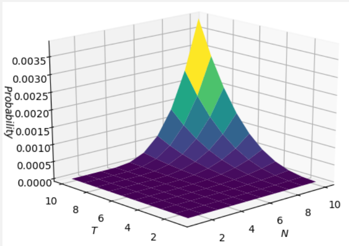
(b) $\mathbb{P}(Z(t) = 2)$



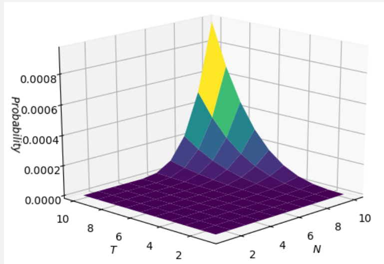
(c) $\mathbb{P}(Z(t) = 3)$



(d) $\mathbb{P}(Z(t) = 4)$



(e) $\mathbb{P}(Z(t) = 5)$



(f) $\mathbb{P}(Z(t) = 6)$

Figure 5.4: Clayton - Probability of having n defaults against time T and N entities

Figure 5.4 showed the results obtained, and we observed that the probabilities became extremely small in comparison with the Gaussian probabilities. This further asserts that as the number of at least n defaults goes up, the corresponding probabilities declines drastically.

Table 5.8: Comparison of n2D BDS prices using FT under the one-factor Clayton copula

Rank	n2D swap premium via Fourier Transform ($\theta = 0.1938$)						
	$\lambda = 0.01$	$\lambda = 0.015$	$\lambda = 0.02$	$\lambda = 0.025$	$\lambda = 0.03$	$\lambda = 0.035$	$\lambda = 0.04$
1	256.9671	417.5711	587.0253	763.7078	946.7921	1135.8055	1330.4752
2	74.0353	143.4878	225.2093	315.9714	413.6235	516.6302	623.8508
3	23.6040	54.1627	95.1930	145.0698	202.3589	265.8131	334.3513
4	7.2595	19.6489	38.7389	64.4075	96.2804	133.8511	176.5486
5	2.0313	6.4822	14.3657	26.1523	42.0887	62.2469	86.5653
6	0.4933	1.8606	4.6480	9.2969	16.1731	25.5603	37.6588
7	0.0986	0.4416	1.2483	2.7534	5.1971	8.8124	13.8145
8	0.0151	0.0806	0.2591	0.6329	1.3012	2.3762	3.9787
9	0.0016	0.0100	0.0367	0.0998	0.2245	0.4432	0.7955
10	0.0001	0.0006	0.0027	0.0081	0.0200	0.0427	0.0824

Table 5.8 gives the corresponding swap values with regards to varying intensity default and the rank of the portfolio default. The same characteristics were exhibited by the swap values in comparison with the Gaussian copula, but we observed that the Clayton swap values were significantly lesser than that of the Gaussian swap. This could result from the choice of the discrepancies between the θ-Clayton dependency and the ρ-Gaussian correlation structure. Furthermore, it could result from the the fact that the Clayton aims at modelling the extreme values for tail dependency which makes the values to be restricted in some sense, and finally the varied distribution functions of both copula families could equally account for the different spread values. However, the question still remains on which copula seems to be the best?, and the answer lies in the category of default events we are modelling. For instance, the Clayton copula remains the best in comparison to the Gaussian copula, especially the modelling of some extreme joint events in systematic risk, in the joint tail or the fat-tailed functions. For modelling events in the low tailed distributions or operational risk, or high-stress forecasting under credit risk scenarios, the Gaussian copula model outperforms the Clayton (Staudt (2010), Demarta and McNeil (2005), Koziol et al. (2015)).

5.6 Conclusion

The price estimation of the BCDS is linked directly to the computation of the joint probability of default, and this research successfully obtained such probability without simulation. Here, we employed the quasi-analytic techniques which involved a combination of the copula modelling and discrete Fourier transform to compute the probabilities of default, thereby connecting it to the price estimation of the swap payment stream. We introduced the concept of one-factor Gaussian copula, the student-t copula and the Clayton copula in describing the probability distribution functions. This concept further led to some semi-analytic expressions for the conditional and the unconditional portfolio loss distribution functions, and the corresponding expressions were solved efficiently with numerical integration via the discrete Fourier transform approach and its inverse. Data analysis with the inclusion of statistical and empirical analysis were equally conducted on the CDS spread quotes of 10 entities in order to estimate the correlation structures and the chances of survival of the basket entities.

6. Conclusion and Future Works

The overall theme of this research lies in the pricing of financial derivative instruments, especially the basket credit default swaps, which are financial contracts that payoff upon pre-defined default/s in a portfolio of credit assets. This thesis aims at computing the values of swaps which are written and defined on a portfolio of entities whose probability of default is estimated using copulas in incorporation with the Monte-Carlo simulation techniques and the discrete Fourier transform. The paramount issue connected in the pricing of these derivatives and modelling the level of their risk has to deal with the computation of the joint probability of default or their correlated default times. Thus, this thesis has investigated the valuations of these financial contracts in a numerical scheme, as well as in a quasi-analytical approach.

6.1 Concluding Summary and Recommendations

The major findings from this thesis, as presented in Chapters 3, 4 and 5 include the following:

- In the first part of Chapter 3, we conducted an investigative study into the idea of the BCDS valuation using the stochastic default intensity models. The hazard rate process, which is a function that measures the default probability for a given time interval conditioned on no prior default, was modelled using the one-factor Vasicek and the one-factor CIR models. The explicit formulas for the corresponding JSPD functions of both the homogeneous and the heterogeneous portfolios were derived. Applying the concept of the Monte-Carlo simulations and the one-factor Gaussian copula models, we estimated the default time for asset i in a basket of N entities. Furthermore, these default times were utilized in the estimation of the default legs, accrued premium legs, the premium legs, and finally the n-to-default basket prices. We equally considered and estimated the survival probabilities, the swap spreads and the equal-weighted portfolio values defined in the frameworks of the Vasicek model and the CIR model. The chapter ended by investigating the impacts of default intensities and default correlations on basket default swap prices.

  From the results section, it was observed that as the seniority of the default protection increased, a direct proportionality existed between the swap premium prices and the hazard rates. However, this was not the case with the default correlation, which played a huge role in the pricing, as we observed a non-uniform pattern in the n2D basket prices. Only for the F2D basket price did we observe a trend of inverse proportionality with regards to the correlation and the swap spread, and a non-convex trend for higher-order basket defaults with the exception of bigger intensity default for the second-to-default basket swap.

Secondly, from the data analysis part of this chapter, we employed the MLE concept to numerically estimate the default intensities of some selected corporate entities, using their OAS bid values. It was observed that firms that have higher credit ratings have a lesser intensity default in comparison with the lower-rated firms. The lower-rated firms accounted for higher bond spreads, which in turn, increased their likelihood of defaults. We further observed that as the higher-rated entities gradually become downgraded to lower credit ratings, the expected values of their hazard rates, together with their probabilities of default increased, thereby rendering them as high-risk entities.

Overall, we recommend the use of the CIR model in modelling the default intensities, as compared to the Vasicek counterpart because the former resulted in a more efficient basket pricing technique. Furthermore, we suggest that appropriate calculations for the swap spread sensitivities with respect to the corresponding parameters should be ensured in order to effectively hedge the credit risks evident in the basket default swaps. Lastly, financial investors who are interested in trading F2D basket swap with the highest premium amount should focus on selling protections on entities having low correlations, and vice versa.

- In Chapter 4, we reviewed different copula models which are important quantitative methods in finance. Here, we presented different algorithms for the computation of the default times, using elliptical (Gaussian and the Student-t) and some Archimedean copula models (Clayton, Frank and Gumbel), and then applied them to the price valuation of the basket credit default swaps. The MCS method was the criterion for the simulation of the random components associated with these copula models. Thus, the final swap premium, computed from the values of the swap payment legs, were ideally estimated, and this was followed by comparative sensitivity studies with regards to the default swap parameters.

  Furthermore, in order to ensure less ambiguous comparative studies, we considered a homogeneous portfolio and ensured that the swap pricing parameters are all constant, notwithstanding the type of copula models being used in the pricing process. The study equally assumed that the dependence structures, in accordance with the equivalent Kendall's tau, remain unchanged throughout the course of the n2D basket valuation. From our results, we inferred that the Gaussian copula (under the elliptical family) and the Gumbel copula (under the Archimedean family) performed best during the pricing process when their computation times are being used as the basis for the selection criteria.

To a large extent, the corresponding numerical experiments were conducted and results presented clearly depicted that the choice of a relevant copula model tremendously affected the portfolio's risk profile, and appropriate care must be ensured during the valuation process in order to avoid under- or over-pricing of these financial contracts. Thus, having a keen understanding of these studies can equip financial participants to hedge and speculate the direction of the credit derivatives market.

- In Chapter 5, we introduced the semi-analytic technique, which comprises of the combination of the copula models and the Fourier transform, in the valuation of the basket credit derivatives. The discrete Fourier transform was introduced in this pricing process because it has the capacity of overcoming the problems of the huge Monte-Carlo simulation numbers, as well as the ability to speed up the convergence analysis results. The first part of this chapter focused on some empirical and statistical analysis of the CDS data for some randomly selected firms from highly-rated to lower-rated categories. We computed the survival probabilities of all the entities in our dataset, as well as estimated the correlation coefficients pertinent in the portfolio of those entities.

  The latter aspect of the chapter focused on estimating the conditional probabilities of default, as well as the dependent defaults associated with the one-factor Clayton and one-factor Gaussian copula models, under the discrete Fourier transform approach. Here, we first employed the approach of the Fourier transform to estimate the conditional characteristic function of the loss distribution. Next, the unconditional characteristic function was retrieved from the conditional characteristic function via the techniques of the inverse fast Fourier transform using numerical integration. As a result, we derived a semi-analytic expression for the estimation of the default and the premium payment legs, as well as the pricing formula for the n2D basket swap premium. These formulas were tested numerically to get the explicit fair value of the n2D swap premium under both the Gaussian and the Clayton copula models.

  From the study, we observed that a quasi-analytic technique which comprises of the Fourier transform and the copula models could be utilized in modelling the default times, and the swap premiums correspondingly. Thus, in the absence of the random default time simulations, a technique involving the concept of the counting process and the portfolio loss distribution can be employed in effectively pricing of the basket credit derivatives.

6.2 Scope for Future Work

Future work under the BCDS valuation will be broadly classified into three components:

- In this current research, we considered the concept of credit risk, and in the future work, we shall focus on the case of counterparty risk on the BCDS pricing. Normally, when a counterparty A decides to buy a BCDS contract from counterparty B in order to protect the bonds issued on a basket of reference entities (say companies 1 to N) from credit events; credit risk, as referred to in this research, reflects the possibility that the companies will not be able to reimburse the bond values or their equivalent fully. The counterparty risk here reflects the financial risk that the counterparty B will be unable to repay the values of the bonds if the companies become bankrupt. Thus, future investigation could be carried out on the valuation of the BCDS following this assumption.
- Furthermore, more studies can be done on the extension of the semi-analytic techniques of BCDS pricing to other copula models, with much focus on the Archimedean class. In this work, we only considered the cases of the Gaussian copula and the Clayton copula model, in connection to the DFT techniques to obtain the quasi-analytic expressions of the swap price. This concept can be extended to other Archimedean copula families, and appropriate generalizations could be made based on this methodology.
- Part of this research modelled the intensity process and estimated the portfolio value under the CIR and the Vasicek model, and here, we assumed an equally-weighted portfolio value. More research can be focused on the optimal portfolio computation of the BCDS pricing, whilst maintaining the stochastic hazard rate modelling.

Bibliography

Aas, K. (2004). 'Modelling the dependence structure of financial assets: A survey of four copulas', *SAMBA* **22**(4), 18.

Abate, J. and Whitt, W. (1992). 'The Fourier-series method for inverting transforms of probability distributions', *Queueing Systems* **10**(1-2), 5–87.

Ackerer, D. and Vatter, T. (2017). 'Dependent defaults and losses with factor copula models', *Dependence Modeling* **5**(1), 375–399.

Agresti, A. (2010). *Analysis of Ordinal Categorical Data*, Vol. 656, John Wiley & Sons.

Andersen, L., Sidenius, J. and Basu, S. (2003). 'All your hedges in one basket', *Risk-London-Risk Magazine Limited* **16**(11), 67–72.

Bartle, R. G. (1976). *The Elements of Real Analysis*, Second Edition. John Wiley & Sons.

Bastide, D., Benhamou, E. and Ciucă, M. (2008). 'A comparative analysis of basket default swaps pricing using the Stein method.', *ICFAI Journal of Derivatives Markets* **5**(2).

Beirlant, J., Goegebeur, Y., Segers, J. and Teugels, J. L. (2004). *Statistics of Extremes: Theory and Applications*, Wiley Series in Probability and Statistics.

Bibby, B. M., Jacobsen, M. and Sørensen, M. (2010). Estimating functions for discretely sampled diffusion-type models, *in* 'Handbook of Financial Econometrics: Tools and Techniques', Elsevier, pp. 203–268.

Bielecki, T. R., Crépey, S., Jeanblanc, M. and Rutkowski, M. (2007). Valuation of basket credit derivatives in the credit migrations environment, *in* 'Handbooks in Operations Research and Management Science', Vol. 15, Elsevier, pp. 471–507.

Bielecki, T. R., Jeanblanc, M. and Rutkowski, M. (2004). Hedging of defaultable claims, *in* 'Paris-Princeton Lectures on Mathematical Finance 2003', Springer, Berlin, Heidelberg, pp. 1–132.

Björk, T. (2009). *Arbitrage Theory in Continuous Time*, Oxford University Press.

Bluhm, C. and Overbeck, L. (2006). *Structured Credit Portfolio Analysis, Baskets and CDOs*, Chapman and Hall/CRC.

Bluhm, C., Overbeck, L. and Wagner, C. (2002). *Introduction to Credit Risk Modeling*, Chapman and Hall/CRC.

Bomfim, A. N. (2015). *Understanding Credit Derivatives and Related Instruments*, Academic Press.

Bouyé, E., Durrleman, V., Nikeghbali, A., Riboulet, G. and Roncalli, T. (2000). 'Copulas for finance- A reading guide and some applications', *Available at SSRN 1032533* .

Brigo, D. and Mercurio, F. (2007). *Interest Rate Models-Theory and Practice: with Smile, Inflation and Credit*, Springer Science & Business Media.

Burtschell, X., Gregory, J. and Laurent, J. P. (2005). 'A comparative analysis of CDO pricing models', *Working Paper, BNP Paribas* .

Bush, N., Hambly, B. M., Haworth, H., Jin, L. and Reisinger, C. (2011). 'Stochastic evolution equations in portfolio credit modelling', *SIAM Journal on Financial Mathematics* **2**(1), 627–664.

Carr, P. and Madan, D. (1999). 'Option valuation using the fast Fourier transform', *Journal of Computational Finance* **2**(4), 61–73.

Černỳ, A. (2004). 'Introduction to fast Fourier transform in finance', *The Journal of Derivatives* **12**(1), 73–88.

Chan, K. C., Karolyi, G. A., Longstaff, F. A. and Sanders, A. B. (1992). 'An empirical comparison of alternative models of the short-term interest rate', *The Journal of Finance* **47**(3), 1209–1227.

Charpentier, A. (2003). Tail distribution and dependence measures, *in* 'Proceedings of the 34th ASTIN Conference', pp. 1–25.

Chen, Z. and Glasserman, P. (2008). 'Fast pricing of basket default swaps', *Operations Research* **56**(2), 286–303.

Cherubini, U., Della Lunga, G., Mulinacci, S. and Rossi, P. (2010). *Fourier Transform Methods in Finance*, Vol. 524, John Wiley & Sons.

Cherubini, U., Luciano, E. and Vecchiato, W. (2004). *Copula Methods in Finance*, John Wiley & Sons.

Choe, G. H. and Jang, H. J. (2011a). 'The k-th default time distribution and basket default swap pricing', *Quantitative Finance* **11**(12), 1793–1801.

Choe, G. H. and Jang, H. J. (2011b). 'Efficient algorithms for basket default swap pricing with multivariate Archimedean copulas', *Insurance: Mathematics and Economics* **48**(2), 205–213.

Choi, Y. and Wirjanto, T. S. (2007). 'An analytic approximation formula for pricing zero-coupon bonds', *Finance Research Letters* **4**(2), 116–126.

Cifuentes, A. and O'Connor, G. (1996). 'The binomial expansion method applied to CBO/-CLO analysis', *Moody's Investor Services, December* **13**.

Cooley, J. W. and Tukey, J. W. (1965). 'An algorithm for the machine calculation of complex Fourier series', *Mathematics of Computation* **19**(90), 297–301.

Cossin, D., Schellhorn, H., Song, N. and Tungsong, S. (2010). 'A theoretical argument why the t-copula explains credit risk contagion better than the Gaussian copula', *Advances in Decision Sciences* **2010**.

Cremers, M., Driessen, J., Maenhout, P. and Weinbaum, D. (2008). 'Individual stock-option prices and credit spreads', *Journal of Banking & Finance* **32**(12), 2706–2715.

Das, S. R. and Geng, G. (2003). 'Simulating correlated default processes using copulas: A criterion-based approach', *Working Paper, University of Santa Clara and Gifford Fong Associates* .

Davis, M. and Lo, V. (2001). 'Infectious defaults', *Quantitative Finance* **1**(4), 382–387.

Debuysscher, A., Szegö, M., Freydefront, M. and Tabe, H. (2003). 'The Fourier transform method–Technical document', *Working Report, Moody's Investors Service* .

Demarta, S. and McNeil, A. J. (2005). 'The t-copula and related copulas', *International Statistical Review* **73**(1), 111–129.

Devroye, L. (2006). 'Nonuniform random variate generation', *Handbooks in Operations Research and Management Science* **13**, 83–121.

Duffie, D. and Garleanu, N. (2001). 'Risk and valuation of collateralized debt obligations', *Financial Analysts Journal* **57**(1), 41–59.

Duffie, D. and Singleton, K. J. (2012). *Credit Risk: Pricing, Measurement, and Management*, Princeton University Press.

Durrleman, V., Nikeghbali, A. and Roncalli, T. (2000). 'Which copula is the right one?', *Available at SSRN 1032545* .

El Karoui, N. and Jiao, Y. (2009). 'Stein's method and zero bias transformation for CDO tranche pricing', *Finance and Stochastics* **13**(2), 151–180.

Elouerkhaoui, Y. (2017). *Credit Correlation: Theory and Practice*, Springer.

Embrechts, P., Lindskog, F. and McNeil, A. (2001). 'Modelling dependence with copulas and applications to risk management. Zürich'.

Embrechts, P., McNeil, A. and Straumann, D. (2002). 'Correlation and dependence in risk management: properties and pitfalls', *Risk Management: Value at Risk and Beyond* **1**, 176–223.

Fadel, S. M. (2010). Pricing basket of credit default swaps and collateralised debt obligation by Lévy linearly correlated, stochastically correlated, and randomly loaded factor copula models and evaluated by the fast and very fast Fourier transform, PhD thesis, University of Bradford.

Fang, Y., Madsen, L. and Liu, L. (2014). 'Comparison of two methods to check copula fitting.', *IAENG International Journal of Applied Mathematics* **44**(1), 53–61.

Fathi, A. and Nader, N. (2007a). 'Price calibration of basket default swap: Evidence from Japanese market', *Retrieved from MPRA-Munich Personal RePEc Archive* .

Fathi, A. and Nader, N. (2007b). 'Copula based simulation procedures for pricing basket credit derivatives', *Retrieved from MPRA-Munich Personal RePEc Archive* .

Flavell, R. R. (2010). *Swaps and Other Derivatives*, Vol. 480, John Wiley & Sons.

Focardi, S. M. and Fabozzi, F. J. (2004). *The Mathematics of Financial Modeling and Investment Management*, Vol. 138, John Wiley & Sons.

Frees, E. W. and Wang, P. (2005). 'Credibility using copulas', *North American Actuarial Journal* **9**(2), 31–48.

Frey, R. and Backhaus, J. (2008). 'Pricing and hedging of portfolio credit derivatives with interacting default intensities', *International Journal of Theoretical and Applied Finance* **11**(06), 611–634.

Frey, R. and McNeil, A. J. (2003). 'Dependent defaults in models of portfolio credit risk', *Journal of Risk* **6**, 59–92.

Fusai, G. (2004). 'Pricing Asian options via Fourier and Laplace transforms', *Journal of Computational Finance* **7**(3), 87–106.

Fusai, G. and Roncoroni, A. (2007). *Implementing Models in Quantitative Finance: Methods and Cases*, Springer Science & Business Media.

Galiani, S. (2003). 'Copula functions and their application in pricing and risk managing multiname credit derivative products', *University of London MSc Project* .

Genest, C. and Nešlehová, J. G. (2014). 'On tests of radial symmetry for bivariate copulas', *Statistical Papers* **55**(4), 1107–1119.

Glasserman, P. (2013). *Monte Carlo Methods in Financial Engineering*, Vol. 53, Springer Science & Business Media.

Goegebeur, Y., Hoedemakers, T. and Tistaert, J. (2007). Synthetic CDO pricing using the student t factor model with random recovery, *in* 'Third Brazilian Conference on Statistical Modelling in Insurance and Finance'.

Gregory, J. and Laurent, J. P. (2003). 'I will survive', *Risk-London-Risk Magazine Limited* **16**(6), 103–108.

Grønneberg, S. and Hjort, N. L. (2014). 'The copula information criteria', *Scandinavian Journal of Statistics* **41**(2), 436–459.

Gunawan, D., Khaled, M. A. and Kohn, R. (2017). 'Mixed marginal copula modeling', *ArXiv:1605.09101v3 [Stat.ME] 4 Sep 2017* .

Haworth, H. and Reisinger, C. (2007). 'Modeling basket credit default swaps with default contagion', *Journal of Credit Risk* **3**(4), 31–68.

Herbertsson, A. (2008). 'Pricing synthetic CDO tranches in a model with default contagion using the matrix-analytic approach', *Available at SSRN 962392* .

Hirsa, A. and Neftci, S. N. (2013). Pricing derivatives via Fourier transform technique, *in* ‘An Introduction to the Mathematics of Financial Derivatives’, pp. 361–372.

Hofert, M., Mächler, M. and Mcneil, A. J. (2012). ‘Likelihood inference for Archimedean copulas in high dimensions under known margins’, *Journal of Multivariate Analysis* **110**, 133–150.

Hull, J. C., Predescu, M. and White, A. (2005). ‘Bond prices, default probabilities and risk premiums’, *Default Probabilities and Risk Premiums (March 9, 2005)* .

Hull, J. C. and White, A. D. (2004). ‘Valuation of a CDO and an n-th to default CDS without Monte Carlo simulation’, *The Journal of Derivatives* **12**(2), 8–23.

Hyrš, M. and Schwarz, J. (2015). Elliptical and Archimedean copulas in estimation of distribution algorithm with model migration, *in* ‘2015 7th International Joint Conference on Computational Intelligence (IJCCI)’, Vol. 1, IEEE, pp. 212–219.

Iscoe, I. and Kreinin, A. (2006). ‘Recursive valuation of basket default swaps’, *Journal of Computational Finance* **9**(3), 95–116.

Jabbour, G. M., Kramin, M. V. and Young, S. D. (2009). ‘Nth-to-default swaps: valuation and analysis’, *Managerial Finance* **35**(1), 25–47.

Jäckel, P. (2002). *Monte Carlo Methods in Finance*, Vol. 71, John Wiley & Sons.

Jiao, Y., El Karoui, N. and Kurtz, D. (2008). ‘Gaussian and Poisson approximation: applications to CDOs tranche pricing’, *Journal of Computational Finance* **12**(2), 31–58.

Joe, H. (1997). *Multivariate Models and Dependence Concepts*, Springer-Science+Business Media, B.V.

Joshi, M. S. and Kainth, D. (2004). ‘Rapid and accurate development of prices and Greeks for nth to default credit swaps in the Li model’, *Quantitative Finance* **4**(3), 266–275.

Jouanin, J. F., Riboulet, G. and Roncalli, T. (2002). ‘Beyond conditionnally independent defaults’, *Available at SSRN 1032564* .

Kijima, M. (2000). ‘Valuation of a credit swap of the basket type’, *Review of Derivatives Research* **4**(1), 81–97.

Kijima, M. and Muromachi, Y. (2000). ‘Credit events and the valuation of credit derivatives of basket type’, *Review of Derivatives Research* **4**(1), 55–79.

Kilgore, R. T. and Thompson, D. B. (2011). 'Estimating joint flow probabilities at stream confluences by using copulas', *Transportation Research Record* **2262**(1), 200–206.

Kolev, N., Anjos, U. D. and Mendes, B. V. D. M. (2006). 'Copulas: a review and recent developments', *Stochastic Models* **22**(4), 617–660.

Koziol, P., Schell, C. and Eckhardt, M. (2015). Credit risk stress testing and copulas: Is the Gaussian copula better than its reputation?, Technical report, Deutsche Bundesbank.

Kroese, D. P., Taimre, T. and Botev, Z. I. (2013). *Handbook of Monte Carlo Methods*, Vol. 706, John Wiley & Sons.

Kumar, P. (2010). 'Probability distributions and estimation of Ali-Mikhail-Haq copula', *Applied Mathematical Sciences* **4**(14), 657–666.

Laurent, J. P. and Gregory, J. (2005). 'Basket default swaps, CDOs and factor copulas', *Journal of Risk* **7**(4), 103–122.

Li, D. X. (2000). 'On default correlation: A copula function approach', *The Journal of Fixed Income* **9**(4), 43–54.

Li, P., Liu, J., Zhang, X. and Huang, G. (2015). 'Pricing of basket default swaps based on factor copulas and NIG', *Procedia Computer Science* **55**, 566–574.

Liang, J., Ma, J. M., Wang, T. and Ji, Q. (2011). 'Valuation of portfolio credit derivatives with default intensities using the Vasicek model', *Asia-Pacific Financial Markets* **18**(1), 33–54.

Lin, J., Liang, G., Wu, S. and Zheng, H. (2011). 'The valuation of the basket CDS in a primary-subsidiary model', *Asia-Pacific Journal of Operational Research* **28**(02), 213–238.

Mashal, R. and Naldi, M. (2002). 'Pricing multiname credit derivatives: heavy tailed hybrid approach', *Available at SSRN 296402* .

Melchiori, M. R. (2006). 'Tools for sampling multivariate Archimedean copulas', *YieldCurve, April* .

Merino, S. and Nyfeler, M. (2002). 'Credit portfolio modelling calculating portfolio loss', *Risk-London-Risk Magazine Limited* **15**(8), 82–86.

Muroi, Y. (2006). *Pricing of Credit Derivatives with the Asymptotic Expansion Approach*, Institute for Monetary and Economic Studies, Bank of Japan.

Naifar, N. (2011). 'Modelling dependence structure with Archimedean copulas and applications to the itraxx CDS index', *Journal of Computational and Applied Mathematics* **235**(8), 2459–2466.

Nelsen, R. B. (2007). *An Introduction to Copulas*, Springer Science & Business Media.

Nolan, J. (2003). *Stable Distributions: Models for Heavy-Tailed Data*, Birkhauser New York.

O'Kane, D. (2011). *Modelling Single-Name and Multi-Name Credit Derivatives*, Vol. 573, John Wiley & Sons.

O'Kane, D. and Turnbull, S. (2003). 'Valuation of credit default swaps', *Lehman Brothers Quantitative Credit Research Quarterly* **2003**(Q1–Q2).

Parodi, P. (2014). *Pricing in General Insurance*, Chapman and Hall/CRC.

Patton, A. J. (2012). 'A review of copula models for economic time series', *Journal of Multivariate Analysis* **110**, 4–18.

Peng, L. and Qi, Y. (2017). *Inference for Heavy-Tailed Data: Applications in Insurance and Finance*, Academic Press.

Press, W. H., Teukolsky, S. A., Vetterling, W. T. and Flannery, B. P. (1992). *Numerical Recipes in C: The Art of Scientific Computing*, Cambridge University Press.

Press, W. H., Teukolsky, S. A., Vetterling, W. T. and Flannery, B. P. (2002). *Numerical Recipes in C++.*, Cambridge University Press.

Rule, D. (2001). 'The credit derivatives market: its development and possible implications for financial stability', *Financial Stability Review* **10**(June), 117–140.

Rüschendorf, L. (2013). 'Mathematical risk analysis', *Springer Series in Operations Research and Financial Engineering. Springer, Heidelberg* .

Scherer, M. and Mai, J. F. (2017). *Simulating Copulas: Stochastic Models, Sampling Algorithms, and Applications*, Vol. 4, World Scientific.

Schmidt, T. (2007). 'Coping with copulas', *Copulas-From Theory to Application in Finance* pp. 3–34.

Schönbucher, P. J. (2003). *Credit Derivatives Pricing Models: Models, Pricing and Implementation*, John Wiley & Sons.

Schönbucher, P. J. and Schubert, D. (2001). Copula-dependent default risk in intensity models, *in* 'Working paper', Department of Statistics, Bonn University.

Schröter, A. and Heider, P. (2013a). 'An analytical formula for pricing m-th to default swaps', *Journal of Applied Mathematics and Computing* **41**(1-2), 229–255.

Schröter, A. and Heider, P. (2013b). 'Numerical methods to quantify the model risk of basket default swaps', *Journal of Computational and Applied Mathematics* **251**, 117–132.

Shelton, D. (2004). 'Back to normal proxy integration: a fast accurate method for CDO and CDO-squared pricing', *Citigroup Structured Credit Research* .

Staudt, A. (2010). Tail risk, systemic risk and copulas, *in* 'Casualty Actuarial Society E-Forum', Vol. 2, pp. 1–23.

Trivedi, P. K. and Zimmer, D. M. (2007). 'Copula modeling: an introduction for practitioners', *Foundations and Trends® in Econometrics* **1**(1), 1–111.

Umeorah, N., Ehrhardt, M. and Mashele, P. (2019). Valuation of basket credit default swaps under stochastic default intensity models, *in* 'Preprint 19/17', University of Wuppertal.

Umeorah, N., Mashele, P. and Ehrhardt, M. (2019). Elliptical and Archimedean copula models: an application to the price estimation of portfolio credit derivatives, *in* 'Preprint 19/24', University of Wuppertal.

Venter, G. G. (2002). 'Tails of copulas', *Proceedings of the Casualty Actuarial Society* **89**(171), 68–113.

von Kleist, K., Mallo, C. and Groucho, S. (2010). 'Triennial and semiannual surveys-positions in global over-the-counter (OTC) derivatives markets at end-June 2010', *Bank for International Settlements, Monetary and Economic Department* .

Vose, D. (2008). *Risk Analysis: A Quantitative Guide*, John Wiley & Sons.

Watts, S. (2016). 'The Gaussian copula and the financial crisis: A recipe for disaster or cooking the books?', *Retrieved from http://samueldwatts.com/wp-content/uploads/2016/08/Watts-Gaussian-Copula_Financial_Crisis.pdf* .

Weistroffer, C. (2009). 'Credit default swaps: Heading towards a more stable system', *Deutsche Bank Research* .

Zheng, H. (2006). 'Efficient hybrid methods for portfolio credit derivatives', *Quantitative Finance* **6**(4), 349–357.

Zhou, C. (2001). 'An analysis of default correlations and multiple defaults', *The Review of Financial Studies* **14**(2), 555–576.

Appendix A. Algorithm for default time modelling

In this section which is fully applicable to Chapter 4, we describe the algorithms employed in the generation of the default times for all the copula models applied in this study. The Gaussian and the student-t copula models for generating the default times involve the use of the Cholesky decomposition of the specified correlation matrix Σ, so as to obtain a lower triangular matrix L, such that $\Sigma = LL^\top$. We employ the techniques of (Scherer and Mai 2017). For the generation of an α-stable distribution used in the Gumbel copula, we employ the methodologies proposed in both Melchiori (2006) and Nolan (2003). For generating random Logarithmic series utilized in the Frank copula, we used the `ipython numpy` inbuilt function `logseries(p)`, with the parameter $p = 1 - \mathrm{e}^{-\theta}$. Otherwise, consider the full algorithm proposed by Melchiori (2006). Here, random variables, which are logarithmically distributed were obtained from Kemp's second generator with acceleration. For further details, we refer to (Devroye 2006).

Algorithm 1 Pseudocode for simulating default time using Gaussian copula model

1: Obtain the Cholesky decomposition L of Σ, such that $\Sigma = LL^\top$.
2: Generate independent random variates $Z \sim \mathcal{N}(0,1)$, where $Z = (z_1, z_2, \dots, z_n)^\top$.
3: Compute $s_i = LZ$, where $s_i \sim \mathcal{N}(0, \Sigma)$.
4: Return the vector $X_i = \big(\psi(s_i)\big)^\top$, with ψ denoting the distribution function associated with univariate standard normal distribution.
5: Finally compute $\tau_i = F_i^{-1}(\Phi(s_i))$ with $X_i = \psi(s_i)$ in equation (4.4.1).

Algorithm 2 Pseudocode for simulating default time using Student-t copula model

1: Obtain the Cholesky decomposition L of Σ, such that $\Sigma = LL^\top$.
2: Generate independent random variates $Z \sim \mathcal{N}(0,1)$, where $Z = (z_1, z_2, \dots, z_n)^\top$.
3: Simulate an independent random variate $r_i \sim \chi^2(\beta)$, where β is the degrees of freedom.
4: Compute $s_i = \sqrt{\frac{\beta}{r_i}} LZ$.
5: Return the vector $X_i = \big(t_\beta(s_i)\big)^\top$, with t_β denoting the distribution function associated with univariate t-distribution, with zero mean.
6: Finally compute $\tau_i = F_i^{-1}(\Phi(s_i))$ with $X_i = \psi(s_i)$ in equation (4.4.1).

Algorithm 3 Pseudocode for simulating default time using Clayton copula model

1: Simulate a uniform random variable $k_i \sim \mathcal{U}([0,1])$.
2: Generate gamma random variables $M \sim \Gamma\left(\frac{1}{\theta}, 1\right)$.
3: Calculate X_i from the equation (4.4.2) and then, the default time from equation (4.4.1).

Algorithm 4 Pseudocode for simulating default time using Gumbel copula model

1: Set $\alpha = 1/\theta, \beta = 1, \gamma = (\cos(\pi/2\theta))^{\theta}$ and $\delta = 0$.
2: Simulate a uniform random variable $p \sim \mathcal{U}\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
3: Simulate Q from an exponential distribution, with mean 1. Both Q and p must be independent.
4: Set $\Psi = 1/\alpha(\arctan(\beta \tan(\pi\alpha)/2))$.
5: Compute $M \sim \text{Stable}(\alpha, \beta, \gamma, \delta)$. Here, $M = \gamma Y + \delta$, where

$$Y = \frac{\sin \alpha(\Psi + p)}{(\cos \alpha\Psi \cos p)^{\frac{1}{\alpha}}} \left[\frac{\cos(\alpha\Psi + (\alpha - 1)p}{Q}\right]^{\frac{1-\alpha}{\alpha}}, \qquad \text{since } \alpha \neq 1\,.$$

6: Finally, calculate the random variables X_i from equation (4.4.2) and then the default time τ_i follows from equations (4.4.1).

Algorithm 5 Pseudocode for simulating default time using Frank copula model

1: Simulate a uniform random variable $k_i \sim \mathcal{U}([0,1])$.
2: Generate log-series random variables $M \sim \text{Logarithmic}(1 - \mathrm{e}^{-\theta})$. Both M and k_i must be independent.
3: Calculate X_i from the equation (4.4.2) and then, the default time from equation (4.4.1).