

**NOETHER SYMMETRY
CLASSIFICATION AND REDUCTIONS
OF THE GENERALIZED
LANE-EMDEN EQUATION**

by

BEN MUATJETJEJA

Submitted in partial fulfillment of the requirements for the degree of
Master of Science in Applied Mathematics in the Department of
Mathematical Sciences in the Faculty of Agriculture, Science and
Technology at North West University, Mafikeng Campus

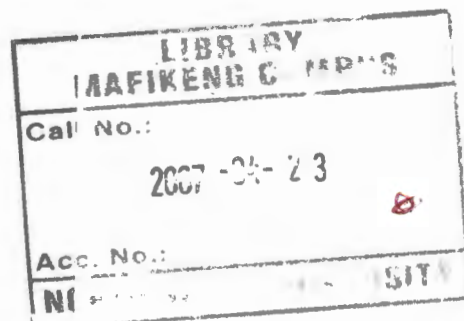
October 2006

Supervisor: Professor C M Khalique



060000758Q

North-West University
Mafikeng Campus Library



Contents

Declaration	3
Dedication	4
Acknowledgements	5
Abstract	6
Introduction	7
1 One-parameter transformation groups	10
1.1 Introduction	10
1.2 One-parameter groups on the plane	11
1.3 Infinitesimal transformations	12
1.4 Group invariants	13
1.5 Construction of a symmetry group	14
1.5.1 Prolongation of point transformations	14
1.5.2 Group admitted by an ODE	15
1.6 Lie Algebra	16
1.7 Conclusion	17
2 Noether point symmetries and first integrals	18
2.1 First integrals, Noether theorem and double reductions	18

2.2	An illustrative example of double reduction using a Noether symmetry	21
2.3	Conclusion	23
3	Integration of the generalized Lane-Emden equation	24
3.1	Introduction	24
3.2	Classification of the generalised Lane-Emden equation for functions $f(y)$	24
3.3	Integration of GLEE for different $f(y)$'s	27
3.4	Conclusion	30
4	Concluding remarks	31
	Bibliography	32

Declaration

I declare that the dissertation for the degree of Master of Science at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other university, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.



BEN MUATJETJEJA

16 October 2006

Dedication

To my parents

Acknowledgements

I would like to express my sincere thanks to Professor C M Khaliq for his guidance, patience and support throughout this research project.

I would also like to thank North-West University, Mafikeng Campus for the much needed financial support during my MSc studies. Thanks are also due to AMMSI for their small bursary which I really appreciate.

Finally, thanks to my parents for their everlasting support.

Above all, I would like to thank the Almighty God, who made this programme successful.

Abstract

In this dissertation we perform a Noether point symmetry classification of the generalized Lane-Emden equation which models several phenomena in mathematical physics and astrophysics. We then obtain first integrals of the various cases which admit Noether point symmetries. Moreover, we obtain reduction to quadratures for the cases that admit Noether symmetries. Three new cases are found and these correspond to cases 4.2, 5.1 and 5.2 of chapter 3. We also obtain a two-parameter family of solutions for case 5.1 (when $r = 5$ and $n = 2$), whereas in the literature only one-parameter family of solutions is usually given.

Introduction

The Lane-Emden-type equations

$$\frac{d^2y}{dx^2} + \frac{2}{x} \frac{dy}{dx} + f(y) = 0, \quad (1)$$

for various forms of $f(y)$ have been used to model several phenomena in mathematical physics and astrophysics. The most widely studied form of f is when $f(y) = y^r$, where r is a constant. In this case eqn (1) takes the form

$$\frac{d^2y}{dx^2} + \frac{2}{x} \frac{dy}{dx} + y^r = 0. \quad (2)$$

This equation was first proposed by Lane (see Thompson [1]) and studied in more detail by Emden [2]. It was used to model the thermal behaviour of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics. Because of the many applications of eqn (2) researchers have used numerical and perturbation approaches to solve eqn (2). See, for example, the works of Horedt [3, 4], Bender [5] and Lima [6]. Other methods have also been used to solve eqn (2) (see Roxbough and Stocken [7], Adomian et al [8], Shawagfeh [9], Burt [10], Wazwaz [11] and Liao [12]). It is known that for $r = 0, 1$ and 5 , eqn (2) has exact solutions (Chandrasekhar [13], Davis [14], Datta [15] and Wrubel [16]). In fact for $r = 5$, a one-parameter family of solutions of eqn (2) is normally given. The Lane-Emden equation (2) appears not only in the study of stellar structure but in other applications as well. The interested reader is referred to the works of Meerson et al [17], Gnutzmann and Ritschel [18] and Bahcall [19, 20].

A more general form of (2), in which the coefficient of y' is considered an arbitrary

function of x , is investigated for first integrals by Leach [21].

Another form of $f(y)$ is given by

$$f(y) = (y^2 - C)^{3/2}. \quad (3)$$

Inserting this value of f into eqn (1) gives us the "white-dwarf" equation which was introduced by Chandrasekhar [13] in his study of the gravitational potential of the degenerate white-dwarf stars. It is interesting to note that when $C = 0$ this equation reduces to Lane-Emden equation with index $r = 3$.

Another nonlinear form of $f(y)$ is the exponential function

$$f(y) = e^y. \quad (4)$$

Substituting $f(y) = e^y$ into eqn (1) results in a model that describes isothermal gas spheres where the temperature remains constant.

Eqn (1) with

$$f(y) = e^{-y}$$

gives a model that appears in the theory of thermionic currents when one seeks to determine the density and electric force of an electron gas in the neighbourhood of a hot body in thermal equilibrium, thoroughly investigated by Richardson [22].

Furthermore, the function $f(y)$ appears in eight additional cases. The interested reader is referred to Davis [14] for more details.

The so-called generalized Lane-Emden equation of the first kind

$$x \frac{d^2 y}{dx^2} + \alpha \frac{dy}{dx} + \beta x^\nu y^n = 0, \quad (5)$$

where α , β , ν and n are real, have been recently studied in Goenner and Havas [23] and Goenner [24]. In Goenner [24], the author uncovers symmetries of eqn (5) to explain integrability of (5) for certain values of the parameters considered in Goenner and Havas [23].

In this research project we investigate the Noether point symmetries of the generalized Lane-Emden equation

$$\frac{d^2 y}{dx^2} + \frac{n}{x} \frac{dy}{dx} + f(y) = 0, \quad (6)$$

where n is a real constant. For $n = 2$, Wazwaz [11] considered eqn (6) for some particular functions $f(y)$ using an algorithm based on the Adomian decomposition method. Our approach is completely different and provides a Noether point symmetry classification of eqn (6) for different forms of $f(y)$. Moreover, we also give reductions of eqn (6) in terms of quadratures.

In more detail, the outline of the research project is as follows:

In chapter one, we recall the basic definitions and theorems on the one-parameter groups of transformations and present the notation that we will use in this project.

Chapter two deals with the preliminaries of the Noether point symmetry approach.

In chapter three we provide the Noether point symmetry classification of eqn (6) for various functions $f(y)$. Then in the same chapter, we determine the double reductions of eqn (6) for the functions for which eqn (6) has Noether point symmetries. The results of this chapter are new and have been submitted for publication. See Khalique et al [25].

Lastly, in chapter four, we summarize the results of our dissertation.

Bibliography is given at the end.

Chapter 1

One-parameter transformation groups

In this chapter we present some basic results from the theory of Lie group analysis of the second-order ordinary differential equations that will be needed for this project.

1.1 Introduction

Lie group analysis was developed in the 1870s by an outstanding mathematician of the nineteenth century Sophus Lie (1842-1899). He showed that the majority of known methods of integration of ordinary differential equations, which until then had seemed artificial, could be derived in a unified manner using his *theory of continuous transformation groups*. In particular, Lie reduced the classical four hundred types of ordinary differential equations to four types only. One can safely say that Lie group analysis is the only universal and effective method for solving nonlinear differential equations analytically.

There exists a vast literature on Lie's symmetry theory. See for example, Lie [26], Ovsiannikov [27], Olver [28], Bluman and Kumei [29], Stephani [30], Ibragimov [31, 32, 33, 34, 35], Hydon [36], Cantwell [37] and the paper by Mahomed [38]. Most of the calculations involved in the Lie's theory can also be performed using symbolic

computer packages. See, for example, Baumann [39], Head [40, 41], Sherring and Head [42], Diaz [43].

1.2 One-parameter groups on the plane

Let x be the independent variable and y be the dependent variable. Consider a change of the variables x and y :

$$T_a : \bar{x} = f(x, y, a), \bar{y} = g(x, y, a), \quad (1.1)$$

where a is a real parameter which continuously ranges in values from a neighbourhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$ and f and g are differentiable functions.

Definition 1.1 A set G of transformations (1.1) is called a *continuous one-parameter (local) Lie group of transformations* in the space of variables x and y if

- (i) For $T_b, T_a \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then $T_b T_a = T_c \in G$, $c = \phi(a, b) \in \mathcal{D}$
(Closure)
- (ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0 T_a = T_a T_0 = T_a$ (Identity)
- (iii) For $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that
 $T_a T_{a^{-1}} = T_{a^{-1}} T_a = T_0$ (Inverse)

The associativity property follows from (i). Also, if the identity transformation occurs at $a = a_0 \neq 0$, i.e., T_{a_0} is the identity, then a shift of the parameter $a = \bar{a} + a_0$ will give T_0 as above. The group property (i) can be written as

$$\begin{aligned} \bar{x} &\equiv f(\bar{x}, \bar{y}, b) = f(x, y, \phi(a, b)), \\ \bar{y} &\equiv g(\bar{x}, \bar{y}, b) = g(x, y, \phi(a, b)). \end{aligned} \quad (1.2)$$

The function ϕ is termed as the *group composition law*. A group parameter a is called *canonical* if $\phi(a, b) = a + b$.

Theorem 1.1 For any $\phi(a, b)$, there exists the canonical parameter $\tilde{a}$ defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)}, \text{ where } w(s) = \left. \frac{\partial \phi(s, b)}{\partial b} \right|_{b=0}.$$

We now give the definition of a symmetry group for ODE by considering the second-order ordinary differential equation, namely

$$y'' = F(x, y, y'). \quad (1.3)$$

Definition 1.2 (Symmetry group) A one-parameter group G of transformations (1.1) is called a symmetry group of equation (1.3) if (1.3) is form-invariant (has the same form) in the new variables $\bar{x}$ and $\bar{y}$, i.e.,

$$\bar{y}'' = F(\bar{x}, \bar{y}, \bar{y}') \quad (1.4)$$

where the function F is the same as in equation (1.3).

1.3 Infinitesimal transformations

According to the Lie's theory, the construction of the symmetry group G is equivalent to the determination of the corresponding *infinitesimal transformations* :

$$\bar{x} \approx x + a \xi(x, y), \quad \bar{y} \approx y + a \eta(x, y) \quad (1.5)$$

obtained from (1.1) by expanding the functions f and g into Taylor series in a about $a = 0$ and also taking into account the initial conditions

$$f|_{a=0} = x, \quad g|_{a=0} = y.$$

Thus, we have

$$\xi(x, y) = \left. \frac{\partial f}{\partial a} \right|_{a=0}, \quad \eta(x, y) = \left. \frac{\partial g}{\partial a} \right|_{a=0}. \quad (1.6)$$

The vector (ξ, η) with components (1.6) is the tangent vector at the point (x, y) to the curve described by the transformed point $(\bar{x}, \bar{y})$ and is termed *tangent vector field* of the group G .

One can now introduce the *symbol* of the infinitesimal transformations by writing (1.5) as

$$\bar{x} \approx (1 + aX)x, \quad \bar{y} \approx (1 + aX)y,$$

where

$$X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}. \quad (1.7)$$

This first-order differential operator X behaves as a *scalar* under an arbitrary change of variables, unlike the *vector* (ξ, η) . It is known as the *symbol* (or *infinitesimal generator*) of the infinitesimal transformation (1.5) or the corresponding group G .

If the group G is admitted by (1.3), we say that X is an *admitted operator* of (1.3) or X is an *infinitesimal symmetry* of the equation (1.3).

1.4 Group invariants

Definition 1.3 A function $F(x, y)$ is called an *invariant of the group of transformations* (1.1) if

$$F(\bar{x}, \bar{y}) \equiv F(f(x, y, a), g(x, y, a)) = F(x, y) \quad (1.8)$$

identically in x, y and for all values of the group parameter a .

Theorem 1.2 (Infinitesimal criterion of invariance) A function $F(x, y)$ is an invariant if and only if it satisfies the PDE

$$X F \equiv \xi(x, y) \frac{\partial F}{\partial x} + \eta(x, y) \frac{\partial F}{\partial y} = 0. \quad (1.9)$$

It follows from the above theorem that every one-parameter group of point transformations (1.1) in the plane has one independent invariant, which can be taken to be the left-hand side of any first integral

$$J(x, y) = C$$

of the characteristic equation

$$\frac{dx}{\xi(x, y)} = \frac{dy}{\eta(x, y)}.$$

Theorem 1.3 (Lie equations) Given an infinitesimal transformation (1.5) or its symbol X , the transformations (1.1) of the corresponding one-parameter group G are obtained by solving the Lie equations

$$\frac{d\bar{x}}{da} = \xi(\bar{x}, \bar{y}), \quad \frac{d\bar{y}}{da} = \eta(\bar{x}, \bar{y}) \quad (1.10)$$

subject to the initial conditions

$$\bar{x}|_{a=0} = x, \quad \bar{y}|_{a=0} = y.$$

1.5 Construction of a symmetry group

In this section we briefly describe the algorithm to determine a symmetry group for a given ODE. First we need to recall some basic definitions.

1.5.1 Prolongation of point transformations

Consider a second-order ODE

$$E(x, y, y', y'') = 0 \quad (1.11)$$

where x is an independent variable and y is the dependent variable. Let

$$X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}, \quad (1.12)$$

be the infinitesimal generator of the one-parameter group G of transformations (1.1). The *first prolongation* of the group G is denoted by $G^{[1]}$ and the symbol of $G^{[1]}$ is given by

$$X^{[1]} = X + \zeta_1(x, y, y') \frac{\partial}{\partial y'}, \quad (1.13)$$

where

$$\zeta_1 = D(\eta) - y'D(\xi)$$

and the *total differential operator* D is given by

$$D = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + \dots \quad (1.14)$$

Likewise, the symbol of the *second prolongation* of the group G is given by

$$X^{[2]} = X + \zeta_1(x, y, y') \frac{\partial}{\partial y'} + \zeta_2(x, y, y', y'') \frac{\partial}{\partial y''}, \quad (1.15)$$

where

$$\zeta_2 = D(\zeta_1) - y''D(\xi).$$

Using the definition of the total differential operator D , one arrives at

$$\zeta_1 = \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2, \quad (1.16)$$

$$\zeta_2 = \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 + (\eta_y - 2\xi_x - 3\xi_y y')y''. \quad (1.17)$$

1.5.2 Group admitted by an ODE

The operator

$$X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} \quad (1.18)$$

is said to be a (generator) *point symmetry* of the second-order ODE

$$E(x, y, y', y'') = 0 \quad (1.19)$$

if

$$X^{[2]}(E) = 0 \quad (1.20)$$

whenever $E = 0$. This can also be written as (symmetry condition)

$$X^{[2]} E|_{E=0} = 0 \quad (1.21)$$

where the symbol $|_{E=0}$ means evaluated on the equation $E = 0$.

Definition 1.4 Equation (1.21) is called the *determining equation* of (1.19), because it determines all the infinitesimal symmetries of equation (1.19).

The theorem below enables us to construct some solutions of (1.19) from known ones.

Theorem 1.4 A symmetry of equation (1.19) transforms any solution of (1.19) into another solution of the same equation.

Proof : It follows from the fact that a symmetry of an equation leaves invariant that equation.

1.6 Lie Algebra

Let X_1 and X_2 be any two operators defined by

$$X_1 = \xi_1(x, y) \frac{\partial}{\partial x} + \eta_1(x, y) \frac{\partial}{\partial y}$$

and

$$X_2 = \xi_2(x, y) \frac{\partial}{\partial x} + \eta_2(x, y) \frac{\partial}{\partial y}.$$

Definition 1.5 (Commutator) The *commutator* of X_1 and X_2 , written as $[X_1, X_2]$, is defined by the formula $[X_1, X_2] = X_1(X_2) - X_2(X_1)$.

Definition 1.6 (Lie algebra) A *Lie algebra* is a vector space L of operators with the following property : For all $X_1, X_2 \in L$, the commutator $[X_1, X_2] \in L$.

Theorem 1.5 The set of all solutions of any determining equation forms a Lie algebra.

Theorem 1.6 For a second-order ODE $y'' = F(x, y, y')$, the symmetry Lie algebra L has the dimension $r \leq 8$. The maximal dimension $r = 8$ is attained if and only if the ODE is either linear or can be linearized by a change of variables.

1.7 Conclusion

In this chapter we provided a brief introduction to the Lie group analysis of second-order ODEs and included the necessary results which will be used throughout this project.

Chapter 2

Noether point symmetries and first integrals

In this chapter we first give some basic definitions and theorems concerning Noether point symmetries and first integrals which we utilise in the remaining part of the dissertation. Thereafter, we perform some calculations for finding the Noether symmetries and first integral for a second-order ordinary differential equation. We then also illustrate the importance of a Noether symmetry by showing how a single Noether symmetry can be used for double reduction.

2.1 First integrals, Noether theorem and double reductions

Consider a vector field

$$X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}, \quad (2.1)$$

which has first prolongation

$$X^{[1]} = X + [\eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2] \frac{\partial}{\partial y'}. \quad (2.2)$$

Now consider an arbitrary second-order ODE

$$y'' = F(x, y, y') \quad (2.3)$$

which has a Lagrangian $L(x, y, y')$, i.e., equation (2.3) is equivalent to the Euler-Lagrange equation

$$\frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) - \frac{\partial L}{\partial y} = 0. \quad (2.4)$$

Let us now consider the variational principle

$$\delta \int_{x_1}^{x_2} L(x, y(x), y'(x)) dx = 0, \quad (2.5)$$

which gives rise to Euler-Lagrange equation (2.4) and the infinitesimal transformations (1.5) in (x, y) -space, viz.,

$$\bar{x} \approx x + a\xi(x, y), \quad \bar{y} \approx y + a\eta(x, y). \quad (2.6)$$

Using equations (2.6), each curve $x \rightarrow y(x)$, defined on an interval $[\alpha, \beta]$ is transformed for, sufficiently small a into a parameter-dependent curve $\bar{x} \rightarrow \bar{y}(\bar{x})$ in the new variables (see [44], [45]).

The infinitesimal transformations (2.6) defines a Noether point symmetry generator of (2.4) if there exists a gauge function $B(x, y)$, such that for each differentiable curve $x \rightarrow y(x)$, we obtain

$$\int_{x_1}^{\bar{x}_2} L(\bar{x}, \bar{y}(\bar{x}), \frac{d\bar{y}}{d\bar{x}}(\bar{x})) d\bar{x} = \int_{x_1}^{x_2} L(x, y(x), y'(x)) dx + a \int_{x_1}^{x_2} \frac{dB}{dx}(x, y(x)) dx + O(a^2), \quad (2.7)$$

where $[x_1, x_2]$ is any subinterval of $[\alpha, \beta]$ on which $y(x)$ is defined.

Using equation (2.7) together with the formula for the change of variables in integrals, we obtain

$$L(\bar{x}, \bar{y}(\bar{x}), \frac{d\bar{y}}{d\bar{x}}(\bar{x})) = L(x, y(x), y'(x)) + a \frac{dB}{dx}(x, y(x)) + O(a^2). \quad (2.8)$$

Expanding the left-hand side of equation (2.8) about $a = 0$ and equating the coefficients of like powers of a yields

$$X^{[1]}L + D(\xi)L = D(B),$$

where $X^{[1]}$ and D are defined by equations (1.13) and (1.14).

Thus we have the following:

Definition 2.1 X is a Noether point symmetry corresponding to a Lagrangian $L(x, y, y')$ of equation (2.3) if there exists a gauge function $B(x, y)$ such that

$$X^{[1]}L + D(\xi)L = D(B). \quad (2.9)$$

Remark. Equation (2.9) in fact plays the role of the *determining equation* for Noether point symmetries. It is in general an overdetermined system of linear PDEs which is solved for ξ , η and B to find Noether point symmetries.

We now state three theorems which demonstrate the power of the Noether point symmetries.

Theorem 2.1 (Noether [46]) If X is a Noether point symmetry corresponding to a Lagrangian $L(x, y, y')$ of the equation (2.3), then

$$I = \xi L + (\eta - y'\xi) \frac{\partial L}{\partial y'} - B \quad (2.10)$$

is a first integral of equation (2.3) associated with the operator X .

Proof. See, for instance, [27, 47].

Theorem 2.2 The first integral I associated with the Noether point symmetry X satisfies

$$X^{[1]}I = 0, \quad (2.11)$$

that is, X is a point symmetry generator of the first integral I of the equation (2.3).

Proof. See, for example, [45, 48]. References [47, 48] contain more general proofs in more general settings.

Theorem 2.3 If for a Lagrangian $L(x, y, y')$ of equation (2.3) there corresponds a Noether point symmetry generator, then equation (2.3) is solvable by means of quadratures.

Proof. See, e.g., [48] (or use theorems 2.1 and 2.2).

2.2 An illustrative example of double reduction using a Noether symmetry

Let $L(x, y, y')$ be a Lagrangian of the second-order ODE

$$y'' = E(x, y, y'). \quad (2.12)$$

Suppose X is a Noether symmetry of the equation (2.12). According to the Noether theorem this equation has a first integral $I(x, y, y')$ and so it is reducible to

$$I(x, y, y') = C, \quad C \text{ a constant.} \quad (2.13)$$

By theorem 2.2 it follows that X is a point symmetry of the first-order equation (2.13) and so it is integrable by means of a single quadrature. Thus we infer that by the use of a single Noether point symmetry one is able to reduce the order of a second-order ODE twice.

Example 2.1 Consider the equation

$$y'' + \frac{1}{x}y' + e^y = 0. \quad (2.14)$$

It can be easily verified that its standard Lagrangian is

$$L = \frac{1}{2}xy'^2 - xe^y. \quad (2.15)$$

To obtain the Noether point symmetries of equation (2.14) with respect to the standard Lagrangian L , we substitute L into the equation (2.9). This gives

$$\begin{aligned} & \frac{1}{2}\xi y'^2 - \xi e^y - x\eta e^y + x\eta_x y' + x\eta_y y'^2 - x\xi_x y'^2 - x\xi_y y'^3 + \frac{1}{2}x\xi_x y'^2 + \frac{1}{2}x\xi_y y'^3 - \\ & x\xi_x e^y - x\xi_y e^y y' - B_x - y'B_y = 0. \end{aligned} \quad (2.16)$$

Since the left-hand side of this equation is a cubic polynomial in y' , we equate the coefficients of y'^3, y'^2, y', y^0 to zero and obtain the following four equations:

$$y'^3 : \frac{1}{2}x\xi_y = 0. \quad (2.17)$$

$$y'^2 : \frac{1}{2}\xi + x\eta_y - \frac{1}{2}x\xi_x = 0. \quad (2.18)$$

$$y' : x\eta_x - x\xi_y e^y = B_y. \quad (2.19)$$

$$y^0 : -\xi e^y - x\eta e^y - x\xi_x e^y = B_x. \quad (2.20)$$

Equation (2.17) yields

$$\xi = a(x), \quad (2.21)$$

where $a(x)$ is an arbitrary function of x . Using this value of ξ in equation (2.18) and integrating with respect to y , we obtain

$$\eta = \frac{1}{2}(a' - x^{-1}a)y + b(x),$$

where $b(x)$ is an arbitrary function of x . Substituting the value of η in equation (2.19) and integrating with respect to y , we obtain

$$B(x, y) = \frac{1}{4}(xa'' - a' + x^{-1}a)y^2 + xyb' + c(x)$$

where $c(x)$ is an arbitrary function of x . Differentiating the above equation with respect to x and comparing it with equation (2.20), we obtain

$$\frac{1}{4}(xa''' + x^{-1}a' - x^{-2}a)y^2 - yb' + xyb'' + c'(x) = -ae^y - \frac{1}{2}xe^y(a' - x^{-1}a)y - xe^yb - xa'e^y. \quad (2.22)$$

Performing the usual splitting of the above equation and solving for a , b and c , we obtain

$$a(x) = C_1x, \quad b(x) = -2C_1, \quad c(x) = C_2,$$

where C_1 and C_2 are arbitrary constants. Thus

$$\xi = C_1x, \quad \eta = -2C_1, \quad B = C_2,$$

and so we obtain one Noether point symmetry

$$X = x\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}. \quad (2.23)$$

Applying theorem 2.1, we deduce that

$$I = -\frac{1}{2}x^2y'^2 - x^2e^y - 2xy' - C_2, \quad (2.24)$$

is a first integral of equation (2.14) associated with the operator (2.23). Hence the reduced equation can be written as

$$\frac{1}{2}x^2y'^2 + x^2e^y + 2xy' = C \quad (2.25)$$

where C is an arbitrary constant. Now according to theorem 2.2, X is also a Lie point symmetry of the reduced first-order equation (2.25) and so we solve the characteristic equation associated with X , viz.,

$$\frac{dx}{x} = \frac{dy}{-2}$$

in order to obtain the invariant $u = x^2 e^y$, and use this invariant as the new dependent variable. Thus $y = \ln\left(\frac{u}{x^2}\right)$ and substituting this in equation (2.25), we obtain

$$\frac{x^2 u'^2}{2u^2} + u - 2 = C. \quad (2.26)$$

This equation is variables separable and its solution follows by means of quadrature.

Thus, we see that a single Noether point symmetry leads to quadrature.

2.3 Conclusion

In this chapter we gave some basic definitions and theorems concerning Noether point symmetries and first integrals which we will use later in the project. We then worked out Noether point symmetries of a second-order ODE with respect to its standard Lagrangian and obtained a first integral of the ODE. By applying theorem 2.2 the reduced first-order equation was solved by means of quadrature. This example illustrated the fact that a single Noether point symmetry gives double reduction. In the usual approach, a single point symmetry of a second-order ODE, in general, does not result in a quadrature.

Chapter 3

Integration of the generalized Lane-Emden equation

3.1 Introduction

In this chapter we perform a Noether point symmetry classification of the generalized Lane-Emden equation (GLEE) which models several phenomena in mathematical physics and astrophysics. We then obtain first integrals of the various cases which admit Noether point symmetries. Moreover, we obtain reduction to a quadrature for the cases that admit Noether symmetries. This work is new and has been submitted for publication. See [25].

3.2 Classification of the generalised Lane-Emden equation for functions $f(y)$

Consider the generalized Lane-Emden equation

$$\frac{d^2y}{dx^2} + \frac{n}{x} \frac{dy}{dx} + f(y) = 0, \quad (3.1)$$

where n is a constant and $f(y)$ is an arbitrary function of y .

It can be easily verified that the standard Lagrangian of (3.1) is given by

$$L = \frac{1}{2} x^n y'^2 - x^n \int f(y) dy. \quad (3.2)$$

The substitution of the Lagrangian (3.2) into the determining equation (2.9), namely

$$X^{[1]}(L) + D(\xi)L = D(B),$$

gives us

$$\begin{aligned} & \left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + (\eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2) \frac{\partial}{\partial y'} \right) \left(\frac{1}{2} x^n y'^2 - x^n \int f(y) dy \right) + \\ & \left(\frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} \right) (\xi) L = \left(\frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + \dots \right) (B). \end{aligned} \quad (3.3)$$

Expanding (3.3), we obtain

$$\begin{aligned} & \left(\frac{1}{2} n x^{n-1} y'^2 - n x^{n-1} \int f(y) dy \right) \xi - \eta x^n f(y) + x^n y' \eta_x + x^n y'^2 \eta_y \\ & - x^n y'^2 \xi_x - x^n y'^3 \xi_y + \frac{1}{2} x^n y'^2 \xi_x - x^n \xi_x \int f(y) dy + \frac{1}{2} x^n y'^3 \xi_y \\ & - x^n y' \xi_y \int f(y) dy - B_x - y' B_y = 0. \end{aligned} \quad (3.4)$$

Separation with respect to the powers of y' yields the linear overdetermined system of four partial differential equations:

$$y'^3 : \xi_y = 0 \quad (3.5)$$

$$y'^2 : \frac{1}{2} n x^{n-1} \xi + x^n \eta_y - \frac{1}{2} x^n \xi_x = 0 \quad (3.6)$$

$$y' : x^n \eta_x = B_y \quad (3.7)$$

$$y'^0 : n x^{n-1} \xi \int f(y) dy + x^n \eta f(y) + x^n \xi_x \int f(y) dy + B_x = 0 \quad (3.8)$$

Integrating (3.5) with respect to y , we obtain

$$\xi = a(x) \quad (3.9)$$

where $a(x)$ is an arbitrary function of x . Substituting the values of ξ and ξ_x in equation (3.6) and integrating with respect to y gives

$$\eta = \frac{1}{2} (a' - n x^{-1} a) y + b(x) \quad (3.10)$$

where $b(x)$ is an arbitrary function of x . Substituting the value of η_x in equation (3.7) and integrating with respect to y , we obtain

$$B = \frac{1}{4} x^n \left[a'' - n \left(\frac{a}{x} \right)' \right] y^2 + x^n b' y + c(x) \quad (3.11)$$

where $c(x)$ is an arbitrary function of x . Differentiating equation (3.11) with respect to x gives

$$B_x = \frac{1}{4} x^n a''' y^2 - \frac{1}{4} n^2 x^{n-2} a' y^2 + \frac{1}{2} n x^{n-2} a' y^2 + \frac{1}{4} n^2 x^{n-3} a y^2 - \frac{1}{2} n x^{n-3} a y^2 + n x^{n-1} b' y + x^n b'' y + c'. \quad (3.12)$$

Substituting the values of ξ , ξ_x and η in equation (3.8), we obtain

$$B_x = - \left[n a x^{n-1} \int f(y) dy + \left(\frac{1}{2} [a' - n a x^{-1}] y + b \right) x^n f(y) + a' x^n \int f(y) dy \right]. \quad (3.13)$$

Equating equations (3.12) and (3.13) gives us the classifying relation

$$\begin{aligned} & (-n x^{n-1} a - a' x^n) \int f(y) dy + \left(-\frac{1}{2} x^n a' y + \frac{1}{2} n x^{n-1} a y - x^n b \right) f(y) \\ &= \frac{1}{4} x^n a''' y^2 + \frac{1}{2} n x^{n-2} a' y^2 - \frac{1}{2} n x^{n-3} a y^2 - \frac{1}{4} n^2 x^{n-2} a' y^2 + \frac{1}{4} n^2 x^{n-3} a y^2 \\ &+ n x^{n-1} b' y + x^n b'' y + c'. \end{aligned} \quad (3.14)$$

The use of this classifying relation enables us to classify all cases for which Noether point symmetries for equation (3.1) may exist.

We obtain the following eight cases for $f(y)$:

Case 1. $n \neq 0$, $f(y)$ arbitrary.

Case 2. $n = 0$, $f(y)$ arbitrary.

Case 3. $f(y)$ is linear in y .

Case 4. $f(y) = \alpha y^2 + \beta y + \gamma$, where α , β and γ are constants with $\alpha \neq 0$.

Case 5. $f(y) = \alpha y^r$, where α and r are constants with $\alpha \neq 0$ and $r \neq 0, 1$.

Case 6. $f(y) = \alpha \exp(\beta y) + \gamma y + \delta$, where α , β , γ and δ are constants with $\alpha \neq 0$ and $\beta \neq 0$.

Case 7. $f(y) = \alpha \ln y + \gamma y + \delta$, where α , γ and δ are constants with $\alpha \neq 0$.

Case 8. $f(y) = \alpha y \ln y + \gamma y + \delta$, where α , γ and δ are constants with $\alpha \neq 0$.

3.3 Integration of GLEE for different $f(y)$'s

In this section we first investigate the Noether point symmetries of the generalized Lane-Emden equation for eight different forms of $f(y)$ obtained in the previous section. After obtaining the Noether point symmetries for each case, we then utilize the three theorems of chapter 2 to calculate the first integrals and obtain reduction to quadrature of the corresponding generalized Lane-Emden equation.

Case 1. $n \neq 0$, $f(y)$ is arbitrary.

In this case we find that $\xi = 0$, $\eta = 0$, and $B = k$, k a constant. Hence, there is no Noether point symmetry for this case.

Noether point symmetries exist in the following cases.

Case 2. $n = 0$, $f(y)$ is arbitrary.

We obtain $\xi = 1$, $\eta = 0$ and $B = k$, k a constant.

Therefore we have a single Noether symmetry generator $X = \frac{\partial}{\partial x}$.

The integration for this case is trivial even without a Noether symmetry. The use of the Noether integral (2.10) results in

$$I = \frac{1}{2}y^2 + \int f(y)dy$$

from which, setting $I = C$, one easily gets quadrature. Note that this case includes the autonomous Ermakov equation [49] ($f(y) = \alpha y^{-3} + \beta y$) which has $sl(2, \mathbb{R})$ symmetry algebra. It turns out that $sl(2, \mathbb{R})$ is also the Noether symmetry algebra.

Case 3. $f(y)$ is linear in y

This case is well-known and the corresponding eqn (3.1) has $sl(3, \mathbb{R})$ symmetry algebra and five Noether point symmetries associated with the standard Lagrangian of the differential equation (3.1) for this case (see [50, 51]).

Case 4. $f(y) = \alpha y^2 + \beta y + \gamma$, where α , β and γ are constants, with $\alpha \neq 0$.

The following three subcases arise:

Case 4.1 $n = 5$, $\beta = 0$ and $\gamma = 0$. We obtain

$\xi = x$, $\eta = -2y$ and $B = k$, k a constant. This is contained in Case 5.1 below.

Case 4.2 $n = 5$, $\beta^2 = 4\alpha\gamma$. We get

$$\xi = x, \eta = -\left(2y + \frac{\beta}{\alpha}\right) \text{ and } B = \frac{\beta\gamma}{6\alpha}x^6$$

and so we have that

$$X = x \frac{\partial}{\partial x} - \left(2y + \frac{\beta}{\alpha}\right) \frac{\partial}{\partial y}.$$

The application of Theorem 2.1, due to Noether, results in

$$I = -\frac{1}{2}x^6y'^2 - \frac{1}{3}\alpha x^6y^3 - \frac{1}{2}\beta x^6y^2 - \gamma x^6y - 2x^5yy' - \frac{\beta}{\alpha}x^5y' - \frac{1}{6}\frac{\beta\gamma}{\alpha}x^6.$$

Thus, the reduced equation is

$$\frac{1}{2}x^6y'^2 + \frac{1}{3}\alpha x^6y^3 + \frac{1}{2}\beta x^6y^2 + \gamma x^6y + 2x^5yy' + \frac{\beta}{\alpha}x^5y' + \frac{1}{6}\frac{\beta\gamma}{\alpha}x^6 = C, \quad (3.15)$$

where C is an arbitrary constant. By Theorem 2.2, X is also a symmetry generator of I as well as the reduced eqn (3.15). Invoking Theorem 2.3, we can solve eqn (3.15). In order to solve the first-order ODE (3.15), we use an invariant of X as the dependent variable. This invariant is obtained by solving the characteristic equation associated with X , viz.,

$$\frac{dx}{x} = \frac{dy}{-(2y + \beta/\alpha)}.$$

The solution of this ODE gives the invariant

$$u = x^2y + \frac{\beta}{2\alpha}x^2.$$

Eqn (3.15) in terms of u , after some calculations, is

$$C = 2u^2 - \frac{1}{2}x^2u'^2 - \frac{\alpha}{3}u^3.$$

This last first-order ODE is variables separable as

$$\frac{du}{\pm \sqrt{4u^2 - (2/3)\alpha u^3 - 2C}} = \frac{dx}{x}.$$

Hence we have quadrature or double reduction of our equation for the f given.

Case 4.3 $n = 5/3$, $\beta = 0$ and $\gamma = 0$. We find

$\xi = x^{1/3}$, $\eta = -\frac{2}{3}x^{-2/3}y$ and $B = \frac{2}{9}y^2 + k$, k a constant. This is subsumed in Case 5.2 below.

Case 5. $f(y) = \alpha y^r$, α and r are constants with $\alpha \neq 0$ and $r \neq 0, 1$.

Here we have two subcases.

Case 5.1 $n = \frac{r+3}{r-1}$. We obtain

$$\xi = x, \eta = \frac{2}{1-r}y \text{ and } B = k, k \text{ a constant.}$$

By the use of Theorems 2.1, 2.2 and 2.3, we find that the solution of eqn (3.1) for the above n and $f(y)$ is

$$y = u^{\frac{2}{r-1}} x^{\frac{2}{1-r}}, \quad (3.16)$$

where u is given by

$$\int \frac{du}{\pm \sqrt{4(1-r)^{-2}u^2 - 2\alpha(1+r)^{-1}u^{1+r} - C_1}} = C_2 \ln x, \quad (3.17)$$

in which C_1 and C_2 are arbitrary constants of integration. Note that for $r = 5$, one gets $n = 2$ and we have the general solution given in eqn (3.17). Only a one-parameter family of solutions is known in the literature (see, e.g., [52]). Here we have determined the two-parameter family of solutions.

Case 5.2 $n = \frac{r+3}{r+1}$, with $r \neq -1$. We have

$$\xi = x^{\frac{r-1}{r+1}}, \eta = -\frac{2}{r+1}x^{-\frac{2}{r+1}}y \text{ and } B = \frac{2}{(r+1)^2}y^2 + k, k \text{ a constant.}$$

Again we invoke Theorems 2.1, 2.2 and 2.3. In this subcase we deduce that the solution of eqn (3.1) is

$$y = ux^{-\frac{2}{r+1}}, \quad (3.18)$$

where u is defined by

$$\int \frac{du}{\pm \sqrt{-2\alpha(r+1)^{-1}u^{r+1} + C_1}} = \frac{r+1}{2}x^{\frac{2}{r+1}} + C_2, \quad (3.19)$$

in which C_1 and C_2 are arbitrary constants.

Case 6. $f(y) = \alpha \exp(\beta y) + \gamma y + \delta$, α, β, γ and δ are constants with $\alpha, \beta \neq 0$.

For $n = 1$, $\gamma = 0$ and $\delta = 0$, we deduce $\xi = x$, $\eta = -2/\beta$ and $B = k$, k a constant.

The invocation of Theorems 2.1, 2.2 and 2.3 gives rise to the solution of eqn (3.1) for this case to be

$$y = \frac{2}{\beta} \ln \left(\frac{u}{x} \right), \quad (3.20)$$

where u is given by

$$\int \frac{du}{\pm u \sqrt{1 - (1/2)\alpha\beta u^2 + C_1}} = \ln x C_2, \quad (3.21)$$

in which C_1 and C_2 are integration constants.

Case 7. $f(y) = \alpha \ln y + \gamma y + \delta$, where α , γ and δ are constants with $\alpha \neq 0$.

If $n = 0$ and $\delta = 0$, we obtain $\xi = 1$, $\eta = 0$ and $B = k$, k a constant. This reduces to Case 2.

Case 8. $f(y) = \alpha y \ln y + \gamma y + \delta$, where α , γ and δ are constants with $\alpha \neq 0$.

If $n = 0$, we obtain $\xi = 1$, $\eta = 0$ and $B = k$, k a constant. This reduces to Case 2.

3.4 Conclusion

We have completely classified the Noether point symmetries of the generalized Lane-Emden equation (3.1) with respect to the standard Lagrangian (3.2). Eight cases arose, out of which seven cases resulted in Noether point symmetries. For each of these we obtained the first integral and also reduction to quadrature of the corresponding Lane-Emden equation (3.1). Three new cases were found. These correspond to Cases 4.2, 5.1 and 5.2. It is interesting to mention that we obtained a two-parameter family of solutions in Case 5.1 (when $n = 2$ and $r = 5$) for which only a one-parameter family of solutions is known in the literature.

Chapter 4

Concluding remarks

In this research project we first reviewed some useful definitions and results of Lie group analysis which were later used in our work. In Chapter 2 we gave the preliminaries of the Noether point symmetry approach and stated three theorems which demonstrated the power of a Noether point symmetry. In the main chapter of our report, that is, Chapter 3, we provided the complete Noether point symmetry classification of the generalized Lane-Emden eqn (3.1) for various functions $f(y)$. Then in Chapter 3 we also determined the double reductions of eqn (3.1) for the functions $f(y)$ for which eqn (3.1) has Noether point symmetries. It is worth mentioning here that three new cases were found and these corresponded to Cases 4.2, 5.1 and 5.2. Also, we obtained a two-parameter family of solutions in the Case 5.1 (when $n = 2$ and $r = 5$) for which only a one-parameter family of solutions is known in the literature.

Bibliography

- [1] Thomson, W. (1911) Collected Papers, Vol. 5, p. 266. Cambridge: University Press.
- [2] Emden, R. (1907) *Gaskugeln*, Leipzig and Berlin: Teubner.
- [3] Horedt, G. P. (1986) Seven-digit tables of Lane-Emden functions, *Astrophys and Space Science*, 126, 357-408.
- [4] Horedt, G. P. (1987) Approximate analytical solutions of the Lane-Emden equation in N-dimensional space, *Astron. Astrophys*, 172, 359-367.
- [5] Bender, C. M. *et al.*, (1989) A new perturbative approach to nonlinear problems, *J. Math. Phys*, 30, 1447-1455.
- [6] Lima, P. M. (1996) Numerical methods and asymptotic error expansions for the Emden-Fowler equations, *J. Comput. Appl. Mathem.*, 70, 245-266; Numerical solution of a singular boundary-value problem in non-newtonian fluid mechanics, *Appl. Num. Mathem.*, (1999), 30, 93-111.
- [7] Roxburgh, I. W. and Stockman, L.M. (1999) Power series solutions of the polytrope equations, *Monthly Not. Roy. Astron. Soc.*, 303, 466-470.
- [8] Adomian, G., Rach, R., and Shawagfen, N.T. (1995) On the analytic solution of the Lane-Emden equation, *Found. Phys. Lett.*, 8, 161-181.
- [9] Shawagfeh, N. T. (1993) Nonperturbative approximate solution for Lane-Emden equation, *J. Math. Phys.*, 34, 4364-4369.

- [10] Burt, P. B. (1987) *Nuov. Cim.*, 100B, 43-52.
- [11] Wazwaz, A. M. (2001) A new algorithm for solving differential equations of Lane-Emden type *Applied Math. and Computation*, 118, 287-310.
- [12] Liao, S. (2003) A new analytic algorithm of Lane-Emden type equations *Applied Math. and Computation*, 142, 1-16.
- [13] Chandrasekhar, S. (1957) *An Introduction to the Study of Stellar Structure* (Dover Publications Inc., New York).
- [14] Davis, H. T. (1962) *Introduction to Nonlinear Differential and Integral Equations* (Dover Publications Inc., New York).
- [15] Datta, B. K. (1996) Analytic solution to the Lane-Emden equation, *Nuovo Cimento* 111B, 1385-1388.
- [16] Wrubel, M. H. (1958) *Stellar Interiors. In Encyclopedia of Physics*, S. Flugge, ed. (Springer Verlag, Berlin), p. 53.
- [17] Meerson, E., Megged E., and Tajima, T. (1996) On the Quasi-hydrostatic Flows of Radiatively Cooling Self-gravitating Gas Clouds, *Astrophysical Journal* 457, 321-331.
- [18] Gnutzmann S Ritschel U (1995) Analytic solution of Emden-Fowler equation and critical adsorption in spherical geometry, *Z. Phys. B: Condens Matter* 96, 391.
- [19] Bahcall, N. A. (1973) The Galaxy Distribution in the Cluster Abell 2199, *Ap. J.* 186, 1179.
- [20] Bahcall, N. A. (1975) Core radii and central densities of 15 rich clusters of galaxies, *Ap. J.* 198, 249.
- [21] Leach, P. G. L. (1985) First integrals for the modified Emden equation $\ddot{q} + \alpha(t)\dot{q} + q^n = 0$. *J. Math. Phys.* 26, 2510-2514.

- [22] Richardson, O. W. (1921) *The Emission of Electricity from Hot Bodies*. 2nd edition, London.
- [23] Goenner, H., and Havas, P. (2000) Exact solutions of the generalized Lane-Emden equation. *J. Math. Phys.* 41, 7029-7042.
- [24] Goenner, H. (2001) Symmetry transformations for the generalized Lane-Emden equation. *Gen. Relativity and Gravitation* 33, 833-841.
- [25] Khalique, C. M., Mahomed, F. M., and Muatjetjeja, B. Lagrangian formulation of a generalized Lane-Emden equation and double reduction. Submitted to *J. Phys. A: Math. Gen.*
- [26] Lie, S. (1891) *Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen*. (Bearbeitet und herausgegeben von Dr. Georg Scheffers) B. G. Teubner, Leipzig.
- [27] Ovsiannikov, L. V. (1982) *Group Analysis of Differential Equations*. Academic Press, New York, (English translation by W. F. Ames).
- [28] Olver, P. J. (1993) *Applications of Lie Groups to Differential Equations*, Springer, New York, Second Edition.
- [29] Bluman, G. W., and Kumei, S. (1989) *Symmetries and Differential Equations*, Springer, New York.
- [30] Stephani, H. (1989) *Differential Equations. Their solutions using symmetries*, Cambridge University Press, Cambridge.
- [31] Ibragimov, N. H. (1999) *Elementary Lie Group Analysis and Ordinary Differential Equations*, John Wiley and Sons, Chichester.
- [32] Ibragimov, N. H. (2000) *Introduction to Modern Group Analysis*, Tay, Ufa, Russia.

- [33] Ibragimov, N. H., Ed. (1994) *CRC Handbook of Lie Group Analysis of Differential Equations Vol 1; Symmetries, exact solutions and conservation laws*. CRC Press Inc., Boca Raton.
- [34] Ibragimov, N. H., Ed. (1995) *CRC Handbook of Lie Group Analysis of Differential Equations Vol 2; Applications in engineering and physical sciences*. CRC Press Inc., Boca Raton.
- [35] Ibragimov, N. H., Ed. (1996) *CRC Handbook of Lie Group Analysis of Differential Equations Vol 3; New trends in theoretical developments and computational methods*. CRC Press Inc., Boca Raton.
- [36] Hydon, P. E. (2000) *Symmetry methods for differential equations; a beginner's guide*, Cambridge University Press, Cambridge.
- [37] Cantwell, B. J. (2002) *Introduction to Symmetry Analysis*, Cambridge University Press, Cambridge.
- [38] Mahomed, F. M. (2002) Recent trends in symmetry analysis of differential equations, *Notices of SAMS*, 33, 11-40
- [39] Baumann, G. (2000) *Symmetry Analysis of Differential Equations with Mathematica*, Springer-Verlag, Berlin.
- [40] Head, A. K. (1993) LIE, a PC program for Lie analysis of differential equations, *Comput. Phys. Commun.* 77, 241-248.
- [41] Head, A. K., (1996) LIE, a PC program for Lie analysis of differential equations, *Comput. Phys. Commun.* 96, 311-313.
- [42] Sherring, J., Head, A. K. and Prince, G. E. (1997) Dimsym and LIE: symmetry determining packages, *Math. Comput. Model.* 25, 153-164.
- [43] Diaz, J. M. (2000) <http://www.matsource.com/Mathsource/Applications/Mathematics/0211-778/Yalie.ps>.

- [44] Logan, J. D. (1977) *Invariant Variational Principles* , Academic Press, New York.
- [45] Sarlet, W. and Cantrijn, F. (1981) Generalizations of Noether's Theorem in Classical Mechanics, *SIAM Review*, 23, 467-494.
- [46] Noether, E. (1918) Invariante Variationsprobleme. *König Gesell Wissen Göttingen, Math-Phys Kl*, Heft 2:235-269.
- [47] Ibragimov, N.H., Kara, A.H. and Mahomed, F.M. (1998) Lie-Bäcklund and Noether Symmetries with Applications, *Nonlinear Dynamics*, 15, 115-136.
- [48] Kara, A.H., Vawda, F. and Mahomed, F.M. (1994) Symmetries of first integrals and solutions of differential equations *Lie Groups and Their Applications*, 1, (2), 27-48.
- [49] Ermakov, V. (1880) Second order differential equations. Conditions of complete integrability *Univ Izvestia Kiev Series III* 9 1-25. (translated by A O Hahn)
- [50] Lie, S. (1967) *Differential Equations*, Chelsea, New York. (in German).
- [51] Mahomed, F.M., Kara, A.H. and Leach, P.G.L. (1993) Lie and Noether Counting Theorems for One-Dimensional Systems *J. Math. Anal. Applic.*, 178, 116-129.
- [52] Dresner, L. (1983) *Similarity solutions of nonlinear partial differential equations*. Pitman Advanced Publishing Program, London.