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APPLICATIONS OF LIE GROUP METHODS TO PROBLEMS IN FLUID MECHANICS

by

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Dissertation submitted for the degree of Master of Science in Applied
Mathematics in the Department of Mathematical Sciences in the
Faculty of Agriculture, Science and Technology at the North-West
University, Mafikeng Campus

June 2011

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Declaration

I declare that the dissertation for the degree of Master of Science at the North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other university, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

GABRIEL MAGALAKWE

June 2011

Dedication

To my family, friends and everyone who gave me support

Acknowledgements

I would like to thank my supervisor Professor CM Khalique for his guidance, patience and support throughout this research project and Mr AR Adem for his assistance. I would also like to thank Professor T Hayat and Dr B Mutjetjeja for their fruitful discussions. Thanks also to the SANHARP for their financial assistance.

Above all, I would like not to forget God, who guided and gave me strength throughout this project.

Abstract

This research studies two nonlinear problems arising in fluid mechanics. Firstly the combined sinh-cosh-Gordon equation is considered. Lie symmetry analysis along with simplest equation method is used to carry out its integration. Also conservation laws are obtained for this equation using the direct method and the new conservation theorem.

Secondly, the nonlinear flow problem of an incompressible viscous fluid is considered. The fluid is taken in a channel having two porous walls. An incompressible fluid fills the porous space inside the channel. Lie group method is applied along with perturbation method in the derivation of an analytical solution which is compared to numerical solution. Salient features are taken into account when the fluid saturates the porous medium. The effects of porous medium, permeation Reynolds number and wall dilation rate on the self-axial velocity are shown and discussed. Lastly temperature distribution is illustrated.

Introduction

In applied mathematics and theoretical physics, fluid mechanics is one of the most important field of study. The study of fluid flow has many applications in scientific and engineering fields, such as aerodynamics, hydrodynamics, convection heat transfer, oceanography and dynamics of multi-phase flows among others. Most problems that arise in fluids are modelled by nonlinear differential equations (see for example [1–5]).

There are a number of approaches for solving nonlinear partial differential equations, which range from completely analytical to completely numerical ones.

One approach is Lie group analysis. It is based on symmetry and invariance principles, and is a systematic method for solving nonlinear differential equations analytically. Originally developed by Sophus Lie (1842-1899), the philosophy of Lie groups has become an essential part of mathematical culture for anyone investigating mathematical models of physical, engineering and natural problems. Lie group analysis embodies and synthesizes symmetries of differential equations. A symmetry is described roughly as a change or a transformation that leaves an object apparently unchanged. Symmetries permeate many mathematical models, in particular those formulated in terms of differential equations.

Firstly, the combined sinh-cosh-Gordon equation in (1+1)-dimensions given by

$$u_{tt} - ku_{xx} + \alpha \sinh u + \beta \cosh u = 0, \quad (1)$$

where k, α and β are constants is studied. This equation arises in a wide range of scientific applications that range from chemical reactions to water surface gravity

waves. Its solitary wave solutions were obtained in [6, 7]. Recently, exact solutions of (1) were obtained in [8] using function transformation method. In this research equation (1) is studied and exact solutions are obtained using Lie symmetry methods along with the simplest equation method. Also, conservation laws are determined for this equation.

In the second part of the research, two-dimensional flow between slowly expanding or contracting walls is considered.

The two-dimensional flow of viscous fluid in a porous channel appears very useful in many applications, hence in the past many experimental and theoretical attempts have been made. Such studies have been presented under the various assumptions like small Reynolds number Re , intermediate Re , large Re and arbitrary Re . The steady flow in a channel with stationary walls and small Re has been studied by Berman [9]. Dauenhaver and Majdalani [10] numerically discussed the two-dimensional viscous flow in a deformable channel when $-50 < Re < 200$ and $-100 < \alpha < 100$ (α denotes the wall expansion ratio).

In another study, Majdalani *et al* [11] analyzed the channel flow of slowly expanding-contracting walls which leads to the transport of biological fluids. They first derived the analytic solution for small Re and α and then compared it with the numerical solution. Boutros *et al* [12] studied the flow problem [11] and obtained the analytical solution when Re and α vary in the ranges $-5 < Re < 5$ and $-1 < \alpha < 1$. Mahmood *et al* [13] discussed the homotopy perturbation and numerical solutions for viscous flow in a deformable channel with porous medium. Asghar *et al* [14] computed exact solution for the flow of viscous fluid through expanding-contracting channels.

Noor *et al* [15] studied heat and mass transfer in a square cavity. Recently, Khalique *et al* [16] studied two dimensional flow in deformable channel with porous medium and variable magnetic field. In this proposed research, heat and mass transfer will be studied in the rectangular domain. The salient features will be taken into account when the fluid saturates the porous medium.

The flow of [12] is generalized with the influence of porous medium and heat transfer

given by

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0, \quad (2)$$

$$\frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{1}{\rho} \frac{\partial \bar{P}}{\partial \bar{x}} + \nu \left[\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \right] - \nu \frac{\phi}{k} \bar{u}, \quad (3)$$

$$\frac{\partial \bar{v}}{\partial t} + \bar{u} \frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{v}}{\partial \bar{y}} = -\frac{1}{\rho} \frac{\partial \bar{P}}{\partial \bar{y}} + \nu \left[\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{y}^2} \right] - \nu \frac{\phi}{k} \bar{v} + g\beta(T - T_w), \quad (4)$$

$$\frac{\partial T}{\partial t} + \bar{u} \frac{\partial T}{\partial \bar{x}} + \bar{v} \frac{\partial T}{\partial \bar{y}} = \alpha \left[\frac{\partial^2 T}{\partial \bar{x}^2} + \frac{\partial^2 T}{\partial \bar{y}^2} \right], \quad (5)$$

with the following conditions

$$\begin{aligned} \text{(i)} \quad & \bar{u} = 0, \quad \bar{v} = -V_w, \quad T = T_w \quad \text{at } \bar{y} = h(t), \\ \text{(ii)} \quad & \frac{\partial \bar{u}}{\partial \bar{y}} = 0, \quad \bar{v} = 0, \quad \frac{\partial T}{\partial \bar{y}} = 0 \quad \text{at } \bar{y} = 0, \\ \text{(iii)} \quad & \bar{u} = 0 \quad \text{at } \bar{x} = 0. \end{aligned} \quad (6)$$

In the above expressions, $\bar{u}$ and $\bar{v}$ are velocity component in the $\bar{x}$ and $\bar{y}$ direction respectively and T is temperature. The fluid properties are assumed to be constant, ρ is the fluid density, μ is the dynamic viscosity and $\bar{k}$ is the thermal conductivity of the incompressible fluid. Thus the kinematic viscosity is $\nu = \frac{\mu}{\rho}$, g is the acceleration due to gravity, β is the coefficient of the thermal expansion and the thermal diffusivity is $\alpha = \frac{\bar{k}}{\rho c_p}$ where c_p is the specific heat capacity, $\bar{P}$ is the pressure, t is time, ϕ and k are porosity and permeability of porous medium respectively.

The outline of the research project is as follows:

In Chapter 1 the basic definitions and theorems concerning the Lie group methods are recalled.

Chapter 2 deals with the construction of exact solutions of the combined sinh-cosh-Gordon equation using Lie group methods along with the simplest equation method. Furthermore, conservation laws are constructed for the combined sinh-cosh-Gordon equation by using the direct method, Noether theorem and the new conservation theorem.

In Chapter 3 Lie group analysis is applied along with perturbation method to obtain

analytical solution for two-dimensional viscous flow between slowly or contracting walls with the influence of porous medium and heat transfer. This solution is then compared with the numerical solution.

Chapter 4 summarizes the results of the dissertation.

Bibliography is given at the end.

Chapter 1

Lie group analysis of PDEs

In this chapter a brief introduction to the Lie group analysis of partial differential equations (PDEs) is given. The algorithm to determine the Lie point symmetries of partial differential equations is also presented.

1.1 Introduction

In the nineteenth century, the Norwegian mathematician Marius Sophus Lie (1842-1899) realized that the majority of the methods for solving differential equations could be unified using group theory. Lie developed a symmetry-based technique to obtain exact solutions of differential equations. Nearly all the well-known methods for solving differential equations are special cases of Lie's methods. Many books have been written recently on this topic. Some of them are Ovsiannikov [17], Bluman and Kumei [18], Olver [19], Ibragimov [20, 21].

The definitions and results presented in this chapter are taken from the books mentioned above.

1.2 Local continuous one-parameter Lie group

Now consider $x = (x^1, \dots, x^n)$ to be the independent variables with coordinates x^i and $q = (q^1, \dots, q^m)$ be the dependent variables with coordinates q^α (n and m finite). Consider a change of the variables x and q involving a real parameter a :

$$T_a : \bar{x}^i = f^i(x, q, a), \quad \bar{q}^\alpha = \phi^\alpha(x, q, a), \quad (1.1)$$

where a continuously ranges in values from a neighborhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$, and f^i and ϕ^α are differentiable functions.

Definition 1.1 A set G of transformations (1.1) is called a *local continuous one-parameter Lie group of transformations* in the space of variables x and q if

- (i) For $T_a, T_b \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then $T_b T_a = T_c \in G$, $c = \phi(a, b) \in \mathcal{D}$ (Closure)
- (ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0 T_a = T_a T_0 = T_a$ (Identity), and
- (iii) For $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that $T_a T_{a^{-1}} = T_{a^{-1}} T_a = T_0$ (Inverse).

We see that the associativity property follows from (i). The group property (i) can be written as

$$\begin{aligned} \bar{\bar{x}}^i &\equiv f^i(\bar{x}, \bar{q}, b) = f^i(x, q, \phi(a, b)), \\ \bar{\bar{q}}^\alpha &\equiv \phi^\alpha(\bar{x}, \bar{q}, b) = \phi^\alpha(x, q, \phi(a, b)) \end{aligned} \quad (1.2)$$

and the function ϕ is called the *group composition law*. A group parameter a is called *canonical* if $\phi(a, b) = a + b$.

Theorem 1.1 For any $\phi(a, b)$, there exists the canonical parameter $\tilde{a}$ defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)}, \quad \text{where } w(s) = \left. \frac{\partial \phi(s, b)}{\partial b} \right|_{b=0}.$$

1.3 Prolongation formulas and Group generator

The derivatives of q with respect to x are defined as

$$q_i^\alpha = D_i(q^\alpha), \quad q_{ij}^\alpha = D_j D_i(q_i), \dots, \quad (1.3)$$

where

$$D_i = \frac{\partial}{\partial x^i} + q_i^\alpha \frac{\partial}{\partial q^\alpha} + q_{ij}^\alpha \frac{\partial}{\partial q_j^\alpha} + \dots, \quad i = 1, \dots, n \quad (1.4)$$

is the operator of total differentiation. The collection of all first derivatives q_i^α is denoted by $q_{(1)}$, i.e.,

$$q_{(1)} = \{q_i^\alpha\} \quad \alpha = 1, \dots, m, \quad i = 1, \dots, n.$$

Similarly

$$q_{(2)} = \{q_{ij}^\alpha\} \quad \alpha = 1, \dots, m, \quad i, j = 1, \dots, n$$

and $q_{(3)} = \{q_{ijk}^\alpha\}$ and likewise $q_{(4)}$ etc. Since $q_{ij}^\alpha = q_{ji}^\alpha$, $q_{(2)}$ contains only q_{ij}^α for $i \leq j$. In the same manner $q_{(3)}$ has only terms for $i \leq j \leq k$. There is natural ordering in $q_{(4)}, q_{(5)} \dots$.

In group analysis all variables $x, q, q_{(1)} \dots$ are considered functionally independent variables connected only by the differential relations (1.3). Thus the q_s^α are called differential variables and a p th-order partial differential equation (PDE) is given as

$$E(x, q, q_{(1)}, \dots, q_{(p)}) = 0. \quad (1.5)$$

Prolonged or extended groups

If $z = (x, q)$, one-parameter group of transformations G is

$$\begin{aligned} \bar{x}^i &= f^i(x, q, a), \quad f^i|_{a=0} = x^i, \\ \bar{q}^\alpha &= \phi^\alpha(x, q, a), \quad \phi^\alpha|_{a=0} = q^\alpha. \end{aligned} \quad (1.6)$$

According to Lie's theory, the construction of the symmetry group G is equivalent to the determination of the corresponding *infinitesimal transformations* :

$$\bar{x}^i \approx x^i + a \xi^i(x, q), \quad \bar{q}^\alpha \approx q^\alpha + a \eta^\alpha(x, q) \quad (1.7)$$

obtained from (1.1) by expanding the functions f^i and ϕ^α into Taylor series in a about $a = 0$ and also taking into account the initial conditions

$$f^i|_{a=0} = x^i, \quad \phi^\alpha|_{a=0} = q^\alpha.$$

Thus, we have

$$\xi^i(x, q) = \left. \frac{\partial f^i}{\partial a} \right|_{a=0}, \quad \eta^\alpha(x, q) = \left. \frac{\partial \phi^\alpha}{\partial a} \right|_{a=0}. \quad (1.8)$$

One can now introduce the *symbol* of the infinitesimal transformations by writing (1.7) as

$$\bar{x}^i \approx (1 + a X)x^i, \quad \bar{q}^\alpha \approx (1 + a X)q^\alpha,$$

where

$$X = \xi^i(x, q) \frac{\partial}{\partial x^i} + \eta^\alpha(x, q) \frac{\partial}{\partial q^\alpha}. \quad (1.9)$$

This differential operator X is known as the infinitesimal operator or generator of the group G . If the group G is admitted by (1.5), we say that X is an *admitted operator* of (1.5) or X is an *infinitesimal symmetry* of equation (1.5).

We now see how the derivatives are transformed.

The D_i transforms as

$$D_i = D_i(f^j) \bar{D}_j, \quad (1.10)$$

where $\bar{D}_j$ is the total differentiations in transformed variables $\bar{x}^i$. So

$$\bar{q}_i^\alpha = \bar{D}_j(q^\alpha), \quad \bar{q}_{ij}^\alpha = \bar{D}_j(\bar{q}_i^\alpha) = \bar{D}_i(\bar{q}_j^\alpha), \dots,$$

and

$$\begin{aligned} D_i(\phi^\alpha) &= D_i(f^j) \bar{D}_j(\bar{q}^\alpha) \\ &= D_i(f^j) \bar{q}_j^\alpha, \end{aligned} \quad (1.11)$$

and so

$$\left(\frac{\partial f^j}{\partial x^i} + q_i^\beta \frac{\partial f^j}{\partial q^\beta} \right) \bar{q}_j^\alpha = \frac{\partial \phi^\alpha}{\partial x^i} + q_i^\beta \frac{\partial \phi^\alpha}{\partial q^\beta}. \quad (1.12)$$

The quantities $\bar{q}_j^\alpha$ can be represented as functions of $x, q, q_{(i)}, a$ for small a , ie., (1.12) is locally invertible:

$$\bar{q}_i^\alpha = \psi_i^\alpha(x, q, q_{(1)}, a), \quad \psi_i^\alpha|_{a=0} = q_i^\alpha. \quad (1.13)$$

The transformations in $x, q, q_{(1)}$ space given by (1.6) and (1.13) form a one-parameter group (one can prove this but we do not consider the proof) called the first prolongation or just extension of the group G and denoted by $G^{[1]}$.

We let

$$\bar{q}_i^\alpha \approx q_i^\alpha + a \zeta_i^\alpha \quad (1.14)$$

be the infinitesimal transformation of the first derivatives so that the infinitesimal transformation of the group $G^{[1]}$ is (2.71) and (1.14).

Higher-order prolongations of G , viz. $G^{[2]}, G^{[3]}$ can be obtained by derivatives of (1.11).

Prolonged generators

Using (1.11) together with (2.71) and (1.14) we get

$$\begin{aligned} D_i(f^j)(\bar{q}_j^\alpha) &= D_i(\phi^\alpha) \\ D_i(x^j + a\xi^j)(q_j^\alpha + a\zeta_j^\alpha) &= D_i(q^\alpha + a\eta^\alpha) \\ (\delta_i^j + aD_i\xi^j)(q_j^\alpha + a\zeta_j^\alpha) &= q_i^\alpha + aD_i\eta^\alpha \\ q_i^\alpha + a\zeta_i^\alpha + aq_j^\alpha D_i\xi^j &= q_i^\alpha + aD_i\eta^\alpha \\ \zeta_i^\alpha &= D_i(\eta^\alpha) - q_j^\alpha D_i(\xi^j), \quad (\text{sum on } j). \end{aligned} \quad (1.15)$$

This is called the first prolongation formula. Likewise, one can obtain the second prolongation, viz.,

$$\zeta_{ij}^\alpha = D_j(\eta_i^\alpha) - q_{ik}^\alpha D_j(\xi^k), \quad (\text{sum on } k). \quad (1.16)$$

By induction (recursively)

$$\zeta_{i_1, i_2, \dots, i_p}^\alpha = D_{i_p}(\zeta_{i_1, i_2, \dots, i_{p-1}}^\alpha) - q_{i_1, i_2, \dots, i_{p-1} j}^\alpha D_{i_p}(\xi^j), \quad (\text{sum on } j). \quad (1.17)$$

The first and higher prolongations of the group G form a group denoted by $G^{[1]}, \dots, G^{[p]}$.

The corresponding prolonged generators are

$$\begin{aligned} X^{[1]} &= X + \zeta_i^\alpha \frac{\partial}{\partial q_i^\alpha} \quad (\text{sum on } i, \alpha), \\ &\vdots \\ X^{[p]} &= X^{[p-1]} + \zeta_{i_1, \dots, i_p}^\alpha \frac{\partial}{\partial q_{i_1, \dots, i_p}^\alpha} \quad p \geq 1, \end{aligned}$$

where

$$X = \xi^i(x, q) \frac{\partial}{\partial x^i} + \eta^\alpha(x, q) \frac{\partial}{\partial q^\alpha}.$$

1.4 Group admitted by a PDE

Definition 1.2 The vector field

$$X = \xi^i(x, q) \frac{\partial}{\partial x^i} + \eta^\alpha(x, q) \frac{\partial}{\partial q^\alpha}, \quad (1.18)$$

is a *point symmetry* of the p th-order PDE (1.5), if

$$X^{[p]}(E) = 0 \quad (1.19)$$

whenever $E = 0$. This can also be written as

$$X^{[p]} E \Big|_{E=0} = 0, \quad (1.20)$$

where the symbol $|_{E=0}$ means evaluated on the equation $E = 0$.

Definition 1.3 Equation (1.19) is called the *determining equation* of (1.5) because it determines all the infinitesimal symmetries of (1.5).

Definition 1.4 (Symmetry group) A one-parameter group G of transformations (1.1) is called a symmetry group of equation (1.5) if (1.5) is form-invariant (has the same form) in the new variables $\bar{x}$ and $\bar{q}$, i.e.,

$$E(\bar{x}, \bar{q}, q_{(1)}^-, \dots, q_{(p)}^-) = 0, \quad (1.21)$$

where the function E is the same as in equation (1.5).

1.5 Group invariants

Definition 1.5 A function $F(x, q)$ is called an *invariant of the group of transformation* (1.1) if

$$F(\bar{x}, \bar{q}) \equiv F(f^i(x, q, a), \phi^\alpha(x, q, a)) = F(x, q), \quad (1.22)$$

identically in x, q and a .

Theorem 1.2 (Infinitesimal criterion of invariance) A necessary and sufficient condition for a function $F(x, q)$ to be an invariant is that

$$X F \equiv \xi^i(x, q) \frac{\partial F}{\partial x^i} + \eta^\alpha(x, q) \frac{\partial F}{\partial q^\alpha} = 0. \quad (1.23)$$

It follows from the above theorem that every one-parameter group of point transformations (1.1) has n functionally independent invariants, which can be taken to be the left-hand side of any first integrals

$$J_1(x, q) = c_1, \dots, J_n(x, q) = c_n$$

of the characteristic equations

$$\frac{dx^1}{\xi^1(x, q)} = \dots = \frac{dx^n}{\xi^n(x, q)} = \frac{dq^1}{\eta^1(x, q)} = \dots = \frac{dq^n}{\eta^n(x, q)}.$$

Theorem 1.3 If the infinitesimal transformation (2.71) or its symbol X is given, then the corresponding one-parameter group G is obtained by solving the Lie equations

$$\frac{d\bar{x}^i}{da} = \xi^i(\bar{x}, \bar{q}), \quad \frac{d\bar{q}^\alpha}{da} = \eta^\alpha(\bar{x}, \bar{q}) \quad (1.24)$$

subject to the initial conditions

$$\bar{x}^i|_{a=0} = x, \quad \bar{q}^\alpha|_{a=0} = q.$$

1.6 Lie algebra

Let us consider two operators X_1 and X_2 defined by

$$X_1 = \xi_1^i(x, q) \frac{\partial}{\partial x^i} + \eta_1^\alpha(x, q) \frac{\partial}{\partial q^\alpha}$$

and

$$X_2 = \xi_2^i(x, q) \frac{\partial}{\partial x^i} + \eta_2^\alpha(x, q) \frac{\partial}{\partial q^\alpha}.$$

Definition 1.6 The *commutator* of X_1 and X_2 , written as $[X_1, X_2]$, is defined by $[X_1, X_2] = X_1(X_2) - X_2(X_1)$.

Definition 1.7 A Lie algebra is a vector space L (over the field of real numbers) of operators $X = \xi^i(x, q) \frac{\partial}{\partial x^i} + \eta^\alpha(x, q) \frac{\partial}{\partial q^\alpha}$ with the following property. If the operators

$$X_1 = \xi_1^i(x, q) \frac{\partial}{\partial x^i} + \eta_1^\alpha(x, q) \frac{\partial}{\partial q^\alpha}, \quad X_2 = \xi_2^i(x, q) \frac{\partial}{\partial x^i} + \eta_2^\alpha(x, q) \frac{\partial}{\partial q^\alpha}$$

are any elements of L , then their commutator

$$[X_1, X_2] = X_1(X_2) - X_2(X_1)$$

is also an element of L . It follows that the commutator is

1. Bilinear: for any $X, Y, Z \in L$ and $a, b \in \mathbb{R}$,

$$[aX + bY, Z] = a[X, Z] + b[Y, Z], \quad [X, aY + bZ] = a[X, Y] + b[X, Z];$$

2. Skew-symmetric: for any $X, Y \in L$,

$$[X, Y] = -[Y, X];$$

3. and satisfies the Jacobi identity: for any $X, Y, Z \in L$,

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

1.7 Conclusion

In this chapter some basic definitions and results of the Lie group analysis of PDEs were recalled. The algorithm to determine the Lie point symmetries of PDEs was also presented.

Chapter 2

Solutions and conservation laws of the combined sinh-cosh-Gordon equation

In this chapter, we first use the Lie symmetry analysis along with simplest equation method to find the solutions of the combined sinh-cosh-Gordon equation in (1+1)-dimensions given by [8]

$$u_{tt} - ku_{xx} + \alpha \sinh u + \beta \cosh u = 0,$$

where k , α and β are constants. It should be noted here that in [8], a method based on a transformed Painlevé property and the variable separated ODE method was used to obtain exact solutions of the combined sinh-cosh-Gordon equation.

Subsequently, we will construct conservation laws of the combined sinh-cosh-Gordon equation using three methods; direct method, Noether theorem and new conservation theorem.

This work is new and has been submitted for publication. See [22].

2.1 Symmetry analysis

In this section Lie point symmetries of the combined sinh-cosh-Gordon equation are first calculated. These will then be used to construct exact solutions.

2.1.1 Lie point symmetries

A Lie point symmetry of a differential equation is an invertible transformation of the dependent and independent variables that leaves the equation unchanged.

The symmetry group of the combined sinh-cosh-Gordon equation, viz.,

$$u_{tt} - ku_{xx} + \alpha \sinh u + \beta \cosh u = 0 \quad (2.1)$$

will be generated by vector field of the form

$$X = \tau(t, x, u) \frac{\partial}{\partial t} + \xi(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u}.$$

Applying the second prolongation $\text{pr}^{(2)}X$ to (2.1) we obtain the following overdetermined system of linear partial differential equations

$$\xi_u = 0, \quad (2.2)$$

$$\tau_u = 0, \quad (2.3)$$

$$\eta_{uu} = 0, \quad (2.4)$$

$$-\tau_t + \xi_x = 0, \quad (2.5)$$

$$-\xi_t + k\tau_x = 0, \quad (2.6)$$

$$-2k\eta_{xu} - \xi_{tt} + k\xi_{xx} = 0, \quad (2.7)$$

$$-\tau_{tt} + 2\eta_{tu} + k\tau_{xx} = 0, \quad (2.8)$$

$$\begin{aligned} &\beta\eta \sinh u - k\eta_{xx} + 2\alpha\tau_t \sinh u + 2\beta\tau_t \cosh u - \alpha\eta_u \sinh u \\ &\quad - \beta\eta_u \cosh u + \eta_{tt} + \alpha\eta \cosh u = 0. \end{aligned} \quad (2.9)$$

From (2.2), we get

$$\xi = A(t, x), \quad (2.10)$$

where $A(t, x)$ is an arbitrary function of t and x . From (2.3), we obtain

$$\tau = B(t, x), \quad (2.11)$$

where $B(t, x)$ is an arbitrary function of t and x . From (2.4), we get

$$\eta = C(t, x)u + D(t, x), \quad (2.12)$$

where $C(t, x)$ and $D(t, x)$ are arbitrary functions of t and x . Substituting (2.11) and (2.12) in (2.9) yields

$$\begin{aligned} & \beta \left(C(t, x)u + D(t, x) \right) \sinh u - kC_{xx}(t, x)u - kD_{xx}(t, x) + 2\alpha B_t(t, x) \sinh u \\ & + 2\beta B_t(t, x) \cosh u - \alpha C(t, x) \sinh u - \beta C(t, x) \cosh u + C_{tt}(t, x)u \\ & + D_{tt}(t, x) + \alpha C(t, x)u \cosh u + \alpha D(t, x) \cosh u = 0. \end{aligned} \quad (2.13)$$

On splitting (2.13) we obtain the following system of PDEs:

$$-kC_{xx}(t, x) + C_{tt}(t, x) = 0, \quad (2.14)$$

$$\beta D(x, t) + 2\alpha B_t(x, t) - \alpha C(x, t) = 0, \quad (2.15)$$

$$2\beta B_t(x, t) - \beta C(x, t) + \alpha D(x, t) = 0, \quad (2.16)$$

$$\beta C(x, t) = 0, \quad (2.17)$$

$$\beta C(x, t) = 0, \quad (2.18)$$

$$-kD_{xx}(t, x) + D_{tt}(t, x) = 0. \quad (2.19)$$

From (2.17) and (2.18) we get

$$C(t, x) = 0,$$

which satisfy (2.14). Solving (2.15) and (2.16) simultaneously when $C(t, x) = 0$ we obtain

$$D(t, x) = 0 \quad \text{and} \quad B_t(t, x) = 0. \quad (2.20)$$

From (2.20) we get

$$B(t, x) = b(x), \quad (2.21)$$

where $b(x)$ is an arbitrary function of x . Also (2.19) is satisfied since $D(t, x) = 0$. Thus we have

$$\xi = A(t, x), \quad \tau = b(x), \quad \eta = 0. \quad (2.22)$$

Using (2.22) in (2.5) we get

$$A(t, x) = a(t), \quad (2.23)$$

where $a(t)$ is an arbitrary function of t . Substituting (2.22) in (2.8) we obtain

$$b''(x) = 0,$$

which on integration yields

$$b(x) = c_1 x + c_2, \quad (2.24)$$

where c_1 and c_2 are arbitrary constants. Substituting (2.23) in (2.22) then using (2.6) yields

$$a'(t) = kb'(x). \quad (2.25)$$

Substituting (2.24) in (2.25) and integrating with respect to t , we obtain

$$a(t) = c_1 kt + c_3, \quad (2.26)$$

where c_3 is an arbitrary constant. Hence we obtain

$$\xi = c_1 kt + c_3, \quad \tau = c_1 x + c_2, \quad \eta = 0.$$

We note that for these values of ξ, τ and η equation (2.7) is also satisfied. Hence the general solutions of the system (2.2)-(2.9) is

$$\xi = c_1 kt + c_3, \quad \tau = c_1 x + c_2, \quad \eta = 0.$$

Thus the Lie point symmetries of the combined sinh-cosh-Gordon equation are given by

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = x \frac{\partial}{\partial t} + kt \frac{\partial}{\partial x}.$$

2.1.2 Exact solutions

One of the main purposes for calculating symmetries of a differential equation is to use them for obtaining symmetry reductions and finding exact solutions. In this subsection we will use the symmetries computed in the previous subsection to obtain exact solutions of (2.1).

The linear combination $\nu X_1 + X_2$ (ν is a constant) of the two symmetries X_1 and X_2 yields the following invariants

$$z = x - \nu t, \quad \psi = u$$

and consequently using these invariants (2.1) is transformed into the nonlinear second order ordinary differential equation (ODE)

$$(\nu^2 - k)\psi''(z) + \alpha \sinh \psi(z) + \beta \cosh \psi(z) = 0. \quad (2.27)$$

By using the transformation

$$H(z) = e^{\psi(z)},$$

or equivalently $\psi(z) = \ln H(z)$, where $z = x - \nu t$, equation (2.27) becomes

$$-kHH'' + \nu^2 HH'' + kH'^2 - \nu^2 H'^2 + \frac{1}{2}\alpha H^3 - \frac{1}{2}\alpha H + \frac{1}{2}\beta H^3 + \frac{1}{2}\beta H = 0. \quad (2.28)$$

We now solve this nonlinear ODE using the simplest equation method.

2.2 Simplest equation method

First we recall that the simplest equation method was introduced by Kudryashov [23, 24]. See also [25, 26]. The simplest equations that will be used are the Bernoulli and Ricatti equations.

Let us consider the solution of (2.28) in the form

$$H(z) = \sum_{i=0}^M A_i (G(z))^i, \quad (2.29)$$

where $G(z)$ satisfies the Bernoulli and Riccati equations, M is a positive integer that can be determined by balancing procedure as in [25] and $A_0, \dots, A_M$ are parameters to be determined. We note that the Bernoulli and Riccati equations are well-known nonlinear ODEs whose solutions can be expressed in terms of elementary functions.

We consider the Bernoulli equation

$$G'(z) = aG(z) + bG^s(z), \quad (2.30)$$

where s is an integer with $s > 1$ and for simplicity we let $s = 2$. As a result the solution of the Bernoulli equation is [26]

$$G(z) = a \left\{ \frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b \cosh[a(z+C)] - b \sinh[a(z+C)]} \right\}.$$

For the Riccati equation

$$G'(z) = aG^2(z) + bG(z) + c \quad (2.31)$$

we shall use the solutions [26]

$$G(z) = -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left[\frac{1}{2}\theta(z+C) \right]$$

and

$$G(z) = -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left(\frac{1}{2}\theta z \right) + \frac{\operatorname{sech} \left(\frac{\theta z}{2} \right)}{C \cosh \left(\frac{\theta z}{2} \right) - \frac{2a}{\theta} \sinh \left(\frac{\theta z}{2} \right)},$$

where $\theta^2 = b^2 - 4ac > 0$ and C is a constant of integration.

2.2.1 Solutions of (2.28) using the equations of Bernoulli as the simplest equation

The balancing procedure with $s = 2$ [25] yields $M = 2$, so the solutions in (2.29) are of the form

$$H(z) = A_0 + A_1 G + A_2 G^2. \quad (2.32)$$

Substituting (2.32) in (2.28) and equating all coefficients of the functions G^i to zero, we obtain an algebraic system of equations in terms of A_0 , A_1 and A_2 . These algebraic equations are

$$\begin{aligned}
\frac{\alpha A_0^3}{2} - \frac{\alpha A_0}{2} + \frac{\beta A_0^3}{2} + \frac{\beta A_0}{2} &= 0, \\
-a^2 A_0 A_1 k + a^2 A_0 A_1 \nu^2 + \frac{3}{2} \alpha A_0^2 A_1 - \frac{\alpha A_1}{2} + \frac{3}{2} \beta A_0^2 A_1 + \frac{\beta A_1}{2} &= 0, \\
-4a^2 A_0 A_2 k + 4a^2 A_0 A_2 \nu^2 - 3ab A_0 A_1 k + 3ab A_0 A_1 \nu^2 + \\
\frac{3}{2} \alpha A_0 A_1^2 + \frac{3}{2} \alpha A_0^2 A_2 - \frac{\alpha A_2}{2} + \frac{3}{2} \beta A_0 A_1^2 + \frac{3}{2} \beta A_0^2 A_2 + \frac{\beta A_2}{2} &= 0, \\
-a^2 A_2 A_1 k + a^2 A_2 A_1 \nu^2 - ab A_1^2 k - 10ab A_0 A_2 k + ab A_1^2 \nu^2 + 10ab A_0 A_2 \nu^2 + \\
\frac{\alpha A_1^3}{2} + 3\alpha A_0 A_2 A_1 - 2b^2 A_0 A_1 k + 2b^2 A_0 A_1 \nu^2 + \frac{\beta A_1^3}{2} + 3\beta A_0 A_2 A_1 &= 0, \\
-5ab A_1 A_2 k + 5ab A_1 A_2 \nu^2 + \frac{3}{2} \alpha A_0 A_2^2 + \frac{3}{2} \alpha A_1^2 A_2 - b^2 A_1^2 k - \\
6b^2 A_0 A_2 k + 6b^2 A_0 A_2 \nu^2 + b^2 A_1^2 \nu^2 + \frac{3}{2} \beta A_0 A_2^2 + \frac{3}{2} \beta A_1^2 A_2 &= 0, \\
-2ab A_2^2 k + 2ab A_2^2 \nu^2 + \frac{3}{2} \alpha A_1 A_2^2 - 4b^2 A_1 A_2 k + 4b^2 A_1 A_2 \nu^2 + \frac{3}{2} \beta A_1 A_2^2 &= 0, \\
\frac{\alpha A_2^3}{2} - 2b^2 A_2^2 k + 2b^2 A_2^2 \nu^2 + \frac{\beta A_2^3}{2} &= 0.
\end{aligned}$$

Solving the system of algebraic equations with the aid of Mathematica, we obtain

$$\begin{aligned}
a &= \frac{A_1 b}{A_2}, \\
\alpha &= \frac{b^2 (16k A_2^2 + A_1^4 k - A_1^4 \nu^2 - 16\nu^2 A_2^2)}{8A_2^3}, \\
\beta &= -\frac{b^2 (A_1^4 k - A_1^4 \nu^2 - 16k A_2^2 + 16\nu^2 A_2^2)}{8A_2^3}, \\
A_0 &= \frac{A_1^2}{4A_2}.
\end{aligned}$$

As a result, a solution of (2.1) is

$$\begin{aligned}
u(t, x) = \ln \left(A_0 + A_1 a \left\{ \frac{\cosh[a(z + C)] + \sinh[a(z + C)]}{1 - b \cosh[a(z + C)] - b \sinh[a(z + C)]} \right\} \right. \\
\left. + A_2 a^2 \left\{ \frac{\cosh[a(z + C)] + \sinh[a(z + C)]}{1 - b \cosh[a(z + C)] - b \sinh[a(z + C)]} \right\}^2 \right), \quad (2.33)
\end{aligned}$$

where $z = x - \nu t$ and C is a constant of integration.

The profile of the solution is given in Figure 2.1.

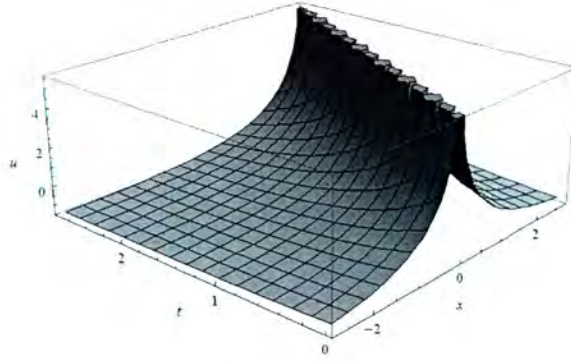


Figure 2.1: Profile of (2.33)

2.2.2 Solutions of (2.28) using the equations of Ricatti as the simplest equation

In this case the balancing procedure with $s = 2$, [25] yields $M = 2$, so the solutions in (2.29) are of the form

$$H(z) = A_0 + A_1 G + A_2 G^2. \quad (2.34)$$

Substituting (2.34) in (2.28) and equating all coefficients of the functions G^i to zero, we obtain an algebraic system of equations in terms of A_0, A_1 and A_2 . The

corresponding algebraic equations are

$$\begin{aligned}
& \frac{\alpha A_0^3}{2} - \frac{\alpha A_0}{2} - bcA_1A_0k + bcA_1A_0\nu^2 + \frac{\beta A_0^3}{2} + \frac{\beta A_0}{2} \\
& \quad - 2c^2A_2A_0k + A_1^2c^2k + 2c^2A_2A_0\nu^2 - A_1^2c^2\nu^2 = 0, \\
& -2acA_0A_1k + 2acA_0A_1\nu^2 + \frac{3}{2}\alpha A_0^2A_1 - \frac{\alpha A_1}{2} - b^2A_0A_1k + b^2A_0A_1\nu^2 + bcA_1^2k \\
& -6bcA_0A_2k - bcA_1^2\nu^2 + 6bcA_0A_2\nu^2 + \frac{3}{2}\beta A_0^2A_1 + \frac{\beta A_1}{2} + 2c^2A_1A_2k - 2c^2A_1A_2\nu^2 = 0, \\
& -3abA_0A_1k + 3abA_0A_1\nu^2 - 8acA_0A_2k + 8acA_0A_2\nu^2 + \frac{3}{2}\alpha A_0A_1^2 + \frac{3}{2}\alpha A_0^2A_2 \\
& \quad - \frac{\alpha A_2}{2} - 4b^2A_0A_2k + 4b^2A_0A_2\nu^2 + bcA_1A_2k - bcA_1A_2\nu^2 + \frac{3}{2}\beta A_0A_1^2 \\
& \quad + \frac{3}{2}\beta A_0^2A_2 + \frac{\beta A_2}{2} + 2c^2A_2^2k - 2c^2A_2^2\nu^2 = 0, \\
& -2a^2A_0A_1k + 2a^2A_0A_1\nu^2 - abA_1^2k - 10abA_0A_2k + abA_1^2\nu^2 + 10abA_0A_2\nu^2 \\
& -2acA_2A_1k + 2acA_2A_1\nu^2 + \frac{\alpha A_1^3}{2} + 3\alpha A_0A_2A_1 - b^2A_2A_1k + b^2A_2A_1\nu^2 \\
& \quad + 2bcA_2^2k - 2bcA_2^2\nu^2 + \frac{\beta A_1^3}{2} + 3\beta A_0A_2A_1 = 0, \\
& -a^2A_1^2k - 6a^2A_0A_2k + a^2A_1^2\nu^2 + 6a^2A_0A_2\nu^2 - 5abA_1A_2k \\
& + 5abA_1A_2\nu^2 + \frac{3}{2}\alpha A_0A_2^2 + \frac{3}{2}\alpha A_1^2A_2 + \frac{3}{2}\beta A_0A_2^2 + \frac{3}{2}\beta A_1^2A_2 = 0, \\
& -4a^2A_1A_2k + 4a^2A_1A_2\nu^2 - 2abA_2^2k + 2abA_2^2\nu^2 + \frac{3}{2}\alpha A_1A_2^2 + \frac{3}{2}\beta A_1A_2^2 = 0, \\
& \quad -2a^2A_2^2k + 2a^2A_2^2\nu^2 + \frac{\alpha A_2^3}{2} + \frac{\beta A_2^3}{2} = 0.
\end{aligned}$$

Solving the system of algebraic equations with the aid of Mathematica, one obtains

$$\begin{aligned}
\alpha &= \frac{2a^2(k - \nu^2)}{A_2}, \\
b &= \frac{A_1a}{A_2}, \\
\beta &= \frac{2a^2(k - \nu^2)}{A_2}, \\
A_0 &= \frac{cA_2}{a}.
\end{aligned}$$

Hence solutions of (2.1) are given by

$$u(t, x) = \ln \left(A_0 + A_1 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left[\frac{1}{2} \theta (z + C) \right] \right\} + A_2 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left[\frac{1}{2} \theta (z + C) \right] \right\}^2 \right) \quad (2.35)$$

and

$$u(t, x) = \ln \left(A_0 + A_1 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left(\frac{1}{2} \theta z \right) + \frac{\operatorname{sech} \left(\frac{\theta z}{2} \right)}{C \cosh \left(\frac{\theta z}{2} \right) - \frac{2a}{\theta} \sinh \left(\frac{\theta z}{2} \right)} \right\} + A_2 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left(\frac{1}{2} \theta z \right) + \frac{\operatorname{sech} \left(\frac{\theta z}{2} \right)}{C \cosh \left(\frac{\theta z}{2} \right) - \frac{2a}{\theta} \sinh \left(\frac{\theta z}{2} \right)} \right\}^2 \right), \quad (2.36)$$

where $z = x - \nu t$ and C is a constant of integration.

The solution of (2.36) is plotted in Figure 2.2.

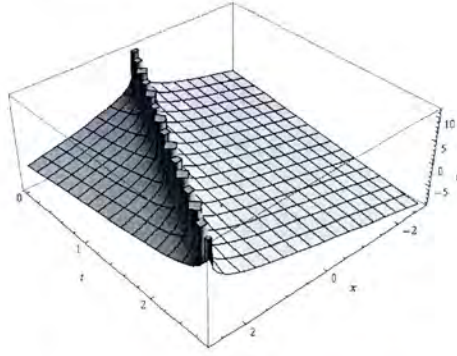


Figure 2.2: Profile of (2.36)

2.3 Conservation laws of the combined sinh-cosh-Gordon equation

The notion of conserved quantity plays an important role in the solution process. The first step towards finding solution of a system of differential equations is by finding its conservation laws. Indeed, the existence of a large number of conservation laws of a system of partial differential equations (PDEs) is a strong indication of its integrability [18].

The famous Noether theorem [27] provides an elegant and constructive way of finding conservation laws. It gives an explicit formula for determining a conservation law once a Noether symmetry associated with a Lagrangian is known for an Euler-Lagrange equation.

In this section conservation laws will be constructed for the combined sinh-cosh-Gordon equation

$$u_{tt} - ku_{xx} + \alpha \sinh u + \beta \cosh u = 0$$

by three methods but first we give some notation which we will utilize later in the section.

2.3.1 Fundamental operators and their relationship

In this subsection we briefly present the notation and pertinent results that will be used in the subsequent work.

Consider a k th-order system of PDEs of n independent variables $x = (x^1, x^2, \dots, x^n)$ and m dependent variables $u = (u^1, u^2, \dots, u^m)$, viz.,

$$E_\alpha(x, u, u_{(1)}, \dots, u_{(k)}) = 0, \quad \alpha = 1, \dots, m, \quad (2.37)$$

where $u_{(1)}, u_{(2)}, \dots, u_{(k)}$ denote the collections of all first, second, $\dots$, k th-order partial derivatives, that is, $u_i^\alpha = D_i(u^\alpha)$, $u_{ij}^\alpha = D_j D_i(u^\alpha)$, $\dots$, respectively, with the *total derivative operator* with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots, \quad i = 1, \dots, n, \quad (2.38)$$

and the summation convention is used whenever appropriate.

The following are known (see for e.g., [28] and the references therein).

The *Euler-Lagrange operator*, for each α , is given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial u_{i_1 i_2 \dots i_s}^\alpha}, \quad \alpha = 1, \dots, m, \quad (2.39)$$

and the *Lie-Bäcklund operator* is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \quad \xi^i, \eta^\alpha \in \mathcal{A}, \quad (2.40)$$

where $\mathcal{A}$ is the space of *differential functions*. The operator (2.40) is an abbreviated form of infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} \zeta_{i_1 i_2 \dots i_s}^\alpha \frac{\partial}{\partial u_{i_1 i_2 \dots i_s}^\alpha}, \quad (2.41)$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} \zeta_i^\alpha &= D_i(W^\alpha) + \xi^j u_{ij}^\alpha, \\ \zeta_{i_1 \dots i_s}^\alpha &= D_{i_1} \dots D_{i_s}(W^\alpha) + \xi^j u_{j i_1 \dots i_s}^\alpha, \quad s > 1, \end{aligned} \quad (2.42)$$

in which W^α is the *Lie characteristic function* defined by

$$W^\alpha = \eta^\alpha - \xi^i u_i^\alpha. \quad (2.43)$$

One can write the Lie-Bäcklund operator (2.41) in characteristic form as

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} D_{i_1} \dots D_{i_s}(W^\alpha) \frac{\partial}{\partial u_{i_1 i_2 \dots i_s}^\alpha}. \quad (2.44)$$

The *Noether operators* associated with a Lie-Bäcklund symmetry operator X are given by

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s \geq 1} D_{i_1} \dots D_{i_s}(W^\alpha) \frac{\delta}{\delta u_{i i_1 i_2 \dots i_s}^\alpha}, \quad i = 1, \dots, n, \quad (2.45)$$

where the Euler-Lagrange operators with respect to derivatives of u^α are obtained from (2.39) by replacing u^α by the corresponding derivatives. For example,

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s \geq 1} (-1)^s D_{j_1} \dots D_{j_s} \frac{\partial}{\partial u_{i j_1 j_2 \dots j_s}^\alpha}, \quad i = 1, \dots, n, \quad \alpha = 1, \dots, m, \quad (2.46)$$

and the Euler-Lagrange, Lie-Bäcklund and Noether operators are connected by the operator identity

$$X + D_i(\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i N^i. \quad (2.47)$$

The n -tuple vector $T = (T^1, T^2, \dots, T^n)$, $T^j \in \mathcal{A}$, $j = 1, \dots, n$, is a *conserved vector* of (2.37) if T^i satisfies

$$D_i T^i|_{(2.37)} = 0. \quad (2.48)$$

The equation (2.48) defines a *local conservation law* of system (2.37).

A Lie-Bäcklund operator X is said to be a *Noether symmetry* generator associated with a Lagrangian $L \in \mathcal{A}$ if there exists a vector $B = (B^1, \dots, B^n)$, $B^i \in \mathcal{A}$, such that

$$XL + LD_i(\xi^i) = D_i(B^i). \quad (2.49)$$

We now state the Noether theorem [27].

Noether theorem [27]. For any Noether symmetry generator X associated with a given Lagrangian $L \in \mathcal{A}$, there corresponds a vector $T = (T^1, \dots, T^n)$, $T^i \in \mathcal{A}$, defined by

$$T^i = N^i(L) - B^i, \quad i = 1, \dots, n \quad (2.50)$$

which is a conserved vector of the Euler-Lagrange equations $\delta L / \delta u^\alpha = 0$, $\alpha = 1, \dots, m$, where $\delta / \delta u^\alpha$ is the Euler-Lagrange operator given by (2.39) and the Noether operator associated with X is defined by (2.45) in which the Euler-Lagrange operators with respect to derivatives of u^α are obtained from (2.39) by replacing u^α by the corresponding derivatives [29].

2.3.2 Variational method for a system and its adjoint

The system of *adjoint equations* to the system of k th-order differential equations (2.37) is defined by [30]

$$E_\alpha^*(x, u, v, \dots, u_{(k)}, v_{(k)}) = 0, \quad \alpha = 1, \dots, m, \quad (2.51)$$

where

$$E_\alpha^*(x, u, v, \dots, u_{(k)}, v_{(k)}) = \frac{\delta(v^\beta E_\beta)}{\delta u^\alpha}, \quad \alpha = 1, \dots, m, \quad v = v(x) \quad (2.52)$$

and $v = (v^1, v^2, \dots, v^m)$ are new dependent variables.

The following results are recalled as given in Ibragimov [31].

A system of equations (2.37) is said to be *self-adjoint* if the substitution of $v = u$ into the system of adjoint equations (2.51) yields the same system (2.37).

Assume the system of equations (2.37) admits the symmetry generator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}. \quad (2.53)$$

Then the system of adjoint equations (2.51) admits the operator

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \eta_*^\alpha \frac{\partial}{\partial v^\alpha}, \quad \eta_*^\alpha = -[\lambda_\beta^\alpha v^\beta + v^\alpha D_i(\xi^i)], \quad (2.54)$$

where the operator (2.54) is an extension of (2.53) to the variable v^α and the λ_β^α are obtainable from

$$X(E_\alpha) = \lambda_\alpha^\beta E_\beta. \quad (2.55)$$

Theorem 3.1. [31] Every Lie point, Lie-Bäcklund and non local symmetry (2.53) admitted by the system of equations (2.37) gives rise to a conservation law for the system consisting of the equation (2.37) and the adjoint equation (2.51), where the components T^i of the conserved vector $T = (T^1, \dots, T^n)$ are determined by

$$T^i = \xi^i L + W^\alpha \frac{\delta L}{\delta u_i^\alpha} + \sum_{s \geq 1} D_{i_1} \dots D_{i_s}(W^\alpha) \frac{\delta L}{\delta u_{i_1 i_2 \dots i_s}^\alpha}, \quad i = 1, \dots, n, \quad (2.56)$$

with Lagrangian given by

$$L = v^\alpha E_\alpha(x, u, \dots, u_{(k)}). \quad (2.57)$$

the differentiation of L up to second-order derivative only then the equation (2.56) can be written as

$$\begin{aligned} T^i = & \xi^i L + W^\alpha \left[\frac{\partial L}{\partial u_i^\alpha} - D_j \left(\frac{\partial L}{\partial u_{ij}^\alpha} \right) \right] \\ & + D_j(W^\alpha) \left(\frac{\partial L}{\partial u_{ij}^\alpha} \right). \end{aligned} \quad (2.58)$$

2.4 Construction of conservation Laws of the combined sinh-cosh-Gordon equation

We recall that the combined sinh-cosh-Gordon equation

$$u_{tt} - ku_{xx} + \alpha \sinh u + \beta \cosh u = 0$$

admits the following Lie point symmetry generators:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = x \frac{\partial}{\partial t} + kt \frac{\partial}{\partial x}.$$

2.4.1 Application of the direct method

There exists a fundamental relationship between the point symmetries admitted by a given equation and the conservation laws of that given equation. Kara and Mahomed [29] showed that the conservation law

$$D_t T^1 + D_x T^2 = 0, \tag{2.59}$$

which must be evaluated on the partial differential equation, can be considered together with the following requirements

$$X^{[n]}(T^1) + T^1 D_x(\xi) - T^2 D_x(\tau) = 0, \tag{2.60}$$

$$X^{[n]}(T^2) + T^2 D_t(\tau) - T^1 D_t(\xi) = 0, \tag{2.61}$$

where $X^{[n]}$ is the n th extension/prolongation of a point symmetry of the original equation. The order of the extension equals to the order of the highest derivative in T^1 and T^2 . Thus, for given X , (2.59)-(2.61) can be solved to obtain the tuple $T = (T^1, T^2)$.

The condition (2.59) on the equation yields

$$D_t T^1 + D_x T^2 = \frac{\partial T^1}{\partial t} + u_t \frac{\partial T^1}{\partial u} + u_{tt} \frac{\partial T^1}{\partial u_t} + u_{tx} \frac{\partial T^1}{\partial u_x} + \frac{\partial T^2}{\partial x} + u_x \frac{\partial T^2}{\partial u} + u_{tx} \frac{\partial T^2}{\partial u_t} + u_{xx} \frac{\partial T^2}{\partial u_x} \quad (2.62)$$

$$= \frac{\partial T^1}{\partial t} + u_t \frac{\partial T^1}{\partial u} + (k u_{xx} - \alpha \sinh u - \beta \cosh u) \frac{\partial T^1}{\partial u_t} + u_{tx} \frac{\partial T^1}{\partial u_x} + \frac{\partial T^2}{\partial x} + u_x \frac{\partial T^2}{\partial u} + u_{tx} \frac{\partial T^2}{\partial u_t} + u_{xx} \frac{\partial T^2}{\partial u_x} = 0. \quad (2.63)$$

Since T^1 and T^2 cannot depend on second derivatives of u it follows that the coefficients of u_{tt} , u_{tx} and u_{xx} must be zero. Thus,

$$\frac{\partial T^1}{\partial u_x} + \frac{\partial T^2}{\partial u_t} = 0, \quad (2.64)$$

$$k \frac{\partial T^1}{\partial u_t} + \frac{\partial T^2}{\partial u_x} = 0, \quad (2.65)$$

$$\frac{\partial T^1}{\partial t} + u_t \frac{\partial T^1}{\partial u} - (\alpha \sinh u + \beta \cosh u) \frac{\partial T^1}{\partial u_t} + \frac{\partial T^2}{\partial x} + u_x \frac{\partial T^2}{\partial u} = 0. \quad (2.66)$$

Given conditions (2.64)-(2.65) we consider each of the point symmetries admitted by combined sinh-cosh-Gordon equation (2.1) in order to obtain the conservation laws.

We begin with

$$X_1 = \frac{\partial}{\partial t}, \quad (2.67)$$

which is already written in its extended form. The symmetry conditions (2.60)-(2.61) yield

$$\frac{\partial T^1}{\partial t} = 0, \quad (2.68)$$

$$\frac{\partial T^2}{\partial t} = 0, \quad (2.69)$$

respectively. It follows from (2.64)-(2.66) and (2.68)-(2.69) that the components of the conserved vector for the combined sinh-cosh-Gordon equation associated with X_1 are given by

$$\begin{aligned} T^1 &= \frac{c_4 u_t^2}{k} + c_4 u_x^2 + c_5 u_x + \frac{2c_4}{k} (\alpha \cosh u + \beta \sinh u) + c_6 + j(x), \\ T^2 &= -2c_4 u_t u_x - c_5 u_t + c_7, \end{aligned}$$

where c_4, c_5, c_6 and c_7 are constants and $j(x)$ is a function of x . If we proceed in a similar manner for the remaining two symmetries we obtain the following result:

For

$$X_2 = \frac{\partial}{\partial x},$$

the components of the conservation laws are given by

$$\begin{aligned} T^1 &= \frac{c_4 u_t^2}{k} + c_4 u_x^2 + c_5 u_x + \frac{2c_4}{k}(\alpha \cosh u + \beta \sinh u) + c_8 \\ T^2 &= -2c_4 u_t u_x - c_5 u_t + p(t), \end{aligned}$$

where c_4, c_5 and c_8 are constants and $p(t)$ is a function of t .

Finally, for the third symmetry

$$X_3 = x \frac{\partial}{\partial t} + kt \frac{\partial}{\partial x},$$

we obtain

$$\begin{aligned} T^1 &= c_5 u_x + ax, \\ T^2 &= -c_5 u_t + akt, \end{aligned}$$

where a and c_5 , are constants, which are now the components of the conserved vector T for the equation (2.1). We note that this is a trivial conserved vector.

Remark. For each case, we have verified that the equation $D_i T^i|_{(2.1)} = 0$ is satisfied.

2.4.2 Application of Noether theorem

In this subsection we use Noether theorem to construct conservation laws for the combined sinh-cosh-Gordon equation (2.1). It can be easily verified that equation (2.1) has a standard Lagrangian given by

$$L = \frac{1}{2} u_t^2 - \frac{k}{2} u_x^2 - (\alpha \cosh u + \beta \sinh u). \quad (2.70)$$

Substitution of (2.70) into the Noether operator determining equation (2.49) yields

$$\begin{aligned}
& -\alpha\eta\sinh u - \beta\eta\cosh u + u_t\{\eta_t + u_t\eta_u - u_t(\tau_t + u_t\tau_u) - u_x(\xi_t + u_t\xi_u)\} \\
& -ku_x\{\eta_x + u_x\eta_u - u_t(\tau_x + u_x\tau_u) - u_x(\xi_x + u_x\xi_u)\} \\
& + \{(1/2)u_t^2 - (k/2)u_x^2 - \alpha\cosh u - \beta\sinh u\}\{\tau_t + u_t\tau_u + \xi_x + u_x\xi_u\} \\
& = B_t^1 + u_tB_u^1 + B_x^2 + u_xB_u^2.
\end{aligned}$$

Splitting the above equation on the derivatives of u , we obtain

$$\begin{aligned}
\tau_u &= 0, \\
\xi_u &= 0, \\
-\xi_t + k\tau_x &= 0, \\
-\tau_t + \xi_x + 2\eta_u &= 0, \\
-k\eta_u + k\xi_x - \frac{k}{2}\tau_t - \frac{k}{2}\xi_x &= 0, \\
\xi_x - \tau_t - 2\eta_u &= 0, \\
\eta_t &= B_u^1, \\
-k\eta_x &= B_u^2, \\
-\alpha\eta\sinh u - \beta\eta\cosh u - \alpha\tau_t\cosh u - \alpha\xi_x\cosh u - \beta\tau_t\sinh u - \beta\xi_x\sinh u &= B_t^1 + B_x^2.
\end{aligned}$$

After some long and tedious calculations, we have

$$\tau = a_1, \quad \xi = a_2, \quad \eta = 0, \quad B^1 = 0 \quad B^2 = 0,$$

where a_1 and a_2 are constants.

Hence we obtain two Noether point symmetries, namely

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}.$$

The application of the Noether theorem (2.50) with $X_1 = \frac{\partial}{\partial t}$, yields the conserved vector $T = (T^1, T^2)$, where

$$T^1 = -\frac{1}{2}u_t^2 - \frac{k}{2}u_x^2 - \alpha\cosh u - \beta\sinh u, \quad T^2 = ku_tu_x.$$

Using $X_2 = \frac{\partial}{\partial x}$ and invoking the Noether theorem, we get

$$T^1 = -u_t u_x, \quad T^2 = \frac{1}{2} u_t^2 + \frac{k}{2} u_x^2 - \alpha \cosh u - \beta \sinh u,$$

Remark. It can easily be verified that $D_i T^i|_{(2.1)} = 0$ for both the cases.

2.4.3 Application of the new conservation theorem

We now use the new conservation theorem given in [31] and obtain conservation laws for the combined sinh-cosh-Gordon equation (2.1). The adjoint equation of (2.1), by invoking (2.52), is

$$E^*(t, x, u, v, \dots, u_{xx}, v_{xx}) = \frac{\delta}{\delta u} \left[u_{tt} - k u_{xx} + \alpha \sinh u + \beta \cosh u \right] = 0, \quad (2.71)$$

where $v = v(t, x)$ is a new dependent variable and (2.71) gives

$$v_{tt} - k v_{xx} + v(\alpha \cosh u + \beta \sinh u) = 0. \quad (2.72)$$

It is obvious from the adjoint equation (2.72) that equation (2.59) is not self-adjoint. By recalling (2.57), we get the following Lagrangian for the system of equations (2.1) and (2.72):

$$L = v(u_{tt} - k u_{xx} + \alpha \sinh u + \beta \cosh u). \quad (2.73)$$

(i) We first consider the Lie point symmetry generator $X_1 = \frac{\partial}{\partial t}$. It can be verified from (2.54) that the operator Y_1 is the same as X_1 and hence the Lie characteristic function is $W = -u_t$. Thus, by using (2.58), the components T^i , $i = 1, 2$, of the conserved vector $T = (T^1, T^2)$ are given by

$$T^1 = v(-k u_{xx} + \alpha \sinh u + \beta \cosh u) + u_t v_t, \quad T^2 = -k u_t v_x + k v u_{tx}.$$

Remark. The conserved vector T contains the arbitrary solution v of the adjoint equation (2.72) and hence gives an infinite number of conservation laws.

The same remark applies to all the following cases where we use the conservation theorem.

(ii) For the symmetry generator $X_2 = \frac{\partial}{\partial x}$, we have $W = -u_x$. Hence, by invoking (2.58), the symmetry generator X_2 gives rise to the following components of the conserved vector:

$$T^1 = v_t u_x - v u_{tx}, \quad T^2 = v(u_{tt} + \alpha \sinh u + \beta \cosh u) - k v_x u_x.$$

(iii) The symmetry generator $X_3 = x \frac{\partial}{\partial t} + kt \frac{\partial}{\partial x}$ has the Lie characteristic function $W = -xu_t - ktu_x$. Hence, using (2.58), one can obtain the conserved vector T whose components are given by

$$\begin{aligned} T^1 &= xv(-ku_{xx} + \alpha \sinh u + \beta \cosh u) + xv_t u_t + ktv_t u_x - kvu_x - ktvu_{tx}, \\ T^2 &= ktv(u_{tt} + \alpha \sinh u + \beta \cosh u) - kxv_x u_t - k^2 t v_x u_x + kvu_t + kxvu_{tx}, \end{aligned}$$

which are now the components of the conserved vector T for the equation (2.1).

Remark. One can easily verify, for each of the three cases, that the equation $D_i T^i|_{(2.1)} = 0$ is satisfied.

2.5 Conclusion

Solutions and conservation laws were constructed for the combined sinh-cosh-Gordon equation. Three methods were employed to search for conservation laws, namely, the direct method, Noether approach and the new conservation theorem. The direct method and the Noether approach gave two conserved quantities. However, infinitely many conservation laws were obtained using the new conservation theorem.

Chapter 3

Flow and heat transfer between slowly expanding or contracting walls

In this chapter a nonlinear flow problem of an incompressible viscous fluid is studied. An analysis has been carried out for the flow and heat transfer of an incompressible laminar and viscous fluid in a rectangular domain bounded by two moving porous walls which enable the fluid to enter or exit during successive expansions or contractions. The basic equations governing the flow are reduced to the ordinary differential equations using Lie group analysis. Effects of the permeation Reynolds number R_e , porosity R and the dimensionless wall dilation rate α on the self-axial velocity are studied both analytically and numerically. The analytical procedure is based on double perturbation in the permeation Reynolds number R_e and the wall expansion ratio α where as the numerical solution is obtained using Runge-kutta method with shooting technique. Results are correlated and compared for some values of the physical parameters. Lastly we look at the temperature distribution.

This work is new and has been submitted for publication. See [32].

3.1 Introduction

Both the flow and heat transfer in a viscous fluid over a stretching surface have been extensively investigated during the past decades owing to its importance in industrial and engineering applications. For example, Berman [9] studied the steady flow in a channel with stationary walls and small Reynolds. Majdalani *et al* [11] considered the two-dimensional viscous flow between slowly expanding and contracting walls with weak permeability. Their study focused on the viscous flow driven by small wall contractions and expansions of two weakly permeable walls. Based on double perturbations in the permeation Reynolds number R_e and wall dilation rate α , they carried out their analytical procedure. Boutros *et al* [12] studied the solution of the Navier-Stokes equations which described the unsteady incompressible laminar flow in a semi-infinite porous circular pipe with injection or suction through the pipe wall whose radius varies with time. The resulting fourth-order nonlinear differential equation was then solved using small-parameter perturbations. Asghar *et al* [14] used the Lie group analysis to compute exact solution for the flow of viscous fluid through expanding-contracting channels.

The purpose of this research is to generalize the flow analysis of ref [12] in two directions. The first generalization is concerned with heat distribution while the second accounts for the features of porous medium. The salient features have been taken into account when the fluid saturates the porous medium. Like in [12], the analytic solution for the arising nonlinear flow problem is given by employing the Lie group method (with R_e and α as the perturbation quantities). Finally, the graphs of velocity and temperature are plotted and discussed.

3.2 Mathematical formulation of the problem

Let us consider a rectangular domain bounded by two walls of equal permeability that enable the fluid to enter or exit during successive expansions or contractions. The walls expand or contract uniformly at the time-dependent rate $\dot{h}$. The continuous

sheet aligned with the x -axis at $y = 0$ means the wall is impulsively stretched with the velocity U_w along the x -axis and $T_w(x, t)$ as our surface temperature. At $y = h(t)$ it is assumed that the fluid inflow velocity V_w is independent of the position. A thin fluid film with uniform thickness $h(t)$ rests on the horizontal wall. The governing time-dependent equations for mass, momentum and energy are given by

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0, \quad (3.1)$$

$$\frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{1}{\rho} \frac{\partial \bar{P}}{\partial \bar{x}} + \nu \left[\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \right] - \frac{\nu \phi}{k} \bar{u}, \quad (3.2)$$

$$\frac{\partial \bar{v}}{\partial t} + \bar{u} \frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{v}}{\partial \bar{y}} = -\frac{1}{\rho} \frac{\partial \bar{P}}{\partial \bar{y}} + \nu \left[\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{y}^2} \right] - \frac{\nu \phi}{k} \bar{v} + g\beta(T - T_w), \quad (3.3)$$

$$\frac{\partial T}{\partial t} + \bar{u} \frac{\partial T}{\partial \bar{x}} + \bar{v} \frac{\partial T}{\partial \bar{y}} = \alpha \left[\frac{\partial^2 T}{\partial \bar{x}^2} + \frac{\partial^2 T}{\partial \bar{y}^2} \right], \quad (3.4)$$

where $\bar{u}$ and $\bar{v}$ are the velocity components in the $\bar{x}$ and $\bar{y}$ directions respectively and T is temperature. We assume that the fluid properties are constant. Here ρ is the fluid density, μ is the dynamic viscosity and $\bar{k}$ is the thermal conductivity of an incompressible fluid. Thus, the kinematic viscosity is $\nu = \frac{\mu}{\rho}$, g is the acceleration due to gravity, β is the coefficient of the thermal expansion and the thermal diffusivity is $\alpha = \frac{k}{\rho c_p}$, where c_p is the specific heat, $\bar{P}$ is the pressure, t is time, ϕ and k are the porosity and permeability of porous medium respectively.

The appropriate boundary conditions are:

$$\begin{aligned} \text{(i)} \quad & \bar{u} = 0, \quad \bar{v} = -V_w, \quad T = T_w \quad \text{at } \bar{y} = h(t), \\ \text{(ii)} \quad & \frac{\partial \bar{u}}{\partial \bar{y}} = 0, \quad \bar{v} = 0, \quad \frac{\partial T}{\partial \bar{y}} = 0 \quad \text{at } \bar{y} = 0, \\ \text{(iii)} \quad & \bar{u} = 0 \quad \text{at } \bar{x} = 0, \end{aligned} \quad (3.5)$$

where $h(t)$ is the film thickness. Boundary condition reflect that the fluid motion within the liquid film is caused by the viscous shear arising from the stretching of the elastic wall.

Now we shall express the axial velocity, normal velocity and boundary conditions in terms of the stream function $\bar{\Psi}$. From the continuity equation (3.1), there exists a

dimensional stream function $\bar{\Psi}(\bar{x}, \bar{y}, t)$ such that

$$\bar{u} = \frac{\partial \bar{\Psi}}{\partial \bar{y}}, \quad \bar{v} = -\frac{\partial \bar{\Psi}}{\partial \bar{x}}, \quad (3.6)$$

which satisfies (3.1) identically. Introducing the dimensionless normal coordinate $y = \bar{y}/h(t)$, equation (3.6) becomes

$$\bar{u} = \frac{1}{h} \frac{\partial \bar{\Psi}}{\partial y}, \quad \bar{v} = -\frac{\partial \bar{\Psi}}{\partial \bar{x}}. \quad (3.7)$$

Substituting (3.7) into (3.2)-(3.4) we obtain

$$\begin{aligned} h^2 \bar{\Psi}_{yt} - h \dot{h} y \bar{\Psi}_{yy} - h \dot{h} \bar{\Psi}_y + h \bar{\Psi}_y \bar{\Psi}_{xy} - h \bar{\Psi}_x \bar{\Psi}_{yy} = -\frac{h^3}{\rho} \bar{P}_x + \nu [h^2 \bar{\Psi}_{xxy} \\ + \bar{\Psi}_{yyy}] - \frac{h^2 \nu \phi}{k} \bar{\Psi}_y, \end{aligned} \quad (3.8)$$

$$\begin{aligned} -h^2 \bar{\Psi}_{xt} + h \dot{h} y \bar{\Psi}_{xy} - h \bar{\Psi}_y \bar{\Psi}_{xx} + h \bar{\Psi}_x \bar{\Psi}_{xy} = -\frac{h}{\rho} \bar{P}_y + \nu [-h^2 \bar{\Psi}_{xxy} - \bar{\Psi}_{xyy}] \\ + \frac{h^2 \nu \phi}{k} \bar{\Psi}_x + g\beta(T - T_w)h^2, \end{aligned} \quad (3.9)$$

$$\frac{\partial T}{\partial t} + \frac{1}{h} \bar{\Psi}_y \frac{\partial T}{\partial \bar{x}} - \bar{\Psi}_x \frac{\partial T}{\partial \bar{y}} = \alpha \left[\frac{\partial^2 T}{\partial \bar{x}^2} + \frac{\partial^2 T}{\partial \bar{y}^2} \right], \quad (3.10)$$

where a dot denotes the derivative with respect to t . The variables in the equations (3.8)-(3.10) are dimensionless according to

$$\begin{aligned} u = \frac{\bar{u}}{V_w}, \quad v = \frac{\bar{v}}{V_w}, \quad x = \frac{\bar{x}}{h(t)}, \quad y = \frac{\bar{y}}{h(t)}, \quad \Psi = \frac{\bar{\Psi}}{hV_w}, \quad P = \frac{\bar{P}}{\rho V_w^2}, \\ t = \frac{\bar{t}}{hV_w}, \quad \alpha = \frac{h\dot{h}}{\nu}, \quad \theta = \frac{T - T_h}{T_w - T_h}, \quad \frac{1}{R} = \frac{\nu \phi a}{kV_w}. \end{aligned} \quad (3.11)$$

Substituting (3.11) into (3.8)-(3.10) we have

$$\begin{aligned} \Psi_{yt} + \Psi_y \Psi_{xy} - \Psi_x \Psi_{yy} + P_x - \frac{1}{R_e} [\alpha \Psi_y + \alpha y \Psi_{yy} + \Psi_{xxy} + \Psi_{yyy}] \\ - \frac{1}{R} \Psi_y = 0, \end{aligned} \quad (3.12)$$

$$\begin{aligned} \Psi_{xt} + \Psi_y \Psi_{xx} - \Psi_x \Psi_{xy} - P_y - \frac{1}{R_e} [\alpha y \Psi_{xy} + \Psi_{xyy} + \Psi_{xxx}] \\ + \frac{1}{R} \Psi_x + \frac{1}{h^2} G_r \theta = 0, \end{aligned} \quad (3.13)$$

$$\frac{\partial \theta}{\partial \bar{t}} + \Psi_y \frac{\partial \theta}{\partial x} - \Psi_x \frac{\partial \theta}{\partial y} = \frac{1}{P_r R_e} \left[\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right], \quad (3.14)$$

in which $R_e = hV_w/\nu$ is the permeation Reynolds number, $G_r = \frac{g\beta(T_w - T_h)h^3}{v^2}$ is the Grashof number, $P_r = \frac{\nu}{\alpha}$ is the Prandtl number and $\dot{h} = \frac{\alpha\nu}{h}$. Through (3.7) and (3.11), we have

$$u = \frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x} \quad (3.15)$$

and thus the boundary conditions take the following forms

$$\begin{aligned} \text{(i)} \quad & \Psi_y = 0, \quad \Psi_x = 1, \quad \theta = 1 \quad \text{at } y = 1, \\ \text{(ii)} \quad & \Psi_{yy} = 0, \quad \Psi_x = 0, \quad \theta_y = 0 \quad \text{at } y = 0, \\ \text{(iii)} \quad & \Psi_y = 0 \quad \text{at } x = 0. \end{aligned} \quad (3.16)$$

3.3 Solution of the problem

This section derives the similarity solutions using Lie-group method under which (3.12)-(3.14) are invariant.

3.3.1 Lie symmetry analysis

We consider the one-parameter (ε) Lie group of infinitesimal transformation in $(x, y, \bar{t}, \Psi, P, \theta)$ given by

$$\begin{aligned} x^* &= x + \varepsilon \phi(x, y, \bar{t}, \Psi, P, \theta) + 0(\varepsilon^2), \\ y^* &= y + \varepsilon \zeta(x, y, \bar{t}, \Psi, P, \theta) + 0(\varepsilon^2), \\ \bar{t}^* &= \bar{t} + \varepsilon F(x, y, \bar{t}, \Psi, P, \theta) + 0(\varepsilon^2), \\ \Psi^* &= \Psi + \varepsilon \eta(x, y, \bar{t}, \Psi, P, \theta) + 0(\varepsilon^2), \\ P^* &= P + \varepsilon g(x, y, \bar{t}, \Psi, P, \theta) + 0(\varepsilon^2), \\ \theta^* &= \theta + \varepsilon H(x, y, \bar{t}, \Psi, P, \theta) + 0(\varepsilon^2), \end{aligned} \quad (3.17)$$

with ε as a small parameter. In view of Lie's algorithm, the vector field

$$X = \phi \frac{\partial}{\partial x} + \zeta \frac{\partial}{\partial y} + F \frac{\partial}{\partial \bar{t}} + \eta \frac{\partial}{\partial \Psi} + g \frac{\partial}{\partial P} + H \frac{\partial}{\partial \theta}, \quad (3.18)$$

if it is left variant by the transformation $(x, y, \bar{t}, \Psi, P, \theta) \rightarrow (x^*, y^*, t^*, \Psi^*, P^*, \theta^*)$.

The solutions $\Psi = \Psi(x, y, \bar{t})$, $P = P(x, y, \bar{t})$ and $\theta = \theta(x, y, \bar{t})$, are invariant under the symmetry (3.18) if

$$\Phi_\Psi = X(\Psi - \Psi(x, y, \bar{t})) = 0, \quad \text{where} \quad \Psi = \Psi(x, y, \bar{t}), \quad (3.19)$$

$$\Phi_P = X(P - P(x, y, \bar{t})) = 0, \quad \text{where} \quad P = P(x, y, \bar{t}), \quad (3.20)$$

$$\Phi_\theta = X(\theta - \theta(x, y, \bar{t})) = 0, \quad \text{where} \quad \theta = \theta(x, y, \bar{t}). \quad (3.21)$$

We set

$$\begin{aligned} \Delta_1 &= \Psi_{y\bar{t}} + \Psi_y \Psi_{xy} - \Psi_x \Psi_{yy} + P_x - \frac{1}{R_e} [\alpha \Psi_y + \alpha y \Psi_{yy} + \Psi_{xxy} + \Psi_{yyy}] \\ &\quad - \frac{1}{R} \Psi_y, \\ \Delta_2 &= \Psi_{x\bar{t}} + \Psi_y \Psi_{xx} - \Psi_x \Psi_{xy} - P_y - \frac{1}{R_e} [\alpha y \Psi_{xy} + \Psi_{xyy} + \Psi_{xxx}] \\ &\quad + \frac{1}{R} \Psi_x + \frac{1}{h^2} G_r \theta, \\ \Delta_3 &= \frac{\partial \theta}{\partial \bar{t}} + \Psi_y \frac{\partial \theta}{\partial x} - \Psi_x \frac{\partial \theta}{\partial y} - \frac{1}{P_r R_e} \left[\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right]. \end{aligned}$$

The vector field X given by (3.18) is a symmetry generator of equations (3.12)-(3.14) if and only if

$$X^{[3]}(\Delta_j) \Big|_{\Delta_j=0} = 0, \quad j = 1, 2, 3, \quad (3.22)$$

in which

$$\begin{aligned} X^{[3]} &= \phi \frac{\partial}{\partial x} + \zeta \frac{\partial}{\partial y} + F \frac{\partial}{\partial \bar{t}} + \eta \frac{\partial}{\partial \Psi} + g \frac{\partial}{\partial P} + H \frac{\partial}{\partial \theta} + \eta^x \frac{\partial}{\partial \Psi_x} + \eta^y \frac{\partial}{\partial \Psi_y} \\ &\quad + g^x \frac{\partial}{\partial P_x} + g^y \frac{\partial}{\partial P_y} + H^x \frac{\partial}{\partial \theta_x} + H^y \frac{\partial}{\partial \theta_y} + H^{\bar{t}} \frac{\partial}{\partial \theta_{\bar{t}}} + \eta^{xy} \frac{\partial}{\partial \Psi_{xy}} \\ &\quad + \eta^{x\bar{t}} \frac{\partial}{\partial \Psi_{x\bar{t}}} + \eta^{y\bar{t}} \frac{\partial}{\partial \Psi_{y\bar{t}}} + \eta^{xx} \frac{\partial}{\partial \Psi_{xx}} + \eta^{yy} \frac{\partial}{\partial \Psi_{yy}} + H^{xx} \frac{\partial}{\partial \theta_{xx}} + H^{yy} \frac{\partial}{\partial \theta_{yy}} \\ &\quad + \eta^{xxy} \frac{\partial}{\partial \Psi_{xxy}} + \eta^{xyy} \frac{\partial}{\partial \Psi_{xyy}} + \eta^{xxx} \frac{\partial}{\partial \Psi_{xxx}} + \eta^{yyy} \frac{\partial}{\partial \Psi_{yyy}} \end{aligned} \quad (3.23)$$

is the third prolongation of X . We now introduce the total derivatives by differentiating (3.17) with respect to $x, y, \bar{t}$ and construct

$$\begin{aligned}
D_x &= \partial_x + \Psi_x \partial_\Psi + P_x \partial_P + \theta_x \partial_\theta + \Psi_{xx} \partial_{\Psi_x} + P_{xx} \partial_{P_x} + \theta_{xx} \partial_{\theta_x} + \Psi_{xy} \partial_{\Psi_y} \\
&\quad + \theta_{xy} \partial_{\theta_y} + \cdots, \\
D_y &= \partial_y + \Psi_y \partial_\Psi + P_y \partial_P + \theta_y \partial_\theta + \Psi_{yy} \partial_{\Psi_y} + P_{yy} \partial_{P_y} + \theta_{yy} \partial_{\theta_y} + \Psi_{xy} \partial_{\Psi_x} \\
&\quad + \theta_{xy} \partial_{\theta_x} + \cdots, \\
D_{\bar{t}} &= \partial_{\bar{t}} + \Psi_{\bar{t}} \partial_\Psi + P_{\bar{t}} \partial_P + \theta_{\bar{t}} \partial_\theta + \Psi_{\bar{t}t} \partial_{\Psi_{\bar{t}}} + P_{\bar{t}t} \partial_{P_{\bar{t}}} + \theta_{\bar{t}t} \partial_{\theta_{\bar{t}}} + \Psi_{x\bar{t}} \partial_{\Psi_x} \\
&\quad + \theta_{x\bar{t}} \partial_{\theta_x} + \cdots.
\end{aligned} \tag{3.24}$$

Choosing G_r small when $T_h \approx T_w$, the system of equations (3.12)-(3.14) has the six parameter Lie group point of symmetries generated by

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial \bar{t}}, \quad X_2 = \theta \frac{\partial}{\partial \bar{t}}, \quad X_3 = \frac{\partial}{\partial \theta}, \quad X_4 = \frac{\partial}{\partial \Psi}, \\
X_5 &= F_2(\bar{t}) \frac{\partial}{\partial y}, \quad X_6 = F_1(\bar{t}) \frac{\partial}{\partial x}.
\end{aligned} \tag{3.25}$$

3.3.2 Invariant solution

When calculating invariant solutions under the group generators X_3 and X_4 , we found that there are no invariant solutions. Then X_5 and X_6 give solutions of (3.1)-(3.3) and this contradicts the boundary conditions. For X_1 and X_2 , the characteristic $\Phi = (\Phi_\Psi, \Phi_P, \Phi_\theta)$ has the components

$$\Phi_\Psi = -\Psi_{\bar{t}}, \quad \Phi_P = -P_{\bar{t}}, \quad \Phi_\theta = -\theta_{\bar{t}}.$$

The general solutions of invariant surface conditions (3.19)-(3.21) are given by

$$\Psi = h(y)H(x, y), \quad P = \Gamma(x, y), \quad \theta = \tau(x, y). \tag{3.26}$$

Invoking (3.26) into (3.12) we have

$$\begin{aligned}
& -K \frac{d^3 h}{dy^3} + \left[-\alpha K y - h K_1 - 3K K_2 \right] \frac{d^2 h}{dy^2} \\
& + \left[-\alpha K - 2\alpha K y K_2 - h K_3 + h K_4 - K K_5 - 3K K_6 + \frac{1}{R} \right] \frac{dh}{dy} \\
& K_1 \left(\frac{dh}{dy} \right)^2 + \left[-\alpha K K_2 + \frac{1}{R} K_2 - \alpha K K_6 y - K K_9 - K K_{10} \right] h \\
& + \left[K_7 - K_8 \right] h^2 + \frac{1}{H} \frac{d\Gamma}{dx}, \tag{3.27}
\end{aligned}$$

$$\begin{aligned}
K_1 &= H_x, \quad K_2 = \frac{H_y}{H}, \quad K_3 = \frac{H_x H_y}{H}, \quad K_4 = H_{xy}, \\
K_5 &= \frac{H_{xx}}{H}, \quad K_6 = \frac{H_{yy}}{H}, \quad K_7 = \frac{H_y H_{xy}}{H}, \quad K_8 = \frac{H_x H_{yy}}{H}, \\
K_9 &= \frac{H_{xxy}}{H}, \quad K_{10} = \frac{H_{yyy}}{H}, \quad K = R_c. \tag{3.28}
\end{aligned}$$

Integration of $H_x = K_1$ from (3.28) leads to the following expression

$$H(x, y) = x K_1(y) + K_{11}(y). \tag{3.29}$$

The above equation when used into $\Psi = h(y)H(x, y)$ (from (3.26)) we get

$$\Psi = (x K_1(y) + K_{11}(y)) h(y), \tag{3.30}$$

which after differentiating with respect to y and using (3.16) (iii) yields

$$K_{11}(y) h(y) = K_{12}, \tag{3.31}$$

where K_{12} is a constant of integration and hence (3.30) reads

$$\Psi = x G(y) + K_{12} \tag{3.32}$$

with $G(y) = K_1(y) h(y)$.

Putting $P = \Gamma(x, y)$ from (3.26) and (3.29) into the last term of (3.27) yields

$$K_{11} = 0. \tag{3.33}$$

With the help of (3.29) and (3.33) one obtains

$$H(x, y) = x K_1(y), \tag{3.34}$$

while (3.31)-(3.33) yields

$$\Psi = xG(y). \quad (3.35)$$

Due to (3.15) and (3.35) one may write

$$u = x \frac{dG}{dy}, \quad v = -G. \quad (3.36)$$

Using (3.35) in (3.13) and then differentiating with respect to x , one arrives at the following result

$$P_{xy} = \frac{1}{h^2} G_r \theta_x. \quad (3.37)$$

Putting (3.35) in (3.12), differentiating with respect to y and then using (3.37) we obtain

$$\begin{aligned} \frac{d^4 G}{dy^4} x + \alpha \left[y \frac{d^3 G}{dy^3} + 2 \frac{d^2 G}{dy^2} \right] x + R_e G \frac{d^3 G}{dy^3} x - R_e \frac{dG}{dy} \frac{d^2 G}{dy^2} x - R_e \frac{d^2 G}{dy^2} \frac{1}{R} x \\ + \frac{1}{h^2} G_r \theta_x = 0. \end{aligned} \quad (3.38)$$

Using (3.35) and $\theta = \tau(x, y)$ from (3.26) in (3.14), we can write

$$x \frac{dG}{dy} \frac{\partial \tau}{\partial x} - G \frac{\partial \tau}{\partial y} - \frac{1}{P_r R_e} \left[\frac{\partial^2 \tau}{\partial x^2} + \frac{\partial^2 \tau}{\partial y^2} \right] = 0, \quad (3.39)$$

and the boundary conditions (3.16) become

$$\begin{aligned} \text{(i)} \quad \frac{dG(1)}{dy} = 0, \quad \text{(ii)} \quad G(1) = 1, \quad \text{(iii)} \quad \frac{d^2 G(0)}{dy^2} = 0, \\ \text{(iv)} \quad G(0) = 0, \quad \text{(v)} \quad \tau(x, 1) = 0, \quad \text{(vi)} \quad \tau(x, 0) = 1. \end{aligned} \quad (3.40)$$

Using $\theta = \tau(x, y)$ and Equating like powers of h , equation (3.38) helps in writing the following equations

$$\begin{aligned} \frac{d^4 G}{dy^4} + \alpha \left[y \frac{d^3 G}{dy^3} + 2 \frac{d^2 G}{dy^2} \right] + R_e G \frac{d^3 G}{dy^3} - R_e \frac{dG}{dy} \frac{d^2 G}{dy^2} - R_e \frac{d^2 G}{dy^2} \frac{1}{R} = 0, \\ G_r \tau_x = 0. \end{aligned}$$

The above equation implies that $\tau = E(y)$ and $G_r = 0$ which satisfy our assumption that G_r is very small. Now equations (3.38)-(3.40) yields

$$\frac{d^4 G}{dy^4} + \alpha \left[y \frac{d^3 G}{dy^3} + 2 \frac{d^2 G}{dy^2} \right] + R_e G \frac{d^3 G}{dy^3} - R_e \frac{dG}{dy} \frac{d^2 G}{dy^2} - R_e \frac{d^2 G}{dy^2} \frac{1}{R} = 0, \quad (3.41)$$

$$G(y) \frac{\partial E}{\partial y} + \frac{1}{P_r R_e} \left[\frac{\partial^2 E}{\partial y^2} \right] = 0, \quad (3.42)$$

$$\begin{aligned}
& \text{(i)} \quad \frac{dG(1)}{dy} = 0, & \text{(ii)} \quad G(1) = 1, & \text{(iii)} \quad \frac{d^2G(0)}{dy^2} = 0, \\
& \text{(iv)} \quad G(0) = 0, & \text{(v)} \quad E(1) = 1, & \text{(vi)} \quad E'(0) = 0.
\end{aligned} \tag{3.43}$$

3.3.3 Analytical solution

The aim of this section is to find the solutions of equations (3.41)-(3.43) using double perturbation [11, 12]. For small R_e and α we expand

$$\begin{aligned}
G &= G_1 + R_e G_2 + O(R_e^2), & G_1 &= G_{10} + \alpha G_{11} + O(\alpha^2), \\
G_2 &= G_{20} + \alpha G_{21} + O(\alpha^2).
\end{aligned} \tag{3.44}$$

Using (3.44) into (3.41)-(3.43) and then solving the resulting problems for small R_e and α we obtain

$$G_{10}(y) = -\frac{1}{2}y^3 + \frac{3}{2}y, \quad G_{11}(y) = \frac{3}{40}y^5 - \frac{3}{20}y^3 + \frac{3}{40}y, \tag{3.45}$$

$$G_{20}(y) = \frac{1}{280}y^7 - \frac{3}{280}y^3 + \frac{1}{140}y + \frac{1}{R} \left(-\frac{1}{40}y^5 + \frac{1}{20}y^3 - \frac{1}{40}y \right), \tag{3.46}$$

$$\begin{aligned}
G_{21}(y) &= -\frac{13}{20160}y^9 - \frac{9}{2800}y^7 + \frac{9}{5600}y^5 + \frac{227}{25200}y^3 + \frac{227}{33600}y \\
&\quad + \frac{1}{R} \left(\frac{1}{210}y^7 - \frac{3}{200}y^5 + \frac{11}{700}y^3 - \frac{23}{4200}y \right),
\end{aligned} \tag{3.47}$$

$$G_1(y) = -\frac{1}{2}y^3 + \frac{3}{2}y + \alpha \left[\frac{3}{40}y^5 - \frac{3}{20}y^3 + \frac{3}{40}y \right], \tag{3.48}$$

$$\begin{aligned}
G_2(y) &= \frac{1}{280}y^7 - \frac{3}{280}y^3 + \frac{1}{140}y + \frac{1}{R} \left(-\frac{1}{40}y^5 + \frac{1}{20}y^3 - \frac{1}{40}y \right) \\
&\quad + \alpha \left[-\frac{13}{20160}y^9 - \frac{9}{2800}y^7 + \frac{9}{5600}y^5 + \frac{227}{25200}y^3 - \frac{227}{33600}y \right. \\
&\quad \left. + \frac{1}{R} \left(\frac{1}{210}y^7 - \frac{3}{200}y^5 + \frac{11}{700}y^3 - \frac{23}{4200}y \right) \right].
\end{aligned} \tag{3.49}$$

and

$$\begin{aligned}
G(y) = & \left(-\frac{1}{2}y^3 + \frac{3}{2}y + \alpha \left[\frac{3}{40}y^5 - \frac{3}{20}y^3 + \frac{3}{40}y \right] \right) \\
& + R_e \left(\frac{1}{280}y^7 - \frac{3}{280}y^3 + \frac{1}{140}y + \frac{1}{R} \left(-\frac{1}{40}y^5 + \frac{1}{20}y^3 - \frac{1}{40}y \right) \right. \\
& + \alpha \left[-\frac{13}{20160}y^9 - \frac{9}{2800}y^7 + \frac{9}{5600}y^5 + \frac{227}{25200}y^3 - \frac{227}{33600}y \right. \\
& \left. \left. + \frac{1}{R} \left(\frac{1}{210}y^7 - \frac{3}{200}y^5 + \frac{11}{700}y^3 - \frac{23}{4200}y \right) \right] \right). \tag{3.50}
\end{aligned}$$

It is noted that for $R \rightarrow \infty$, the expression of $G(y)$ in [12] is recovered.

Let

$$E = E_1 + R_e E_2 + O(R_e^2). \tag{3.51}$$

From (3.42), (3.50) and (3.51) we obtain

$$\frac{d^2 E_1}{dy^2} = 0, \quad E_1(1) = 1, \quad E_1'(0) = 0, \tag{3.52}$$

$$Pr G(y) \frac{dE_1(y)}{dy} + \frac{d^2 E_2(y)}{dy^2}, \quad E_2(1) = 0, \quad E_2'(0) = 0. \tag{3.53}$$

Solving above problems and using (3.51) one obtains

$$E(y) = 1. \tag{3.54}$$

3.3.4 Numerical solution

Now numerical solution of equations (3.41)-(3.43) have been obtained using shooting method with Runge-Kutta scheme.

3.4 Results and discussion

Figures 3.1-3.4 illustrate the behaviour of self-axial velocity over a range of R with R_e and α fixed. Figures 3.1 and 3.2 illustrate the behaviour of self-axial velocity u/x for permeation Reynolds number $R_e = 1$ (injection) and $\alpha = 0.5, -0.5$ (expansion

and contraction respectively) over a range of porosity parameter R . For $R > 0$, these Figures show that the higher the porosity R leads to higher self-axial velocity near the center and lower near the wall. The results for $R < 0$ are quite opposite to that of $R > 0$. A comparative study of these Figures further indicates that the self-axial velocity near the center in case of injection with expanding wall and high porosity is higher than injection with contracting wall and high porosity.

The plots of self-axial velocity u/x for permeation Reynolds number $R_e = -1$ (suction) and $\alpha = 0.5, -0.5$ (expansion and contraction respectively) over a range of R have been displayed in Figures 3.3 and 3.4. In case of $R > 0$, these graphs depict that the higher the porosity R leads to lower self-axial velocity near the center and higher near the wall. For $R < 0$, these Figures depict that the lower porosity R leads to higher self-axial velocity near the center and lower near the wall. By comparing Figures 3.3 and 3.4 we note that the self-axial velocity near the center in case of suction with expanding wall and high porosity is higher than suction with contracting wall and high porosity.

The behaviour of the self-axial velocity u/x for wall dilation rate $\alpha = -0.5$ (contraction) and $R_e = 1, -1$ (injection and suction) over a range of R has been displayed in the Figures 3.2 and 3.3. For $R > 0$, Figure 3.3 shows that the higher the porosity R , the lower the self-axial velocity at the center and higher near the wall. Figure 3.2 shows that the higher the porosity R , the higher the self-axial velocity at the center and lower near the wall. When $R < 0$, Figure 3.3 elucidates that the lower the porosity R gives a higher self-axial velocity near the center and a lower one near the wall. Figure 3.2 elucidates that the lower the porosity R gives a lower self-axial velocity near the center and higher near the wall. A comparative study of Figures 3.2 and 3.3 indicates that the self-axial velocity near the center in case of injection with contracting wall and high porosity is higher than suction with contracting wall and high porosity.

The variations of self-axial velocity u/x for wall dilation rate $\alpha = 0.5$ (expansion) and $R_e = 1, -1$ (injection and suction) over a range of R have been plotted in the

Figures 3.1 and 3.4. When $R > 0$, then Figure 3.1 shows that the higher the porosity R , the higher the self-axial velocity at the center and lower near the wall. Figure 3.4 shows that the higher the porosity R , the lower the self-axial velocity at the center and higher near the wall. When $R < 0$, Figure 3.1 describes that the lower the porosity R gives a lower self-axial velocity near the center and higher near the wall. Figure 3.4 provides that the lower the porosity R yields a higher self-axial velocity near the center and lower near the wall. Comparison of Figures 3.1 and 3.4 leads to the conclusion that the axial-velocity near the center for suction with expanding wall and high porosity is higher than injection with expanding wall and high porosity.

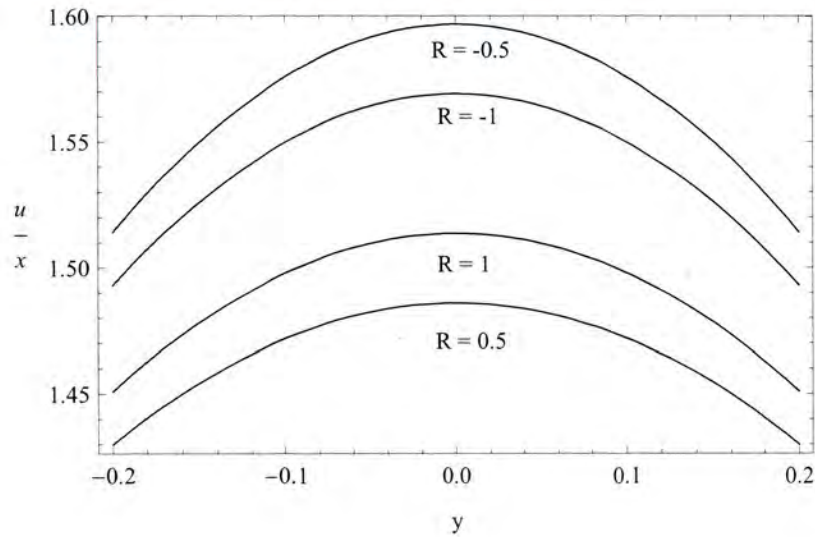


Figure 3.1: Self-axial velocity profiles over a range of R at $R_e = 1$ and $\alpha = 0.5$

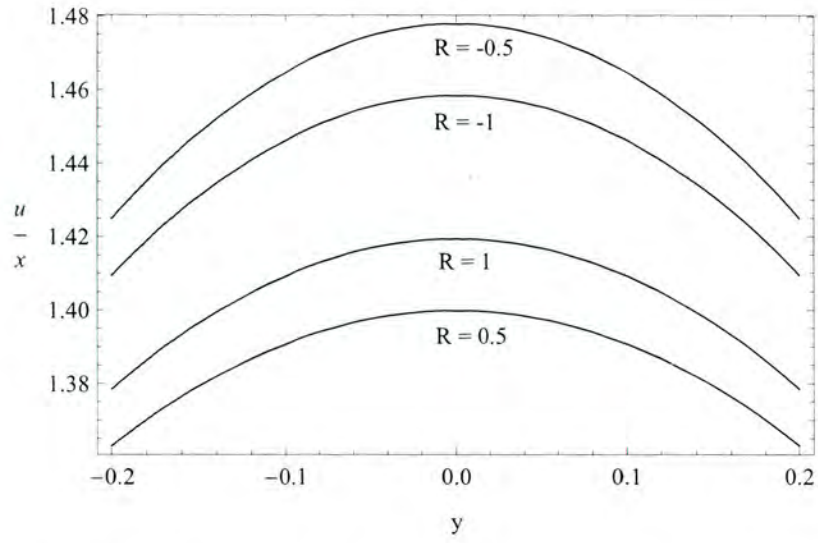


Figure 3.2: Self-axial velocity profiles over a range of R at $R_e = 1$ and $\alpha = -0.5$

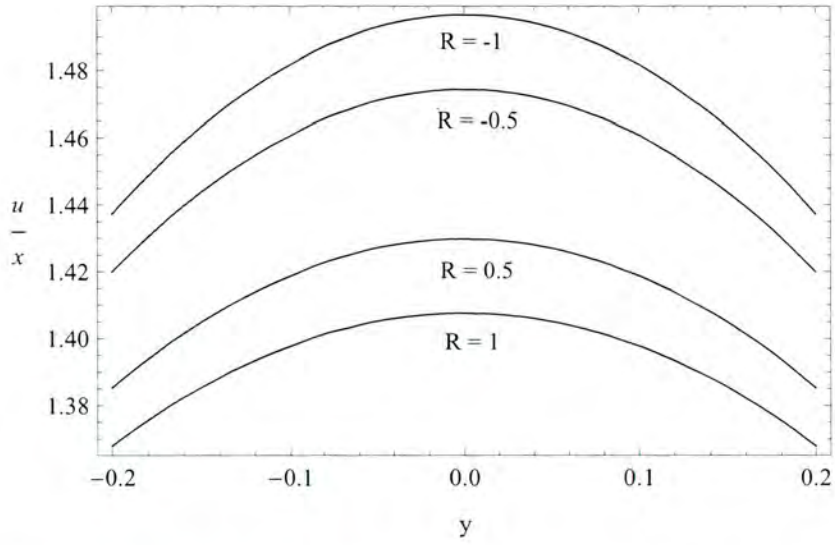


Figure 3.3: Self-axial velocity profiles over a range of R at $R_e = -1$ and $\alpha = -0.5$

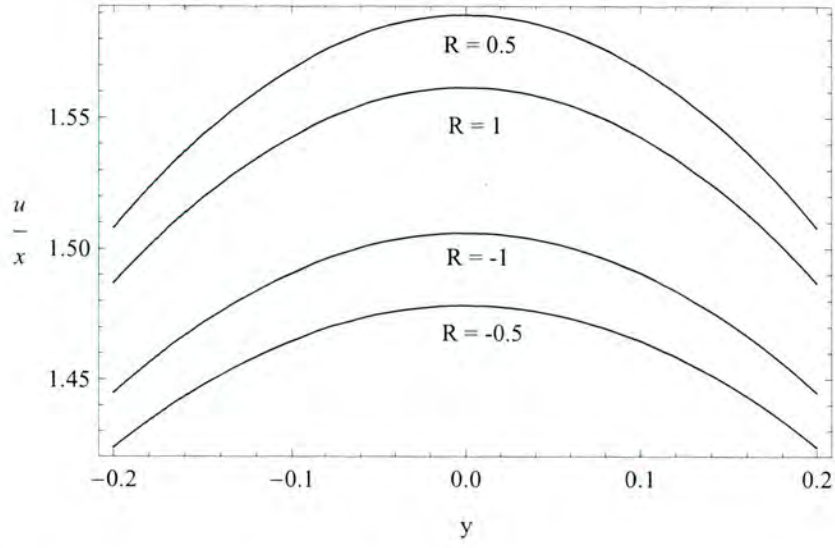


Figure 3.4: Self-axial velocity profiles over a range of R at $R_e = -1$ and $\alpha = 0.5$

Tables 3.1-3.4 depicts that the percentage error is small for different values of R .

Table 3.1: Comparison between analytical and numerical solutions for self-axial velocity u/x at $y = 0.1$ for $R_e = 1, \alpha = 0.5$.

	Analytical Method	Numerical Method	Percentage Error (%)
$R = -1$	1.549755	1.549040	0.046138
$R = -0.5$	1.575774	1.576305	0.033691
$R = 0.5$	1.471699	1.480956	0.625085
$R = 1$	1.497718	1.501653	0.262055

Table 3.2: Comparison between analytical and numerical solutions for self-axial velocity u/x at $y = 0.1$ for $R_e = 1, \alpha = -0.5$.

	Analytical Method	Numerical Method	Percentage Error (%)
$R = -1$	1.480690	1.480513	0.012005
$R = -0.5$	1.501697	1.503207	0.100457
$R = 0.5$	1.417671	1.423694	0.423027
$R = 1$	1.438678	1.440991	0.160530

Table 3.3: Comparison between analytical and numerical solutions for self-axial velocity u/x at $y = 0.1$ for $R_e = -1, \alpha = 0.5$.

	Analytical Method	Numerical Method	Percentage Error (%)
$R = -1$	1.490782	1.490460	0.012660
$R = -0.5$	1.464764	1.466093	0.090696
$R = 0.5$	1.568838	1.583640	0.934648
$R = 1$	1.542820	1.548592	0.372749

Table 3.4: Comparison between analytical and numerical solutions for self-axial velocity u/x at $y = 0.1$ for $R_e = -1, \alpha = -0.5$.

	Analytical Method	Numerical Method	Percentage Error (%)
$R = -1$	1.418772	1.419039	0.018778
$R = -0.5$	1.397766	1.399917	0.153681
$R = 0.5$	1.481791	1.491195	0.630644
$R = 1$	1.460785	1.464232	0.235434

Figures 3.5-3.8 plot the behaviour of self-axial velocity over a range of α with fixed R_e and R .

For $\alpha > 0$, Figures 3.5-3.8 witness that the greater α leads to higher self-axial velocity at the center and lower near the wall. For $\alpha < 0$, these Figures show that an increase in contraction ratio leads to lower self-axial velocity near the center and higher near the wall. By comparing Figures 3.5 and 3.6 we note that the self-axial velocity near the center in case of suction with expanding wall and high porosity is higher than injection with expanding wall and high porosity.

Comparison of Figures 3.5 and 3.8 shows that the self-axial velocity near the center in case of injection with expanding wall and low porosity is higher than injection with expanding wall and high porosity. Comparative study of Figures 3.6 and 3.7 reveals that the self-axial velocity near the center in case of suction with expanding wall and high porosity is higher than suction with expanding wall and low porosity.

By comparing Figures 3.7 and 3.8, the self-axial velocity near the center in case of injection with expanding wall and low porosity is higher than suction with expanding wall and low porosity.

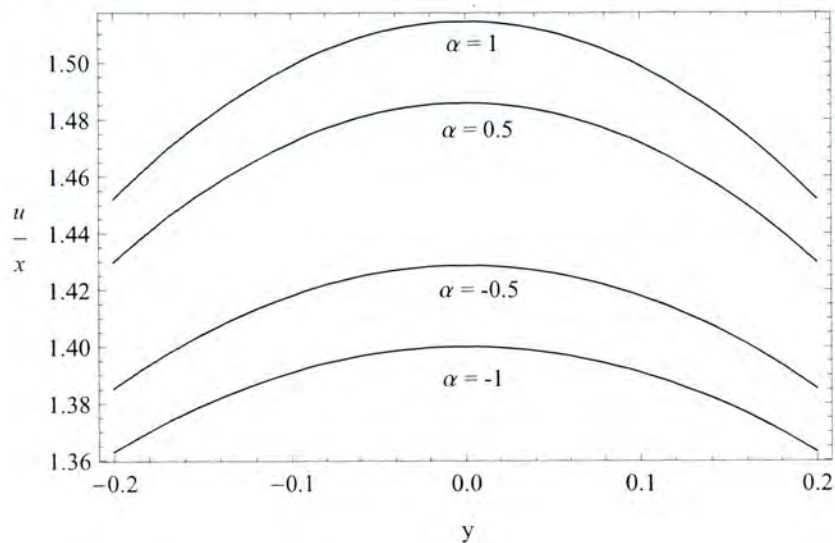


Figure 3.5: Self-axial velocity profiles over a range of α at $Re = 1$ and $R = 0.5$

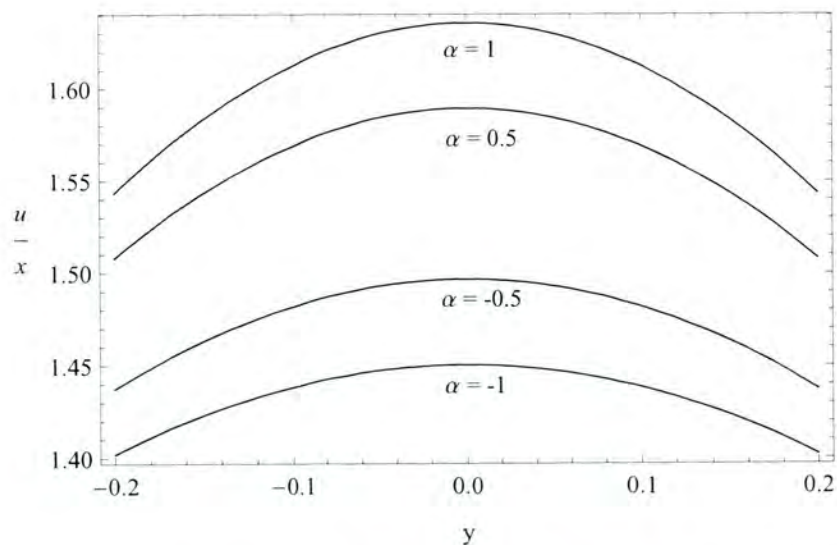


Figure 3.6: Self-axial velocity profiles over a range of α at $Re = -1$ and $R = 0.5$

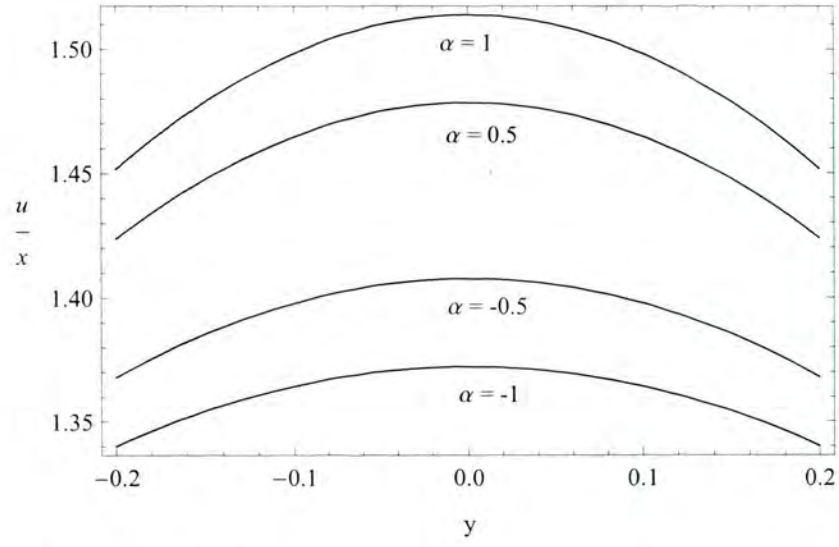


Figure 3.7: Self-axial velocity profiles over a range of α at $R_e = -1$ and $R = -0.5$

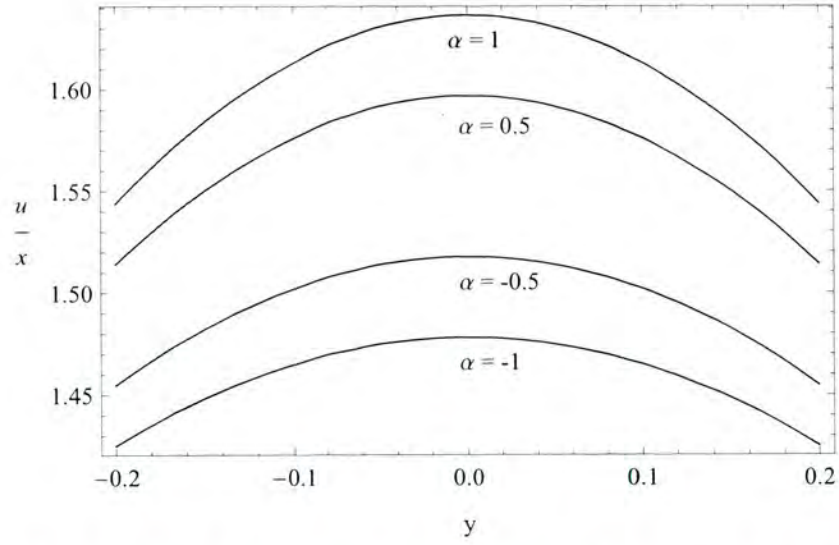


Figure 3.8: Self-axial velocity profiles over a range of α at $R_e = 1$ and $R = -0.5$

Tables 3.5-3.8 indicate the percentage error is small for different values of α .

Table 3.5: Comparison between analytical and numerical solutions for self-axial velocity u/x at $y = 0.1$ for $R = 1, R_e = 1$.

	Analytical Method	Numerical Method	Percentage Error (%)
$\alpha = -1$	1.409157	1.412797	0.257641
$\alpha = -0.5$	1.438678	1.440991	0.160530
$\alpha = 0.5$	1.497718	1.501653	0.262055
$\alpha = 1$	1.527238	1.534003	0.440984

Table 3.6: Comparison between analytical and numerical solutions for self-axial velocity u/x at $y = 0.1$ for $R = 1, R_e = -1$.

	Analytical Method	Numerical Method	Percentage Error (%)
$\alpha = -1$	1.419768	1.426770	0.490778
$\alpha = -0.5$	1.460785	1.464232	0.235434
$\alpha = 0.5$	1.542820	1.548592	0.372749
$\alpha = 1$	1.583837	1.595620	0.738487

Table 3.7: Comparison between analytical and numerical solutions for self-axial velocity u/x at $y = 0.1$ for $R = -1, R_e = -1$.

	Analytical Method	Numerical Method	Percentage Error (%)
$\alpha = -1$	1.382767	1.387131	0.314608
$\alpha = -0.5$	1.418772	1.419039	0.018778
$\alpha = 0.5$	1.490782	1.490460	0.021660
$\alpha = 1$	1.526788	1.530071	0.214601

Table 3.8: Comparison between analytical and numerical solutions for self-axial velocity u/x at $y = 0.1$ for $R = -1, R_e = 1$.

	Analytical Method	Numerical Method	Percentage Error (%)
$\alpha = -1$	1.446158	1.448653	0.172244
$\alpha = -0.5$	1.480690	1.480513	0.012005
$\alpha = 0.5$	1.549755	1.549040	0.046138
$\alpha = 1$	1.584287	1.585538	0.078889

Tables 3.5-3.8 indicate the percentage error is small for different values of α .

Figures 3.9-3.12 illustrate the behaviour of self-axial velocity over a range of R_e with fixed α and R .

The self-axial velocity u/x for porosity parameter $R = 0.5$ (high porosity) and wall dilation rate $\alpha = 0.5, -0.5$ (expansion and contraction respectively) over a range of R_e have been sketched in the Figures 3.9 and 3.10. For $R_e > 0$, we found that increasing injection R_e leads to lower self-axial velocity at the center and a higher one near the wall. When $R_e < 0$, Figures 3.9 and 3.10 indicate that increasing suction ratio leads to higher self-axial velocity near the center and a lower one near the wall. Comparison of Figures 3.9 and 3.10 shows that the self-axial velocity near the center in case of injection with expanding wall and high porosity is higher than injection with contracting wall and high porosity.

Figures 3.11 and 3.12 provide the variation of self-axial velocity u/x for porosity parameter $R = -0.5$ (low porosity) and wall dilation rate $\alpha = 0.5, -0.5$ (expansion and contraction respectively) over a range of R_e . In case of $R_e > 0$, Figures 3.11 and 3.12 show that increasing injection leads to a higher self-axial velocity near the center and a lower one near the wall. For $R_e < 0$, Figures 3.11 and 3.12 show that increasing suction ratio leads to lower self-axial velocity at the center and a higher one near the wall. A comparison between Figures 3.11 and 3.12 shows that the self-axial velocity near the center in case of injection with expanding wall and low porosity is higher than injection with contracting wall and low porosity.

The self-axial velocity u/x for porosity parameter $R = -0.5, 0.5$ (low and high porosity respectively) and wall dilation rate $\alpha = -0.5$ (contraction) over a range of R_e have been explained in Figures 3.10 and 3.11. When $R_e > 0$, then Figure 3.10 shows that increasing injection leads to lower self-axial velocity near the center and a higher one near the wall. Figure 3.11 shows that increasing injection leads to a higher self-axial velocity near the center and a lower one near the wall. In case of $R_e < 0$, Figure 3.10 shows that increasing suction ratio leads to a higher self-axial velocity at the center and a lower one near the wall. Increasing suction ratio leads to lower self-axial velocity at the center and a higher one near the wall (Figure 3.11). A comparison shows that self-axial velocity near the center in case of injection with contracting wall and low porosity is higher than injection with contracting wall and high porosity (Figures 3.10 and 3.11).

Figures 3.9 and 3.12 indicate the behaviour of self-axial velocity u/x for porosity parameter $R = -0.5, 0.5$ (low and high porosity respectively) and wall dilation rate $\alpha = 0.5$ (expansion) over a range of R_e . In case of $R_e > 0$, Figure 3.9 shows that increasing injection leads to a lower self-axial velocity near the center and a higher one near the wall. Figure 3.12 shows that increasing injection leads to a higher self-axial velocity near the center and a lower one near the wall. In case of $R_e < 0$, Figure 3.9 depicts that increasing suction ratio leads to higher self-axial velocity at the center and a lower one near the wall. Figure 3.12 shows that increasing suction ratio leads to lower self-axial velocity at the center and a higher one near the wall. By comparing Figures 3.9 and 3.12, the self-axial velocity near the center in case of injection with expansion wall and low porosity is higher than injection with expansion wall and large porosity.

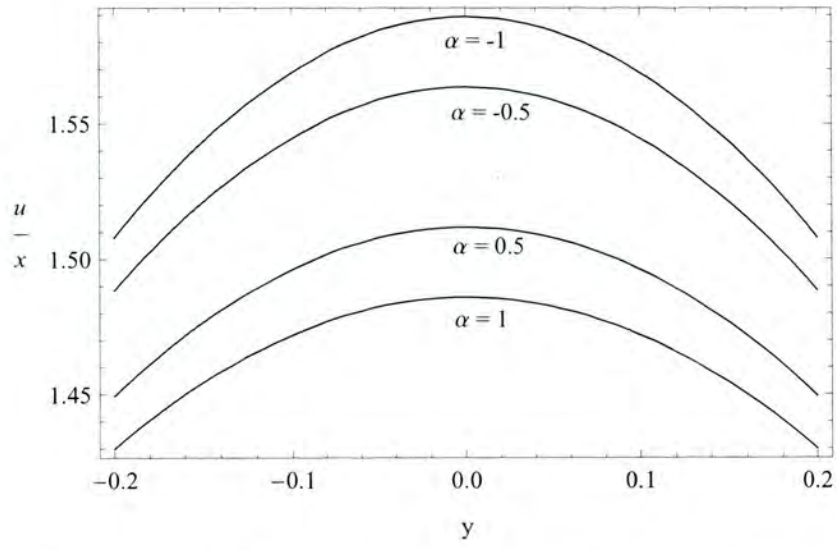


Figure 3.9: Self-axial velocity profiles over a range of R_e at $\alpha = 0.5$ and $R = 0.5$

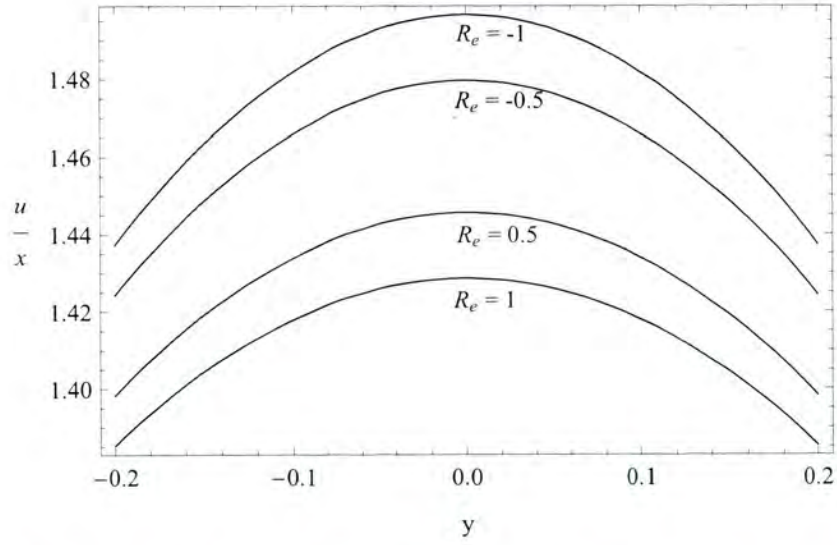


Figure 3.10: Self-axial velocity profiles over a range of R_e at $\alpha = -0.5$ and $R = 0.5$

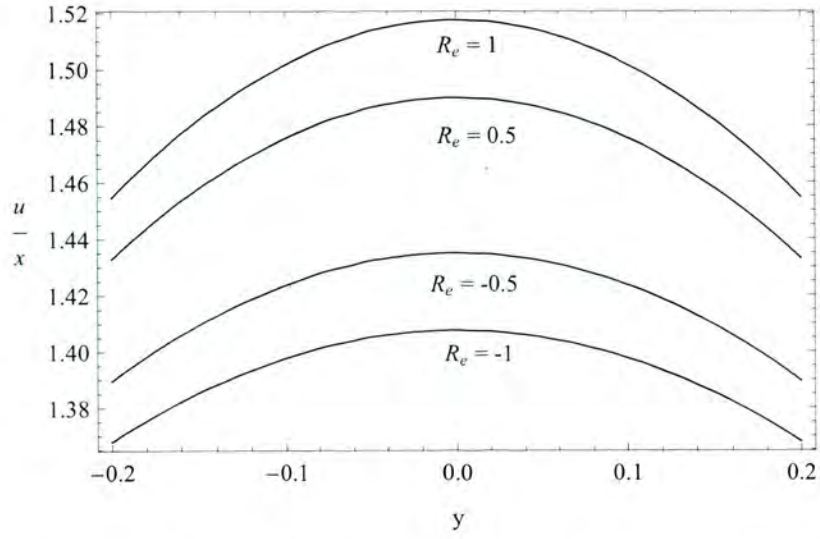


Figure 3.11: Self-axial velocity profiles over a range of R_e at $\alpha = -0.5$ and $R = -0.5$

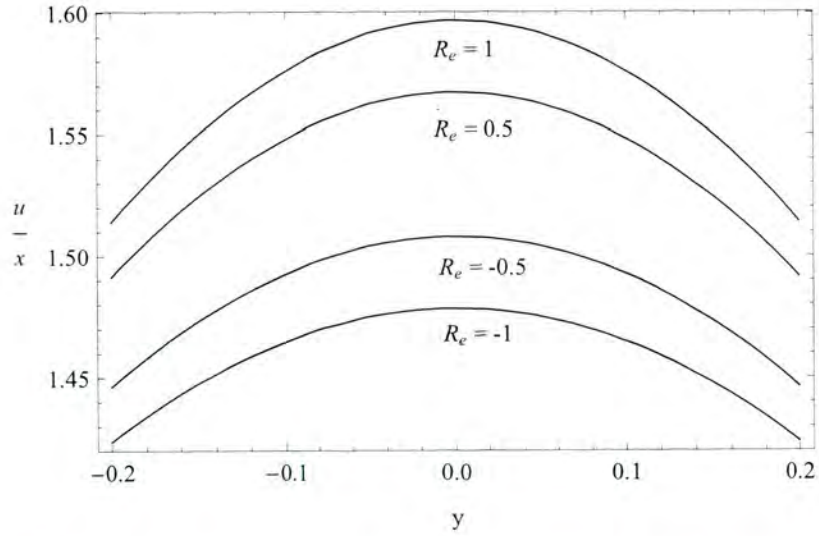


Figure 3.12: Self-axial velocity profiles over a range of R_e at $\alpha = 0.5$ and $R = -0.5$

Tables 3.9-3.12 shows the percentage error decrease for a small R_e .

Table 3.9: Comparison between analytical and numerical solutions for self-axial velocity u/x at $y = 0.1$ for $R = 1, \alpha = -0.5$.

	Analytical Method	Numerical Method	Percentage Error (%)
$R_e = -1$	1.460785	1.464232	0.235434
$R_e = -0.5$	1.455288	1.456930	0.114768
$R_e = 0.5$	1.444204	1.445507	0.090102
$R_e = 1$	1.438678	1.440991	0.160530

Table 3.10: Comparison between analytical and numerical solutions for self-axial velocity u/x at $y = 0.1$ for $R = 1, \alpha = 0.5$.

	Analytical Method	Numerical Method	Percentage Error (%)
$R_e = -1$	1.542820	10548592	0.372749
$R_e = -0.5$	1.531544	1.533740	0.143138
$R_e = 0.5$	1.508993	1.510721	0.114334
$R_e = 1$	1.497718	1.501653	0.262055

Table 3.11: Comparison between analytical and numerical solutions for self-axial velocity u/x at $y = 0.1$ for $R = -1, \alpha = -0.5$.

	Analytical Method	Numerical Method	Percentage Error (%)
$R_e = -1$	1.418772	1.419039	0.018778
$R_e = -0.5$	1.434252	1.435114	0.060097
$R_e = 0.5$	1.465211	1.465912	0.048020
$R_e = 1$	1.480690	1.480513	0.012005

Table 3.12: Comparison between analytical and numerical solutions for self-axial velocity u/x at $y = 0.1$ for $R = -1, \alpha = 0.5$.

	Analytical Method	Numerical Method	Percentage Error (%)
$R_e = -1$	1.490782	1.490460	0.021660
$R_e = -0.5$	1.505526	1.506243	0.047638
$R_e = 0.5$	1.535012	1.535561	0.035738
$R_e = 1$	1.523217	1.524206	0.046138

The plots in Figure 3.13 elucidate that the temperature distribution is constant through out and it is independent of physical parameter. Numerical solution for temperature is similar to our analytical solution and therefore temperature distribution has no error.

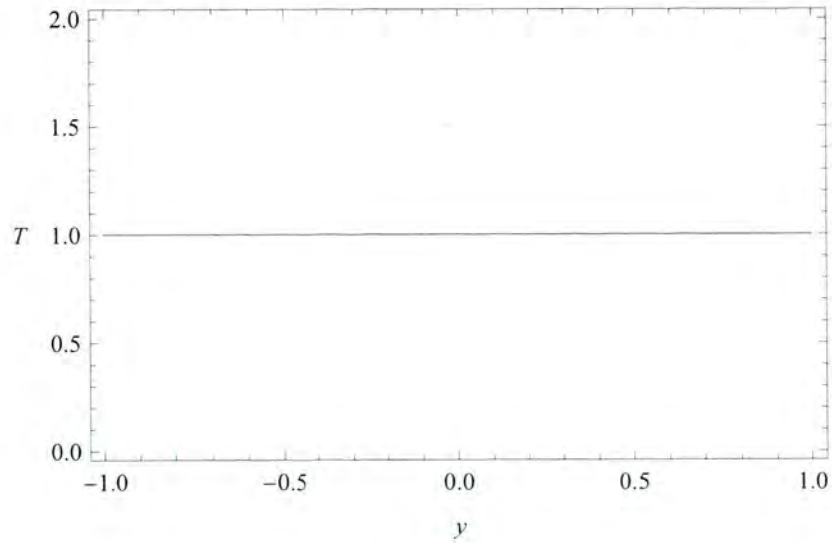


Figure 3.13: Temperature distribution profile

3.5 Conclusions

In this chapter, we have generalized the flow analysis of [12] with the influence of porous medium and heat transfer. Analytical solution for the arising nonlinear

problem is obtained by using Lie symmetry technique in conjunction with a second-order double perturbation method. We have studied the effects of porous medium (R), permeation Reynolds R_e and wall dilation rate α on the self-axial velocity and temperature distribution within the fluid. We compared the analytical solution with the numerical solution for self-axial velocity for the different values of R , R_e and α . It was found that the temperature distribution has no error since analytical solution is similar to numerical solution and both are equal to one. Also the percentage errors were found to be small for different values of R . The temperature distribution was found to be constant through out. Here we have noticed that the obtained analytical results match quite well with the numerical results for a good range of these parameters. We also noticed that in all cases, the self-axial velocity has similar trend as in [12], that is, the self-axial velocity approaches a cosine profile. Finally, we observed that when R approaches infinity our problem reduces to the problem in [12] and our results (analytical and numerical) also reduce to the results in [12].

Chapter 4

Concluding remarks

In this research project Lie group method was applied to study two nonlinear partial differential equations arising in fluids.

In Chapter 1, a brief introduction to the Lie group theory of partial differential equations was given.

In Chapter 2, the Lie symmetry method along with the simplest equation method were used to carry out the integration of the combined sinh-cosh-Gordon equation (2.1). Also the conservation laws of (2.1) were obtained by using three approaches, the direct method, application of Noether theorem and the application of new conservation theorem.

In Chapter 3, the flow analysis of [12] was generalized with the influence of heat distribution and porous medium. Lie symmetry analysis along with second-order double perturbation was applied to obtain the analytical solution. In addition a numerical simulation was performed to supplement the analytical solution. The effects of porous medium on axial velocity and temperature distribution were shown graphically and discussed.

In the future, flow and heat transfer between slowly expanding or contracting walls will be studied when there is a big temperature difference between the lower and the upper walls.

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