

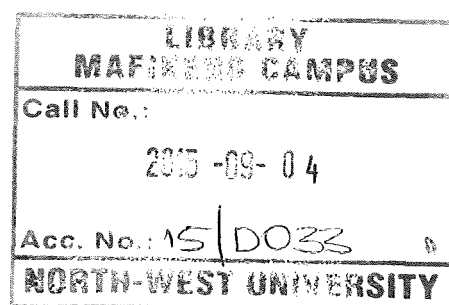
NORTH-WEST UNIVERSITY
DEPARTMENT OF MATHEMATICAL SCIENCES

The δ -completion of quasi-pseudometric spaces

by
Mukongo K.D.

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Dr Olela Otafudu O.
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Abstract

Over many years much progress has been made in the investigation of completion theory of quasi-pseudometric spaces. In particular, Doitchinov, Kunzi, Salbany and others have published several articles concerning the concept of completion for quasi-pseudometric spaces.

Recently, Andrikopoulos introduced the theory of κ -completion which uses the pair of family of right κ -Cauchy and left κ -Cauchy sequences that he called κ -cut.

The aim of this dissertation is to begin a similar investigation by using the pair of family of right K-Cauchy and left K-Cauchy filters. It starts off with a summary of results obtained for the theory of bicompletion, B-completion and κ -completion, which has been investigated in the past.

We conclude by commencing an investigation of δ -completion. Here several results obtained for κ -completion are generalized.

Introduction

The study of the completion of quasi-pseudometric space was started in the early 1970s with the PhD thesis of Salbany that was published as monograph [21]. In 1988 Doitchinov [7] developed an interesting completion theory for balanced T_0 -quasi-metric spaces. What is interesting in Doitchinov's work is that he has considered a quasi-pseudometric space as bitopological space and introduced the concept of cosequence of a sequence. Since the 1990s, Künzi has emerged as a leading researcher in this area of asymmetric topology with many papers dedicated on completion of quasi-uniform and quasi-pseudometric spaces. In [13, 14], Künzi and Kivuvu have extended the theory of balanced quasi-pseudometric spaces to arbitrary T_0 , denoted B-completion.

In parallel, Andrikopoulos has raised a concern about B-completion of arbitrary quasi-pseudometric spaces in his 2013 paper [2], and has introduced a new technique, inspired from Dedekind-MacNeille completion of rational numbers. The technique stands on the construction of cut, using Doitchinov's concept of Cauchy pair of sequences.

In the light of the above, it is natural to start an investigation to extend the theory of cut in the framework of Cauchy filter pairs. We define a δ -cut with use of Cauchy filter pair; motivated by Künzi and Kivuvu in [14], where they observed that a convergence of filter pair need not be balanced. In Lemma 3.1.5, we show that any filter \mathcal{F} which is d^s -Cauchy is also δ -Cauchy filter. We also show that each quasi-pseudometric space that is δ -complete is bicomplete.

It is interesting to note various resemblances between the κ -completion and the δ -completion, such as the limit of a K-Cauchy sequence and δ -Cauchy filter, if they exist they are both unique. We bring it to the reader's attention that some of the most interesting new results on the δ -completions of a quasi-pseudometric space obtained during this investigation are collected in [18] for

possible publication. The proofs given in this dissertation and in [18] may sometimes differ.

In the next, we describe the contents of each chapter.

This dissertation starts with some preliminary definitions that are listed in the next chapter.

In chapter 2, we give an overview of the different construction of completions of T_0 -quasimetric spaces.

Chapter 3 is our main and own work. It presents the construction of δ -completion. It starts with the construction of δ -cut that uses Cauchy filter pairs. We explain in detail the process of δ -completion of a quasi-pseudometric space. The advantage of the method is that it can be applied to arbitrary quasi-pseudometric spaces.

The last chapter of this dissertation is the conclusion that sets the way for our next investigation.

Chapter 1

Preliminaries

In this chapter, we recall some basic concepts from the theory of quasi-pseudometric spaces and give some examples related to their topologies.

1.1 Some basic concepts

The following concepts can be found in most recent articles on asymmetric topology.

Definition 1.1.1. *A quasi-pseudometric on a set X is a nonnegative real-valued function d on $X \times X$ such that for all $x, y, z \in X$:*

1. $d(x, x) = 0$

2. $d(x, y) \leq d(x, z) + d(z, y)$.

In addition,

3. *If a quasi-pseudometric d on a set X , satisfies the condition $d(x, y) = 0 = d(y, x)$ implies $x = y$ where $x, y \in X$, then d is said to be a T_0 -quasi-metric on X .*

A set X endowed with a quasi-pseudometric d is a *quasi-pseudometric space* denoted (X, d) . In particular, if d satisfies the symmetric condition, then (X, d) is the well-known metric space. So, metric spaces are a particular case of quasi-pseudometric spaces.

Remark 1.1.1. *In T_0 -quasi-metric spaces we can have $d(x, y) = 0$ with $x \neq y$.*

The next definition is an obvious consequence of the lack of symmetry of the distance from a point to another.

Definition 1.1.2. Let d be quasi-pseudometric on a set X . Thus $d^{-1} : X \times X \rightarrow \mathbb{R}$ defined by $d^{-1}(x, y) = d(y, x)$ whenever $x, y \in X$ is also a quasi-pseudometric, called the conjugate quasi-pseudometric of d .

Definition 1.1.3. Let d be quasi-pseudometric on a set X . Thus $d^s : X \times X \rightarrow \mathbb{R}$ defined by $d^s(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$ whenever $x, y \in X$ is a pseudometric of d .

The following describe some properties of maps between two quasi-pseudometric spaces.

A map $f : (X, d_X) \rightarrow (Y, d_Y)$ between two quasi-pseudometric spaces (X, d_X) and (Y, d_Y) is called *nonexpansive* provided that $d_Y(f(x), f(y)) \leq d_X(x, y)$ whenever $x, y \in X$.

A map $f : (X, d_X) \rightarrow (Y, d_Y)$ between two quasi-pseudometric spaces (X, d_X) and (Y, d_Y) is called *isometry* provided that $d_Y(f(x), f(y)) = d_X(x, y)$ whenever $x, y \in X$.

Two quasi-pseudometric spaces (X, d_X) and (Y, d_Y) will be called *isometric* provided that there exists a bijective isometry $f : (X, d_X) \rightarrow (Y, d_Y)$ between two quasi-pseudo-metric spaces (X, d_X) and (Y, d_Y) .

A map $f : (X, d_X) \rightarrow (Y, d_Y)$ between two quasi-pseudometric spaces (X, d_X) and (Y, d_Y) will be called *quasi-uniformly continuous* provided that for each $\varepsilon > 0$ there is $\delta > 0$ such that for all $x, y \in X, d_X(x, y) < \delta$ implies that $d_Y(f(x), f(y)) < \varepsilon$.

The topology τ_d of a quasi-pseudometric space (X, d) can be defined starting from the family $\mathcal{V}_d(x)$ of neighborhoods of an arbitrary point $x \in X$: for any $V \subseteq X$, we have $V \in \mathcal{V}_d(x)$ if and only if there exists $r > 0$ such that $B_d(x, r) = \{y \in X : d(x, y) < r\} \subset V$ if and only if there exists $r' > 0$ such that $C_d(x, r') = \{y \in Y : d(x, y) \leq r'\} \subseteq V$. A set $U \subset X$ is τ_d -open if and only if for every $x \in U$ there exists $r = r_x > 0$ such that $B_d(x, r) \subset U$. We shall say that U is a d -neighborhood of x or that the set U is d -open.

Taking in consideration the lack of symmetry, a quasi-pseudometric d , can generates tree different topologies (see [6]), that we recall next.

Definition 1.1.4. Let (X, d) be a quasi-pseudometric space. The topology τ_d is generated by the quasi-pseudometric d , where the open balls are described as follows: given $x \in X$ and $\varepsilon > 0$, we have,

$$B_d(x, \varepsilon) \subseteq X, \text{ where } B_d(x, \varepsilon) := \{y \in X \mid d(x, y) < \varepsilon\},$$

and the closed balls are described as follows: given $x \in X$ and $\varepsilon > 0$, we have,

$$C_d(x, \varepsilon) \subseteq X, \text{ where } C_d(x, \varepsilon) := \{y \in X \mid d(x, y) \leq \varepsilon\},$$

the balls with respect to d are called forward balls and the topology τ_d is called the forward topology.

Definition 1.1.5. Let (X, d) be a quasi-pseudometric space. The topology $\tau_{d^{-1}}$ is generated by the conjugate quasi-pseudometric d^{-1} , where the open balls are described as follow: given $x \in X$ and $\varepsilon > 0$, we have,

$$B_{d^{-1}}(x, \varepsilon) \subseteq X, \text{ where } B_{d^{-1}}(x, \varepsilon) := \{y \in X \mid d(y, x) < \varepsilon\},$$

and the closed balls are described as follow: given $x \in X$ and $\varepsilon > 0$, we have,

$$C_{d^{-1}}(x, \varepsilon) \subseteq X, \text{ where } C_{d^{-1}}(x, \varepsilon) := \{y \in X \mid d(y, x) \leq \varepsilon\},$$

the balls with respect to d^{-1} are called backward balls and the topology $\tau_{d^{-1}}$ is called the backward topology.

Definition 1.1.6. Let (X, d) be a quasi-pseudometric space. The topology τ_{d^s} is generated by the pseudometric d^s , where the open balls are described as follow: given $x \in X$ and $\varepsilon > 0$, we have,

$$B_{d^s}(x, \varepsilon) \subseteq X, \text{ where } B_{d^s}(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}.$$

Remark 1.1.2. Note that if d is a T_0 -quasi-metric on X , then $d^s = \max\{d, d^{-1}\} = d \vee d^{-1}$ is a metric on X . Furthermore, $B_{d^s}(x, \varepsilon) \subseteq B_{d^{-1}}(x, \varepsilon)$ and $B_{d^s}(x, \varepsilon) \subseteq B_d(x, \varepsilon)$, whenever $x \in X$ and $\varepsilon > 0$.

Example 1.1.1. (Sorgenfrey line) For $x, y \in \mathbb{R}$ define a quasi-metric d by $d(x, y) = y - x$, if $x \leq y$ and $d(x, y) = 1$, if $x > y$. A basis of open d -neighborhoods of a point $x \in \mathbb{R}$ is formed by the family $[x; x + \varepsilon)$, $0 < \varepsilon < 1$. The family of intervals $(x - \varepsilon; x]$, $0 < \varepsilon < 1$, forms a basis of open d^{-1} -neighborhoods of x . $d^s(x, y) = 1$ for $x \neq y$, so that τ_{d^s} is the discrete topology of \mathbb{R} .

Example 1.1.2. For any $x, y \in \mathbb{R}$, define $d(x, y) = \max\{x - y, 0\}$. Then d is a T_0 -quasimetric on \mathbb{R} .

A basis for open d -neighborhoods of a point $x \in \mathbb{R}$ is formed by the family $[x; \varepsilon)$, $0 < \varepsilon < 1$. The family of intervals $(x - \varepsilon; x]$, $0 < \varepsilon < 1$, forms a basis of open d^{-1} -neighborhoods of x . Obviously, $d^s(x, y) = |x - y|$ for $x, y \in \mathbb{R}$ so that τ_{d^s} is the usual Euclidean topology of \mathbb{R} .

Example 1.1.3. Let $X = \{\frac{-1}{n+1}, \frac{1}{n+1}, n \in \mathbb{N}\}$. For each $x, y \in X$, let $d(x, y) = 1$, if $x < 0 < y$, $d(x, y) = 0$, if $y \leq x$, and $d(x, y) = \min\{1, |x - y|\}$ otherwise. It is easy to check that (X, d) is a T_0 -quasimetric space.

1.2 Convergence in T_0 -quasi-metric spaces

We next recall some basic concepts related to the convergence of sequence and filters.

Definition 1.2.1. [2, Definition 2] A sequence (x_n) in a quasi-pseudometric space (X, d) is called right K -Cauchy, if for any $\varepsilon > 0$ there is an $n_\varepsilon \in \mathbb{N}$ such that $d(x_n, x_{n'}) < \varepsilon$ whenever $n > n' > n_\varepsilon$.

Definition 1.2.2. [2, Definition 2] A sequence (y_n) in a quasi-pseudometric space (X, d) is called left K -Cauchy, if for any $\varepsilon > 0$ there is an $n_\varepsilon \in \mathbb{N}$ such that $d(y_n, y_{n'}) < \varepsilon$ whenever $n > n' > n_\varepsilon$.

This notion of K -Cauchy sequence was first introduced by Kelly in [10].

Proposition 1.2.1. [6, Proposition 1.1.2.] If (X, d) is a quasi-pseudometric space, then a sequence (x_n) in X is τ_{d^s} -convergent to $x \in X$ if and only if it is d -convergent and d^{-1} -convergent to x .

In order to work with two or a family of K -Cauchy sequences, we need next to discuss the relationship between two K -Cauchy sequences.

Definition 1.2.3. ([7]) Any sequence (y_m) is called cosequence to a sequence (x_n) if for any $\varepsilon > 0$ there are $m_\varepsilon, n_\varepsilon \in \mathbb{N}$ such that $d(y_m, x_n) < \varepsilon$ whenever $m > m_\varepsilon, n > n_\varepsilon$ i.e for which $\lim_{n,m} d(y_m, x_n) = 0$.

If (y_m) is a cosequence of sequence (x_n) in a quasi-pseudometric space (X, d) , we shall call the pair $((x_n), (y_m))$ a *Cauchy pair of sequences* in (X, d) . The following concepts were first introduced by Romaguera in [20].

Definition 1.2.4. A left K -Cauchy filter in a quasi-metric space (X, d) is a filter \mathcal{F} such that for all $\varepsilon > 0$, there is $F_\varepsilon \in \mathcal{F}$ such that $B_d(x, \varepsilon) \in \mathcal{F}$, whenever $x \in F_\varepsilon$.

Definition 1.2.5. A right K -Cauchy filter in a quasi-metric space (X, d) is a filter \mathcal{F} such that for all $\varepsilon > 0$, there is $F_\varepsilon \in \mathcal{F}$ such that $B_{d^{-1}}(x, \varepsilon) \in \mathcal{F}$, whenever $x \in F_\varepsilon$.

The following is the filter version of K -sequentially convergence.

Definition 1.2.6. Let (X, d) be a quasi-pseudometric space. A filter \mathcal{F} is said to be d -convergent to $x \in X$, denoted $\mathcal{F} \xrightarrow{d} x$, if and only if every d -neighborhood of x (with respect to the topology τ_d) belong to \mathcal{F} . Equivalently, if there exists $F \in \mathcal{F}$, for all $\varepsilon > 0$, $B_d(x, \varepsilon) \subseteq F$.

Definition 1.2.7. Let (X, d) be a quasi-pseudometric space. A filter \mathcal{F} is said to be d^{-1} -convergent to $x \in X$, denoted $\mathcal{F} \xrightarrow{d^{-1}} x$, if and only if every d^{-1} -neighborhood of x (with respect to the topology $\tau_{d^{-1}}$) belong to \mathcal{F} . Equivalently, if there exists $F \in \mathcal{F}$, for all $\varepsilon > 0$, $B_{d^{-1}}(x, \varepsilon) \subseteq F$.

Definition 1.2.8. Let (X, d) be a quasi-pseudometric space. A filter \mathcal{F} is said τ_{d^*} -convergent to $x \in X$ if and only if it is d -convergent and d^{-1} -convergent to x .

Chapter 2

Some concepts of completeness in T_0 -quasimetric spaces

In this chapter, we summarize some important concepts of completions for quasi-pseudometric spaces, and present the advantages of κ -completion with examples that failed to satisfy the requirements for balanced quasi-metric spaces (see [7]).

2.1 Bicompletion

In this section, we summarize the construction of bicompletion of a T_0 -quasi-metric space.

The following definition is due to Salbany (see [21]).

Definition 2.1.1. ([21]) *Let (X, d) be a quasi-pseudometric space. The sequence (x_n) in X is a Cauchy sequence if $\lim_{n, m \rightarrow \infty} d(x_n, x_m) = 0$. A quasi-pseudometric space (X, d) is bicomplete if every Cauchy sequence (x_n) converges with respect to τ_d and with respect to $\tau_{d^{-1}}$ to a point x_0 .*

Note that (x_n) is a Cauchy sequence in (X, d) in this sense if and only if (x_n) is a Cauchy sequence in the pseudometric space (X, d^s) .

Proposition 2.1.1. ([21]) *A T_0 -quasi-pseudometric space (X, d) is bicomplete if and only if the metric space (X, d^s) is complete.*

Definition 2.1.2. ([21]) *The bicompletion of a T_0 -quasimetric space (X, d_X) is a complete T_0 -quasimetric space (Y, d_Y) such that (X, d_X) is isometric to a τ_{d_Y} -dense subspace of (Y, d_Y) .*

The following describes the construction of the bicompletion of a given quasi-pseudometric space.

Proposition 2.1.2. ([21]) *Let (X, d) be a quasi-pseudometric space. Define an equivalence relation \sim on X by $x \sim y$ if and only if $d(x, y) = 0 = d(y, x)$. Let \hat{X} be the set of all equivalence classes \hat{x} with respect to \sim where $x \in X$. Then the function \hat{d} on $\hat{X} \times \hat{X}$ defined by $\hat{d}(\hat{x}, \hat{y}) = d(x, y)$ is a T_0 -quasi-metric on \hat{X} .*

Proof. The proof is taken from [21]. It is clear that \sim is reflexive and symmetric. We now show that \sim is transitive. Let $x \sim y$ and $y \sim z$, so we have that $d(x, y) = 0 = d(y, x)$ and $d(y, z) = 0 = d(z, y)$. By using the triangle inequality as $d(x, z) \leq d(x, y) + d(y, z)$ we get that $d(x, z) = 0 = d(z, x)$. That is $x \sim z$. Then \sim is transitive. The quotient set is denoted by \hat{X} .

We next show that \hat{d} is well-defined on \hat{X} . Suppose that $x, x', y, y' \in X$, $x \sim x'$ and $y \sim y'$. By the triangle inequality we see that $d(x', y') \leq d(x', x) + d(x, y) + d(y, y')$ thus, $d(x', y') \leq 0 + d(x, y) + 0$. Similarly, we get that $d(x, y) \leq d(x, x') + d(x', y') + d(y', y)$, that is $d(x, y) \leq 0 + d(x', y') + 0$, hence $d(x, y) = d(y, x)$ and we have shown that \hat{d} is well-defined. we now show that \hat{d} is T_0 . If $\hat{d}(\hat{x}, \hat{y}) = \hat{d}(\hat{y}, \hat{x}) = 0$, then $d(x, y) = d(y, x) = 0$ which implies that $\hat{x} = \hat{y}$. □

We next discuss the bicompletion process of a quasi-pseudometric space. The following lemmas prepare the proof of the main theorem of bicompletion.

Lemma 2.1.1. ([21]) *Let (X, d) be a quasi-pseudometric space. Then for all $a, b, x, y \in X$, we have that*

$$|d(x, y) - d(a, b)| \leq d^s(x, a) + d^s(y, b).$$

Proof. If $x, y, a, b \in X$, by the triangle inequality, we have that

$$d(x, y) - d(a, b) \leq d(x, a) + d(b, y)$$

and

$$d(a, b) - d(x, y) \leq d(a, x) + d(y, b)$$

that implies

$$|d(x, y) - d(a, b)| \leq d^s(x, a) + d^s(y, b).$$

□

Corollary 2.1.1. ([21]) *Let (X, d) be a quasi-pseudometric space and $(x_n), (y_n)$ sequences in (X, d) . If $(x_n) \rightarrow x$ and $(y_n) \rightarrow y$ with respect to τ_{d^s} , then $\lim_{n \rightarrow \infty} d(x_n, y_n) = d(x, y)$.*

Lemma 2.1.2. ([21]) *The space (Y, d') is a quasi-pseudometric space.*

Proof. This proof comes from Salbany [21]. Let $(x_n), (y_n) \in Y$ where (x_n) and (y_n) are two Cauchy sequences in X . We observe that $(d(x_n, y_n))$ is a Cauchy sequence of real numbers. For $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$, such that $d^s(x_n, x_m) < \frac{\varepsilon}{2}$ and $d^s(y_n, y_m) < \frac{\varepsilon}{2}$ whenever $n, m \in n_\varepsilon$. It follows from the above lemma that

$$|d(x_n, y_n) - d(x_m, y_m)| \leq d^s(x_n, x_m) + d^s(y_n, y_m) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $n, m \geq n_\varepsilon$. Hence we get that $\lim_{n \rightarrow \infty} d(x_n, y_n)$ exists. Moreover we have that:

1. $d'((x_n), (x_n)) = \lim_{n \rightarrow \infty} d(x_n, x_n) = 0$.
2. Let $(x_n), (y_n)$ and $(z_n) \in Y$ suppose that $d'((x_n), (y_n)) = a$ and $d'((y_n), (z_n)) = b$. For any $\varepsilon > 0$, there are m_1, m_2 such that $d(x_n, y_n) < a + \frac{\varepsilon}{2}$ whenever $n \geq m_1$ and $d(y_n, z_n) < b + \frac{\varepsilon}{2}$ whenever $n \geq m_2$. It follows that $d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n) < a + \frac{\varepsilon}{2} + b + \frac{\varepsilon}{2} = a + b + \varepsilon$ whenever $n \geq m_1, m_2$. Hence we get that $d'((x_n), (z_n)) \leq d'((x_n), (y_n)) \leq d'((x_n), (y_n)) + d'((y_n), (z_n))$, and so d' is a quasi-pseudometric on Y .

□

We define an equivalence relation on Y as follow. Let $(x_n), (y_n) \in Y$, we have $(x_n) \sim (y_n)$ if $(d')^s((x_n), (y_n)) = 0$. We denote the quotient set by \tilde{X} and $[(x_n)]$ denotes an equivalence class. For each pair $[(x_n)], [(y_n)] \in \tilde{X}$, let

$$\tilde{d}([(x_n)], [(y_n)]) = d'((x_n), (y_n))$$

that means

$$\tilde{d}([(x_n)], [(y_n)]) = \lim_{n \rightarrow \infty} d((x_n), (y_n)).$$

The next step is to show that X can be isometrically embedded to \tilde{X} .

Lemma 2.1.3. ([21]) *The T_0 -quasi-pseudometric space (X, d) can be isometrically embedded into (\tilde{X}, \tilde{d}) .*

Proof. Let X_0 be the subspace of \tilde{X} consisting of those equivalence classes which contain a Cauchy sequence (x_n) for which $x_n = x$ where $x \in X$ whenever $n \in \mathbb{N}$. Denote by $[(\bar{x})]$ an element in X_0 . If $[(\bar{x})], [(\bar{y})] \in X_0$, define the map $i : X \rightarrow \tilde{X}$ by $i(x) = [(\bar{x})]$. Then $\tilde{d}(i(x), i(y)) = [(\bar{x}), [(\bar{y})]] = d(x, y)$. It is then clear that i is an isometry from X into \tilde{X} . Since (X, d) is T_0 -space, i is injective because $x \neq y$ implies that $i(x) \neq i(y)$, that is we cannot find two different Cauchy sequences of this kind in the same equivalence class and $i(X) = X_0$ can be identified with X . Then X can be regarded as a subspace of (\tilde{X}, \tilde{d}) . \square

Lemma 2.1.4. ([21]) X_0 is a \tilde{d}^s -dense subspace of the T_0 -quasimetric space (\tilde{X}, \tilde{d}) .

Proof. This proof comes from [21]. Let $[(x_n)] \in \tilde{X}$ where (x_n) be a sequence in (X, d) . For every $\varepsilon > 0$ there exists \mathbb{N} such that $d(x_n, x_m) < \varepsilon$ whenever $n, m \geq \mathbb{N}$. Then $i(x_m) \in X_0$ for fixed m ; letting $m \rightarrow \infty$, we get that $i(x_m) \rightarrow [(x_m)]$ in (\tilde{X}, \tilde{d}) . Hence X_0 is $\tau_{\tilde{d}^s}$ -dense in \tilde{X} . \square

Lemma 2.1.5. ([21]) The space (\tilde{X}, \tilde{d}) is bicomplete.

Proof. Let (ξ_n) be a Cauchy sequence in (\tilde{X}, \tilde{d}) . for each n , let us choose $i(z_n) \in X_0$ such that $\tilde{d}^s(\xi_n, i(z_n)) < \frac{1}{n}$. we first need to show that (z_n) is Cauchy sequence in (X, d) . We have that: $d(z_n, z_m) = \tilde{d}(i(z_n), i(z_m)) \leq \tilde{d}(i(z_n), \xi_n) + \tilde{d}(\xi_n, \xi_m) + \tilde{d}(\xi_m, i(z_m)) \leq \frac{1}{n} + \frac{1}{m} + \tilde{d}(\xi_n, \xi_m)$. So (z_n) is a Cauchy sequence in (X, d) . Hence $[(z_n)] \in \tilde{X}$. It follows from the above lemma that $(\tilde{d})^s(i(z_n), [(z_n)]) \rightarrow 0$. We have that $(\tilde{d})^s(\xi_n, i(z_n)) \leq (\tilde{d})^s(\xi_n, i(z_n)) + (\tilde{d})(i(z_n), [(z_n)]) \leq \frac{1}{n} + (\tilde{d})^s(i(z_n), [(z_n)])$ which implies that $(\tilde{d})(\xi_n, [(z_n)]) \rightarrow 0$, hence (ξ_n) converges in $(\tilde{X}, (\tilde{d})^s)$ and (\tilde{X}, \tilde{d}) is bicomplete. \square

Theorem 2.1.1. [21] Each T_0 -quasimetric space (X, d) has a bicompletion denoted by (\tilde{X}, \tilde{d}) which is a T_0 -quasimetric space.

Proof. The proof of this theorem follows from Proposition 2.1.2 and the preceding lemmas. \square

We refer the reader to the last part for the discussion on the extension map and structure preserving map between two quasi-pseudometric spaces in the sense of Salbany.

2.2 B-completion

In [13, 14] Künzi and Kivuvu have extended Doitchinov's completion theory for balanced quasi-pseudometric spaces. They replaced the Cauchy pairs of sequences by balanced Cauchy filter pairs, and constructed a completion of T_0 -quasimetric space which they called the *B-completion of T_0 -quasi-pseudometric spaces*. They have shown that each T_0 -quasimetric space admits a B-completion which is larger than the bicompletion of the original space.

In this section we present the summary of the construction of the B-completion of a T_0 quasimetric space.

We start the discussion on the distance between two Cauchy filter pairs.

Let (X, d) be a quasi-pseudometric space and let A, B be nonempty subsets of X . then we define the *2-diameter* from A to B by

$$\Phi_d(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$$

Note that usually $\Phi_d(A, A)$ is called the diameter of A .

Of course ∞ is a possible value of a 2-diameter. For a singleton $\{x\}$ we write $\Phi_d(x, A)$ and $\Phi_d(B, x)$ instead of $\Phi_d(\{x\}, A)$ and $\Phi_d(B, \{x\})$, respectively.

Note that d^{-1} is the conjugate of d , then $\Phi_{d^{-1}}(A, B) = \Phi_d(B, A)$.

We recall the definition of Cauchy filter pair $(\mathcal{F}, \mathcal{G})$ on (X, d) .

Definition 2.2.1. ([14, Definition 2]) *Let (X, d) be a quasi-pseudometric space. We say that a pair $(\mathcal{F}, \mathcal{G})$ of filters \mathcal{F} and \mathcal{G} on X is a Cauchy filter pair on (X, d) if $\inf_{F \in \mathcal{F}, G \in \mathcal{G}} \Phi_d(F, G) = 0$.*

Lemma 2.2.1. ([14, Lemma 2]) *Let $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{F}', \mathcal{G}')$ be two Cauchy filter pairs on a quasi-pseudometric space (X, d) . Then $\inf_{F \in \mathcal{F}, G' \in \mathcal{G}'} \Phi_d(F, G')$ is a non-negative real number.*

For the proof of this lemma see [14, Lemma 2].

Lemma 2.2.2. ([14, Lemma 1])

Let (X, d) be a quasi-pseudometric space. Define an equivalence relation \sim on X by $x \sim y$ if $d(x, y) = 0 = d(y, x)$. Let \hat{X} be the set of equivalence classes $q_X(x)$ with respect to \sim where $x \in X$. Then \hat{d} on \hat{X} defined by $\hat{d}(q_X(x), q_X(y)) = d(x, y)$ defines a T_0 -quasimetric \hat{d} on \hat{X} . In the following, $q_X : X \rightarrow \hat{X}$ whenever $x \in X$. Let $f : (X, d) \rightarrow (Y, e)$ be a quasi-uniformly

continuous map between quasi-pseudometric spaces (X, d) and (Y, e) . Then $\hat{f} : (\hat{X}, \hat{d}) \rightarrow (\hat{Y}, \hat{e})$ defined by $\hat{f}(q_X(x)) : +(q_Y \circ f)(x)$ whenever $x \in X$ is a well-defined quasi-uniformly continuous map. It is an isometry provided that f is an isometry.

The following is the definition of the distance between two Cauchy filter pairs.

Definition 2.2.2. ([14, Definition 3]) Let (X, d) be a quasi-pseudometric space and let $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{F}', \mathcal{G}')$ be two Cauchy filter pairs on X . Then we define the distance from $(\mathcal{F}, \mathcal{G})$ to $(\mathcal{F}', \mathcal{G}')$ by:

$$d^+((\mathcal{F}, \mathcal{G}), (\mathcal{F}', \mathcal{G}')) = \inf_{F \in \mathcal{F}, G \in \mathcal{G}'} \Phi_d(F, G') = \inf_{F \in \mathcal{F}, G' \in \mathcal{G}'} \sup_{f \in F, g' \in G'} d(f, g').$$

Note that this distance belongs to \mathbb{R}^+ and a filter pair $(\mathcal{G}, \mathcal{H})$ on a quasi-pseudometric space (X, d) is a Cauchy filter pair if and only if

$$d^+((\mathcal{G}, \mathcal{H}), (\mathcal{G}, \mathcal{H})) = 0.$$

The following definition is a key notion in the study of the B-completion.

Definition 2.2.3. ([14, Definition 4]) Let (X, d) be a quasi-pseudometric space. A Cauchy filter pair $(\mathcal{F}, \mathcal{G})$ on (X, d) is said to be balanced on (X, d) if for each $x, y \in X$ we have

$$d(x, y) \leq \inf_{G \in \mathcal{G}} \Phi_d(x, G) + \inf_{F \in \mathcal{F}} \Phi_d(F, y).$$

Definition 2.2.4. (Compare [14, Definition 4]) Let $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{F}', \mathcal{G}')$ be two filter pairs on a set X . Then $(\mathcal{F}, \mathcal{G})$ is called coarser than $(\mathcal{F}', \mathcal{G}')$ ($(\mathcal{F}', \mathcal{G}')$ is finer than $(\mathcal{F}, \mathcal{G})$) provided that both $\mathcal{F} \subseteq \mathcal{F}'$ and $\mathcal{G} \subseteq \mathcal{G}'$.

Let (X, d) be a quasi-pseudometric space. Let $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{F}', \mathcal{G}')$ be two Cauchy filter pairs on (X, d) such that $(\mathcal{F}, \mathcal{G})$ is coarser than $(\mathcal{F}', \mathcal{G}')$. Then $(\mathcal{F}, \mathcal{G})$ is balanced if $(\mathcal{F}', \mathcal{G}')$ is balanced.

We now explain the construction of B-completion of a T_0 -quasi-metric space (X, d) .

Proposition 2.2.1. ([14, Theorem 1]) Let (X, d) be a quasi-pseudometric space and let X^+ be the set of all balanced Cauchy filter pairs $(\mathcal{F}, \mathcal{G})$ on (X, d) . Define $d^+ : X^+ \times X^+ \rightarrow [0, \infty)$ as above. Then (X^+, d^+) is a quasi-pseudometric space.

Proof. We first notice that $d^+((\mathcal{F}, \mathcal{G}), (\mathcal{F}, \mathcal{G})) = 0$ whenever $(\mathcal{F}, \mathcal{G}) \in X^+$, and verify the triangle inequality. Let $\varepsilon > 0$, find $F_\varepsilon \in \mathcal{F}, G'_\varepsilon \in \mathcal{G}'$ such that $\Phi_d(F_\varepsilon, G'_\varepsilon) \leq d^+((\mathcal{F}, \mathcal{G}), (\mathcal{F}', \mathcal{G}')) + \frac{\varepsilon}{2}$ and similarly $F'_\varepsilon \in \mathcal{F}', G''_\varepsilon \in \mathcal{G}''$ such that $\Phi_d(F'_\varepsilon, G''_\varepsilon) \leq d^+((\mathcal{F}', \mathcal{G}'), (\mathcal{F}'', \mathcal{G}'')) + \frac{\varepsilon}{2}$. For each $f \in F_\varepsilon, g'' \in G''_\varepsilon$ we have

$$d(f, g'') \leq \Phi_d(f, G'_\varepsilon) + \Phi_d(F'_\varepsilon, g'') \leq \Phi_d(F_\varepsilon, G'_\varepsilon) + \Phi_d(F'_\varepsilon, G''_\varepsilon),$$

because $(\mathcal{F}', \mathcal{G}')$ is balanced on (X, d) . It follows that

$$d(f, g) \leq d^+((\mathcal{F}, \mathcal{G}), (\mathcal{F}', \mathcal{G}')) + d^+((\mathcal{F}', \mathcal{G}'), (\mathcal{F}'', \mathcal{G}'')) + \varepsilon$$

whenever $f \in F_\varepsilon$ and $g'' \in G''_\varepsilon$.

Therefore $\Phi_d(F_\varepsilon, G''_\varepsilon) \leq d^+((\mathcal{F}, \mathcal{G}), (\mathcal{F}', \mathcal{G}')) + d^+((\mathcal{F}', \mathcal{G}'), (\mathcal{F}'', \mathcal{G}'')) + \varepsilon$. Hence $d^+((\mathcal{F}, \mathcal{G}), (\mathcal{F}'', \mathcal{G}'')) = \inf_{F \in \mathcal{F}, g'' \in G''} \Phi_d(F, G'') \leq d^+((\mathcal{F}, \mathcal{G}), (\mathcal{F}', \mathcal{G}')) + d^+((\mathcal{F}', \mathcal{G}'), (\mathcal{F}'', \mathcal{G}''))$, since $\varepsilon > 0$ was arbitrary. The triangle inequality is verified. \square

Lemma 2.2.3. ([14, Lemma 4]) *An isometry $g : (X, d) \rightarrow (Y, e)$ from a T_0 -quasi-pseudometric space (X, d) to a quasi-pseudometric space (Y, e) is injective.*

Proof. For any $x, y \in X, g(x) = g(y)$ implies that $e(g(x), g(y)) = 0 = e(g(y), g(x))$ and thus $d(x, y) = 0 = d(y, x)$, since g is an isometry. Hence $x = y$, because (X, d) is a T_0 -quasi-metric space. \square

Let X be a set. For each $x \in X$, we denote by \underline{x} the filter generated by the filter base $\{\{x\}\}$ on X .

Remark 2.2.1. *If (X, d) is a T_0 -quasi-metric space, then $\alpha_X : X \rightarrow X^+$ is an isometric embedding of (X, d) into (X^+, d^+) . Indeed, $d(x, y) = d^+(\alpha_X(x), \alpha_X(y))$ whenever $x, y \in X$.*

Lemma 2.2.4. ([14, Lemma 5]) *Let (X, d) be a quasi-pseudometric space and let $(\mathcal{F}, \mathcal{G})$ be a balanced Cauchy filter pair on (X, d) . Then there exists a unique minimal (balanced) Cauchy filter pair coarser than $(\mathcal{F}, \mathcal{G})$ on (X, d) . It can be described as the 2-intersection of all balanced Cauchy filter pairs belonging to the equivalence class of $(\mathcal{F}, \mathcal{G})$. Moreover it belongs to the equivalence class of $(\mathcal{F}, \mathcal{G})$ and has a countable base.*

Proof. The proof can be found in [14]. \square

Definition 2.2.5. ([14, Definition 6]) Let (X, d) be quasi-pseudometric space. An arbitrary Cauchy filter pair $(\mathcal{F}, \mathcal{G})$ on X is said to converge to $x \in X$ provided that

$$\inf_{F \in \mathcal{F}} \Phi_d(x, F) = 0$$

and

$$\inf_{G \in \mathcal{G}} \Phi_d(G, x) = 0.$$

Equivalently, we say \mathcal{F} converge to $x \in X$ with respect to τ_d and \mathcal{G} converge to $x \in X$ with respect to $\tau_{d^{-1}}$.

A quasi-pseudometric space (X, d) is called *B-complete* provided that each balanced Cauchy filter pair $(\mathcal{F}, \mathcal{G})$ converge in X .

Definition 2.2.6. ([14, Definition 7]) Let (X, d) be a T_0 -quasi-metric space. Then the T_0 -quasimetric space (X^b, d^b) will be called the (standard) B-completion of (X, d) . We set $\beta_X = q_{X^+} \circ \alpha_X$ where $q_{X^+} : (X^+, d^+) \rightarrow (X^b, d^b)$ is the T_0 -quotient map according to Lemma 2.2.2.

Corollary 2.2.1. ([14, Corollary 1]) If (X, d) is a T_0 -quasimetric space, then $\beta_X : (X, d) \rightarrow (X^b, d^b)$ is an (isometric) embedding.

In what follows, we discuss the properties of maps between two quasi-pseudometric spaces, in particular the extension map.

Definition 2.2.7. [14, Definition 8] A quasi-uniformly continuous map $f : (X, d) \rightarrow (Y, e)$ between quasi-pseudometric spaces (X, d) and (Y, e) is called *balanced* provided that for each balanced Cauchy filter pair $(\mathcal{F}, \mathcal{G})$ on (X, d) , the Cauchy filter pair $(f(\mathcal{F}), f(\mathcal{G}))$ is balanced on (Y, e) .

Lemma 2.2.5. ([14, Lemma 6]) Let (X, d) and (Y, e) be quasi-pseudometric spaces and let $f : (X, d) \rightarrow (Y, e)$ be a surjective isometry.

1. Then f is balanced.
2. If $(\mathcal{F}, \mathcal{G})$ is balanced Cauchy filter pair on (Y, e) , then $(f^{-1}\mathcal{F}, f^{-1}\mathcal{G})$ is balanced Cauchy filter pair on (X, d) .

Lemma 2.2.6. ([14, Lemma 7]) Let $(\mathcal{F}, \mathcal{G})$ be a Cauchy filter pair on a quasi-pseudometric space (X, d) . Then for each $x \in X$ and $m \in \mathbb{N}$, there is $g \in X$ such that

$$d^+(\alpha_X(x), (\mathcal{F}, \mathcal{G})) \leq d(x, g) + \frac{1}{m}$$

and

$$d^+((\mathcal{F}, \mathcal{G}), \alpha_X(g)) < \frac{1}{m}.$$

Proof. there are $F_m \in \mathcal{F}$ and $G_m \in \mathcal{G}$ such that $\Phi_d(F_m, G_m) < \frac{1}{m}$. Furthermore for some $g \in G_m$, $d^+(\alpha_X(x), (\mathcal{F}, \mathcal{G})) = \inf_{G \in \mathcal{G}} \Phi_d(x, G) \leq \Phi_d(x, G_m) \leq d(x, g) + \frac{1}{m}$. here we have used the fact that $\Phi_d(x, G_m)$ is bounded. Furthermore $d^+((\mathcal{F}, \mathcal{G}), \alpha_X(g)) < \frac{1}{m}$, since $G_m \in \underline{g}$ and $F_m \in \mathcal{F}$. Hence the assertion holds. \square

Corollary 2.2.2. ([14, Corollary 3]) *Let $(\mathcal{F}, \mathcal{G})$ be a Cauchy filter pair on a quasi-pseudometric space (X, d) . Then for each $y \in X$ and $m \in \mathbb{N}$ there is $f \in X$ such that*

$$d^+((\mathcal{F}, \mathcal{G}), \alpha_X(y)) \leq d(f, y) + \frac{1}{m}$$

and

$$d^+(\alpha_X(f), (\mathcal{F}, \mathcal{G})) < \frac{1}{m}.$$

Proposition 2.2.2. ([11, Proposition 1.0.2]) *Given two Cauchy filter pairs $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{F}', \mathcal{G}')$ on a quasi-pseudometric space (X, d) , we have that*

$$\inf_{G' \in \mathcal{G}'} \Phi_{d^+}((\mathcal{F}, \mathcal{G}), \alpha_X(G')) \leq d^+((\mathcal{F}, \mathcal{G}), (\mathcal{F}', \mathcal{G}'))$$

with equality if $(\mathcal{F}, \mathcal{G})$ is balanced; similarly we have that

$$\inf_{F \in \mathcal{F}} \Phi_{d^+}(\alpha_X(F), (\mathcal{F}', \mathcal{G}')) \leq d^+((\mathcal{F}, \mathcal{G}), (\mathcal{F}', \mathcal{G}'))$$

with equality if $(\mathcal{F}', \mathcal{G}')$ is balanced.

Proof. The statements follow from Lemmas 9,10 and 11 of [14]. \square

Theorem 2.2.1. ([14, Theorem 2]) *Let (X, d) be a T_0 -quasimetric space. Then (X^b, d^b) is B-complete.*

Proof. Suppose that (Ξ, Υ) is a balanced Cauchy filter pair on (X^+, d^+) . For each $n \in \mathbb{N}$ choose $X_n \in \Xi$ and $Y_n \in \Upsilon$ such that $\Phi_{d^+}(X_n, Y_n) < \frac{1}{n}$. Without loss of generality, we can assume that both sequences (X_n) and (Y_n) are decreasing.

For each $n \in \mathbb{N}$ and $x \in X$ we find $\eta_x^n \in Y_n$ such that $\Phi_{d^+}(\alpha_X(x), Y_n) \leq d^+(\alpha_X(x), \eta_x^n) + \frac{1}{n}$. Here we have considered the boundedness of $\Phi_{d^+}(\alpha_X(x), Y_n)$.

Similarly, for each $y \in X$ and $n \in \mathbb{N}$ choose $\xi_y^n \in X_n$ such that $\Phi_{d^+}(X_n, \alpha_X(y)) \leq d^+(\xi_y^n, \alpha_X(y)) + \frac{1}{n}$. For all $x \in X$ and $n \in \mathbb{N}$ each ξ_y^n is a balanced Cauchy filter pair on (X, d) . By lemma 2.2.6 for each $n \in \mathbb{N}$ and $x \in X$ we find $h^{x,n} \in X$ such that $d^+(\alpha_X(x), \eta_x^n) \leq d(x, h^{x,n}) + \frac{1}{n}$ and $d^+(\eta_x^n, \alpha_X(h^{x,n})) < \frac{1}{n}$. Similarly by Corollary 2.2.2 for each $n \in \mathbb{N}$ and $y \in X$ we find $g^{n,y} \in X$ such that $d^+(\xi_y^n, \alpha_X(y)) \leq d(g^{n,y}, y) + \frac{1}{n}$ and $d^+(\alpha_X(g^{n,y}), \xi_y^n) < \frac{1}{n}$. For each $n \in \mathbb{N}$, set $G_n = \{g^{m,y} : m \geq n, m \in \mathbb{N} \text{ and } y \in Y\}$ and for each $n \in \mathbb{N}$ set $H_n = \{h^{m,x} : m \geq n, m \in \mathbb{N} \text{ and } x \in X\}$.

Note that the sequences (G_n) and (H_n) of X are decreasing. Let \mathcal{G} be the filter on X generated by the filter base $\{G_n : n \in \mathbb{N}\}$ and \mathcal{H} be the filter on X generated by the filter base $\{H_n : n \in \mathbb{N}\}$. One checks that $(\mathcal{G}, \mathcal{H})$ is a Cauchy filter pair on (X, d) . Let $x, y \in X$, since (Ξ, Υ) is balanced on (X^+, d^+) , we have

$$d(x, y) = d^+(\alpha_X(x), \alpha_X(y)) \leq \inf_{n \in \mathbb{N}} \Phi_{d^+}(\alpha_X(x), Y_n) + \inf_{n \in \mathbb{N}} \Phi_{d^+}(X_n, \alpha_X(y)).$$

Consequently

$$d(x, y) \leq \left(\inf_{n \in \mathbb{N}} d^+(\alpha_X(x), \eta_x^n) + \frac{1}{n} \right) + \left(\inf_{n \in \mathbb{N}} (d^+(\xi_y^n, \alpha_X(y)) + \frac{1}{n}) \right)$$

by our choices of the Cauchy filter pairs η_x^n and ξ_y^n on X .

It follows that $d(x, y) \leq \inf_{n \in \mathbb{N}} d(x, h^{n,x}) + \frac{2}{n} + \inf_{n \in \mathbb{N}} (d(g^{n,y}, y) + \frac{2}{n})$. We conclude that $d(x, y) \leq \inf_{n \in \mathbb{N}} \Phi_d(x, H_n) + \inf_{n \in \mathbb{N}} \Phi_d(G_n, y)$, because $h^{n,x} \in H_n$ and $g^{n,y} \in G_n$. Hence $(\mathcal{G}, \mathcal{H})$ is balanced Cauchy filter pair on (X, d) .

It remains to show that (Ξ, Υ) converges to the point $(\mathcal{G}, \mathcal{H})$ in X^+ . Let $n \in \mathbb{N}$ and let $\xi = (,) \in X_n \subseteq X^+$. there are $A_n \in \mathcal{G}$ and $B_n \in \mathcal{H}$ such that $\Phi_d(A_n, B_n) < \frac{1}{n}$. Let $a \in A_n$. Then $d^+(\alpha_X(a), \xi) = \inf_{B \in \mathcal{H}} d(a, B) < \frac{1}{n}$. Furthermore for each $m \in \mathbb{N}$ with $m \geq n$ and each $x \in X$, $d^+(\xi, \eta_x^m) < \frac{1}{n}$ and $d^+(\eta_x^m, \alpha_X(h^{m,x})) < \frac{1}{n}$. Thus for each $a \in A_n$, any $m \in \mathbb{N}$ with $m \geq n$ and any $x \in X$ we have $d(a, h^{m,x}) < \frac{3}{n}$ and $d^+(\xi, (\mathcal{G}, \mathcal{H})) \leq \frac{3}{n}$. Therefore $\Phi_{d^+}(X_n, (\mathcal{G}, \mathcal{H})) \leq \frac{3}{n}$.

Analogously, we conclude that $\Phi_{d^+}((\mathcal{G}, \mathcal{H}), Y_n) \leq \frac{3}{n}$. Hence (Ξ, Υ) converges on (X^+, d^+) to the point $(\mathcal{G}, \mathcal{H})$ in X^+ . We have shown that (X^+, d^+) is B-complete. □

The next corollary shows that if (X, d) is B-complete, the isometric embedding is bijective.

Corollary 2.2.3. ([14, Corollary 4]) *Let (X, d) be a B-complete T_0 -quasimetric space. Then the isometric embedding $\beta_X : (X, d) \rightarrow (X^b, d^b)$ is bijective. (Therefore (X, d) and (X^b, d^b) can be identified under these conditions).*

This example is taken from C.M. Kivuvu [11].

Example 2.2.1. *Let $X = \{\frac{-1}{n+1}, \frac{1}{n+1}, n \in \mathbb{N}\}$. For each $x, y \in X$, let $d(x, y) = 1$, if $x < 0 < y$, $d(x, y) = 0$, if $y \leq x$, and $d(x, y) = \min\{1, |x - y|\}$ otherwise. It is readily checked that (X, d) is a T_0 -quasi-metric space. Let $(\mathcal{F}, \mathcal{G})$ be the filter pair on X generated by $(\langle \frac{-1}{n+1} \rangle, \langle \frac{1}{n+1} \rangle)$. Observe that $(\mathcal{F}, \mathcal{G})$ is a Cauchy filter pair on (X, d) , which is not balanced, since*

$$1 = d(\frac{1}{4}, \frac{-1}{4}) \not\leq \inf_{G \in \mathcal{G}} \Phi_d(\frac{1}{4}, G) + \inf_{F \in \mathcal{F}} \Phi_d(F, \frac{-1}{4}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

On the other hand the non-convergent Cauchy filter pair $(\mathcal{F}, \mathcal{F})$ and $(\mathcal{G}, \mathcal{G})$ show that (X, d) is not bicomplete.

We leave it to the reader to check the following additional facts: the B-completion of (X, d) is obtained by adding two new distinct points 0^- and 0^+ to X which represent the equivalence classes of $(\mathcal{F}, \mathcal{F})$ resp. $(\mathcal{G}, \mathcal{G})$. Then, d^b extends d as follows: $d^b(0^-, x) = d^b(x, 0^+) = |x|$ if $x \in X$; $d^b(x, 0^-) = 1$ if $x > 0$; $d^b(x, 0^-) = |x|$ if $x < 0$; $d^b(0^+, x) = 1$ if $x < 0$; $d^b(0^+, x) = x$ if $x > 0$; and $d^b(0^-, 0^+) = 0$; $d^b(0^+, 0^-) = 1$. Of course $d^b(0^-, 0^-) = d^b(0^+, 0^+) = 0$.

This ends the section on B-completion. We next summarize the theory of κ -completion due to Andrikopoulos.

2.3 κ -completion

The essential information of this section is taken from Andrikopoulos's paper[2].

In [2] Andrikopoulos has extended Doitchinov's completion theory for arbitrary quasi-pseudometric spaces. The resulting completion is called the κ -completion. He has shown that each T_0 -quasimetric space admits a κ -completion even when the original space is not balanced.

We present the summary of the construction of the κ -completion of a T_0 -quasimetric space with emphasis on some propositions that help to understand this theory.

Let us first start with some useful concepts of sequences that will be used in the sequel.

Definition 2.3.1. ([2, Definition 3]) Let (X, d) be a quasi-pseudometric space and let $(x_n), (y_m)$ be two sequences on it. One says that (x_n) is right d-cofinal to (y_m) , if for each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ satisfying the following property: for each $n > n_\varepsilon$ there exists $m_n \in \mathbb{N}$ such that $d(y_m, x_n) < \varepsilon$ whenever $m > m_n$.

Definition 2.3.2. (Compare [2, Definition 3]) Let (X, d) be a quasi-pseudometric space and let $(x_n), (y_m)$ be two sequences on it. One says that (y_m) left d-cofinal to (x_n) , if for each $\varepsilon > 0$ there exists $m_\varepsilon \in \mathbb{N}$ satisfying the following property: for each $m > m_\varepsilon$ there exists $n_m \in \mathbb{N}$ such that $d(y_m, x_n) < \varepsilon$ whenever $n > n_m$.

The sequences (x_n) and (y_m) are right (resp. left) d-cofinal if (x_n) is right (resp. left) d-cofinal to (y_m) and vice-versa.

The following propositions prepare us for the definition of a family of some K-Cauchy sequences, and the relation between two members of that family.

Proposition 2.3.1. ([2, Proposition 4]) Let (x_n) be a right K-Cauchy sequence in a quasi-pseudometric space (X, d) with a subsequence (x_{n_g}) . Then (x_n) and (x_{n_g}) are right d-cofinal.

Proof. (x_n) right K-Cauchy means for any $\varepsilon > 0$, there exist $n_\varepsilon, n > n' > n_\varepsilon$ such that $d(x_n, x_{n'}) < \varepsilon/2$. On the other hand, (x_{n_g}) subsequence of (x_n) , hence is also a right K-Cauchy sequence. If $n > n_\varepsilon$, there is N_g such that for any $n_g > N_g, d(x_{n_g}, x_{N_g}) < \varepsilon/2$ by definition of right K-Cauchy sequence (x_{n_g}) , and $d(x_{N_g}, x_n) < \varepsilon/2$ by the fact that (x_n) is a right K-Cauchy sequence. Therefore, $d(x_{n_g}, x_n) \leq d(x_{n_g}, x_{N_g}) + d(x_{N_g}, x_n) < \varepsilon$. Hence, for each $n > n_\varepsilon$ there exists $N_g \in \mathbb{N}$ such that $d(x_{n_g}, x_n) < \varepsilon$ whenever $n_g > N_g$. It is nothing else than the definition of d-Cofinality. \square

Corollary 2.3.1. Let (y_m) be a left K-Cauchy sequence in a quasi-pseudometric space (X, d) with a subsequence (y_{m_g}) . Then (y_m) and (y_{m_g}) are left d-cofinal.

Proof. (y_m) left K-Cauchy means for any $\varepsilon > 0$, there exist $m_\varepsilon, m' > m > m_\varepsilon$ such that $d(y_m, y_{m'}) < \varepsilon/2$. On the other hand, (y_{m_g}) subsequence of (y_m) , hence is also a right K-Cauchy sequence. If $m > m_\varepsilon$, there is M_g such that for any $m_g > M_g, d(y_{M_g}, y_{m_g}) < \varepsilon/2$ by definition of right K-Cauchy

sequence (y_{m_g}) , and $d(y_{M_g}, y_m) < \varepsilon/2$ by the fact that (y_m) is a right K-Cauchy sequence. Therefore, $d(y_m, y_{m_g}) \leq d(y_m, y_{M_g}) + d(y_{M_g}, y_{m_g}) < \varepsilon$. Hence, for each $m > m_\varepsilon$ there exists $M_\varepsilon \in \mathbb{N}$ such that $d(y_m, y_{M_g}) < \varepsilon$ whenever $m_g > M_g$. It is nothing else than the definition of d-cofinality. \square

Proposition 2.3.2. ([2, Proposition 5]) *In every quasi-pseudometric space (X, d) , two right d-cofinal sequences have the same cosequence.*

Proof. We consider two sequences in (X, d) , (x_n) and (x'_m) being right K-Cauchy and right d-cofinal to each other. Firstly, the fact (y_s) is a cosequence of (x'_m) means, for any $\varepsilon > 0$ there is $s_\varepsilon, m_\varepsilon \in \mathbb{N}$ such that $d(y_s, x'_m) < \varepsilon/2$ whenever $s > s_\varepsilon$ and $m > m_\varepsilon$. In another word, $\lim_{n,m}(y_s, x'_m) = 0$.

By d-cofinality of (x_n) and (x'_m) , we have $n_\varepsilon \in \mathbb{N}$, with $n > n_\varepsilon$, there is m_n such that $d(x'_m, x_n) < \varepsilon/2$, for $m > m_n$. When we combine the two properties, we get $m(n) = \max\{m_\varepsilon, m_n\}$, the maximum number between m_ε from the fact that (x'_m) has (y_s) as cosequence, m_n from the other fact related to the d-finality of (x_n) and (x'_m) . This leads to the following inequality:

$$d(y_s, x_n) \leq d(y_s, x'_{m(n)}) + d(x'_{m(n)}, x_n) < \varepsilon/2 + \varepsilon/2 = \varepsilon, \text{ when } s > s_\varepsilon$$

Hence, we get: for any $\varepsilon > 0$ there are $s_\varepsilon, n_\varepsilon \in \mathbb{N}$ such that $d(y_s, x_n) < \varepsilon$ when $s > s_\varepsilon$ and $n > n_\varepsilon$, we conclude that (y_s) is a cosequence of (x_n) , meaning $\lim_{s,n} d(y_s, x_n) = 0$. \square

Proposition 2.3.3. ([2, Proposition 6]) *In every quasi-pseudo metric space (X, d) , two left d-cofinal left K-Cauchy sequences are cosequences of the same sequence.*

Proof. Similarly to the previous proposition, we consider the fact that (y_s) and (y'_t) left d-cofinal, therefore for any $\varepsilon > 0$, there is $s_\varepsilon \in \mathbb{N}$ such that $d(y_s, y'_t) < \varepsilon/2$ when $t > t_s$. On the other side, we have (y_s) is a cosequence of (x_n) , meaning that for any $\varepsilon > 0$ there are $t_\varepsilon, n_\varepsilon \in \mathbb{N}$ such that $d(y'_t, x_n) < \varepsilon/2$, when $t > t_\varepsilon$ and $n > n_\varepsilon$. By taking, $t = \max\{t_\varepsilon, t_s\}$ from the two properties related to (y_s) . We get the following inequality $d(y_s, x_n) \leq d(y_s, y'_{t(n)}) + d(y'_{t(n)}, x_n) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Hence, for any $\varepsilon > 0$ there are $s_\varepsilon, n_\varepsilon$ such that $d(y_s, x_n) < \varepsilon$, when $s > s_\varepsilon$ and $n > n_\varepsilon$. \square

Following the above propositions, Andrikopoulos has defined a new object κ -cut, that contains a collection of K-Cauchy sequences sharing something in common.

This concept will be extended in the next chapter, where we will replace sequences with filters.

Definition 2.3.3. ([2, Definition 8]) Let (X, d) be a quasi-pseudometric space. We call κ -cut in X an ordered pair $\xi = (\mathcal{R}, \mathcal{L})$ of families of right K -Cauchy sequences and left K -Cauchy sequences respectively, with the following properties.

1. For any $(x_n) \in \mathcal{R}$ and $(y_\sigma) \in \mathcal{L}$ there holds $\lim_{\sigma, n} d(y_\sigma, x_n) = 0$.
2. Any two members of the family \mathcal{R} (resp. \mathcal{L}) are right (resp. left) d -cofinal.
3. The class \mathcal{R} and \mathcal{L} are maximal with respect to set inclusion.

We call \mathcal{R} (resp. \mathcal{L}) *first* (resp. *second*) class of ξ . In what follows, for simplicity of the proofs, we call the elements of \mathcal{R} cosequences of the elements of \mathcal{L} .

Definition 2.3.4. ([2, Definition 9]) For every $x \in X$ one chooses a κ -cut $\phi(x) = (\mathcal{R}, \mathcal{L})$, where \mathcal{R} consists of right K -Cauchy sequences which converges to x with respect to τ_d and \mathcal{L} consists of left K -Cauchy cosequences which converges to x with respect to $\tau_{d^{-1}}$. The sequence $(x) = (x, x, x, \dots)$ itself belongs to both of the classes. If there are not right K -Cauchy sequences (resp. left K -Cauchy sequences) converging to x , then $\phi(x) = ((x), \mathcal{L})$ (resp. $\phi(x) = (\mathcal{R}, (x))$). If x is an isolated point for τ_d and $\tau_{d^{-1}}$, respectively, then $\phi(x) = ((x), (x))$.

In order to work with sequences, Andrikopoulos has taken a representative of each κ -cut that he called κ -Cauchy sequence.

Definition 2.3.5. ([2, Definition 11]) One calls κ -Cauchy sequence any right K -Cauchy sequence which is member of the first class of a κ -cut.

Definition 2.3.6. ([2, Definition 12]) Two right K -Cauchy sequences (x_n) and (x'_m) defined in a quasi-pseudometric space (X, d) are called κ -equivalent if every left K -Cauchy cosequence to (x_n) is also cosequence to (x'_m) and vice-versa.

The next step is to define the distance between two κ -cuts, then provide the proof that this distance function is a quasi-pseudometric.

We next define the distance between two κ -cuts.

Definition 2.3.7. ([2, Definition 17]) Let (X, d) be a quasi-pseudometric space. Suppose that r is a nonnegative real number, $\xi', \xi'' \in \hat{X}$, $(x'_n) \in \xi'$, and $(x''_m) \in \xi''$. We put $\hat{d}(\xi', \xi'') < r$ if

1. $\xi' = \xi''$ or

2. for each $\varepsilon > 0$ there are $n'_\varepsilon, m''_\varepsilon \in \mathbb{N}$ such that $d(x'_n, x''_m) < r + \varepsilon$

when $n \geq n'_\varepsilon$ and $m \geq m''_\varepsilon$. If $\xi' = \phi(x)$ for some $x \in X$, then the arbitrary sequence (x'_n) always coincides with the fixed sequence, for which x'_n for all $n \in \mathbb{N}$. That is, $\hat{d}(\phi(x), \xi'') \leq r$ if $d(x, x''_m) < r + \varepsilon$ when $m \geq m''_\varepsilon$. Then we let $\hat{d}(\xi', \xi'') = \inf\{r \mid \hat{d}(\xi', \xi'') \leq r\}$.

Proposition 2.3.4. ([2, Proposition 18]) Let $\xi', \xi'' \in \widehat{X}$, $(x'_n) \in \xi'$ and $(x''_m) \in \xi''$. Suppose that $\hat{d}(\xi', \xi'') = 0$. Then, (x''_m) is right d-cofinal to (x'_n) .

Proof. For each $\varepsilon > 0$ there are $n'_\varepsilon, m''_\varepsilon \in \mathbb{N}$ such that $d(x'_n, x''_m) < \varepsilon$ whenever $n \geq n'_\varepsilon$ and $m \geq m''_\varepsilon$. By taking $n_m = n'_\varepsilon$, we see that $d(x'_n, x''_m) < \varepsilon$ whenever $n \geq n_m$. By definition of d-cofinality, we get that (x''_m) is d-cofinal to (x'_n) . \square

The next proposition shows that the distance between two κ -cuts does not depend on the choice of representatives of these κ -cuts.

Proposition 2.3.5. ([2, Proposition 19]) The truth of $\hat{d}(\xi', \xi'') \leq r$ in the definition depends only on ξ', ξ'' , and r ; it does not depend on the choice of the sequences (x'_n) and (x''_m) .

Proof. Let $\xi', \xi'' \in \widehat{X}$, $(x'_n) \in \xi'$, $(x''_m) \in \xi''$. Let $\hat{d}(\xi', \xi'') \leq r$. We have, for each $\varepsilon > 0$ there are $n'_\varepsilon, m''_\varepsilon \in \mathbb{N}$ such that $d(x'_n, x''_m) < r + \varepsilon/3$ when $n \geq n'_\varepsilon$ and $m \geq m''_\varepsilon$. We take two arbitrary sequences $(x'_s) \in \xi'$ and $(x''_t) \in \xi''$. Since (x'_n) is right d-cofinal to (x'_s) , there is $N'_\varepsilon \in \mathbb{N}$ satisfying the following property:

\square

Proposition 2.3.6. ([2, Proposition 20]) Let $\xi', \xi'' \in \widehat{X}$, $x'_n \in \xi'$ and $x''_m \in \xi''$. Suppose that $\hat{d}(\xi', \xi'') = 0$. Then, (x''_m) is right d-cofinal to (x'_n) .

Proof. By suppositions, for each $\varepsilon > 0$ there are $n'_\varepsilon, m''_\varepsilon \in \mathbb{N}$ such that $\hat{d}(x'_n, x''_m) < \varepsilon$ whenever $n \geq n'_\varepsilon$ and $m \geq m''_\varepsilon$. By taking $m_n = m''_\varepsilon$, we get (x''_m) is right d-cofinal to (x'_n) . \square

Proposition 2.3.7. ([2, Proposition 21]) Let (X, d) be a quasi-pseudometric space, and consider \hat{d} as defined on Definition 2.3.7, then \hat{d} is quasi-metric.

Proof. 1. $\widehat{d}(\xi, \xi) = 0$, by definition if we take two arbitrary sequences $(x_n), (x_m) \in \xi$, in fact they are d-cofinal therefore $r = 0$.

2. $\widehat{d}(\xi, \xi') \geq 0$, obvious by definition of \widehat{d} .

3. To prove the triangle inequality, we will proceed with three cases:

(a) $\xi \neq \xi'$ and $\xi' \neq \xi''$. Suppose $\widehat{d}(\xi, \xi') = r_1$ and

$$\widehat{d}(\xi', \xi'') = r_2, (x_n) \in \xi, (x') \in \xi' \text{ and } (x'') \in \xi''$$

Then, by definition for any $\varepsilon > 0$ there are $n_\varepsilon, m'_\varepsilon > 0$ such that $d(X_n, X'_m) < r_1 + \varepsilon/2$ whenever $n > n_\varepsilon$ and $m > m'_\varepsilon$. In similar way, there are $\tilde{m}'_\varepsilon, s''_\varepsilon > 0$ such that $d(x'_m, s''_t) < r_2 + \varepsilon/2$ whenever $m \geq \tilde{m}'_\varepsilon$ and $s \geq s''_\varepsilon$. Let us consider $M_o = \max\{m_\varepsilon, \tilde{m}'_\varepsilon\}$. Then, $d(x_n, x'') \leq d(x_n, x'_{M_o}) + d(x'_{M_o}, x''_s) < r_1 + r_2 + \varepsilon$ for each $n \geq n_\varepsilon, s \geq s''_\varepsilon$. Hence, we have by definition $\widehat{d}(\xi, \xi'') \leq r_1 + r_2 = \widehat{d}(\xi, \xi') + \widehat{d}(\xi', \xi'')$.

(b) $\xi \neq \xi'$ and $\xi' = \xi''$. Suppose $\widehat{d}(\xi, \xi') = r$ and $\widehat{d}(\xi', \xi'') = 0$ due to the above equality,

$$\widehat{d}(\xi, \xi'') \leq r + 0 = \widehat{d}(\xi, \xi') + \widehat{d}(\xi', \xi'').$$

(c) $\xi = \xi'$ and $\xi' \neq \xi''$. The proof is similar to b).

□

As consequence of the above, $(\widehat{X}, \widehat{d})$ is a quasi-pseudometric space. Now, we can discuss on the completeness of this quasi-pseudometric space. To do so, we relate on the properties of the extension map ϕ .

Proposition 2.3.8. ([2, Proposition 22]) *For any $x, y \in X$ there holds $\widehat{d}(\phi(x), \phi(y)) = d(x, y)$.*

Proof. Refer to [2].

□

This proposition has shown that ϕ is an isometric embedding. The next proposition is on the density of $\phi(X)$ in \widehat{X} . Let us look on the property of the image of the map ϕ .

Proposition 2.3.9. ([2, Proposition 23]) *For any $\xi \in \widehat{X}$,*

1. if $(x_n) \in \xi$ then $\widehat{d}(\xi, \phi(x_n)) \rightarrow 0$;

2. if $(y_\sigma) \in \xi$ then $\widehat{d}(\phi(y_\sigma), \xi) \rightarrow 0$.

the set $\phi(X)$ is dense in $(\widehat{X}, \widehat{d})$.

Proposition 2.3.10. ([2, Proposition 25]) Let (ξ_γ) be a nonconstant right K -Cauchy sequence of $(\widehat{X}, \widehat{d})$ without last element. Then there exists a right K -Cauchy sequence (x_n) of (X, d) such that the sequences (ξ_γ) and $(\phi(x_n))$ are right \widehat{d} -cofinal right K -Cauchy sequences.

Proof. Refer to [2]. □

Proposition 2.3.11. ([2, Proposition 26]) Let (ν_δ) be a nonconstant left K -Cauchy sequence of $(\widehat{X}, \widehat{d})$ without last element. Then there exists a left K -Cauchy sequence (y_σ) of (X, d) such that the sequences (ν_δ) and (ϕ_σ) are left \widehat{d} -cofinal.

Proof. Refer to [2]. □

Theorem 2.3.1. ([2, Theorem 27]) Every quasi-pseudometric space has a κ -completion.

Proof. Refer to [2]. □

Proposition 2.3.12. ([2, Proposition 29]) Let (X, d) be a T_0 -quasi-metric space and let $\widehat{X}, \widetilde{X}$ be as above. Suppose that $\widetilde{d} : \widetilde{X} \times \widetilde{X} \rightarrow \mathbb{R}$ is a function mapping defined by $\widetilde{d}(\Xi, \Xi') = \widehat{d}(\xi, \xi')$ whenever $\xi, \xi' \in \widehat{X}$. then, \widetilde{d} determines a T_0 -quasi-pseudometric on \widetilde{X} .

Proof. To prove that \widetilde{d} is well-defined suppose that $\xi, \xi_1, \xi', \xi'_1 \in \widehat{X}$ and $\xi \approx \xi'$ and $\xi_1 \approx \xi'_1$. By triangle inequality, we see that $\widehat{d}(\xi_1, \xi'_1) \leq \widehat{d}(\xi_1, \xi) + \widehat{d}(\xi, \xi') + \widehat{d}(\xi', \xi'_1)$ and hence $\widehat{d}(\xi_1, \xi'_1) \leq 0 + \widehat{d}(\xi, \xi') + 0$. Similarly, $\widehat{d}(\xi, \xi') \leq \widehat{d}(\xi_1, \xi'_1)$ which implies that $\widehat{d}(\xi, \xi') = \widehat{d}(\xi_1, \xi'_1)$. Hence, \widetilde{d} is well-defined. It is obvious that \widetilde{d} is a quasi-pseudometric. To prove that $(\widetilde{X}, \widetilde{d})$ is a T_0 -quasi-metric space, suppose that $\widetilde{d}(\Xi, \Xi') = \widetilde{d}(\Xi', \Xi) = 0$. Then, $\widetilde{d}(\xi, \xi') = \widetilde{d}(\xi', \xi) = 0$. Suppose that $\xi, \xi' \in \widehat{X}$. Let $(x_n) \in \xi$ and $(x'_m) \in \xi'$. Then, by proposition 2.3.4 we conclude that (x_n) and (x'_m) are right d -cofinal. therefore, $\xi = \xi'$. If $\xi = \phi(x)$ and $\xi' = \phi(x')$ or some $x, x' \in X$, then $\widehat{d}(\phi(x), \phi(x')) = \widehat{d}(\phi(x'), \phi(x)) = 0$ which implies that $d(x, x') = d(x', x) = 0$. Since (X, d) is T_0 , we conclude that $x = x'$. Hence, in any case we have $\Xi = \Xi'$. Consequently, $(\widetilde{X}, \widetilde{d})$ is a T_0 -quasi-metric space. □

The next proposition shows that the extension map is an isometric embedding.

Proposition 2.3.13. (*[2, Proposition 30]*) *If (X, d) is a T_0 -quasi-metric space, then $\varphi = \phi_q \circ \phi : (X, d) \rightarrow (\tilde{X}, \tilde{d})$ is an isometric embedding.*

Proof. First, we notice that $\varphi = \phi_q \circ \phi$ is an isometry, since both ϕ_q and ϕ are isometries. On the other hand, for any $x, y \in X$, $\phi(x) = \phi(y)$ implies that $\hat{d}(\phi(x), \phi(y)) = 0 = \hat{d}(\phi(y), \phi(x))$. Then, Proposition 2.3.8 implies that $d(x, y) = 0 = d(y, x)$. Hence, $x = y$ because (X, d) is a T_0 -quasi-metric space. Therefore, ϕ is injective which implies that φ is a isometric embedding. \square

Theorem 2.3.2. (*[2, Theorem 31]*) *Let (X, d) be a T_0 -quasi-metric space. Then (\tilde{X}, \tilde{d}) is a T_0 κ -completion of (X, d) .*

Proof. Let (Ξ_γ) be a κ -Cauchy sequence in (\tilde{X}, \tilde{d}) . Therefore, since ϕ_q is a surjective isometry, $\phi_q^{-1}((\Xi_\gamma))$ is a κ -Cauchy sequence in (\hat{X}, \hat{d}) . Thus for some $\xi \in \hat{X}$, $\phi_q^{-1}((\Xi_\gamma))$ converges to ξ . It follows that (Ξ_γ) converges to $\phi_q(\xi) = \Xi$ since ϕ_q is an isometry. So (\tilde{X}, \tilde{d}) is κ -complete T_0 -quasi-metric space. It remains to prove that the set $\varphi(X)$ is dense in (\tilde{X}, \tilde{d}) . Indeed, suppose that $\Xi \in \tilde{X}$. Then, since ϕ_q is a surjective isometry, there exists $\xi \in \hat{X}$ such that $\phi_q(\xi) = \Xi$. Let $(x_n) \in \xi$. Then, by Definition 2.3.7 we have that $\phi(x_n) \rightarrow \xi$. Therefore, $\phi_q(\phi(x_n))$ converges to $\phi_q(\xi) = \Xi$ with respect to $\tau_{\tilde{d}}$. It follows that $\varphi(X)$ is dense in (\tilde{X}, \tilde{d}) which implies that (\tilde{X}, \tilde{d}) is a κ -completion of (X, d) . \square

This theorem has shown the existence of a T_0 κ -completion whenever we have a T_0 -quasi-metric space.

Definition 2.3.8. (*[2, Definition 32]*) *Let (X, d) be a T_0 -quasi-metric space. Then the T_0 -quasi-pseudometric space (\tilde{X}, \tilde{d}) previously defined will be called the standard κ -completion of (X, d) .*

With this section, we have shown that given an arbitrary quasi-pseudometric space, Andrikopoulos has succeeded in constructing a completion of the space by means of cut technique. In the next chapter, we will extend this technique using the concept of K -Cauchy filters on a quasi-pseudometric space.

Chapter 3

The δ -completion of quasi-pseudometric space

In this main chapter, we introduce the concept of the δ -completion on a quasi-pseudometric space.

We will first define δ -cut, the elements of the δ -complete space. Then we define the distance between two members of the set \widehat{X} . Finally, we will prove that any quasi-pseudometric space has a δ -completion.

In the last section, we will show that the δ -completion of a quasi-pseudometric space is idempotent to δ -completion of a quasi-pseudometric space and we will make connections between δ -completion and other completions.

3.1 The δ -cut construction

In this section we construct the new set defined by its members. We start by extending some definitions of Cauchy pair of sequences to filter.

The following well-known definition was introduced by Romaguera [20].

Definition 3.1.1. *A filter \mathcal{F} on a quasi-pseudometric space (X, d) is called left K -Cauchy if for any $\varepsilon > 0$, there exists $F_\varepsilon \in \mathcal{F}$ such that $B_d(x, \varepsilon) \in \mathcal{F}$, whenever $x \in F_\varepsilon$.*

Definition 3.1.2. *A filter \mathcal{F} on a quasi-pseudometric space (X, d) is called right K -Cauchy if for any $\varepsilon > 0$, there exists $F_\varepsilon \in \mathcal{F}$ such that $B_{d^{-1}}(x, \varepsilon) \in \mathcal{F}$, whenever $x \in F_\varepsilon$.*

Definition 3.1.3. Let (X, d) be a quasi-pseudometric space. We say that a filter \mathcal{F} is a cofilter to a filter \mathcal{G} if $(\mathcal{F}, \mathcal{G})$ is Cauchy filter pair.

We next define the concept of cofinality for filter.

Definition 3.1.4. Let (X, d) be a quasi-pseudometric space and \mathcal{F} and \mathcal{G} be two filters on X . We say that \mathcal{F} is right d-cofinal to \mathcal{G} , if $(\mathcal{G}, \mathcal{F})$ is a Cauchy filter pair on (X, d) . Similarly, we say that \mathcal{F} is left d-cofinal to \mathcal{G} , if $(\mathcal{F}, \mathcal{G})$ is Cauchy filter pair on (X, d) . The filters \mathcal{F} and \mathcal{G} are d-cofinal if \mathcal{F} is right d-cofinal to \mathcal{G} and \mathcal{G} is also right d-cofinal to \mathcal{F} .

Lemma 3.1.1. Let (X, d) be a quasi-pseudometric space. Consider $F \in \mathcal{F}$ (resp. $G \in \mathcal{G}$), with \mathcal{F} and \mathcal{G} two left K-Cauchy filters on (X, d) . If \mathcal{G} is right d-cofinal to \mathcal{F} , then there exists $x \in F$, $\inf_{G \in \mathcal{G}} \Phi_d(\{x\}, G) = 0$.

Proof. Since \mathcal{G} is right d-cofinal to \mathcal{F} , we have

$$\inf_{F \in \mathcal{F}, G \in \mathcal{G}} \Phi_d(F, G) = 0$$

In other word, we have for any $\varepsilon > 0$, there exists $F_\varepsilon \in \mathcal{F}$ and $G_\varepsilon \in \mathcal{G}$ such that

$$\Phi_d(F_\varepsilon, G_\varepsilon) = \sup\{d(x, y) : x \in F_\varepsilon, y \in G_\varepsilon\} = \varepsilon,$$

by the fact that $(\mathcal{F}, \mathcal{G})$ is Cauchy filter pair. This implies for any $\varepsilon > 0$ we have $C_d(x, \varepsilon) \subseteq F_\varepsilon, C_d(y, \varepsilon) \subseteq G_\varepsilon$, such that $\sup\{d(x, y) : x \in F_\varepsilon, y \in G_\varepsilon\} = 0 < \varepsilon$. Since \mathcal{G} is left K-Cauchy filter we have $x \in F$, for any $\varepsilon > 0, C_d(x, \varepsilon) \subseteq G_\varepsilon$. Hence,

$$\inf_{G \in \mathcal{G}} \Phi_d(\{x\}, G) = 0$$

□

Lemma 3.1.2. Let (X, d) be a quasi-pseudometric space. Consider $F \in \mathcal{F}$ (resp. $G \in \mathcal{G}$), with \mathcal{F} and \mathcal{G} two right K-Cauchy filter on (X, d) . If \mathcal{F} is left d-cofinal to \mathcal{G} , then there exists $y \in G$, $\inf_{F \in \mathcal{F}} \Phi_d(F, \{y\}) = 0$.

Proof. Since \mathcal{F} is left d-cofinal to \mathcal{G} , we have

$$\inf_{F \in \mathcal{F}, G \in \mathcal{G}} \Phi_d(F, G) = 0$$

In other word, we have for any $\varepsilon > 0$ there exist $F_\varepsilon \in \mathcal{F}, G_\varepsilon \in \mathcal{G}$ such that

$$\Phi_d(F_\varepsilon, G_\varepsilon) = \sup\{d(x, y) : x \in F_\varepsilon, y \in G_\varepsilon\} = \varepsilon,$$

by definition of Cauchy filter pair. This implies for any $\varepsilon > 0$, $C_{d^{-1}}(x, \varepsilon) \subseteq F_\varepsilon$, $C_{d^{-1}}(y, \varepsilon) \subseteq G_\varepsilon$, such that $\sup\{d(x, y) : x \in F_\varepsilon, y \in G_\varepsilon\} = 0 < \varepsilon$, since \mathcal{F} is right K-Cauchy filter we have there exists $y \in G$ such that for any $\varepsilon > 0$, $C_{d^{-1}}(y, \varepsilon) \subseteq F_\varepsilon$. Hence,

$$\inf_{F \in \mathcal{F}} \Phi_d(F, \{y\}) = 0$$

□

We next prove that two right d -cofinal filters have the same cofilter. The following proposition can be compared with Proposition 2.3.2.

Lemma 3.1.3. *Let (X, d) be a quasi-metric space. If the filter \mathcal{F} is right d -cofinal to the filter \mathcal{G} , then the filter \mathcal{G} is left d^{-1} -cofinal to the filter \mathcal{F} . Moreover if the filter \mathcal{F} is left d -cofinal to the filter \mathcal{G} , then the filter \mathcal{G} is right d^{-1} -cofinal to the filter \mathcal{F} .*

Proof. It follows from the definition of Cauchyness of filter pair and d -cofinality definition filters. □

Proposition 3.1.1. *Consider a quasi-pseudometric space (X, d) . Let \mathcal{F} and \mathcal{G} two filters on (X, d) . If \mathcal{F} is right d -cofinal to \mathcal{G} , then \mathcal{F} and \mathcal{G} have the same cofilters.*

Proof. Let \mathcal{F} and \mathcal{G} be two filters on (X, d) and \mathcal{F} is right d -cofinal filter to \mathcal{G} . Suppose \mathcal{H} be cofilter to \mathcal{G} . Then $(\mathcal{H}, \mathcal{G})$ is a Cauchy filter pair, we have that

$$\inf_{H \in \mathcal{H}, G \in \mathcal{G}} \Phi_d(H, G) = 0.$$

Since \mathcal{F} is right d -cofinal to \mathcal{G} , we have that $(\mathcal{G}, \mathcal{F})$ is a Cauchy filter pair, then

$$\inf_{G \in \mathcal{G}, F \in \mathcal{F}} \Phi_d(G, F) = 0.$$

Moreover, we have

$$\Phi_d(H, F) \leq \Phi_d(H, G) + \Phi_d(G, F),$$

whenever $H \in \mathcal{H}$, $G \in \mathcal{G}$ and $F \in \mathcal{F}$. Hence

$$\inf_{H \in \mathcal{H}, F \in \mathcal{F}} \Phi_d(H, F) = 0.$$

Furthermore $(\mathcal{H}, \mathcal{F})$ is Cauchy filter pair. Therefore, \mathcal{H} is a cofilter to \mathcal{F} . □

One uses Lemma 3.1.3 and Proposition 3.1.1 to prove the following lemma.

Proposition 3.1.2. *Let \mathcal{F}, \mathcal{G} be two filters on a quasi-pseudometric space (X, d) . If \mathcal{F} is left d^{-1} -cofinal to \mathcal{G} , then \mathcal{F} and \mathcal{G} have the same cofilters.*

Lemma 3.1.4. *Consider two right K -Cauchy filters \mathcal{F}, \mathcal{G} on a quasi-pseudometric space (X, d) . If \mathcal{G} is right d -cofilter to \mathcal{F} (respectively d^{-1} -cofilter), we have that \mathcal{G} converges to x with respect to $\tau_{d^{-1}}$ (respectively τ_d) and \mathcal{F} converges to x with respect to $\tau_d(\tau_{d^{-1}})$.*

Proof. This is a consequence of Definition 3.1.4 and [14, Definition 6]. \square

By similar arguments, this shows that two left d -cofinal filters are cofilters of the same filters.

Corollary 3.1.1. *In every quasi-pseudometric space (X, d) , two d -cofinal K -Cauchy filters have the same limit point with respect to τ_d and $\tau_{d^{-1}}$.*

To illustrate the concept of cofinality of filters in a quasi-pseudometric space (X, d) , let us consider the following examples.

Example 3.1.1. *Let (X, d) be a quasi-pseudometric space. Let \mathcal{F} be the filter generated by the filter base $\{B_d(x, r) : x \in X, r > 0\}$ and \mathcal{F}' the filter generated by the filter base $\{B_d(x, r') : x \in X, r' > 0\}$ such that $r < r'$, we see that \mathcal{F}' is coarser than \mathcal{F} . Moreover, The two filters are d -cofinal.*

Example 3.1.2. *Let (X, d) be a quasi-pseudometric space. Let \mathcal{G} be the filter generated by the filter base $\{B_{d^{-1}}(y, r) : y \in X, r > 0\}$ and \mathcal{G}' the filter generated by the filter base $\{B_{d^{-1}}(y, r') : y \in X, r' > 0\}$ such that $r < r'$, we see that \mathcal{G}' is coarser than \mathcal{G} . Moreover, The two filters are d^{-1} -cofinal.*

Our next definition extends Definition 2.3.3.

Definition 3.1.5. *Let (X, d) be a quasi-pseudometric space. We call δ -cut on (X, d) an ordered pair $\xi = (\cdot, \cdot)$ of families of right K -Cauchy filters and left K -Cauchy filters, respectively, with the following properties.*

1. For any $\mathcal{F} \in \cdot$ and $\mathcal{G} \in \cdot$ there holds

$$\inf_{F \in \mathcal{F}, G \in \mathcal{G}} \phi_d(F, G) = 0$$

2. Any two members of family \cdot are right d -cofinal

3. The classes \mathcal{C}_x and \mathcal{C}_y are maximal with respect to set inclusion.

Definition 3.1.6. For any $x \in X$, one chooses a δ -cut $\psi(x) = (\mathcal{C}_x, \mathcal{D}_x)$, where \mathcal{C}_x is a collection of right K -Cauchy filters which converges to x with respect to τ_d and \mathcal{D}_x for left K -Cauchy cofilters which converges to x with respect to $\tau_{d^{-1}}$. If there are no right K -Cauchy filters (resp. left K -Cauchy cofilters) converging to x , then $\psi(x) = (\{\mathcal{C}_x\}, \mathcal{D}_x)$ (resp. $\psi(x) = (\mathcal{C}_x, \{\mathcal{D}_x\})$).

We call \mathcal{C}_x the first class of $\psi(x)$ and \mathcal{D}_x the second class. We define \widehat{X} as the set of all δ -cut on X . If there are not right Cauchy filters (resp. left Cauchy cofilters) converging to x , then $\psi(x) = (\{\mathcal{C}_x\}, \mathcal{D}_x)$ resp. $\psi(x) = (\mathcal{C}_x, \{\mathcal{D}_x\})$.

Proposition 3.1.3. If (X, d) is a T_0 -quasi-metric space, then the function ψ in Definition 3.1.6 is an injective function.

Proof. Let $x, y \in X$ be such that $\psi(x) = \psi(y)$. Then,

$$\{\{\mathcal{C}_x\}, \{\mathcal{D}_y\}\} \in \psi(x) \cap \psi(y) \cap \psi(x) \cap \psi(y).$$

Thus

$$\Phi_d(\{\mathcal{C}_x\}, \{\mathcal{D}_y\}) = 0 = \Phi_d(\{\mathcal{D}_y\}, \{\mathcal{C}_x\}),$$

hence $d(x, y) = 0 = d(y, x)$ which implies that $x = y$ since (X, d) is a T_0 -quasi-metric space. Therefore ψ is an injective function. \square

Definition 3.1.7. Let (X, d) be a quasi-pseudometric space. A filter \mathcal{F} is called δ -Cauchy filter if $\mathcal{F} \in \mathcal{C}_x$ for any δ -cut $(\mathcal{C}_x, \mathcal{D}_x)$ on (X, d) .

Definition 3.1.8. Suppose two right K -Cauchy filters \mathcal{F} and \mathcal{H} on quasi-pseudometric space (X, d) . We say that \mathcal{F} is δ -equivalent to \mathcal{G} if any left K -Cauchy cofilter to \mathcal{F} is a left K -Cauchy cofilter to \mathcal{G} .

Proposition 3.1.4. Consider a quasi-pseudometric space (X, d) . Let $(\mathcal{C}_x, \mathcal{D}_x)$ be a δ -cut on (X, d) . Then δ -equivalence is an equivalence relation on \mathcal{C}_x .

Proof. Observe that if $\mathcal{F} \in \mathcal{C}_x$, then \mathcal{F} is δ -equivalent to itself. So δ -equivalence relation is reflexive.

Consider $\mathcal{F}, \mathcal{G} \in \mathcal{C}_x$, if \mathcal{F} is δ -equivalent to \mathcal{G} , then for any \mathcal{H} cofilter to \mathcal{F} , we have that \mathcal{H} is cofilter to \mathcal{G} since \mathcal{F} and \mathcal{G} have the same cofilter by Proposition 3.1.2. Therefore \mathcal{G} is δ -equivalent to \mathcal{F} .

Similarly, if \mathcal{F} is δ -equivalent to \mathcal{G} and \mathcal{G} is δ -equivalent to \mathcal{H} , then \mathcal{F} has the same cofilter with \mathcal{H} , since \mathcal{F} and \mathcal{G} have the same cofilter. Thus \mathcal{F} is δ -equivalent to \mathcal{H} . Therefore, δ -equivalence is an equivalent relation. \square

Corollary 3.1.2. *For any quasi-pseudometric space (X, d) . Two δ -Cauchy filters which belong to the same δ -cut are δ -equivalent.*

Definition 3.1.9. *A quasi-pseudometric space (X, d) is called δ -complete if any δ -Cauchy filter converges.*

Proposition 3.1.5. *In a T_0 -quasi-metric space (X, d) the limit of a δ -Cauchy filter is unique if it exists.*

Proof. Consider a δ -Cauchy filter \mathcal{F} in (X, d) and suppose that \mathcal{F} converges to x and y , with $x, y \in X$. Then there exists a δ -cut $\eta = (,)$ such that $\mathcal{F} \in \eta$ and for some $\mathcal{G} \in \eta$ with $(\mathcal{F}, \mathcal{G})$ converges x and y . Furthermore, $\inf_{F \in \mathcal{F}} \Phi_d(F, \{x\}) = 0$, $\inf_{G \in \mathcal{G}} \Phi_d(\{x\}, G) = 0$, $\inf_{F \in \mathcal{F}} \Phi_d(F, \{y\}) = 0$ and $\inf_{G \in \mathcal{G}} \Phi_d(\{y\}, G) = 0$.

Then by Definition 3.1.5, we have

$$0 \leq d(x, y) \leq \inf_{F \in \mathcal{F}} \Phi_d(\{x\}, F) + \inf_{F \in \mathcal{F}, G \in \mathcal{G}} \Phi_d(F, G) + \inf_{G \in \mathcal{G}} \Phi_d(\{x\}, G) = 0.$$

Moreover, since \mathcal{G} is cofilter to \mathcal{F} , it follows

$$0 \leq d(y, x) \leq \inf_{G \in \mathcal{G}} \Phi_d(\{y\}, G) + \inf_{F \in \mathcal{F}, G \in \mathcal{G}} \Phi_d(G, F) + \inf_{G \in \mathcal{G}} \Phi_d(F, \{x\}) = 0.$$

Consequently, $x = y$ since (X, d) is a T_0 -quasi-metric space. \square

Remark 3.1.1. *For any quasi-pseudometric space (X, d) , we observe that the quasi-pseudometric (X, d^{-1}) is δ -complete if and only if (X, d) is δ -complete.*

Lemma 3.1.5. *Let \mathcal{F} be a d^s -Cauchy filter on a quasi-pseudometric space (X, d) . Then \mathcal{F} is δ -Cauchy filter*

Proof. Indeed \mathcal{F} is right K-Cauchy filter, since \mathcal{F} is d^s -Cauchy filter on (X, d) . Moreover \mathcal{F} is left K-Cauchy filter on (X, d^{-1}) . Observe that

$$\inf_{F \in \mathcal{F}} \Phi_d(F, F) = 0.$$

Therefore, there exists a δ -cut $\eta = (,)$ such that $(\mathcal{F}, \mathcal{F}) \in \eta$. \square

Our next result makes connections between δ -completion and bicompletion.

Proposition 3.1.6. *Each quasi-pseudometric (X, d) that is δ -complete is bicomplete.*

Proof. Let \mathcal{F} be d^s -Cauchy filter on (X, d) . Then \mathcal{F} is δ -Cauchy filter on (X, d) . Furthermore, \mathcal{F} converges to x with respect to $\tau(d)$ and \mathcal{F} converges to x with respect to $\tau(d^{-1})$ since (X, d) is δ -complete. Therefore \mathcal{F} converges to x with respect to $\tau(d^s)$. \square

In the following example, we construct the δ -cut of a concrete quasi-pseudometric space, for instance, the example of a non balanced T_0 -quasi-metric space.

Example 3.1.3. *Let $X = \{\frac{-1}{n+1}, \frac{1}{n+1}, n \in \mathbb{N}\}$. For each $x, y \in X$, let $d(x, y) = 1$, if $x < 0 < y$, $d(x, y) = 0$, if $y \leq x$, and $d(x, y) = \min\{1, |x - y|\}$ otherwise. It is easy to check that (X, d) is a T_0 -quasimetric space.*

Let \mathcal{F} be the filter generated by the filter base $\{B_d(\frac{1}{n+1}, \frac{\varepsilon}{2}) : \varepsilon > 0\}$ and \mathcal{F}' the filter generated by the filter base $\{B_d(\frac{1}{n+1}, \varepsilon) : \varepsilon > 0\}$ with $\frac{\varepsilon}{2} < \varepsilon$, we see that \mathcal{F}' is coarser than \mathcal{F} . Moreover, for any $m \in \mathbb{N}$, \mathcal{F}_m a left K -Cauchy filter generated by $\{B_d(\frac{1}{n+1}, \frac{\varepsilon}{m}) : \varepsilon > 0\}$ is finer than \mathcal{F} . This filter \mathcal{F}_m is δ -equivalent to \mathcal{F} because it is a member of $\psi(\frac{1}{n+1})$.

Let \mathcal{G} be the filter generated by the filter base $\{B_{d^{-1}}(\frac{1}{n+1}, \frac{\varepsilon}{2}) : \varepsilon > 0\}$ and \mathcal{G}' the filter generated by the filter base $\{B_{d^{-1}}(\frac{1}{n+1}, \varepsilon) : \varepsilon > 0\}$ with $\frac{\varepsilon}{2} < \varepsilon$, we see that \mathcal{G}' is coarser than \mathcal{G} . Moreover, for any $m \in \mathbb{N}$, \mathcal{G}_m a right K -Cauchy filter generated by $\{B_{d^{-1}}(\frac{1}{n+1}, \frac{\varepsilon}{m}) : \varepsilon > 0\}$ is finer than \mathcal{G} . This filter \mathcal{G}_m is δ -equivalent to \mathcal{G} because it is a member of $\psi(\frac{1}{n+1})$. Hence, we have a δ -cut $\psi(\frac{1}{n+1}) = (\psi(\frac{1}{n+1}), \psi(\frac{1}{n+1}))$.

Similarly to the above example, we can describe also the δ -cut $\psi(\frac{-1}{n+1}) = (\psi(\frac{-1}{n+1}), \psi(\frac{-1}{n+1}))$ as follow.

Example 3.1.4. *Let \mathcal{F}' be the filter generated by the filter base $\{B_d(\frac{-1}{n+1}, \varepsilon) : \varepsilon > 0\}$. For any $m \in \mathbb{N}$, \mathcal{F}_m a left K -Cauchy filter generated by $\{B_d(\frac{-1}{n+1}, \frac{\varepsilon}{m}) : \varepsilon > 0\}$ is finer than \mathcal{F}' . This filter \mathcal{F}_m is δ -equivalent to \mathcal{F}' because it is a member of $\psi(\frac{-1}{n+1})$.*

Let \mathcal{G}' be the filter generated by the filter base $\{B_{d^{-1}}(\frac{-1}{n+1}, \varepsilon) : \varepsilon > 0\}$. For any $m \in \mathbb{N}$, \mathcal{G}_m a right K -Cauchy filter generated by $\{B_{d^{-1}}(\frac{-1}{n+1}, \frac{\varepsilon}{m}) : \varepsilon > 0\}$ is finer than \mathcal{G}' . This filter \mathcal{G}_m is δ -equivalent to \mathcal{G}' because it is a member of $\psi(\frac{-1}{n+1})$. Hence, we have a δ -cut $\psi(\frac{-1}{n+1}) = (\psi(\frac{-1}{n+1}), \psi(\frac{-1}{n+1}))$.

As we have defined the set of δ -cuts, we devote the next section to the construction of the δ -completion.

3.2 The δ -completion

In this section, we show that for any quasi-pseudometric space, its T_0 -quasi-metric quotient space is δ -complete.

We start our investigation by defining a distance function between two δ -cuts, then we prove that the defined distance is a quasi-pseudometric on the set of δ -cuts.

Definition 3.2.1. *A δ -completion of a quasi-pseudometric space (X, d) is a δ -complete quasi-metric space $(\widehat{X}, \widehat{d})$ in which (X, d) can be isometrically embedded as a dense subspace.*

Definition 3.2.2. *If a δ -cut $\eta = (,)$ does not belong to $\bigcup_{x \in X} \psi(x)$ we say that $(,)$ is a δ -gap. The set of all δ -gap of X is denoted by \widetilde{X} . We set*

$$\widehat{X} = \psi(X) \cup \widetilde{X}.$$

In the following, for any $\eta \in \widehat{X}$, then we denote by ${}_{\eta, \eta}$ the two classes of the δ -cut η . Therefore, we denote $\eta = ({}_{\eta, \eta})$.

Definition 3.2.3. *Let (X, d) be a quasi-metric space. Suppose that r is nonnegative real number. Let $\eta_1, \eta_2 \in \widehat{X}$. For any $\mathcal{F}_1 \in {}_{\eta_1}$ and $\mathcal{F}_2 \in {}_{\eta_2}$.*

Then, we set $\widehat{d}(\eta_1, \eta_2) \leq r$ if

1. $\eta_1 = \eta_2$ or
2. For any $\epsilon > 0$, there exists $F_\epsilon \in \mathcal{F}_1$ and $G_\epsilon \in \mathcal{F}_2$ such that

$$\Phi_d(F_\epsilon, G_\epsilon) < r + \epsilon. \quad (3.1)$$

If $\eta_1 = \psi(x)$ for some $x \in X$, then the arbitrary filter \mathcal{F}_1 coincides with the filter base $\{\{x\}\}$. Moreover,

$\widehat{d}(\psi(x), \eta_2) \leq r$ if $\Phi_d(\{\{x\}\}, G) < r + \epsilon$ whenever $G \in \mathcal{F}_2$.

Furthermore, we set

$$\widehat{d}(\eta_1, \eta_2) = \inf\{r : \widehat{d}(\eta_1, \eta_2) \leq r\}. \quad (3.2)$$

Proposition 3.2.1. *Consider a quasi-pseudometric space (X, d) . Let $\eta_1, \eta_2 \in \widehat{X}$. If $\mathcal{F}_1 \in \eta_1$ and $\mathcal{F}_2 \in \eta_2$ with $\widehat{d}(\eta_1, \eta_2) = 0$, then \mathcal{F}_2 is right d -cofinal to \mathcal{F}_1 .*

Proof. For any $\varepsilon > 0$, there exists $\mathcal{F}_\varepsilon^1 \in \mathcal{F}_1$ and $\mathcal{F}_\varepsilon^2 \in \mathcal{F}_2$ such that

$$\Phi_d(\mathcal{F}_\varepsilon^1, \mathcal{F}_\varepsilon^2) < \varepsilon,$$

whenever $\mathcal{F}_1 \in \eta_1$ and $\mathcal{F}_2 \in \eta_2$. then

$$\inf_{F^1 \in \mathcal{F}_1, F^2 \in \mathcal{F}_2} \Phi_d(F_1, F_2) = 0.$$

therefore \mathcal{F}_2 is right d -cofinal to \mathcal{F}_1 . □

Remark 3.2.1. *Observe that if $\eta_1, \eta_2 \in \widehat{X}$, $\mathcal{F}_1 \in \eta_1$, $\mathcal{F}_2 \in \eta_2$, then*

$$\widehat{d}(\eta_1, \eta_2) = \inf_{F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2} \Phi_d(F_1, F_2).$$

Proposition 3.2.2. *If (X, d) is a quasi-pseudometric space, then \widehat{d} is a quasi-pseudometric on \widehat{X} .*

Proof. For any $\eta_1, \eta_2 \in \widehat{X}$, we have that $\widehat{d}(\eta_1, \eta_1) = 0$ and $\widehat{d}(\eta_1, \eta_2) \geq 0$ by Definition 3.2.3.

In order to prove the triangle inequality, let $\eta_1, \eta_2, \eta_3 \in \widehat{X}$. We have four cases:

Case 1. $\eta_1 \neq \eta_2 \neq \eta_3$.

Suppose that $\widehat{d}(\eta_1, \eta_2) = r_1$ and $\widehat{d}(\eta_2, \eta_3) = r_2$. Then for any $\varepsilon > 0$, there exists $F_\varepsilon^1 \in \mathcal{F}_1$, $F_\varepsilon^2 \in \mathcal{F}_2$, $F_\varepsilon^3 \in \mathcal{F}_3$ such that

$$\Phi_d(F_\varepsilon^1, F_\varepsilon^2) < r + \frac{\varepsilon}{3}$$

and

$$\Phi_d(F_\varepsilon^2, F_\varepsilon^3) < r + \frac{\varepsilon}{3},$$

whenever $\mathcal{F}_1 \in \eta_1$, $\mathcal{F}_2 \in \eta_2$ and $\mathcal{F}_3 \in \eta_3$.

Furthermore,

$$\Phi_d(F_\varepsilon^1, F_\varepsilon^3) \leq \Phi_d(F_\varepsilon^1, F_\varepsilon^2) + \Phi_d(F_\varepsilon^2, F_\varepsilon^3) < r_1 + r_2 + \varepsilon,$$

whenever $\varepsilon > 0$. Therefore

$$\widehat{d}(\eta_1, \eta_3) \leq r_1 + r_2 = \widehat{d}(\eta_1, \eta_2) + \widehat{d}(\eta_2, \eta_3).$$

Case 2. $\eta_1 \neq \eta_2$ and $\eta_2 = \eta_3$. Suppose that $\widehat{d}(\eta_1, \eta_2) = r$ and $\widehat{d}(\eta_2, \eta_3) = 0$ by Definition 3.2.3 since $\eta_2 = \eta_3$. Then

$$\widehat{d}(\eta_1, \eta_3) \leq r = \widehat{d}(\eta_1, \eta_2) + \widehat{d}(\eta_2, \eta_3).$$

Case 3. $\eta_1 = \eta_2$ and $\eta_2 \neq \eta_3$. This case is analogue to Case 2.

Case 4. $\eta_1 = \eta_2 = \eta_3$. This case is obvious. \square

Proposition 3.2.3. *Let (X, d) be quasi-metric space. Then the function ψ in Definition 3.1.6 is an isometric embedding of (X, d) into $(\widehat{X}, \widehat{d})$.*

Proof. For any $x, y \in X$, then $\{\{x\}\} \in \psi(x)$ and $\{\{y\}\} \in \psi(y)$. From Remark 3.2.1, we have

$$\widehat{d}(\psi(x), \psi(y)) = \inf_{F \in \{\{x\}\}, G \in \{\{y\}\}} \phi_d(F, G).$$

For any $f \in F$ and $g \in G$, we have $f = x$ and $g = y$. Hence

$$\widehat{d}(\psi(x), \psi(y)) = \inf_{F \in \{\{x\}\}, G \in \{\{y\}\}} \sup_{f \in F, g \in G} d(f, g) = d(x, y).$$

Therefore ψ is isometric embedding of (X, d) into $(\widehat{X}, \widehat{d})$. \square

Proposition 3.2.4. *Given two δ -cuts η and η' with their Cauchy filter pairs $(\mathcal{F}, \mathcal{G}) \in \eta$, $(\mathcal{F}', \mathcal{G}') \in \eta'$, we have that*

$$\inf_{f' \in F'} \Phi_{\widehat{d}}(\eta, \psi(f')) \leq \widehat{d}(\eta, \eta')$$

with $F' \in \mathcal{F}'$. Similarly, we have that

$$\inf_{f \in F} \Phi_{\widehat{d}}(\psi(f), \eta') \leq \widehat{d}(\eta, \eta')$$

with $F \in \mathcal{F}$.

Proof. The proof follows from proposition 3.2.3 \square

Proposition 3.2.5. *Let (X, d) be a quasi-pseudometric space. For any $\eta \in \widehat{X}$ and $x \in X$, we have*

(a) if $\{\{x\}\} \in \eta$, then $\widehat{d}(\eta, \psi(x)) \rightarrow 0$;

(b) if $\{\{x\}\} \in \eta$, then $\widehat{d}(\psi(x), \eta) \rightarrow 0$.

Proof. (a) Suppose $\eta \in \widehat{X}$ and $x \in X$. For any $\mathcal{F} \in \eta$ and since $\{\{x\}\} \in \eta$. We have that $\{\{x\}\}$ is d -cofinal to \mathcal{F} . Then for any $\epsilon > 0$, there exists $F_\epsilon \in \mathcal{F}$ such that

$$\Phi_d(F_\epsilon, \{x\}) < \frac{\epsilon}{2}.$$

For any $\mathcal{G} \in \psi(x)$, \mathcal{G} is d -cofinal to $\{\{x\}\}$ since $\{\{x\}\} \in \psi(x)$ we have

$$\Phi_d(\{x\}, G_\epsilon) < \frac{\epsilon}{2}.$$

Furthermore

$$\Phi_d(F_\epsilon, G_\epsilon) \leq \Phi_d(F_\epsilon, \{x\}) + \Phi_d(\{x\}, G_\epsilon) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

whenever $\epsilon > 0$ and $\mathcal{F} \in \eta$ and $\mathcal{G} \in \psi(x)$. Therefore $\widehat{d}(\eta, \psi(x)) < \epsilon$.

(b) Can be proved by analogous arguments to (a). \square

Theorem 3.2.1. *Any quasi-pseudometric space has a δ -completion.*

Proof. Let (X, d) be a quasi-pseudometric space and Υ be a δ -Cauchy filter in the quasi-pseudometric space $(\widehat{X}, \widehat{d})$. Then by Definition 3.1.5, there exists a δ -cut $\widehat{\eta} = (\widehat{\eta}, \widehat{\eta}) \in \widehat{X}$ such that $\Upsilon \in \widehat{\eta}$. Suppose that $\widehat{\eta} = \{\mathcal{F}_i : i \in I\}$ and $\widehat{\eta} = \{\mathcal{G}_i : i \in I\}$. Then, $\Upsilon = \mathcal{F}_{i_0}$ for some $i_0 \in I$. We have to prove that there exists a δ -cut η in (X, d) such that Υ converges to η .

We define $\eta = (\eta, \eta)$ by $\eta = \{\{\{x\}\} : x \in X \text{ such that } \psi(x) \in \widehat{\eta}\}$ and $\eta = \{\{\{x\}\} : x \in X \text{ such that } \psi(x) \in \widehat{\eta}\}$.

We show that η defined above is δ -cut on (X, d) . It means that we have to show that $\eta \neq \emptyset$ and $\eta \neq \emptyset$ and then we show that the pair (η, η) satisfies the conditions of Definition 3.1.5.

To show that $\eta \neq \emptyset$, we have two cases.

Case 1. $\Upsilon_{\widehat{i}} \neq \psi(x)$ for some $\widehat{i} \in I$. Since $\{\{x\}\}$ is a right K-Cauchy filter for any $x \in X$ and $\Upsilon_{\widehat{i}}$ and $\psi(x)$ are \widehat{d} -cofinal, then $\psi(x) \in \widehat{\eta}$, so $\{\{x\}\} \in \eta$.

Case 2. $\Upsilon = \psi(x)$, obviously $\{\{x\}\} \in \eta$. Therefore, in both cases $\eta \neq \emptyset$. By similar arguments we have $\eta \neq \emptyset$.

To show that (η, η) satisfies the conditions of Definition 3.1.5. Let $\{\{x\}\} \in \eta$ and $\{\{y\}\} \in \eta$. Then by construction of η and η , we have

$$\inf_{R \in \psi(x), T \in \psi(y)} \Phi_{\widehat{d}}(R, T) = 0.$$

Hence, we have $\inf_{F \in \{\{x\}\}, G \in \{\{x\}\}} \Phi_d(F, G) = 0$. The condition 2 of Definition 3.1.5 follows, since for any $\{\{x\}\}$ right K-Cauchy filter of η , $\psi(x) \in \hat{\eta}$ since $\{\{x\}\}$ is right d -cofinal to itself. Furthermore, the maximality of η and $\hat{\eta}$ follow from the maximality of $\hat{\eta}$ and $\hat{\eta}$, respectively. Therefore the conditions (i), (ii) and (iii) of the definition of δ -cut are satisfied.

We show that Υ converges to η with respect to $\tau(\hat{d})$. If $\Upsilon = \psi(x)$ for any $x \in X$, then $\inf_{T \in \Upsilon} \Phi_{\hat{d}}(T, \eta) = 0$ and $\inf_{R \in \mathcal{G}_{i_0}} \Phi_{\hat{d}}(\eta, R) = 0$ for some $i_0 \in I$ and $\mathcal{G}_{i_0} \in \hat{\eta}$. So Υ converges to η with respect to $\tau(\hat{d})$.

If $\Upsilon \neq \psi(x)$, since $\{\{x\}\} \in \eta$ and $\psi(x)$ and Υ are \hat{d} -cofinal, we have by Propostion 3.2.5 that $\psi(x) \rightarrow \eta$ and moreover, by Lemma 3.1.4 we have $\Upsilon \rightarrow \eta$ since Υ and $\psi(x)$ are \hat{d} -cofinal. \square

In the next, we define the quotient space of δ -cuts. Let first define the equivalence relation between two δ -cuts on (\hat{X}, \hat{d}) .

Definition 3.2.4. *Let (X, d) be a T_0 -quasi-metric space. We define a relation denoted \cong on \hat{X} by $\eta \cong \eta'$ if and only if $\eta = \eta'$ whenever $\eta, \eta' \in \hat{X}$ or $\eta = \psi_i(x)$ and $\eta' = \psi_j(x)$ for some $i, j \in I_x$ whenever $\eta, \eta' \in \psi(x)$, $x \in X$.*

Observe that the relation \cong defined on \hat{X} is an equivalence relation. In the following \bar{X} will be denoted the set of all equivalence classes. Therefore, the quotient map $\psi_q : \hat{X} \rightarrow \bar{X}$ with $\psi_q(\eta) = \Xi$ whenever $\eta \in \hat{X}$ is surjective.

The next proposition can be compared with Proposition 2.3.12.

Proposition 3.2.6. *Let (X, d) be a T_0 -quasi-metric space. We define $\bar{d} : \bar{X} \times \bar{X} \rightarrow \mathbb{R}$ by $\bar{d}(\Xi, \Xi') = \hat{d}(\eta, \eta')$ whenever $\eta, \eta' \in \hat{X}$. Then, \bar{d} is a T_0 -quasi-metric space on \bar{X} .*

Proof. Indeed, \cong is an equivalence relation on \hat{X} . To prove that \bar{d} is well defined, consider η, η_1, η' and $\eta'_1 \in \hat{X}$ and $\eta \cong \eta'$ and $\eta_1 \cong \eta'_1$. Then we have

$$\hat{d}(\eta, \eta_1) \leq \hat{d}(\eta, \eta') + \hat{d}(\eta', \eta'_1) + \hat{d}(\eta'_1, \eta') \leq 0 + \hat{d}(\eta', \eta'_1) + 0 = \hat{d}(\eta', \eta'_1).$$

Similarly, we have

$$\hat{d}(\eta', \eta'_1) \leq \hat{d}(\eta, \eta_1).$$

Moreover, $\hat{d}(\eta, \eta_1) = \hat{d}(\eta', \eta'_1)$, hence \bar{d} is well defined.

Now, we show that \bar{d} is a T_0 -quasi-metric on \bar{X} . It is clear that \bar{d} is quasi-pseudometric. We have to show that \bar{d} is T_0 . Suppose that

$$\bar{d}(\Xi, \Xi') = 0 = \bar{d}(\Xi', \Xi).$$

Then $\widehat{d}(\eta, \eta') = 0 = \widehat{d}(\eta', \eta)$. Suppose that $\eta, \eta' \in \tilde{X}$. Let $\mathcal{F} \in \eta$ and $\mathcal{G} \in \eta'$. Then by Proposition 3.2.1, \mathcal{G} is right d -cofinal to \mathcal{F} , therefore $\eta = \eta'$. If $\eta = \psi(x)$ and $\eta' = \psi(y)$ for some $x, y \in X$, then

$$\widehat{d}(\psi(x), \psi(y)) = 0 = \widehat{d}(\psi(y), \psi(x))$$

which implies that $d(x, y) = 0 = d(y, x)$ since ψ is an isometric embedding of (X, d) into $(\widehat{X}, \widehat{d})$. Hence $x = y$ since d is T_0 -quasi-metric on X . Consequently, in any case $\Xi = \Xi'$. Therefore, \bar{d} is a T_0 -quasi-metric on \bar{X} . \square The following proposition is similar to Propostion 2.3.13.

Proposition 3.2.7. *If (X, d) is a T_0 -quasi-metric space, then $\phi = \psi_q \circ \psi : (X, d) \rightarrow (\bar{X}, \bar{d})$ is an isometric embedding.*

Proof. Observe that ψ_q is an isometric embedding of $(\widehat{X}, \widehat{d})$ onto (\bar{X}, \bar{d}) , then ϕ is an isometric embedding of (X, d) into (\bar{X}, \bar{d}) as a composition of isometrics.

For any $x, y \in X$, if $\psi(x) = \psi(y)$ then $\widehat{d}(\psi(x), \psi(y)) = 0 = \widehat{d}(\psi(y), \psi(x))$. Once again, we have $d(x, y) = 0 = d(y, x)$. Hence $x = y$ by T_0 property of d . Hence ψ is injective. Therefore ϕ is an isometric embedding. \square

Theorem 3.2.2. *Let (X, d) be a T_0 -quasi-metric space. Then (\bar{X}, \bar{d}) is a T_0 - δ -completion of (X, d) .*

Proof. Consider a δ -Cauchy sequence Ξ in (\bar{X}, \bar{d}) . Since ψ_q is a surjective isometry, then $\psi_q^{-1}(\Xi)$ is a δ -Cauchy filter in $(\widehat{X}, \widehat{d})$. Thus, there exists $\eta \in \widehat{X}$ such that $\psi_q^{-1}(\Xi)$ converges to η . It follows that Ξ converges to $\psi_q(\eta) = \Xi$ since ψ_q is an isometry. So (\bar{X}, \bar{d}) is a δ -complete T_0 -quasi-metric space.

To complete the proof, we have to prove that the set $\phi(X) = (\psi_q \circ \psi)(X)$ is dense in (\bar{X}, \bar{d}) . Let $\Xi \in \bar{X}$, then there exists $\eta \in \widehat{X}$ such that $\psi_q(\eta) = \Xi$ since ψ_q is a surjective isometry. Let $\{\{x\}\} \in \eta$, then by Proposition 3.2.5, we have $\psi(x) \rightarrow \eta$. Hence, $\psi_q(\psi(x))$ converges to $\psi_q(\eta) = \Xi$ with respect to $\tau(\bar{d})$. It follows that $\phi(X)$ is dense in (\bar{X}, \bar{d}) which implies that (\bar{X}, \bar{d}) is a δ -completion of (X, d) . \square

Definition 3.2.5. If (X, d) is a T_0 -quasi-metric space, then the T_0 -quasi-metric space $(\overline{X}, \overline{d})$ in theorem 3.2.2 is called standard δ -completion of (X, d) .

Proposition 3.2.8. Every convergent right K -Cauchy filter on a quasi-pseudometric space (X, d) is a δ -Cauchy filter.

Proof. Consider a right K -Cauchy filter \mathcal{F} which converges to $x \in X$, then there exists a δ -cut $\eta = (\eta, \eta)$ such that $\mathcal{F} \in \eta$. Therefore \mathcal{F} is δ -Cauchy filter. \square

Proposition 3.2.9. The concept of a δ -Cauchy filter coincides with that of Cauchy filter in a pseudometric space.

Proof. Let \mathcal{F} be a δ -Cauchy filter on a pseudometric space (X, d) . Then \mathcal{F} is a Cauchy filter on (X, d) and for some left K -Cauchy filter \mathcal{G} on (X, d) with $(\mathcal{F}, \mathcal{G}) \in \eta$, where η is a δ -cut on (X, d) . Hence the first class and the second class of the δ -cut consist of Cauchy filters on (X, d) . \square

Definition 3.2.6. Let \mathcal{F} and \mathcal{G} be two filters on a T_0 -quasi-metric space (X, d) .

1. We say that the pair $(\mathcal{F}, \mathcal{G})$ forms a δ -Cauchy filter pair if there exists a δ -cut $\eta = (\eta, \eta)$ on (X, d) such that $\mathcal{F} \in \eta$ and $\mathcal{G} \in \eta$.
2. We say that filters are δ -compatibles if they belong to the same δ -cut.

Definition 3.2.7. Consider a uniformly continuous map $f : (X, d) \rightarrow (Y, e)$ between two T_0 -quasi-metric spaces (X, d) and (Y, e) . Let $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{F}', \mathcal{G}')$ be two δ -Cauchy filter pairs in (X, d) . We say that $(\mathcal{F}', \mathcal{G}')$ is (δ, f) -compatible to $(\mathcal{F}, \mathcal{G})$ in (X, d) if it satisfies the following conditions:

1. $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{F}', \mathcal{G}')$ are δ -compatible;
2. $(f(\mathcal{F}'), f(\mathcal{G}'))$ is a δ -Cauchy filter pair in Y .

Definition 3.2.8. A uniformly continuous map $f : (X, d) \rightarrow (Y, e)$ between two T_0 -quasi-metric spaces (X, d) and (Y, e) is called δ -cut preserving if it satisfies the following conditions:

1. For each δ -Cauchy pair $(\mathcal{F}, \mathcal{G})$ in (X, d) , there exists a δ -Cauchy filter pair $(\mathcal{F}', \mathcal{G}')$ in Y such that $(\mathcal{F}', \mathcal{G}')$ is (δ, f) -compatible to $(\mathcal{F}, \mathcal{G})$.

2. If $(\mathcal{F}_1, \mathcal{G}_1), (\mathcal{F}'_1, \mathcal{G}'_1), (\mathcal{F}_2, \mathcal{G}_2)$ and $(\mathcal{F}'_2, \mathcal{G}'_2)$ are δ -Cauchy filter pairs such that $(\mathcal{F}_1, \mathcal{G}_1)$ is (δ, f) -compatible to $(\mathcal{F}_2, \mathcal{G}_2)$ and $(\mathcal{F}'_1, \mathcal{G}'_1)$ is (δ, f) -compatible to $(\mathcal{F}'_2, \mathcal{G}'_2)$, then the δ -compatibility (resp. non- δ -compatibility) $(\mathcal{F}_1, \mathcal{G}_1)$ and $(\mathcal{F}'_1, \mathcal{G}'_1)$ implies the δ -compatibility (non- δ -compatibility) of $(f(\mathcal{F}_2), f(\mathcal{G}_2))$ and $(f(\mathcal{F}'_2), f(\mathcal{G}'_2))$, respectively.

Notice, for each $\eta = (\eta, \eta)$ on (X, d) , the δ -cut $f(\eta) = (f(\eta), f(\eta))$ is a δ -cut on (Y, e) .

The following result shows that the standard δ -completion is the smallest δ -complete quasi-pseudometric space containing (X, d) .

Lemma 3.2.1. *Let (X, d) and (Y, e) be quasi-pseudometric spaces and let $f : (X, d) \rightarrow (Y, e)$ be a δ -cut preserving surjective isometry. If η' is a δ -cut on (Y, e) , then $f^{-1}(\eta')$ is a δ -cut on (X, d) .*

The categorical notion of a T_0 -quasi-metric space is as follows: a complete T_0 -quasi-metric space is called a δ -completion of a given T_0 -quasi-metric space (X, d) if there exists a δ -cut preserving function $\psi : (X, d) \rightarrow (\bar{X}, \bar{d})$ such that

1. $\psi(X)$ is dense in (\bar{X}, \bar{d}) ;
2. the quasi-pseudometric space (X, d) is the inverse image of (\bar{X}, \bar{d}) under ψ ;
3. for any δ -cut preserving function $f : (X, d) \rightarrow (Y, e)$, where (Y, e) is a unique δ -cut preserving function $f^* : (\bar{X}, \bar{d}) \rightarrow (Y, e)$ with $f = f^* \circ \psi$.

The map ψ is called the *canonical quasi-pseudometric embedding* of (X, d) into (\bar{X}, \bar{d}) .

Proposition 3.2.10. *(Compare [2, Proposition 40])*

Let (X, d) and (Y, e) be two T_0 -quasi-metric spaces. If (Y, e) is δ -complete and $f : (X, d) \rightarrow (Y, e)$ be a δ -cut preserving function, then there exists a unique δ -cut preserving function $f^ : (\bar{X}, \bar{d}) \rightarrow (Y, e)$ such that $f = f^* \circ \psi$.*

Proof. We first prove the existence of f^* . Consider $f : (X, d) \rightarrow (Y, e)$ a δ -cut preserving map. Let $\eta = (\eta, \eta)$ be a δ -cut on (X, d) with $\mathcal{F} \in \eta$ such that \mathcal{F} converges to $x \in X$ with respect to τ_d and $\psi_q(\eta) = \Xi \in \bar{X}$. Then we define $f^* : (\bar{X}, \bar{d}) \rightarrow (Y, e)$ by $f^*(\Xi) := f(x)$. Observe that for any two distinct points in \bar{X} , their images under f^* are distinct points in Y .

Let $\Xi = \phi(x)$, where $\phi = \psi_q \circ \psi : (X, d) \rightarrow (\bar{X}, \bar{d})$ is the isometric embedding in Proposition 3.2.7. By uniform continuity of f we have $f^*(\phi(x)) = f(x)$.

secondly we have to prove that f^* is δ -cut preserving map. Let (Ξ, Ψ) be a δ -Cauchy filter pair on (\bar{X}, \bar{d}) . Suppose that $\psi_q(\eta) = \xi$ and $\psi_q(\xi) = \Psi$. Then, let (η, ξ) be a δ -Cauchy filter pair on (\hat{X}, \hat{d}) . Then (η, ξ) and $(\{\{x\}\}, \{\{x\}\})$ are δ -compatible. Since f is a δ -cut preserving map, there exists a δ -Cauchy filter pair $(\mathcal{H}, \mathcal{M})$ satisfying the conditions (i) and (ii) of Definition 3.2.8. Since $\phi = \psi_q \circ \psi$ is an isometric embedding and $f = f^* \circ \phi$, it follows that f^* satisfies the condition of a δ -cut preserving map. \square

Theorem 3.2.3. *Let (X, d) and (Y, e) be two T_0 -quasi-metric spaces, and let (\bar{X}, \bar{d}) and (\bar{Y}, \bar{e}) be their standard δ -completion with canonical maps $\phi_X : (X, d) \rightarrow (\bar{X}, \bar{d})$ and $\phi_Y : (Y, e) \rightarrow (\bar{Y}, \bar{e})$. If $g : (X, d) \rightarrow (Y, e)$ be a cut-preserving map, then there is a unique cut-preserving map $g^* : (\bar{X}, \bar{d}) \rightarrow (\bar{Y}, \bar{e})$ such that*

$$\bar{f} \circ \phi_X = \phi_Y \circ g.$$

If f is also an isometry, then \bar{f} is an isometry.

Proof. This is a consequence of Proposition 3.2.10, by taking $g^* : (\bar{X}, \bar{d}) \rightarrow (\bar{Y}, \bar{e})$ such that $g^* \circ \phi_X = \phi_Y \circ g$ \square

Theorem 3.2.4. *Let (X, d) be a subspace of the δ -complete T_0 -quasi-metric space (Y, e) . Suppose that the embedding $i : (X, d) \rightarrow (Y, e)$ is cut-preserving and that for each $y \in Y$ there is a Cauchy filter pair $(i(\mathcal{F}), i(\mathcal{G}))$ converge to y . Then the δ -completion (\bar{X}, \bar{d}) of (X, d) is isometric to (Y, e) under the isometric extension \bar{i} of i to \bar{X} .*

Proposition 3.2.11. *Let (X, d) be a T_0 -quasi-metric space and let A be a subset of \bar{X} such that $X \subseteq A \subseteq \bar{X}$. Then the extension \bar{j} of cut-preserving isometric embedding $j : (A, \bar{d}|_A) \rightarrow (\bar{X}, \bar{d})$ to \bar{A} yields a bijective cut-preserving isometry between $(\bar{A}, \bar{d}|_{\bar{A}})$ and (\bar{X}, \bar{d}) .*

3.3 Connection between δ -completion and κ -completion

In order to establish the connection between the sequence and filter related completions, we need to recall some facts that link sequences to filters.

We next recall the definition of a *net*.

Definition 3.3.1. [19, Definition II.11.] Let S be a non-empty directed set and (X, d) a quasi-pseudometric space. A mapping φ of S into X is called a net on S . We may regard a net as a set $\{\varphi(\delta) | \delta \in S\}$ of points of X indexed by the elements of S . When we have $S \subseteq \mathbb{N}$, we call this indexed set a sequence. For convenience, we use notation $\varphi(S | >)$ to denote a net on S and $\varphi(\delta)$ to denote the individual point of the net, whenever $\delta \in S$.

Let $\varphi(S | >)$ be a net of a quasi-pseudometric space (X, d) and A a subset of X . If there is $\delta_0 \in S$ such that $\varphi(\delta) \in A$ for every $\delta > \delta_0$, then $\varphi(S | >)$ is said to be *residual* in A (or *eventually* in A). If for every $\delta_0 \in S$, there is $\delta \in S$ such that $\delta > \delta_0 \in A$, then $\varphi(S | >)$ is said to be *cofinal* in A . If for every subset A of X , $\varphi(S | >)$ is residual either in A or in $X-A$, then it is called a *maximal net* (or *ultra-net*).

Definition 3.3.2. [19, Definition II.12] Let a collection of residual nets $\varphi(S | >)$, putting

$$\mathcal{F} = \{A | \varphi(S | >) \text{ is residual in } A\},$$

we obtain a filter \mathcal{F} which is called the filter derived from the net $\varphi(S | >)$.

The next two propositions are an asymmetric adaptation of the general one. First, let us define the filter derived from a sequence of quasi-pseudometric space (X, d) .

Definition 3.3.3. Let (X, d) be a quasi-pseudometric space and $A \subseteq X$. Take \mathcal{K} a collection of K -Cauchy sequences in A , putting

$$\mathcal{F} = \{A \in \mathcal{K} | \varphi(S \subseteq \mathbb{N}) \text{ is residual in } A\},$$

we obtain a filter \mathcal{F} which is called the filter derived from the K -Cauchy sequence $\varphi(S \subseteq \mathbb{N})$.

The filter derived from (x_n) is the set of subsets M of X such that $x_n \in M$ except for a finite number of values of n . If S_n denotes the set of all x_p such that $p \geq n$, then the sets S_n form a *filter base* of the filter by the sequence (x_n) . The filter derived from an infinite subsequence of a sequence (x_n) is finer than the filter derived from (x_n) .

As Nagata stated in [19], one can discuss convergence of filters and that of nets in parallel. The following is the asymmetric version that translates the relation between convergence of net or sequence and the one of filter.

Lemma 3.3.1. *Let (X, d) be a quasi-pseudometric space and (x_n) a K-Cauchy sequence on (X, d) . A filter derived from (x_n) , is a K-Cauchy filter.*

Proof. Take the set of $U_n \subset X$ such that U_n is d -neighborhood of x_n , except for a finite number of values of n . If $S_n := \bigcup_{p>n} U_n$, set of all x_p such that, $p > n$, then the sets form a filter base of the K-Cauchy filter derived from the K-Cauchy sequence (x_n) . \square

Proposition 3.3.1. *(Compare [19, Proposition D]) Let (x_n) a K-Cauchy sequence on a quasi-pseudometric space (X, d) . A K-Cauchy filter \mathcal{F} derived from (x_n) converges to x if and only if every K-Cauchy sequence (net) derived from \mathcal{F} converges to x .*

Proof. The proof follows from Lemma 3.3.1 \square

Proposition 3.3.2. *(Compare [19, Proposition E]) Let (X, d) be a quasi-pseudometric space. A K-Cauchy sequence (x_n) converges to a point x if and only if the K-Cauchy filter associated with (x_n) converges to x .*

Proof. The proof follows from Lemma 3.3.1 \square

The next lemma is the consequence of the previous one. It was established by Romaguera in [20].

Lemma 3.3.2. *For a quasi-pseudometric space (X, d) the following are equivalent.*

1. (X, d) is left K-sequentially complete.
2. Every left K-Cauchy filter on X converges with respect to τ_d .
3. Every left K-Cauchy net in X converges with respect to τ_d .

Proof. For the proof, we refer the reader to [20, Lemma]. \square

This proposition links the notion of Cauchy filter pair and the one of pair of sequences in the sense of Doitchinov.

Proposition 3.3.3. *Let $((x_n), (y_m))$ is the pair of sequences as defined previously. Suppose \mathcal{F} the filter generated by (x_n) and \mathcal{G} , the filter generated by (y_m) , then the pair of filters generated by the above pair of sequences form a Cauchy filter pair.*

Proof. This is a consequence of the Lemma 3.3.1, Lemma 3.3.2 and Doitchinov's definition of pair of Cauchy sequences. \square

We can see that if we are given a pair of Cauchy sequences, we can embed it in a Cauchy filter pair. Moreover, the embedding map preserves the Cauchiness property.

Let $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{F}', \mathcal{G}')$ be arbitrary Cauchy filter pairs on X , then the distance from $(\mathcal{F}, \mathcal{G})$ to $(\mathcal{F}', \mathcal{G}')$ is determined by the formula below

$$\widehat{d}((\mathcal{F}, \mathcal{G}), (\mathcal{F}', \mathcal{G}')) := \inf_{F \in \mathcal{F}, F' \in \mathcal{F}'} \Phi_d(F, F') = \inf_{F \in \mathcal{F}, F' \in \mathcal{F}'} \sup_{f \in F, f' \in F'} d(f, f').$$

The conjugate distance is obtained as such:

$$(\widehat{d})^{-1}((\mathcal{F}, \mathcal{G}), (\mathcal{F}', \mathcal{G}')) := \inf_{F \in \mathcal{F}, F' \in \mathcal{F}'} \Phi_{d^{-1}}(F, F') = \inf_{F \in \mathcal{F}, F' \in \mathcal{F}'} \sup_{f \in F, f' \in F'} d(f', f)$$

The distance from $\eta_1 = (1, 1)$ to $\eta_2 = (2, 2)$, two δ -cuts on a quasi-pseudometric space (X, d) can be defined in two ways. One chooses to work with the first classes of the δ -cuts, as such

$$\widehat{d}((1, 1), (2, 2)) = \inf_{F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2} \Phi_d(F_1, F_2) = r$$

Another one chooses to work with the second classes as follow,

$$\widehat{d}((1, 1), (2, 2)) = \inf_{G_1 \in \mathcal{G}_1, G_2 \in \mathcal{G}_2} \Phi_d(G_1, G_2) = t,$$

whenever $\mathcal{G}_1 \in 1, \mathcal{G}_2 \in 2$.

The following proposition states the equality between the two defined distances. Moreover, this proposition provides the relation between the elements of 2 the first class of left d -cofinal filters to \mathcal{G}_2 and the elements of 1 , the second class of right d -cofinal filters to \mathcal{F}_1 .

Proposition 3.3.4. *Let $\eta_1 = (1, 1), \eta_2 = (2, 2) \in \widehat{X}$ be two δ -cuts on (X, d) a quasi-pseudometric space such that $\mathcal{F}_1 \in 1, \mathcal{F}_2 \in 2, \mathcal{G}_1 \in 1$ and $\mathcal{G}_2 \in 2$. Then*

$$\widehat{d}((1, 2), (2, 2)) = \widehat{d}((1, 1), (2, 2))$$

Proof. Let consider two filters $\mathcal{G}_2 \in 2$, and $\mathcal{F}_2 \in 2$, from the same δ -cut, one get $\Phi_d(F_2, G_2) = 0$, by definition. This can be added as such

$$\widehat{d}((1, 1), (2, 2)) = \inf_{F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2} \Phi_d(F_1, F_2),$$

which is $\Phi_d(F_1, F_2) + 0 = \Phi_d(F_1, F_2) + \Phi_d(F_2, G_2) = \Phi_d(F_1, G_2)$ whenever $F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2, G_2 \in \mathcal{G}_2$. □

Corollary 3.3.1. *The extension of the conjugate of a quasi-pseudometric, is the same as the conjugate of the extension of that quasi-pseudometric.*

Proof. The conjugate of \widehat{d} is obtained by

$$(\widehat{d})^{-1}((1, 1), (2, 2)) = \inf_{F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2} \Phi_d(F_2, F_1) \quad (3.3)$$

On the other hand, the $\widehat{d^{-1}}$ is given as follow

$$(\widehat{d^{-1}})((1, 1), (2, 2)) = \inf_{G_1 \in \mathcal{G}_1, G_2 \in \mathcal{G}_2} \Phi_d(G_2, G_1) \quad (3.4)$$

whenever $\mathcal{G}_1 \in \mathcal{G}_1$ and $\mathcal{G}_2 \in \mathcal{G}_2$.

By applying the Proposition 3.3.4 to the equation 3.3, we get the equality. □

This is to say that the extension commutes with the conjugate of quasi-pseudometric, $(\widehat{d})^{-1} = \widehat{(d^{-1})}$.

The following remark sets a link between δ -completion and B-completion for balanced quasi-pseudometric spaces. We need more investigation to formulate or conclude the existence of such connection.

Remark 3.3.1. *One notices that for balanced quasi-pseudometric spaces, we have,*

$$\widehat{d}'((1, 1), (2, 2)) = \inf_{G_1 \in \mathcal{G}_1, G_2 \in \mathcal{G}_2} \Phi_d(G_1, G_2),$$

which is $\Phi_d(G_1, G_2) + 0 = \Phi_d(F_1, G_1) + \Phi_d(G_1, G_2) \geq \Phi_d(F_1, G_2)$, whenever $F_1 \in \mathcal{F}_1, G_1 \in \mathcal{G}_1, G_2 \in \mathcal{G}_2$ and $\mathcal{G}_1 \in \mathcal{G}_1, \mathcal{G}_2 \in \mathcal{G}_2$. The distance \widehat{d} is greater than d^+ (see [14]). Furthermore, to avoid the balancedness requirement, the first takes two representatives of the same type, for instance two left δ -Cauchy filters \mathcal{F}_1 and \mathcal{F}_2 .

As a conclusion to this chapter, we have constructed a new theory of completion of quasi-pseudometric spaces that we have called δ -completion. We still need to provide some examples with added points that complete quasi-pseudometric spaces.

Chapter 4

Conclusion

We next summarize what we have done so far in our investigation on the completion of quasi-pseudometric space. We first have given an overview of some completion theories in the recent past. Then, we have recalled the concept of Cauchy pair filters introduced by Künzi and Kivuvu. We introduced the concept of δ -cut that makes use of Cauchy filter pairs. This notion is an expansion of κ -cut where we replaced the sequence-cosequence pairs by Cauchy filter pairs.

We have defined a distance function \hat{d} between two δ -cut and proved that it is a quasi-pseudometric on the set \hat{X} of all δ -cut. Then, we have shown that the map ψ from (X, d) into (\hat{X}, \hat{d}) is isometric embedding. After proving that $\psi(X)$ is dense in \hat{X} , we have defined the δ -completion of T_0 -quasi-metric spaces which is a δ -complete containing $\psi(X)$.

This investigation leads us to list some open problems encountered throughout the present investigation. We hope to study these questions in future work.

Problem 1 Andrikopoulos has extend his theory of κ -completion of quasi-pseudometric spaces to a related theory of quasi-uniform spaces. Can the completion theory presented in this dissertation be generalized to quasi-uniform spaces?

Problem 2 Are there simple conditions which characterize these balanced Cauchy filter pairs which are δ -Cauchy filter pairs? Is there a connection between the δ -completion and the hyperconvex hull of a B-completion in the sense of Künzi?

Problem 3 As the problem of completion occurs in theoretical Computer Science, can we find some practical examples or contra-examples to test the

robustness and the computability of the theory developed?

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