Accelerated life testing using the Eyring model for the Weibull and Birnbaum-Saunders distributions

N Smit

orcid.org/0000-0002-4570-033X

Thesis accepted in fulfilment of the requirements for the degree *Doctor of Philosophy in Science with Statistics* at the North-West University

Promoter: Prof L Raubenheimer

Graduation May 2021

22730656

Declaration

I, the undersigned, declare that the work contained in this thesis is my own work, except for references specifically indicated in the text, and that I have not previously submitted it elsewhere for degree purposes.

N. Smit

17/12/2020

Date

Abstract

In this thesis, we present a novel approach to new Bayesian dual-stress accelerated life testing models. The generalised Eyring model, with one thermal stressor and one non-thermal stressor, is utilised as the time transformation function. The new models use the Weibull and Birnbaum-Saunders distributions as the life distributions. General likelihood formulations are given for the models, which can accommodate uncensored, type-I censored and type-II censored data. Variations for the generalised Eyring-Weibull model are presented via different choices of prior distributions, which include uniform, gamma, and log-normal priors. For the generalised Eyring-Birnbaum-Saunders model, gamma priors are imposed on the model parameters. The full conditional and joint posterior distributions for the models are presented. The models have mathematically intractable posterior distributions, which means that Markov chain Monte Carlo methods need to be employed to generate posterior samples for inference. The log-concavity of the generalised Eyring-Weibull models is assessed to determine which Markov chain Monte Carlo methods are appropriate to use.

The new models are applied to a real data set, where temperature and relative humidity are the accelerated stressors. The sensitivity of the models is investigated by specifying various values for the hyperparameters. The models are implemented in OpenBUGS to generate posterior samples. The convergence of the Markov chains is monitored using trace plots and the Brooks-Gelman-Rubin approach. The Monte Carlo error is used to determine if an adequate number of samples have been generated by the Markov chains. The fit of the models is assessed via the deviance information criterion. Inferential results, such as summary statistics, marginal posterior distributions and the predictive reliability, are presented and compared between the models. It is found that both models are sensitive to the spescific choice of subjective priors, specifically when the prior variance is small. It is recommended that flat priors should ideally be used if no prior information is available.

The use of Bayes factors for model selection in accelerated life testing is also explored. Due to the mathematically intractable posterior distributions of the new models, the marginal likelihood for Bayes factors must be estimated. The focus is on methods that can approximate the marginal likelihood from the samples generated by a Markov chain Monte Carlo algorithm. These methods include a simple Monte Carlo estimator, the harmonic mean estimator, the Laplace-Metropolis estimator, and a posterior predictive density estimate for posterior Bayes factors. The new models are applied to another real data set and implemented in OpenBUGS to generate posterior samples. The Bayes factors and posterior model probabilities are calculated using different estimators. Model selection is carried out by comparing the Bayes factors and the deviance information criterion.

Keywords: Accelerated life testing; Adaptive rejection sampling; Bayes factors; Bayesian methods; Birnbaum-Saunders distribution; Deviance information criterion; Generalised Eyring model; Markov chain Monte Carlo; Model selection; Slice sampling; Weibull distribution.

Contents

List of Figures viii

List of Tables x

Acknowledgements xii

List of Research Outputs xiii

List of Abbreviations xiv

1 Introduction 1

1.1 Overview 1
1.2 Objectives 2
1.3 Contributions 2
1.4 Outline 3

2 Background for Accelerated Life Testing 5

2.1 Introduction 5
2.2 Core Reliability Concepts 6
2.2.1 Reliability Function 6
2.2.2 Hazard Function 7
2.3 Censoring 7
2.3.1 Right Censoring 7
2.3.2 Left Censoring 8
2.3.3 Interval Censoring 8
2.4 Life Distributions 9
2.4.1 Exponential Distribution 9
2.4.2 Weibull Distribution 11
2.4.3 Log-normal Distribution 13
2.4.4 Birnbaum-Saunders Distribution 15

2.5 Acceleration Models . . . 17
2.5.1 Stress Loading . . . 18
2.5.2 Arrhenius Model . . . 19
2.5.3 Inverse Power Law Model . . . 19
2.5.4 Eyring Model . . . 20
2.5.5 Generalised Eyring Model . . . 21
2.6 Bayesian Considerations . . . 21
2.7 Markov Chain Monte Carlo Methods . . . 22
2.7.1 Metropolis-Hastings Sampler . . . 23
2.7.2 Gibbs Sampler . . . 24
2.7.3 Adaptive Rejection Sampling . . . 25
2.7.4 Adaptive Rejection Metropolis Sampling . . . 26
2.7.5 Slice Sampling . . . 28
2.8 MCMC Convergence in OpenBUGS . . . 30
2.8.1 Trace Plots . . . 30
2.8.2 Brooks-Gelman-Rubin (BGR) Approach . . . 32
2.8.3 Monte Carlo Error . . . 33
2.9 Deviance Information Criterion . . . 33
2.10 Literature Review of Bayesian Accelerated Life Testing . . . 34
2.11 Motivation for New Models . . . 38

3 The Generalised Eyring-Weibull Model 40
3.1 Introduction . . . 40
3.2 The GEW Model Specification . . . 40
3.3 Priors and Posteriors . . . 42
3.3.1 GEW_1 Model . . . 43
3.3.2 GEW_2 Model . . . 45
3.3.3 GEW_3 Model . . . 47
3.3.4 Log-concavity . . . 49
3.3.5 MCMC Implementation . . . 57
3.4 Application . . . 58
3.5 Conclusion . . . 70

4 The Generalised Eyring-Birnbaum-Saunders Model 72
4.1 Introduction . . . 72
4.2 The GEBS Model Specification . . . 72
4.3 Priors and Posteriors . . . 74
4.4 Application . . . 78

4.5 Conclusion 92

5 Bayes Factors in Accelerated Life Testing 93

5.1 Introduction 93

5.2 Bayes Factors 94

5.3 Estimating the Marginal Likelihood 96

5.3.1 Simple Monte Carlo Estimator 97

5.3.2 Laplace Approximation 97

5.3.3 Harmonic Mean Estimator 98

5.3.4 Posterior Predictive Density Estimate 99

5.4 Application 99

5.5 Conclusion 115

6 Conclusion 117

6.1 Summary of Conclusions 117

6.2 Future Research 120

References 121

Appendix A: MCMC Convergence Diagnostics 129

A.1 Convergence for the GEW Models 131

A.1.1 GEW_1 131

A.1.2 $GEW_{2,1}$ 133

A.1.3 $GEW_{2,2}$ 135

A.1.4 $GEW_{2,3}$ 137

A.1.5 $GEW_{3,1}$ 139

A.1.6 $GEW_{3,2}$ 141

A.2 Convergence for the GEBS Models 143

A.2.1 $GEBS_1$ 143

A.2.2 $GEBS_2$ 145

A.2.3 $GEBS_3$ 147

A.2.4 $GEBS_4$ 149

A.2.5 $GEBS_5$ 151

A.2.6 $GEBS_6$ 153

Appendix B: Additional Results 155

B.1 OpenBUGS® Code: MCMC Samples for GEW and GEBS Models 155

B.2 R® Code: Predictive Reliability of GEW and GEBS Models 156

B.3 R® Code: Simple Monte Carlo Estimation of Bayes Factors 158

B.4 R® Code: Estimation of Posterior Bayes Factors 160
B.5 R® Code: Laplace Approximation of Bayes Factors 163
B.6 R® Code: Harmonic Mean Estimator for Bayes Factors 166

List of Figures

2.1 PDF of the exponential distribution for various values of λ. 10
2.2 Reliability function of the exponential distribution for various values of λ. 10
2.3 PDF of the Weibull distribution for various values of α and β. 12
2.4 Reliability function of the Weibull distribution for various values of α and β. 12
2.5 Reliability function of the Weibull distribution for $\alpha = 1$ and various values of β. . . . 13
2.6 PDF of the log-normal distribution for various values of μ and σ. 14
2.7 Reliability function of the log-normal distribution for various values of μ and σ. 15
2.8 PDF of the Birnbaum-Saunders distribution for various values of α and β. 16
2.9 Reliability function of the Birnbaum-Saunders distribution for various values of α and β. 17
2.10 Single chain trace plots (Images from Spiegelhalter et al., 2014). 31
2.11 Multiple chain trace plots (Images from Spiegelhalter et al., 2014). 31
2.12 Example of a BGR plot, where convergence occurs around 250 iterations (Image from Spiegelhalter et al., 2014). 33

3.1 Marginal posterior distributions for the GEW_1 model. 63
3.2 Marginal posterior distributions for the $\text{GEW}_{2,1}$ model. 64
3.3 Marginal posterior distributions for the $\text{GEW}_{2,2}$ model. 65
3.4 Marginal posterior distributions for the $\text{GEW}_{2,3}$ model. 66
3.5 Marginal posterior distributions for the $\text{GEW}_{3,1}$ model. 67
3.6 Marginal posterior distributions for the $\text{GEW}_{3,2}$ model. 68
3.7 Predictive reliability for the GEW models at use stress $T_u = 350$, $S_u = 0.3$. 70

4.1 Marginal posterior distributions for the GEBS_1 model. 83
4.2 Marginal posterior distributions for the GEBS_2 model. 84
4.3 Marginal posterior distributions for the GEBS_3 model. 85
4.4 Marginal posterior distributions for the GEBS_4 model. 86
4.5 Marginal posterior distributions for the GEBS_5 model. 87
4.6 Marginal posterior distributions for the GEBS_6 model. 88

4.7 Predictive reliability for the GEBS models at use stress $T_u = 350$, $S_u = 0.3$. 90
4.8 Predictive vs empirical reliability of the GEBS models. 91

5.1 Marginal posterior distributions for the GEW$_{BF1}$ model. 105
5.2 Marginal posterior distributions for the GEW$_{BF2}$ model. 106
5.3 Marginal posterior distributions for the GEW$_{BF3}$ model. 107
5.4 Marginal posterior distributions for the GEBS$_{BF1}$ model. 108
5.5 Marginal posterior distributions for the GEBS$_{BF2}$ model. 109
5.6 Marginal posterior distributions for the GEBS$_{BF3}$ model. 110

A.1 BGR plots for the GEW_1 model. 131
A.2 Trace plots for the GEW_1 model. 132
A.3 BGR plots for the $GEW_{2,1}$ model. 133
A.4 Trace plots for the $GEW_{2,1}$ model. 134
A.5 BGR plots for the $GEW_{2,2}$ model. 135
A.6 Trace plots for the $GEW_{2,2}$ model. 136
A.7 BGR plots for the $GEW_{2,3}$ model. 137
A.8 Trace plots for the $GEW_{2,3}$ model. 138
A.9 BGR plots for the $GEW_{3,1}$ model. 139
A.10 Trace plots for the $GEW_{3,1}$ model. 140
A.11 BGR plots for the $GEW_{3,2}$ model. 141
A.12 Trace plots for the $GEW_{3,2}$ model. 142
A.13 BGR plots for the $GEBS_1$ model. 143
A.14 Trace plots for the $GEBS_1$ model. 144
A.15 BGR plots for the $GEBS_2$ model. 145
A.16 Trace plots for the $GEBS_2$ model. 146
A.17 BGR plots for the $GEBS_3$ model. 147
A.18 Trace plots for the $GEBS_3$ model. 148
A.19 BGR plots for the $GEBS_4$ model. 149
A.20 Trace plots for the $GEBS_4$ model. 150
A.21 BGR plots for the $GEBS_5$ model. 151
A.22 Trace plots for the $GEBS_5$ model. 152
A.23 BGR plots for the $GEBS_6$ model. 153
A.24 Trace plots for the $GEBS_6$ model. 154

List of Tables

3.1 Failure times for an electronics epoxy packaging ALT. 58
3.2 Prior specifications for the GEW models. 59
3.3 Monte Carlo error as a percentage of the standard deviation for GEW_1. 59
3.4 Monte Carlo error as a percentage of the standard deviation for $GEW_{2,1}$. 59
3.5 Monte Carlo error as a percentage of the standard deviation for $GEW_{2,2}$. 60
3.6 Monte Carlo error as a percentage of the standard deviation for $GEW_{2,3}$. 60
3.7 Monte Carlo error as a percentage of the standard deviation for $GEW_{3,1}$. 60
3.8 Monte Carlo error as a percentage of the standard deviation for $GEW_{3,2}$. 60
3.9 Deviance information criterion for the GEW models. 61
3.10 Summary statistics for the GEW models. 62
3.11 Predictive reliability for the GEW models at use stress $T_u = 350$, $S_u = 0.3$. 69

4.1 Prior specifications for the GEBS models. 79
4.2 Monte Carlo error as a percentage of the standard deviation for $GEBS_1$. 80
4.3 Monte Carlo error as a percentage of the standard deviation for $GEBS_2$. 80
4.4 Monte Carlo error as a percentage of the standard deviation for $GEBS_3$. 80
4.5 Monte Carlo error as a percentage of the standard deviation for $GEBS_4$. 80
4.6 Monte Carlo error as a percentage of the standard deviation for $GEBS_5$. 81
4.7 Monte Carlo error as a percentage of the standard deviation for $GEBS_6$. 81
4.8 Deviance information criterion for the GEBS models. 81
4.9 Summary statistics for the GEBS models. 82
4.10 Predictive reliability for the GEBS models at use stress $T_u = 350, S_u = 0.3$. 89

5.1 Bayes factor interpretations. 95
5.2 Failure times for some device. 100
5.3 Prior specifications for the Bayes factors application. 100
5.4 Deviance information criterion for the Bayes factors application. 111
5.5 Natural log of the marginal likelihood estimates for the models. 112
5.6 Laplace-Metropolis estimates for the Bayes factors. 112

5.7 Laplace-Metropolis estimates for the posterior model probabilities. 113
5.8 Harmonic mean estimates for the Bayes factors. 114
5.9 Harmonic mean estimates for the posterior model probabilities. 114
5.10 Posterior predictive density estimates for posterior Bayes factors. 115
5.11 Posterior model probabilities using the estimated posterior Bayes factors 115

Acknowledgements

I would like to acknowledge and express my gratitude to the following people for their wonderful support while I was working on this thesis:

- My supervisor, Prof. Lizanne Raubenheimer, for all her help, advice, guidance, encouragement and patience throughout the past three years.
- Prof. Thomas Mazzuchi and Prof. Refik Soyer, for providing valuable suggestions and insight into the work in this thesis.
- My family and friends, especially my parents, for all their love and support. Without you, I would not have been able to come this far.

The financial assistance of the National Research Foundation (NRF) is hereby acknowledged, where my PhD studies have been funded by the *NRF/SASA: Crisis in Academic Statistics* bursary, grant UID 114629. Any opinion, finding and conclusion or recommendation expressed in this material is that of the author(s) and the NRF does not accept any liability in this regard.

List of Research Outputs

The following revised paper has been submitted to the journal *Quality and Reliability Engineering International*, and is currently under review:

- Smit, N. and Raubenheimer, L. (2020). Bayesian accelerated life testing: A generalized Eyring-Birnbaum-Saunders model.

The following paper has been submitted to the journal *Computational Statistics*:

- Smit, N. and Raubenheimer, L. (2020). Bayes factors for accelerated life testing models.

The following presentations at local and international conferences also stem from the work in this thesis:

- Accelerated life testing for the generalised Eyring-Weibull model. *60th Annual Conference of the South African Statistical Association*, hosted by the University of South Africa, Johannesburg, South Africa, November 2018.
- A Bayesian Eyring-Birnbaum-Saunders ALT model. *39th Conference on Applied Statistics in Ireland*, hosted by Trinity College Dublin, Dundalk, Ireland, May 2019.
- A comparison of two Bayesian accelerated life testing models. *32nd European Meeting of Statisticians*, hosted by the University of Palermo, Palermo, Italy, July 2019.
- A dual-stress log-normal accelerated life testing model. *61st Annual Conference of the South African Statistical Association*, hosted by Nelson Mandela University, Port Elizabeth, South Africa, November 2019.
- Accelerated life testing: A Bayesian Eyring-log-normal model. *2020 Joint Statistical Meeting*, virtual conference, August 2020.

List of Abbreviations

AIC	Akaike information criterion
ALT	Accelerated life testing
ARMS	Adaptive rejection Metropolis sampling
ARS	Adaptive rejection sampling
BGR	Brooks-Gelman-Rubin
BIC	Bayes information criterion
CDF	Cumulative distribution function
DIC	Deviance information criterion
GEBS	Generalised Eyring-Birnbaum-Saunders
GEW	Generalised Eyring-Weibull
MCMC	Markov chain Monte Carlo
PDF	Probability density function
TTF	Time transformation function

Chapter 1

Introduction

1.1 Overview

The quantification of life characteristics for high-reliability products, components or systems can be a strenuous task. Many products are designed to operate without failure for years, and obtaining failure data to perform reliability analysis for such items may not be feasible due to budget or time constraints. At the rate at which new products are developed and manufactured in our modern society, a great number of durable products can even become obsolete by the time reliability measures have been established.

A possible solution to this problem, stemming from reliability engineering, is the use of accelerated life testing (ALT). In ALT, products are tested under more severe than normal use environments, to induce early failures (Nelson, 1990). These accelerated testing environments are obtained by applying or fluctuating one or more stressors (stress variables), such as temperature, voltage or humidity, for example, at accelerated levels. The failure data acquired during ALT can be extrapolated to estimate the life characteristics of the products at the normal operating conditions (Pan, 2009). A functional relationship is assumed between the parameters of the life distribution and the accelerated stressors, in order to estimate reliability measures for the products during normal use (Singpurwalla et al., 1975).

In recent years, there has been a greater interest in the development of Bayesian ALT methods. These methods, however, have mostly been concerned with models where only one accelerated stressor is used. In this thesis, Bayesian models that make use of the generalised Eyring model are developed. The generalised Eyring model allows for two stressors, one thermal and one non-thermal. The Weibull and Birnbaum-Saunders distributions are used as life distributions due to their flexibility and versatility over the exponential distribution. The new models are applied to real data sets and model selection in the Bayesian ALT paradigm is also addressed.

1.2 Objectives

The main objectives of this thesis are summarised as follows:

- provide an overview of some important aspects of reliability engineering and ALT.
- provide relevant information regarding ALT in the Bayesian paradigm, including Markov chain Monte Carlo methods (MCMC) and model selection.
- provide a review of the Bayesian ALT literature and identify gaps in this literature.
- formulate a dual-stress, Bayesian ALT model that utilises the generalised Eyring time transformation function (TTF), where lifetimes follow a Weibull distribution.
- formulate a dual-stress, Bayesian ALT model that utilises the generalised Eyring TTF, where lifetimes follow a Birnbaum-Saunders distribution.
- determine which MCMC methods are appropriate to generate posterior samples for the new models.
- apply these models to a real data set using the Bayesian data analysis software OpenBUGS (version 3.2.3 rev 1012), investigate the sensitivity of the models, assess the convergence of the Markov chains, and provide some inferential results.
- explore model selection in the Bayesian ALT paradigm via the deviance information criterion (DIC) and Bayes factors.
- apply the models to a real data set in order to illustrate the model selection criteria practically.

1.3 Contributions

Given the objectives above, the contributions of this thesis in the field of Bayesian ALT are summarised as follows:

- develop a novel approach to formulating two dual-stress, Bayesian ALT models where the generalised Eyring model is the TTF. The models use the Weibull and Birnbaum-Saunders distributions and can accommodate uncensored, type-I censored and type-II censored data.
- determine appropriate MCMC methods that can be used to obtain posterior samples for the models.
- demonstrate the models via an application to a real data set, where the Bayesian data analysis software OpenBUGS is used to generate posterior samples.

- provide a framework for model selection in the Bayesian ALT paradigm using the DIC and Bayes factors.

1.4 Outline

Two Bayesian ALT models, which make use of the generalised Eyring model, are presented. This follows on the work of Singpurwalla et al. (1975), where classical least square estimators are determined for an ALT model with the exponential distribution and generalised Eyring model. The formulation and application of the new models follow the same thought process of Soyer et al. (2008), where a Weibull ALT model is presented with the Arrhenius or power law models.

In **Chapter 2**, the necessary background on ALT is provided. The objectives of reliability and life testing, as well as the need for ALT, are discussed. Some important aspects of life testing are addressed, such as the types of censoring and different life distributions. The most commonly used TTFs and the types of stress loading for ALT designs are presented. Due to the mathematically intractable posteriors of the new models, MCMC methods are employed to generate posterior samples to make statistical inference. The MCMC methods discussed are the Metropolis-Hastings algorithm, based on the work by Metropolis et al. (1953) and Hastings (1970), the Gibbs sampler, introduced by Geman and Geman (1984), adaptive rejection sampling (ARS), presented in Gilks and Wild (1992), adaptive rejection Metropolis sampling (ARMS), introduced by Gilks et al. (1995), and slice sampling, presented in Neal (2003). Methods to assess the convergence of the Markov chains in OpenBUGS are also presented. A review of the current Bayesian ALT literature is undertaken, which leads to the motivation for the new Bayesian models formulated in this thesis.

In **Chapter 3**, the generalised Eyring-Weibull (GEW) model is formulated. A general likelihood formulation is given, which can accommodate uncensored, type-I censored and type-II censored data. Variations on the model are defined through the choice of different prior distributions. The posterior distributions and full conditional posterior distributions, up to proportionality, are given. The log-concavity of the full conditional posterior distributions are assessed to determine which MCMC algorithms are suitable. The models are applied to a data set, where summary statistics, marginal posterior plots and the predictive reliability are presented. The fit of the models to the data is assessed via the DIC statistics.

In **Chapter 4**, the generalised Eyring-Birnbaum-Saunders (GEBS) model is developed. Again, a general likelihood formulation is provided that can incorporate censoring. Gamma priors are imposed on the model parameters and the mathematically intractable posterior is presented. The full conditional posterior distributions for this model are not log-concave on their entire domains, meaning that the slice sampling algorithm in OpenBUGS must be utilised to generate posterior samples. The model is applied to a real data set, where the sensitivity of the model is investigated via the choice of hyperparameters. Similar inferential results to those of the GEW model in Chapter 3, are presented.

In **Chapter 5**, Bayesian model selection in the ALT paradigm is discussed. The use of Bayes factors for model selection is explored. Due to the mathematically intractable posterior distributions of the new models, the marginal likelihoods must be estimated. Methods that can estimate the marginal likelihood easily from the MCMC output, without further complicating the sampling process, are focussed on. These methods include a simple Monte Carlo estimator, harmonic mean estimator, Laplace-Metropolis estimator, and the use of posterior Bayes factors. The new models are applied to a real data set, where model selection is then examined in terms of Bayes factors and the DIC. The posterior model probabilities for the models in the application are also presented.

In **Chapter 6,** the conclusions from the thesis and some remarks on future research, are presented.

Appendix A contains the MCMC convergence diagnostics for applying the new models in Chapter 3 and Chapter 4. Multiple chains are initiated in OpenBUGS for the models, where the convergence is assessed via trace plots and the approach presented in Brooks and Gelman (1998).

Appendix B contains most of the important code used in the applications of the new ALT models. The code to implement our two models in OpenBUGS and generate posterior samples for inference, is presented. Also included, is the R code which is used to calculate the predictive reliability. Lastly, the R code for the different estimators of the marginal likelihood are presented.

Chapter 2

Background for Accelerated Life Testing

2.1 Introduction

Reliability and life testing are vitally important manufacturing consumer and capital goods, particularly in the engineering fields. It is crucial to determine whether a product will perform its prescribed function, without failure, for a desired period of time. Reliability and life testing provides a theoretical as well as practical framework by which the life characteristics of a product, component or system can be quantified. Kececioglu (2002) highlights the objectives of reliability and life testing as follows:

- Determine whether components, equipment, and systems will perform their function, either in a normal use environment or in a laboratory testing environment, for the prescribed time period.
- Identify causes of failure and recurring patterns in failures for certain stress levels.
- Determine the underlying failure distribution, failure rate, mean life, confidence limits and other reliability measures for obtained failure data.
- Provide guidelines and, if necessary, suggest remedial actions based on the reliability results.
- Re-evaluate the reliability of products to determine whether remedial actions were effective or whether further or alternative remedial actions are necessary.
- Determine the growth in reliability during the product research and development phases, in order to ensure that reliability requirements are met before the product enters the final testing and production phases.
- Statistically evaluate whether, with a specified confidence, a redesign has actually improved the reliability of a product.

- Statistically determine which design, manufacturer or supplier should be preferred, based on the reliability of components and equipment, and assuming all other factors to be economically and practically equivalent.
- Identify the best reliability tests to use for a product, based on the deadline, budget, available personnel, and equipment.
- Present the reliability results in an easily understood and suitable format to management, in order to ensure that the right decisions are made.

In this day and age, manufacturers are pressured to provide durable products, timely and at competitive prices. Performing life tests on high-reliability materials, components and systems can be a tedious and unproductive task. High-reliability components can include items such as semi-conductors, microchips, connectors, cables, filters, embedded software, etc. Many modern products are made to endure for years. Obtaining sufficient failure time data for these products at normal use conditions can prove to be very costly and time-consuming. This longevity obstacle has lead to a rising interest in reliability engineering and the development of ALT models. In ALT, products are tested in a more severe than normal use environment in order to induce early failures (Nelson, 1990). This failure data is then extrapolated to predict the reliability characteristics of the products at normal use conditions (Pan, 2009).

2.2 Core Reliability Concepts

In this section, some core concepts regarding reliability and life testing are discussed.

2.2.1 Reliability Function

Let $X \geq 0$ be a continuous random variable that denotes the time to failure of an item. Suppose X has a probability density function (PDF) denoted by $f(x)$, and a cumulative distribution function (CDF) denoted by $F(x)$. The reliability function, denoted by $R(x)$, is defined as the probability that an item will not fail during the time period leading up to time x, and is given by

$$\begin{aligned} R(x) &= P(X > x) \\ &= \int_x^\infty f(s)ds \\ &= 1 - F(x). \end{aligned} \tag{2.1}$$

The reliability function is a monotonically decreasing function with $R(x) \geq 0$, $R(0) = 1$, and $\lim_{x\to\infty} R(x) = 0$.

2.2.2 Hazard Function

The hazard function, also known as the instantaneous failure rate function, is also commonly used in reliability and life testing (Hamada et al., 2008). Suppose an item has survived up to time x and we want to know the probability that the item will fail within a time of length Δx. This probability can be written as

$$\begin{aligned} P(x < X \leq x + \Delta x | X > x) &= \frac{P(x < X \leq x + \Delta x)}{P(X > x)} \\ &= \frac{F(x + \Delta x) - F(x)}{R(x)}. \end{aligned} \tag{2.2}$$

The hazard function, denoted by $h(x)$, can be found by dividing (2.2) by the interval length Δx and making it very small. Thus,

$$\begin{aligned} h(x) &= \lim_{\Delta x \to 0} \frac{P(x < X \leq x + \Delta x | X > x)}{\Delta t} \\ &= \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} \times \frac{1}{R(x)} \\ &= F'(x) \times \frac{1}{R(x)} \\ &= \frac{f(x)}{R(x)}. \end{aligned} \tag{2.3}$$

2.3 Censoring

Often in reliability and life testing one finds that not all items under investigation fail during the testing process. According to Klein and Moeschberger (2003), this event, known as censoring, occurs when lifetimes are only observed within certain intervals. It is important to identify the type of censoring applied to failure data, as this will affect the likelihood function, which is fundamental in Bayesian inference (Klein and Moeschberger, 2003). We distinguish between right, left, and interval censoring, emphasising right censoring.

2.3.1 Right Censoring

Right censoring is the most commonly used censoring scheme and consists of several different types of censoring. Suppose we have a specific number of items, say n, that are observed during a life test experiment. *Type-I censoring* (also known as time censoring) occurs when the experiment is terminated after a predetermined length of time, say $\tau < \infty$, has elapsed (Singpurwalla, 2006). Suppose that k failures were observed during the experiment, then the observed failure times can be indicated by $x_1 < x_2 < \ldots < x_k$. Mann and Singpurwalla (1980) state that the main advantage of type-I censoring is that the researcher knows exactly when the experiment will terminate, and can choose this termination time to suit the budget and deadline for the study. A drawback of using type-I censoring is the possibility of very few or even no failures occuring before time τ.

When *type-II censoring* (also known as item censoring) is applied, the life test is terminated after a specified number, say $r \leq n$, of the n items have failed (Singpurwalla, 2006). The failure times are indicated by $x_1 < x_2 < \ldots < x_r$, and note that the time at which the experiment terminates x_r is random. The choice of r is important to the researcher since more failure data could lead to more reliable results for the life test, but the termination time x_r must also fall within the time constraints for the study (Kalbfleisch and Prentice, 2002; Singpurwalla, 2006).

A mixture of type-I and type-II censoring (better known as hybrid censoring) can also be used. With *type-I hybrid censoring* the experiment is terminated if either r items have failed or time τ has been reached, where both r and τ are set in advance by the researcher (Balakrishnan and Kundu, 2013). The time of termination is thus a random variable expressed as $\min(x_r, \tau)$. Utilising *type-II hybrid censoring*, the researcher chooses fixed values for r and τ, but the experiment is terminated either at time τ, given that at least r failures have occurred, or continues past time τ until r items have failed. In this case, the termination time can be expressed by $\max(x_r, \tau)$.

For further reading on other types of right censoring, such as progressive, progressive hybrid, and random censoring, refer to Balakrishnan and Kundu (2013), Childs et al. (2008), Cohen (1963), and Efron (1967).

2.3.2 Left Censoring

Left censoring occurs when items have failed before a chosen censoring time (alternatively called an inspection time) τ_L, but the exact failure times are unknown (Klein and Moeschberger, 2003). It is only evident that the items failed at some time during the interval $[0, \tau_L)$. In medical studies, for example, data can be left censored if a certain event has occured before the time of the study. A study concerning chickenpox may contain some participants who have already had the disease in the past (left censored), and others who have not. The exact age at which the participants contracted the disease may be unknown, but they contracted the disease before entering the study.

2.3.3 Interval Censoring

In some cases, the exact failure times for items are not known, but it is known that the failures took place at or after time τ_L and before time τ_R. This is called interval censoring, since the failures happened during the interval $[\tau_L, \tau_R)$. For example, suppose a number of transistors are studied in a life test and an engineer inspects the transistors every 24 hours to document failures. If a transistor fails, the time interval in which it failed, for example $[96, 120)$, is recorded and not the exact time of the failure.

2.4 Life Distributions

Harris and Singpurwalla (1968) describe a life distribution as an endeavour to mathematically represent the lifetime of some material or item. Life distributions are a group of probability distributions that are used in reliability engineering and survival analysis. In this section, some of the most commonly used life distributions, i.e. the exponential, Weibull, log-normal and Birnbaum-Saunders distributions, are discussed. In the chapters that follow, new ALT models are formulated for the Weibull and Birnbaum-Saunders distributions. Many other distributions, such as the normal, mixed Weibull, gamma, generalised gamma, logistic, log-logistic and Gumbel distributions, can also be used as life distributions in certain cases (see, for example, Cordeiro et al., 2012; Gupta and Groll, 1961; Kantam et al., 2001; Mann et al., 1974; Nelson and Kielpinski, 1976).

2.4.1 Exponential Distribution

Let X be a continuous random variable that follows an exponential distribution with parameter $\lambda > 0$. The PDF is given by

$$f(x) = \lambda \exp(-\lambda x), \quad x > 0, \tag{2.4}$$

and the reliability function by

$$R(x) = \exp(-\lambda x). \tag{2.5}$$

The exponential distribution has a mean of

$$\mathrm{E}(X) = \frac{1}{\lambda},$$

and a variance of

$$\mathrm{Var}(X) = \frac{1}{\lambda^2}.$$

The effect of the parameter λ on the PDF and reliability function of the exponential distribution is shown in Figure 2.1 and Figure 2.2, respectively.

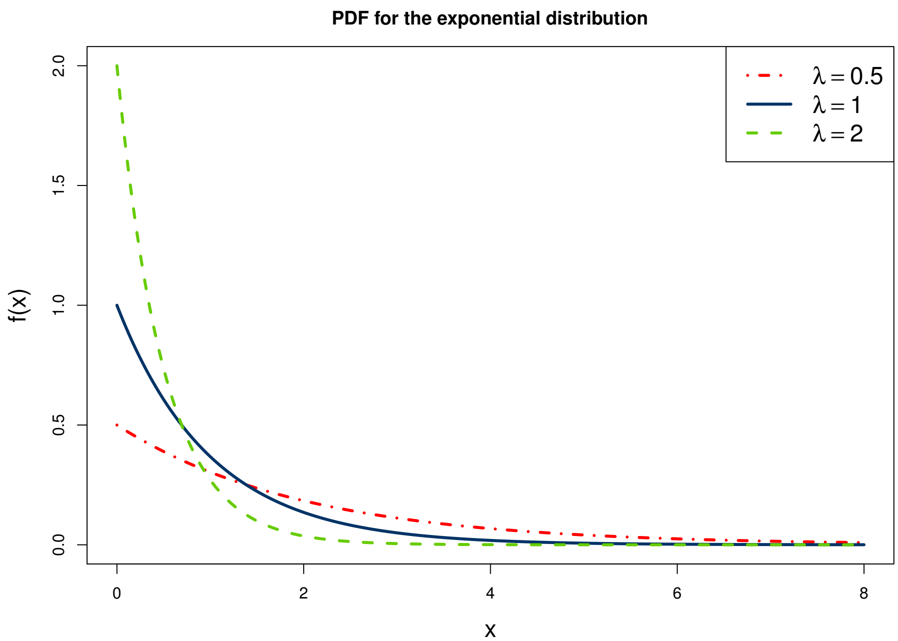

Figure 2.1: PDF of the exponential distribution for various values of λ.

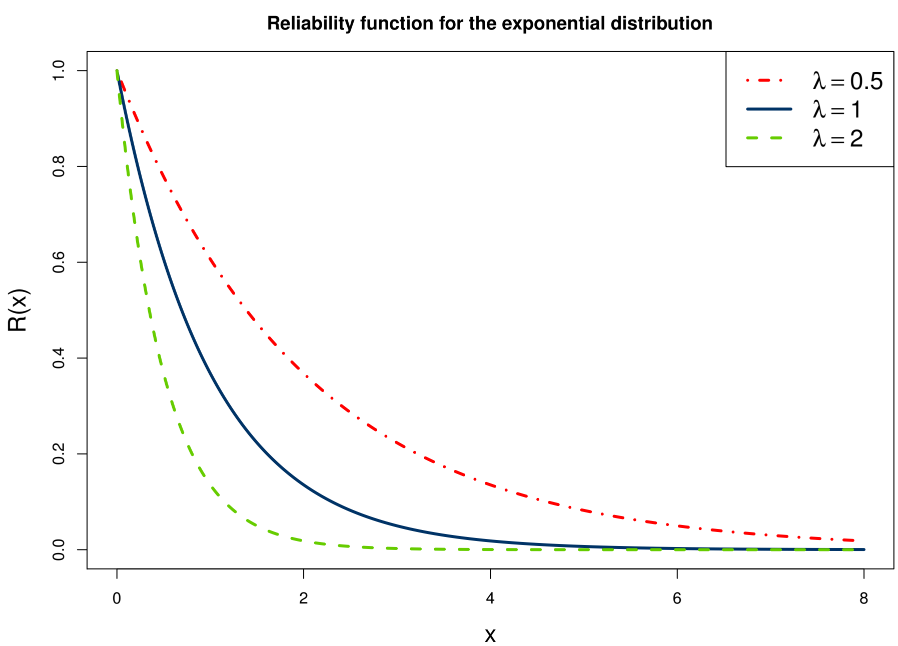

Figure 2.2: Reliability function of the exponential distribution for various values of λ.

The exponential distribution is often used in reliability and life testing due to its simplicity. An important characteristic of this distribution is that it has a constant hazard rate, meaning the instantaneous rate of failure for an item remains the same throughout its whole life. Therefore, the effects of aging and degradation are assumed to have no impact on the item (Singpurwalla, 2006). Due to this property, the exponential distribution is unsuitable for most practical situations, though it is still useful in theoretical applications, but should be implemented with caution.

2.4.2 Weibull Distribution

Let X be a continuous random variable following a Weibull distribution with scale parameter $\alpha > 0$ and shape parameter $\beta > 0$. The corresponding PDF is given by

$$f(x) = \alpha\beta x^{\beta-1} \exp\left(-\alpha x^{\beta}\right), \quad x > 0. \tag{2.6}$$

The reliability function is given by

$$R(x) = \exp\left(-\alpha x^{\beta}\right). \tag{2.7}$$

The mean and variance of the Weibull distribution are given, respectively, by

$$\mathrm{E}[X] = \frac{1}{\alpha^{\frac{1}{\beta}}}\Gamma\left(1+\frac{1}{\beta}\right),$$

and

$$\mathrm{Var}[X] = \frac{1}{\alpha^{\frac{2}{\beta}}}\left[\Gamma\left(1+\frac{2}{\beta}\right) - \left(\Gamma\left(1+\frac{1}{\beta}\right)\right)^2\right].$$

Figure 2.3 shows possible shapes for the PDF of the Weibull distribution for various values of the parameters α and β, and the corresponding reliability functions are shown in Figure 2.4. The effect of the shape parameter β on the reliability function is shown in Figure 2.5.

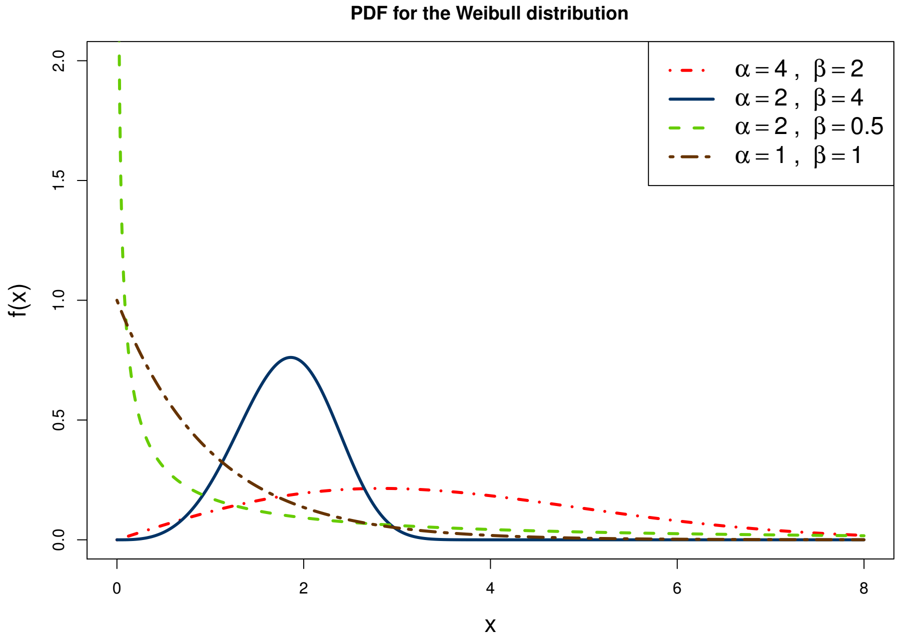

Figure 2.3: PDF of the Weibull distribution for various values of α and β.

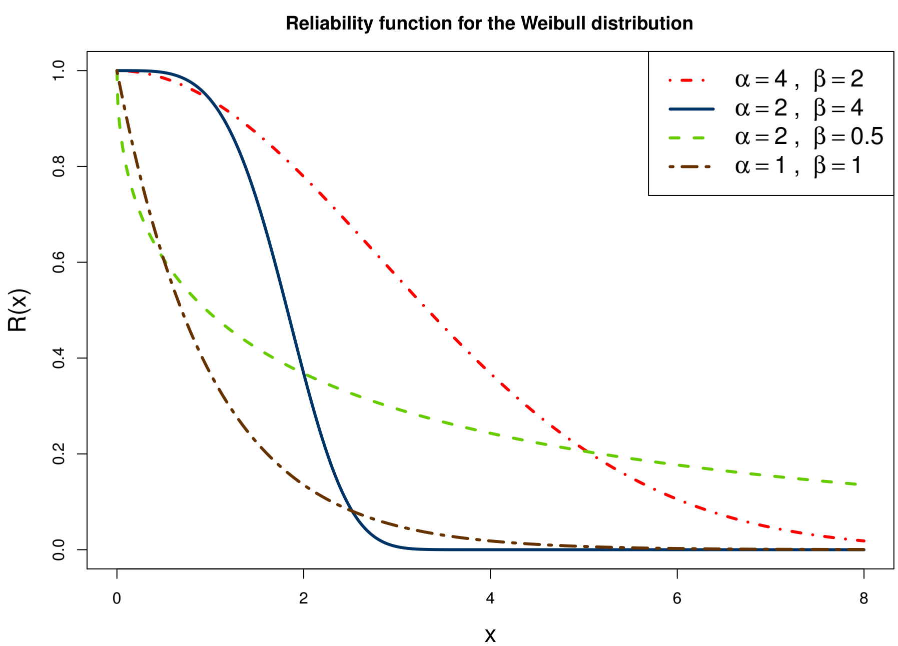

Figure 2.4: Reliability function of the Weibull distribution for various values of α and β.

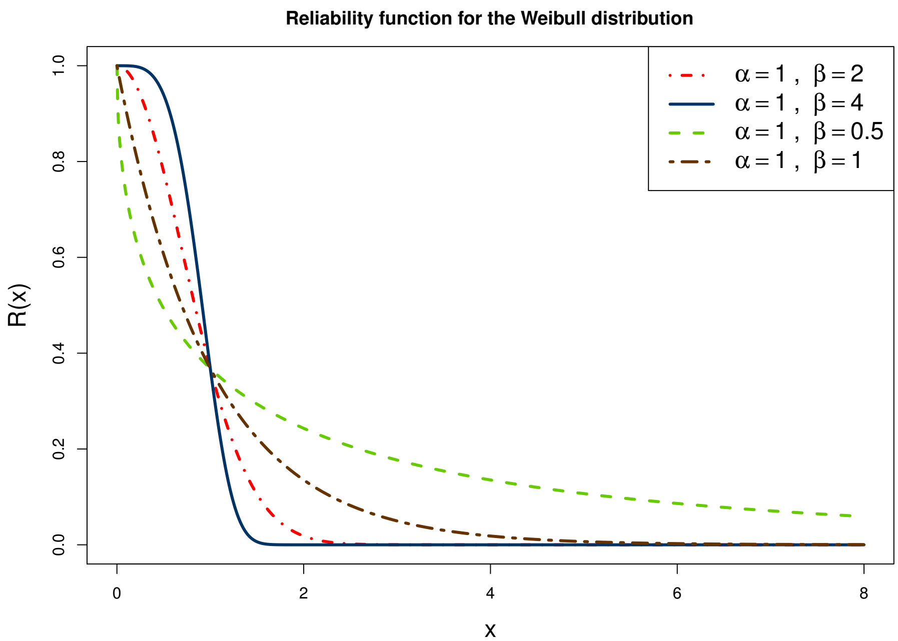

Figure 2.5: Reliability function of the Weibull distribution for $\alpha = 1$ and various values of β.

The Weibull distribution was first formally defined in Weibull (1951) and is one of the most popular life distributions. Due to its versatility and flexibility, it has numerous applications in reliability analysis and various other fields. Lai et al. (2006) list several examples of where the Weibull distribution has been used in reliability studies. These include testing the fracture strength of glass, the yield strength of steel, the failure of brittle materials, and many more. Several other areas, such as geophysics, food science, social science and medical science, where the Weibull distribution has been applied, are also provided in Lai et al. (2006). Various parameterisations and modifications of the Weibull distribution can be found in Almalki and Nadarajah (2014), Hamada et al. (2008), and Lai et al. (2006). These other versions of the Weibull distribution is, however, beyond the scope of this study.

2.4.3 Log-normal Distribution

Let X be a continuous random variable that follows a log-normal distribution with parameters $-\infty < \mu < \infty$ and $\sigma > 0$. The PDF is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left[-\frac{(\ln(x)-\mu)^2}{2\sigma^2}\right], \quad x > 0, \tag{2.8}$$

and the reliability function by

$$R(x) = 1 - \Phi\left(\frac{\ln(x)-\mu}{\sigma}\right), \tag{2.9}$$

where $\Phi(\cdot)$ denotes the CDF of the standard normal distribution. The log-normal distribution has a mean of

$$\mathrm{E}(X) = \exp\left(\mu + \frac{1}{2}\sigma^2\right),$$

and a variance of

$$\mathrm{Var}(X) = \exp\left(2\mu + \sigma^2\right)\left[\exp\left(\sigma^2\right) - 1\right].$$

The PDF and reliability funtion of the log-normal distribution for various values of μ and σ are shown in Figure 2.6 and Figure 2.7.

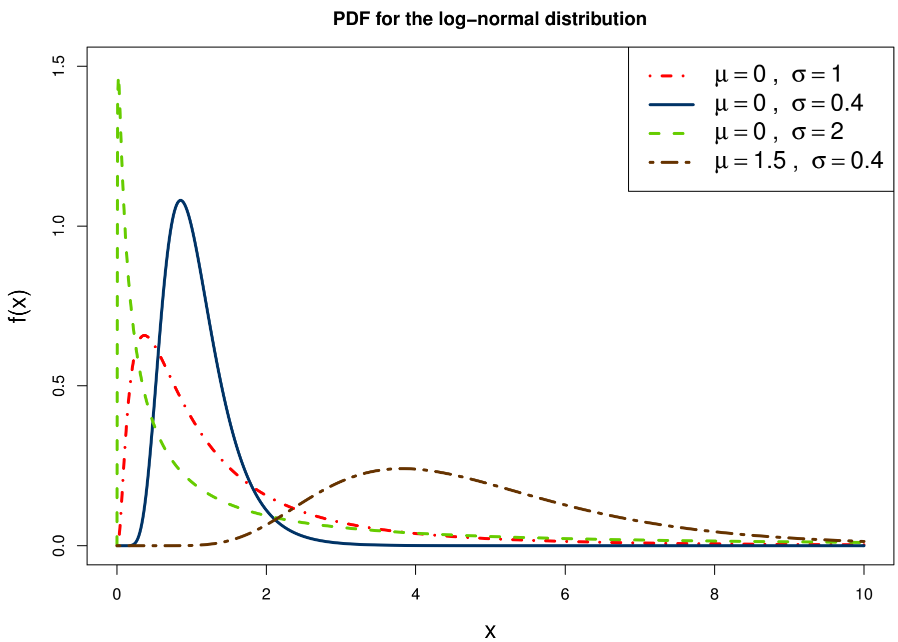

Figure 2.6: PDF of the log-normal distribution for various values of μ and σ.

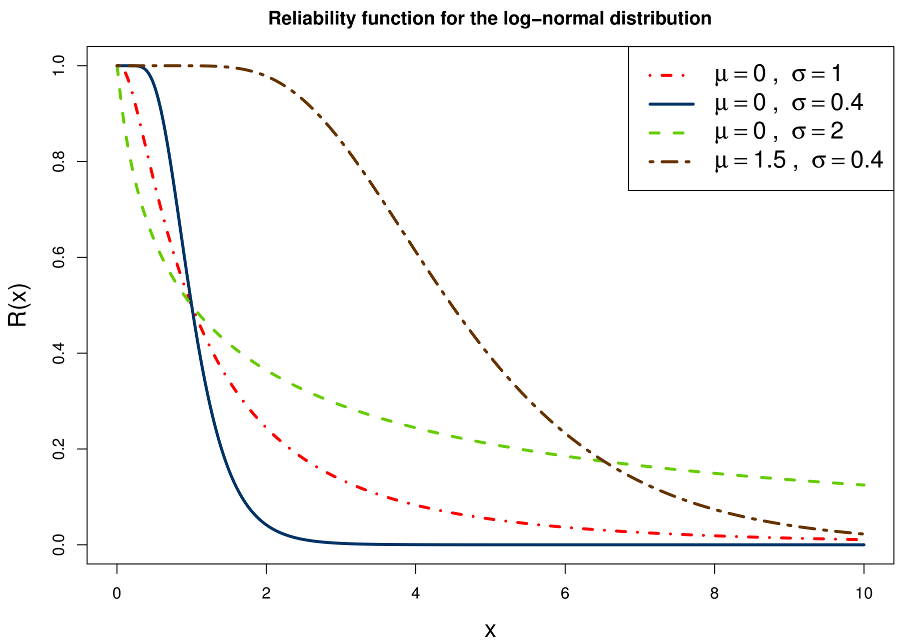

Figure 2.7: Reliability function of the log-normal distribution for various values of μ and σ.

Yang (2000) describes the log-normal distribution as a widely used distribution with applications in agriculture, geology, reliability, quality control, and numerous other disciplines. Concerning reliability and life testing, the log-normal distribution poses some competition for the Weibull distribution (Yang, 2000). An attractive feature of the log-normal distribution is that its hazard function makes an initial peak and then gradually decreases. According to John and Chen (2006), this feature is important when modelling failure times of shock absorbers, semi-conductors and other components that can be work-hardened. John and Chen (2006), Jordan (1978), and Mullen (1998) are but a few examples of where the log-normal distribution has been applied in reliability and life testing.

2.4.4 Birnbaum-Saunders Distribution

Let X be a continuous random variable following a Birnbaum-Saunders distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$. The PDF and reliability function are given, respectively, by

$$f(x) = \frac{x+\beta}{2\sqrt{2\pi}\alpha\sqrt{\beta}\sqrt{x^3}} \exp\left[-\frac{1}{2\alpha^2}\left(\frac{x}{\beta}+\frac{\beta}{x}-2\right)\right], \quad x > 0, \tag{2.10}$$

and

$$R(x) = 1 - \Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{x}{\beta}} - \sqrt{\frac{\beta}{x}}\right)\right], \tag{2.11}$$

where $\Phi(\cdot)$ denotes the CDF of the standard normal distribution. The Birnbaum-Saunders distribution has a mean of

$$E(X) = \beta\left(1 + \frac{\alpha^2}{2}\right),$$

and a variance of

$$\mathrm{Var}(X) = (\alpha\beta)^2\left(1 + \frac{5\alpha^2}{4}\right).$$

Figure 2.8 and Figure 2.9 show, respectively, the effects that the parameters α and β have on the PDF and reliability function of the Birnbaum-Saunders distribution.

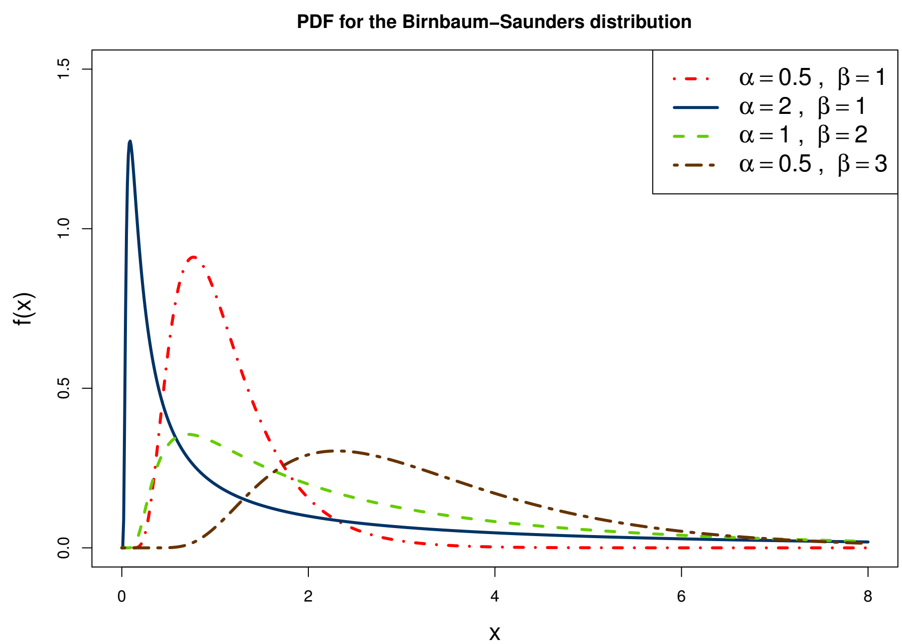

Figure 2.8: PDF of the Birnbaum-Saunders distribution for various values of α and β.

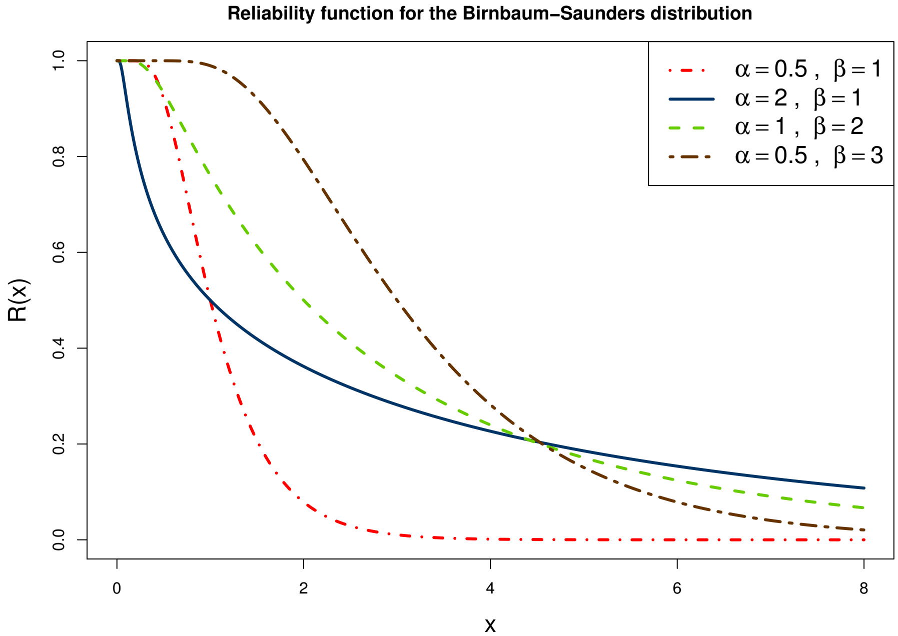

Figure 2.9: Reliability function of the Birnbaum-Saunders distribution for various values of α and β.

This distribution, also known as the fatigue life distribution, was first introduced by Birnbaum and Saunders (1969). This distribution was specifically developed by considering the characteristics of the fatigue process. A fatigue failure of an item under cyclic loading is observed, where a failure is caused by the origination, growth, and finally a substantial crack (Birnbaum and Saunders, 1969). A more general derivation, using a biological model, obtained by relaxing some of the assumptions made by Birnbaum and Saunders (1969), is provided in Desmond (1985).

2.5 Acceleration Models

ALT models are used to obtain failure data and perform reliability analysis about an item in a shorter amount of time than it would normally fail within. In order to do this, items are subjected to a more severe environment than the normal functioning environment. One or more stressors are applied or fluctuated at higher than normal use stress levels to induce early failures. Stressors can include temperature, pressure, voltage, wattage, humidity, loads, vibration amplitude, usage rates, etc. (Escobar and Meeker, 2006; Kececioglu, 2002).

An acceleration model is then used to make conclusions about the lifetimes of the items within their normal operating conditions. A functional relationship, known as an acceleration model or TTF, is assumed between the stressors and the parameters of the failure distribution (see, for example, Singpurwalla, 1971a; Singpurwalla, 1971b; Singpurwalla, 1973; Singpurwalla et al., 1975). This section focuses on the most commonly used models for acceleration: the Arrhenius, inverse power law, Eyring, and generalised Eyring models. Various other acceleration models are discussed in Escobar and Meeker (2006), Kececioglu (2002), and Thiraviam (2010), but many of these have only been implemented in isolated cases.

2.5.1 Stress Loading

Stress loading refers to the approach used when applying higher levels of stress to an item during ALT. According to Nelson (1990), how an item operates and other practical and theoretical limitations, may influence the type of stress loading used during the ALT process. Stress loading is discussed in terms of a single stressor, which can easily be generalised to multiple stressors.

Constant stress is the most frequently used stress loading, and is in most cases preferred above other types of stress loading. Each item is exposed to a constant application of some accelerated stressor levels until the item fails or the life test is terminated. This type of stress loading applies to items that also operate under constant use of stress. It is easy to maintain a constant stress level, and acceleration models for constant stress are well established and empirically verified (Nelson, 1990). A possible problem with constant stress is identifying the accelerated levels of stress that will be used (Hakim-Mashhadi, 1992). If stress levels are set too low, not many failures might occur during the testing period, and when stress levels are set too high, all items might instantly fail when the test starts.

In *step-stress* tests, items are subjected to increasing levels of stress as the life test runs. An item would be exposed to an initial constant level of stress for a certain time period. If the item did not fail, an increased constant level of stress is applied for the next time interval. The process of periodically increasing the applied stress step by step continues until all items have failed or the experiment is terminated. The advantage of step-stress loading is that it can yield more failures due to higher stress levels, which can be used to model lifetimes (Nelson, 1990). A disadvantage of using step-stress is that it may complicate the acceleration model and reliability estimation to a great degree (Hakim-Mashhadi, 1992; Mann and Singpurwalla, 1980; Nelson, 1990). The cumulative effect of the increasing stresses should be handled with care, especially if the items normally operate under constant stress.

Items to which *progressive stress* (also called ramp stress) loading is applied are exposed to continuously increasing levels of stress. For each item or group of items, a different rate at which the stress level increases may be used. According to Mann and Singpurwalla (1980), the stress level increases in line with some chosen function, where a linear increase is the elementary case. This type of stress loading should provide ample failure data, but it may be very difficult to properly control the progressively increasing application of stress. Fitting a model to this failure data and obtaining inference is complicated (Nelson, 1990).

Cyclic stress can also be used, provided that an item normally operates under a cyclic stress pattern. A sinusoidal (or wavelike) stress cycle is used in many cases, but the repeating cycle can also take on other shapes (Nelson, 1990). Some items might experience randomly changing stress levels in their normal functioning environment, for example, bridges or aeroplanes exposed to wind buffeting. It would then be appropriate to apply a *random stress* loading when performing ALT on such items.

2.5.2 Arrhenius Model

The Arrhenius acceleration model is a widely used TTF in ALT, which incorporates temperature as the accelerated stressor. This model is based on the Arrhenius law for chemical reaction rates, which is given by

$$\delta(T) = A \exp\left(-\frac{E}{KT}\right),$$

where
δ is the rate of reaction,
T is the absolute temperature (K),
K is Boltzman's constant (8.6171×10^{-5} eV/K),
A is a non-thermal constant,
E is the activation energy (eV).

According to Mann et al. (1974), the Arrhenius acceleration model describes the degradation rate of some item parameter as a function of the environmental temperature. The model is given by

$$\lambda_i = \theta_1 \exp\left(\frac{\theta_2}{T_i}\right),$$

(ReliaSoft, 2015) where λ_i is some life measure, T_i is the i^{th} level of the temperature stressor, θ_1 and θ_2 are unknown parameters that need to be estimated in order to make inferences about lifetimes at the normal use temperature T_u. Applications for this model are mostly associated with item failures due to degradation caused by metal diffusion or chemical reactions, and include battery cells, lubricants, plastics, electrical insulations, semi-conductors and incandescent lamp filaments (Hakim-Mashhadi, 1992; Nelson, 1990). The Arrhenius model has been used in conjunction with various life distributions, such as the exponential, normal, log-normal, Weibull and exponentiated Weibull distributions (see, for example, Barriga et al., 2008; Nelson and Kielpinski, 1976; Nelson, 1990).

2.5.3 Inverse Power Law Model

The inverse power law acceleration model, derived from kinetic theory and activation energy considerations, has been studied to a great extent. This is an acceleration model with a single non-thermal stressor, where typically voltage or pressure is used (Escobar and Meeker, 2006). The inverse power law model is given by

$$\lambda_i = \frac{1}{\theta_1 V_i^{\theta_2}},$$

(ReliaSoft, 2015) where λ_i is a life measure, V_i is the i^{th} level of the non-thermal stressor, θ_1 and θ_2 are model parameters to be estimated. Inferences can then be made about an item's life at the normal operating stress level V_u. Examples where the inverse power law model has been applied include metal fatigue testing, flash lamps, voltage endurance tests on electrical insulation, and capacitor failures (Ahmad, 1990; Nelson, 1990). The model has also been employed with various other stresses, such as thermal cycling in the Coffin-Manson relationship, load in Palmgren's equation, and velocity in Taylor's model (Nelson, 1990). The exponential, Weibull, log-normal and Birnbaum-Saunders distributions are examples of life distributions where the inverse power law model has been applied (Nelson, 1990; Owen and Padgett, 2000).

2.5.4 Eyring Model

The Eyring relationship is derived from quantum mechanic principles and describes a reaction rate for chemical degradation, which is influenced by the environmental temperature. This relationship is given by

$$\delta(T) = \frac{A}{T} \exp\left(-\frac{E}{KT}\right),$$

where
δ is the rate of reaction,
T is the absolute temperature (K),
K is Boltzman's constant (8.6171×10^{-5} eV/K),
A is a constant which takes into account the item and testing methodology,
E is the activation energy (eV).

The Eyring acceleration model, which is based on the Eyring relationship, has temperature as the stressor. The model is given by

$$\lambda_i = \frac{1}{T_i} \exp\left(-\theta_1 + \frac{\theta_2}{T_i}\right),$$

(ReliaSoft, 2015) where λ_i is some life measure, T_i is the i^{th} level of the temperature stressor, θ_1 and θ_2 unknown parameters. Nelson (1990) describes the Eyring model as an alternative to Arrhenius model for temperature acceleration, and states that both models fit the data equally well in most applications. The Arrhenius model is, however, the more popular choice between the two. See, for example, Mann et al. (1974), Nelson (1990), and Thiraviam (2010) for various other parameterisations of the Eyring model. This model has also been used with various life distributions, in particular the exponential, Weibull and log-normal distributions.

2.5.5 Generalised Eyring Model

The generalised Eyring model is an extension of the Eyring model, which allows for an acceleration model with more than one stressor. In most cases only one additional stress is added, since the model becomes very complex with more stressors. The generalised Eyring model used in this thesis incorporates two stressors, one thermal and one non-thermal, and is given by

$$\lambda_i = \frac{1}{T_i} \exp\left(\theta_1 + \frac{\theta_2}{T_i} + \theta_3 V_i + \frac{\theta_4 V_i}{T_i}\right),$$

(ReliaSoft, 2015) where λ_i is some life measure, θ_1, θ_2, θ_3, θ_4 are unknown parameters, T_i is the i^{th} level of the temperature stressor, and V_i is a function of the i^{th} level of the non-thermal stressor, S_i (Escobar and Meeker, 2006). The four parameters are estimated to make inferences under normal use conditions T_u and S_u. Examples where the generalised Eyring model has been used for accelerated testing include capacitors, failures caused by electro-migration, epoxy packaging for electronics, and rupture of solids (Hakim-Mashhadi, 1992). The non-thermal stressors in each of the aforementioned examples are, respectively, voltage, current density, relative humidity, and tensile stress. Refer to Kececioglu (2002), Mann et al. (1974) and Nelson (1990) for other parameterisations of the generalised Eyring model. The exponential, Weibull and log-normal distibutions have been used with the generalised Eyring model, but papers on this model are not as common as for the other acceleration models, due to its complexity.

2.6 Bayesian Considerations

Bayesian methods for ALT are the main focal point in this thesis. Bayesian statistics is built on the work by Thomas Bayes (1701-1761), which was published posthumously in 1763. In this work, he described that a researcher's understanding or subjective beliefs could be used in statistical inference. Prior knowledge or beliefs are stated concerning an event, and then changed or updated to posterior beliefs as more information is gathered. In frequentist inference, model parameters are fixed, whereas, with Bayesian inference, probabilities are assigned to the parameters. Hamada et al. (2008) describe that Bayesians combine prior knowledge of the parameters (usually defined by a probability density function) with information gathered from an experiment to update their beliefs about the parameters, which is expressed as a probability density function. Using Bayes' theorem, the posterior distribution, denoted by $\pi(\boldsymbol{\theta}|\underline{x})$, is given by

$$\pi(\boldsymbol{\theta}|\underline{x}) = \frac{\pi(\boldsymbol{\theta}) L(\underline{x}|\boldsymbol{\theta})}{\int \pi(\boldsymbol{\theta}) L(\underline{x}|\boldsymbol{\theta}) \, d\boldsymbol{\theta}},$$

where $\boldsymbol{\theta}$ is the parameter vector, $\underline{x}$ is the data, $\pi(\boldsymbol{\theta})$ denotes the prior distribution, $L(\underline{x}|\boldsymbol{\theta})$ is the likelihood function of a model $f(x|\boldsymbol{\theta})$, and the denominator is the normalising constant. The unnormalised posterior distribution can be written as

$$\begin{aligned}\pi(\boldsymbol{\theta}|\underline{x}) &\propto \pi(\boldsymbol{\theta})L(\underline{x}|\boldsymbol{\theta}) \\ \text{posterior} &\propto \text{prior} \times \text{likelihood}.\end{aligned}$$

One of the most debated topics in Bayesian statistics is selecting prior distributions for the model parameters. An expert's opinion or historical field data can be used to impose subjective priors. Experienced engineers, for example, have prior knowledge regarding the life of an item (whether that knowledge is based on the physics of failure or on historical failure data), and this can be reflected in the model by choosing an appropriate subjective prior (Meeker and Escobar, 1998). The alternative is to use objective priors, also in many texts referred to as non-informative, flat or vague priors. Examples of objective priors include the uniform, reference, Jeffreys and maximal data information priors. Refer to Berger et al. (2015), Ghosh (2011), and Irony and Singpurwalla (1997) for more information and a general discussion on objective priors. Yang and Berger (1998) provide a catalogue of non-informative priors for various distributions and models, favouring the reference and Jeffreys priors. Note that many objective priors are improper, meaning that is it extremely important to assess the propriety of the posterior in such cases.

2.7 Markov Chain Monte Carlo Methods

In many complicated Bayesian models, one finds an intractable posterior distribution that can not be written in closed form. Determining marginal posterior distributions or even the normalising constant might prove difficult in such cases, making Bayesian inference challenging. This has led to the development of MCMC methods, which are computational algorithms that can be used to generate samples from complex posterior distributions (Hamada et al., 2008). A Markov chain is a sequence of random variables $X^{(1)}, X^{(2)}, \ldots$, where for some t, the next state $X^{(t+1)}$ only depends on the current state $X^{(t)}$, and not on all the previous states $X^{(1)}, X^{(2)}, \ldots, X^{(t-1)}$. The Markov property can be expressed as

$$P\left(X^{(t+1)} = x \middle| X^{(1)} = x^{(1)}, X^{(2)} = x^{(2)}, \ldots, X^{(t)} = x^{(t)}\right) = P\left(X^{(t+1)} = x \middle| X^{(t)} = x^{(t)}\right).$$

For some arbitrary starting value $X^{(0)}$, a Markov chain $\left\{X^{(t)}\right\}$ can be generated using a transition distribution, that has stationary distribution f, to guarantee that $\left\{X^{(t)}\right\}$ converges in distribution to a random variable from f (Robert and Casella, 1999). In general, an MCMC method which simulates values from some distribution f is any method that generates an ergodic Markov chain $\left\{X^{(t)}\right\}$ which has the stationary distribution f. This implies that calculating the expected value for some function

$g(x)$, given by

$$\mathrm{E}\left[g(x)\right] = \int g(x) f(x)\, dx,$$

can be approximated by the Monte Carlo average resulting from an MCMC algorithm as

$$\mathrm{E}\left[g(x)\right] \approx \frac{1}{M} \sum_{m=1}^{M} g(X^{(m)}),$$

for a large value M. The MCMC methods discussed in this thesis include the Metropolis-Hastings sampler, Gibbs sampler, ARS, ARMS, and slice sampling.

2.7.1 Metropolis-Hastings Sampler

The Metropolis-Hastings algorithm is a frequently used type of MCMC sampler due to its simplicity and generality. This algorithm is based on the work of Metropolis et al. (1953) and Hastings (1970), where parameter values are drawn from approximate distributions and then adjusted to better approximate the posterior distribution of interest. The approximate distributions eventually converge to the posterior distribution. Sequential sampling is applied, where each subsequent drawing only depends on the previous draw, so that the draws form a Markov chain.

The basic idea behind MCMC algorithms is to simulate parameter values from the posterior distribution. Inferences are then based on these simulated samples. Let $\pi(\boldsymbol{\theta}\,|\underline{x})$ be the posterior distribution (also known as the target density) with a d-dimensional parameter vector $\boldsymbol{\theta} = \{\theta_1, \theta_2, \ldots, \theta_d\}$. Denote the t^{th} draw in the sequence by $\boldsymbol{\theta}^{(t)}$. As t becomes large, the distribution of the sampled values converges to the target density. The Metropolis-Hastings algorithm is performed as follows:

1. Select a starting value, denoted by $\boldsymbol{\theta}^{(0)}$, for which $\pi\left(\boldsymbol{\theta}^{(0)}\,|\underline{x}\right) > 0$. This can be done by drawing from a starting distribution or simply choosing some rough estimate (see, for example, Gelman et al. (2014) for more information on selecting the starting value).

2. Generate a candidate point (or proposal) $\boldsymbol{\theta}^{*}$ from the previous state in the sequence $\boldsymbol{\theta}^{(t-1)}$ with a jumping distribution (or proposal distribution), which is a conditional density denoted by $q\left(\boldsymbol{\theta}^{*}\middle|\boldsymbol{\theta}^{(t-1)}\right)$, at time t. Denote by $q\left(\boldsymbol{\theta}^{(t-1)}\,|\boldsymbol{\theta}^{*}\right)$ the probability of jumping from $\boldsymbol{\theta}^{*}$ back to $\boldsymbol{\theta}^{(t-1)}$. The proposal distribution should be easy to simulate from and must satisfy certain conditions to be a valid choice, that is, the resulting Markov chain must be irreducible and aperiodic.

3. Calculate the probability that the candidate point will be accepted as the next state in the se-

quence. This is known as the acceptance probability, denoted by ρ, and defined as

$$\rho = \min\left(1, \frac{\pi\left(\boldsymbol{\theta}^* \left| \underline{x}\right.\right) q\left(\boldsymbol{\theta}^{(t-1)} \left| \boldsymbol{\theta}^*\right.\right)}{\pi\left(\boldsymbol{\theta}^{(t-1)} \left| \underline{x}\right.\right) q\left(\boldsymbol{\theta}^* \middle| \boldsymbol{\theta}^{(t-1)}\right)}\right).$$

Hamada et al. (2008) state that the ratio $\frac{\pi\left(\boldsymbol{\theta}^* | \underline{x}\right)}{\pi\left(\boldsymbol{\theta}^{(t-1)} | \underline{x}\right)}$ drives the algorithm to jump to parameter values resulting in high posterior probabilities, and the ratio $\frac{q\left(\boldsymbol{\theta}^{(t-1)} | \boldsymbol{\theta}^*\right)}{q\left(\boldsymbol{\theta}^* \middle| \boldsymbol{\theta}^{(t-1)}\right)}$ reflects how the proposal distribution might prefer certain values for the parameters over others.

4. Simulate a value u from a $U(0,1)$ distribution. If $u \leq \rho$, the candidate point is accepted as the next state in the sequence. The state of the sequence remains unchanged (the candidate point is rejected) if $u > \rho$. This can be expressed as

$$\boldsymbol{\theta}^{(t)} = \begin{cases} \boldsymbol{\theta}^* & \text{with probability } \rho \\ \boldsymbol{\theta}^{(t-1)} & \text{with probability } 1-\rho. \end{cases}$$

5. Move to the next state in the sequence by setting $t = t+1$ and returning to Step 2. Repeat a large number, say N_{MH}, of times, resulting in the Markov chain $\boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}, \ldots, \boldsymbol{\theta}^{(N_{MH})}$.

For the proofs regarding the convergence of the Markov chain in the Metropolis-Hastings algorithm, and for adaptations and optimisation of the algorithm, the reader is referred to Robert and Casella (1999).

2.7.2 Gibbs Sampler

The Gibbs sampler was first introduced by Geman and Geman (1984) and is one of the most popular MCMC algorithms. The Gibbs sampler creates a Markov chain by separating the parameter vector into subvectors, and then sampling each subvector conditional only on the current values of all the other subvectors. Let $\pi\left(\boldsymbol{\theta} \left| \underline{x}\right.\right)$ be the posterior distribution with a d-dimensional parameter vector $\boldsymbol{\theta} = \{\theta_1, \theta_2, \ldots, \theta_d\}$, and denote the full conditional posterior distributions by $\pi\left(\theta_i \left| \underline{x}, \theta_1, \theta_2, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_d\right.\right)$. Both the posterior and full conditional posterior distributions need only be defined at least up to proportionality. The Gibbs sampler is performed as follows:

1. Select arbitrary starting values $\boldsymbol{\theta}^{(0)} = \left\{\theta_1^{(0)}, \theta_2^{(0)}, \ldots, \theta_d^{(0)}\right\}$ for each parameter.

2. Sample new values for each parameter from the corresponding full conditional posterior distribution. The full conditional distributions are updated for each new parameter value sampled.

This is done in succession as

$$
\begin{array}{lll}
\text{Sample } \theta_1^{(1)} & \text{from} & \pi\left(\theta_1 \left| \underline{x}, \theta_2^{(0)}, \theta_3^{(0)}, \ldots, \theta_d^{(0)}\right.\right) \\
\text{Sample } \theta_2^{(1)} & \text{from} & \pi\left(\theta_2 \left| \underline{x}, \theta_1^{(1)}, \theta_3^{(0)}, \ldots, \theta_d^{(0)}\right.\right) \\
\cdots & \cdots & \cdots \\
\text{Sample } \theta_i^{(1)} & \text{from} & \pi\left(\theta_i \left| \underline{x}, \theta_1^{(1)}, \theta_2^{(1)}, \ldots, \theta_{i-1}^{(1)}, \theta_{i+1}^{(0)}, \ldots, \theta_d^{(0)}\right.\right) \\
\cdots & \cdots & \cdots \\
\text{Sample } \theta_d^{(1)} & \text{from} & \pi\left(\theta_d \left| \underline{x}, \theta_1^{(1)}, \theta_2^{(1)}, \ldots, \theta_{d-1}^{(1)}\right.\right).
\end{array}
$$

After a new value is sampled for each parameter to find $\boldsymbol{\theta}^{(1)} = \left\{\theta_1^{(1)}, \theta_2^{(1)}, \ldots, \theta_d^{(1)}\right\}$, one iteration of the Gibbs sampler has been completed.

3. Repeat Step 2 a large number of times, say N_G times, conditioning only on the most recent values of all other parameters, to obtain a Markov chain of simulated parameter values $\boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}, \ldots, \boldsymbol{\theta}^{(N_G)}$.

The Gibbs sampler reduces the difficulty of working with multi-parameter problems, but requires that the full conditional posterior distributions are known and easy to sample from. Refer to Robert and Casella (1999) for complete proofs on the convergence of the Markov chain generated by the Gibbs sampler.

2.7.3 Adaptive Rejection Sampling

ARS, introduced by Gilks and Wild (1992), is an extension to the Gibbs sampler which can be employed when it is impossible to directly sample from the full conditional posterior distributions. This technique incorporates rejection sampling (together with the Gibbs sampler) by drawing candidate points from an uncomplicated proposal distribution, and then accepting or rejecting the candidate points based on some decision rule. Furthermore, an adaptive strategy is used, in which the proposal distribution is updated with the most recently generated sample, and an improved proposal distribution is constructed via more support points when a candidate point is rejected (Martino et al., 2015). The limitation of ARS is that it can only be applied in cases where the full conditional posterior distributions are log-concave.

In order to emphasise the rejection sampling portion of this algorithm, some simplified notation is adopted. Denote by $p(\theta)$ a full conditional posterior distribution $\pi\left(\theta_i \left| \underline{x}, \theta_1^{(t)}, \theta_2^{(t)}, \ldots, \theta_{i-1}^{(t)}, \theta_{i+1}^{(t-1)}, \right.\right.$ $\left.\left.\ldots, \theta_d^{(t-1)}\right)\right.$, which is the target density. The adaptive algorithm creates a chain of proposal distributions, denoted by $\{\pi_{t_A}(\theta)\}$, with $t_A = 1, 2, \ldots, N_A$ and N_A is the number of iterations of the adaptive algorithm. Denote by N_{ARS} the number of samples generated by the ARS algorithm, where $N_A \geq N_{ARS}$. The support points at the t_A^{th} iteration are given by $S_{t_A} = \{s_1, s_2, \ldots, s_{m_{t_A}}\}$, where $s_1 < s_2 < \ldots < s_{m_{t_A}}$, and m_{t_A} is the number of support points. Define $l(\theta) = \ln[p(\theta)]$ and $w_h(\theta)$ as the tangent line of $l(\theta)$ at support point s_h for $h = 1, 2, \ldots, m_{t_A}$. A piecewise linear function can be constructed as

$$W_{t_A}(\theta) = \min\left[w_1(\theta), w_2(\theta), \ldots, w_{m_{t_A}}(\theta)\right], \tag{2.12}$$

so that the proposal distribution $\pi_{t_A}(\theta) = \exp[W_{t_A}(\theta)]$. The proposal distribution is thus piecewise exponential and is set up in such a way that $W_{t_A}(\theta) \geq l(\theta)$ when $l(\theta)$ is concave, or equivalently, $\pi_{t_A}(\theta) \geq p(\theta)$ when $p(\theta)$ is log-concave. Refer to Gilks and Wild (1992) for more information on how the proposal distribution is approximated by a piecewise exponential function. Finally, the ARS algorithm can be performed as follows:

1. Set $t = 0$ and $t_A = 0$. Select a set of initial support points $S_0 = \{s_1, s_2, \ldots, s_{m_0}\}$.

2. Construct a proposal distribution $\pi_{t_A}(\theta)$ for the support points $S_{t_A} = \{s_1, s_2, \ldots, s_{m_{t_A}}\}$ conforming to (2.12).

3. Generate a candidate point θ^* from $\pi_{t_A}(\theta)$, and generate a value u from a $U(0,1)$ distribution.

4. If $u > \frac{p(\theta^*)}{\pi_{t_A}(\theta^*)}$, reject the canditate point θ^*. Update the support points $S_{t_A+1} = S_{t_A} \cup \{\theta^*\}$, $m_{t_A+1} = m_{t_A} + 1$. Increment the algorithm iteration $t_A = t_A + 1$ and go back to Step 2.

5. Otherwise, if $u \leq \frac{p(\theta^*)}{\pi_{t_A}(\theta^*)}$, accept the candidate point θ^* by setting $\theta^{(t)} = \theta^*$. The support points are left unchanged, $S_{t_A+1} = S_{t_A}$, $m_{t_A+1} = m_{t_A}$. Increment the algorithm iteration $t_A = t_A + 1$ and the sample count $t = t + 1$. Stop the algorithm if N_{ARS} samples have been generated, otherwise return to Step 2.

Parameter values are sampled in succession as in the Gibbs sampler, where the proposal distribution is updated every time a candidate point is accepted or rejected. An attractive property of ARS is that the proposal distribution is improved when a candidate point is rejected. According to Martino et al. (2015), the sequence of proposal distributions $\{\pi_{t_A}(\theta)\}$ converges very quickly to the target density.

2.7.4 Adaptive Rejection Metropolis Sampling

Gilks et al. (1995) propose ARMS, a generalisation of ARS that can be used to sample from full conditional posterior distributions even when they are non-log-concave. This sampling algorithm combines ARS with the Metropolis-Hastings sampler to present a universal algorithm operating within the Gibbs sampler (Martino et al., 2015). With ARMS, rejection sampling is performed, where rejected candidate points are used to construct an improved proposal distribution via additional support points (as in ARS), and accepted candidate points undergo a Metropolis-Hastings step.

Again, for the sake of generality, simplified notation is used to discuss the ARMS algorithm. Let $p(\theta)$ denote a full conditional posterior distribution (the target density), and denote the chain of proposal distributions generated with the algorithm by $\{\pi_{t_A}(\theta)\}$, with $t_A = 1, 2, \ldots, N_A$ and N_A being the number of iterations of the algorithm. Denote by N_{ARMS} the number of samples generated by the ARMS algorithm, where $N_A \geq N_{ARMS}$, and let $l(\theta) = \ln[p(\theta)]$. The support points at the t_A^{th} iteration is given by $S_{t_A} = \{s_1, s_2, \ldots, s_{m_{t_A}}\}$, where $s_1 < s_2 < \ldots < s_{m_{t_A}}$, and m_{t_A} is the number of support points. Define intervals based on the support points as $I_0 = (-\infty, s_1]$, $I_h = (s_h, s_{h+1}]$ for $h = 1, 2, \ldots, m_{t_A} - 1$, and $I_{m_{t_A}} = \left(s_{m_{t_A}}, \infty\right)$. Let $w_{h,h+1}(\theta)$ be the straight line that passes through the coordinates $\left\{s_h, l(s_{h)}\right\}$ and $\left\{s_{h+1}, l(s_{h+1)}\right\}$ for $h = 1, 2, \ldots, m_{t_A} - 1$. Afterwards, construct a piecewise linear function $W_{t_A}(\theta)$ as

$$W_{t_A}(\theta) = \begin{cases} w_{1,2}(\theta) & \theta \in I_0 \\ \max\left[w_{1,2}(\theta), w_{2,3}(\theta)\right] & \theta \in I_1 \\ \max\left[w_{h,h+1}(\theta), \min\left(w_{h-1,h}(\theta), w_{h+1,h+2}(\theta)\right)\right] & \theta \in I_h \\ \max\left[w_{m_{t_A}-2, m_{t_A}-1}(\theta), w_{m_{t_A}-1, m_{t_A}}\right] & \theta \in I_{m_{t_A}-1} \\ w_{m_{t_A}-1, m_{t_A}} & \theta \in I_{m_{t_A}}, \end{cases} \tag{2.13}$$

which is used to approximate $l(\theta)$. This means that the proposal distribution $\pi_{t_A}(\theta) = \exp[W_{t_A}(\theta)]$ is expressed as a piecewise exponential function. For more clarity on how to construct the above piecewise exponential function, refer to Gilks et al. (1995), and Martino et al. (2015). The ARMS algorithm is performed as follows:

1. Set $t = 0$ and $t_A = 0$. Select a starting value $\theta^{(0)}$ and a set of initial support points $S_0 = \{s_1, s_2, \ldots, s_{m_0}\}$.

2. Construct a proposal distribution $\pi_{t_A}(\theta)$ for the support points $S_{t_A} = \{s_1, s_2, \ldots, s_{m_{t_A}}\}$ according to (2.13).

3. Generate a candidate point θ^* from $\pi_{t_A}(\theta)$, and generate a value u_1 from a $U(0,1)$ distribution.

4. If $u_1 > \frac{p(\theta^*)}{\pi_{t_A}(\theta^*)}$, reject the canditate point θ^*. Update the support points $S_{t_A+1} = S_{t_A} \cup \{\theta^*\}$, $m_{t_A+1} = m_{t_A} + 1$. Increment the algorithm iteration $t_A = t_A + 1$ and go back to Step 2.

5. Otherwise, if $u_1 \leq \frac{p(\theta^*)}{\pi_{t_A}(\theta^*)}$ simulate a value u_2 from a $U(0,1)$ distribution and calculate the Metropolis-Hastings step acceptance probability ρ given by

$$\rho = \min\left(1, \frac{p(\theta^*) \min\left[p(\theta^{(t)}), \pi_{t_A}\left(\theta^{(t)}\right)\right]}{p(\theta^{(t)}) \min\left[p(\theta^*), \pi_{t_A}(\theta^*)\right]}\right).$$

   If $u_2 \leq \rho$ the candidate point is accepted by setting $\theta^{(t+1)} = \theta^*$. Otherwise, if $u_2 > \rho$, reject the candidate point and set $\theta^{(t+1)} = \theta^{(t)}$.

6. The support points remain unchanged with $S_{t_A+1} = S_{t_A}$, $m_{t_A+1} = m_{t_A}$.Afterwards, increment the sample count $t = t+1$ and the algorithm iteration $t_A = t_A + 1$. Stop the algorithm if N_{ARMS} samples have been generated, otherwise return to Step 2.

ARMS is executed within the Gibbs sampler, meaning that parameter values are sampled sequentially and not as a parameter vector. Martino et al. (2015) state that the convergence of the proposal distribution to the target density is not always guaranteed, there are certain cases where convergence is not achieved. This mainly occurs when candidate points are recurrently accepted in the rejection sampling step but rejected in the Metropolis-Hastings step, resulting in the proposal distribution never being updated. A large number of initial support points and other considerations discussed in Gilks et al. (1995) can be used to reduce the Metropolis-Hastings rejection probability. This, in turn, could help to prevent the occurrence of non-convergence.

2.7.5 Slice Sampling

Neal (2003) presents the slice sampling algorithm, which is an MCMC algorithm that can sample from a wide range of distributions. The slice sampler is based on the principle that it is possible to sample from a distribution by sampling uniformly from the region under the curve of the PDF. By alternating between uniform sampling vertically and from horizontal slices defined by the vertical point, a Markov chain can be constructed which will converge to the particular uniform distribution (Neal, 2003). The author states that the slice sampling algorithm is an alternative to the Gibbs sampler and is more efficient than Metropolis procedures.

Here, the slice sampling algorithm is discussed in its simplest form, by considering only one variable. For sampling from a specific PDF, let $f(\theta)$ denote a function that is proportional to this PDF. The slice sampling algorithm can be performed as follows:

1. Evaluate the point $f(\theta^{(t)})$. If this is the first iteration, a starting value $\theta^{(0)}$ must be selected.

2. Generate a value u_1 from a $U(0, f(\theta^{(t)}))$ distribution and define a horizontal slice, denoted by S, as $S = \{\theta : u_1 < f(\theta)\}$. It is noted that $\theta^{(t)}$ is always included in S.

3. Construct an interval, $I_{Slice} = (L, R)$, around $\theta^{(t)}$ such that the interval contains all, or at least much, of the slice S. Including as much of the slice as possible allows a new point generated $\theta^{(t+1)}$ to differ more from the current point $\theta^{(t)}$, but defining an interval considerably larger than the slice may lead to less efficient sampling for the next point $\theta^{(t+2)}$. A stepping out procedure can be utilised to construct a suitable interval as follows:

   - Set a value for w, which is an estimate for the width of the slice.
   - Generate a value u_2 from a $U(0,1)$ distribution.

- Set the initial values for the interval bounds as $L = \theta^{(t)} - wu_2$ and $R = L + w$. As a result, an interval of width w is positioned randomly around the point $\theta^{(t)}$.
- While $u_1 < f(L)$, set $L = L - w$. This implies that the lower bound is moved to the left by a step size of w, until the bound is outside the slice.
- While $u_1 < f(R)$, set $R = R + w$. This implies that the upper bound is moved to the right by a step size of w, until the bound is outside the slice.

4. The new point $\theta^{(t+1)}$ is now generated from the part of the slice that is included in the interval I_{Slice}. In order to find the new point, more efficiently than simply sampling uniformly from I_{Slice} until a point is generated within S, a shrinkage procedure can be applied as follows:

   - Define interval bounds L' and R' for the procedure by setting their initial values equal to that of the interval I_{Slice}. Thus, $L' = L$ and $R' = R$.
   - Generate a value u_3 from a $U(0,1)$ distribution.
   - Generate a candidate point, denoted by θ^*, randomly from the interval (L', R') by setting $\theta^* = L' + u_3(R' - L')$.
   - If $u_1 < f(\theta^*)$, accept the candidate point by setting $\theta^{(t+1)} = \theta^*$.
   - Otherwise, shrink the sampling interval by either setting $L' = \theta^*$ if $\theta^* < \theta^{(t)}$, or $R' = \theta^*$ if $\theta^* > \theta^{(t)}$. Repeat from the second point until a candidate point is accepted.

5. Move to the next state in the sequence by setting $t = t + 1$ and return to Step 1. Repeat a large number, say N_{Slice}, of times, resulting in the Markov chain of generated values $\theta^{(1)}$, $\theta^{(2)}$, ..., $\theta^{(N_{Slice})}$.

Neal (2003) also discusses two situational procedures and a doubling procedure to find a suitable interval, I_{Slice}, in Step 3. In the doubling procedure, a sequence of intervals is created by doubling the width of I_{Slice} until an interval is produced where both bounds are outside the slice. The bounds of I_{Slice} are not equally increased, but rather the bound that will be increased is randomly chosen (even if that bound is already outside the slice). The interval I_{Slice} may be obtained faster when using the doubling procedure, in comparison with using the stepping out procedure, if the estimate of the slice width w is too small.

The slice sampling algorithm can also be used to sample from multivariate distributions by simply sampling each variable in turn, similar to Gibbs sampling. Alternatively, Neal (2003) presents generalisations of the algorithm which can simultaneously sample values for all variables directly from the multivariate distribution. The author also discusses approaches that suppress random walks in the univariate and multivariate case, which can improve the efficiency of the sampling algorithm.

The main advantage of the slice sampling algorithm is that it can be used to sample from any continuous distribution, given that a function that is proportional to the PDF can be evaluated. The strengths and weaknesses of slice sampling methods, compared to other widely used MCMC methods such as Metropolis methods, ARS and ARMS, are also discussed in Neal (2003)

2.8 MCMC Convergence in OpenBUGS

The Markov chain generated by most MCMC methods is guaranteed, under some conditions, to converge to the posterior distribution. However, when working with complex models, it may be unclear in some cases whether these conditions are satisfied (Lesaffre and Lawson, 2012). There is also uncertainty with regards to when the convergence of the Markov chain occurs.

When using MCMC techniques to generate posterior samples for inference, only samples obtained from a converged Markov chain are valid. Usually, a Markov chain is initiated to generate a sufficiently large number of samples, after which the first portion of the samples are discarded. The discarded samples are called the *burn-in*, which consists of the invalid samples where the Markov chain has not yet converged to the posterior distribution. It is thus important to be able to assess the convergence of a Markov chain, in order to set an appropriate burn-in for the chain. In this thesis, we focus on some MCMC convergence diagnostics available in the OpenBUGS software. Various other graphical approaches and formal diagnostic tests for convergence are discussed in Lesaffre and Lawson (2012).

2.8.1 Trace Plots

Trace plots offer an intuitive graphical approach to assess the convergence of a Markov chain. Trace plots monitor the movement of the chain from one iteration to the next, individually for each parameter. When a chain has converged, the trace plot will display a horizontal strip with few specific moves outside of this strip. This method of assessing convergence is sometimes referred to as a *thick pen test*, in the sense that when the chain has converged it can be covered when holding a think pen over it (Lesaffre and Lawson, 2012). Figure 2.10 shows examples of assessing convergence with trace plots. In the first trace plot the chain has achieved convergence and the thick pen test is passed. The chain in the second trace plot has not converged, seeing that the movement of the chain does not occur in a horizontal strip.

Although the thick pen test is an informal method of evaluating convergence, it is widely used in practice and considered a starting point when conducting research. Very valuable information can also be acquired from trace plots. A Markov chain that has not converged can easily be identified when there is a trend present in the movement of the chain. The trace plot can also show how efficiently the chain is exploring the posterior distribution, referred to as the *mixing rate* of the chain (Lesaffre and Lawson, 2012).

The use of multiple chains in assessing convergence using trace plots is recommended in Gelman et al. (2014), and also discussed in Ntzoufras (2009) and Spiegelhalter et al. (2014). The idea is to run multiple chains simultaneously, where overdispersed starting values are chosen for the different chains. The chains are represented on the same trace plot, where convergence of the Markov chain is achieved when the different chains converge, mix and pass the thick pen test. Figure 2.11 shows examples of examining convergence with trace plots, where multiple chains are used. In the first trace plot the chains are mixing and are said to have converged. The chains in this plot pass the thick pen test. The chains in the second trace plot have not converged. The blue chain is moving towards the red chain, but mixing has not occured yet.

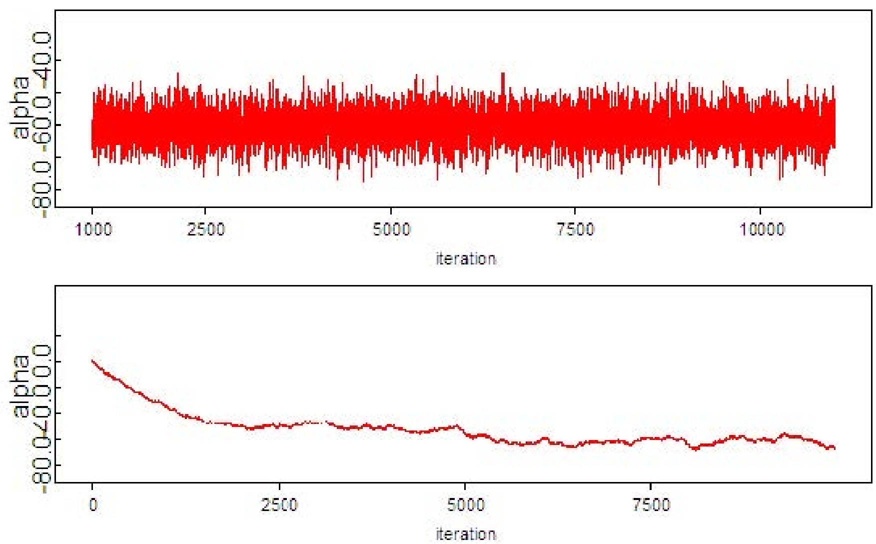

Figure 2.10: Single chain trace plots (Images from Spiegelhalter et al., 2014).

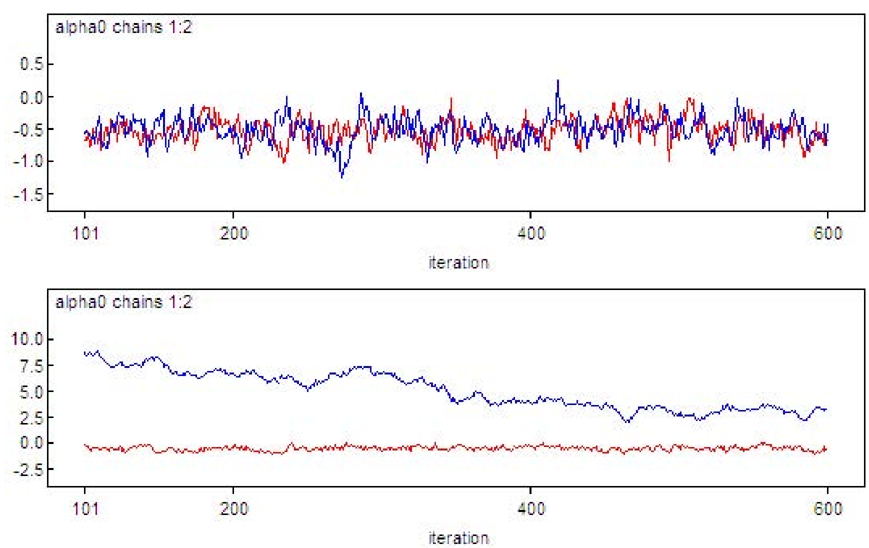

Figure 2.11: Multiple chain trace plots (Images from Spiegelhalter et al., 2014).

2.8.2 Brooks-Gelman-Rubin (BGR) Approach

Brooks and Gelman (1998) present a modified version of the Gelman-Rubin statistic, introduced in Gelman and Rubin (1992), to more formally assess the convergence of Markov chains generated by MCMC methods. For this method, multiple Markov chains are generated, using overdispersed starting values, and divided into subchains. The convergence is then assessed by comparing the within-chain and between-chain variability of the subchains.

The BGR approach is an interval diagnostics tool, and is implemented by considering the variability in the second half of the chains as follows (Lesaffre and Lawson, 2012; Spiegelhalter et al., 2014).

1. Generate M Markov chains for the parameter of interest, where each chain has a length of $2T$ iterations.

2. Calculate the empirical $100(1-\alpha)\%$ credible interval for each chain from the last T iterations of the chain.

3. Calculate the average width of the M within-chain intervals from Step 2 and denote this by W.

4. Pool all the last T iterations of the M chains together. Calculate the width of the empirical credible interval based on these pooled MT iterations, and denote this by B.

5. The BGR convergence statistic is then given by

$$R = \frac{B}{W},$$

   where $R \to 1$ as convergence is approached.

A dynamic version of R is implemented in OpenBUGS, where the chains are divided into cumulative subchains and then R is calculated for these subchains. Figure 2.12 presents an example of the BGR plot generated by OpenBUGS, which shows the values of R in red, the normalised values of B in green, and the normalised values of W in blue. The dotted line indicates the value 1. In order for convergence to be reached, it is not only important to verify that R converges to 1, but also to confirm that B and W converge and stabilise (Brooks and Gelman, 1998).

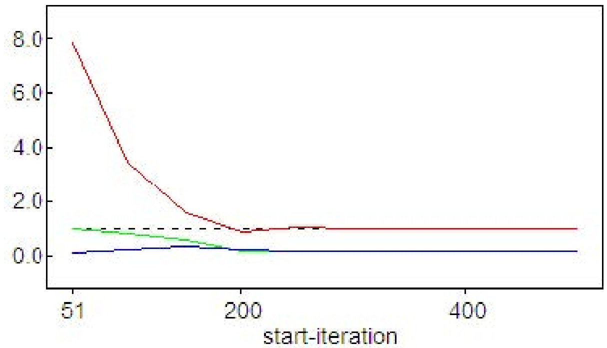

Figure 2.12: Example of a BGR plot, where convergence occurs around 250 iterations (Image from Spiegelhalter et al., 2014).

2.8.3 Monte Carlo Error

After convergence has been achieved, samples are generated to perform statistical inference. The accuracy of the posterior estimates then depend on the number of samples generated. It is important to monitor this accuracy, which can be done with the Monte Carlo error. According to Spiegelhalter et al. (2014), the rule of thumb is that the Markov chain should continue running at least until the Monte Carlo error is less than 5% of the sample standard deviation for each parameter of interest.

The sample standard and the Monte Carlo error are given in OpenBUGS as part of the summary statistics. The Monte Carlo error is estimated in OpenBUGS by the method of batch means, but a time series approach can also be used.

2.9 Deviance Information Criterion

The DIC, proposed by Spiegelhalter et al. (2002), is a widely used measure for Bayesian model comparison. It is used to assess both the goodness-of-fit and the complexity of the model. The model with a significantly lower DIC value will usually be the preferred model to use, but there are other considerations. The authors state that there is no specific rule of thumb on what constitutes a significant difference in DIC, but that guidelines proposed by Burnham and Anderson (1998) on the Akaike information criterion (AIC) also seem to work well for the DIC.

The formulation of the DIC follows on the work of Akaike (1973) and is based on using the posterior mean deviance as a measure of fit, and a new complexity measure called the *effective number of parameters*. For a parameter vector $\boldsymbol{\theta}$, with likelihood function $L(\underline{x}|\boldsymbol{\theta})$, the deviance can be defined as

$$D(\boldsymbol{\theta}) = -2\ln[L(\underline{x}|\boldsymbol{\theta})].$$

Denote the posterior mean of the deviance by $\overline{D}$, and let

$$\hat{D}\left(\overline{\theta}\right) = -2\ln\left[L\left(\underline{x}\,|\,\overline{\theta}\right)\right],$$

be a point estimate for the deviance, with $\overline{\theta}$ being the posterior mean of $\boldsymbol{\theta}$. The DIC can then be calculated as

$$\text{DIC} = \overline{D} + p_D,$$

where p_D is the effective number of parameters given by $p_D = \overline{D} - \hat{D}\left(\overline{\theta}\right)$. It is shown in Spiegelhalter et al. (2002) that the DIC is approximately equivalent to the AIC for models with very weak prior information.

The DIC is a prevalent choice for model comparison in the Bayesian ALT setup, particularly when working with complicated models where MCMC methods are used to obtain posterior samples for inference (see, for example, Barriga et al., 2008, Soyer et al., 2008, and Upadhyay and Mukherjee, 2010). The reason for this is that the DIC can easily be obtained from the MCMC output (see Spiegelhalter et al., 2002), and it is also provided as standard output in some well-known Bayesian data analysis software such as WinBUGS/OpenBUGS and JAGS.

2.10 Literature Review of Bayesian Accelerated Life Testing

Considering the main focus of this thesis, we review some of the most prominent papers on Bayesian ALT models and design.

Ahmad (1990) provides a general overview of early Bayesian methods in ALT. These methods include general parametic inference for an accelerated model, reducing high dimensional integration problems via semi-sufficient statistics, the use of semi-parametric inference, a Kalman filter approach, and an approach based on failure rates.

Achcar and Louzada-Neto (1991) consider a Bayesian ALT model, based on the Eyring model with one stressor, where lifetimes follow an exponential distribution. Type-II censored data are assumed and Jeffreys' priors are used for the model parameters. The authors make use of Laplace approximations to calculate integrals that are difficult to solve analytically, in order to find the marginal posterior distributions. The posterior predictive distribution is also derived and used to develop a quality control test.

Chaloner and Larntz (1992) examine experimental Bayesian design for ALT, where lifetimes follow either a Weibull or log-normal distribution. The approach allows for a linear or quadratic relationship between the log-lifetimes and the stressor.

Mazzuchi and Soyer (1992) present a dynamic general linear model setup for ALT, that uses linear Bayesian methods for inference. The approach is suitable for both censored and uncensored data. Lifetimes are assumed to be exponential with the power law as the TTF. The authors state that the approach can easily be extended to other TTFs.

Achcar (1993) explores the use of Laplace's method to approximate marginal posterior distributions in Bayesian ALT. The approximations are presented for the exponential, Weibull, Birnbaum-Saunders and inverse Gaussian distributions. A general TTF is assumed, which allows for the Arrhenius, Eyring, and power law models.

Van Dorp et al. (1996) develop a Bayesian model for step-stress ALT where the lifetimes are exponentially distributed. A multivariate ordered Dirichlet distribution is motivated as the prior distribution. Bayesian point estimates and credibility intervals are provided for the use-stress life parameters.

Dietrich and Mazzuchi (1996) discuss the design of experiments in ALT, pointing out some problems that emerge from typical regression and ANOVA techniques. An alternative analysis procedure is proposed based on Bayesian considerations.

Mazzuchi et al. (1997) present a Bayesian approach, based on general linear models, for inference from ALT models. The approach can handle censored or uncensored data, and any linearisable TTF can be incorporated. The lifetimes are assumed to be Weibull distributed and the power law is used as TTF on the scale parameter.

Erkanli and Soyer (2000) provide simulation-based designs for ALT models, using a Bayesian decision theory approach. An ALT model is considered where lifetimes are assumed to follow an exponential distribution and the power law is used as the TTF. MCMC methods are used to present inference for this model. A curve-fitting approach is utilised to find the optimal design for fixed and sequential ALT designs.

Perdona and Louzada-Neto (2005) propose a Bayesian ALT model where lifetimes are exponential and a general log-non-linear TTF is used. This TTF can also accommodate the general log-linear TTF, including the Arrhenius, Eyring, and power law models. A uniform prior is assumed and Laplace approximations are utilised to find marginal posterior distributions. This model with the log-non-linear TTF is also compared to a model with a log-linear TTF, via the Bayesian information criterion (BIC).

Van Dorp and Mazzuchi (2004) develop a general Bayes inference model for ALT, where the lifetimes are exponentially distributed. The inference procedure can accommodate interval censored and type-I censored data, and is compatible with constant stress, step-stress, profile-stress, and mixtures of stress loadings. The restriction of a parametric TTF is removed by rather imposing a multivariate prior distribution on the failure rates over an ordered region, which can be interpreted as a non-parametric TTF. MCMC methods are used for inference and to derive posterior quantities.

Van Dorp and Mazzuchi (2005) extend the general Bayes inference model for ALT, presented in Van Dorp and Mazzuchi (2004), to the Weibull distribution. The same strategy is followed to define a model where no parametric TTF is used. This model is again compatible with various stress loadings and can accommodate type-I and interval censored data. MCMC techniques are utilised for posterior inference.

Van Dorp et al. (2006) compare constant stress, step-stress and profile stress ALT models within a single Bayesian inferencial framework. The model presented in Van Dorp and Mazzuchi (2004) is used to compare the ALT designs via two optimality criteria.

León et al. (2007) present Bayesian inferences from ALT models, where items originate from different groups with random effects. This is a scale-accelerated ALT model, where time acceleration is defined by an acceleration factor with a special TTF. Independent vague priors are imposed on model parameters and MCMC methods are employed to generate posterior samples for inference. The authors state that the approach can incorporate multiple random effects and acceleration factors.

Barriga et al. (2008) consider Bayesian methods for ALT where the exponentiated Weibull is used as the life distribution and the Arrhenius model is the TTF. The model is formulated under type-II censoring, where normal and gamma priors are imposed on the parameters. MCMC techniques are used for an application of the model, where the hyperparameters give rise to vague prior distributions.

Soyer (2008) reviews Bayesian designs for ALT in a Bayesian decision theory setup. Difficulties regarding the evaluation of pre-posterior losses are discussed. Monte Carlo and linear Bayesian methods are presented to address some of these difficulties. An exponential life distribution with the power law as TTF is used throughout the review.

Soyer et al. (2008) present a basic parametric bayesian ALT model, where lifetimes follow a Weibull distribution with the power law or Arrhenius model as TTF. The model can accommodate uncensored, type-I censored and type-II censored data. Two extensions of the basic model, namely a hierarchical Bayesian model and a Markov model, are also presented. The three models are compared in two applications, where MCMC methods are employed to generate posterior samples for inference. The models are compared via the DIC and some inferences are presented.

Pan (2009) provides a Bayesian approach, and introduces a calibration factor, which allows for a combination of field data and ALT data to be used for reliability prediction. The author states that this calibration factor should be considered when similar products are designed and manufactured. The exponential, Weibull and log-normal distributions are considered with a log-linear TTF. Bayesian analysis is discussed in terms of the use of conjugate priors, as well as MCMC methods. The exponential model, incorporating the calibration factor, is demonstrated in two examples where conjugate priors and MCMC methods are used, respectively.

Upadhyay et al. (2009) consider a Bayesian approach to accelerated test system strength models. Two models based on cumulative damage arguments, presented in Owen and Padgett (1999), are explored in the Bayesian paradigm. Vague independent priors are selected for the model parameters and MCMC methods are employed for posterior analysis. The models are applied to a data set and Bayes point estimates are compared with the maximum likelihood estimates. Model comparison is performed using the BIC, DIC, and a posterior predictive loss approach.

Upadhyay and Mukherjee (2010) present a Bayesian comparison between accelerated Weibull and Birnbaum-Saunders models, where the TTF is the inverse power law. Independent vague priors are chosen for the model parameters and MCMC techniques are utilised to generate posterior samples. The models are applied to two data sets, where posterior modes, means and point estimates are presented. Model compatibility is addressed via empirical distribution function and empirical hazard function plots. Model comparison is evaluated by means of the models' DIC and BIC, as well as the calculation of fractional Bayes factors. The authors conclude that the Birnbaum-Saunders model is preferred above the Weibull model, but that the Weibull model is considerably easier to implement.

Yuan et al. (2014) propose a semi-parametric Bayesian approach for ALT where the TTF is log-linear and no assumption is made on the form of the life distribution. The authors employ a Dirichlet process mixture model, using a Weibull kernel, to model the life distribution at specific stress levels. Model fitting is performed via a simulation-based algorithm that incorporates Gibbs sampling. The predicted CDF of the proposed model and a parametric Weibull ALT model with log-linear TTF is compared, in an application, to the empirical distribution function.

Mukhopadhyay and Roy (2016) consider Bayesian analysis of an ALT model where log-lifetimes follow a distribution from the log-concave log-location-scale family. These distributions include the exponential, Weibull, generalised gamma, log-normal, log-Laplace, log-logistic and log-sech distributions. A general linear stress translation function is used in the model formulation, which can accommodate various single and multiple stressor acceleration models. The authors then focus on using the log-normal life distribution with the Arrhenius model as TTF in a simulation study. MCMC techniques are employed to obtain posterior samples for inference. The exponential, Weibull and log-normal models, still using the Arrhenius model, are compared in a application to real data. Model comparison is conducted via the AIC, BIC, and Bayes factors where Laplace approximations and the method in Carlin and Chib (1995) are utilised.

Sun and Shi (2016) propose a simple step-stress ALT model for the Birnbaum-Saunders distribution, utilising the power law as the TTF. The model is constructed under type-II censoring and MCMC methods are employed to find parameter estimates. In a simulation study, Bayes estimates and maximum likelihood estimates are compared in terms of the square root of mean squared error, the relative error, and the relative absolute bias. The average confidence interval lengths and coverage probabilities for approximate confidence intervals for the maximum likelihood estimates, percentile bootstrap confidence intervals, and highest posterior density credible intervals are also compared in the simulation study.

Polson and Soyer (2017) develop a Bayesian augmented probability simulation approach to determine an optimal ALT design. The optimal ALT design is determined in terms of maximising the expected pre-posterior utility. Conjugate utility functions, presented in Lindley (1976), are used, which allow the construction of a simulation approach that simultaneously calculates and optimises the expected utility.

Sha (2018) considers a progressive step-stress ALT where lifetimes follow a Birnbaum-Saunders distribution and the inverse power law is used as the TTF. Through this model, the author introduces a generalised Birnbaum-Saunders distribution. Some properties of this distribution as well as classical and Bayesian inference methods are discussed. A simulation study is conducted to compare the average bias, mean squared error, average confidence or credible interval length and coverage probability for maximum likelihood and Bayesian parameter estimates. Model comparison is carried out via the BIC and the chi-squared goodness-of-fit test in a real data set application.

Upadhyay and Sharma (2018) considers the scenario in ALT where the exact failure times are not known, and an item's status is rather respresented by a binary outcome indicating whether the item is surviving or has failed. Weibull and log-normal models are presented, where general log-linear TTFs are used on both the scale and shape parameters of the models. These models are also adapted to be able to deal with missing data. Independent normal priors are imposed on the model parameters and MCMC techniques are utilised to obtain samples for posterior inference. Model comparison is conducted by comparing the models' DIC and expected posterior predictive loss values. A thorough Bayesian analysis is performed by applying the models to a real data set.

2.11 Motivation for New Models

As seen from the previous discussion, the development of Bayesian methods to draw inferences from ALT has experienced a surge in recent years. However, ALT models incorporating TTFs that allow the use of more than one stressor have primarily been implemented only in frequentistic ALT setups. Even in papers where a log-linear TTF is defined, which in theory includes models that allow multiple stressors, the Bayesian ALT model is demonstrated via single stressor TTFs such as the inverse power law or Arrhenius model. The generalised Eyring model has not been used as a TTF for the Weibull and Birnbaum-Saunders distributions (and also various other two-parameter distributions), in the Bayesian setup. This is a critical gap in the literature, seeing that these two parameter life distributions offer a wider applicability in practice.

In the chapters that follow, Bayesian ALT models are formulated and applied that make use of the generalised Eyring TTF with two stressors. The utility and necessity of ALT models with multiple accelerated stressors are discussed in Nelson (1990). First, many products operate under multiple stressors simultaneously in their normal use environment. It is thus important to not only investigate one of these stressors, as there may also be interactions between the stressors. Electronic devices, for example, operate concurrently under stressors such as temperature, voltage and current. It may not be possible to fully isolate and examine only one of these stressors in an ALT experiment, as there are inherent interactions and relationships between these stressors. For this reason, more complex acceleration models that allow the use of multiple stressors are needed. Such models include the generalised Eyring model, Peck's model, Black's model, and Zhurkov's model (Nelson, 1990). Second, it may not be practically possible to increase a single stressor to high enough levels to induce failures within the time limit of the experiment. In some cases the testing equipment itself might also not allow high enough levels of a single stressor. For this reason, additional stressors can be incorporated for further acceleration. Third, under normal operating conditions, many products have various modes of failure. The use of one stressor may only accelerate failure modes linked to the specific stressor. Using multiple stressors at various combinations of stress levels may be necessary to explore a wider range of failure modes.

In this thesis, two Bayesian ALT models are explored, where the Weibull and Birnbaum-Saunders distributions are assumed as the life distributions. The Weibull distribution is one of the most widely used and studied life distributions in reliability theory. This distribution has applications in various engineering fields, and is specifically important in assessing product reliability (see, for example, Lai et al., 2006, Luko, 1999, and Nelson, 1990). Inference and analysis on the Weibull distribution are readily available and the distribution is incorporated in most statistical software packages. The Weibull distribution is very flexible and has the attractive feature that it allows for constant, increasing and decreasing failure rates (Hamada et al., 2008). The Birnbaum-Saunders distribution was specifically developed by considering the fatigue process, which many common life distributions fail to do in certain cases (see Lemonte, 2016; Upadhyay and Mukherjee, 2010, for further details). According to Birnbaum and Saunders (1969), most two parameter life distributions, such as the Weibull, the log-normal, and the gamma distributions, can be fitted reasonably well to fatigue data. When working with relatively small samples, as is often the case in ALT, it is unlikely that any of these distributions would produce a bad fit and be rejected by a goodness-of-fit test. These life distributions may even result in similar fits in the region of central tendency, while large discrepancies can occur in the lower and upper percentiles. This poses a challenge in the manufacturing and design processes of products, since the prediction of specifically the lower percentiles are of interest to the reliability engineer (Upadhyay and Mukherjee, 2010). It is for this reason that Birnbaum and Saunders (1969) motivate using the Birnbaum-Saunders distribution, which is derived from considerations of the characteristics of the fatigue process.

Chapter 3

The Generalised Eyring-Weibull Model

3.1 Introduction

In this chapter, a Bayesian ALT model using the generalised Eyring model, where lifetimes follow a Weibull distribution, is explored. To our knowledge, the generalised Eyring model has only been fully implemented in frequentistic ALT setups, and is still less common than other TTFs. The generalised Eyring model with two stressors, one thermal and one non-thermal is used. The GEW model is an intricate model due to the number of unknown parameters, which complicates both classical and Bayesian inferences.

First, we formulate a general likelihood for the GEW model that can accommodate uncensored, type-I censored and type-II censored data. Second, several variations of the GEW model are defined using choices of the prior distributions. The corresponding posterior distributions for the variations are given, up to at least proportionality. Due to the mathematically intractable posterior distributions, MCMC methods are employed to generate posterior samples to base inference on. The log-concavity of the full conditional posterior distributions are evaluated in order to determine which MCMC techniques are appropriate. Third, the GEW model is applied to a real data set, where summary statistics are provided together with the predictive reliability and model comparison statistics. The chapter is concluded with some final comments and remarks on the GEW model.

3.2 The GEW Model Specification

Let X be a continuous random variable that follows a Weibull distribution with scale parameter α and shape parameter β $(\alpha > 0, \beta > 0)$. The PDF is then given by

$$f(x|\alpha,\beta) = \alpha\beta x^{\beta-1}\exp\left(-\alpha x^{\beta}\right), \quad x \geq 0, \tag{3.1}$$

and the reliability function by

$$R(x) = 1 - F(x) = \exp\left(-\alpha x^{\beta}\right). \tag{3.2}$$

Consider two stressors, one thermal and one non-thermal. Indicate the k distinct accelerated levels of the stressors by $\{T_i, S_i\}, i = 1, \ldots, k$, where $T_i, i = 1, \ldots, k$, is the accelerated levels of the thermal stressor and $S_i, i = 1, \ldots, k$ is the accelerated levels of the non-thermal stressor. An item is exposed to the constant application of a specific stress level combination $\{T_i, S_i\}$. A common assumption in the literature is that the Weibull scale parameter α is then dependent on the stress levels, whereas the shape parameter β is not (see, for example, Mazzuchi et al., 1997; Soyer et al., 2008; Upadhyay and Mukherjee, 2010). This translates into assuming that the failure mechanism remains constant for all stressor levels. The reparameterisation of α given by the generalised Eyring model is

$$\alpha_i = T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right), \tag{3.3}$$

where θ_1, θ_2, θ_3, and θ_4 are unknown model parameters, and V_i is a function of the non-thermal stressor S_i (Escobar and Meeker, 2006). For the formulation of the Weibull PDF given in (3.1), the generalised Eyring TTF from Section 2.5.5 is substituted for the scale parameter as $\alpha_i = \frac{1}{\lambda_i}$ (see, for example, Upadhyay and Mukherjee, 2010; Mukhopadhyay and Roy, 2016). For a lifetime subjected to the i^{th} level of the stressors, it follows from (3.1) and (3.3) that the Weibull PDF can be written as

$$\begin{aligned} f(x_i | \theta_1, \theta_2, \theta_3, \theta_4, \beta) &= T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right) \beta x_i^{\beta - 1} \\ &\quad \times \exp\left[-T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right) x_i^{\beta}\right]. \end{aligned} \tag{3.4}$$

From (3.2) and (3.3), the Weibull reliability function at some time τ can be written as

$$R(\tau) = \exp\left[-T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right) \tau^{\beta}\right]. \tag{3.5}$$

Suppose that n_i items are tested at each of the k different stress levels and the test is truncated at time τ_i. Denote the failure times by $x_{ij}, j = 1, \ldots, n_i, i = 1, \ldots, k$. The likelihood function, in general, is then given by

$$L(\underline{x} | \theta_1, \theta_2, \theta_3, \theta_4, \beta) = \prod_{i=1}^{k} \left[\prod_{j=1}^{r_i} f\left(x_{ij} | \theta_1, \theta_2, \theta_3, \theta_4, \beta\right)\right] \left[R(\tau_i)\right]^{n_i - r_i}.$$

Note that for complete samples $r_i = n_i$. For type-I censoring r_i is the number of failures that occur before censoring time τ_i, where $\tau_i < \infty, i = 1, \ldots, k$ is predetermined censoring times for the k different stress levels. For type-II censoring $\tau_i = x_{i(r_i)}$, where $x_{i(r_i)}$ is the r_i^{th} ordered failure time, and $r_i, i = 1, \ldots, k$ is the prespecified number of failures after which censoring occurs for the k different stress levels. From (3.4) and (3.5) it follows that the likelihood function for the GEW model is given by

$$
\begin{aligned}
L(\underline{x}|\theta_1,\theta_2,\theta_3,\theta_4,\beta) &= \prod_{i=1}^{k}\left[\prod_{j=1}^{r_i} f\left(x_{ij}|\theta_1,\theta_2,\theta_3,\theta_4,\beta\right)\right][R(\tau_i)]^{n_i-r_i} \\
&= \prod_{i=1}^{k}\exp\left[-(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\right] \\
&\quad\times\beta^{r_i}T_i^{r_i}\exp\left(-\theta_1r_i-\frac{\theta_2r_i}{T_i}-\theta_3r_iV_i-\frac{\theta_4r_iV_i}{T_i}\right) \\
&\quad\times\prod_{j=1}^{r_i}x_{ij}^{\beta-1}\exp\left[-T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}\right] \\
&= \beta^{\sum_{i=1}^{k}r_i}\exp\left(-\theta_1\sum_{i=1}^{k}r_i-\theta_2\sum_{i=1}^{k}\frac{r_i}{T_i}-\theta_3\sum_{i=1}^{k}r_iV_i-\theta_4\sum_{i=1}^{k}\frac{r_iV_i}{T_i}\right) \\
&\quad\times\exp\left[-\sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\right] \qquad (3.6) \\
&\quad\times\exp\left[-\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}\right]\left[\prod_{i=1}^{k}\prod_{j=1}^{r_i}T_ix_{ij}^{\beta-1}\right].
\end{aligned}
$$

3.3 Priors and Posteriors

In this section, several GEW models are defined to investigate the sensitivity of the GEW model. This is conducted via the selection of different prior distributions. Due to the mathematically intractable likelihood function, the derivation of non-informative priors such as the Jeffreys, reference and probability matching priors are complicated. Soland (1969) affirms that there is no conjugate family of continuous joint prior distributions for the two-parameter Weibull distribution, and proposes a gamma prior for the scale parameter and a discrete prior for the shape parameter. Tsokos (1972) suggests using an inverse gamma prior for the scale parameter and a uniform prior for the shape parameter. A compound inverse gamma prior on the scale parameter, with either a discrete, inverse gamma, or uniform prior on the shape parameter is proposed in Papadopoulos and Tsokos (1976). Kundu (2008) assumes a gamma prior for the scale parameter and a non-spesific log-concave prior density with support $(0,\infty)$ for the shape parameter. Banerjee and Kundu (2008) consider gamma priors on both the scale and shape parameters.

Taking the above into consideration, we formulate three variations of the GEW model. Assume that the priors on the unknown parameters θ_1, θ_2, θ_3, θ_4 and β are independent. By doing so, no assumption is made that the priors are a priori informative about one another. This is a common assumption in Bayesian ALT models (see, for example, Soyer et al., 2008; Upadhyay and Mukherjee, 2010). The joint prior distribution is then given by

$$\pi(\theta_1,\theta_2,\theta_3,\theta_4,\beta) = \pi(\theta_1)\,\pi(\theta_2)\,\pi(\theta_3)\,\pi(\theta_4)\,\pi(\beta)\,.$$

The joint posterior distribution is then given by

$$\pi(\theta_1,\theta_2,\theta_3,\theta_4,\beta\,|\underline{x}) \;\propto\; L(\underline{x}|\theta_1,\theta_2,\theta_3,\theta_4,\beta)\,\pi(\theta_1,\theta_2,\theta_3,\theta_4,\beta)\,.$$

In the case of mathematically intractable posterior distributions, MCMC methods must be utilised to draw posterior samples for inference. Many widely applicable MCMC methods require the specification of the full conditional posterior distributions, up to at least proportionality. The full conditional posterior distributions for the GEW models are provided in the next section, and the log-concavity of these distributions is assessed in Section 3.3.4.

3.3.1 GEW$_1$ Model

The GEW model where uniform priors are imposed on all the parameters, thus

$$\begin{aligned}
\theta_1 &\sim U(c_0,c_1) \quad, c_1 > c_0 \quad, \pi_1(\theta_1) \propto \text{constant}\\
\theta_2 &\sim U(c_2,c_3) \quad, c_3 > c_2 \quad, \pi_1(\theta_2) \propto \text{constant}\\
\theta_3 &\sim U(c_4,c_5) \quad, c_5 > c_4 \quad, \pi_1(\theta_3) \propto \text{constant}\\
\theta_4 &\sim U(c_6,c_7) \quad, c_7 > c_6 \quad, \pi_1(\theta_4) \propto \text{constant}\\
\beta &\sim U(c_8,c_9) \quad, c_9 > c_8 \quad, \pi_1(\beta) \propto \text{constant},
\end{aligned}$$

is denoted by GEW$_1$. The joint prior for GEW$_1$ is then given by

$$\pi_1(\theta_1,\theta_2,\theta_3,\theta_4,\beta) \propto \text{constant}, \tag{3.7}$$

and using (3.6) and (3.7) the joint posterior is given by

$$\begin{aligned}
\pi_1(\theta_1,\theta_2,\theta_3,\theta_4,\beta\,|\underline{x}) \;\propto\;& \beta^{\sum_{i=1}^{k} r_i} \exp\left(-\theta_1\sum_{i=1}^{k} r_i - \theta_2\sum_{i=1}^{k}\frac{r_i}{T_i} - \theta_3\sum_{i=1}^{k} r_i V_i - \theta_4\sum_{i=1}^{k}\frac{r_i V_i}{T_i}\right)\\
&\times \exp\left[-\sum_{i=1}^{k}(n_i - r_i)\,T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\tau_i^{\beta}\right]\\
&\times \exp\left[-\sum_{i=1}^{k}\sum_{j=1}^{r_i} T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)x_{ij}^{\beta}\right]\left[\prod_{i=1}^{k}\prod_{j=1}^{r_i} T_i x_{ij}^{\beta-1}\right].
\end{aligned}$$

The posterior can not be written in closed-form, therefore MCMC methods are employed to generate posterior samples for inferences. The full conditional posteriors for the GEW_1 model are given by

$$\begin{aligned}\pi_1(\theta_1\,|\underline{x},\theta_2,\theta_3,\theta_4,\beta) &\propto \exp(-\theta_1\sum_{i=1}^{k} r_i)\exp\left[-\sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\right]\\ &\quad\times\exp\left[-\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}\right]\end{aligned}$$

$$\begin{aligned}\pi_1(\theta_2\,|\underline{x},\theta_1,\theta_3,\theta_4,\beta) &\propto \exp\left(-\theta_2\sum_{i=1}^{k}\frac{r_i}{T_i}\right)\exp\left[-\sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\right]\\ &\quad\times\exp\left[-\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}\right]\end{aligned}$$

$$\begin{aligned}\pi_1(\theta_3\,|\underline{x},\theta_1,\theta_2,\theta_4,\beta) &\propto \exp\left(-\theta_3\sum_{i=1}^{k}r_iV_i\right)\exp\left[-\sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\right]\\ &\quad\times\exp\left[-\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}\right]\end{aligned}$$

$$\begin{aligned}\pi_1(\theta_4\,|\underline{x},\theta_1,\theta_2,\theta_3,\beta) &\propto \exp\left(-\theta_4\sum_{i=1}^{k}\frac{r_iV_i}{T_i}\right)\exp\left[-\sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\right]\\ &\quad\times\exp\left[-\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}\right]\end{aligned}$$

$$\begin{aligned}\pi_1(\beta\,|\underline{x},\theta_1,\theta_2,\theta_3,\theta_4) &\propto \beta^{\sum_{i=1}^{k}r_i}\exp\left[-\sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\right]\\ &\quad\times\exp\left[-\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}\right]\left[\prod_{i=1}^{k}\prod_{j=1}^{r_i}x_{ij}^{\beta-1}\right].\end{aligned}$$

3.3.2 GEW$_2$ Model

Let GEW$_2$ denote the GEW model where gamma priors on all the parameters are assumed, with

$$\begin{aligned}
\theta_1 &\sim \Gamma(c_{10}, c_{11}) \quad , \; c_{10}, c_{11} > 0 \quad , \; \pi_2(\theta_1) \propto \theta_1^{c_{10}-1} \exp(-c_{11}\theta_1) \\
\theta_2 &\sim \Gamma(c_{12}, c_{13}) \quad , \; c_{12}, c_{13} > 0 \quad , \; \pi_2(\theta_2) \propto \theta_2^{c_{12}-1} \exp(-c_{13}\theta_2) \\
\theta_3 &\sim \Gamma(c_{14}, c_{15}) \quad , \; c_{14}, c_{15} > 0 \quad , \; \pi_2(\theta_3) \propto \theta_3^{c_{14}-1} \exp(-c_{15}\theta_3) \\
\theta_4 &\sim \Gamma(c_{16}, c_{17}) \quad , \; c_{16}, c_{17} > 0 \quad , \; \pi_2(\theta_4) \propto \theta_4^{c_{16}-1} \exp(-c_{17}\theta_4) \\
\beta &\sim \Gamma(c_{18}, c_{19}) \quad , \; c_{18}, c_{19} > 0 \quad , \; \pi_2(\beta) \propto \beta^{c_{18}-1} \exp(-c_{19}\beta) .
\end{aligned}$$

The joint prior for GEW$_2$ is then given by

$$\pi_2(\theta_1, \theta_2, \theta_3, \theta_4, \beta) \propto \theta_1^{c_{10}-1} \theta_2^{c_{12}-1} \theta_3^{c_{14}-1} \theta_4^{c_{16}-1} \beta^{c_{18}-1} \exp(-c_{11}\theta_1 - c_{13}\theta_2 - c_{15}\theta_3 - c_{17}\theta_4 - c_{19}\beta), \tag{3.8}$$

and, using (3.6) and (3.8), the joint posterior is given by

$$\begin{aligned}
\pi_2(\theta_1, \theta_2, \theta_3, \theta_4, \beta \,|\underline{x}) \;\propto\; & \theta_1^{c_{10}-1} \theta_2^{c_{12}-1} \theta_3^{c_{14}-1} \theta_4^{c_{16}-1} \beta^{c_{18}-1} \exp(-c_{11}\theta_1 - c_{13}\theta_2 - c_{15}\theta_3 - c_{17}\theta_4 - c_{19}\beta) \\
& \times \beta^{\sum_{i=1}^{k} r_i} \exp\left(-\theta_1 \sum_{i=1}^{k} r_i - \theta_2 \sum_{i=1}^{k} \frac{r_i}{T_i} - \theta_3 \sum_{i=1}^{k} r_i V_i - \theta_4 \sum_{i=1}^{k} \frac{r_i V_i}{T_i}\right) \\
& \times \exp\left[-\sum_{i=1}^{k} (n_i - r_i) T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right) \tau_i^{\beta}\right] \\
& \times \exp\left[-\sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right) x_{ij}^{\beta}\right] \left[\prod_{i=1}^{k} \prod_{j=1}^{r_i} T_i x_{ij}^{\beta-1}\right] .
\end{aligned}$$

Due to the complexity of the posterior, MCMC methods are used to draw posterior samples to base inferences on. For GEW$_2$, the full conditional posteriors are given by

$$\begin{aligned}
\pi_2(\theta_1 \,|\underline{x}, \theta_2, \theta_3, \theta_4, \beta) \;\propto\; & \theta_1^{c_{10}-1} \exp(-c_{11}\theta_1) \exp(-\theta_1 \sum_{i=1}^{k} r_i) \\
& \times \exp\left[-\sum_{i=1}^{k} (n_i - r_i) T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right) \tau_i^{\beta}\right] \\
& \times \exp\left[-\sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right) x_{ij}^{\beta}\right]
\end{aligned}$$

$$\begin{aligned}\pi_2(\theta_2|\underline{x},\theta_1,\theta_3,\theta_4,\beta) \propto\ & \theta_2^{c_{12}-1}\exp(-c_{13}\theta_2)\exp\left(-\theta_2\sum_{i=1}^{k}\frac{r_i}{T_i}\right)\\ & \times\exp\left[-\sum_{i=1}^{k}(n_i-r_i)T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\right]\\ & \times\exp\left[-\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}\right]\end{aligned}$$

$$\begin{aligned}\pi_2(\theta_3|\underline{x},\theta_1,\theta_2,\theta_4,\beta) \propto\ & \theta_3^{c_{14}-1}\exp(-c_{15}\theta_3)\exp\left(-\theta_3\sum_{i=1}^{k}r_iV_i\right)\\ & \times\exp\left[-\sum_{i=1}^{k}(n_i-r_i)T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\right]\\ & \times\exp\left[-\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}\right]\end{aligned}$$

$$\begin{aligned}\pi_2(\theta_4|\underline{x},\theta_1,\theta_2,\theta_3,\beta) \propto\ & \theta_4^{c_{16}-1}\exp(-c_{17}\theta_4)\exp\left(-\theta_4\sum_{i=1}^{k}\frac{r_iV_i}{T_i}\right)\\ & \times\exp\left[-\sum_{i=1}^{k}(n_i-r_i)T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\right]\\ & \times\exp\left[-\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}\right]\end{aligned}$$

$$\begin{aligned}\pi_2(\beta|\underline{x},\theta_1,\theta_2,\theta_3,\theta_4) \propto\ & \beta^{c_{18}-1}\exp(-c_{19}\beta)\,\beta^{\sum_{i=1}^{k}r_i}\\ & \times\exp\left[-\sum_{i=1}^{k}(n_i-r_i)T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\right]\\ & \times\exp\left[-\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}\right]\left[\prod_{i=1}^{k}\prod_{j=1}^{r_i}x_{ij}^{\beta-1}\right].\end{aligned}$$

3.3.3 GEW_3 Model

Consider log-normal priors for the model parameters, and denote this model by GEW_3, where

$$\begin{aligned}
\theta_1 &\sim LN(c_{20}, c_{21}^2) \quad , \ c_{20} \in \mathbb{R}, c_{21} > 0 \quad , \ \pi_3(\theta_1) \propto \frac{1}{\theta_1} \exp\left[-\frac{1}{2c_{21}^2}\ln^2\theta_1 + \frac{c_{20}}{c_{21}^2}\ln\theta_1\right] \\
\theta_2 &\sim LN(c_{22}, c_{23}^2) \quad , \ c_{22} \in \mathbb{R}, c_{23} > 0 \quad , \ \pi_3(\theta_2) \propto \frac{1}{\theta_2} \exp\left[-\frac{1}{2c_{23}^2}\ln^2\theta_2 + \frac{c_{22}}{c_{23}^2}\ln\theta_2\right] \\
\theta_3 &\sim LN(c_{24}, c_{25}^2) \quad , \ c_{24} \in \mathbb{R}, c_{25} > 0 \quad , \ \pi_3(\theta_3) \propto \frac{1}{\theta_3} \exp\left[-\frac{1}{2c_{25}^2}\ln^2\theta_3 + \frac{c_{24}}{c_{25}^2}\ln\theta_3\right] \\
\theta_4 &\sim LN(c_{26}, c_{27}^2) \quad , \ c_{26} \in \mathbb{R}, c_{27} > 0 \quad , \ \pi_3(\theta_4) \propto \frac{1}{\theta_4} \exp\left[-\frac{1}{2c_{27}^2}\ln^2\theta_4 + \frac{c_{26}}{c_{27}^2}\ln\theta_4\right] \\
\beta &\sim LN(c_{28}, c_{29}^2) \quad , \ c_{28} \in \mathbb{R}, c_{29} > 0 \quad , \ \pi_3(\beta) \propto \frac{1}{\beta} \exp\left[-\frac{1}{2c_{29}^2}\ln^2\beta + \frac{c_{28}}{c_{29}^2}\ln\beta\right].
\end{aligned}$$

The joint prior for GEW_3 is then given by

$$\begin{aligned}
\pi_3(\theta_1, \theta_2, \theta_3, \theta_4, \beta) \ \propto \ & \frac{1}{\theta_1\theta_2\theta_3\theta_4\beta} \exp\left[-\frac{1}{2c_{21}^2}\ln^2\theta_1 + \frac{c_{20}}{c_{21}^2}\ln\theta_1\right] \exp\left[-\frac{1}{2c_{23}^2}\ln^2\theta_2 + \frac{c_{22}}{c_{23}^2}\ln\theta_2\right] \\
& \times \exp\left[-\frac{1}{2c_{25}^2}\ln^2\theta_3 + \frac{c_{24}}{c_{25}^2}\ln\theta_3\right] \exp\left[-\frac{1}{2c_{27}^2}\ln^2\theta_4 + \frac{c_{26}}{c_{27}^2}\ln\theta_4\right] \qquad (3.9) \\
& \times \exp\left[-\frac{1}{2c_{29}^2}\ln^2\beta + \frac{c_{28}}{c_{29}^2}\ln\beta\right]
\end{aligned}$$

and, using (3.6) and (3.9), the joint posterior is given by

$$\begin{aligned}
\pi_3(\theta_1, \theta_2, \theta_3, \theta_4, \beta \,|\underline{x}) \ \propto \ & \frac{1}{\theta_1\theta_2\theta_3\theta_4} \beta^{\left(\Sigma_{i=1}^k r_i\right)-1} \exp\left[-\frac{1}{2c_{21}^2}\ln^2\theta_1 + \frac{c_{20}}{c_{21}^2}\ln\theta_1\right] \\
& \times \exp\left[-\frac{1}{2c_{23}^2}\ln^2\theta_2 + \frac{c_{22}}{c_{23}^2}\ln\theta_2\right] \exp\left[-\frac{1}{2c_{25}^2}\ln^2\theta_3 + \frac{c_{24}}{c_{25}^2}\ln\theta_3\right] \\
& \times \exp\left[-\frac{1}{2c_{27}^2}\ln^2\theta_4 + \frac{c_{26}}{c_{27}^2}\ln\theta_4\right] \exp\left[-\frac{1}{2c_{29}^2}\ln^2\beta + \frac{c_{28}}{c_{29}^2}\ln\beta\right] \\
& \times \exp\left(-\theta_1\sum_{i=1}^k r_i - \theta_2\sum_{i=1}^k \frac{r_i}{T_i} - \theta_3\sum_{i=1}^k r_iV_i - \theta_4\sum_{i=1}^k \frac{r_iV_i}{T_i}\right) \\
& \times \exp\left[-\sum_{i=1}^k (n_i - r_i)\,T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3V_i - \frac{\theta_4V_i}{T_i}\right)\tau_i^\beta\right] \\
& \times \exp\left[-\sum_{i=1}^k\sum_{j=1}^{r_i} T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3V_i - \frac{\theta_4V_i}{T_i}\right)x_{ij}^\beta\right]\left[\prod_{i=1}^k\prod_{j=1}^{r_i} T_i x_{ij}^{\beta-1}\right].
\end{aligned}$$

MCMC methods are used for posterior inference since the posterior is of an unmanageable form. The full conditional posteriors for this model are by

$$\begin{aligned}\pi_3\left(\theta_1\left|\underline{x},\theta_2,\theta_3,\theta_4,\beta\right.\right) &\propto \frac{1}{\theta_1}\exp\left[-\frac{1}{2c_{21}^2}\ln^2\theta_1+\frac{c_{20}}{c_{21}^2}\ln\theta_1\right]\exp(-\theta_1\sum_{i=1}^{k}r_i)\\ &\times\exp\left[-\sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\right]\\ &\times\exp\left[-\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}\right]\end{aligned}$$

$$\begin{aligned}\pi_3\left(\theta_2\left|\underline{x},\theta_1,\theta_3,\theta_4,\beta\right.\right) &\propto \frac{1}{\theta_2}\exp\left[-\frac{1}{2c_{23}^2}\ln^2\theta_2+\frac{c_{22}}{c_{23}^2}\ln\theta_2\right]\exp\left(-\theta_2\sum_{i=1}^{k}\frac{r_i}{T_i}\right)\\ &\times\exp\left[-\sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\right]\\ &\times\exp\left[-\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}\right]\end{aligned}$$

$$\begin{aligned}\pi_3\left(\theta_3\left|\underline{x},\theta_1,\theta_2,\theta_4,\beta\right.\right) &\propto \frac{1}{\theta_3}\exp\left[-\frac{1}{2c_{25}^2}\ln^2\theta_3+\frac{c_{24}}{c_{25}^2}\ln\theta_3\right]\exp\left(-\theta_3\sum_{i=1}^{k}r_iV_i\right)\\ &\times\exp\left[-\sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\right]\\ &\times\exp\left[-\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}\right]\end{aligned}$$

$$\begin{aligned}\pi_3\left(\theta_4\left|\underline{x},\theta_1,\theta_2,\theta_3,\beta\right.\right) &\propto \frac{1}{\theta_4}\exp\left[-\frac{1}{2c_{27}^2}\ln^2\theta_4+\frac{c_{26}}{c_{27}^2}\ln\theta_4\right]\exp\left(-\theta_4\sum_{i=1}^{k}\frac{r_iV_i}{T_i}\right)\\ &\times\exp\left[-\sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\right]\\ &\times\exp\left[-\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}\right]\end{aligned}$$

$$
\begin{aligned}
\pi_3\left(\beta\,|\underline{x},\theta_1,\theta_2,\theta_3,\theta_4\right) \propto\; & \beta^{\left(\sum_{i=1}^{k} r_i\right)-1}\exp\left[-\frac{1}{2c_{29}^2}\ln^2\beta+\frac{c_{28}}{c_{29}^2}\ln\beta\right] \\
& \times\exp\left[-\sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\right] \\
& \times\exp\left[-\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}\right]\left[\prod_{i=1}^{k}\prod_{j=1}^{r_i}x_{ij}^{\beta-1}\right].
\end{aligned}
$$

3.3.4 Log-concavity

A twice-differentiable function $f(x)$ is said to be log-concave if the second derivative of its natural log is non-positive on its domain (see, for example, Bagnoli and Bergstrom, 2005), thus if

$$\frac{\partial^2\ln[f(x)]}{\partial x^2}\leq 0\quad\forall x.$$

Theorem 3.1. *The full conditional posterior distributions of the GEW_1 model are log-concave on their domains.*

Proof. For the GEW_1 model, the second derivatives of the natural logs of the full conditional posteriors are determined as follows

$$
\begin{aligned}
\ell_{1,\theta_1} &= \ln\left[\pi_1\left(\theta_1\,|\underline{x},\theta_2,\theta_3,\theta_4,\beta\right)\right] \\
&= -\theta_1\sum_{i=1}^{k}r_i-\sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad -\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta} \\
\frac{\partial\ell_{1,\theta_1}}{\partial\theta_1} &= -\sum_{i=1}^{k}r_i+\sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad +\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta} \\
\frac{\partial^2\ell_{1,\theta_1}}{\partial\theta_1^2} &= -\sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad -\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}
\end{aligned}
$$

$$\begin{aligned}
\ell_{1,\theta_2} &= \ln\left[\pi_1\left(\theta_2\,|\underline{x},\theta_1,\theta_3,\theta_4,\beta\,\right)\right] \\
&= -\theta_2\sum_{i=1}^{k}\frac{r_i}{T_i}-\sum_{i=1}^{k}\left(n_i-r_i\right)T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad -\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta} \\
\frac{\partial\ell_{1,\theta_2}}{\partial\theta_2} &= -\sum_{i=1}^{k}\frac{r_i}{T_i}+\sum_{i=1}^{k}\left(n_i-r_i\right)\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad +\sum_{i=1}^{k}\sum_{j=1}^{r_i}\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta} \\
\frac{\partial^2\ell_{1,\theta_2}}{\partial\theta_2^2} &= -\sum_{i=1}^{k}\left(n_i-r_i\right)\frac{1}{T_i}\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad -\sum_{i=1}^{k}\sum_{j=1}^{r_i}\frac{1}{T_i}\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}
\end{aligned}$$

$$\begin{aligned}
\ell_{1,\theta_3} &= \ln\left[\pi_1\left(\theta_3\,|\underline{x},\theta_1,\theta_2,\theta_4,\beta\,\right)\right] \\
&= -\theta_3\sum_{i=1}^{k}r_iV_i-\sum_{i=1}^{k}\left(n_i-r_i\right)T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad -\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta} \\
\frac{\partial\ell_{1,\theta_3}}{\partial\theta_3} &= -\sum_{i=1}^{k}r_iV_i+\sum_{i=1}^{k}\left(n_i-r_i\right)T_iV_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad +\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_iV_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta} \\
\frac{\partial^2\ell_{1,\theta_3}}{\partial\theta_3^2} &= -\sum_{i=1}^{k}\left(n_i-r_i\right)T_iV_i^2\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad -\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_iV_i^2\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}
\end{aligned}$$

$$\begin{aligned}
\ell_{1,\theta_4} &= \ln\left[\pi_1\left(\theta_4\,|\underline{x},\theta_1,\theta_2,\theta_3,\beta\right)\right] \\
&= -\theta_4\sum_{i=1}^{k}\frac{r_iV_i}{T_i}-\sum_{i=1}^{k}\left(n_i-r_i\right)T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad -\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta} \\
\frac{\partial\ell_{1,\theta_4}}{\partial\theta_4} &= -\sum_{i=1}^{k}\frac{r_iV_i}{T_i}+\sum_{i=1}^{k}\left(n_i-r_i\right)V_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad +\sum_{i=1}^{k}\sum_{j=1}^{r_i}V_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta} \\
\frac{\partial^2\ell_{1,\theta_4}}{\partial\theta_4^2} &= -\sum_{i=1}^{k}\left(n_i-r_i\right)\frac{V_i^2}{T_i}\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad -\sum_{i=1}^{k}\sum_{j=1}^{r_i}\frac{V_i^2}{T_i}\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}
\end{aligned}$$

$$\begin{aligned}
\ell_{1,\beta} &= \ln\left[\pi_1\left(\beta\,|\underline{x},\theta_1,\theta_2,\theta_3,\theta_4\right)\right] \\
&= \ln(\beta)\sum_{i=1}^{k}r_i-\sum_{i=1}^{k}\left(n_i-r_i\right)T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad -\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}+(\beta-1)\sum_{i=1}^{k}\sum_{j=1}^{r_i}\ln\left(x_{ij}\right) \\
\frac{\partial\ell_{1,\beta}}{\partial\beta} &= \frac{1}{\beta}\sum_{i=1}^{k}r_i-\sum_{i=1}^{k}\left(n_i-r_i\right)T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\ln\left(\tau_i\right) \\
&\quad -\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}\ln\left(x_{ij}\right)+\sum_{i=1}^{k}\sum_{j=1}^{r_i}\ln\left(x_{ij}\right) \\
\frac{\partial^2\ell_{1,\beta}}{\partial\beta^2} &= -\frac{1}{\beta^2}\sum_{i=1}^{k}r_i-\sum_{i=1}^{k}\left(n_i-r_i\right)T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\ln^2\left(\tau_i\right) \\
&\quad -\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}\ln^2\left(x_{ij}\right).
\end{aligned}$$

Since $T_i \geq 0$ (temperature measured in kelvin), $r_i \leq n_i$ (there cannot be more failures than the items tested), $\tau_i \geq 0$ (censoring time), $x_{ij} \geq 0$ (failure time), $\beta > 0$ (shape parameter of the Weibull distribution), $\exp(\cdot) \geq 0$, $V_i^2 \geq 0$ and $\ln^2(\cdot) \geq 0$, the full conditional posteriors for the GEW_1 model are confirmed to be log-concave on their domains. □

Theorem 3.2. *The full conditional posterior distributions of the* GEW_2 *model are log-concave on their domains, subject to* $c_{10}, c_{12}, c_{14}, c_{16}, \sum_{i=1}^{k} r_i \geq 1$.

Proof. The second derivatives of the natural logs of the full conditional posteriors for the GEW_2 model are given by

$$\begin{aligned}
\ell_{2,\theta_1} &= \ln\left[\pi_2\left(\theta_1\left|\underline{x},\theta_2,\theta_3,\theta_4,\beta\right.\right)\right] \\
&= (c_{10}-1)\ln(\theta_1) - c_{11}\theta_1 - \theta_1\sum_{i=1}^{k} r_i - \sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad -\sum_{i=1}^{k}\sum_{j=1}^{r_i} T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta} \\
\frac{\partial\ell_{2,\theta_1}}{\partial\theta_1} &= \frac{c_{10}-1}{\theta_1} - c_{11} - \sum_{i=1}^{k} r_i + \sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad +\sum_{i=1}^{k}\sum_{j=1}^{r_i} T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta} \\
\frac{\partial^2\ell_{2,\theta_1}}{\partial\theta_1^2} &= \frac{1-c_{10}}{\theta_1^2} - \sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad -\sum_{i=1}^{k}\sum_{j=1}^{r_i} T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}
\end{aligned}$$

$$\begin{aligned}
\ell_{2,\theta_2} &= \ln\left[\pi_2\left(\theta_2\left|\underline{x},\theta_1,\theta_3,\theta_4,\beta\right.\right)\right] \\
&= (c_{12}-1)\ln(\theta_2) - c_{13}\theta_2 - \theta_2\sum_{i=1}^{k}\frac{r_i}{T_i} - \sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad -\sum_{i=1}^{k}\sum_{j=1}^{r_i} T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta} \\
\frac{\partial\ell_{2,\theta_2}}{\partial\theta_2} &= \frac{c_{12}-1}{\theta_2} - c_{13} - \sum_{i=1}^{k}\frac{r_i}{T_i} + \sum_{i=1}^{k}(n_i-r_i)\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad +\sum_{i=1}^{k}\sum_{j=1}^{r_i}\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta} \\
\frac{\partial^2\ell_{2,\theta_2}}{\partial\theta_2^2} &= \frac{1-c_{12}}{\theta_2^2} - \sum_{i=1}^{k}(n_i-r_i)\frac{1}{T_i}\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad -\sum_{i=1}^{k}\sum_{j=1}^{r_i}\frac{1}{T_i}\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}
\end{aligned}$$

$$
\begin{aligned}
\ell_{2,\theta_3} &= \ln\left[\pi_2\left(\theta_3 \,|\underline{x}, \theta_1, \theta_2, \theta_4, \beta\right)\right] \\
&= (c_{14}-1)\ln(\theta_3) - c_{15}\theta_3 - \theta_3 \sum_{i=1}^{k} r_i V_i - \sum_{i=1}^{k}(n_i - r_i)\, T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad - \sum_{i=1}^{k}\sum_{j=1}^{r_i} T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right) x_{ij}^{\beta} \\
\frac{\partial \ell_{2,\theta_3}}{\partial \theta_3} &= \frac{c_{14}-1}{\theta_3} - c_{15} - \sum_{i=1}^{k} r_i V_i + \sum_{i=1}^{k}(n_i - r_i)\, T_i V_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad + \sum_{i=1}^{k}\sum_{j=1}^{r_i} T_i V_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right) x_{ij}^{\beta} \\
\frac{\partial^2 \ell_{2,\theta_3}}{\partial \theta_3^2} &= \frac{1-c_{14}}{\theta_3^2} - \sum_{i=1}^{k}(n_i - r_i)\, T_i V_i^2 \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad - \sum_{i=1}^{k}\sum_{j=1}^{r_i} T_i V_i^2 \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right) x_{ij}^{\beta}
\end{aligned}
$$

$$
\begin{aligned}
\ell_{2,\theta_4} &= \ln\left[\pi_2\left(\theta_4 \,|\underline{x}, \theta_1, \theta_2, \theta_3, \beta\right)\right] \\
&= (c_{16}-1)\ln(\theta_4) - c_{17}\theta_4 - \theta_4 \sum_{i=1}^{k} \frac{r_i V_i}{T_i} - \sum_{i=1}^{k}(n_i - r_i)\, T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad - \sum_{i=1}^{k}\sum_{j=1}^{r_i} T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right) x_{ij}^{\beta} \\
\frac{\partial \ell_{2,\theta_4}}{\partial \theta_4} &= \frac{c_{16}-1}{\theta_4} - c_{17} - \sum_{i=1}^{k} \frac{r_i V_i}{T_i} + \sum_{i=1}^{k}(n_i - r_i)\, V_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad + \sum_{i=1}^{k}\sum_{j=1}^{r_i} V_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right) x_{ij}^{\beta} \\
\frac{\partial^2 \ell_{2,\theta_4}}{\partial \theta_4^2} &= \frac{1-c_{16}}{\theta_4^2} - \sum_{i=1}^{k}(n_i - r_i)\, \frac{V_i^2}{T_i} \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad - \sum_{i=1}^{k}\sum_{j=1}^{r_i} \frac{V_i^2}{T_i} \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right) x_{ij}^{\beta}
\end{aligned}
$$

$$\begin{aligned}
\ell_{2,\beta} &= \ln\left[\pi_2\left(\beta\,|\underline{x},\theta_1,\theta_2,\theta_3,\theta_4\right)\right]\\
&= (c_{18}-1)\ln(\beta) - c_{19}\beta + \ln(\beta)\sum_{i=1}^{k} r_i - \sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\\
&\quad -\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta} + (\beta-1)\sum_{i=1}^{k}\sum_{j=1}^{r_i}\ln\left(x_{ij}\right)\\
\frac{\partial\ell_{2,\beta}}{\partial\beta} &= \frac{c_{18}-1}{\beta} - c_{19} + \frac{1}{\beta}\sum_{i=1}^{k} r_i - \sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\ln(\tau_i)\\
&\quad -\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}\ln\left(x_{ij}\right) + \sum_{i=1}^{k}\sum_{j=1}^{r_i}\ln\left(x_{ij}\right)\\
\frac{\partial^2\ell_{2,\beta}}{\partial\beta^2} &= \frac{1-c_{18}}{\beta^2} - \frac{1}{\beta^2}\sum_{i=1}^{k} r_i - \sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\ln^2(\tau_i)\\
&\quad -\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}\ln^2\left(x_{ij}\right).
\end{aligned}$$

Since $T_i \geq 0$ (temperature measured in kelvin), $r_i \leq n_i$ (there cannot be more failures than the items tested), $\tau_i \geq 0$ (censoring time), $x_{ij} \geq 0$ (failure time), $\beta > 0$ (shape parameter of the Weibull distribution), $\exp(\cdot) \geq 0$, $V_i^2 \geq 0$ and $\ln^2(\cdot) \geq 0$, the full conditional posteriors for the GEW$_2$ model are log-concave on their domains, subject to $c_{10}, c_{12}, c_{14}, c_{16}, \sum_{i=1}^{k} r_i \geq 1$. □

Theorem 3.3. *The full conditional posterior distributions of the GEW*$_3$ *model are not log-concave on their entire domains.*

Proof. For the GEW$_3$ model, the second derivatives of the natural logs of the full conditional posteriors are determined as follows

$$\begin{aligned}
\ell_{3,\theta_1} &= \ln\left[\pi_3\left(\theta_1\left|\underline{x},\theta_2,\theta_3,\theta_4,\beta\right.\right)\right] \\
&= -\ln(\theta_1) - \frac{\ln^2(\theta_1)}{2c_{21}^2} + \frac{c_{20}\ln(\theta_1)}{c_{21}^2} - \sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad -\theta_1\sum_{i=1}^{k}r_i - \sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta} \\
\frac{\partial\ell_{3,\theta_1}}{\partial\theta_1} &= -\frac{1}{\theta_1} - \frac{\ln(\theta_1)}{c_{21}^2\theta_1} + \frac{c_{20}}{c_{21}^2\theta_1} - \sum_{i=1}^{k}r_i + \sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad +\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta} \\
\frac{\partial^2\ell_{3,\theta_1}}{\partial\theta_1^2} &= \frac{\ln(\theta_1)+c_{21}^2-c_{20}-1}{c_{21}^2\theta_1^2} - \sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad -\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}
\end{aligned}$$

$$\begin{aligned}
\ell_{3,\theta_2} &= \ln\left[\pi_3\left(\theta_2\left|\underline{x},\theta_1,\theta_3,\theta_4,\beta\right.\right)\right] \\
&= -\ln(\theta_2) - \frac{\ln^2(\theta_2)}{2c_{23}^2} + \frac{c_{22}\ln(\theta_2)}{c_{23}^2} - \sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad -\theta_2\sum_{i=1}^{k}\frac{r_i}{T_i} - \sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta} \\
\frac{\partial\ell_{3,\theta_2}}{\partial\theta_2} &= -\frac{1}{\theta_2} - \frac{\ln(\theta_2)}{c_{23}^2\theta_2} + \frac{c_{22}}{c_{23}^2\theta_2} - \sum_{i=1}^{k}\frac{r_i}{T_i} + \sum_{i=1}^{k}(n_i-r_i)\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad +\sum_{i=1}^{k}\sum_{j=1}^{r_i}\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta} \\
\frac{\partial^2\ell_{3,\theta_2}}{\partial\theta_2^2} &= \frac{\ln(\theta_2)+c_{23}^2-c_{22}-1}{c_{23}^2\theta_2^2} - \sum_{i=1}^{k}(n_i-r_i)\,\frac{1}{T_i}\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad -\sum_{i=1}^{k}\sum_{j=1}^{r_i}\frac{1}{T_i}\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}
\end{aligned}$$

$$\begin{aligned}
\ell_{3,\theta_3} &= \ln\left[\pi_3\left(\theta_3\,|\underline{x},\theta_1,\theta_2,\theta_4,\beta\right)\right]\\
&= -\ln(\theta_3)-\frac{\ln^2(\theta_3)}{2c_{25}^2}+\frac{c_{24}\ln(\theta_3)}{c_{25}^2}-\sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\\
&\quad -\theta_3\sum_{i=1}^{k}r_iV_i-\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}\\
\frac{\partial\ell_{3,\theta_3}}{\partial\theta_3} &= -\frac{1}{\theta_3}-\frac{\ln(\theta_3)}{c_{25}^2\theta_3}+\frac{c_{24}}{c_{25}^2\theta_3}-\sum_{i=1}^{k}r_iV_i+\sum_{i=1}^{k}(n_i-r_i)\,T_iV_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\\
&\quad +\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_iV_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}\\
\frac{\partial^2\ell_{3,\theta_3}}{\partial\theta_3^2} &= \frac{\ln(\theta_3)+c_{25}^2-c_{24}-1}{c_{25}^2\theta_3^2}-\sum_{i=1}^{k}(n_i-r_i)\,T_iV_i^2\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\\
&\quad -\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_iV_i^2\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}
\end{aligned}$$

$$\begin{aligned}
\ell_{3,\theta_4} &= \ln\left[\pi_3\left(\theta_4\,|\underline{x},\theta_1,\theta_2,\theta_3,\beta\right)\right]\\
&= -\ln(\theta_4)-\frac{\ln^2(\theta_4)}{2c_{27}^2}+\frac{c_{26}\ln(\theta_4)}{c_{27}^2}-\sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\\
&\quad -\theta_4\sum_{i=1}^{k}\frac{r_iV_i}{T_i}-\sum_{i=1}^{k}\sum_{j=1}^{r_i}T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}\\
\frac{\partial\ell_{3,\theta_4}}{\partial\theta_4} &= -\frac{1}{\theta_4}-\frac{\ln(\theta_4)}{c_{27}^2\theta_4}+\frac{c_{26}}{c_{27}^2\theta_4}-\sum_{i=1}^{k}\frac{r_iV_i}{T_i}+\sum_{i=1}^{k}(n_i-r_i)\,V_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\\
&\quad +\sum_{i=1}^{k}\sum_{j=1}^{r_i}V_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}\\
\frac{\partial^2\ell_{3,\theta_4}}{\partial\theta_4^2} &= \frac{\ln(\theta_4)+c_{27}^2-c_{26}-1}{c_{27}^2\theta_4^2}-\sum_{i=1}^{k}(n_i-r_i)\,\frac{V_i^2}{T_i}\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\\
&\quad -\sum_{i=1}^{k}\sum_{j=1}^{r_i}\frac{V_i^2}{T_i}\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}
\end{aligned}$$

$$
\begin{aligned}
\ell_{3,\beta} &= \ln\left[\pi_3\left(\beta\,|\underline{x},\theta_1,\theta_2,\theta_3,\theta_4\right)\right] \\
&= -\ln(\beta) - \frac{\ln^2(\beta)}{2c_{29}^2} + \frac{c_{28}\ln(\beta)}{c_{29}^2} - \sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta} \\
&\quad +\ln(\beta)\sum_{i=1}^{k} r_i - \sum_{i=1}^{k}\sum_{j=1}^{r_i} T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta} + (\beta-1)\sum_{i=1}^{k}\sum_{j=1}^{r_i}\ln\left(x_{ij}\right) \\
\frac{\partial \ell_{3,\beta}}{\partial\beta} &= -\frac{1}{\beta} - \frac{\ln(\beta)}{c_{29}^2\beta} + \frac{c_{28}}{c_{29}^2\beta} - \sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\ln(\tau_i) \\
&\quad +\frac{1}{\beta}\sum_{i=1}^{k} r_i - \sum_{i=1}^{k}\sum_{j=1}^{r_i} T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}\ln\left(x_{ij}\right) + \sum_{i=1}^{k}\sum_{j=1}^{r_i}\ln\left(x_{ij}\right) \\
\frac{\partial^2 \ell_{3,\beta}}{\partial\beta^2} &= \frac{\ln(\beta)+c_{29}^2-c_{28}-1}{c_{29}^2\beta^2} - \sum_{i=1}^{k}(n_i-r_i)\,T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\tau_i^{\beta}\ln^2(\tau_i) \\
&\quad -\frac{1}{\beta^2}\sum_{i=1}^{k} r_i - \sum_{i=1}^{k}\sum_{j=1}^{r_i} T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)x_{ij}^{\beta}\ln^2\left(x_{ij}\right).
\end{aligned}
$$

From the first part of each of the second derivatives, it is clear that there will be some value of the parameters of interest for which that part of the second derivatives are positive. It then follows that the full conditional posteriors for the GEW_3 model are not log-concave on their entire domains. □

3.3.5 MCMC Implementation

From Theorem 3.1 and Theorem 3.2 it is evident that the full conditional posterior distributions of the GEW_1 and GEW_2 models are log-concave, subject to the conditions $c_{10}, c_{12}, c_{14}, c_{16} \geq 1$, and at least one failure occurring. Under these conditions, it is possible to use the ARS method of Gilks and Wild (1992) to sample from the full conditional posterior distributions at each iteration of the Gibbs sampler. The slice sampling algorithm of Neal (2003) can also be utilised.

As established in Theorem 3.3, the full conditional posterior distributions for the GEW_3 model are not log-concave over their entire domains. This model requires the implementation of more advanced MCMC techniques such as ARMS, proposed by Gilks et al. (1995), slice sampling, presented in Neal (2003), or independent doubly adaptive rejection Metropolis sampling, introduced by Martino et al. (2015).

The Bayesian data analysis software OpenBUGS is used in this thesis to draw posterior samples to base inference on. MCMC software such as OpenBUGS can work with intricate (non-log-concave) densities that may require the implementation of the slice sampler.

3.4 Application

An ALT data set from ReliaSoft Corporation (2015) is used in this application of the GEW model. The data relates to failure times (in hours) obtained from an electronics epoxy packaging accelerated life test, where temperature and relative humidity are used as the accelerated stressors. The normal use conditions are $T_u = 350$K and $S_u = 0.3$. The data set consists of 17 items and the failure times are presented in Table 3.1.

Table 3.1: Failure times for an electronics epoxy packaging ALT.

#	Temperature (K)	Humidity	Failure time	#	Temperature (K)	Humidity	Failure time
1	406	0.5	248	10	416	0.7	340
2	406	0.5	456	11	416	0.7	543
3	406	0.7	528	12	426	0.5	92
4	406	0.7	731	13	426	0.5	105
5	406	0.7	813	14	426	0.5	184
6	416	0.5	164	15	426	0.7	155
7	416	0.5	176	16	426	0.7	219
8	416	0.5	289	17	426	0.7	235
9	416	0.7	319				

The sensitivity of the GEW model is explored via different choices of hyperparameters for the models in Section 3.3. Table 3.2 contains the specifications for the priors used in this application. Flat uniform and gamma priors are imposed on all the parameters for the GEW_1 and $\text{GEW}_{2,1}$ models. For a vague gamma prior, the hyperparameters are chosen which result in a large prior variance (Park and Casella, 2008). Various combinations of $\{0.01, 0.05, 0.5, 1\}$ for the parameters of vague gamma priors are proposed by Lesaffre and Lawson (2012). The hyperparameters for the flat gamma prior are chosen to be close to those in Lesaffre and Lawson (2012), and other combinations in the proposed range give similar results. Subjective gamma priors, with mean 5 but different variances, are chosen for the $\text{GEW}_{2,2}$ and $\text{GEW}_{2,3}$ models. The choice of these parameters is not based on any prior information and are solely used to investigate the sensitivity of the GEW model. Other choices of subjective priors likewise give variable results, depending on the choice of hyperparameters. The hyperparameters for the $\text{GEW}_{3,1}$ and $\text{GEW}_{3,2}$ models are chosen such that these priors have similar means and variances to the priors of the $\text{GEW}_{2,2}$ and $\text{GEW}_{2,3}$ models, respectively.

Table 3.2: Prior specifications for the GEW models.

Model	θ_1	θ_2	θ_3	θ_4	β
GEW_1	$U(0,100)$	$U(0,100)$	$U(0,100)$	$U(0,100)$	$U(0,100)$
$GEW_{2,1}$	$\Gamma(1,0.001)$	$\Gamma(1,0.001)$	$\Gamma(1,0.001)$	$\Gamma(1,0.001)$	$\Gamma(1,0.001)$
$GEW_{2,2}$	$\Gamma(2.5,0.5)$	$\Gamma(2.5,0.5)$	$\Gamma(2.5,0.5)$	$\Gamma(2.5,0.5)$	$\Gamma(2.5,0.5)$
$GEW_{2,3}$	$\Gamma(5,1)$	$\Gamma(5,1)$	$\Gamma(5,1)$	$\Gamma(5,1)$	$\Gamma(5,1)$
$GEW_{3,1}$	$LN(1.44,0.58^2)$	$LN(1.44,0.58^2)$	$LN(1.44,0.58^2)$	$LN(1.44,0.58^2)$	$LN(1.44,0.58^2)$
$GEW_{3,2}$	$LN(1.52,0.43^2)$	$LN(1.52,0.43^2)$	$LN(1.52,0.43^2)$	$LN(1.52,0.43^2)$	$LN(1.52,0.43^2)$

The models are implemented in OpenBUGS to generate posterior samples. A single Markov chain is initiated for each model with a burn-in of 50000 iterations. The modified Gelman-Rubin statistic, proposed by Brooks and Gelman (1998), and trace plots are used to confirm that the Markov chains for the above models all converge well in advance of 50000 iterations. See Appendix A for the convergence diagnostics for the GEW models.

Each chain is then run for another 200000 iterations to obtain posterior samples to base inference on. Table 3.3 up to Table 3.8 summarise the sample standard deviations, Monte Carlo errors, and the Monte Carlo errors as a percentage of the sample standard deviation for the parameters of the the GEW models. For this number of samples, the Monte Carlo error is less than 5% of the sample standard deviation for all parameters of the GEW models. This is considered the rule of thumb in Spiegelhalter et al. (2014) as to whether enough samples have been generated for inference. The accuracy of the posterior estimates, calculated from the 200000 samples generated for each parameter in this application, is thus considered sufficient.

Table 3.3: Monte Carlo error as a percentage of the standard deviation for GEW_1.

	Standard Deviation	MC Error	%
θ_1	2.8837	0.094780	3.2867%
θ_2	3.4488	0.116900	3.3896%
θ_3	0.5781	0.003169	0.5482%
θ_4	0.5802	0.002951	0.5086%
β	0.3621	0.009556	2.6390%

Table 3.4: Monte Carlo error as a percentage of the standard deviation for $GEW_{2,1}$.

	Standard Deviation	MC Error	%
θ_1	2.7709	0.091310	3.2953%
θ_2	3.3645	0.114800	3.4121%
θ_3	0.5664	0.002825	0.4988%
θ_4	0.5698	0.002908	0.5104%
β	0.3571	0.009575	2.6813%

Table 3.5: Monte Carlo error as a percentage of the standard deviation for $GEW_{2,2}$.

	Standard Deviation	MC Error	%
θ_1	1.9465	0.044300	2.2759%
θ_2	2.3221	0.059740	2.5727%
θ_3	0.6856	0.003050	0.4449%
θ_4	0.6974	0.003105	0.4452%
β	0.3055	0.006984	2.2861%

Table 3.6: Monte Carlo error as a percentage of the standard deviation for $GEW_{2,3}$.

	Standard Deviation	MC Error	%
θ_1	1.5957	0.028950	1.8143%
θ_2	1.8390	0.038800	2.1098%
θ_3	0.7514	0.002920	0.3886%
θ_4	0.7624	0.003023	0.3965%
β	0.2821	0.005478	1.9419%

Table 3.7: Monte Carlo error as a percentage of the standard deviation for $GEW_{3,1}$.

	Standard Deviation	MC Error	%
θ_1	1.7539	0.036930	2.1056%
θ_2	2.0889	0.048490	2.3213%
θ_3	0.6588	0.002541	0.3857%
θ_4	0.6684	0.002626	0.3929%
β	0.2974	0.006337	2.1308%

Table 3.8: Monte Carlo error as a percentage of the standard deviation for $GEW_{3,2}$.

	Standard Deviation	MC Error	%
θ_1	1.4746	0.024710	1.6757%
θ_2	1.7379	0.034980	2.0128%
θ_3	0.6858	0.002463	0.3591%
θ_4	0.6966	0.002556	0.3669%
β	0.2684	0.005103	1.9013%

The DIC is one of the most popular measures that is used to compare various models in Bayesian ALT, particularly where MCMC methods are utilised. The DIC values for the various GEW models are given in Table 3.9. The GEW_1 and $GEW_{2,1}$ models both exhibit the lowest DIC, meaning that these would typically be the best models to use. The $GEW_{2,2}$, $GEW_{2,3}$, $GEW_{3,1}$ and $GEW_{3,2}$ models show an increasingly worse fit to the data as subjective priors with smaller variances are implemented. This may be due to the posterior being dominated by the prior, when the prior variance is very small. Considering the guidelines in Burnham and Anderson (1998), there is considerably less support to choose models $GEW_{2,2}$, $GEW_{2,3}$, $GEW_{3,1}$, and almost no support to use model GEW_4.

Table 3.9: Deviance information criterion for the GEW models.

Model	DIC
GEW_1	228.4
$GEW_{2,1}$	228.4
$GEW_{2,2}$	231.8
$GEW_{2,3}$	236.0
$GEW_{3,1}$	235.4
$GEW_{3,2}$	239.4

The summary statistics for the marginal posterior distributions of the GEW models are provided in Table 3.10. For the models with flat priors, that is GEW_1 and $GEW_{2,1}$, reasonably similar summary statistics are produced. When subjective priors are employed, one can note that the central location measures of the marginal posteriors are progressively, as more certainty is given by the prior, shifted towards the region of central tendency of the prior. As the variance is reduced between the priors imposed on the $GEW_{2,2}$ and $GEW_{2,3}$ models, the posterior is dominated to a greater extent by the prior. The same holds for the $GEW_{3,1}$ and $GEW_{3,2}$ models.

Table 3.10: Summary statistics for the GEW models.

Model	Parameter	Mean	Standard Deviation	2.5th Percentile	Median	97.5th Percentile
GEW_1	θ_1	3.6597	2.8837	0.1155	2.9560	10.3400
	θ_2	7.2650	3.4488	0.6997	7.4680	13.6698
	θ_3	0.5989	0.5781	0.0157	0.4247	2.1460
	θ_4	0.6053	0.5802	0.0157	0.4329	2.1390
	β	1.9474	0.3621	1.2880	1.9300	2.7100
$GEW_{2,1}$	θ_1	3.3681	2.7709	0.1046	2.6440	10.0200
	θ_2	7.7020	3.3645	0.8950	7.8670	13.8800
	θ_3	0.5870	0.5664	0.0156	0.4165	2.1010
	θ_4	0.5973	0.5698	0.0156	0.4270	2.1090
	β	1.9722	0.3571	1.3280	1.9570	2.7190
$GEW_{2,2}$	θ_1	3.8210	1.9465	0.7810	3.5780	8.1060
	θ_2	5.7987	2.3221	1.5860	5.7200	10.4900
	θ_3	1.1275	0.6856	0.1944	0.9976	2.8000
	θ_4	1.1457	0.6974	0.2005	1.0110	2.8460
	β	1.7896	0.3055	1.2260	1.7780	2.4260
$GEW_{2,3}$	θ_1	4.2772	1.5957	1.5950	4.1380	7.7670
	θ_2	5.3606	1.8390	2.1520	5.2430	9.2860
	θ_3	1.7371	0.7514	0.5791	1.6330	3.4850
	θ_4	1.7620	0.7624	0.5882	1.6570	3.5270
	β	1.8549	0.2821	1.3450	1.8410	2.4490
$GEW_{3,1}$	θ_1	4.0476	1.7539	1.3630	3.8080	8.0800
	θ_2	5.3529	2.0889	1.8930	5.1650	9.8590
	θ_3	1.6517	0.6588	0.6656	1.5500	3.2060
	θ_4	1.6703	0.6684	0.6673	1.5660	3.2490
	β	1.8080	0.2974	1.2790	1.7910	2.4460
$GEW_{3,2}$	θ_1	4.3918	1.4746	1.9730	4.2430	7.6040
	θ_2	5.3018	1.7379	2.4260	5.1220	9.1830
	θ_3	2.1814	0.6858	1.0940	2.0960	3.7530
	θ_4	2.1993	0.6966	1.0990	2.1130	3.7990
	β	1.9093	0.2684	1.4270	1.8930	2.4780

The marginal posterior distributions for the GEW models are shown in Figure 3.1 up to Figure 3.6. It can be observed that the GEW_1 and $GEW_{2,1}$ models produce very similar marginal posteriors. For these models, it can also be noted that the marginal posterior for β is somewhat symmetrical, while those of the other parameters are heavily skewed. The marginal posteriors of the $GEW_{2,2}$, $GEW_{2,3}$, $GEW_{3,1}$and $GEW_{3,2}$ models show how the density is increasingly concentrated towards the region of central tendency of the priors as more prior certainty is conveyed via smaller prior variances. The marginal posteriors for these models are less skewed than those produced by the models where flat priors are used.

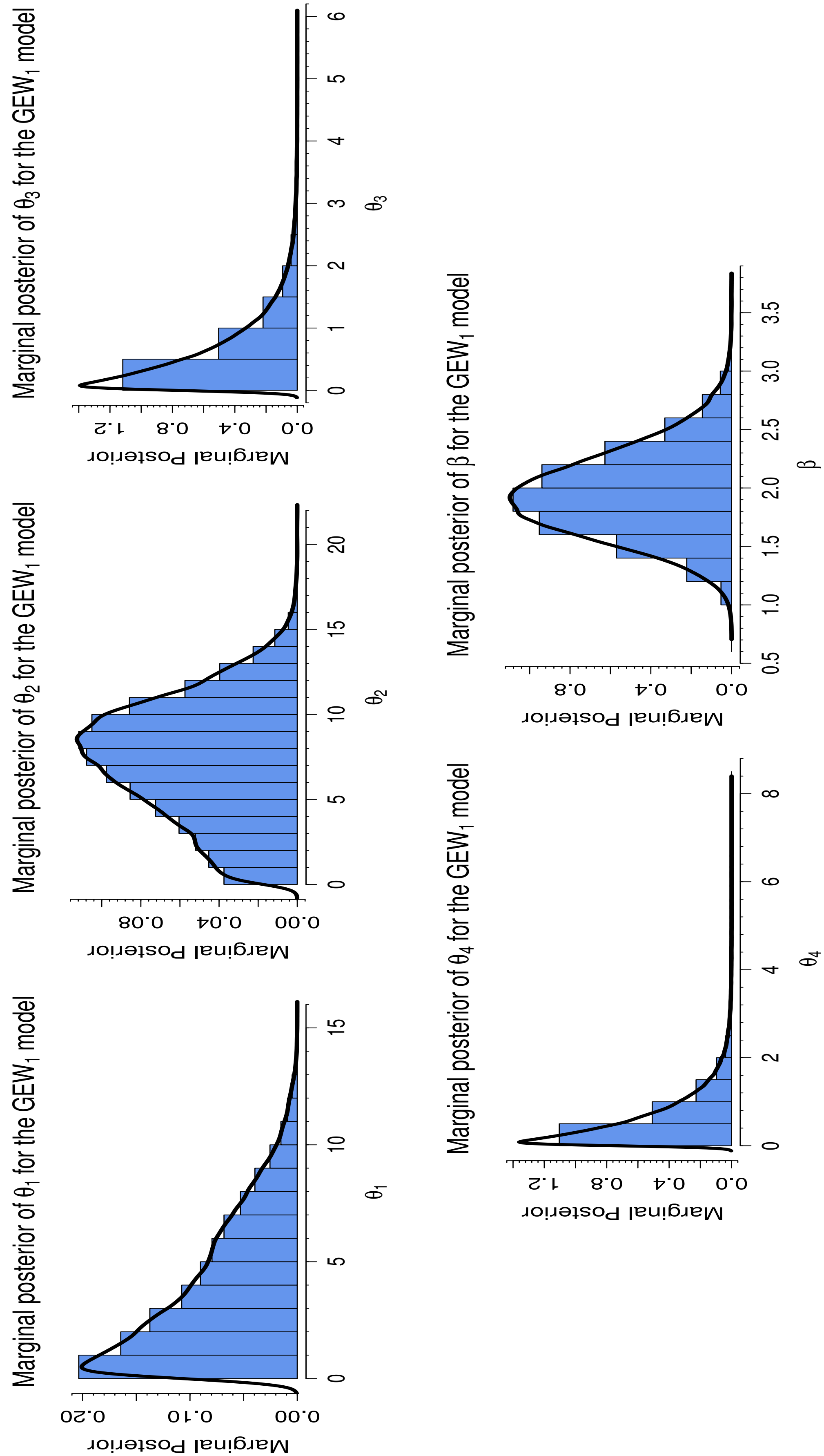

Figure 3.1: Marginal posterior distributions for the GEW_1 model.

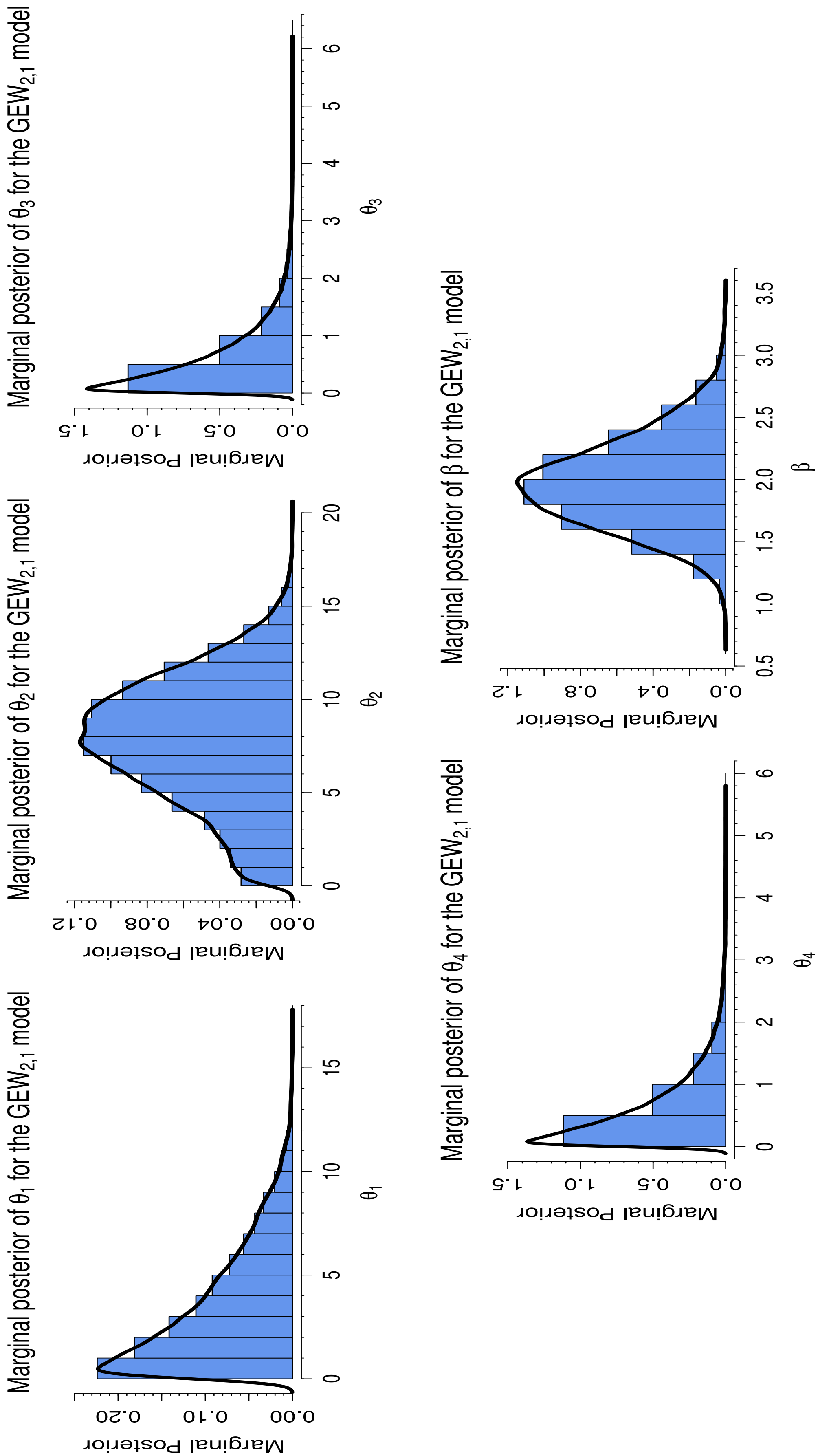

Figure 3.2: Marginal posterior distributions for the $\mathrm{GEW}_{2,1}$ model.

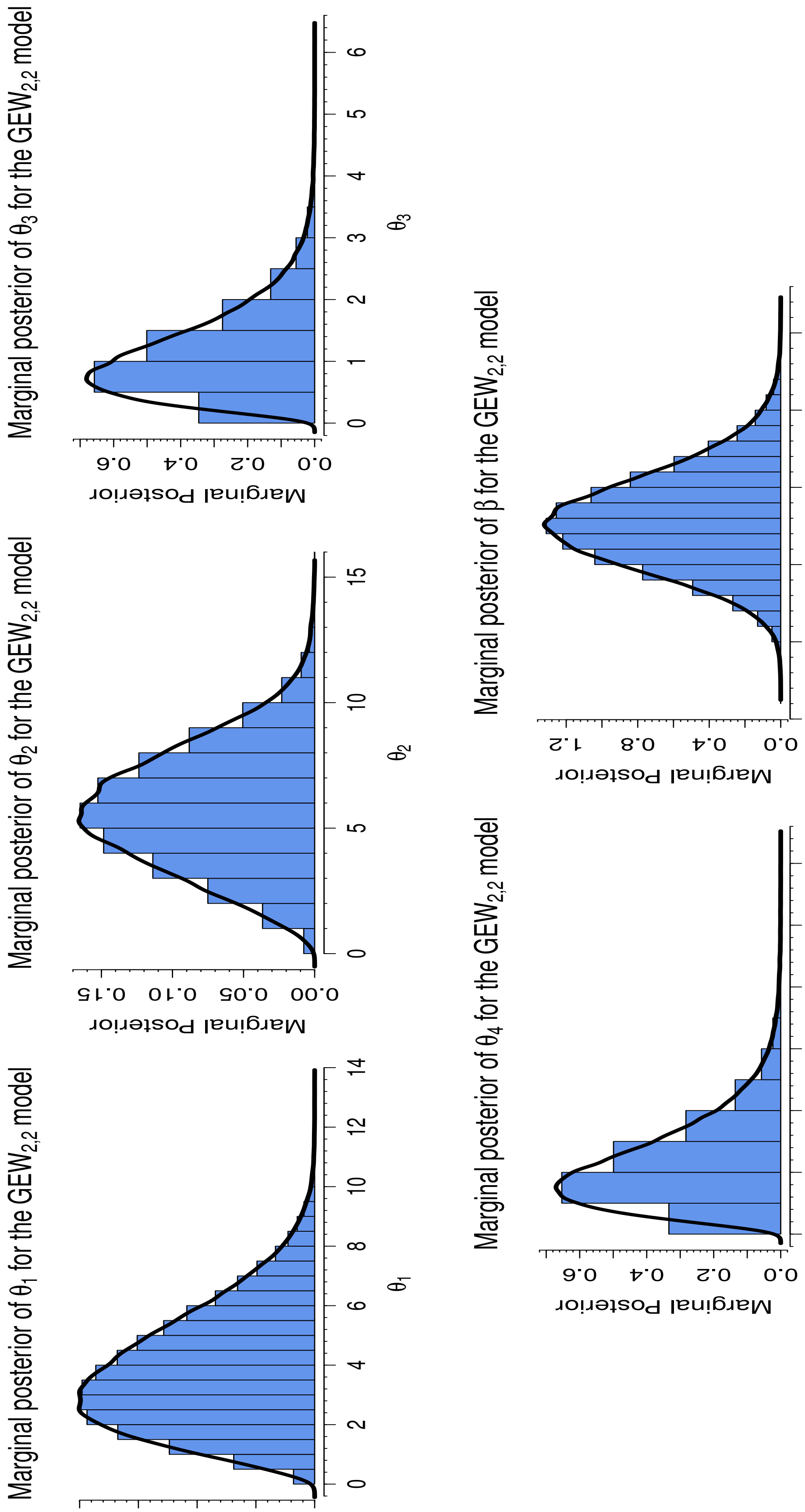

Figure 3.3: Marginal posterior distributions for the $GEW_{2,2}$ model.

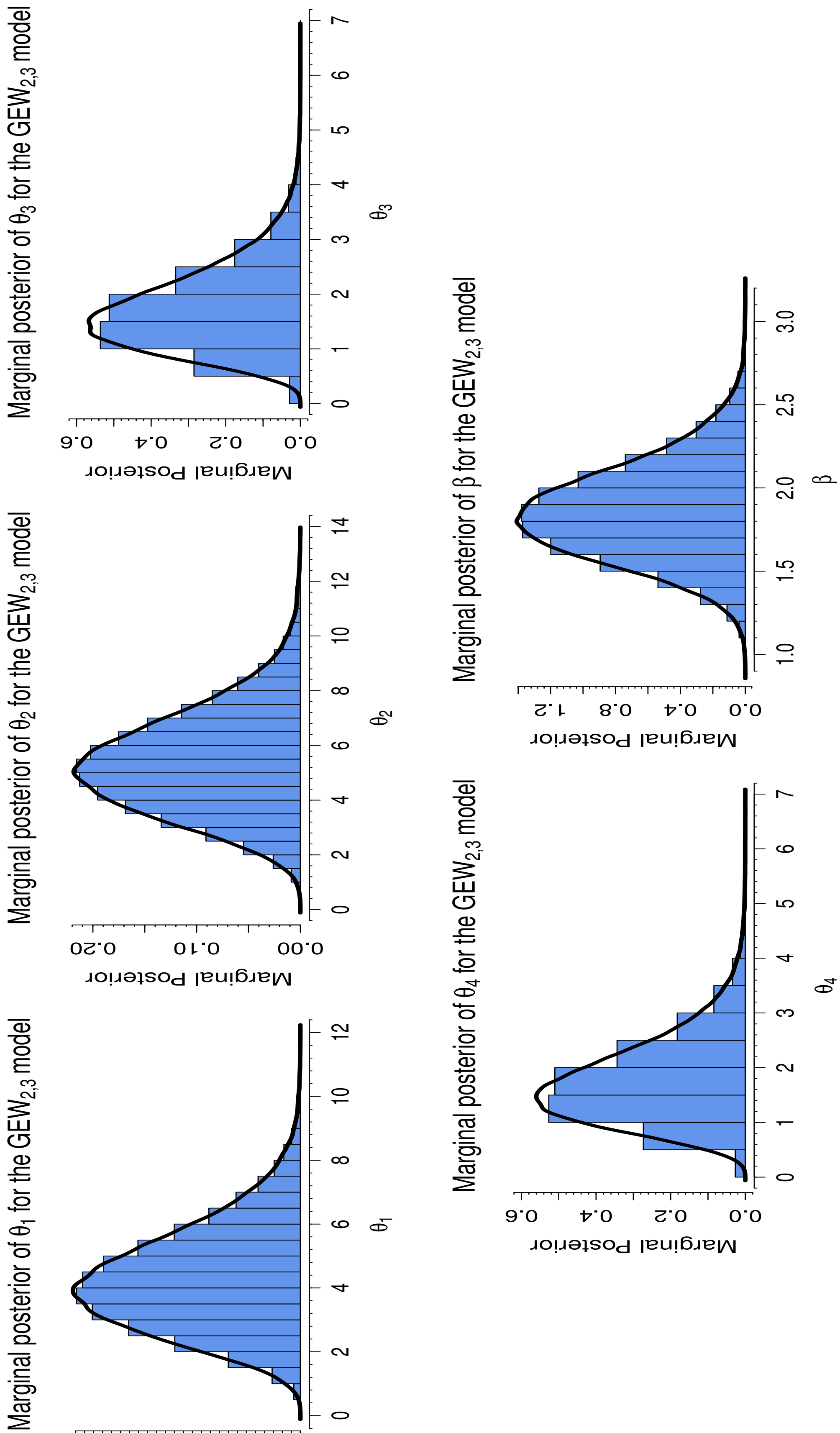

Figure 3.4: Marginal posterior distributions for the $\mathrm{GEW}_{2,3}$ model.

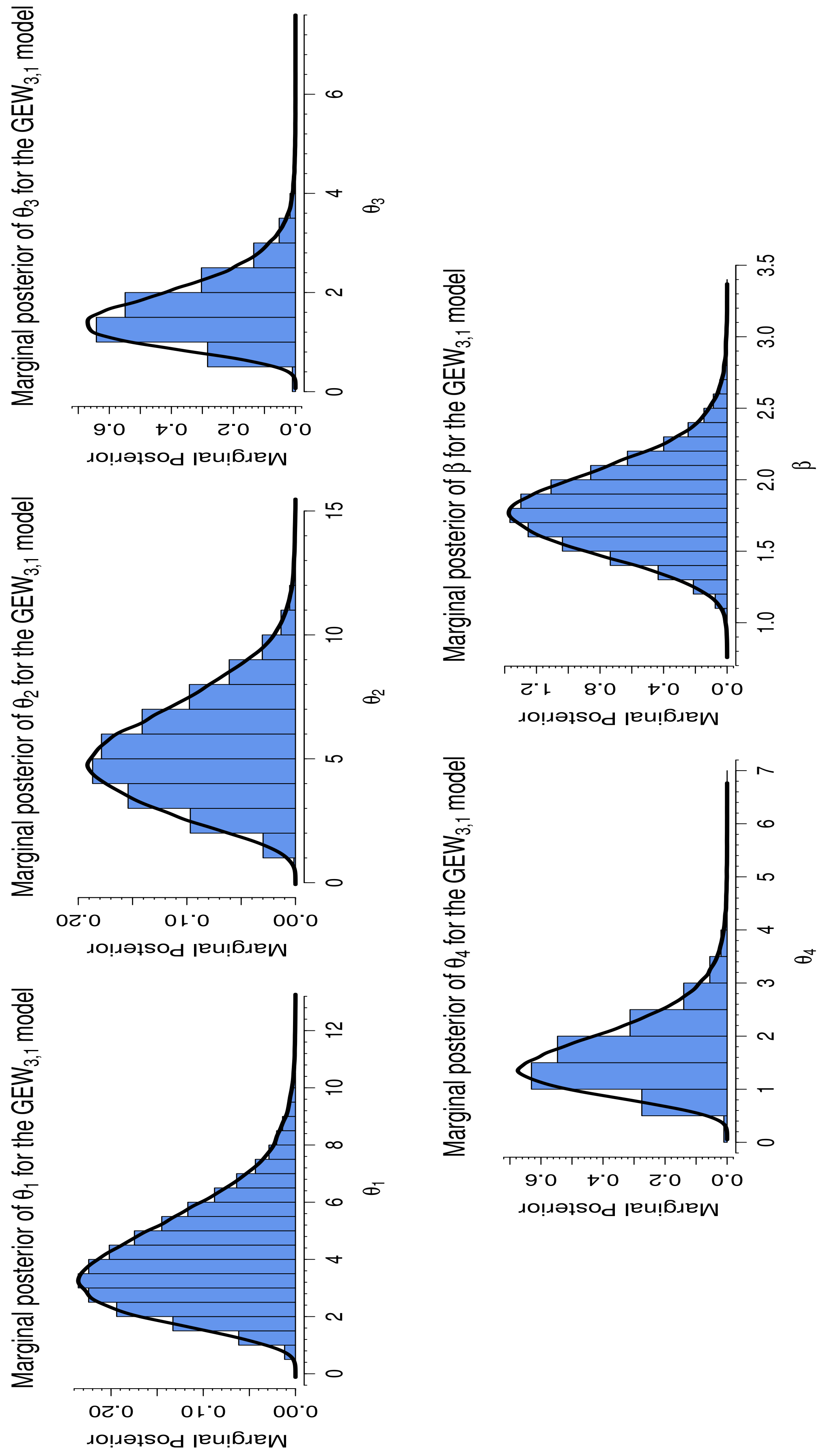

Figure 3.5: Marginal posterior distributions for the $GEW_{3,1}$ model.

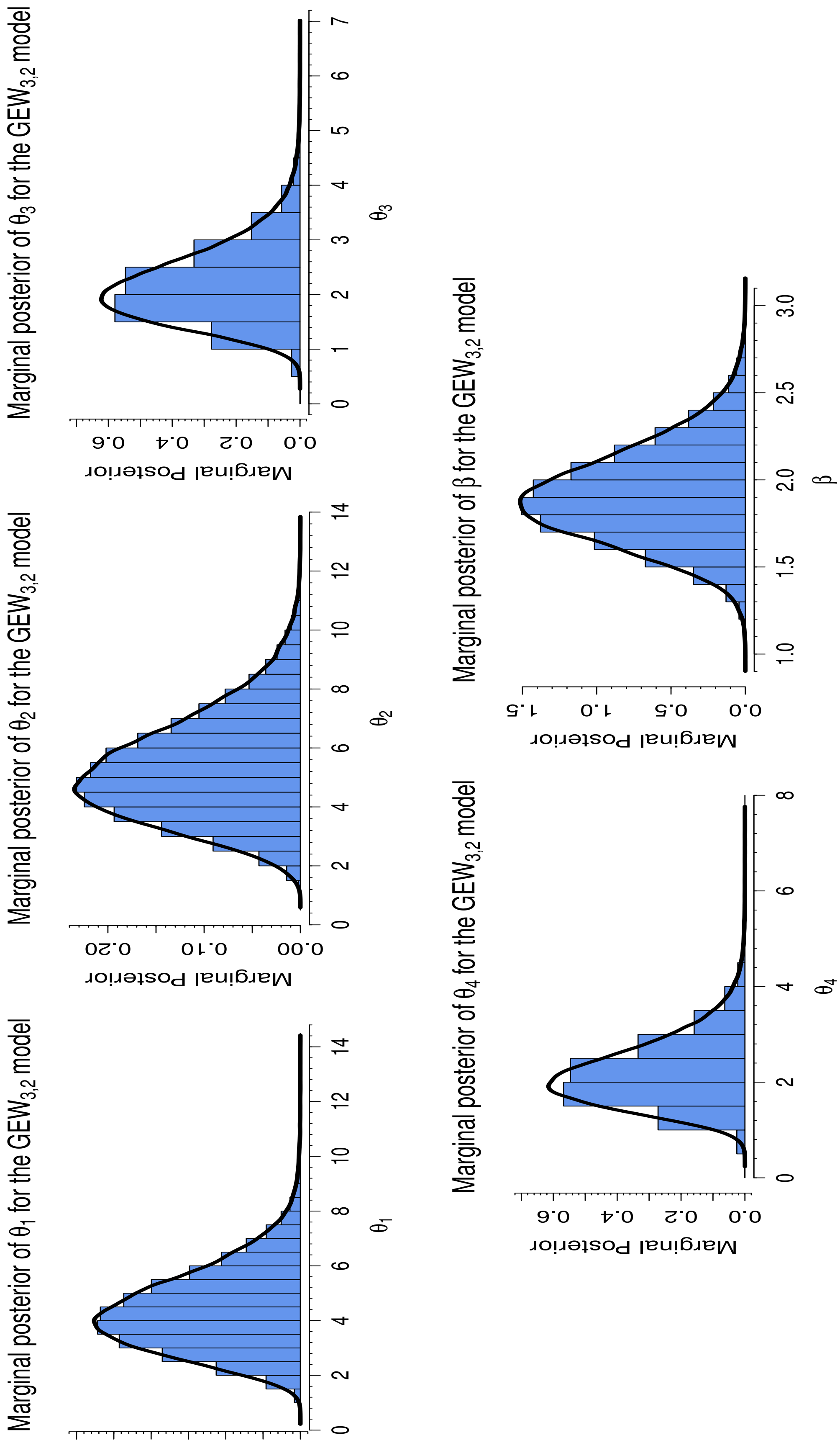

Figure 3.6: Marginal posterior distributions for the $GEW_{3,2}$ model.

Finally, the GEW model aims to obtain the predictive reliability for the item being tested. The predictive reliability at normal use stress levels is given by

$$R(x_u\,|\underline{x}) = \int\int\int\int\int R(x_u\,|\theta_1,\theta_2,\theta_3,\theta_4,\beta)\,\pi(\theta_1,\theta_2,\theta_3,\theta_4,\beta\,|\underline{x})\,d\theta_1 d\theta_2 d\theta_3 d\theta_4 d\beta, \tag{3.10}$$

where $R(x_u\,|\theta_1,\theta_2,\theta_3,\theta_4,\beta)$ is the Weibull reliability function at use stress levels T_u and S_u. In order to evaluate $R(x_u\,|\underline{x})$, the following is carried out:

1. Sample $\theta_1,\theta_2,\theta_3,\theta_4$ and β from the posterior M times, where M is a sufficiently large number.

2. Calculate the integral in (3.10) by the Monte Carlo average

$$R(x_u\,|\underline{x}) \approx \frac{1}{M}\sum_{m=1}^{M} R\left(x_u\,\middle|\,\theta_1^{(m)},\theta_2^{(m)},\theta_3^{(m)},\theta_4^{(m)},\beta^{(m)}\right),$$

which is the expected reliability at time x_u, using the posterior sample

$$\left\{\theta_1^{(m)},\theta_2^{(m)},\theta_3^{(m)},\theta_4^{(m)},\beta^{(m)}\right\}, m = 1,\ldots,M.$$

Table 3.11 and Figure 3.7 show the predictive reliability of the models. Consistent with the previous findings, the predictive reliability results for the GEW$_1$ and GEW$_{2,1}$ models are comparable. The GEW model is not sensitive to different choices of flat priors. The models that make use of subjective priors produce significantly higher predictive reliability results in this case, compared to the models where flat priors are used. The use of subjective priors can result in either an underestimation or overestimation of the predictive reliability, compared to data driven results obtained from employing flat priors, depending on the choice of hyperparameters for the GEW$_2$ and GEW$_3$ models.

Table 3.11: Predictive reliability for the GEW models at use stress $T_u = 350$, $S_u = 0.3$.

Time	GEW$_1$	GEW$_{2,1}$	GEW$_{2,2}$	GEW$_{2,3}$	GEW$_{3,1}$	GEW$_{3,2}$
1	0.999987	0.999990	0.999990	0.999998	0.999996	0.999999
2	0.999968	0.999974	0.999974	0.999994	0.999990	0.999998
3	0.999943	0.999954	0.999956	0.999989	0.999983	0.999996
⋮	⋮	⋮	⋮	⋮	⋮	⋮
500	0.826214	0.836377	0.894309	0.952155	0.947369	0.976127
501	0.825665	0.835848	0.893973	0.951989	0.947192	0.976040
502	0.825115	0.835319	0.893638	0.951823	0.947015	0.975953
⋮	⋮	⋮	⋮	⋮	⋮	⋮
2000	0.179376	0.183626	0.369133	0.586761	0.570840	0.735105
2001	0.179176	0.183418	0.368884	0.586517	0.570593	0.734907
2002	0.178977	0.183210	0.368636	0.586273	0.570346	0.734708

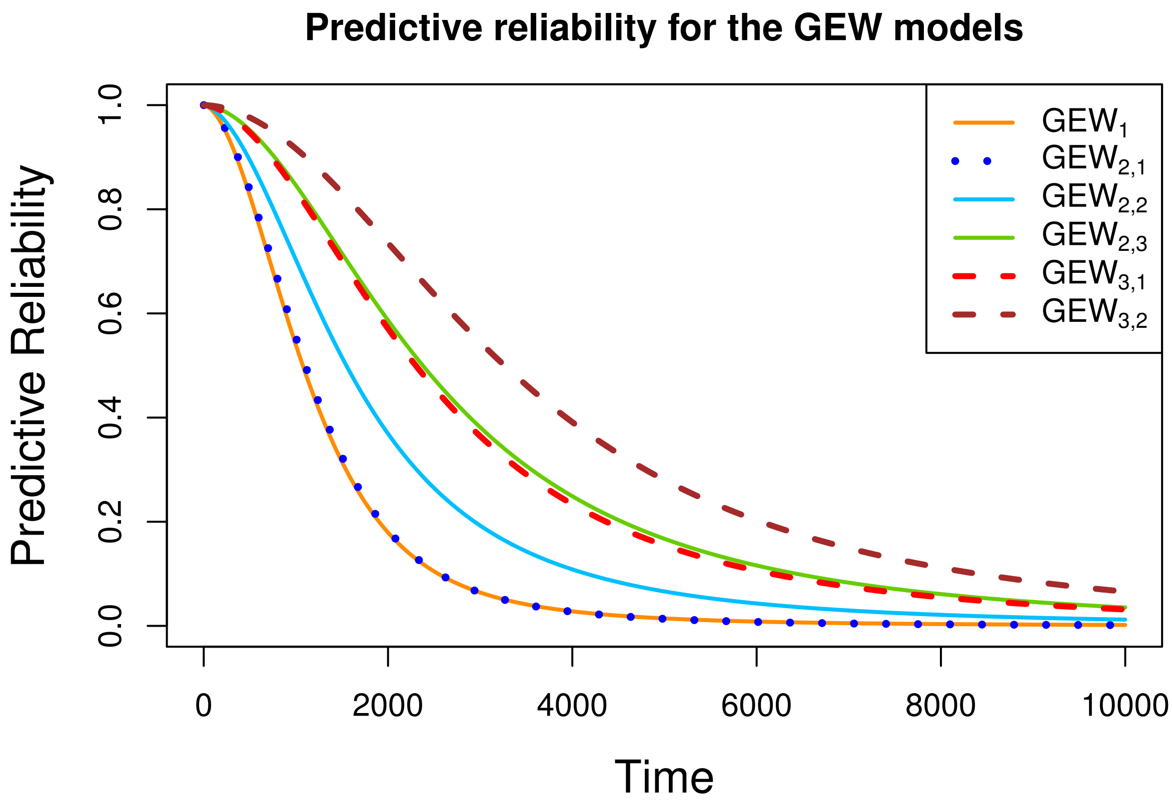

Figure 3.7: Predictive reliability for the GEW models at use stress $T_u = 350$, $S_u = 0.3$.

3.5 Conclusion

In this chapter, a Bayesian ALT model is presented, where the generalised Eyring model is used as the TTF and lifetimes follow a Weibull distribution. The generalised Eyring model allows for more than one accelerated stressor, whereas many other TTFs only permit the use of a single accelerated stressor. A general likelihood formulation is given, which allows for complete samples, type-I censoring and type-II censoring. Variations of the GEW model are defined by means of the choice of prior distributions for the model parameters. The full conditional posterior distributions for each model are given and the log-concavity for each is also assessed. Under no, to very lenient, conditions for the GEW_1 and GEW_2 models, the ARS method within the Gibbs sampler can be used to obtain posterior samples from the full conditional distributions. Alternatively, since these are complex models, slice sampling can also be used to obtain posterior samples. For the GEW_3 model, more advanced MCMC methods such as slice sampling, must implemented for posterior sampling. The models are applied to a data set concerning an electronics epoxy packaging accelerated life test, where temperature and relative humidity are used as the accelerated stressors, and the results are compared.

Different choices of the hyperparameters for the GEW_2 and GEW_3 models are considered. The

Bayesian data analysis software OpenBUGS is used to generate posterior samples for the models. The convergence of the Markov chains are assessed via trace plots and the modified Gelman-Rubin statistic. The fit of the models are compared via the DIC. The GEW_1 and $GEW_{2,1}$ models show the lowest DIC. These two models, where flat priors are imposed on the parameters, produce alike summary statistics, marginal posteriors and predictive reliability results. The use of subjective priors in the $GEW_{2,2}$, $GEW_{2,3}$, $GEW_{3,1}$ and $GEW_{3,2}$ models lead to much different results, as noted in the summary statistics, marginal posteriors and significantly higher predictive reliability. Subjective priors can thus be utilised to adjust reliability estimates if the researcher is of the opinion that the use of flat priors either overestimates or underestimates the predictive reliability. This can be achieved via the choice of hyperparameters for the priors in the GEW_2 and GEW_3 models. It may also be possible to adjust a single parameter that is related to a specific stress, by means of a subjective prior, if the researcher is of the opinion that the effect of that specific accelerated stress is not correctly expressed by a model where only flat priors are used. The use of priors with very small variance in the GEW_2 and GEW_3 models is not recommended and great caution should be exercised by the reliability engineer if doing so.

Chapter 4

The Generalised Eyring-Birnbaum-Saunders Model

4.1 Introduction

In this chapter, a Bayesian ALT model is considered, which uses the Birnbaum-Saunders distribution as the life distribution. The generalised Eyring model with one thermal stressor and one non-thermal stressor, is used as the TTF. To our knowledge, a Bayesian ALT approach with this life distribution and TTF has not been used before. As pointed out in Section 2.11, the Birnbaum-Saunders distribution can be considered a competing model, compared to other two-parameter life distributions such as the Weibull, log-normal and gamma distributions. The Birnbaum-Saunders distribution is motivated by Birnbaum and Saunders (1969), seeing that it is derived by taking the characteristics of the fatigue process into consideration. We believe that the GEBS model will be a valuable contribution to the ALT field and literature.

First, a general likelihood formulation for the GEBS model is specified. This allows for a model that is capable of handling uncensored, type-I censored or type-II censored data. Second, the prior, posterior and full conditional posterior distributions for the model is specified. The GEBS model has a mathematically intractable posterior distribution, thus MCMC methods must me employed to generate posterior samples for inference. Third, the GEBS model is applied to a real data set, where a sensitivity analysis is performed in terms of the values for the hyperparameters. The chapter is concluded with some commentary on the GEBS model.

4.2 The GEBS Model Specification

Let X be a continuous random variable that follows a Birnbaum-Saunders distribution with shape parameter α and scale parameter β $(\alpha > 0, \beta > 0)$. The PDF of X is then given by

$$f(x|\alpha,\beta) = \frac{x+\beta}{2\sqrt{2\pi}\alpha\sqrt{\beta}\sqrt{x_i^3}} \exp\left[-\frac{1}{2\alpha^2}\left(\frac{x}{\beta}+\frac{\beta}{x}-2\right)\right], \quad x>0. \tag{4.1}$$

The Birnbaum-Saunders reliability function at some time τ is given by

$$R(\tau) = 1-\Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{\tau}{\beta}}-\sqrt{\frac{\beta}{\tau}}\right)\right], \tag{4.2}$$

where $\Phi(\cdot)$ is the CDF of the standard normal distribution. Consider one thermal stressor and one non-thermal stressor. Indicate the k distinct accelerated levels of the stressors by $\{T_i, S_i\}, i=1,\ldots,k$, where $T_i, i=1,\ldots,k$, is the accelerated levels of the thermal stressor and $S_i, i=1,\ldots,k$, is the accelerated levels of the non-thermal stressor. In this model, a constant stress loading is used, which means that an accelerated stress combination $\{T_i, S_i\}$ is applied to each item until it fails or the experiment is terminated. A common practice in ALT is to assume that the Birnbaum-Saunders' scale parameter β depends on the stress levels, whereas the shape parameter α does not (see, for example, Owen and Padgett, 2000; Upadhyay and Mukherjee, 2010; Sun and Shi, 2016; Sha, 2018). The reparameterisation of β given by the generalised Eyring model is

$$\beta_i = \frac{1}{T_i}\exp\left(\theta_1+\frac{\theta_2}{T_i}+\theta_3 V_i+\frac{\theta_4 V_i}{T_i}\right), \tag{4.3}$$

where θ_1, θ_2, θ_3, and θ_4 are unknown parameters, and V_i is a function of the non-thermal stressor S_i (Escobar and Meeker, 2006). From (4.1) and (4.3) it follows that the Birnbaum-Saunders PDF of a lifetime subjected to the i^{th} stress level, can be written as

$$\begin{aligned} f(x_i|\alpha,\theta_1,\theta_2,\theta_3,\theta_4) &= \frac{x_i+\frac{1}{T_i}\exp\left(\theta_1+\frac{\theta_2}{T_i}+\theta_3 V_i+\frac{\theta_4 V_i}{T_i}\right)}{2\sqrt{2\pi}\alpha\sqrt{x_i^3}\sqrt{\frac{1}{T_i}\exp\left(\theta_1+\frac{\theta_2}{T_i}+\theta_3 V_i+\frac{\theta_4 V_i}{T_i}\right)}} \\ &\times\exp\left\{-\frac{1}{2\alpha^2}\left[x_i T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3 V_i-\frac{\theta_4 V_i}{T_i}\right)\right.\right. \\ &\left.\left.+\frac{1}{x_i T_i}\exp\left(\theta_1+\frac{\theta_2}{T_i}+\theta_3 V_i+\frac{\theta_4 V_i}{T_i}\right)-2\right]\right\}. \end{aligned} \tag{4.4}$$

The corresponding reliability function at some time τ, resulting from using (4.2) and (4.3), is given by

$$\begin{aligned} R(\tau) &= 1-\Phi\left\{\frac{1}{\alpha}\left[\sqrt{\tau T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3 V_i-\frac{\theta_4 V_i}{T_i}\right)}\right.\right. \\ &\left.\left.-\sqrt{\frac{1}{\tau T_i}\exp\left(\theta_1+\frac{\theta_2}{T_i}+\theta_3 V_i+\frac{\theta_4 V_i}{T_i}\right)}\right]\right\}. \end{aligned} \tag{4.5}$$

For each of the k distinct stress levels, suppose that n_i items are tested and the test is truncated at time τ_i. Further, suppose that $r_i \leq n_i$ of these items fail and denote the failure times by $x_{ij}, j = 1, \ldots, r_i, i = 1, \ldots, k$. The likelihood function, in general, is given by

$$L(\underline{x}|\alpha, \theta_1, \theta_2, \theta_3, \theta_4) = \prod_{i=1}^{k} \left[\prod_{j=1}^{r_i} f\left(x_{ij}|\alpha, \theta_1, \theta_2, \theta_3, \theta_4\right) \right] [R(\tau_i)]^{n_i - r_i}.$$

This likelihood formulation allows for the use of complete samples, type-I censoring and type-II censoring. For type-I censoring, $r_i \leq n_i$ is the number of failures that occur before censoring time τ_i, where $\tau_i < \infty$ is a predetermined censoring time for the i^{th} stress level. For type-II censoring, $\tau_i = x_{i(r_i)}$, where $x_{i(r_i)}$ is the r_i^{th} ordered failure time given that $r_i \leq n_i$ is a pre-chosen number of failures after which censoring occurs for the i^{th} stress level. For complete samples it can be noted that $r_i = n_i$. From (4.4) and (4.5), it follows that the likelihood function for the GEBS model can be written as

$$\begin{aligned}
L(\underline{x}|\alpha, \theta_1, \theta_2, \theta_3, \theta_4) \;=\;& \left(2\sqrt{2\pi}\alpha\right)^{-\sum_{i=1}^{k} r_i} \exp\left(\frac{1}{\alpha^2}\sum_{i=1}^{k} r_i\right) \\
& \times \left[\prod_{i=1}^{k}\prod_{j=1}^{r_i} \frac{x_{ij} + \frac{1}{T_i}\exp\left(\theta_1 + \frac{\theta_2}{T_i} + \theta_3 V_i + \frac{\theta_4 V_i}{T_i}\right)}{x_{ij}^{\frac{3}{2}}}\right] \\
& \times \exp\left\{-\frac{1}{2\alpha^2}\sum_{i=1}^{k}\left[T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\sum_{j=1}^{r_i} x_{ij}\right]\right. \\
& \left. \quad - \frac{1}{2\alpha^2}\sum_{i=1}^{k}\left[\frac{1}{T_i}\exp\left(\theta_1 + \frac{\theta_2}{T_i} + \theta_3 V_i + \frac{\theta_4 V_i}{T_i}\right)\sum_{j=1}^{r_i}\frac{1}{x_{ij}}\right]\right\} \\
& \times \prod_{i=1}^{k}\left\{\left[T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)\right]^{\frac{r_i}{2}}\right. \\
& \quad \times \left[1 - \Phi\left(\frac{1}{\alpha}\left(\sqrt{\tau_i T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)}\right.\right.\right. \\
& \quad \left.\left.\left.\left. - \sqrt{\frac{1}{\tau_i T_i}\exp\left(\theta_1 + \frac{\theta_2}{T_i} + \theta_3 V_i + \frac{\theta_4 V_i}{T_i}\right)}\right)\right)\right]^{n_i - r_i}\right\}.
\end{aligned} \tag{4.6}$$

4.3 Priors and Posteriors

As with the GEW model, assume independent priors on the model parameters θ_1, θ_2, θ_3, θ_4 and α (see, for example, Upadhyay and Mukherjee, 2010), which leads to the joint prior distribution being given by

$$\pi(\alpha, \theta_1, \theta_2, \theta_3, \theta_4) \;=\; \pi(\alpha)\,\pi(\theta_1)\,\pi(\theta_2)\,\pi(\theta_3)\,\pi(\theta_4). \tag{4.7}$$

The joint posterior distribution is then given by

$$\pi\left(\alpha,\theta_1,\theta_2,\theta_3,\theta_4\,|\underline{x}\right) \propto L\left(\underline{x}\,|\alpha,\theta_1,\theta_2,\theta_3,\theta_4\right)\pi\left(\alpha,\theta_1,\theta_2,\theta_3,\theta_4\right).$$

For the GEBS model, gamma priors are imposed on all the parameters, therefore

$$\begin{array}{lll}
\alpha \sim \Gamma\left(c_0,c_1\right) & ,\ c_0,c_1>0 & ,\ \pi\left(\alpha\right) \propto \alpha^{c_0-1}\exp\left(-c_1\alpha\right) \\
\theta_1 \sim \Gamma\left(c_2,c_3\right) & ,\ c_2,c_3>0 & ,\ \pi\left(\theta_1\right) \propto \theta_1^{c_2-1}\exp\left(-c_3\theta_1\right) \\
\theta_2 \sim \Gamma\left(c_4,c_5\right) & ,\ c_4,c_5>0 & ,\ \pi\left(\theta_2\right) \propto \theta_2^{c_4-1}\exp\left(-c_5\theta_2\right) \\
\theta_3 \sim \Gamma\left(c_6,c_7\right) & ,\ c_6,c_7>0 & ,\ \pi\left(\theta_3\right) \propto \theta_3^{c_6-1}\exp\left(-c_7\theta_3\right) \\
\theta_4 \sim \Gamma\left(c_8,c_9\right) & ,\ c_8,c_9>0 & ,\ \pi\left(\theta_4\right) \propto \theta_4^{c_8-1}\exp\left(-c_9\theta_4\right).
\end{array}$$

The joint prior from (4.7) can now be written as

$$\pi\left(\alpha,\theta_1,\theta_2,\theta_3,\theta_4\right) \propto \alpha^{c_0-1}\theta_1^{c_2-1}\theta_2^{c_4-1}\theta_3^{c_6-1}\theta_4^{c_8-1}\exp\left(-c_1\alpha-c_3\theta_1-c_5\theta_2-c_7\theta_3-c_9\theta_4\right). \quad (4.8)$$

The resulting joint posterior, using (4.6) and (4.8), is

$$\begin{aligned}
\pi\left(\alpha,\theta_1,\theta_2,\theta_3,\theta_4\,|\underline{x}\right) \propto\ & \alpha^{c_0-1-\sum_{i=1}^{k} r_i}\theta_1^{c_2-1}\theta_2^{c_4-1}\theta_3^{c_6-1}\theta_4^{c_8-1} \\
& \times\exp\left(\frac{1}{\alpha^2}\sum_{i=1}^{k} r_i\right)\exp\left(-c_1\alpha-c_3\theta_1-c_5\theta_2-c_7\theta_3-c_9\theta_4\right) \\
& \times\left[\prod_{i=1}^{k}\prod_{j=1}^{r_i}\frac{x_{ij}+\frac{1}{T_i}\exp\left(\theta_1+\frac{\theta_2}{T_i}+\theta_3V_i+\frac{\theta_4V_i}{T_i}\right)}{x_{ij}^{\frac{3}{2}}}\right] \\
& \times\exp\left\{-\frac{1}{2\alpha^2}\sum_{i=1}^{k}\left[T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\sum_{j=1}^{r_i}x_{ij}\right]\right. \\
& \quad\left.-\frac{1}{2\alpha^2}\sum_{i=1}^{k}\left[\frac{1}{T_i}\exp\left(\theta_1+\frac{\theta_2}{T_i}+\theta_3V_i+\frac{\theta_4V_i}{T_i}\right)\sum_{j=1}^{r_i}\frac{1}{x_{ij}}\right]\right\} \\
& \times\prod_{i=1}^{k}\left\{\left[T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\right]^{\frac{r_i}{2}}\right. \\
& \quad\times\left[1-\Phi\left(\frac{1}{\alpha}\left(\sqrt{\tau_iT_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)}\right.\right.\right. \\
& \quad\quad\left.\left.\left.\left.-\sqrt{\frac{1}{\tau_iT_i}\exp\left(\theta_1+\frac{\theta_2}{T_i}+\theta_3V_i+\frac{\theta_4V_i}{T_i}\right)}\right)\right)\right]^{n_i-r_i}\right\}.
\end{aligned}$$

The posterior is mathematically intractable, hence MCMC methods must be employed to draw posterior samples to be used for inference. The majority of MCMC methods require the full conditional posterior distributions to be specified up to at least proportionality. The full conditional posterior distributions for the GEBS model are given by

$$
\begin{aligned}
\pi\left(\alpha\,|\underline{x},\theta_1,\theta_2,\theta_3,\theta_4\right) \;\propto\;& \alpha^{c_0-1-\sum_{i=1}^{k} r_i}\exp\left(\frac{1}{\alpha^2}\sum_{i=1}^{k} r_i\right)\exp\left(-c_1\alpha\right)\\
&\times\exp\left\{-\frac{1}{2\alpha^2}\sum_{i=1}^{k}\left[T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\sum_{j=1}^{r_i}x_{ij}\right]\right.\\
&\left.-\frac{1}{2\alpha^2}\sum_{i=1}^{k}\left[\frac{1}{T_i}\exp\left(\theta_1+\frac{\theta_2}{T_i}+\theta_3V_i+\frac{\theta_4V_i}{T_i}\right)\sum_{j=1}^{r_i}\frac{1}{x_{ij}}\right]\right\}\\
&\times\prod_{i=1}^{k}\left\{\left[1-\Phi\left(\frac{1}{\alpha}\left(\sqrt{\tau_iT_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)}\right.\right.\right.\right.\\
&\left.\left.\left.\left.-\sqrt{\frac{1}{\tau_iT_i}\exp\left(\theta_1+\frac{\theta_2}{T_i}+\theta_3V_i+\frac{\theta_4V_i}{T_i}\right)}\right)\right)\right]^{n_i-r_i}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\pi\left(\theta_1\,|\underline{x},\alpha,\theta_2,\theta_3,\theta_4\right) \;\propto\;& \theta_1^{c_2-1}\exp\left(-c_3\theta_1\right)\left[\prod_{i=1}^{k}\prod_{j=1}^{r_i}\frac{x_{ij}+\frac{1}{T_i}\exp\left(\theta_1+\frac{\theta_2}{T_i}+\theta_3V_i+\frac{\theta_4V_i}{T_i}\right)}{x_{ij}^{\frac{3}{2}}}\right]\\
&\times\exp\left\{-\frac{1}{2\alpha^2}\sum_{i=1}^{k}\left[T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\sum_{j=1}^{r_i}x_{ij}\right]\right.\\
&\left.-\frac{1}{2\alpha^2}\sum_{i=1}^{k}\left[\frac{1}{T_i}\exp\left(\theta_1+\frac{\theta_2}{T_i}+\theta_3V_i+\frac{\theta_4V_i}{T_i}\right)\sum_{j=1}^{r_i}\frac{1}{x_{ij}}\right]\right\}\\
&\times\prod_{i=1}^{k}\left\{\left[T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\right]^{\frac{r_i}{2}}\right.\\
&\times\left[1-\Phi\left(\frac{1}{\alpha}\left(\sqrt{\tau_iT_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)}\right.\right.\right.\\
&\left.\left.\left.\left.-\sqrt{\frac{1}{\tau_iT_i}\exp\left(\theta_1+\frac{\theta_2}{T_i}+\theta_3V_i+\frac{\theta_4V_i}{T_i}\right)}\right)\right)\right]^{n_i-r_i}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\pi(\theta_2 \,|\underline{x}, \alpha, \theta_1, \theta_3, \theta_4) \;\propto\; & \theta_2^{c_4-1} \exp(-c_5\theta_2) \left[\prod_{i=1}^{k} \prod_{j=1}^{r_i} \frac{x_{ij} + \frac{1}{T_i} \exp\left(\theta_1 + \frac{\theta_2}{T_i} + \theta_3 V_i + \frac{\theta_4 V_i}{T_i}\right)}{x_{ij}^{\frac{3}{2}}} \right] \\
& \times \exp\left\{ -\frac{1}{2\alpha^2} \sum_{i=1}^{k} \left[T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right) \sum_{j=1}^{r_i} x_{ij} \right] \right. \\
& \left. - \frac{1}{2\alpha^2} \sum_{i=1}^{k} \left[\frac{1}{T_i} \exp\left(\theta_1 + \frac{\theta_2}{T_i} + \theta_3 V_i + \frac{\theta_4 V_i}{T_i}\right) \sum_{j=1}^{r_i} \frac{1}{x_{ij}} \right] \right\} \\
& \times \prod_{i=1}^{k} \left\{ \left[T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right) \right]^{\frac{r_i}{2}} \right. \\
& \times \left[1 - \Phi\left(\frac{1}{\alpha} \left(\sqrt{\tau_i T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)} \right. \right. \right. \\
& \left. \left. \left. \left. - \sqrt{\frac{1}{\tau_i T_i} \exp\left(\theta_1 + \frac{\theta_2}{T_i} + \theta_3 V_i + \frac{\theta_4 V_i}{T_i}\right)} \right) \right) \right]^{n_i - r_i} \right\}
\end{aligned}
$$

$$
\begin{aligned}
\pi(\theta_3 \,|\underline{x}, \alpha, \theta_1, \theta_2, \theta_4) \;\propto\; & \theta_3^{c_6-1} \exp(-c_7\theta_3) \left[\prod_{i=1}^{k} \prod_{j=1}^{r_i} \frac{x_{ij} + \frac{1}{T_i} \exp\left(\theta_1 + \frac{\theta_2}{T_i} + \theta_3 V_i + \frac{\theta_4 V_i}{T_i}\right)}{x_{ij}^{\frac{3}{2}}} \right] \\
& \times \exp\left\{ -\frac{1}{2\alpha^2} \sum_{i=1}^{k} \left[T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right) \sum_{j=1}^{r_i} x_{ij} \right] \right. \\
& \left. - \frac{1}{2\alpha^2} \sum_{i=1}^{k} \left[\frac{1}{T_i} \exp\left(\theta_1 + \frac{\theta_2}{T_i} + \theta_3 V_i + \frac{\theta_4 V_i}{T_i}\right) \sum_{j=1}^{r_i} \frac{1}{x_{ij}} \right] \right\} \\
& \times \prod_{i=1}^{k} \left\{ \left[T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right) \right]^{\frac{r_i}{2}} \right. \\
& \times \left[1 - \Phi\left(\frac{1}{\alpha} \left(\sqrt{\tau_i T_i \exp\left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i}\right)} \right. \right. \right. \\
& \left. \left. \left. \left. - \sqrt{\frac{1}{\tau_i T_i} \exp\left(\theta_1 + \frac{\theta_2}{T_i} + \theta_3 V_i + \frac{\theta_4 V_i}{T_i}\right)} \right) \right) \right]^{n_i - r_i} \right\}
\end{aligned}
$$

$$
\begin{aligned}
\pi\left(\theta_4 \,|\underline{x}, \alpha, \theta_1, \theta_2, \theta_3\right) \;\propto\;& \theta_4^{c_8-1}\exp\left(-c_9\theta_4\right)\left[\prod_{i=1}^{k}\prod_{j=1}^{r_i}\frac{x_{ij}+\frac{1}{T_i}\exp\left(\theta_1+\frac{\theta_2}{T_i}+\theta_3V_i+\frac{\theta_4V_i}{T_i}\right)}{x_{ij}^{\frac{3}{2}}}\right] \\
&\times\exp\left\{-\frac{1}{2\alpha^2}\sum_{i=1}^{k}\left[T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\sum_{j=1}^{r_i}x_{ij}\right]\right. \\
&\left.-\frac{1}{2\alpha^2}\sum_{i=1}^{k}\left[\frac{1}{T_i}\exp\left(\theta_1+\frac{\theta_2}{T_i}+\theta_3V_i+\frac{\theta_4V_i}{T_i}\right)\sum_{j=1}^{r_i}\frac{1}{x_{ij}}\right]\right\} \\
&\times\prod_{i=1}^{k}\left\{\left[T_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)\right]^{\frac{r_i}{2}}\right. \\
&\times\left[1-\Phi\left(\frac{1}{\alpha}\left(\sqrt{\tau_iT_i\exp\left(-\theta_1-\frac{\theta_2}{T_i}-\theta_3V_i-\frac{\theta_4V_i}{T_i}\right)}\right.\right.\right. \\
&\left.\left.\left.\left.-\sqrt{\frac{1}{\tau_iT_i}\exp\left(\theta_1+\frac{\theta_2}{T_i}+\theta_3V_i+\frac{\theta_4V_i}{T_i}\right)}\right)\right)\right]^{n_i-r_i}\right\}.
\end{aligned}
$$

The full conditional posteriors of the GEBS model can not be reduced to any known forms, which prevents the implementation of the simple Gibbs sampler (see Geman and Geman, 1984). Posterior sampling is further complicated by the difficulty experienced to establish the log-concavity of the full conditional posteriors over their domains. As a result, more advanced MCMC methods have to be utilised. ARMS, proposed by Gilks et al. (1995), slice sampling, presented in Neal (2003), and independent doubly adaptive rejection Metropolis sampling, introduced in Martino et al. (2015), are examples of MCMC methods that can sample from intricate, non-log-concave densities.

For the purpose of this thesis, the Bayesian data analysis software OpenBUGS is used. Slice sampling is the only MCMC algorithm available in the OpenBUGS software that can handle complex densities that are not log-concave. In the application that follows, posterior samples are generated using the slice sampler with the default settings regarding the adaptive phase of the sampler.

4.4 Application

The GEBS model will now be applied to the same ALT data set used to demonstrate the GEW model in Chapter 3. The failure time data was obtained using an electronics epoxy packaging ALT. The test investigated the synergistic effect of applying accelerated temperature and humidity stressors. The normal use conditions for the items are a temperature of $T_u = 350K$ and a relative humidity of $S_u = 0.3$. The data set consists of 17 items and no censoring is present in the data.

The sensitivity of the GEBS model is explored in terms of the values of the hyperparameters

$c_l, l = 0, \ldots, 9$. Six specifications for the hyperparameters used in this application are given in Table 4.1. The $GEBS_1$, $GEBS_2$ and $GEBS_3$ models make use of flat gamma priors, whereas subjective gamma priors with mean 5 and differing variances are used for the $GEBS_4$, $GEBS_5$ and $GEBS_6$ models. Park and Casella (2008) suggest that the values of the parameters for the gamma prior should, if one wishes to have a vague prior, result in a large prior variance. Lesaffre and Lawson (2012) use various combinations of $\{0.01, 0.05, 0.5, 1\}$ for the parameters of the vague gamma prior. In this application, we choose the values for the parameters of the flat (vague) priors close to those used in Lesaffre and Lawson (2012), such that the prior variances are large. Other combinations in this range give similar results. The subjective priors are included to see if the model is sensitive to the choice of prior parameters, thus parameter values for these priors are chosen such that the prior variances are much smaller than that of the flat priors. These choices are not based on any expert or prior believe and is merely for illustrative purposes to address model sensitivity when using subjective priors. Other combinations for the prior parameters similarly give variable results, depending on the choice of hyperparameters.

Table 4.1: Prior specifications for the GEBS models.

Model	α	θ_1	θ_2	θ_3	θ_4
$GEBS_1$	$\Gamma(1,0.1)$	$\Gamma(1,0.1)$	$\Gamma(1,0.1)$	$\Gamma(1,0.1)$	$\Gamma(1,0.1)$
$GEBS_2$	$\Gamma(1,0.001)$	$\Gamma(1,0.001)$	$\Gamma(1,0.001)$	$\Gamma(1,0.001)$	$\Gamma(1,0.001)$
$GEBS_3$	$\Gamma(1,0.00001)$	$\Gamma(1,0.00001)$	$\Gamma(1,0.00001)$	$\Gamma(1,0.00001)$	$\Gamma(1,0.00001)$
$GEBS_4$	$\Gamma(2.5.0.5)$	$\Gamma(2.5.0.5)$	$\Gamma(2.5.0.5)$	$\Gamma(2.5.0.5)$	$\Gamma(2.5.0.5)$
$GEBS_5$	$\Gamma(5,1)$	$\Gamma(5,1)$	$\Gamma(5,1)$	$\Gamma(5,1)$	$\Gamma(5,1)$
$GEBS_6$	$\Gamma(25,5)$	$\Gamma(25,5)$	$\Gamma(25,5)$	$\Gamma(25,5)$	$\Gamma(25,5)$

The Bayesian data analysis software OpenBUGS is used to generate posterior samples from the full conditional posterior distributions. For each of the GEBS models defined in Table 4.1, a single Markov chain is initiated with a burn-in of 50000 iterations. The convergence of the chains are analysed by means of trace plots and the approach suggested in Brooks and Gelman (1998). For each parameter considered in the six models, the modified Gelman-Rubin statistic shows satisfactory and fast convergence to 1. The convergence is achieved well before the set burn-in of 50000 iterations. The within-chain variances and between-chain variances also stabilise sufficiently ahead of the end of the burn-in period. See Appendix A for the convergence diagnostics regarding the GEBS models.

The burn-in period for each model is followed by a further 200000 iterations to obtain samples to base posterior inference on. The sample standard deviation, Monte Carlo error, and the Monte Carlo error as a percentage of the standard deviation, for each parameter of the GEBS models, are given in Table 4.2 up to Table 4.7. According to the rule of thumb in Spiegelhalter et al. (2014) regarding the ratio of the Monte Carlo to the sample standard deviation, a sufficient number of posterior samples have been generated to base inference on.

Table 4.2: Monte Carlo error as a percentage of the standard deviation for $GEBS_1$.

	Standard Deviation	MC Error	%
α	0.1749	0.001103	0.6306%
θ_1	1.4409	0.036380	2.5248%
θ_2	1.4364	0.036290	2.5265%
θ_3	0.4404	0.002357	0.5352%
θ_4	0.4492	0.002556	0.5690%

Table 4.3: Monte Carlo error as a percentage of the standard deviation for $GEBS_2$.

	Standard Deviation	MC Error	%
α	0.1776	0.001142	0.6430%
θ_1	1.3982	0.035200	2.5175%
θ_2	1.3983	0.035290	2.5238%
θ_3	0.4449	0.002455	0.5518%
θ_4	0.4612	0.002736	0.5932%

Table 4.4: Monte Carlo error as a percentage of the standard deviation for $GEBS_3$.

	Standard Deviation	MC Error	%
α	0.1801	0.001170	0.6496%
θ_1	1.4348	0.036360	2.5342%
θ_2	1.4342	0.036330	2.5331%
θ_3	0.4599	0.002584	0.5619%
θ_4	0.4690	0.002692	0.5740%

Table 4.5: Monte Carlo error as a percentage of the standard deviation for $GEBS_4$.

	Standard Deviation	MC Error	%
α	0.2520	0.001343	0.5329%
θ_1	0.9729	0.012940	1.3300%
θ_2	0.9800	0.012940	1.3204%
θ_3	0.6682	0.003409	0.5102%
θ_4	0.6784	0.003686	0.5433%

Table 4.6: Monte Carlo error as a percentage of the standard deviation for $GEBS_5$.

	Standard Deviation	MC Error	%
α	0.3775	0.001990	0.5272%
θ_1	0.6829	0.005341	0.7821%
θ_2	0.6786	0.005265	0.7759%
θ_3	0.8087	0.004066	0.5028%
θ_4	0.8042	0.003961	0.4925%

Table 4.7: Monte Carlo error as a percentage of the standard deviation for $GEBS_6$.

	Standard Deviation	MC Error	%
α	1.0912	0.004577	0.4194%
θ_1	0.4905	0.002165	0.4414%
θ_2	0.4894	0.002141	0.4375%
θ_3	0.5985	0.002061	0.3444%
θ_4	0.5965	0.002064	0.3460%

The DIC is again used as a tool for model comparison. The DIC values for the six GEBS models are given in Table 4.8. The $GEBS_1$ model has the lowest DIC, but the other two models where flat priors are used also have very similar DIC values. For the models with subjective priors, the DIC increases as the variance on the prior is decreased. This could be an indication that the posterior is being dominated by the prior if the prior variance is too small. According to the guidelines provided in Burnham and Anderson (1998), models $GEBS_1$, $GEBS_2$ and $GEBS_3$ have substantial support and should primarily be considered for making inference. Substantially less consideration can be given to the $GEBS_4$ model, and there is very little support for the $GEBS_5$ and $GEBS_6$ models.

Table 4.8: Deviance information criterion for the GEBS models.

Model	DIC
$GEBS_1$	203.4
$GEBS_2$	203.6
$GEBS_3$	203.6
$GEBS_4$	209.3
$GEBS_5$	216.4
$GEBS_6$	248.3

Table 4.9 contains the summary statistics for the marginal posterior distributions of the GEBS models. The models where flat priors are used produce very similar summary statistics. For the models where subjective priors are used, the central location measures of the marginal posteriors are pulled more and more towards that of the priors as the variance on the prior is decreased. This shows how

the marginal posteriors are dominated by the prior when hyperparameters are chosen which result in priors with small variances.

Table 4.9: Summary statistics for the GEBS models.

Model	Parameter	Mean	Standard Deviation	2.5th Percentile	Median	97.5th Percentile
$GEBS_1$	α	0.7440	0.1749	0.4918	0.7150	1.1640
	θ_1	2.0002	1.4409	0.0698	1.7170	4.9660
	θ_2	3.1507	1.4364	0.2659	3.4020	5.2110
	θ_3	0.4228	0.4404	0.0102	0.2854	1.6080
	θ_4	0.4325	0.4492	0.0105	0.2932	1.6370
$GEBS_2$	α	0.7481	0.1776	0.4925	0.7185	1.1750
	θ_1	1.9067	1.3982	0.0675	1.6190	4.8990
	θ_2	3.2370	1.3983	0.3039	3.4970	5.2210
	θ_3	0.4315	0.4449	0.0108	0.2928	1.6330
	θ_4	0.4445	0.4612	0.0110	0.3023	1.6860
$GEBS_3$	α	0.7504	0.1801	0.4937	0.7201	1.1860
	θ_1	1.9950	1.4348	0.0699	1.7340	4.9660
	θ_2	3.1480	1.4342	0.2594	3.3740	5.2090
	θ_3	0.4359	0.4599	0.0105	0.2939	1.6600
	θ_4	0.4463	0.4690	0.0111	0.3018	1.6980
$GEBS_4$	α	0.9504	0.2520	0.5920	0.9066	1.5590
	θ_1	2.1571	0.9729	0.4923	2.1010	4.1290
	θ_2	2.5174	0.9800	0.6635	2.5310	4.3340
	θ_3	1.0048	0.6682	0.1609	0.8587	2.6870
	θ_4	1.0196	0.6784	0.1620	0.8688	2.7310
$GEBS_5$	α	1.3011	0.3775	0.7737	1.2360	2.2070
	θ_1	2.0348	0.6829	0.8102	2.0060	3.4300
	θ_2	2.0862	0.6786	0.8550	2.0580	3.4650
	θ_3	1.7680	0.8087	0.5621	1.6420	3.6760
	θ_4	1.7693	0.8042	0.5682	1.6450	3.6690
$GEBS_6$	α	5.6335	1.0912	3.6670	5.5720	7.9350
	θ_1	2.8102	0.4905	1.9080	2.7920	3.8260
	θ_2	2.8045	0.4894	1.9020	2.7870	3.8140
	θ_3	3.2089	0.5985	2.1310	3.1760	4.4680
	θ_4	3.1949	0.5965	2.1180	3.1630	4.4510

The marginal posterior distributions for the GEBS models are displayed in Figure 4.1 up to Figure 4.6. It can be seen that the $GEBS_1$, $GEBS_2$ and $GEBS_3$ models produce approximately the same marginal posteriors. The marginal posteriors for these models are heavily skewed. The marginal posteriors for the remaining models are progressively dominated by the prior as the variance of the prior is reduced. The marginal posteriors for these models are less skew, with almost symmetrical distributions produced by the $GEBS_6$ model.

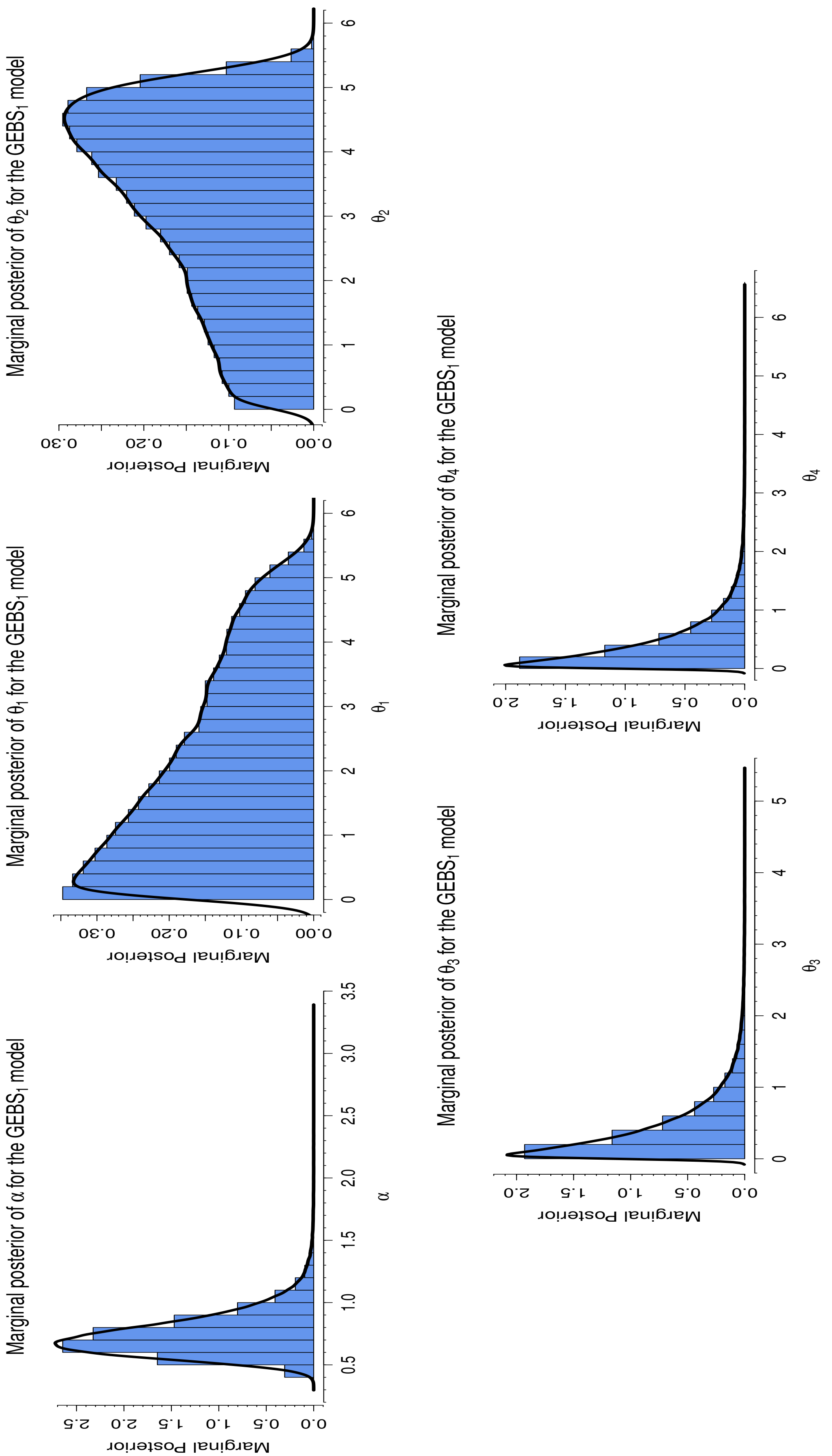

Figure 4.1: Marginal posterior distributions for the $GEBS_1$ model.

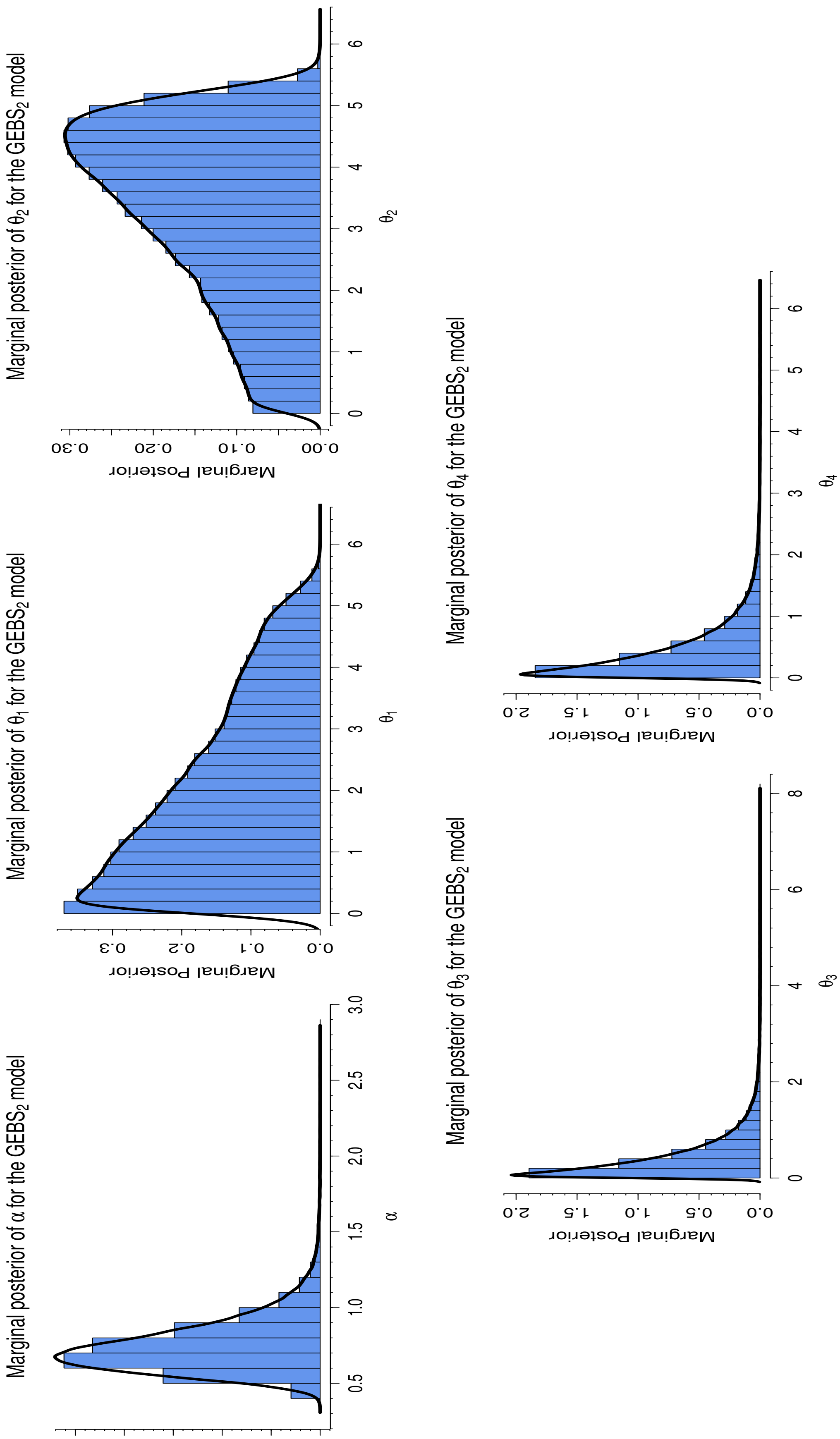

Figure 4.2: Marginal posterior distributions for the $GEBS_2$ model.

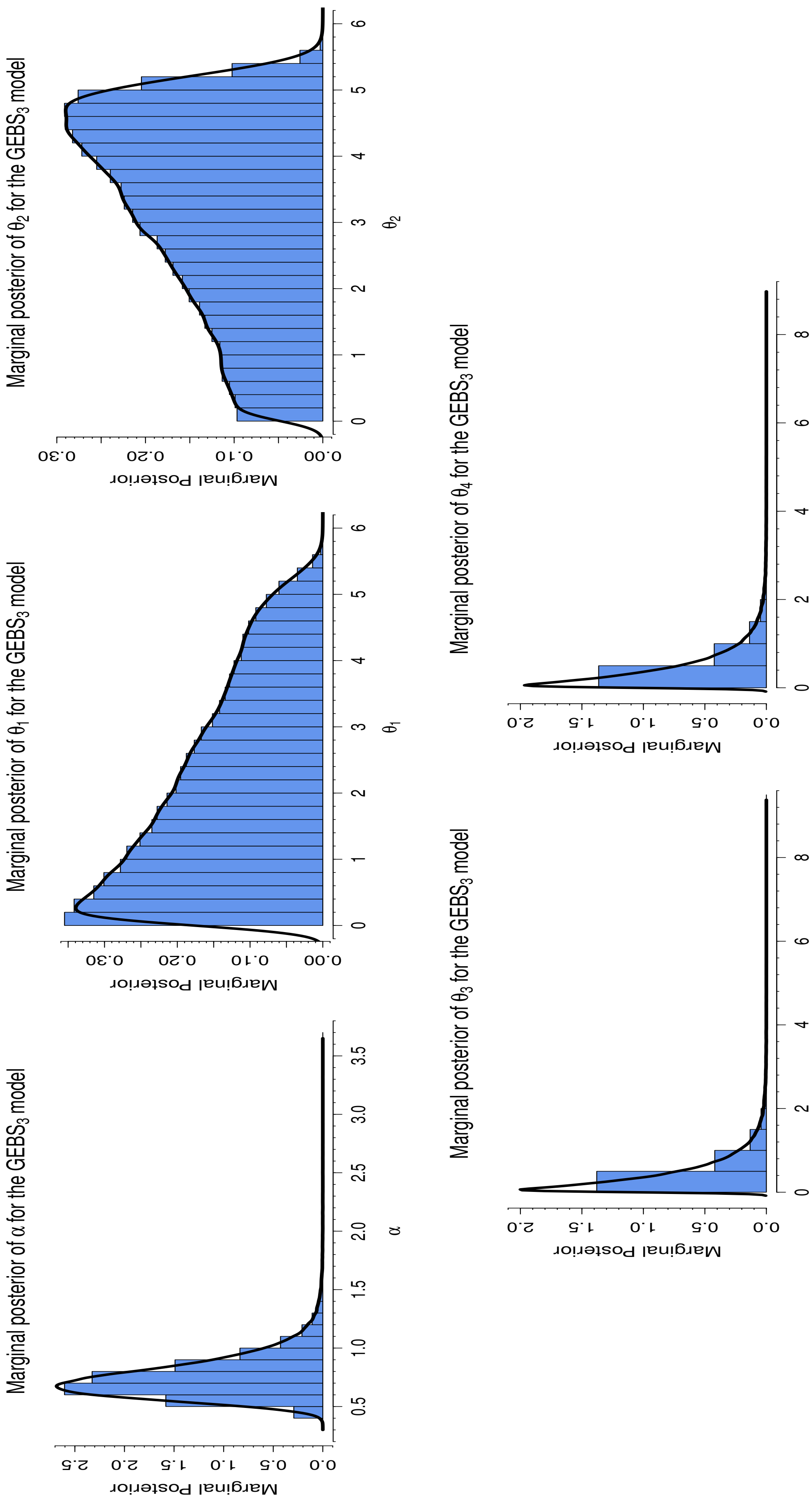

Figure 4.3: Marginal posterior distributions for the $GEBS_3$ model.

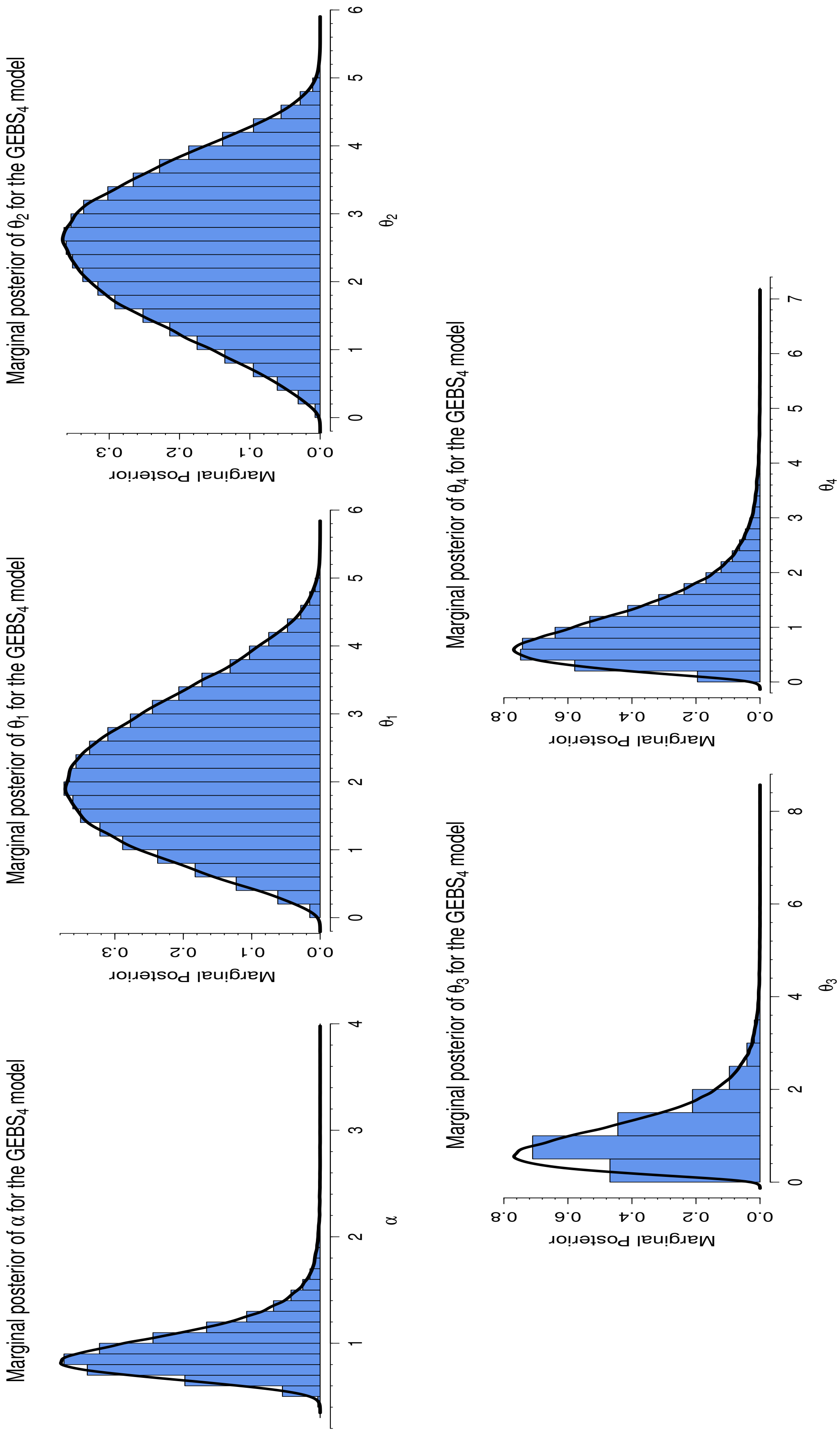

Figure 4.4: Marginal posterior distributions for the $GEBS_4$ model.

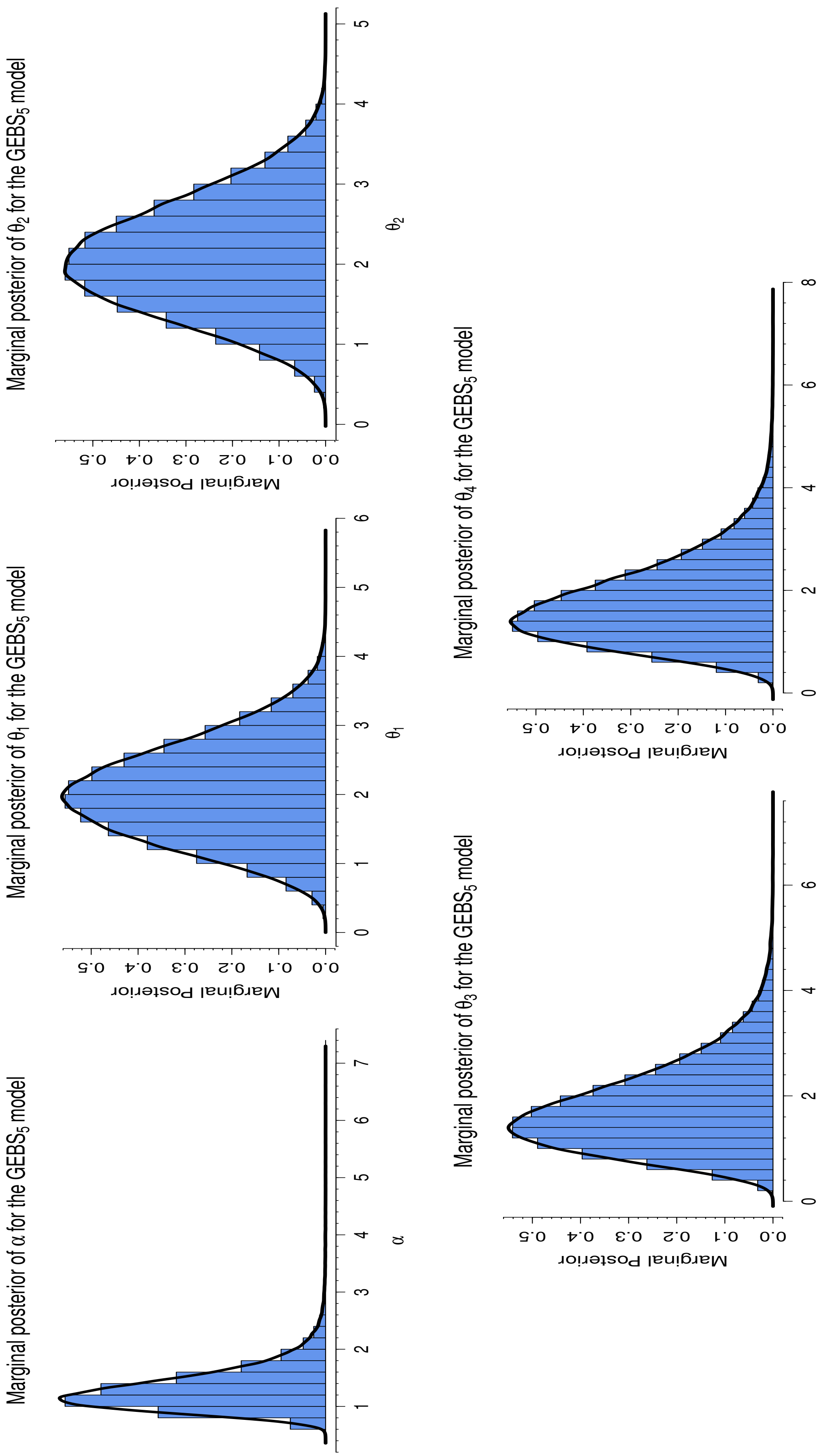

Figure 4.5: Marginal posterior distributions for the $GEBS_5$ model.

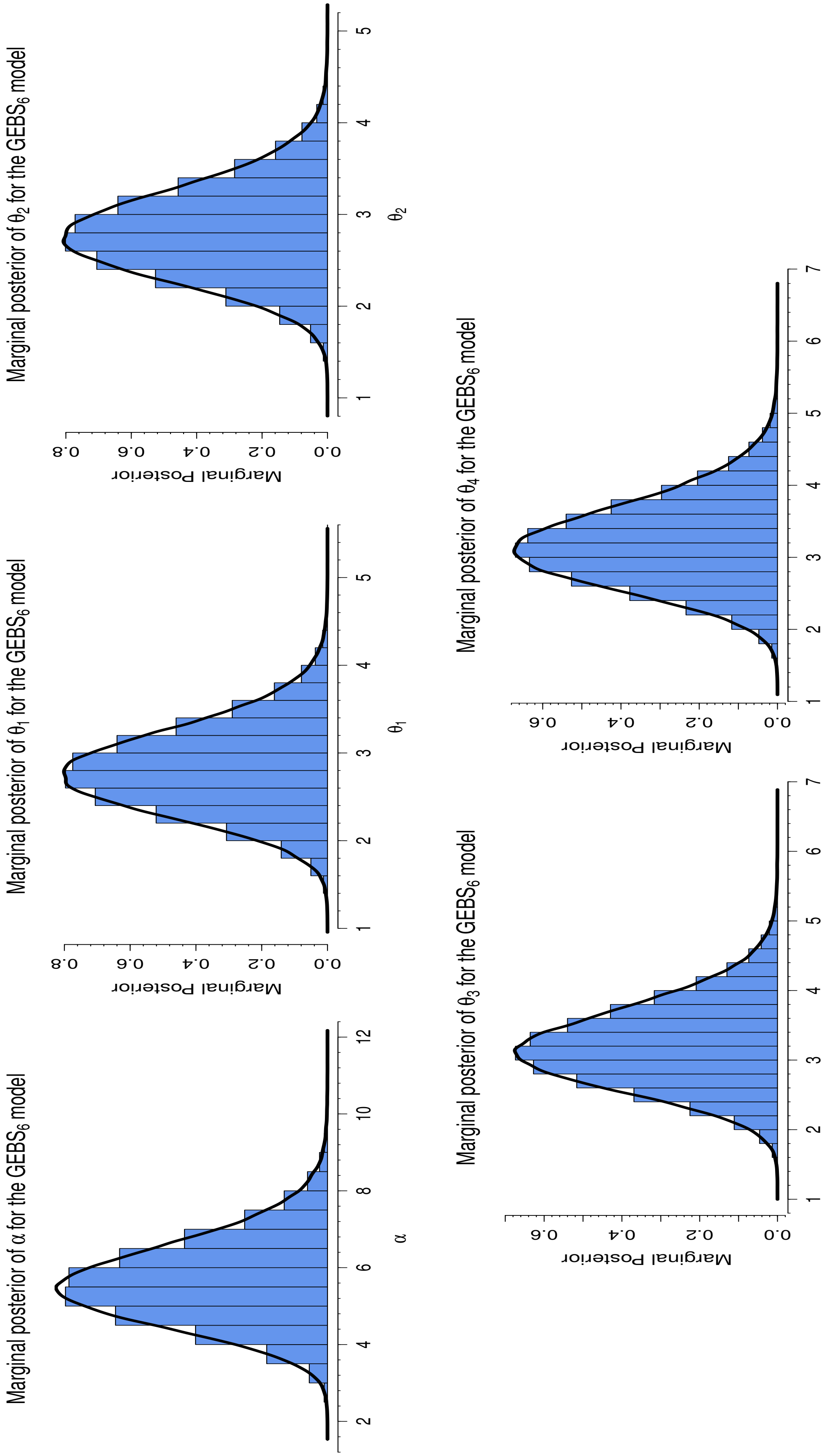

Figure 4.6: Marginal posterior distributions for the GEBS_6 model.

The primary interest of implementing the GEBS model is to calculate the predictive reliability, subject to the normal operating conditions. The predictive reliability at use stress levels is given by

$$R(x_u|\underline{x}) = \int\int\int\int\int R(x_u|\alpha,\theta_1,\theta_2,\theta_3,\theta_4)\,\pi(\alpha,\theta_1,\theta_2,\theta_3,\theta_4|\underline{x})\,d\alpha d\theta_1 d\theta_2 d\theta_3 d\theta_4, \tag{4.9}$$

where $R(x_u|\alpha,\theta_1,\theta_2,\theta_3,\theta_4)$ is the Birnbaum-Saunders reliability function at use stress levels T_u and S_u. $R(x_u|\underline{x})$ can be evaluated as follows:

1. Sample $\alpha,\theta_1,\theta_2,\theta_3$ and θ_4 from the posterior a sufficiently large number, say M, times.
2. Calculate the integral in (4.9) by the Monte Carlo average

$$R(x_u|\underline{x}) \approx \frac{1}{M}\sum_{m=1}^{M} R\left(x_u \middle| \alpha^{(m)},\theta_1^{(m)},\theta_2^{(m)},\theta_3^{(m)},\theta_4^{(m)}\right),$$

which is the expected reliability at time x_u, using the posterior sample

$$\left\{\alpha^{(m)},\theta_1^{(m)},\theta_2^{(m)},\theta_3^{(m)},\theta_4^{(m)}\right\}, m = 1,\ldots,M.$$

The predictive reliability at use stress for each model is displayed in Table 4.10 and Figure 4.7. The models that make use of flat priors produce very similar predictive reliability results, indicating that the GEBS model is not very sensitive to different choices of flat priors. The model is, however, sensitive to the choice of subjective priors with small variances. The GEBS_6 model is seen to heavily overestimate the predictive reliability compared to the models with flat priors.

Table 4.10: Predictive reliability for the GEBS models at use stress $T_u = 350, S_u = 0.3$.

Time	GEBS_1	GEBS_2	GEBS_3	GEBS_4	GEBS_5	GEBS_6
1	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
2	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
3	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
⋮	⋮	⋮	⋮	⋮	⋮	⋮
500	0.81375287	0.82411743	0.81792571	0.91555621	0.97360684	0.99998526
501	0.81309664	0.82348366	0.81728094	0.91522195	0.97349171	0.99998511
502	0.81244030	0.82284972	0.81663604	0.91488741	0.97337643	0.99998496
⋮	⋮	⋮	⋮	⋮	⋮	⋮
2000	0.22505777	0.23707694	0.23309917	0.49172725	0.76744774	0.99697109
2001	0.22489074	0.23690532	0.23293065	0.49154259	0.76732616	0.99696673
2002	0.22472388	0.23673387	0.23276229	0.49135805	0.76720462	0.99696237

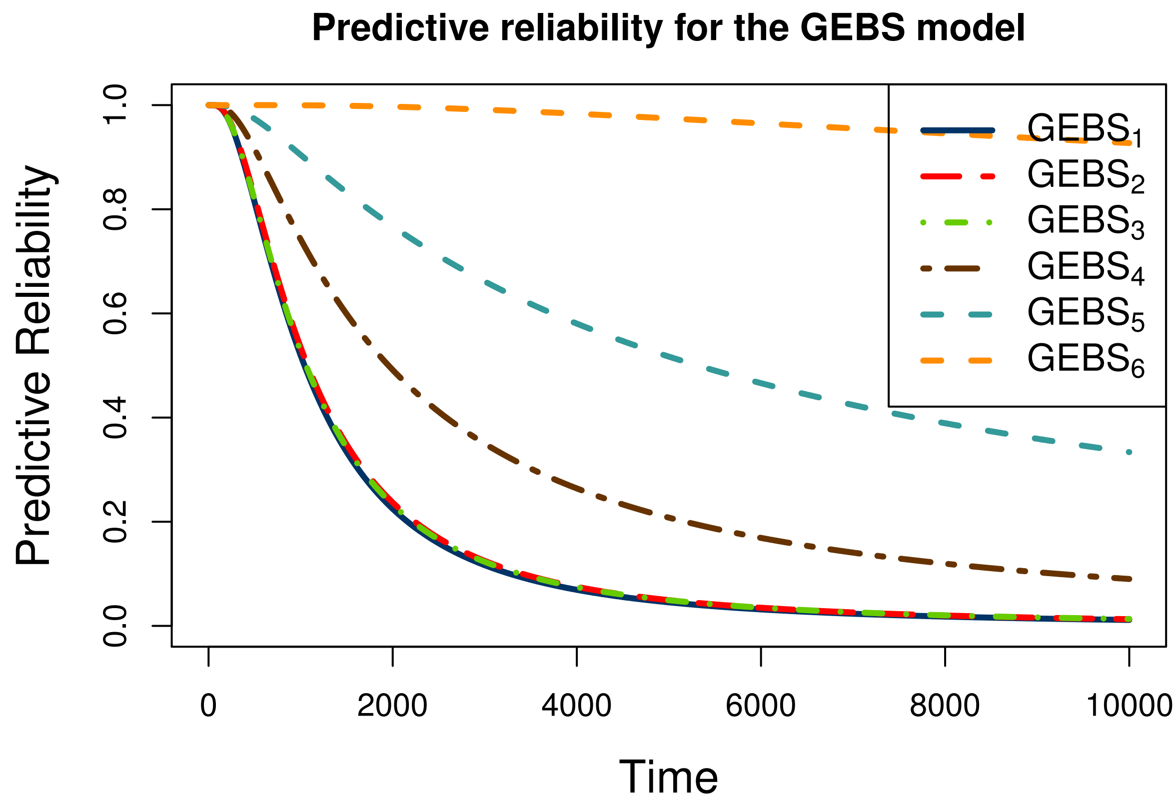

Figure 4.7: Predictive reliability for the GEBS models at use stress $T_u = 350$, $S_u = 0.3$.

The predictive reliability results for each model is also compared to the empirical reliability. Since we are working with a small sample and there is no failure data available at the use stress levels, the empirical reliability for each model is obtained by adjusting the accelerated failure times via the generalised Eyring acceleration factor, which is given by

$$\text{Acceleration Factor} = \frac{\frac{1}{T_i}\exp\left(\theta_1 + \frac{\theta_2}{T_i} + \theta_3 V_i + \frac{\theta_4 V_i}{T_i}\right)}{\frac{1}{T_u}\exp\left(\theta_1 + \frac{\theta_2}{T_u} + \theta_3 V_u + \frac{\theta_4 V_u}{T_u}\right)}.$$

For each model, the posterior means are used as parameter estimates, which is the Bayes estimate under squared error loss, to obtain the empirical reliability. The predictive reliability of each model is compared to the empirical reliability in Figure 4.8. We note that the models that make use of flat priors fit the empirical reliability very well, whereas the models that make use of subjective priors show an increasingly worse fit to the empirical reliability as the prior variance is decreased. These results correspond with those given by the DIC regarding the fit of the models.

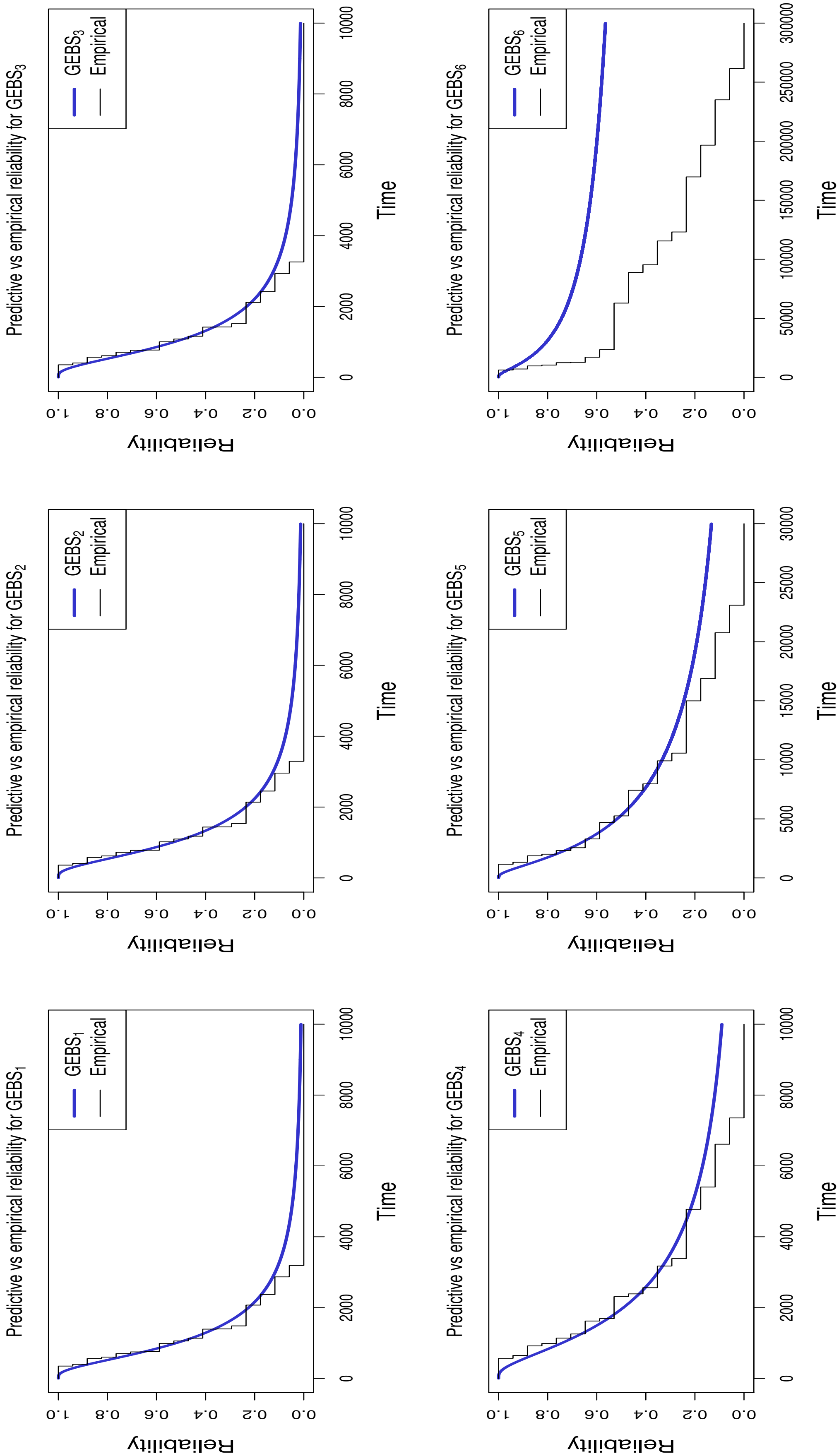

Figure 4.8: Predictive vs empirical reliability of the GEBS models.

4.5 Conclusion

In this chapter, a Bayesian approach to an ALT model with two stressors is considered. Lifetimes are assumed to follow a Birnbaum-Saunders distribution and the generalised Eyring model is used as the TTF. This model explores the acceleration effect of one thermal stressor and one non-thermal stressor, which also incorporates an interaction term. A general likelihood formulation is presented for the GEBS model, which can be utilised for complete samples, type-I censoring and type-II censoring. Gamma priors are imposed on the model parameters. Due to the mathematically intractable posterior distribution, MCMC methods are employed to generate posterior samples from the full conditional posterior distributions. The GEBS model is applied to an electronics epoxy packaging ALT data set, where temperature and relative humidity are the accelerated stressors.

Several choices of hyperparameters are considered, in order to perform a sensitivity analysis. Posterior samples are generated via the slice sampler algorithm available in the Bayesian data analysis software OpenBUGS, and the convergence of the Markov chains are assessed. The fit and complexity of the various GEBS models are compared using the DIC. The $GEBS_1$ model results in the lowest DIC, with the $GEBS_2$ and $GEBS_3$ models also showing very similar DIC values. The $GEBS_6$ model has an overly bad fit to the data and produces radically different results.

The predictive reliability for the models are also calculated. The $GEBS_1$, $GEBS_2$ and $GEBS_3$ models have very similar results in this regard. The remaining models all produce higher predictive reliability results, compared to the models where flat priors are used. The predictive reliability results are also compared to the empirical reliability, which is obtained via the generalised Eyring acceleration factor. The predictive reliability of the models where flat priors are used show a better fit to the empirical reliability. The GEBS model can be very sensitive when making use of subjective priors, thus caution should be exercised when doing so. An expert reliability engineer may however decide to make use of subjective priors if he/she feels that using flat priors lead to either an overestimation or underestimation of the predictive reliability. It is recommended that flat priors are used for the GEBS model if no other concrete prior information is available.

Chapter 5

Bayes Factors in Accelerated Life Testing

5.1 Introduction

Bayesian model selection is an important aspect of any Bayesian analysis and aids the researcher in determining which of the proposed models should be used. Various approaches towards Bayesian model selection are discussed in Kadane and Lazar (2004). The different perspectives to model selection discussed include Bayes factors, Bayesian model averaging, Bayesian linear models, and predictive methods. The authors also state that model selection tools should rather be used to identify inappropriate models, instead of attempting so select a single best model.

Bayesian ALT models with more than one stressor often have mathematically intractable posterior distributions and MCMC methods are employed to obtain posterior samples to make inferences. For this reason, and the fact that it is a standard output in many Bayesian data analysis software packages, the DIC is a very popular tool for model comparison. It can be argued that the more formal and traditional approach for Bayesian model selection would be the use of Bayes factors (Spiegelhalter et al., 2002; Upadhyay and Mukherjee, 2010).

The authors in Spiegelhalter et al. (2002) explain that the DIC is intended as an alternative to Bayes factors and that there are situations where the DIC may be a more appropriate tool for model comparison. Bayes factors are the most suited when the complete set of possible models can be specified and the true model is included in this set, whereas the DIC does not have this requirement (Bernardo and Smith, 1994; Spiegelhalter et al., 2002). In a discussion on the DIC, the authors state that Bayes factors and the DIC are intended for different purposes, and can thus lead to different conclusions regarding model selection. Bayes factors consider how well the prior predicts the observed data, whereas the DIC considers how well the posterior will predict future data by means of the same mechanism that resulted in the observed data (Spiegelhalter et al., 2002).

In this chapter, model selection in the Bayesian ALT setup is considered by comparing the GEW and GEBS models via the DIC and Bayes factors. Due to the mathematically intractable posterior distributions of these models, the computation of the marginal likelihood, needed for the calculation of the Bayes factor, is also complicated. Methods for approximating the marginal likelihood, without further complicating the MCMC sampling process, are discussed. The methods considered include a simple Monte Carlo estimator, the harmonic mean estimator, the Laplace-Metropolis estimator, and a posterior predictive density estimate for posterior Bayes factors.

First, an introduction to the formulation and interpretation of Bayes factors is given. Second, methods for estimating the marginal likelihood are presented. Since MCMC techniques are utilised for the GEW and GEBS models, the focus is placed on estimation methods that can easily be applied in this setup. Third, the GEW and GEBS models are used in an application, where model comparison is discussed by means of the DIC and Bayes factors. The posterior model probabilities for the models are also presented. Some final remarks and comments regarding model selection concludes this chapter.

5.2 Bayes Factors

The foundation for Bayesian hypothesis testing via Bayes factors was developed by Jeffreys (1961). The author referred to his methods as "significance tests" and presented them as an alternative to p-values. The approach compares predictions made by two competing scientific theories by defining statistical models for each theory and calculating the posterior probability that one of the theories is correct (Kass and Raftery, 1995).

Denote by $f\left(m_i\left|\underline{x}\right.\right)$ and $f\left(m_j\left|\underline{x}\right.\right)$ the posterior model probabilities for models m_i and m_j, respectively. The comparison of two models m_i and m_j, in the Bayesian setup, is conducted by means of the posterior odds for model m_i against model m_j. This is given by

$$PO_{ij} = \frac{f\left(m_i\left|\underline{x}\right.\right)}{f\left(m_j\left|\underline{x}\right.\right)} = \frac{f\left(\underline{x}\left|m_i\right.\right)}{f\left(\underline{x}\left|m_j\right.\right)} \times \frac{f\left(m_i\right)}{f\left(m_j\right)} = B_{ij} \times \frac{f\left(m_i\right)}{f\left(m_j\right)},$$

where $f\left(\underline{x}\left|m_g\right.\right)$, $g = i, j$, are the marginal likelihoods and $f\left(m_g\right)$, $g = i, j$, are the prior model probabilities of the two models. The above expression is often summarised as

$$\text{Posterior odds} = \text{Bayes factor} \times \text{Prior odds}.$$

In most situations no prior information will be available regarding the model structure, in which case the prior model probabilities are set equal (see, for example, Ntzoufras, 2009). This results in using the Bayes factor for hypothesis testing, which is also extended to model selection. It is important to note that Bayes factors can be used to evaluate evidence *against* the null hypothesis or *in favour* of the null hypothesis, which is not possible in classical hypothesis testing.

The Bayes factor B_{ij} of model m_i versus model m_j is defined as the ratio of the marginal likelihoods

$f(\underline{x}|m_i)$ and $f\left(\underline{x}\left|m_j\right.\right)$. This can be written as

$$B_{ij} = \frac{f(\underline{x}|m_i)}{f\left(\underline{x}\left|m_j\right.\right)} = \frac{\int f(\underline{x}|\boldsymbol{\theta}_{m_i}, m_i)\, f(\boldsymbol{\theta}_{m_i}|m_i)\, d\boldsymbol{\theta}_{m_i}}{\int f\left(\underline{x}\left|\boldsymbol{\theta}_{m_j}, m_j\right.\right) f\left(\boldsymbol{\theta}_{m_j}\left|m_j\right.\right) d\boldsymbol{\theta}_{m_j}}, \tag{5.1}$$

where $f\left(\underline{x}\left|\boldsymbol{\theta}_{m_g}, m_g\right.\right)$, $g = i, j$, is the likelihood of model m_g with parameter vector $\boldsymbol{\theta}_{m_g}$, and $f\left(\boldsymbol{\theta}_{m_g}\left|m_g\right.\right)$, $g = i, j$, is the prior imposed on $\boldsymbol{\theta}_{m_g}$ under model m_g.

Interpreting Bayes factors in half-units on the $\log_{10}$-scale is suggested in Jeffreys (1961), where the author provides a table for interpreting Bayes factor values. Kass and Raftery (1995) suggest a modified version of these interpretations by rather considering twice the natural logarithm of the Bayes factor. By doing this, Bayes factors are interpreted on the same scale as the likelihood ratio test statistic. The interpretation of Bayes factors suggested in Kass and Raftery (1995) can be summarised as in Table 5.1.

Table 5.1: Bayes factor interpretations.

B_{ij}	B_{ji}	Interpretation
>150	$<0.00\dot{6}$	Very strong evidence for model m_i
20 to 150	$0.00\dot{6}$ to 0.5	Strong evidence for model m_i
3 to 20	0.05 to $0.\dot{3}$	Positive evidence for model m_i
1 to 3	$0.\dot{3}$ to 1	Negligible evidence for model m_i
1	1	No evidence for either model
$0.\dot{3}$ to 1	1 to 3	Negligible evidence for model m_j
0.05 to $0.\dot{3}$	3 to 20	Positive evidence for model m_j
$0.00\dot{6}$ to 0.5	20 to 150	Strong evidence for model m_j
$<0.00\dot{6}$	>150	Very strong evidence for model m_j

In some cases, the marginal likelihoods in (5.1) can be computed analytically (see, for example, DeGroot, 1970; Zellner, 1971). More often than not, the marginal likelihood must be estimated. Ntzoufras (2009) explains that numerous other versions of Bayes factors as well as alternative model selection approaches have also been developed. Other popular types of Bayes factors include pseudo-Bayes factors, resulting from the work of Geisser and Eddy (1979), posterior Bayes factors, presented in Aitkin (1991), fractional Bayes factors, introduced in O'Hagan (1995), and intrinsic Bayes factors, presented in Berger and Pericchi (1996).

Suppose we compare two models, m_i and m_j, where the prior model probabilities are denoted by $f(m_g)$, $g = i, j$. From Bayes' theorem we then have that the posterior model probability for model m_i can be written as

$$\begin{aligned} f(m_i|\underline{x}) &= \frac{f(\underline{x}|m_i)\, f(m_i)}{f(\underline{x}|m_i)\, f(m_i) + f\left(\underline{x}\left|m_j\right.\right) f\left(m_j\right)} \\ &= \frac{f(m_i) \int f(\underline{x}|\boldsymbol{\theta}_{m_i}, m_i)\, f(\boldsymbol{\theta}_{m_i}|m_i)\, d\boldsymbol{\theta}_{m_i}}{f(m_i) \int f(\underline{x}|\boldsymbol{\theta}_{m_i}, m_i)\, f(\boldsymbol{\theta}_{m_i}|m_i)\, d\boldsymbol{\theta}_{m_i} + [1 - f(m_i)] \int f\left(\underline{x}\left|\boldsymbol{\theta}_{m_j}, m_j\right.\right) f\left(\boldsymbol{\theta}_{m_j}\left|m_j\right.\right) d\boldsymbol{\theta}_{m_j}}, \end{aligned}$$

where $f\left(\underline{x}\,\middle|m_g\right)$, $g=i,j$ is the marginal likelihoods, $f\left(\underline{x}\,\middle|\boldsymbol{\theta}_{m_g},m_g\right)$, $g=i,j$ denotes the likelihood of model m_g with parameter vector $\boldsymbol{\theta}_{m_g}$, and $f\left(\boldsymbol{\theta}_{m_g}\middle|m_g\right)$, $g=i,j$ is the prior imposed on $\boldsymbol{\theta}_{m_g}$ under model m_g. The posterior model probability for model m_i can easily be computed by re-writing it in terms of Bayes factors as

$$\begin{aligned}
f\left(m_i\,|\underline{x}\right) &= \frac{f\left(\underline{x}\,|m_i\right)f\left(m_i\right)}{f\left(\underline{x}\,|m_i\right)f\left(m_i\right)+f\left(\underline{x}\,\middle|m_j\right)f\left(m_j\right)}\\
&= \left[\frac{f\left(m_i\right)f\left(\underline{x}\,|m_i\right)+\left[1-f\left(m_i\right)\right]f\left(\underline{x}\,\middle|m_j\right)}{f\left(m_i\right)f\left(\underline{x}\,|m_i\right)}\right]^{-1}\\
&= \left[1+\frac{1-f\left(m_i\right)}{f\left(m_i\right)}\times\frac{f\left(\underline{x}\,\middle|m_j\right)}{f\left(\underline{x}\,|m_i\right)}\right]^{-1}\\
&= \left[1+\frac{1-f\left(m_i\right)}{f\left(m_i\right)}B_{ij}^{-1}\right]^{-1},
\end{aligned}$$

where B_{ij} is the Bayes factor of model m_i versus model m_j.

If equal prior model probabilities are assumed, which is the case when no information is available on the prior model probabilities, the posterior model probability for model m_i simplifies to

$$f\left(m_i\,|\underline{x}\right)=\left[1+B_{ij}^{-1}\right]^{-1}.$$

5.3 Estimating the Marginal Likelihood

There are various methods that can be used to estimate the marginal likelihood. In this section, we focus on methods that can easily estimate Bayes factors from the output of an MCMC algorithm. This includes the simple Monte Carlo estimator, the Laplace-Metropolis estimator, the harmonic mean estimator, and using the predictive posterior density to estimate the posterior Bayes factors.

There is merit for many other methods of estimating the marginal likelihood, but most of these methods either further complicate the MCMC algorithm or require the meticulous selection of some importance density or bridge function (see, for example, Ntzoufras, 2009). The marginal likelihood can also be estimated from the BIC (Kass and Raftery, 1995). Other methods include the bridge sampling estimator, Chib's estimator, and methods based on the arithmetic mean identity as well as the Savage-Dickey density ratio (see, Carlin and Chib, 1995; Chib, 1995; Verdinelli and Wasserman, 1995; Ntzoufras, 2009; Pajor, 2017).

5.3.1 Simple Monte Carlo Estimator

Since the marginal likelihood for a model m_i is given by

$$f(\underline{x}|m_i) = \int f(\underline{x}|\boldsymbol{\theta}_{m_i}, m_i) f(\boldsymbol{\theta}_{m_i}|m_i) \, d\boldsymbol{\theta}_{m_i},$$

a straightforward and simple estimate is provided by the Monte Carlo integration estimate

$$\hat{f}_{MC}(\underline{x}|m_i) = \frac{1}{N}\sum_{t=1}^{N} f\left(\underline{x}\middle|\boldsymbol{\theta}_{m_i}^{*(t)}, m_i\right),$$

where $\boldsymbol{\theta}_{m_i}^{*(1)}, \boldsymbol{\theta}_{m_i}^{*(2)}, \ldots, \boldsymbol{\theta}_{m_i}^{*(N)}$ are samples from the prior distribution of model m_i. According to Kass and Raftery (1995) this estimator can be very inefficient, particularly when the prior distribution and posterior distribution significantly differ. An example of where a considerable difference in the prior and posterior distributions can be expected, is when a flat prior is used (Ntzoufras, 2009). In such a case, very small likelihood values will be produced for most of the samples $\boldsymbol{\theta}_{m_i}^{*(t)}$ and the estimate will be dominated by only a few large likelihood values.

5.3.2 Laplace Approximation

A widely used approximation for the marginal likelihood is the Laplace approximation. The approximation is given by

$$f(\underline{x}|m_i) \approx (2\pi)^{\frac{d_{m_i}}{2}} \left|\tilde{\boldsymbol{\Sigma}}_{m_i}\right|^{\frac{1}{2}} f\left(\underline{x}\middle|\tilde{\boldsymbol{\theta}}_{m_i}, m_i\right) f\left(\tilde{\boldsymbol{\theta}}_{m_i}\middle|m_i\right),$$

where d_{m_i} is the number of parameters for model m_i, $\tilde{\boldsymbol{\theta}}_{m_i}$ is the posterior mode of the parameters for model m_i, and $\tilde{\boldsymbol{\Sigma}}_{m_i} = \left[-\boldsymbol{H}_m\left(\tilde{\boldsymbol{\theta}}_{m_i}\right)\right]^{-1}$. $\boldsymbol{H}_m\left(\tilde{\boldsymbol{\theta}}_{m_i}\right)$ is the Hessian matrix of second derivatives for the log of the posterior density, $\ln\left[f(\boldsymbol{\theta}_{m_i}|\underline{x}, m_i)\right]$, evaluated at the posterior mode $\tilde{\boldsymbol{\theta}}_{m_i}$. Kass and Raftery (1995) state that the Laplace approximation works well for symmetric likelihood functions and for a parameter vector of moderate dimensionality.

Raftery (1996) and Lewis and Raftery (1997) propose an extension of the Laplace approximation, called the Laplace-Metropolis estimator, which avoids the analytical calculation of $\tilde{\boldsymbol{\Sigma}}_{m_i}$ and $\tilde{\boldsymbol{\theta}}_{m_i}$. The posterior mean and variance-covariance matrix of the posterior sample $\boldsymbol{\theta}_{m_i}^{(1)}, \boldsymbol{\theta}_{m_i}^{(2)}, \ldots, \boldsymbol{\theta}_{m_i}^{(N)}$, generated from an MCMC algorithm, are used to estimate $\tilde{\boldsymbol{\theta}}_{m_i}$ and $\tilde{\boldsymbol{\Sigma}}_{m_i}$, respectively. The Laplace-Metropolis estimator if given by

$$\hat{f}_{LM}(\underline{x}|m_i) = (2\pi)^{\frac{d_{m_i}}{2}} |\boldsymbol{S}_{m_i}|^{\frac{1}{2}} f\left(\underline{x}\middle|\overline{\theta}_{m_i}, m_i\right) f\left(\overline{\theta}_{m_i}\middle|m_i\right),$$

where $\overline{\theta}_{m_i} = \frac{1}{N}\sum_{t=1}^{N} \boldsymbol{\theta}_{m_i}^{(t)}$ and $\boldsymbol{S}_{m_i}$ is a weighted variance matrix estimate (see, Lewis and Raftery, 1997, for further details).

Ntzoufras (2009) explains how the Laplace-Metropolis estimator can be calculated from an MCMC output in OpenBUGS as follows:

1. Implement the model m_i in OpenBUGS and produce posterior samples for the parameters of interest via an MCMC algorithm.

2. From the MCMC samples calculate the following estimates:

   - $\overline{\theta}_{m_i} = \left(\overline{\theta}_1, \overline{\theta}_2, \ldots, \overline{\theta}_{d_{m_i}}\right)$, which are the posterior means of the parameters of interest.
   - $\boldsymbol{s}_{\theta_{m_i}} = \left(s_{\theta_1}, s_{\theta_2}, \ldots, s_{\theta_{d_{m_i}}}\right)$, which are the posterior standard deviations of the parameters of interest.
   - $\boldsymbol{R}_{\boldsymbol{\theta}_{m_i}}$, which is the posterior correlation matrix between the parameters of interest.

3. Calculate the expression

$$\ln \hat{f}_{LM}\left(\underline{x}\,|m_i\right) = \frac{1}{2} d_{m_i} \ln(2\pi) + \frac{1}{2} \ln \left|\boldsymbol{R}_{\boldsymbol{\theta}_{m_i}}\right| + \sum_{l=1}^{d_{m_i}} \ln s_l + \sum_{k=1}^{n} \ln f\left(x_k \left|\overline{\theta}_{m_i}, m_i\right.\right) + \ln f\left(\overline{\theta}_{m_i}\,|m_i\right),$$

   and simply take $e^{\ln \hat{f}_{LM}(\underline{x}|m_i)}$ to get the Laplace-Metropolis estimate for the marginal likelihood.

5.3.3 Harmonic Mean Estimator

The harmonic mean estimator for the marginal likelihood, introduced in Newton and Raftery (1994), is given by

$$\hat{f}_{HM}\left(\underline{x}\,|m_i\right) = \left[\frac{1}{N}\sum_{t=1}^{N}\left(f\left(\underline{x}\left|\boldsymbol{\theta}_{m_i}^{(t)}, m_i\right.\right)\right)^{-1}\right]^{-1},$$

where $\boldsymbol{\theta}_{m_i}^{(1)}, \boldsymbol{\theta}_{m_i}^{(2)}, \ldots, \boldsymbol{\theta}_{m_i}^{(N)}$ are posterior samples generated by an MCMC method or other sampling technique. According to Newton and Raftery (1994), $\hat{f}_{HM}\left(\underline{x}\,|m_i\right)$ converges almost surely to $f\left(\underline{x}\,|m_i\right)$, but the harmonic mean estimator does not, in general, satisfy a Gaussian central limit theorem. This estimator is shown to be unstable and sensitive to small likelihood values (Raftery, 1996; Raftery et al., 2007) but it is simple to calculate. The authors also state that the harmonic mean estimator is simulation-consistent and unbiased, but can have infinite variance resulting in unstable behaviour.

A generalised harmonic mean estimator, which is an unbiased, consistent and more stable estimator for the marginal likelihood, is presented in Gelfand and Dey (1994). This estimator, however, requires the specification of an importance density, which must be carefully chosen and be relatively close to the posterior. Raftery et al. (2007) present two more methods for stabilising the harmonic mean estimator, and improvements on the harmonic mean estimator via Lebesgue integration is presented in Weinberg (2012).

5.3.4 Posterior Predictive Density Estimate

A variation on the traditional Bayes factor is the posterior Bayes factor, introduced in Aitkin (1991). The posterior Bayes factor of model m_i versus model m_j is based on the ratio of the posterior predictive densities of these models, for the observed data, and is given by

$$B_{ij} = \frac{f\left(\underline{x}\,|\underline{x}, m_i\right)}{f\left(\underline{x}\,|\underline{x}, m_j\right)},$$

where $f\left(\underline{x}\,|\underline{x}, m_g\right)$, $g = i, j$ is the posterior predictive density of model m_g, evaluated at the observed data. The posterior predictive density can be easily estimated by the posterior mean of the likelihood for the posterior samples obtained from an MCMC algorithm (Ntzoufras, 2009). This estimate is given by

$$\hat{f}_{PPD}\left(\underline{x}\,|\underline{x}, m_i\right) = \frac{1}{N}\sum_{t=1}^{N} f\left(\underline{x}\,\middle|\,\boldsymbol{\theta}_{m_i}^{(t)}, m_i\right),$$

where $\boldsymbol{\theta}_{m_i}^{(1)}, \boldsymbol{\theta}_{m_i}^{(2)}, \ldots, \boldsymbol{\theta}_{m_i}^{(N)}$ are posterior samples for the model parameters.

The use of posterior Bayes factors has been criticised due the double use of the data, which is also the case in the construction of the effective number of parameters p_D, used as a complexity measure in the DIC. Posterior Bayes factors can, however, support more complex models than traditional Bayes factors.

5.4 Application

An ALT data set from ReliaSoft (2020) is used in this application. The data represent failure times for a certain device under accelerated temperature and relative humidity stressors. The normal use conditions are a temperature of $T_u = 313K$ and a relative humidity of $S_u = 0.5$. The data set, presented in Table 5.2, contains failure data on 21 devices that were tested, where testing continued until all the devices had failed. The combined effect of these stressors must be investigated by a dual-stress acceleration model, such as the generalised Eyring model.

Table 5.2: Failure times for some device.

#	Temperature (K)	Humidity	Failure time	#	Temperature (K)	Humidity	Failure time
1	333	0.9	521	12	353	0.8	504
2	333	0.9	561	13	353	0.9	115
3	333	0.9	575	14	353	0.9	119
4	333	0.9	599	15	353	0.9	150
5	333	0.9	609	16	353	0.9	152
6	333	0.9	684	17	353	0.9	153
7	333	0.9	709	18	353	0.9	155
8	333	0.9	713	19	353	0.9	156
9	353	0.8	345	20	353	0.9	164
10	353	0.8	357	21	353	0.9	199
11	353	0.8	439				

The ALT models utilised in this application are the GEW_2 model, from Chapter 3, and the GEBS model, from Chapter 4. The prior specifications for these models are given in Table 5.3. Three choices of hyperparameters are used for each model, in order to investigate model selection. To discern from the models in the previous chapters, the models in this application are denoted by GEW_{BF1}, GEW_{BF2}, GEW_{BF3}, GEBS_{BF1}, GEBS_{BF2}, and GEBS_{BF3}. Flat gamma priors are imposed on the GEW_{BF1} and GEBS_{BF1} models. Subjective gamma priors with mean 5 and different variances are used for the GEW_{BF2} and GEW_{BF3}, as well as for the GEBS_{BF2} and GEBS_{BF3} models. These subjective prior are chosen such that the DIC values for the models differ enough to illustrate more meaningful model selection conclusions.

Table 5.3: Prior specifications for the Bayes factors application.

Model	θ_1	θ_2	θ_3	θ_4	β
GEW_{BF1}	$\Gamma(1,0.001)$	$\Gamma(1,0.001)$	$\Gamma(1,0.001)$	$\Gamma(1,0.001)$	$\Gamma(1,0.001)$
GEW_{BF2}	$\Gamma(5,1)$	$\Gamma(5,1)$	$\Gamma(5,1)$	$\Gamma(5,1)$	$\Gamma(5,1)$
GEW_{BF3}	$\Gamma(125,25)$	$\Gamma(125,25)$	$\Gamma(125,25)$	$\Gamma(125,25)$	$\Gamma(125,25)$
GEBS_{BF1}	$\Gamma(1,0.001)$	$\Gamma(1,0.001)$	$\Gamma(1,0.001)$	$\Gamma(1,0.001)$	$\Gamma(1,0.001)$
GEBS_{BF2}	$\Gamma(2.5,0.5)$	$\Gamma(2.5,0.5)$	$\Gamma(2.5,0.5)$	$\Gamma(2.5,0.5)$	$\Gamma(2.5,0.5)$
GEBS_{BF3}	$\Gamma(5,1)$	$\Gamma(5,1)$	$\Gamma(5,1)$	$\Gamma(5,1)$	$\Gamma(5,1)$

Since no prior information regarding the models are available, the prior model probabilities are set equal. This allows one to perform model selection by using only the Bayes factors. The Bayes factors used in this application are now defined. The Bayes factor for a model GEW_p versus a model GEW_q

is given by

$$
\begin{aligned}
B_{pq} &= \frac{\int f\left(\underline{x}\left|\theta_{1_p},\theta_{2_p},\theta_{3_p},\theta_{4_p},\beta_p,p\right.\right) f\left(\theta_{1_p},\theta_{2_p},\theta_{3_p},\theta_{4_p},\beta_p\left|p\right.\right) d\theta_{1_p} d\theta_{2_p} d\theta_{3_p} d\theta_{4_p} d\beta_p}{\int f\left(\underline{x}\left|\theta_{1_q},\theta_{2_q},\theta_{3_q},\theta_{4_q},\beta_q,q\right.\right) f\left(\theta_{1_q},\theta_{2_q},\theta_{3_q},\theta_{4_q},\beta_q\left|q\right.\right) d\theta_{1_q} d\theta_{2_q} d\theta_{3_q} d\theta_{4_q} d\beta_q} \\
&= \int \beta_p^{\sum_{i=1}^{k} r_i} \exp\left(-\theta_{1_p}\sum_{i=1}^{k} r_i - \theta_{2_p}\sum_{i=1}^{k}\frac{r_i}{T_i} - \theta_{3_p}\sum_{i=1}^{k} r_i V_i - \theta_{4_p}\sum_{i=1}^{k}\frac{r_i V_i}{T_i}\right) \\
&\quad \times \exp\left[-\sum_{i=1}^{k}\left(n_i - r_i\right) T_i \exp\left(-\theta_{1_p} - \frac{\theta_{2_p}}{T_i} - \theta_{3_p} V_i - \frac{\theta_{4_p} V_i}{T_i}\right)\tau_i^{\beta_p}\right] \\
&\quad \times \exp\left[-\sum_{i=1}^{k}\sum_{j=1}^{r_i} T_i \exp\left(-\theta_{1_p} - \frac{\theta_{2_p}}{T_i} - \theta_{3_p} V_i - \frac{\theta_{4_p} V_i}{T_i}\right) x_{ij}^{\beta_p}\right]\left[\prod_{i=1}^{k}\prod_{j=1}^{r_i} T_i x_{ij}^{\beta_p - 1}\right] \\
&\quad \times \frac{c_{11_p}^{c_{10_p}} c_{13_p}^{c_{12_p}} c_{15_p}^{c_{14_p}} c_{17_p}^{c_{16_p}} c_{19_p}^{c_{18_p}}}{\Gamma\left(c_{10_p}\right)\Gamma\left(c_{12_p}\right)\Gamma\left(c_{14_p}\right)\Gamma\left(c_{16_p}\right)\Gamma\left(c_{18_p}\right)} \theta_{1_p}^{c_{10_p}-1}\theta_{2_p}^{c_{12_p}-1}\theta_{3_p}^{c_{14_p}-1}\theta_{4_p}^{c_{16_p}-1}\beta_p^{c_{18_p}-1} \\
&\quad \times \exp\left(-c_{11_p}\theta_{1_p} - c_{13_p}\theta_{2_p} - c_{15_p}\theta_{3_p} - c_{17_p}\theta_{4_p} - c_{19_p}\beta_p\right) d\theta_{1_p} d\theta_{2_p} d\theta_{3_p} d\theta_{4_p} d\beta_p \\
&\quad \div \\
&\quad \int \beta_q^{\sum_{i=1}^{k} r_i} \exp\left(-\theta_{1_q}\sum_{i=1}^{k} r_i - \theta_{2_q}\sum_{i=1}^{k}\frac{r_i}{T_i} - \theta_{3_q}\sum_{i=1}^{k} r_i V_i - \theta_{4_q}\sum_{i=1}^{k}\frac{r_i V_i}{T_i}\right) \\
&\quad \times \exp\left[-\sum_{i=1}^{k}\left(n_i - r_i\right) T_i \exp\left(-\theta_{1_q} - \frac{\theta_{2_q}}{T_i} - \theta_{3_q} V_i - \frac{\theta_{4_q} V_i}{T_i}\right)\tau_i^{\beta_q}\right] \\
&\quad \times \exp\left[-\sum_{i=1}^{k}\sum_{j=1}^{r_i} T_i \exp\left(-\theta_{1_q} - \frac{\theta_{2_q}}{T_i} - \theta_{3_q} V_i - \frac{\theta_{4_q} V_i}{T_i}\right) x_{ij}^{\beta_q}\right]\left[\prod_{i=1}^{k}\prod_{j=1}^{r_i} T_i x_{ij}^{\beta_q - 1}\right] \\
&\quad \times \frac{c_{11_q}^{c_{10_q}} c_{13_q}^{c_{12_q}} c_{15_q}^{c_{14_q}} c_{17_q}^{c_{16_q}} c_{19_q}^{c_{18_q}}}{\Gamma\left(c_{10_q}\right)\Gamma\left(c_{12_q}\right)\Gamma\left(c_{14_q}\right)\Gamma\left(c_{16_q}\right)\Gamma\left(c_{18_q}\right)} \theta_{1_q}^{c_{10_q}-1}\theta_{2_q}^{c_{12_q}-1}\theta_{3_q}^{c_{14_q}-1}\theta_{4_q}^{c_{16_q}-1}\beta_q^{c_{18_q}-1} \\
&\quad \times \exp\left(-c_{11_q}\theta_{1_q} - c_{13_q}\theta_{2_q} - c_{15_q}\theta_{3_q} - c_{17_q}\theta_{4_q} - c_{19_q}\beta_q\right) d\theta_{1_q} d\theta_{2_q} d\theta_{3_q} d\theta_{4_q} d\beta_q.
\end{aligned}
$$

The Bayes factor for comparing a model GEBS_p and a model GEBS_q is given by

$$
\begin{aligned}
B_{pq} &= \frac{\int f\left(\underline{x}\left|\alpha_p,\theta_{1_p},\theta_{2_p},\theta_{3_p},\theta_{4_p},p\right.\right) f\left(\alpha_p,\theta_{1_p},\theta_{2_p},\theta_{3_p},\theta_{4_p}\left|p\right.\right) d\alpha_p d\theta_{1_p} d\theta_{2_p} d\theta_{3_p} d\theta_{4_p}}{\int f\left(\underline{x}\left|\alpha_q,\theta_{1_q},\theta_{2_q},\theta_{3_q},\theta_{4_q},q\right.\right) f\left(\alpha_q,\theta_{1_q},\theta_{2_q},\theta_{3_q},\theta_{4_q}\left|q\right.\right) d\alpha_q d\theta_{1_q} d\theta_{2_q} d\theta_{3_q} d\theta_{4_q}} \\
&= \int \left(2\sqrt{2\pi}\alpha_p\right)^{-\sum_{i=1}^{k} r_i} \exp\left(\frac{1}{\alpha_p^2}\sum_{i=1}^{k} r_i\right)\left[\prod_{i=1}^{k}\prod_{j=1}^{r_i}\frac{x_{ij}+\frac{1}{T_i}\exp\left(\theta_{1_p}+\frac{\theta_{2_p}}{T_i}+\theta_{3_p}V_i+\frac{\theta_{4_p}V_i}{T_i}\right)}{x_{ij}^{\frac{3}{2}}}\right] \\
&\quad \times \exp\left\{-\frac{1}{2\alpha_p^2}\sum_{i=1}^{k}\left[T_i\exp\left(-\theta_{1_p}-\frac{\theta_{2_p}}{T_i}-\theta_{3_p}V_i-\frac{\theta_{4_p}V_i}{T_i}\right)\sum_{j=1}^{r_i}x_{ij}\right]\right. \\
&\quad \left. -\frac{1}{2\alpha_p^2}\sum_{i=1}^{k}\left[\frac{1}{T_i}\exp\left(\theta_{1_p}+\frac{\theta_{2_p}}{T_i}+\theta_{3_p}V_i+\frac{\theta_{4_p}V_i}{T_i}\right)\sum_{j=1}^{r_i}\frac{1}{x_{ij}}\right]\right\} \\
&\quad \times \prod_{i=1}^{k}\left\{\left[T_i\exp\left(-\theta_{1_p}-\frac{\theta_{2_p}}{T_i}-\theta_{3_p}V_i-\frac{\theta_{4_p}V_i}{T_i}\right)\right]^{\frac{r_i}{2}}\right. \\
&\quad \times \left[1-\Phi\left(\frac{1}{\alpha_p}\left(\sqrt{\tau_i T_i\exp\left(-\theta_{1_p}-\frac{\theta_{2_p}}{T_i}-\theta_{3_p}V_i-\frac{\theta_{4_p}V_i}{T_i}\right)}\right.\right.\right. \\
&\quad \left.\left.\left.\left. -\sqrt{\frac{1}{\tau_i T_i}\exp\left(\theta_{1_p}+\frac{\theta_{2_p}}{T_i}+\theta_{3_p}V_i+\frac{\theta_{4_p}V_i}{T_i}\right)}\right)\right)\right]^{n_i-r_i}\right\}. \\
&\quad \times \frac{c_{1_p}^{c_{0_p}} c_{3_p}^{c_{2_p}} c_{5_p}^{c_{4_p}} c_{7_p}^{c_{6_p}} c_{9_p}^{c_{8_p}}}{\Gamma\left(c_{0_p}\right)\Gamma\left(c_{2_p}\right)\Gamma\left(c_{4_p}\right)\Gamma\left(c_{6_p}\right)\Gamma\left(c_{8_p}\right)} \alpha_p^{c_{0_p}-1}\theta_{1_p}^{c_{2_p}-1}\theta_{2_p}^{c_{4_p}-1}\theta_{3_p}^{c_{6_p}-1}\theta_{4_p}^{c_{8_p}-1} \\
&\quad \times \exp\left(-c_{1_p}\alpha_p - c_{3_p}\theta_{1_p} - c_{5_p}\theta_{2_p} - c_{7_p}\theta_{3_p} - c_{9_p}\theta_{4_p}\right) d\alpha_p d\theta_{1_p} d\theta_{2_p} d\theta_{3_p} d\theta_{4_p} \\
&\quad \div \\
&\quad \int \left(2\sqrt{2\pi}\alpha_q\right)^{-\sum_{i=1}^{k} r_i} \exp\left(\frac{1}{\alpha_q^2}\sum_{i=1}^{k} r_i\right)\left[\prod_{i=1}^{k}\prod_{j=1}^{r_i}\frac{x_{ij}+\frac{1}{T_i}\exp\left(\theta_{1_q}+\frac{\theta_{2_q}}{T_i}+\theta_{3_q}V_i+\frac{\theta_{4_q}V_i}{T_i}\right)}{x_{ij}^{\frac{3}{2}}}\right] \\
&\quad \times \exp\left\{-\frac{1}{2\alpha_q^2}\sum_{i=1}^{k}\left[T_i\exp\left(-\theta_{1_q}-\frac{\theta_{2_q}}{T_i}-\theta_{3_q}V_i-\frac{\theta_{4_q}V_i}{T_i}\right)\sum_{j=1}^{r_i}x_{ij}\right]\right. \\
&\quad \left. -\frac{1}{2\alpha_q^2}\sum_{i=1}^{k}\left[\frac{1}{T_i}\exp\left(\theta_{1_q}+\frac{\theta_{2_q}}{T_i}+\theta_{3_q}V_i+\frac{\theta_{4_q}V_i}{T_i}\right)\sum_{j=1}^{r_i}\frac{1}{x_{ij}}\right]\right\} \\
&\quad \times \prod_{i=1}^{k}\left\{\left[T_i\exp\left(-\theta_{1_q}-\frac{\theta_{2_q}}{T_i}-\theta_{3_q}V_i-\frac{\theta_{4_q}V_i}{T_i}\right)\right]^{\frac{r_i}{2}}\right. \\
&\quad \times \left[1-\Phi\left(\frac{1}{\alpha_q}\left(\sqrt{\tau_i T_i\exp\left(-\theta_{1_q}-\frac{\theta_{2_q}}{T_i}-\theta_{3_q}V_i-\frac{\theta_{4_q}V_i}{T_i}\right)}\right.\right.\right. \\
&\quad \left.\left.\left.\left. -\sqrt{\frac{1}{\tau_i T_i}\exp\left(\theta_{1_q}+\frac{\theta_{2_q}}{T_i}+\theta_{3_q}V_i+\frac{\theta_{4_q}V_i}{T_i}\right)}\right)\right)\right]^{n_i-r_i}\right\}. \\
&\quad \times \frac{c_{1_q}^{c_{0_q}} c_{3_q}^{c_{2_q}} c_{5_q}^{c_{4_q}} c_{7_q}^{c_{6_q}} c_{9_q}^{c_{8_q}}}{\Gamma\left(c_{0_q}\right)\Gamma\left(c_{2_q}\right)\Gamma\left(c_{4_q}\right)\Gamma\left(c_{6_q}\right)\Gamma\left(c_{8_q}\right)} \alpha_q^{c_{0_q}-1}\theta_{1_q}^{c_{2_q}-1}\theta_{2_q}^{c_{4_q}-1}\theta_{3_q}^{c_{6_q}-1}\theta_{4_q}^{c_{8_q}-1} \\
&\quad \times \exp\left(-c_{1_q}\alpha_q - c_{3_q}\theta_{1_q} - c_{5_q}\theta_{2_q} - c_{7_q}\theta_{3_q} - c_{9_q}\theta_{4_q}\right) d\alpha_q d\theta_{1_q} d\theta_{2_q} d\theta_{3_q} d\theta_{4_q}.
\end{aligned}
$$

The Bayes factor for the model GEW_p versus the model GEBS_q is given by

$$
\begin{aligned}
B_{pq} &= \frac{\int f\left(\underline{x}\,|\,\theta_{1_p},\theta_{2_p},\theta_{3_p},\theta_{4_p},\beta_p,p\right) f\left(\theta_{1_p},\theta_{2_p},\theta_{3_p},\theta_{4_p},\beta_p\,|p\right) d\theta_{1_p}d\theta_{2_p}d\theta_{3_p}d\theta_{4_p}d\beta_p}{\int f\left(\underline{x}\,|\,\alpha_q,\theta_{1_q},\theta_{2_q},\theta_{3_q},\theta_{4_q},q\right) f\left(\alpha_q,\theta_{1_q},\theta_{2_q},\theta_{3_q},\theta_{4_q}\,|q\right) d\alpha_q d\theta_{1_q}d\theta_{2_q}d\theta_{3_q}d\theta_{4_q}} \\
&= \int \beta_p^{\sum_{i=1}^k r_i} \exp\left(-\theta_{1_p}\sum_{i=1}^k r_i - \theta_{2_p}\sum_{i=1}^k \frac{r_i}{T_i} - \theta_{3_p}\sum_{i=1}^k r_i V_i - \theta_{4_p}\sum_{i=1}^k \frac{r_i V_i}{T_i}\right) \\
&\quad \times \exp\left[-\sum_{i=1}^k (n_i - r_i)\, T_i \exp\left(-\theta_{1_p} - \frac{\theta_{2_p}}{T_i} - \theta_{3_p}V_i - \frac{\theta_{4_p}V_i}{T_i}\right)\tau_i^{\beta_p}\right] \\
&\quad \times \exp\left[-\sum_{i=1}^k\sum_{j=1}^{r_i} T_i \exp\left(-\theta_{1_p} - \frac{\theta_{2_p}}{T_i} - \theta_{3_p}V_i - \frac{\theta_{4_p}V_i}{T_i}\right) x_{ij}^{\beta_p}\right]\left[\prod_{i=1}^k\prod_{j=1}^{r_i} T_i x_{ij}^{\beta_p - 1}\right] \\
&\quad \times \frac{c_{11_p}^{c_{10_p}} c_{13_p}^{c_{12_p}} c_{15_p}^{c_{14_p}} c_{17_p}^{c_{16_p}} c_{19_p}^{c_{18_p}}}{\Gamma\left(c_{10_p}\right)\Gamma\left(c_{12_p}\right)\Gamma\left(c_{14_p}\right)\Gamma\left(c_{16_p}\right)\Gamma\left(c_{18_p}\right)}\theta_{1_p}^{c_{10_p}-1}\theta_{2_p}^{c_{12_p}-1}\theta_{3_p}^{c_{14_p}-1}\theta_{4_p}^{c_{16_p}-1}\beta_p^{c_{18_p}-1} \\
&\quad \times \exp\left(-c_{11_p}\theta_{1_p} - c_{13_p}\theta_{2_p} - c_{15_p}\theta_{3_p} - c_{17_p}\theta_{4_p} - c_{19_p}\beta_p\right) d\theta_{1_p}d\theta_{2_p}d\theta_{3_p}d\theta_{4_p}d\beta_p \\
&\quad \div \\
&\quad \int \left(2\sqrt{2\pi}\alpha_q\right)^{-\sum_{i=1}^k r_i} \exp\left(\frac{1}{\alpha_q^2}\sum_{i=1}^k r_i\right)\left[\prod_{i=1}^k\prod_{j=1}^{r_i} \frac{x_{ij} + \frac{1}{T_i}\exp\left(\theta_{1_q} + \frac{\theta_{2_q}}{T_i} + \theta_{3_q}V_i + \frac{\theta_{4_q}V_i}{T_i}\right)}{x_{ij}^{\frac{3}{2}}}\right] \\
&\quad \times \exp\Bigg\{-\frac{1}{2\alpha_q^2}\sum_{i=1}^k\left[T_i \exp\left(-\theta_{1_q} - \frac{\theta_{2_q}}{T_i} - \theta_{3_q}V_i - \frac{\theta_{4_q}V_i}{T_i}\right)\sum_{j=1}^{r_i} x_{ij}\right] \\
&\qquad - \frac{1}{2\alpha_q^2}\sum_{i=1}^k\left[\frac{1}{T_i}\exp\left(\theta_{1_q} + \frac{\theta_{2_q}}{T_i} + \theta_{3_q}V_i + \frac{\theta_{4_q}V_i}{T_i}\right)\sum_{j=1}^{r_i}\frac{1}{x_{ij}}\right]\Bigg\} \\
&\quad \times \prod_{i=1}^k \Bigg\{\left[T_i \exp\left(-\theta_{1_q} - \frac{\theta_{2_q}}{T_i} - \theta_{3_q}V_i - \frac{\theta_{4_q}V_i}{T_i}\right)\right]^{\frac{r_i}{2}} \\
&\qquad \times \Bigg[1 - \Phi\Bigg(\frac{1}{\alpha_q}\Bigg(\sqrt{\tau_i T_i \exp\left(-\theta_{1_q} - \frac{\theta_{2_q}}{T_i} - \theta_{3_q}V_i - \frac{\theta_{4_q}V_i}{T_i}\right)} \\
&\qquad\quad - \sqrt{\frac{1}{\tau_i T_i}\exp\left(\theta_{1_q} + \frac{\theta_{2_q}}{T_i} + \theta_{3_q}V_i + \frac{\theta_{4_q}V_i}{T_i}\right)}\Bigg)\Bigg)\Bigg]^{n_i - r_i}\Bigg\}. \\
&\quad \times \frac{c_{1_q}^{c_{0_q}} c_{3_q}^{c_{2_q}} c_{5_q}^{c_{4_q}} c_{7_q}^{c_{6_q}} c_{9_q}^{c_{8_q}}}{\Gamma\left(c_{0_q}\right)\Gamma\left(c_{2_q}\right)\Gamma\left(c_{4_q}\right)\Gamma\left(c_{6_q}\right)\Gamma\left(c_{8_q}\right)}\alpha_q^{c_{0_q}-1}\theta_{1_q}^{c_{2_q}-1}\theta_{2_q}^{c_{4_q}-1}\theta_{3_q}^{c_{6_q}-1}\theta_{4_q}^{c_{8_q}-1} \\
&\quad \times \exp\left(-c_{1_q}\alpha_q - c_{3_q}\theta_{1_q} - c_{5_q}\theta_{2_q} - c_{7_q}\theta_{3_q} - c_{9_q}\theta_{4_q}\right) d\alpha_q d\theta_{1_q}d\theta_{2_q}d\theta_{3_q}d\theta_{4_q}.
\end{aligned}
$$

The models are implemented in OpenBUGS to generate posterior samples to base inference on. A single Markov chain is initiated for each model, with a burn-in of 50000 iterations, after which 200000 samples are obtained. Trace plots and the modified Gelman-Rubin statistic, proposed by Brooks and Gelman (1998), are used to verify that the Markov chains have converged before the burn-in iterations end. The Monte Carlo error is less than 5% of the sample standard deviation for all parameters in these models, indicating that enough samples have been generated.

The marginal posterior distributions for the models are given in Figures 5.1 to 5.6. It can be noted that the models where flat priors are used produce fairly skewed marginal posteriors. The marginal posteriors for the GEW$_{BF2}$, GEBS$_{BF2}$ and GEBS$_{BF3}$ models are somewhat less skewed, and those of the GEW$_{BF3}$ model are relatively symmetric.

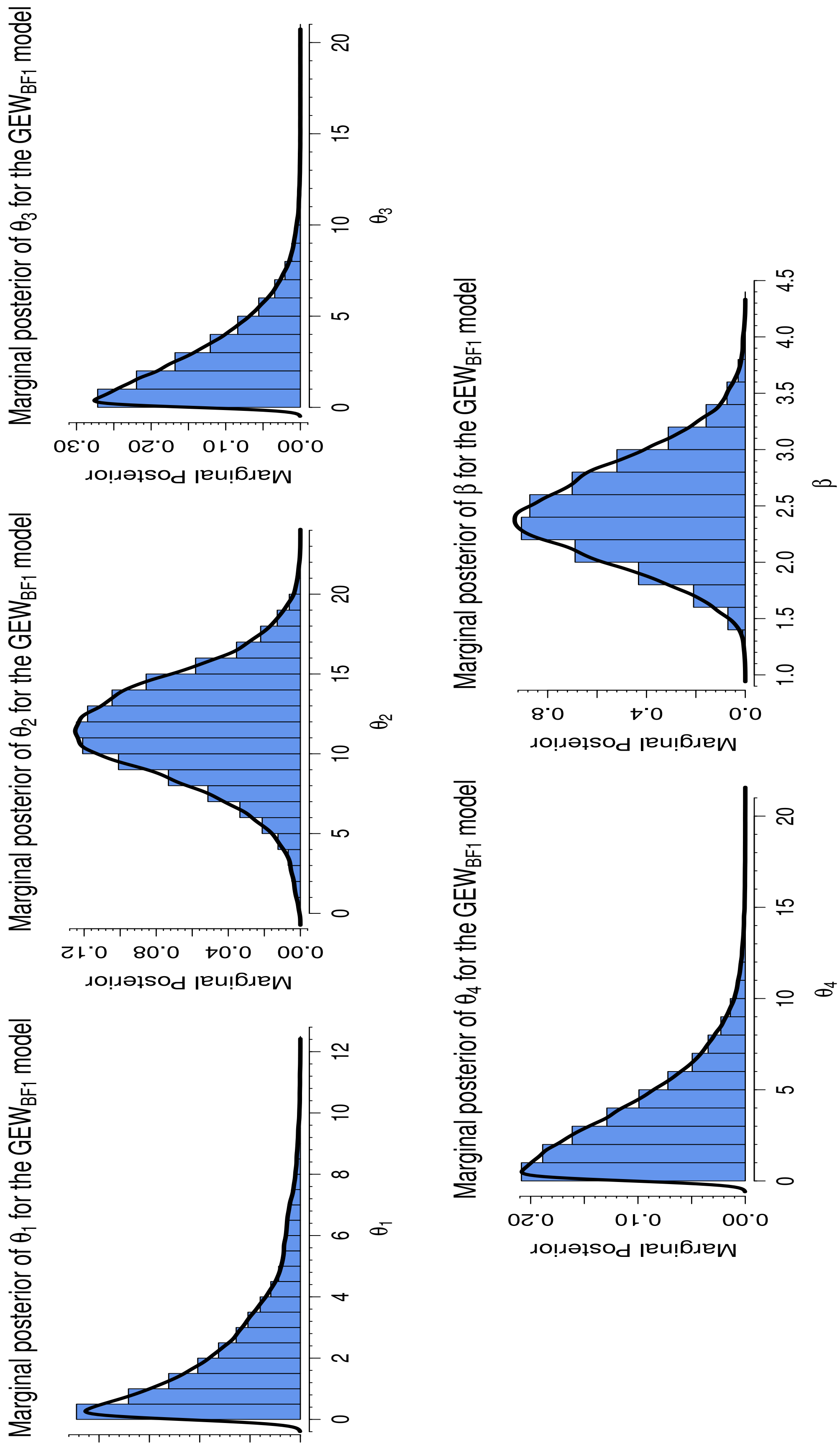

Figure 5.1: Marginal posterior distributions for the GEW$_{BF1}$ model.

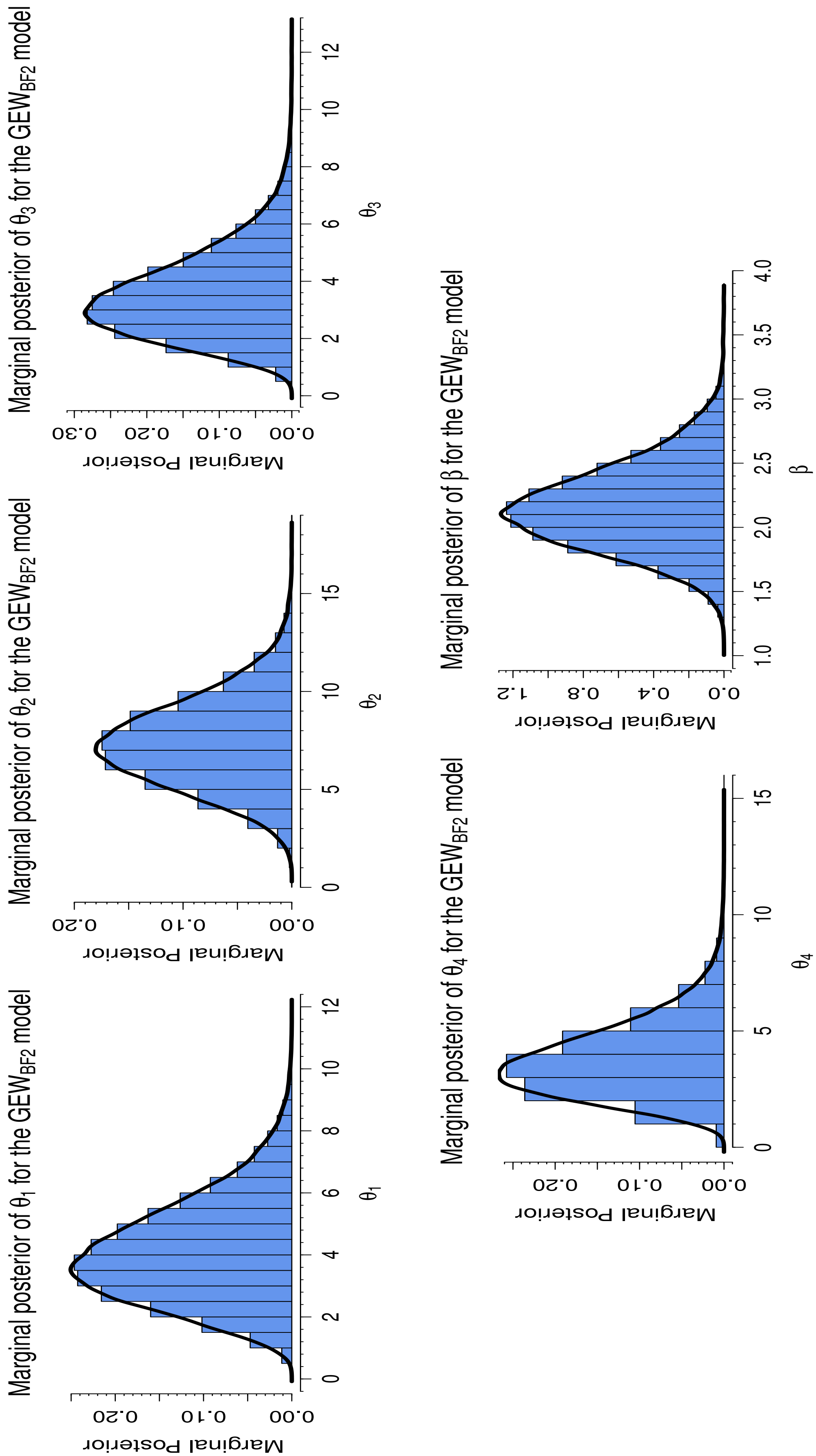

Figure 5.2: Marginal posterior distributions for the GEW_{BF2} model.

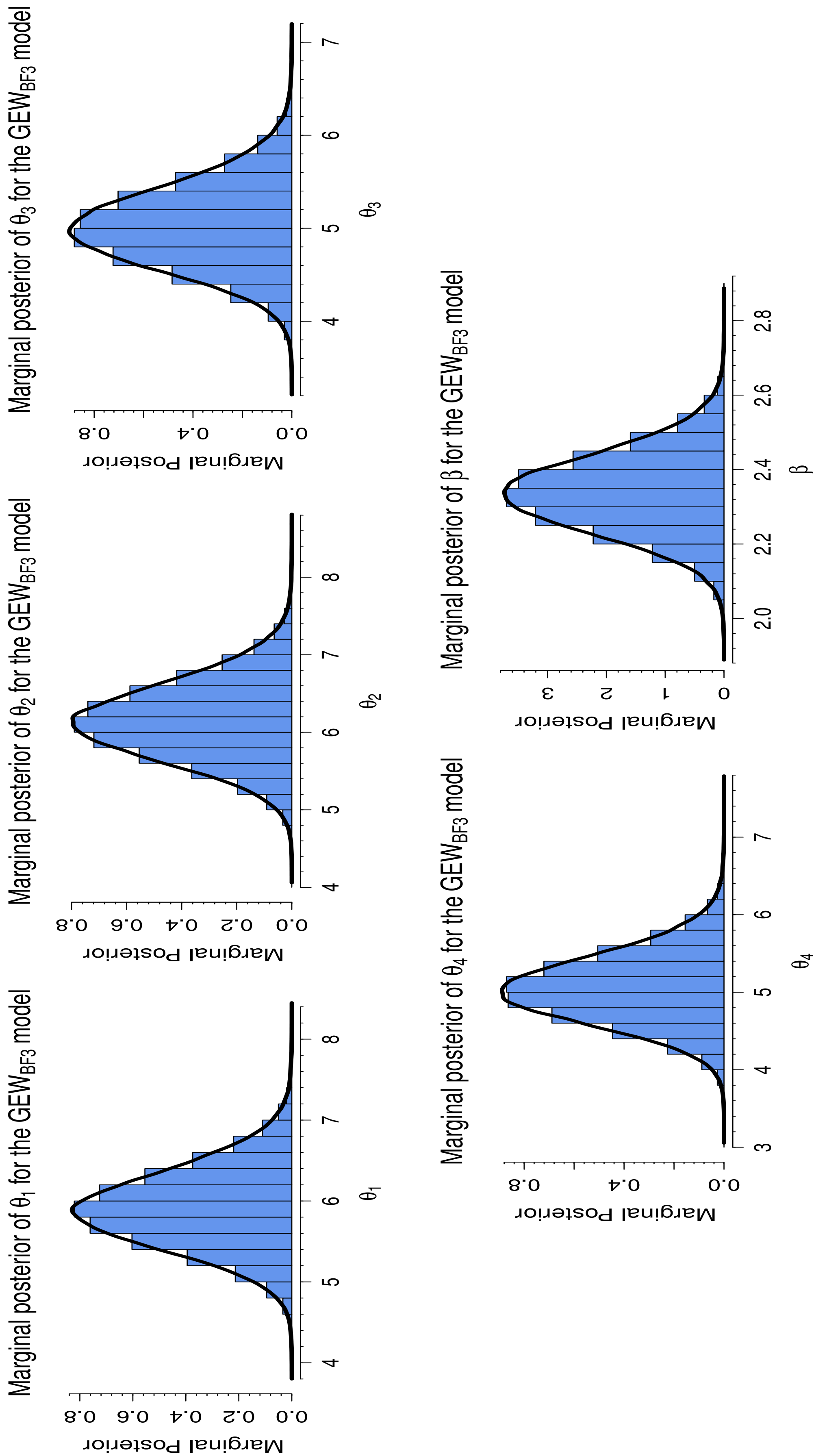

Figure 5.3: Marginal posterior distributions for the GEW_{BF3} model.

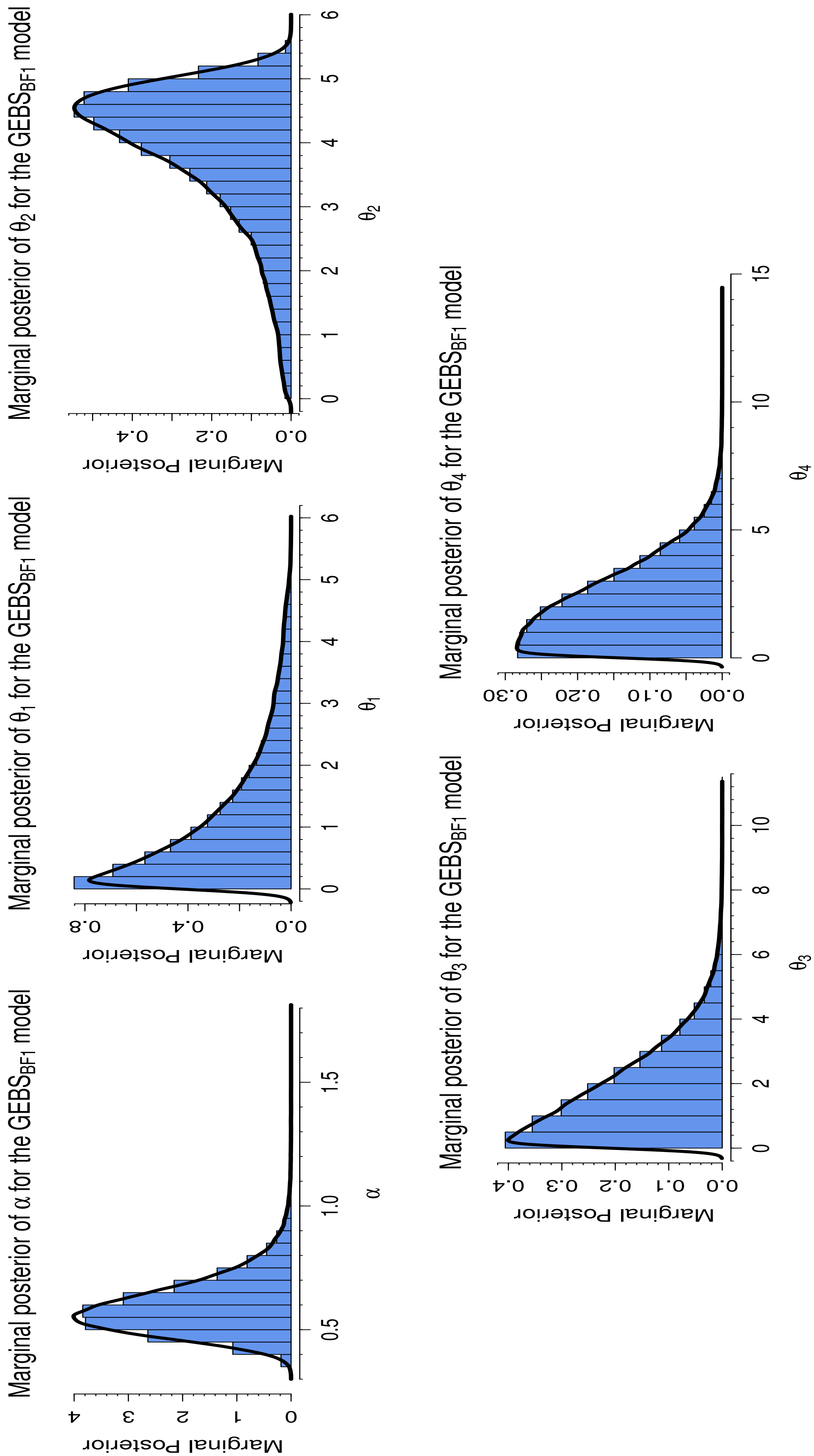

Figure 5.4: Marginal posterior distributions for the GEBS$_{BF1}$ model.

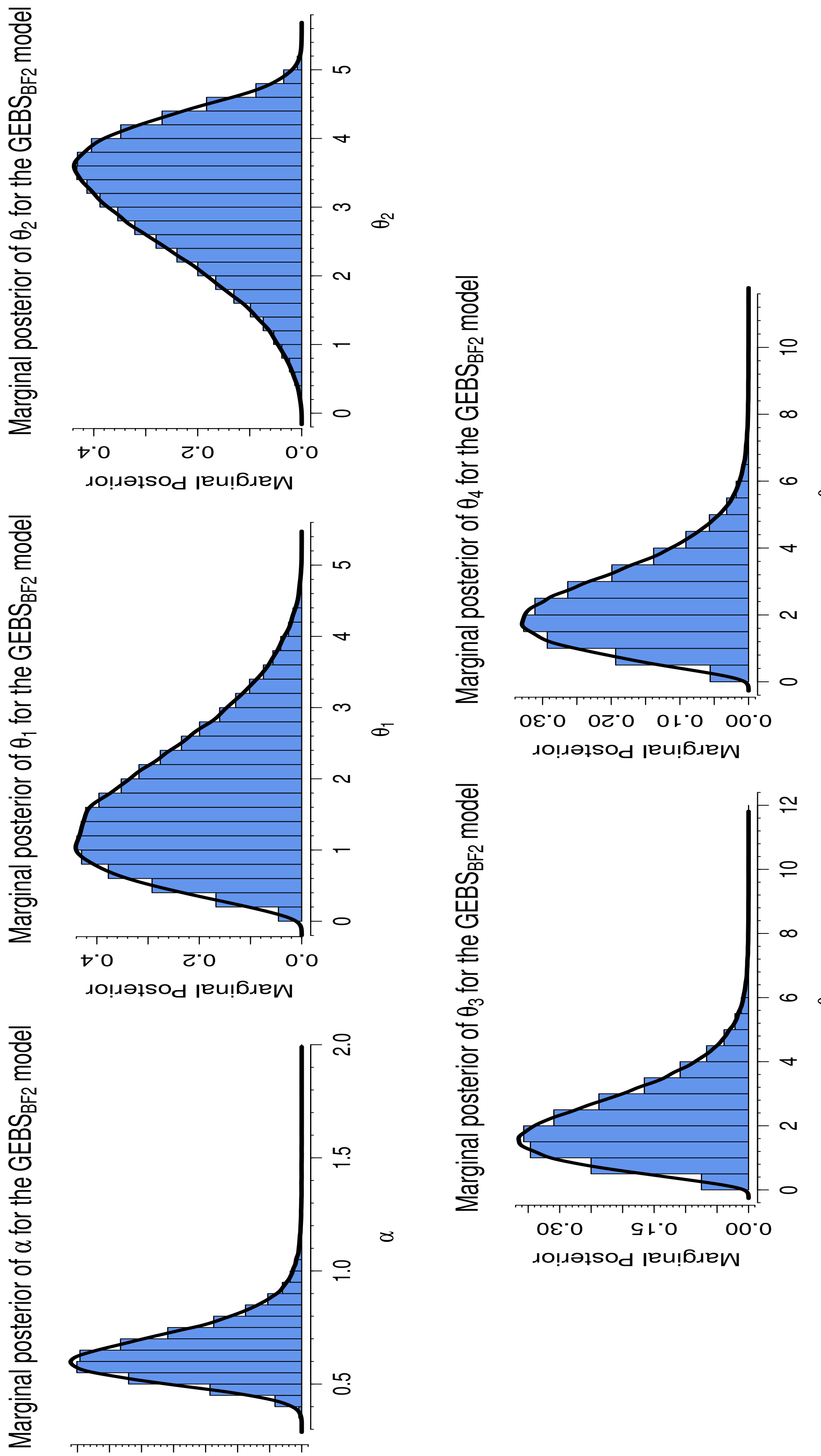

Figure 5.5: Marginal posterior distributions for the GEBS_{BF2} model.

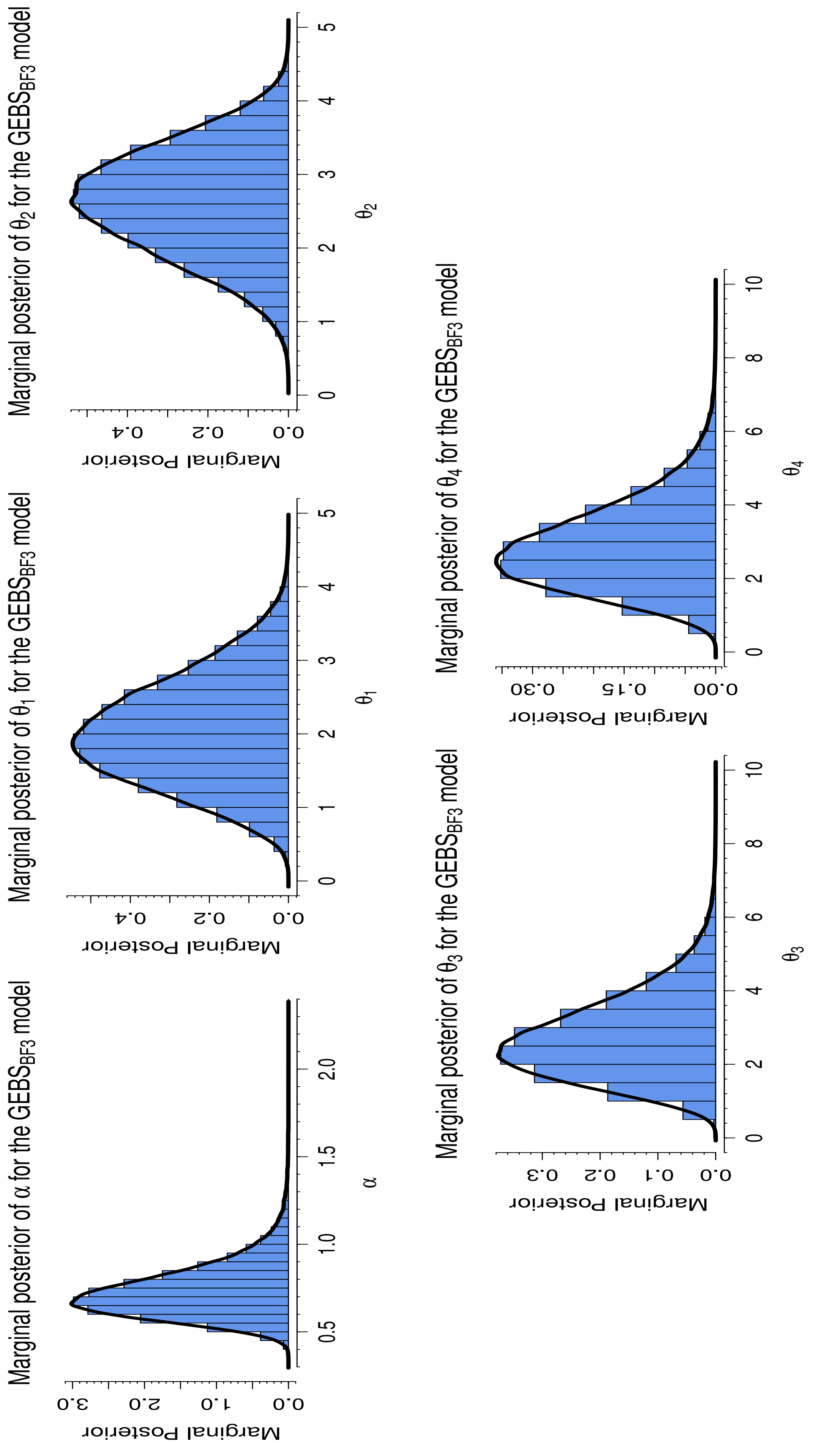

Figure 5.6: Marginal posterior distributions for the GEBS$_{BF3}$ model.

The DIC values for the models considered in this application are given in Table 5.4. The GEBS$_{BF1}$ model exhibits the lowest DIC amongst the six models. The DIC indicates a strong preference for the GEBS models in this case. Among the GEW models, the GEW$_{BF1}$, exibits the lowest DIC. Again, it can be noted that the models with smaller variance subjective priors have higher DIC values than the models with with flat priors or subjective priors with a larger variance.

Considering all six models and keeping in mind that the GEBS$_{BF1}$ model has the lowest DIC, the guidelines in Burnham and Anderson (1998) indicate that GEBS$_{BF2}$ still has substantial support to be used for inference. There is considerably less support to choose GEBS$_{BF3}$, and the remaining three GEW models have almost no support to be chosen. By investigating the GEW models separately, we see that GEW$_{BF1}$ has the lowest DIC, there is fairly substantial support for GEW$_{BF2}$, and considerably less support for GEW$_{BF3}$.

Table 5.4: Deviance information criterion for the Bayes factors application.

Model	DIC
GEW$_{BF1}$	278.4
GEW$_{BF2}$	280.9
GEW$_{BF3}$	284.6
GEBS$_{BF1}$	248.9
GEBS$_{BF2}$	250.8
GEBS$_{BF3}$	253.3

Due to the large discrepancies between many of the prior distributions in Table 5.3 and the marginal posteriors produced from the MCMC algorithm, the simple Monte Carlo estimator is omitted from this application. Sampling from the prior for this estimator causes computational issues for most of the models. However, for the models where the marginal likelihood can be estimated, it is relatively close to the approximation given by the Laplace-Metropolis estimator.

Table 5.5 shows the natural log of the marginal likelihood estimates for the models, given by the Laplace-Metropolis estimator and the harmonic mean estimator. The natural log of the posterior predictive density estimates, which are required to calculate the posterior Bayes factors, are also given. We number the models in this table in order to simplify the notation for the Bayes factors to follow.

The Laplace-Metropolis estimator favours the GEW$_{BF2}$ model, the harmonic mean estimator favours the GEW$_{BF1}$ model, and the posterior predictive density estimate favours the GEBS$_{BF1}$ model. It is interesting to note that the Laplace-Metropolis estimator does not favour the models where flat priors are used. This can partially be explained by the skewed marginal posterior distributions produced by these models, seeing that the Laplace-Metropolis estimator works well for symmetric distributions. For the harmonic mean estimator and posterior predictive density estimate there is not a distinctive preference towards the GEBS models, as we have seen with the DIC. It must again be stressed that the DIC and Bayes factors measure model fit differently, as is explained in Spiegelhalter et al. (2002).

Table 5.5: Natural log of the marginal likelihood estimates for the models.

Model #	Model	Laplace-Metropolis estimator	Harmonic mean estimator	Posterior predictive density estimate
1	GEW_{BF1}	-166.7920	-141.5369	-137.1797
2	GEW_{BF2}	-145.1006	-141.6805	-138.7164
3	GEW_{BF3}	-176.4858	-145.1727	-141.3472
4	$GEBS_{BF1}$	-170.1288	-142.2716	-136.8164
5	$GEBS_{BF2}$	-147.7771	-144.2880	-137.6861
6	$GEBS_{BF3}$	-151.8292	-146.6435	-138.8855

Table 5.6 contains the Bayes factor values, given by the Laplace-Metropolis estimator, for all combinations of the models used in this application. Note that there is positive to very strong evidence in favour of the GEW_{BF2} model. Considering the GEBS models separately, there is very strong evidence for the $GEBS_{BF2}$ model. Table 5.7 contains the posterior model probabilities for each pair of models, given by the Laplace-Metropolis estimator.

Table 5.6: Laplace-Metropolis estimates for the Bayes factors.

	B_{ij}	Interpretation
B_{12}	3.7979×10^{-10}	Very strong evidence for model 2
B_{13}	1.6217×10^{4}	Very strong evidence for model 1
B_{14}	28.1309	Strong evidence for model 1
B_{15}	5.5201×10^{-9}	Very strong evidence for model 5
B_{16}	3.1751×10^{-7}	Very strong evidence for model 6
B_{23}	4.2700×10^{13}	Very strong evidence for model 2
B_{24}	7.4069×10^{10}	Very strong evidence for model 2
B_{25}	14.5346	Positive evidence for model 2
B_{26}	836.0159	Very strong evidence for model 2
B_{34}	0.0017	Very strong evidence for model 4
B_{35}	3.4039×10^{-13}	Very strong evidence for model 5
B_{36}	1.9579×10^{-11}	Very strong evidence for model 6
B_{45}	1.9623×10^{-10}	Very strong evidence for model 5
B_{46}	1.1287×10^{-8}	Very strong evidence for model 6
B_{56}	57.5192	Strong evidence for model 5

Table 5.7: Laplace-Metropolis estimates for the posterior model probabilities.

ij	Posterior model probability for model m_i	Posterior model probability for model m_j
12	0.0000	1.0000
13	0.9999	0.0001
14	0.9657	0.0343
15	0.0000	1.0000
16	0.0000	1.0000
23	1.0000	0.0000
24	1.0000	0.0000
25	0.9356	0.0644
26	0.9988	0.0012
34	0.0017	0.9983
35	0.0000	1.0000
36	0.0000	1.0000
45	0.0000	1.0000
46	0.0000	1.0000
56	0.9829	0.0171

The Bayes factor values, given by the harmonic mean estimator, for all pairs of the models are given in Table 5.8. We observe negligible to very strong evidence for the GEW_{BF1} model. Among only the GEBS models, there is positive to strong evidence in favour of the GEBS_{BF1} model. It is clear that the Bayes factors, using the harmonic mean estimator, also favour the models where flat priors are used. This is in agreement with the conclusions from the DIC. The posterior model probabilities for each pair of models, given by the harmonic mean estimator, are provided in Table 5.9.

Table 5.8: Harmonic mean estimates for the Bayes factors.

	B_{ij}	Interpretation
B_{12}	1.1543	Negligible evidence for model 1
B_{13}	37.9317	Strong evidence for model 1
B_{14}	2.0847	Negligible evidence for model 1
B_{15}	15.6587	Positive evidence for model 1
B_{16}	165.1082	Very strong evidence for model 1
B_{23}	32.8600	Strong evidence for model 2
B_{24}	1.8060	Negligible evidence for model 2
B_{25}	13.5651	Positive evidence for model 2
B_{26}	143.0325	Strong evidence for model 2
B_{34}	0.0550	Positive evidence for model 4
B_{35}	0.4128	Negligible evidence for model 5
B_{36}	4.3528	Positive evidence for model 3
B_{45}	7.5112	Positive evidence for model 4
B_{46}	79.2000	Strong evidence for model 4
B_{56}	10.5442	Positive evidence for model 5

Table 5.9: Harmonic mean estimates for the posterior model probabilities.

ij	Posterior model probability for model m_i	Posterior model probability for model m_j
12	0.5358	0.4642
13	0.9743	0.0257
14	0.6758	0.3242
15	0.9400	0.0600
16	0.9940	0.0060
23	0.9705	0.0295
24	0.6436	0.3564
25	0.9313	0.0687
26	0.9931	0.0069
34	0.0521	0.9479
35	0.2922	0.7078
36	0.8132	0.1868
45	0.8825	0.1175
46	0.9875	0.0125
56	0.9134	0.0866

Table 5.10 shows the posterior Bayes factor values for the models, where the posterior predictive density estimates are used. Overall, the GEBS$_{BF1}$ model is supported with negligible to strong evidence. For the GEW models separately, we see positive to strong evidence for the GEW$_{BF1}$ model. The posterior Bayes factors also favour the models where flat priors are utilised. By using the estimated posterior Bayes factors, the posterior model probabilities are presented in Table 5.11.

Table 5.10: Posterior predictive density estimates for posterior Bayes factors.

	B_{ij}	Interpretation
B_{12}	4.6489	Positive evidence for model 1
B_{13}	64.5548	Strong evidence for model 1
B_{14}	0.6954	Negligible evidence for model 4
B_{15}	1.6592	Negligible evidence for model 1
B_{16}	5.5059	Positive evidence for model 1
B_{23}	13.8860	Positive evidence for model 2
B_{24}	0.1496	Positive evidence for model 4
B_{25}	0.3569	Negligible evidence for model 5
B_{26}	1.1843	Negligible evidence for model 2
B_{34}	0.0108	Strong evidence for model 4
B_{35}	0.0257	Strong evidence for model 4
B_{36}	0.0853	Positive evidence for model 6
B_{45}	2.3862	Negligible evidence for model 4
B_{46}	7.9181	Positive evidence for model 4
B_{56}	3.3183	Positive evidence for model 5

Table 5.11: Posterior model probabilities using the estimated posterior Bayes factors

ij	Posterior model probability for model m_i	Posterior model probability for model m_j
12	0.8230	0.1770
13	0.9847	0.0153
14	0.4102	0.5898
15	0.6240	0.3760
16	0.8463	0.1537
23	0.9328	0.0672
24	0.1301	0.8699
25	0.2630	0.7370
26	0.5422	0.4578
34	0.0107	0.9893
35	0.0251	0.9749
36	0.0786	0.9214
45	0.7047	0.2953
46	0.8879	0.1121
56	0.7684	0.2316

5.5 Conclusion

In this chapter, the use of Bayes factors and the DIC for model selection are compared in a Bayesian ALT setup. Two dual-stress models, namely the GEW and GEBS models with gamma priors, are utilised for this comparison. The posterior distributions for these models can not be written in closed form, which complicates the calculation of the Bayes factors. MCMC methods are employed to generate posterior samples to base inference on. Methods for estimating the marginal likelihood, without further complicating the sampling process, is explored. These methods include a simple Monte Carlo estimator, the Laplace-Metropolis estimator, the harmonic mean estimator, and a posterior predictive density estimate used for calculating posterior Bayes factors.

The models are applied to an ALT data set where the stressors are temperature and relative humidity. Several choices of hyperparameters are used in order to illustrate the use of the DIC and Bayes factors in model selection. It is interesting to note that the DIC shows definitive support for the GEBS models above the GEW models. The models where flat priors are imposed on the model parameters are favoured by the DIC.

The different methods for estimating the marginal likelihood give variable conclusions. The simple Monte Carlo estimator is omitted in the application, due to it causing computational issues for some of the models. The Laplace-Metropolis estimator results in Bayes factors which show virtually no evidence in favour of the models with flat priors. The Bayes factors produced by the harmonic mean estimator and the posterior Bayes factors have results that are more comparable to the DIC. Viewing the GEW and GEBS models separately, these Bayes factors also favour the models with flat priors. The harmonic mean estimator shows more evidence in support of the GEW models, whereas the posterior Bayes factors support the GEBS models to a greater extent.

As described in Spiegelhalter et al. (2002), the DIC and Bayes factors address model selection from different angles. It can be possible to come to different conclusions when using these model selection tools. It is interesting to note that the conclusions from the posterior Bayes factors most closely relate to those made by the DIC. This may be explained by the double use of the data in both of these measures (see, Aitkin, 1991; Spiegelhalter et al., 2002).

Chapter 6

Conclusion

In this chapter, we present a summary of the conclusions for the chapters of the thesis. This chapter is concluded with some remarks on possible future research.

6.1 Summary of Conclusions

This thesis presented a novel approach to Bayesian accelerated life testing, where two new dual-stress models were developed. The models used the generalised Eyring model as TTF, which allowed for one thermal stressor and one non-thermal stressor. The Weibull and Birnbaum-Saunders distributions were used as the life distributions, since they are more versatile and more applicable in practice than the exponential distribution.

In Chapter 2, a background for Bayesian ALT was provided. This included discussions on the objectives of reliability and life testing, distributions commonly used in life testing, censoring, stress loading and acceleration models. Due to the mathematically intractable posteriors of the GEW and GEBS models, a focal point of the thesis was on the implementation of the models in Bayesian data analysis software, such as OpenBUGS, and using MCMC algorithms to generate posterior samples. Some widely used MCMC methods were presented, which included the Metropolis-Hastings algorithm, Gibbs sampling, ARS, ARMS, and slice sampling. Some important aspects of implementing the models in OpenBUGS were also discussed, such as model selection via the DIC and methods to monitor the convergence of the Markov chains. The convergence diagnostics considered the initiation of multiple Markov chains at different starting values. Trace plots were utilised to visually check the convergence and mixing rate of the chains, where the BGR approach was employed to more formally check the convergence of the chains. A review was done on the prominent papers on Bayesian ALT models and designs, from which we identified the need for versatile dual-stress models in the Bayesian ALT paradigm.

Chapter 3 introduced the GEW model. A general likelihood was formulated for the GEW model, which can accommodate uncensored, type-I censored and type-II censored data. The generalised

Eyring model with one thermal stressor and one non-thermal stressor was used as the TTF. Variations on the GEW model were presented using different prior distributions. The posterior distributions and full conditional posterior distributions for these variations were specified. We assessed the log-concavity of the full conditional posterior distributions and determined that ARS can be used to generate posterior samples for the GEW_1 and GEW_2 models, subject to the conditions $c_{10}, c_{12}, c_{14}, c_{16} \geq 1$ and that at least one failure occurred. For the GEW_3 model, more advanced MCMC methods such as slice sampling had to be used to generate posterior samples. The GEW models were implemented in OpenBUGS and applied to a data set relating to failure times from an accelerated electronics epoxy packaging test, where temperature and relative humidity were the accelerated stressors. Different hyperparameters were chosen to investigate the sensitivity of the GEW models. The convergence of the Markov chains for the models was also discussed. The fit of the models was compared via the DIC, where we concluded that the models that made use of flat priors, that was GEW_1 and $GEW_{2,1}$, had the best fit to the data. The models where subjective priors with small variances were used showed the worst fit to the data. Summary statistics, the marginal posterior distributions and the predictive reliability for the models were also presented. We saw that the models where flat priors were used, produced similar results. We noted that the GEW model can be sensitive to the choice of subjective priors, particularly when these priors have small variances. Small variance priors led to the posterior being dominated by the prior, which resulted in very different results. We recommend using flat priors for the GEW model, but an expert reliability engineer may motivate the use of subjective priors if he/she feels that the predictive reliability is being overestimated or underestimated.

Chapter 4 introduced the GEBS model. The Birnbaum-Saunders distribution was used and the generalised Eyring model was chosen as the TTF. The Birnbaum-Saunders distribution is motivated in life testing, seeing that it was derived by taking the characteristics of the fatigue process into account. Again, a general likelihood formulation was provided, which can incorporate censored data. Gamma priors were imposed on the model parameters. The resulting posterior distribution was mathematically intractable, where MCMC techniques that can handle complex non-log-concave densities had to be utilised to generate posterior samples. The GEBS model was applied to the same data set as in Chapter 3, where different choices of hyperparameters were used. The sensitivity of the model was explored by using flat priors and subjective priors with decreasing variance. The models with flat priors, that was $GEBS_1$, $GEBS_2$ and $GEBS_3$, showed a similar fit to the data and also produced similar summary statistics, marginal posterior distributions and predictive reliability results. The models with subjective priors showed an increasingly worse fit as more certainty was conveyed by smaller variance priors. Considering the DIC values, we concluded that the GEBS model was even more sensitive to the choice of prior than the GEW model. We saw this in the predictive reliability as well, where the results were heavily affected by the choice of subjective priors with small variance. The fit of the models was also assessed by comparing the predictive reliability and the empirical reliability. Since no data was available under normal use conditions, the empirical reliability under each model was calculated by adjusting the failure times with the generalised Eyring acceleration factor. The conclusions on model fit corresponded with those made via the DIC. We recommend using flat priors for the GEBS model, since the model can be extremely sensitive to certain choices of subjective priors.

Chapter 5 further explored model selection criteria in the Bayesian ALT paradigm, specifically for the new models in this thesis. While model selection for complicated models is usually carried out with the DIC, we discussed the importance of considering Bayes factors. It is, however, important to stress that the conclusions regarding model fit can be different between these measures. This results from the measures using different approaches to assess model fit. Since the GEW and GEBS models have mathematically intractable posterior distributions, MCMC methods were employed to generate posterior samples for inference. The DIC was easily calculated from the MCMC samples and is provided as a standard output in many Bayesian data analysis software. The calculation of Bayes factors was complicated, since the marginal likelihood for the models could not be solved analytically. As a result, we focused on methods that could estimate the marginal likelihood without further complicating the sampling process, and without the requirement of choosing a suitable importance or bridge function. The methods discussed included a simple Monte Carlo estimator, the Laplace-Metropolis estimator, the harmonic mean estimator, and using the posterior predictive density for the calculation of posterior Bayes factors. The GEW_2 and GEBS models were applied to another data set and implemented in OpenBUGS. Various choices of hyperparameters were selected to illustrate the model selection criteria. The DIC clearly favoured the GEBS models above the GEW models, where the $GEBS_{BF1}$ model, had the lowest DIC. Among the GEW models, the model where flat priors were used, had the lowest DIC. The Bayes factors were calculated to compare the six models in this application. The simple Monte Carlo estimator was omitted due to computational issues created by the large discrepancies between the priors and posteriors for some models. The Laplace-Metropolis estimator gave variable and somewhat unstable results, which could be explained by the skewed marginal posteriors for some models. The GEW_{BF2} model was supported by this estimator, followed by the $GEBS_{BF2}$ model. The harmonic mean estimator favoured the GEW models above the GEBS models, where the most support was given to the GEW_{BF1} model. The estimator seemed to be stable for these models and, in line with the DIC, prefered the models where flat priors were used. Using posterior Bayes factors gave fit results more similar to those of the DIC. This could be explained by the double use of the data in both of these measures. The $GEBS_{BF1}$ model enjoyed the most support from the posterior Bayes factors, and in general, the models using flat priors were favoured in this case. Considering the DIC and Bayes factors, we recommend again the use of flat priors for the GEW and GEBS models. The use of flat priors resulted in a better fit to the data for most of the measures considered.

6.2 Future Research

Since there is very little Bayesian literature on multiple stress ALT models, there are many opportunities for future research in this area. The generalised Eyring model can be extended to include more than two stressors. The main limitation regarding this extention is the availability of data for applications. Data for two stressors are already hard to obtain, and the assistance of reliability engineers would be needed to generate new data sets with more stressors. Another limitation to the thesis is simulation studies for the new models. Since the models are very complex, simulation studies require immense computational power and additional software. Simulation studies will be considered in the articles that follow from the work in this thesis.

The generalised Eyring model can also be investigated for even more complex life distributions, such as generalisations of the Weibull, Birnbaum-Saunders, Pareto and gamma distributions. A limitation on extending the generalised Eyring model to these distribution is the implementation in Bayesian data analysis software. These distributions are not readily available in all of the software packages, and implementing new distributions is also difficult for some of the software packages. Other Bayesian data analysis software packages such as Stan and JAGS will also be investigated for future research.

Model selection can also be further investigated. There are new and innovative estimators available for approximating Bayes factors, but these methods usually further complicate the MCMC sampling process. The posterior model probabilities can also be computed by simultaneously comparing multiple models to some base model.

Another gap in the current literature is on the design of dual-stress or even multiple-stress ALT in the Bayesian paradigm. Park and Yum (1996) discuss the optimal design of ALT models with two stressors in the frequentist paradigm, and made use of the generalised Eyring model. The Bayesian research areas on ALT designs have only dealt with single stressor models.

References

Achcar, J. A. (1993). Approximate Bayesian methods: Some applications in life testing problems. *Brazilian Journal of Probability and Statistics*, 7:135–158.

Achcar, J. A. and Louzada-Neto, F. (1991). Accelerated life tests with one stress variable: A Bayesian analysis of the Eyring model. *Brazilian Journal of Probability and Statistics*, 5:169–179.

Ahmad, N. (1990). *Bayesian and accelerated reliability analysis*. PhD thesis, Aligarh Muslim University.

Aitkin, M. (1991). Posterior Bayes factors. *Journal of the Royal Statistical Society. Series B (Methodological)*, 53(1):111–142.

Akaike, H. (1973). Information theory and an extension of the maximum likelihood principle. In Petrov, B. N. and Csáki, F., editors, *Proceedings of the* 2^{nd} *International Symposium on Information Theory*, pages 267–281, Budapest.

Almalki, S. J. and Nadarajah, S. (2014). Modifications of the Weibull distribution: A review. *Reliability Engineering and System Safety*, 124:32–55.

Bagnoli, M. and Bergstrom, T. (2005). Log-concave probability and its applications. *Economic Theory*, 26:445–469.

Balakrishnan, N. and Kundu, D. (2013). Hybrid censoring: Models, inferencial results and applications. *Computational Statistics and Data Analysis*, 57:166–209.

Banerjee, A. and Kundu, D. (2008). Inference based on type-II hybrid censored data from a Weibull distribution. *IEEE Transactions on Reliability*, 57(2):369–378.

Barriga, G. D. C., Ho, L. L., and Cancho, V. G. (2008). Planning accelerated life tests under exponentiated-Weibull-Arrhenius model. *International Journal of Quality & Reliability Management*, 25(6):636–653.

Berger, J. O., Bernardo, J. M., and Sun, D. (2015). Overall objective priors. *Bayesian Analysis*, 10(1):189–221.

Berger, J. O. and Pericchi, L. R. (1996). The intrinsic Bayes factor for model selection and prediction. *Journal of the American Statistical Association*, 91(433):109–122.

Bernardo, J. M. and Smith, A. F. M. (1994). *Bayesian Theory*. Wiley, New York.

Birnbaum, Z. W. and Saunders, S. C. (1969). A new family of life distributions. *Journal of Applied Probability*, 6(2):319–327.

Brooks, S. P. and Gelman, A. (1998). General methods for monitoring convergence of iterative simulations. *Journal of Computational and Graphical Statistics*, 7(4):434–455.

Burnham, K. P. and Anderson, D. R. (1998). *Model Selection and Inference*. Springer, New York.

Carlin, B. P. and Chib, S. (1995). Bayesian model choice via Markov chain Monte Carlo methods. *Journal of the Royal Statistical Society. Series B (Methodological)*, 57(3):473–484.

Chaloner, K. and Larntz, K. (1992). Bayesian design for accelerated life testing. *Journal of Statistical Planning and Inference*, 33:245–259.

Chib, S. (1995). Marginal likelihood from the Gibbs output. *Journal of the American Statistical Association*, 90(432):1313–1321.

Childs, A., Chandrasekar, B., and Balakrishnan, N. (2008). Exact likelihood inference for an exponential parameter under progressive hybrid censoring schemes. In *Statistical Models and Methods for Biomedical and Technical Systems*, pages 319–330. Birkhäuser, Boston.

Cohen, A. C. (1963). Progressively censored samples in life testing. *Technometrics*, 5(3):327–339.

Cordeiro, G. M., Nadarajah, S., and Ortega, E. M. M. (2012). The Kumaraswamy Gumbel distribution. *Statistical Methods & Applications*, 21(2):139–168.

DeGroot, M. H. (1970). *Optimal Statistical Decisions*. McGraw-Hill, New York.

Desmond, A. (1985). Stochastic models of failure in random environments. *The Canadian Journal of Statistics*, 13(2):171–183.

Dietrich, D. L. and Mazzuchi, T. A. (1996). An alternative method of analyzing multi-stress multi-level life and accelerated life tests. *Proceedings of the 1996 Annual Reliability and Maintainability Symposium*, pages 90–96.

Efron, B. (1967). The two sample problem with censored data. *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability*, 4:831–853.

Erkanli, A. and Soyer, R. (2000). Simulation-based designs for accelerated life tests. *Journal of Statistical Planning and Inference*, 90:335–348.

Escobar, L. A. and Meeker, W. Q. (2006). A review of accelerated test models. *Statistical Science*, 21(4):552–577.

Geisser, S. and Eddy, W. F. (1979). A predictive approach to model selection. *Journal of the American Statistical Association*, 74(365):153–160.

Gelfand, A. E. and Dey, D. K. (1994). Bayesian model choice: Asymptotics and exact calculations. *Journal of the Royal Statistical Society. Series B*, 56:501–514.

Gelman, A., Carlin, J. B., Stern, H. S., Dunson, D. B., Vehtari, A., and Rubin, D. B. (2014). *Bayesian Data Analysis*. Chapman & Hall.

Gelman, A. and Rubin, D. B. (1992). Inference from iterative simulation using multiple sequences. Technical report, University of California.

Geman, S. and Geman, D. (1984). Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 6(6):721–741.

Ghosh, M. (2011). Objective priors: An introduction for frequentists. *Statistical Science*, 26(2):187–202.

Gilks, W. R., Best, N. G., and Tan, K. K. C. (1995). Adaptive rejection Metropolis sampling within Gibbs sampling. *Applied Statistics*, 44(4):455–472.

Gilks, W. R. and Wild, P. (1992). Adaptive rejection sampling for Gibbs sampling. *Applied Statistics*, 41(2):337–348.

Gupta, S. S. and Groll, P. A. (1961). Gamma distribution in acceptance sampling based on life tests. *Journal of the American Statistical Association*, 56(296):942–970.

Hakim-Mashhadi, M. (1992). On the application of design of experiments to accelerated life testing. Master's thesis, Linköping University.

Hamada, M. S., Wilson, A. G., Reese, C. S., and Martz, H. F. (2008). *Bayesian Reliability*. Springer, New York.

Harris, C. M. and Singpurwalla, N. D. (1968). Life distributions derived from stochastic hazard functions. *IEEE Transactions on Reliability*, 17(2):70–79.

Hastings, W. K. (1970). Monte Carlo sampling methods using Markov chains and their applications. *Biometrika*, 57(1):97–109.

Irony, T. Z. and Singpurwalla, N. D. (1997). Noninformative priors do not exist: A discussion with José M. Bernardo. *Journal of Statistical Inference and Planning*, 65(1):159–189.

Jeffreys, H. (1961). *Theory of Probability*. Oxford University Press, Oxford.

John, T. T. and Chen, P. (2006). Lognormal selection with applications to lifetime data. *IEEE Transactions on Reliability*, 55(1):135–148.

Jordan, A. S. (1978). A comprehensive review of the lognormal failure distribution with application to LED reliability. *Microelectronic Reliability*, 18:267–279.

Kadane, J. B. and Lazar, N. A. (2004). Methods and criteria for model selection. *Journal of the American Statistical Association*, 99(465):279–290.

Kalbfleisch, J. D. and Prentice, R. L. (2002). *The Statistical Analysis of Failure Time Data*. Wiley, Hoboken.

Kantam, R. R. L., Rosaiah, K., and Srinibasa Rao, G. (2001). Acceptance sampling based on life tests: Log-logistic model. *Journal of Applied Statistics*, 28(1):121–128.

Kass, R. E. and Raftery, A. E. (1995). Bayes factors. *Journal of the American Statistical Association*, 90(430):773–795.

Kececioglu, D. B. (2002). *Reliability and Life Testing Handbook*. DEStech Publications, Lancaster.

Klein, J. P. and Moeschberger, M. L. (2003). *Survival Analysis: Techniques for Censored and Truncated Data*. Statistics for Biology and Health. Springer, New York.

Kundu, D. (2008). Bayesian inference and life testing plan for the Weibull distribution in presence of progressive censoring. *Technometrics*, 50(20):144–154.

Lai, C., Murthy, D. N. P., and Xie, M. (2006). Weibull distributions and their applications. In *Springer Handbook of Engineering Statistics*, pages 63–78. Springer, London.

Lemonte, A. J. (2016). A note on the Fisher information matrix of the Birnbaum-Saunders distribution. *Journal of Statistical Theory and Applications*, 15(2):196–205.

León, R. V., Ramachandran, R., Ashby, A. J., and Thyagarajan, J. (2007). Bayesian modeling of accelerated life tests with random effects. *Journal of Quality Technology*, 39(1):3–16.

Lesaffre, E. and Lawson, A. B. (2012). *Bayesian Biostatistics*. Wiley, Chichester.

Lewis, S. M. and Raftery, A. E. (1997). Estimating Bayes factors via posterior simulation with the Laplace-Metropolis estimator. *Journal of the American Statistical Association*, 92(438):648–655.

Lindley, D. V. (1976). A class of utility functions. *The Annals of Statistics*, 4(1):1–10.

Luko, S. N. (1999). A review of the Weibull distribution and selected engineering applications. *SAE Transaction*, 108:398–412.

Mann, N. R., Schafer, R. E., and Singpurwalla, N. D. (1974). *Methods for Statistical Analysis of Reliability and Life Data*. Wiley, New York.

Mann, N. R. and Singpurwalla, N. D. (1980). Life Testing. Technical report, Rockwell International Science Center.

Martino, L., Read, J., and Luengo, D. (2015). Independent doubly adaptive rejection Metropolis sampling within Gibbs sampling. *IEEE Transactions on Signal Processing*, 63(12):3123–3138.

Mazzuchi, T. A. and Soyer, R. (1992). A dynamic general linear model for inference from accelerated life tests. *Naval Research Logistics*, 39:757–773.

Mazzuchi, T. A., Soyer, R., and Vopatek, A. L. (1997). Linear Bayesian inference for accelerated Weibull model. *Lifetime Data Analysis*, 3:63–75.

Meeker, W. Q. and Escobar, L. A. (1998). *Statistical Methods for Reliability Data*. Wiley, New York.

Metropolis, N., Rosenbluth, A. W., Rosenbluth, M. N., Teller, A. H., and Teller, E. (1953). Equation of state calculations by fast computing machines. *The Journal of Chemical Physics*, 21(6):1087–1092.

Mukhopadhyay, C. and Roy, S. (2016). Bayesian accelerated life testing under competing log-location-scale family of causes of failure. *Computational Statistics*, 31:89–119.

Mullen, R. E. (1998). The lognormal distribution of software failure rates: Application to software reliability growth modeling. In *Proceedings of the Ninth International Symposium on Software Reliability Engineering*, pages 134–142. IEEE.

Neal, R. M. (2003). Slice sampling. *The Annals of Statistics*, 31(3):705–767.

Nelson, W. B. (1990). *Accelerated Testing: Statistical Models, Test Plans, and Data Analysis*. Wiley, New York.

Nelson, W. B. and Kielpinski, T. J. (1976). Theory for optimum censored accelerated life tests for normal and lognormal life distributions. *Technometrics*, 18(1):105–144.

Newton, M. A. and Raftery, A. E. (1994). Approximate Bayesian inference with the weighted likelihood bootstrap. *Journal of the Royal Statistical Society. Series B (Methodological)*, 56(1):3–48.

Ntzoufras (2009). *Bayesian Modeling Using WinBUGS*. Wiley, Hoboken.

O'Hagan, A. (1995). Fractional Bayes factors for model comparison. *Journal of the Royal Statistical Society. Series B*, 57(1):99–138.

Owen, W. J. and Padgett, W. J. (1999). Accelerated test models for system strength based on Birnbaum-Saunders distributions. *Lifetime Data Analysis*, 5:133–147.

Owen, W. J. and Padgett, W. J. (2000). A Birnbaum-Saunders accelerated life model. *IEEE Transactions on Reliability*, 49(2):224–229.

Pajor, A. (2017). Estimating the marginal likelihood using the arithmetic mean identity. *Bayesian Analysis*, 12(1):261–287.

Pan, R. (2009). A Bayes approach to reliability prediction utilizing data from accelerated life tests and field failure observations. *Quality and Reliability Engineering International*, 25:229–240.

Papadopoulos, A. S. and Tsokos, C. P. (1976). Bayesian analysis of the Weibull failure model with unknown scale and shape parameters. *Statistica*, 36(4):547–560.

Park, J. W. and Yum, B. J. (1996). Optimal design of accelerated life tests with two stresses. *Naval Research Logistics*, 43:863–884.

Park, T. and Casella, G. (2008). The Bayesian Lasso. *Journal of the American Statistical Association*, 103(482):681–686.

Perdona, G. C. and Louzada-Neto, F. (2005). An approximate Bayesian analysis for accelerated tests with log-non-linear stress-response relationship. *Journal of Mathematics and Statistics*, 23(1):59–65.

Polson, N. G. and Soyer, R. (2017). Augmented probability simulation for accelerated life test design. *Applied Stochastic Models in Business and Industry*, 33:322–332.

Raftery, A. E. (1996). Hypothesis testing and model selection. In Gilks, W. R., Richardson, S., and Spiegelhalter, D. J., editors, *Markov Chain Monte Carlo in Practice*, pages 163–188. Chapman & Hall, Suffolk.

Raftery, A. E., Newton, M. A., Satagopan, J. M., and Krivitsky, P. N. (2007). Estimating the integrated likelihood via posterior simulation using the harmonic mean identity. *Bayesian Statistics*, 8:1–45.

ReliaSoft (2015). *Accelerated life testing reference (Accessed March 26, 2019)*. http://www.synthesisplatform.net/references/Accelerated_Life_Testing_Reference.pdf.

ReliaSoft (2020). *Accelerated testing data analysis without a known physical failure model (Accessed October 15, 2020)*. https://www.reliasoft.com/resources/resource-center/accelerated-testing-data-analysis- without-a-known-physical-failure-model.

Robert, C. P. and Casella, G. (1999). *Monte Carlo Statistical Methods*. Springer, New York.

Sha, N. (2018). Statistical inference for progressive stress accelerated life testing with Birnbaum-Saunders distribution. *Stats*, 1:189–203.

Singpurwalla, N. D. (1971a). Inference from accelerated life tests when observations are obtained from censored samples. *Technometrics*, 13(1):161–170.

Singpurwalla, N. D. (1971b). A problem in accelerated life testing. *Journal of the American Statistical Association*, 66(336):841–845.

Singpurwalla, N. D. (1973). Inference from accelerated life tests using Arrhenius type re-parameterizations. *Technometrics*, 15(2):289–299.

Singpurwalla, N. D. (2006). *Reliability and Risk: A Bayesian Perspective*. Wiley, Hoboken.

Singpurwalla, N. D., Castellino, V. C., and Goldschen, D. Y. (1975). Inference from accelerated life tests using Eyring type re-parameterizations. *Naval Research Logistics Quarterly*, 22(2):289–296.

Soland, R. M. (1969). Bayesian analysis of the Weibull process with unknown scale and shape parameters. *IEEE Transactions on Reliability*, 18(4):181–184.

Soyer, R. (2008). Accelerated life tests: Bayesian design. *Encyclopedia of Statistics in Quality and Reliability*, 1:12–20.

Soyer, R., Erkanli, A., and Merrick, J. R. (2008). Accelerated life tests: Bayesian models. *Encyclopedia of Statistics in Quality and Reliability*, 1:20–30.

Spiegelhalter, D. J., Best, N. G., Carlin, B. P., and Van der Linde, A. (2002). Bayesian measures of model complexity and fit. *Journal of the Royal Statistical Society, Series B*, 64(4):583–639.

Spiegelhalter, D. J., Thomas, A., Best, N. G., and Lunn, D. (2014). *OpenBUGS User Manual*.

Sun, T. and Shi, Y. (2016). Estimation for Birnbaum-Saunders distribution in simple step stress-accelerated life test with type-II censoring. *Communications in Statistics - Simulation and Computation*, 45:880–901.

Thiraviam, A. R. (2010). *Accelerated life testing of subsea equipment under hydrostatic pressure*. PhD thesis, University of Central Florida.

Tsokos, C. P. (1972). A Bayesian approach to reliability: Theory and simulation. *Proceedings of the 1972 Annual Reliability and Maintainability Symposium*, pages 78–87.

Upadhyay, S. K. and Mukherjee, B. (2010). Bayes analysis and comparison of accelerated Weibull and accelerated Birnbaum-Saunders models. *Communications in Statistics - Theory and Methods*, 39:195–213.

Upadhyay, S. K., Mukherjee, B., and Gupta, A. (2009). Accelerated test system strength models based on Birnbaum-Saunders distribution: A complete Bayesian analysis and comparison. *Lifetime Data Analysis*, 15:379–396.

Upadhyay, S. K. and Sharma, R. (2018). A Bayes analysis and comparison of Weibull and lognormal based accelerated test models with actual lifetimes unknown. *Reliability: Theory and Applications*, 13(4(51)):55–73.

Van Dorp, J. R. and Mazzuchi, T. A. (2004). A general Bayes exponential inference model for accelerated life testing. *Journal of Statistical Planning and Inference*, 119:55–74.

Van Dorp, J. R. and Mazzuchi, T. A. (2005). A general Bayes Weibull inference model for accelerated life testing. *Reliability Engineering and System Safety*, 90(90):140–147.

Van Dorp, J. R., Mazzuchi, T. A., Fornell, G. E., and Pollock, L. R. (1996). A Bayes approach to step-stress accelerated life testing. *IEEE Transactions on Reliability*, 45(3):491–498.

Van Dorp, J. R., Mazzuchi, T. A., and Garcidueñas, J. E. (2006). A comparison of accelerated life testing designs within a single Bayesian inferential framework. *RAMS'06. Annual Reliability and Maintainability Symposium*, pages 208–214.

Verdinelli, I. and Wasserman, L. (1995). Computing Bayes factors using a generalization of the Savage-Dickey density ratio. *Journal of the American Statistical Association*, 90(430):614–618.

Weibull, W. (1951). A statistical distribution function of wide applicability. *Journal of Applied Mechanics*, 18:293–297.

Weinberg, M. D. (2012). Computing the Bayes Factor from a Markov chain Monte Carlo simulation of the posterior distribution. *Bayesian Analysis*, 7(3):737–770.

Yang, R. and Berger, J. O. (1998). *A Catalog of Noninformative Priors*. Institute of Statistics and Decision Sciences, Duke University.

Yang, Z. (2000). Predictive densities for the lognormal distribution and their applications. *Microelectronics Reliability*, 40:1051–1059.

Yuan, T., Liu, X., Ramadan, S. Z., and Kuo, Y. (2014). Bayesian analysis for accelerated life tests using a Dirichlet process Weibull mixture model. *IEEE Transactions on Reliability*, 63(1):58–67.

Zellner, A. (1971). *An Introduction to Bayesian Inference in Econometrics*. John Wiley, New York.

Appendix A: MCMC Convergence Diagnostics

In this appendix, the convergence of the Markov chains for the applications in Chapter 3 and Chapter 4 is evaluated via trace plots and the BGR diagnostics tool.

In order to assess the convergence of the Markov chains in the applications, multiple chains are needed. For each parameter of the models under investigation, three chains are initiated at different starting values. Each chain is set to run for 50000 iterations, after which the BGR plots are generated in OpenBUGS. On the BGR plot, the dashed line indicates the value 1, the red line indicates the value of R, and the green and blue lines indicate the normalised values of B and W, respectively. The trace plots are also generated for the final 30000 iterations of this run, to examine the convergence and mixing rate of the chains. The generated values for the three chains are indicated seperately by red, blue and green. The BGR diagnostics plots and trace plots for all parameters in the GEW and GEBS applications are presented in Figure A.1 to Figure A.24.

The BGR statistic converges to 1 reasonably fast for most of the parameters under consideration. Convergence in this regard is established in advance of 10000 iterations for essentially all parameters. According to Spiegelhalter et al. (2014), for practical purposes convergence may even be assumed when $R < 1.05$. The convergence of the within-chain and between-chain variability is also fast for all parameters, but the stabilisation of these variabilities take somewhat longer for certain parameters. Stabilisation of the within-chain and between-chain variabilities is achieved for all parameters around 20000 iterations. From the BGR plots one can thus conclude that sufficient convergence is reached, for the Markov chains of all parameters of interest, by around 20000 iterations. As a precaution, using a burn-in for the Markov chains in the applications of 50000 iterations is more than adequate.

The trace plots for all parameters of interest pass the thick pen test to an acceptable standard. For certain parameters there are some iterations that can be considered to fall outside the thick pen test, but this is to a minimal degree. These parameters are $\theta_3$and θ_4 in the GEW models, and α, θ_3 and θ_4 in the GEBS models. A possible reason for this one-sided deviance indicates towards a skewed marginal posterior, and should not be a great cause for concern seeing that the BGR plots have indicated convergence.

The mixing rates for certain parameters are also slower than for other parameters. Compare, for example, the trace plots of θ_1 in the GEW_1 model and θ_1 in the $GEW_{2,3}$ model. The mixing rate of this parameter is much faster in the $GEW_{2,3}$ than in the GEW_1. For the application of the GEW and GEBS models, the Markov chains are set to run for 200000 iterations. By examining the behaviour of the trace plots for 30000 iterations in this appendix, one may conclude that 200000 iterations should allow the adequate exploration of the marginal posteriors for the parameters of interest. This is also confirmed via the assessment of the Monte Carlo error regarding the 200000 samples generated for inference.

A.1 Convergence for the GEW Models

A.1.1 *GEW_1*

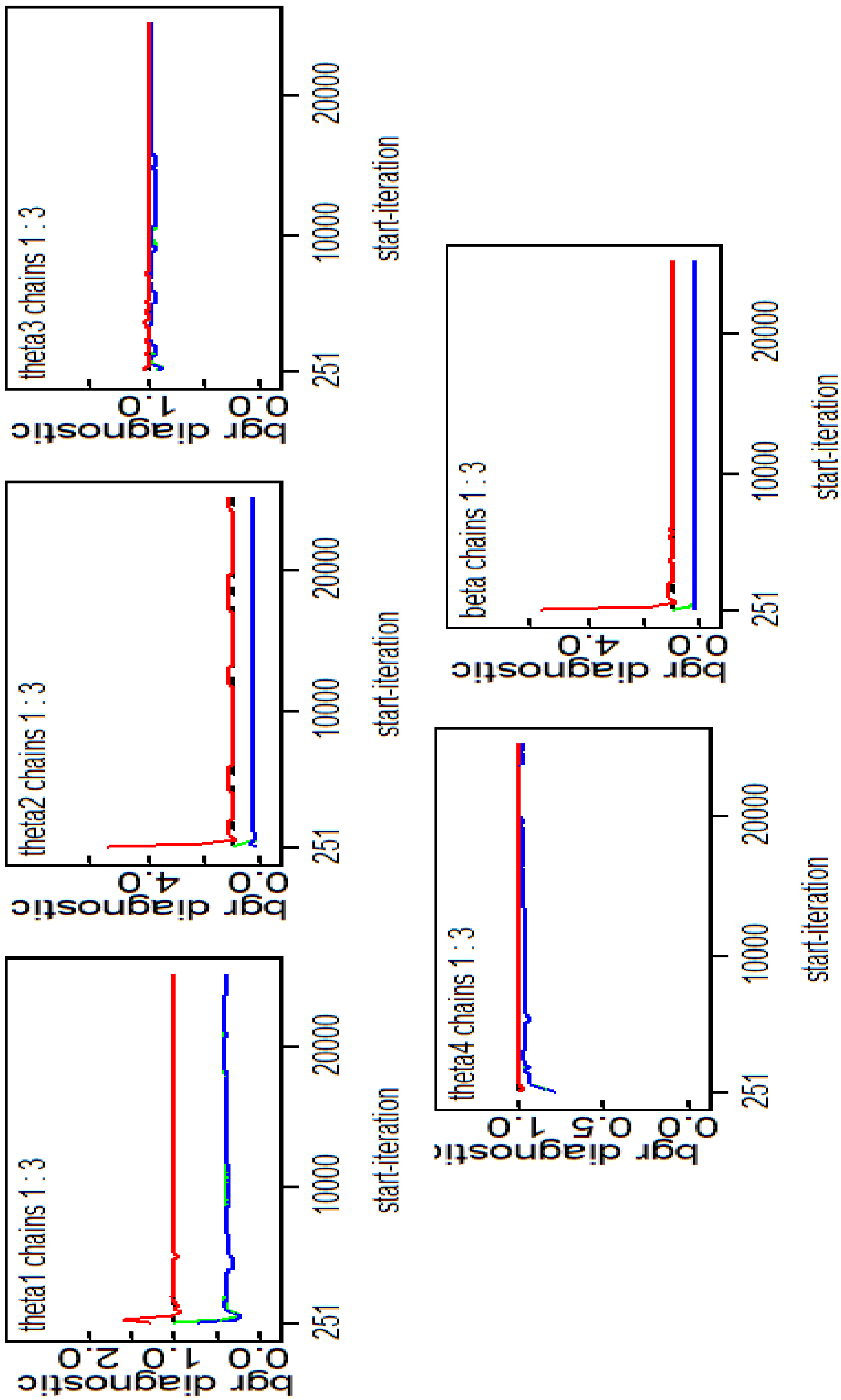

Figure A.1: BGR plots for the *GEW_1* model.

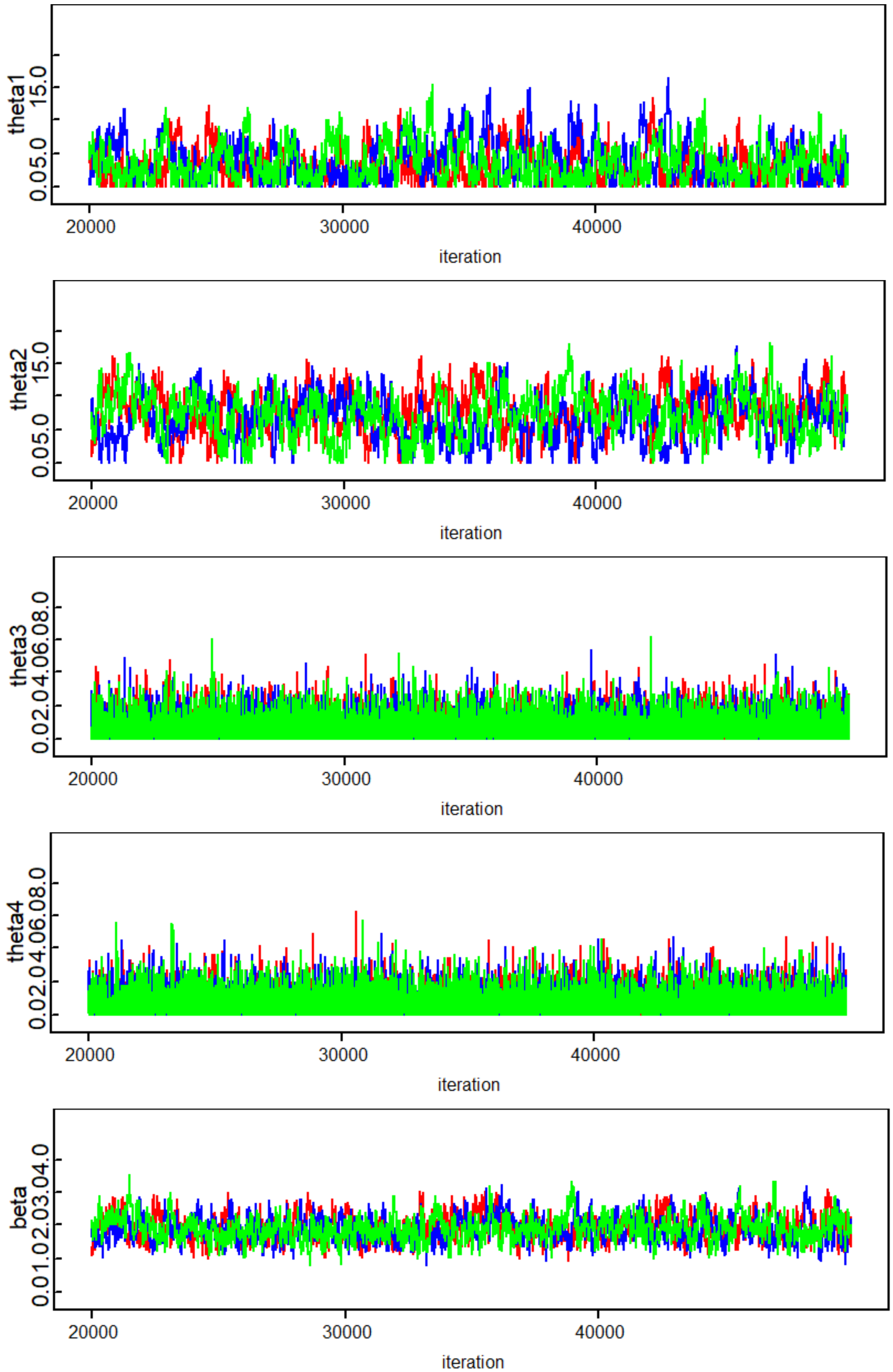

Figure A.2: Trace plots for the GEW_1 model.

A.1.2 $GEW_{2,1}$

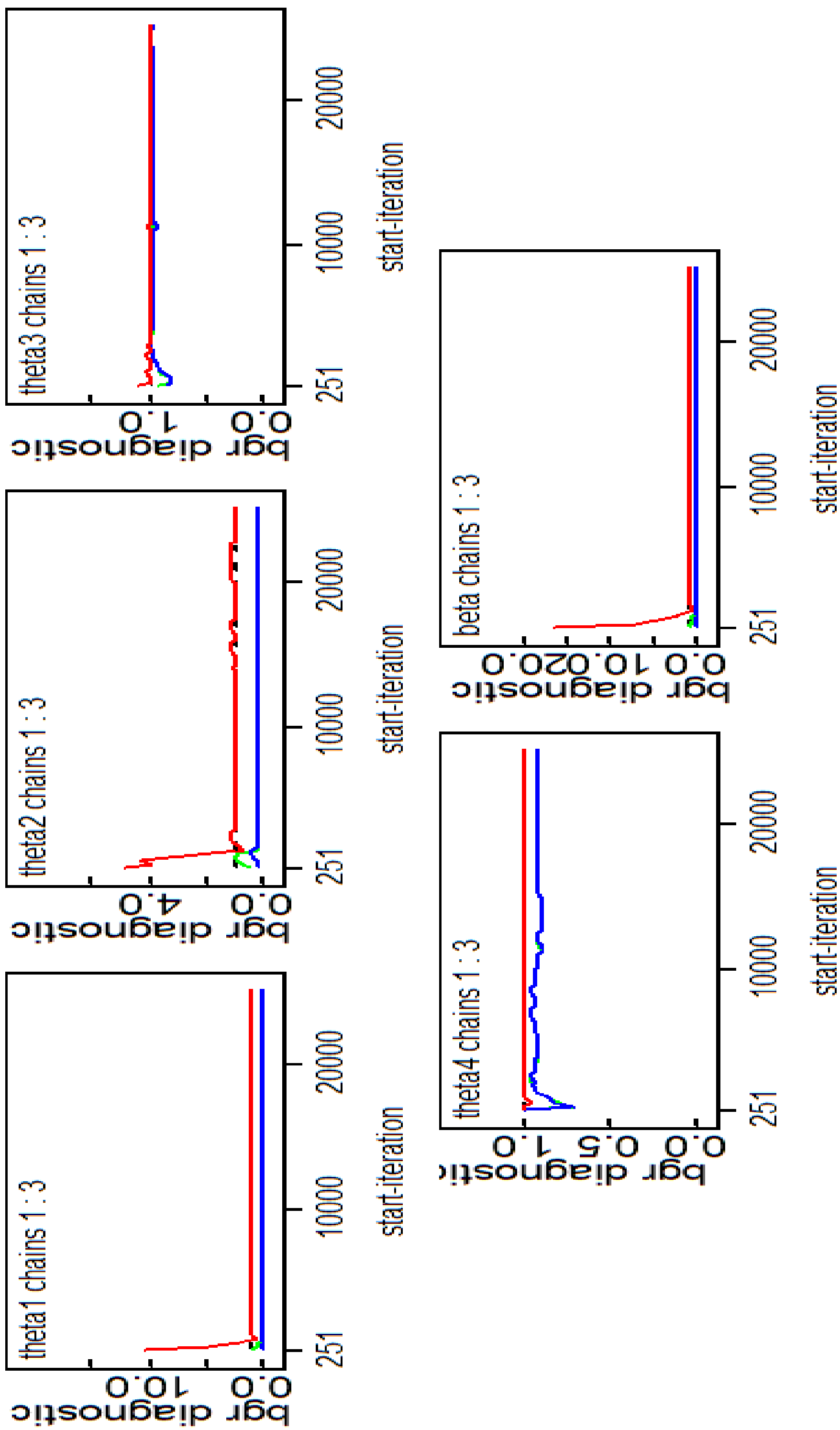

Figure A.3: BGR plots for the $GEW_{2,1}$ model.

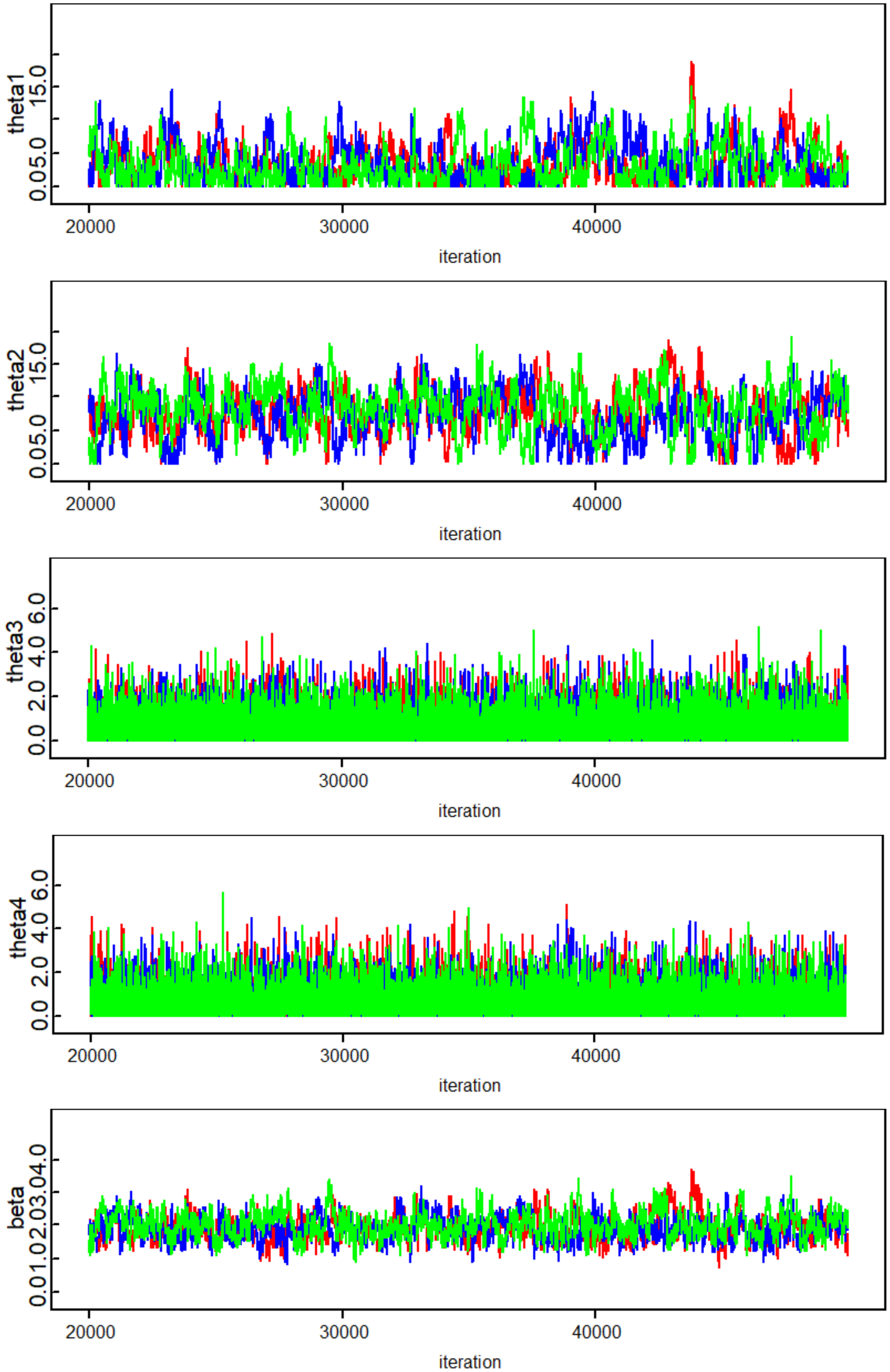

Figure A.4: Trace plots for the $GEW_{2,1}$ model.

A.1.3 $GEW_{2,2}$

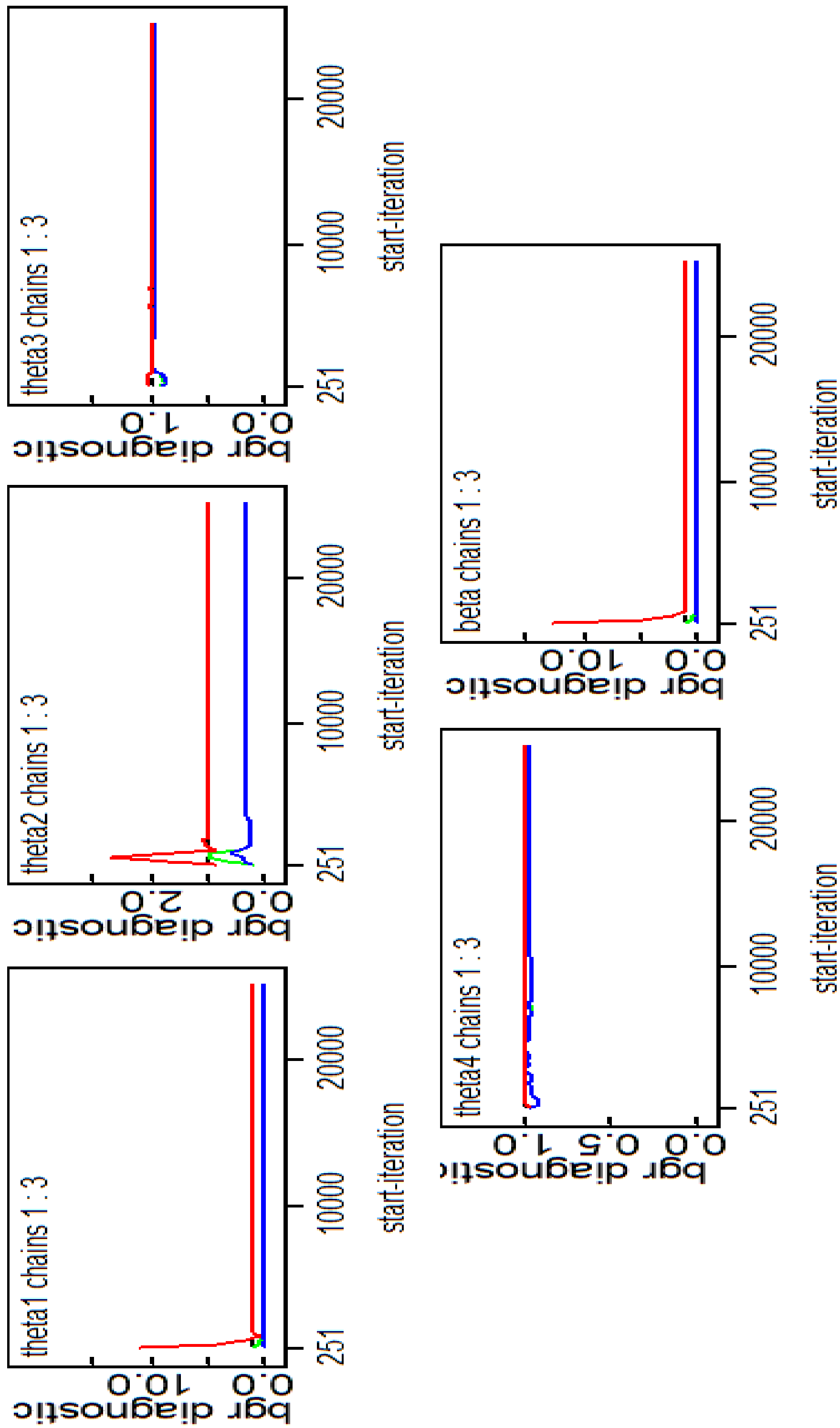

Figure A.5: BGR plots for the $GEW_{2,2}$ model.

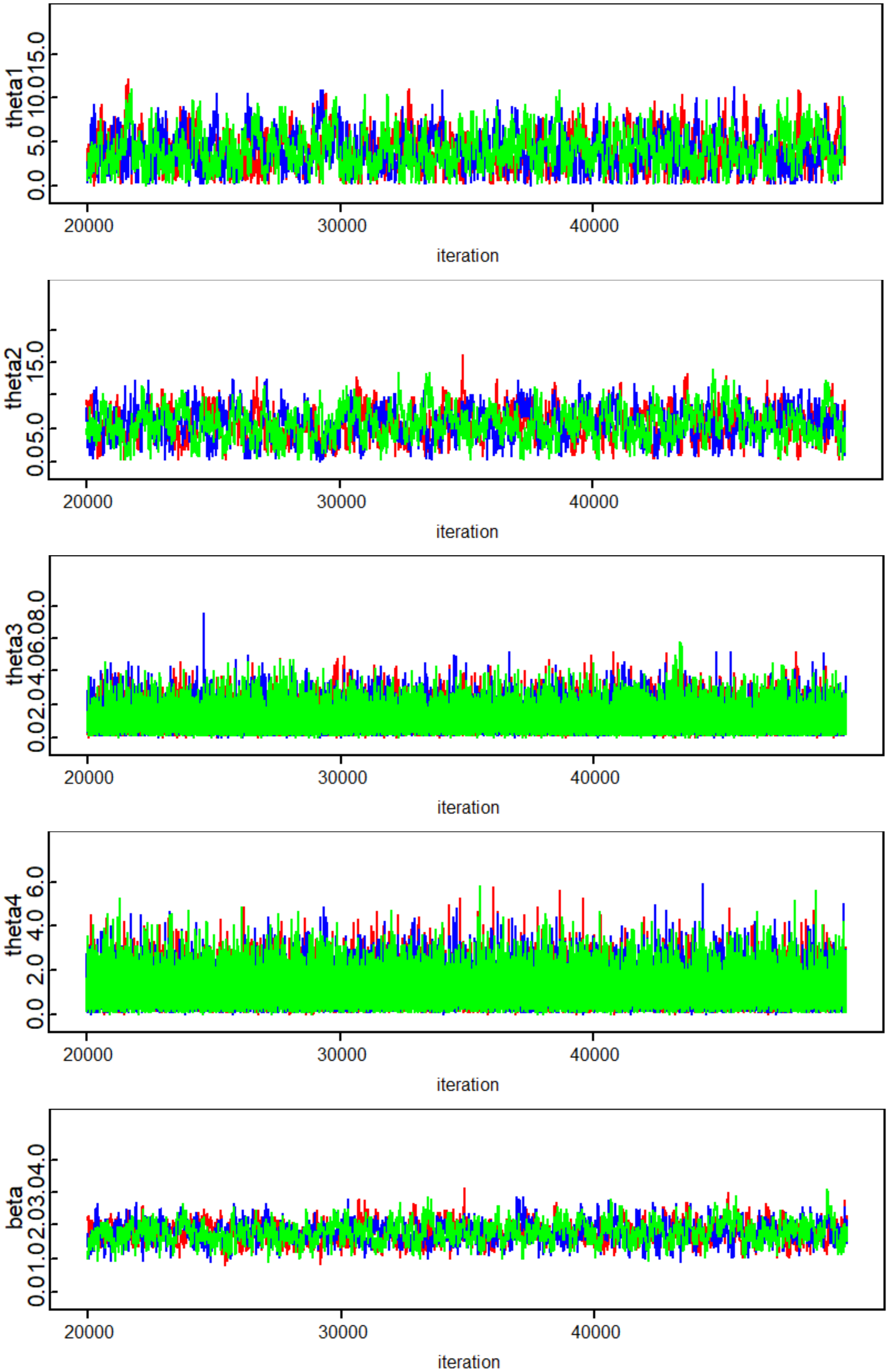

Figure A.6: Trace plots for the $GEW_{2,2}$ model.

A.1.4 $GEW_{2,3}$

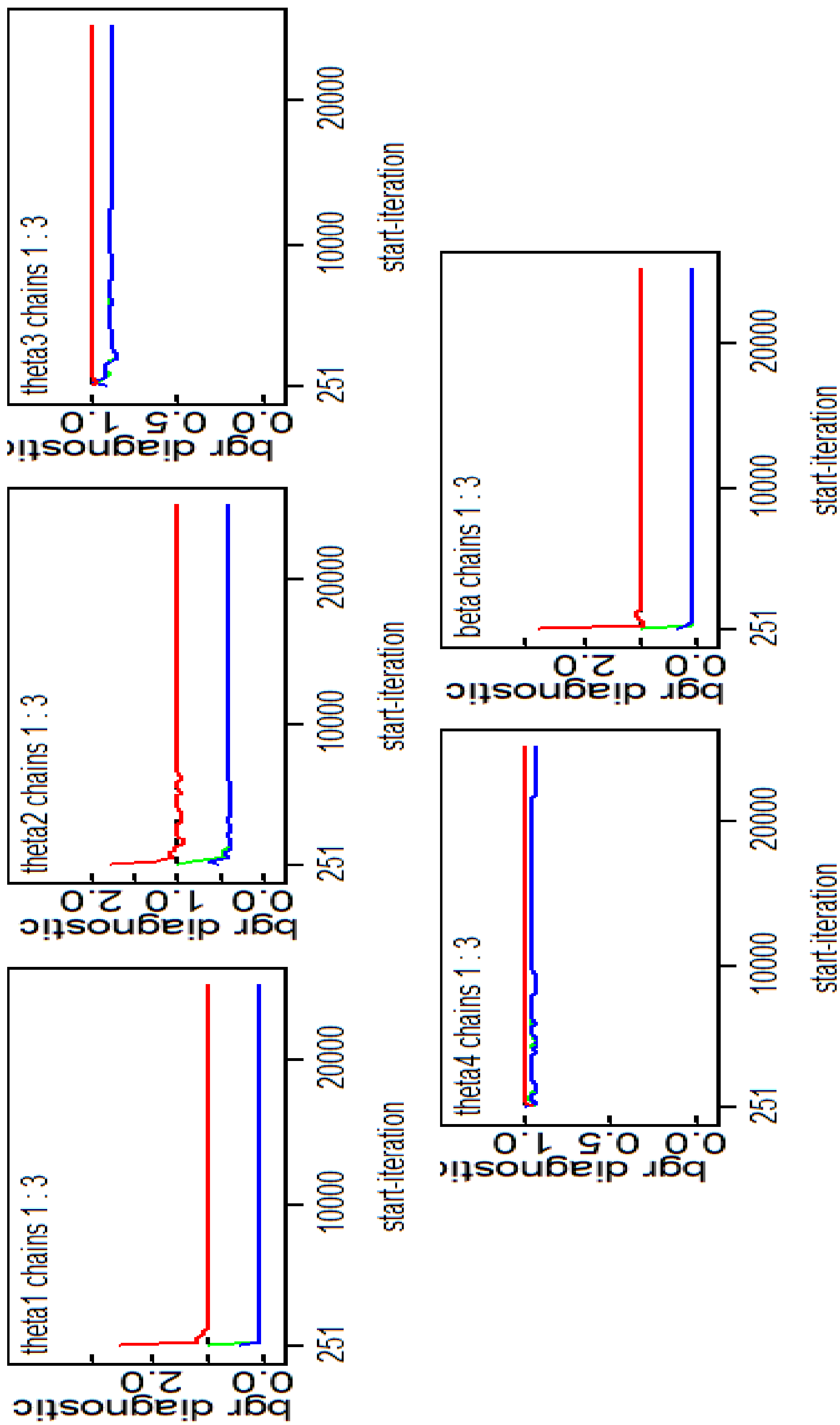

Figure A.7: BGR plots for the $GEW_{2,3}$ model.

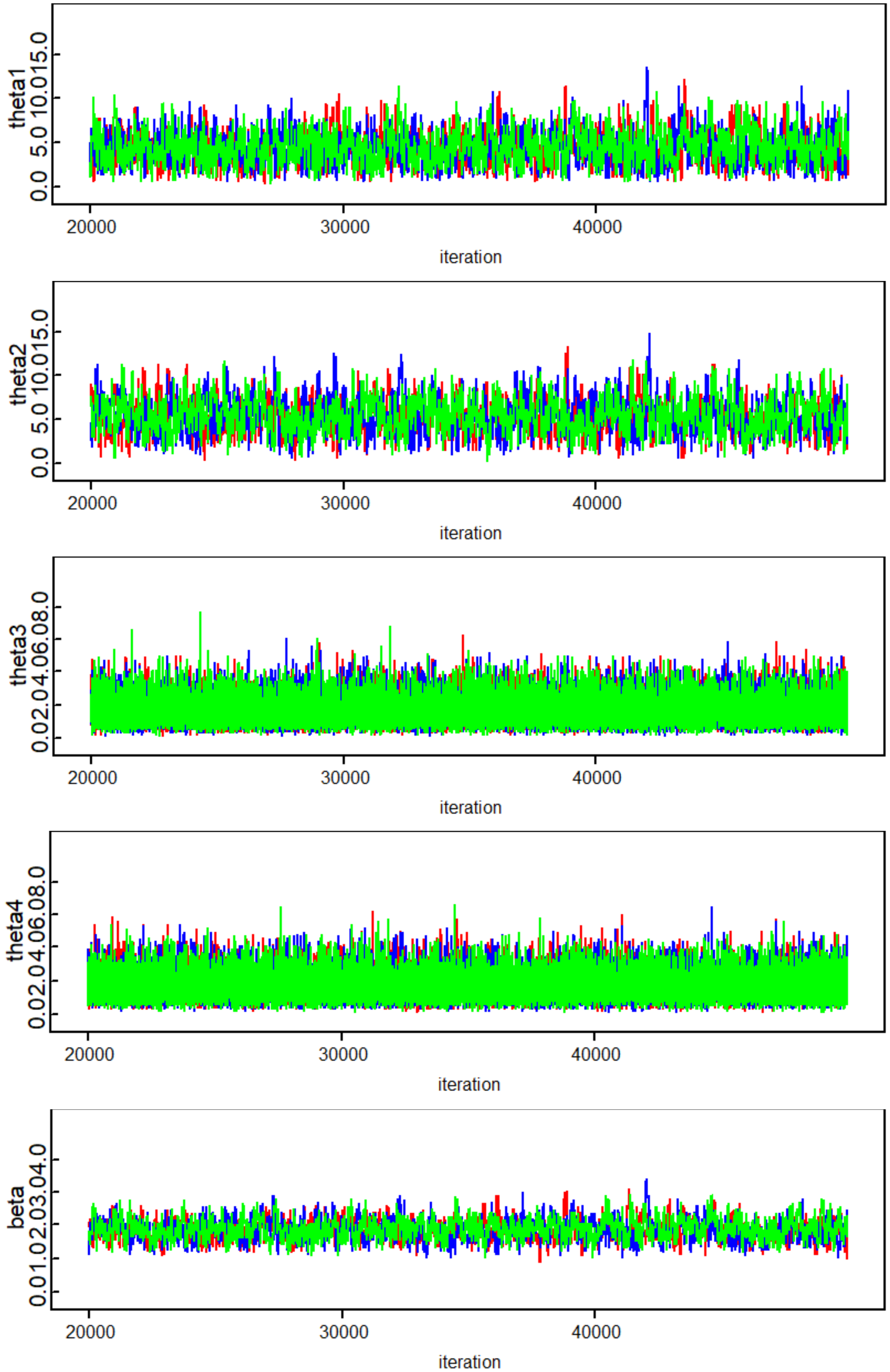

Figure A.8: Trace plots for the $GEW_{2,3}$ model.

A.1.5 $GEW_{3,1}$

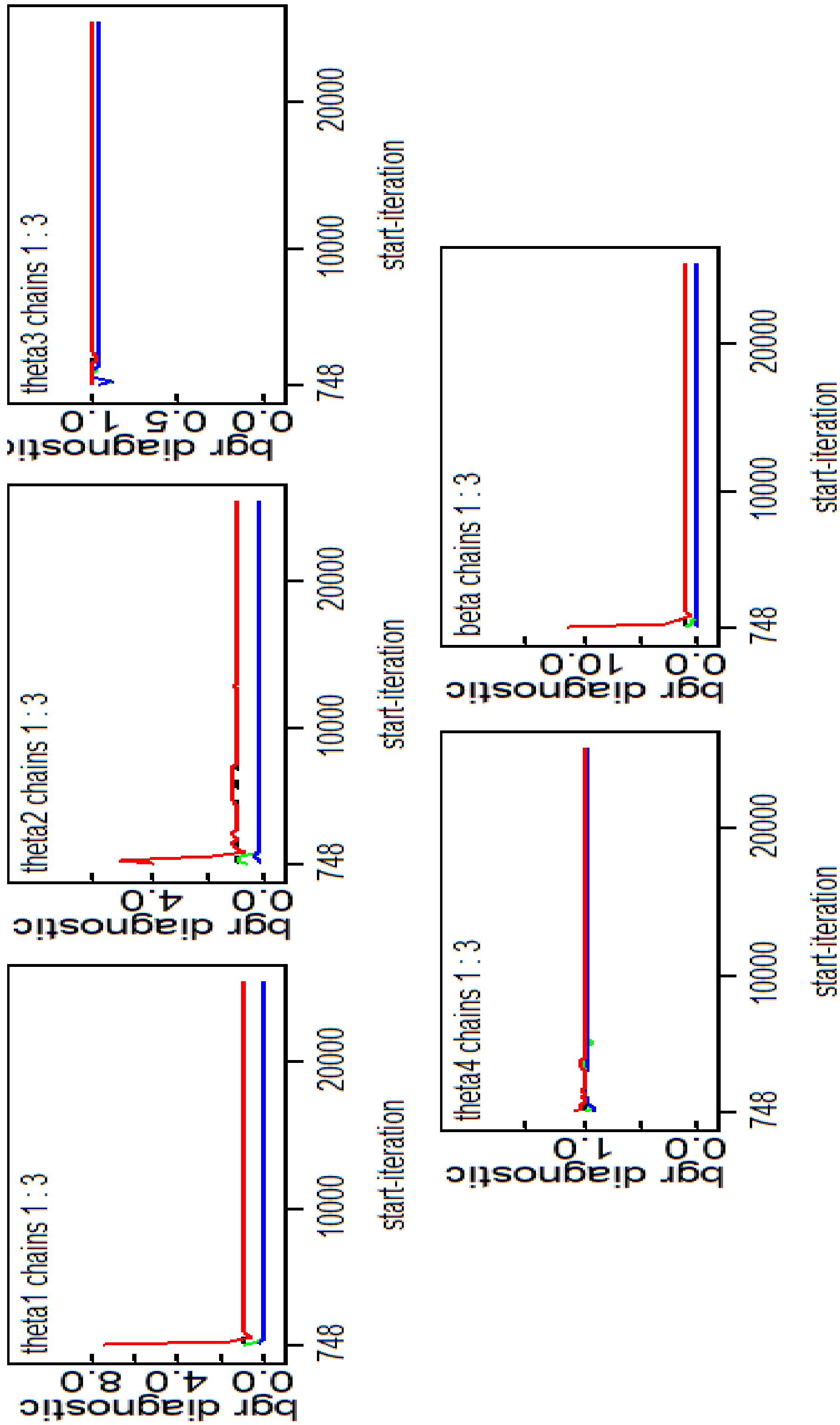

Figure A.9: BGR plots for the $GEW_{3,1}$ model.

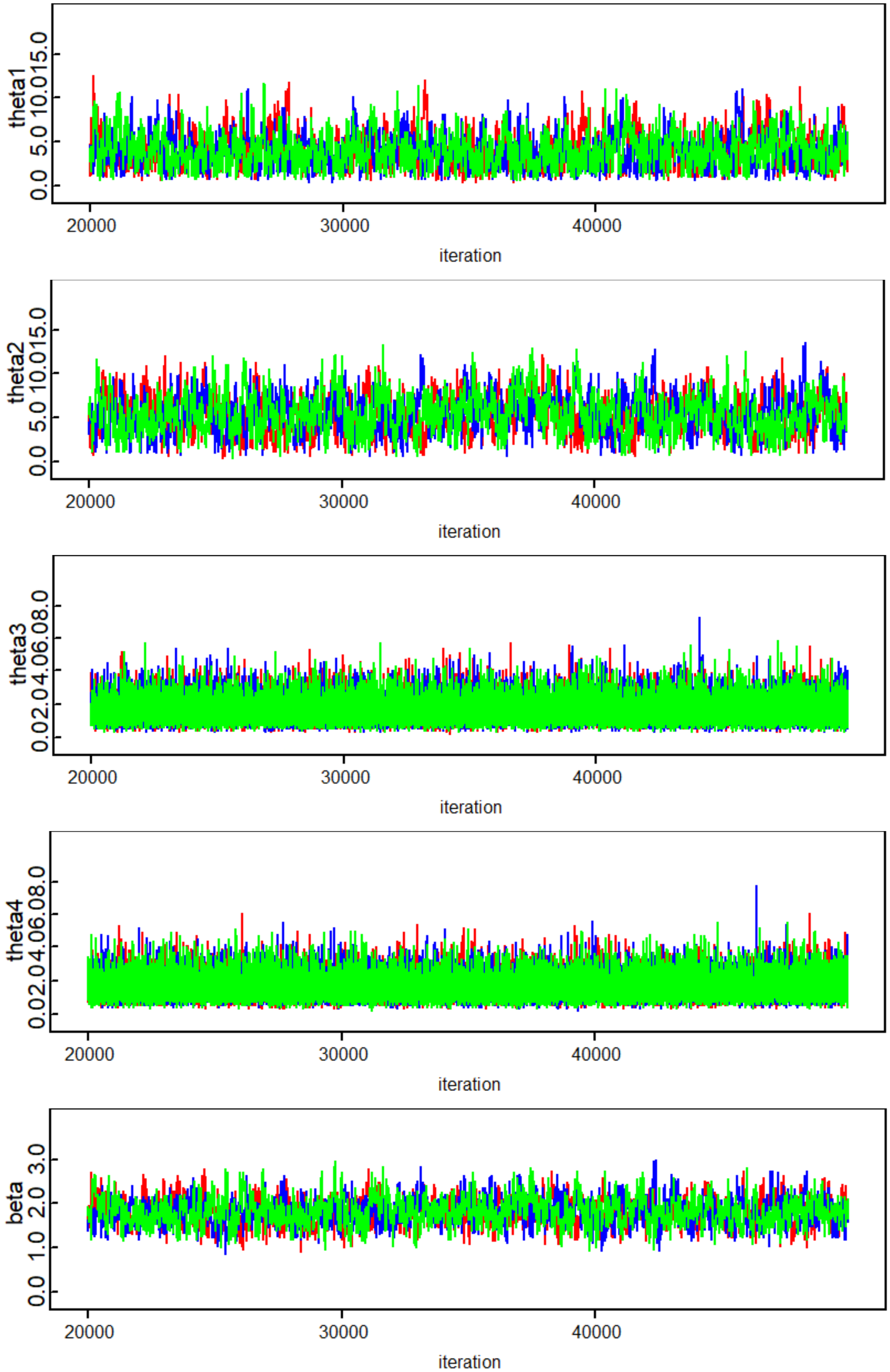

Figure A.10: Trace plots for the $GEW_{3,1}$ model.

A.1.6 $GEW_{3,2}$

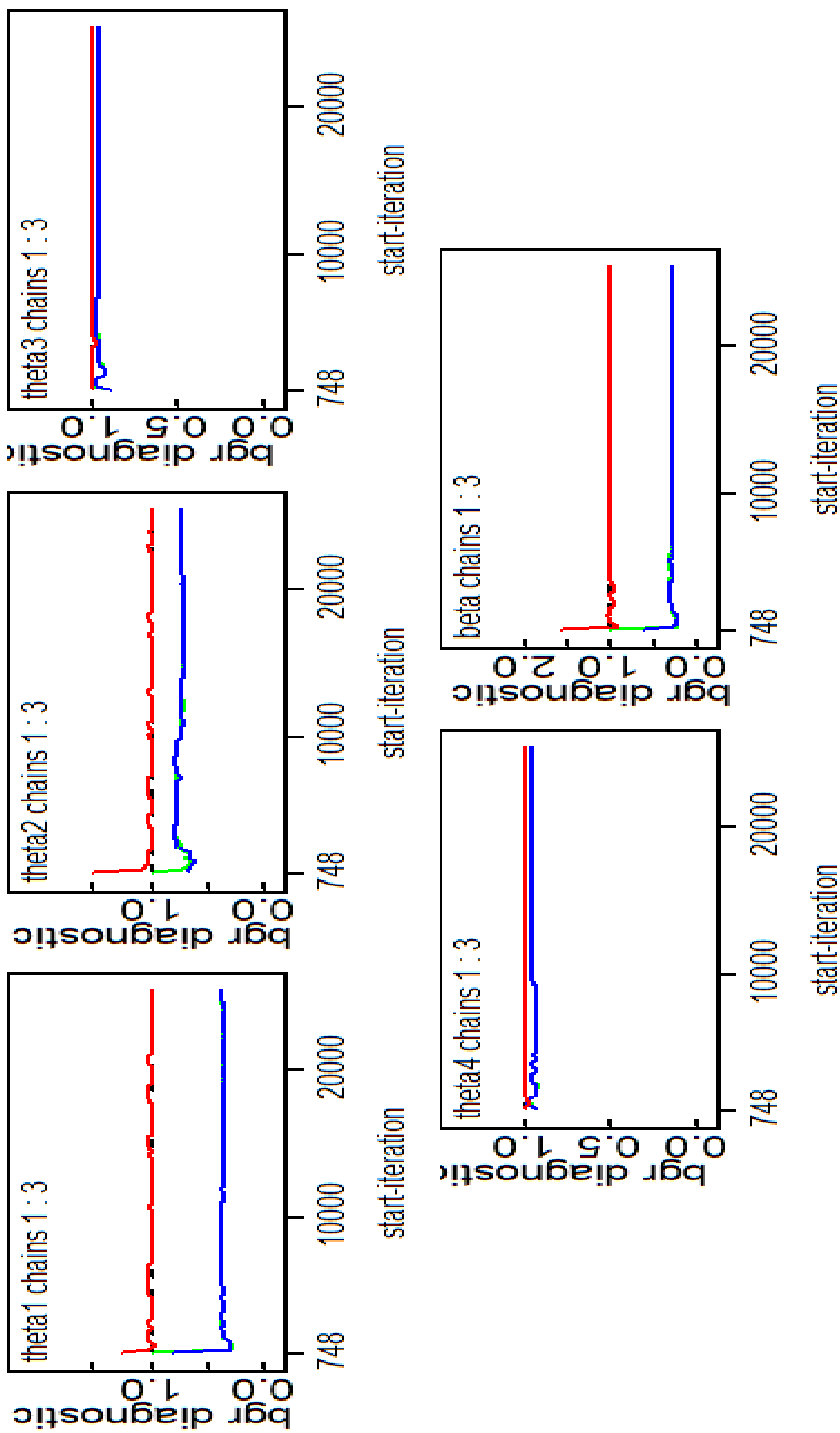

Figure A.11: BGR plots for the $GEW_{3,2}$ model.

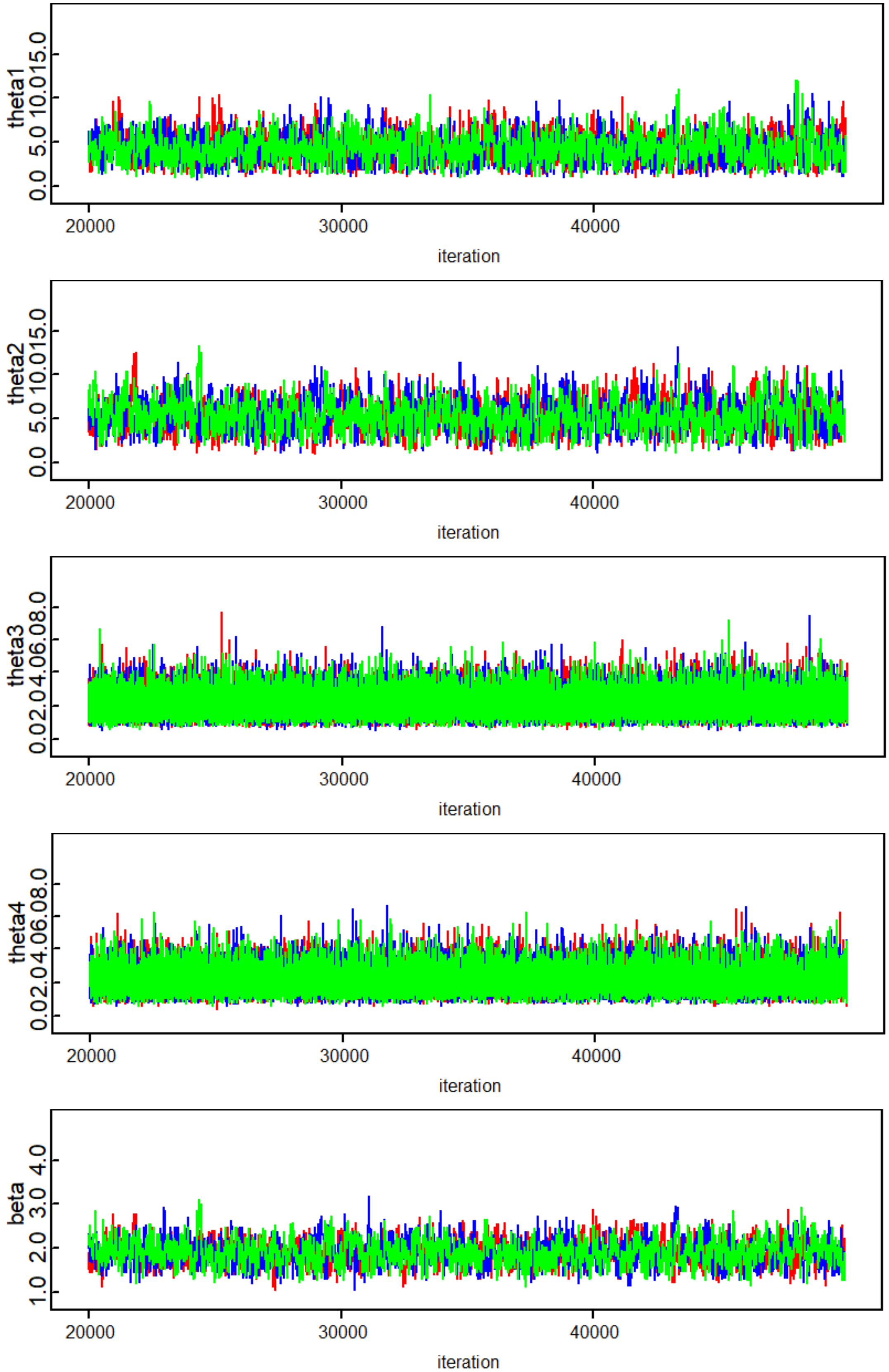

Figure A.12: Trace plots for the $GEW_{3,2}$ model.

A.2 Convergence for the GEBS Models

A.2.1 $GEBS_1$

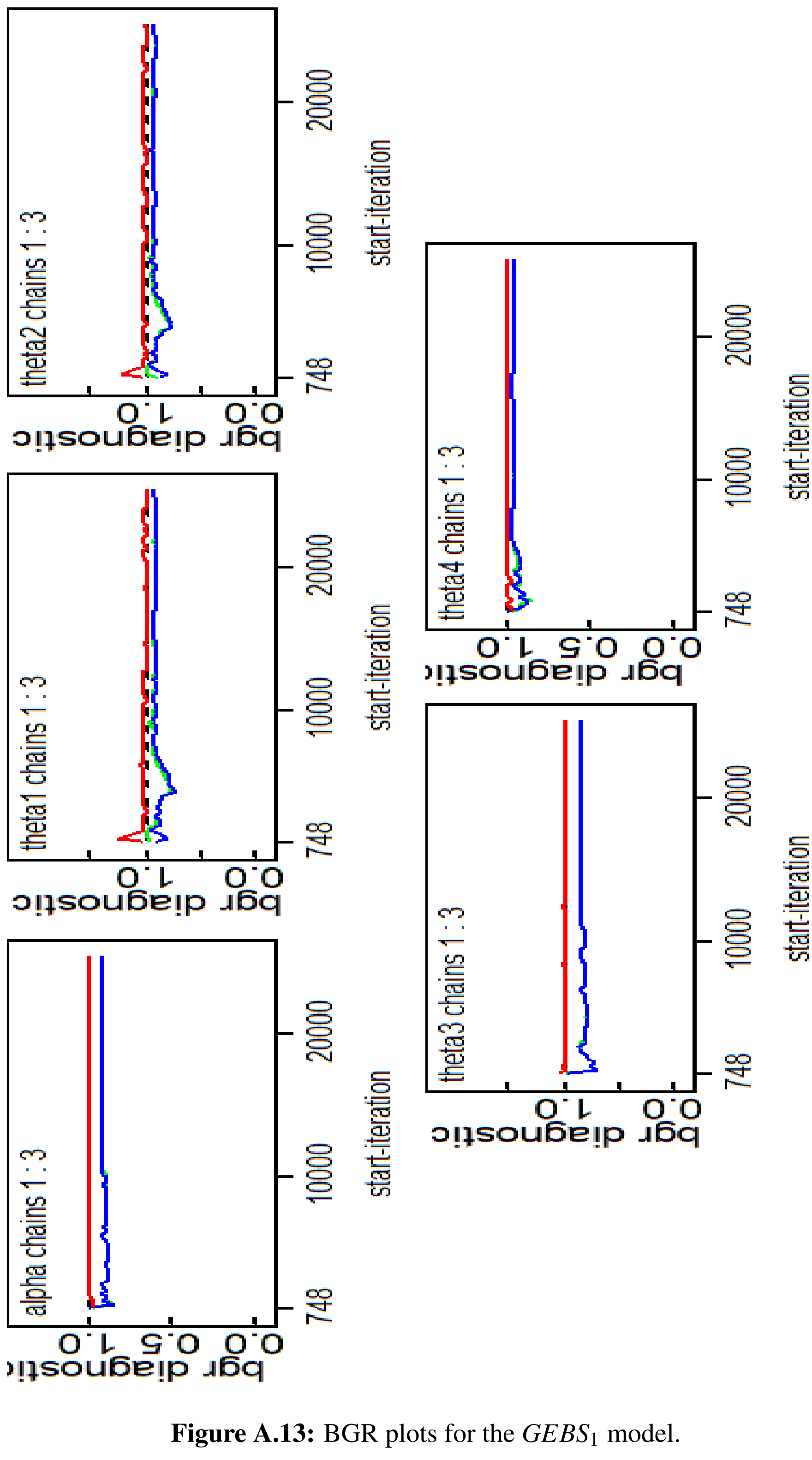

Figure A.13: BGR plots for the $GEBS_1$ model.

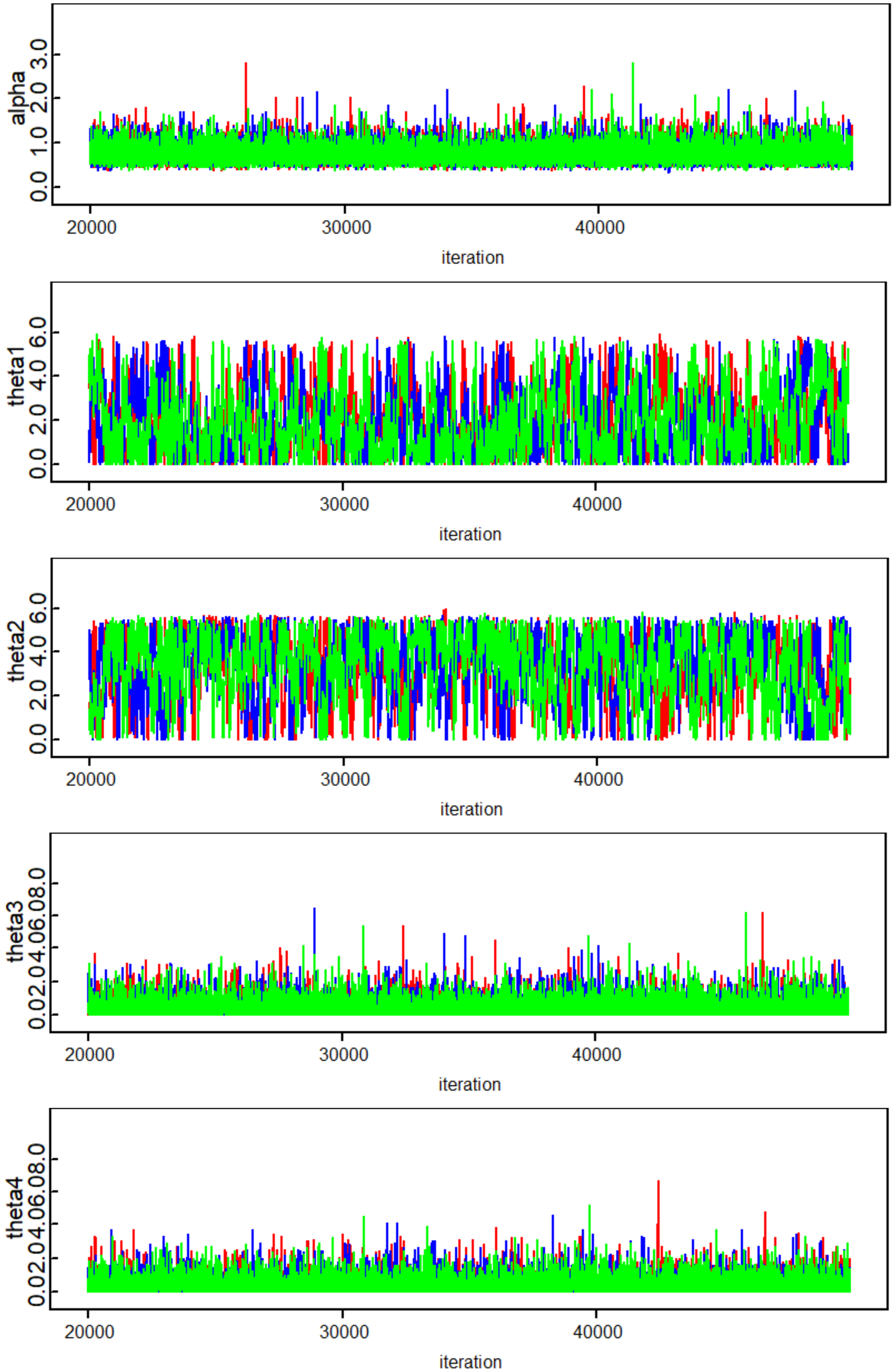

Figure A.14: Trace plots for the $GEBS_1$ model.

A.2.2 $GEBS_2$

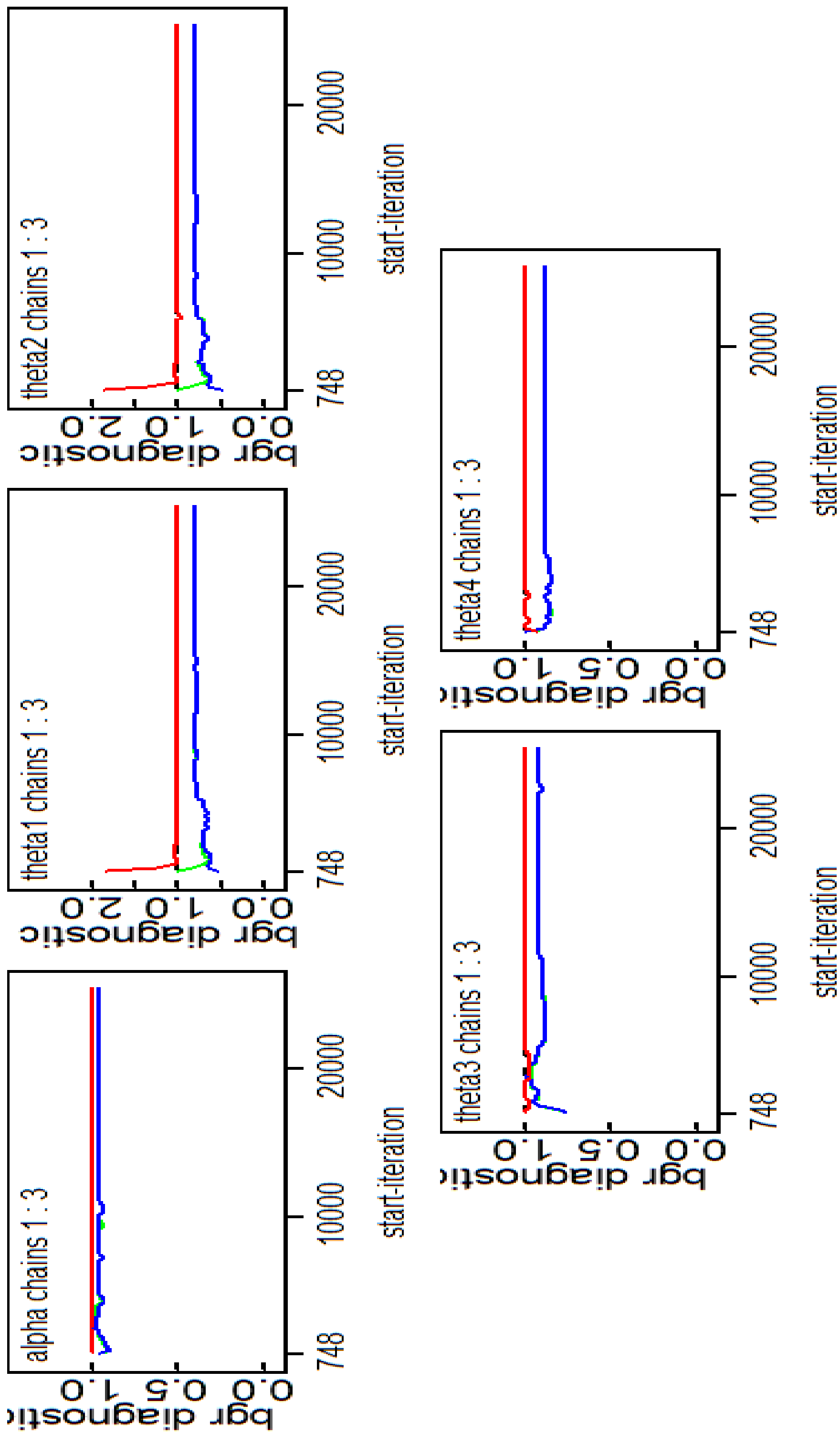

Figure A.15: BGR plots for the $GEBS_2$ model.

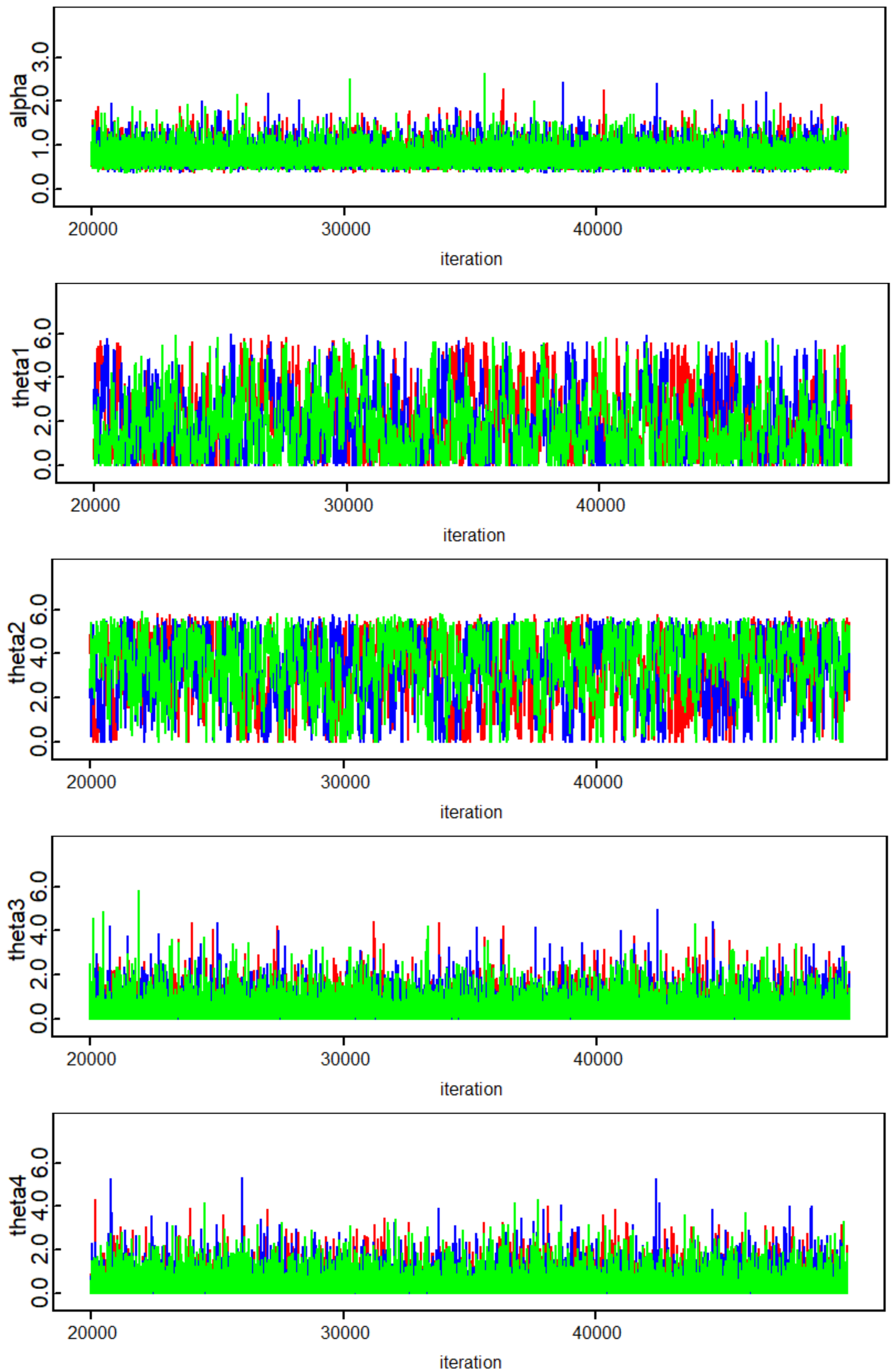

Figure A.16: Trace plots for the $GEBS_2$ model.

A.2.3 $GEBS_3$

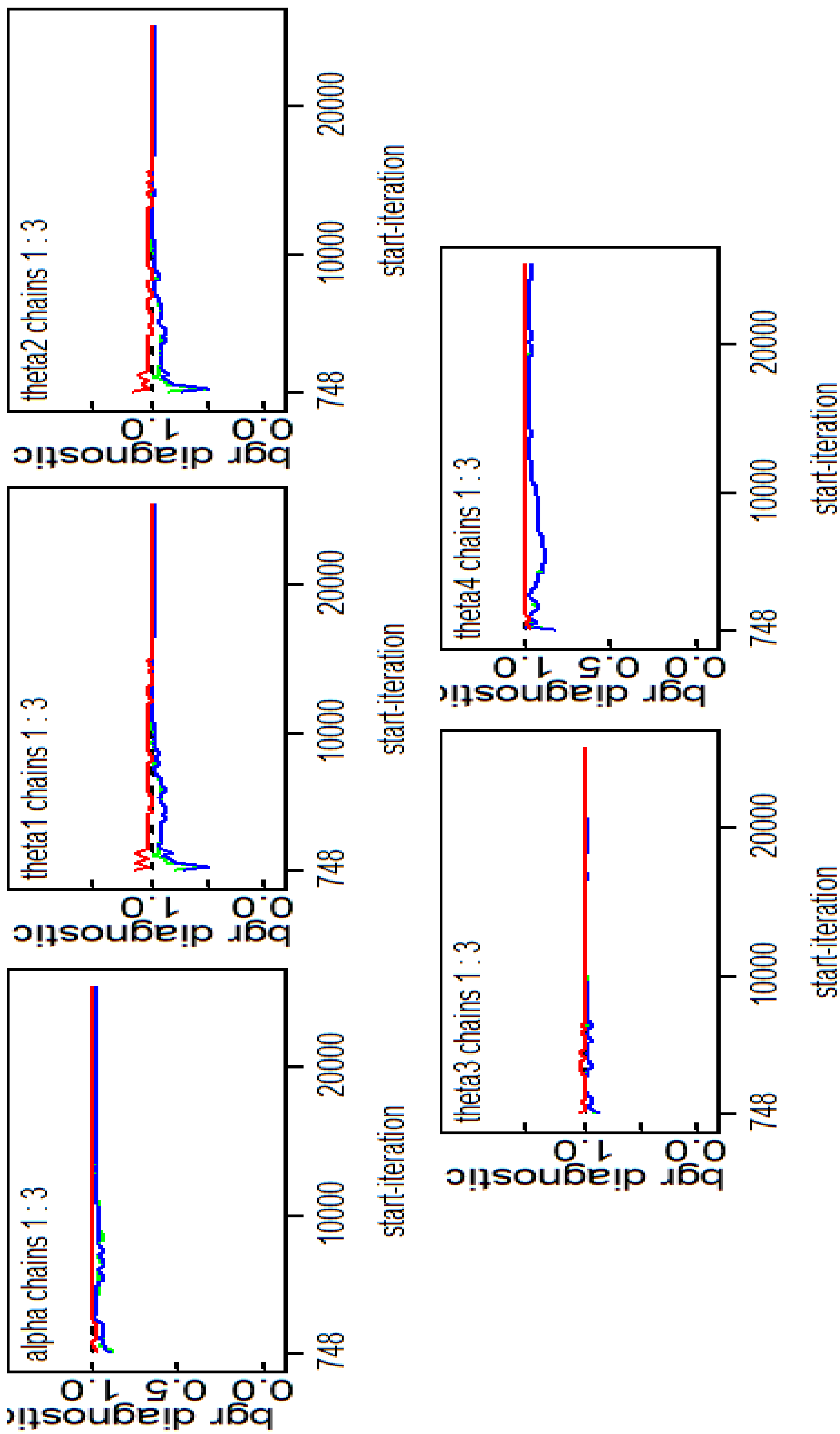

Figure A.17: BGR plots for the $GEBS_3$ model.

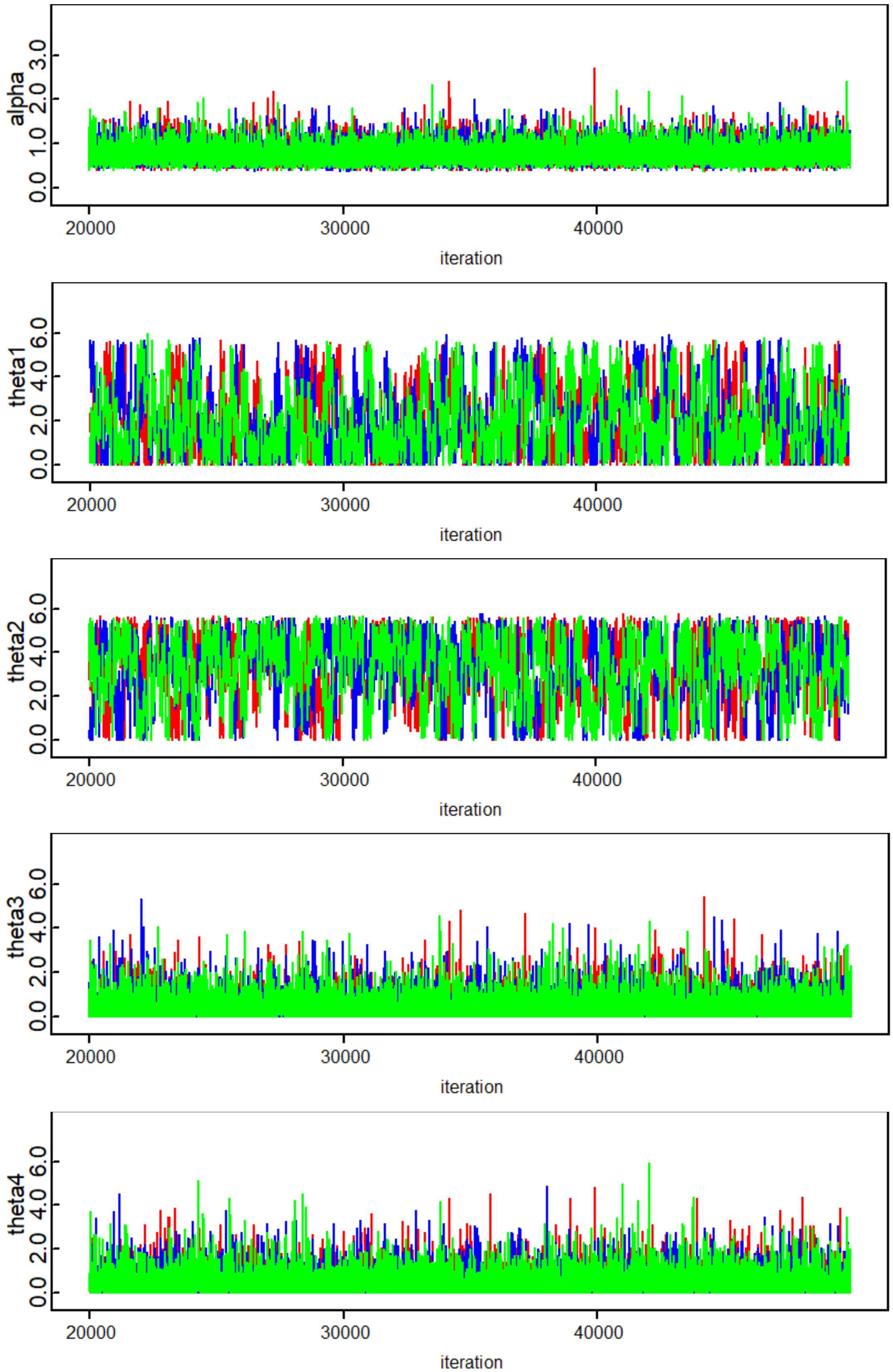

Figure A.18: Trace plots for the $GEBS_3$ model.

A.2.4 $GEBS_4$

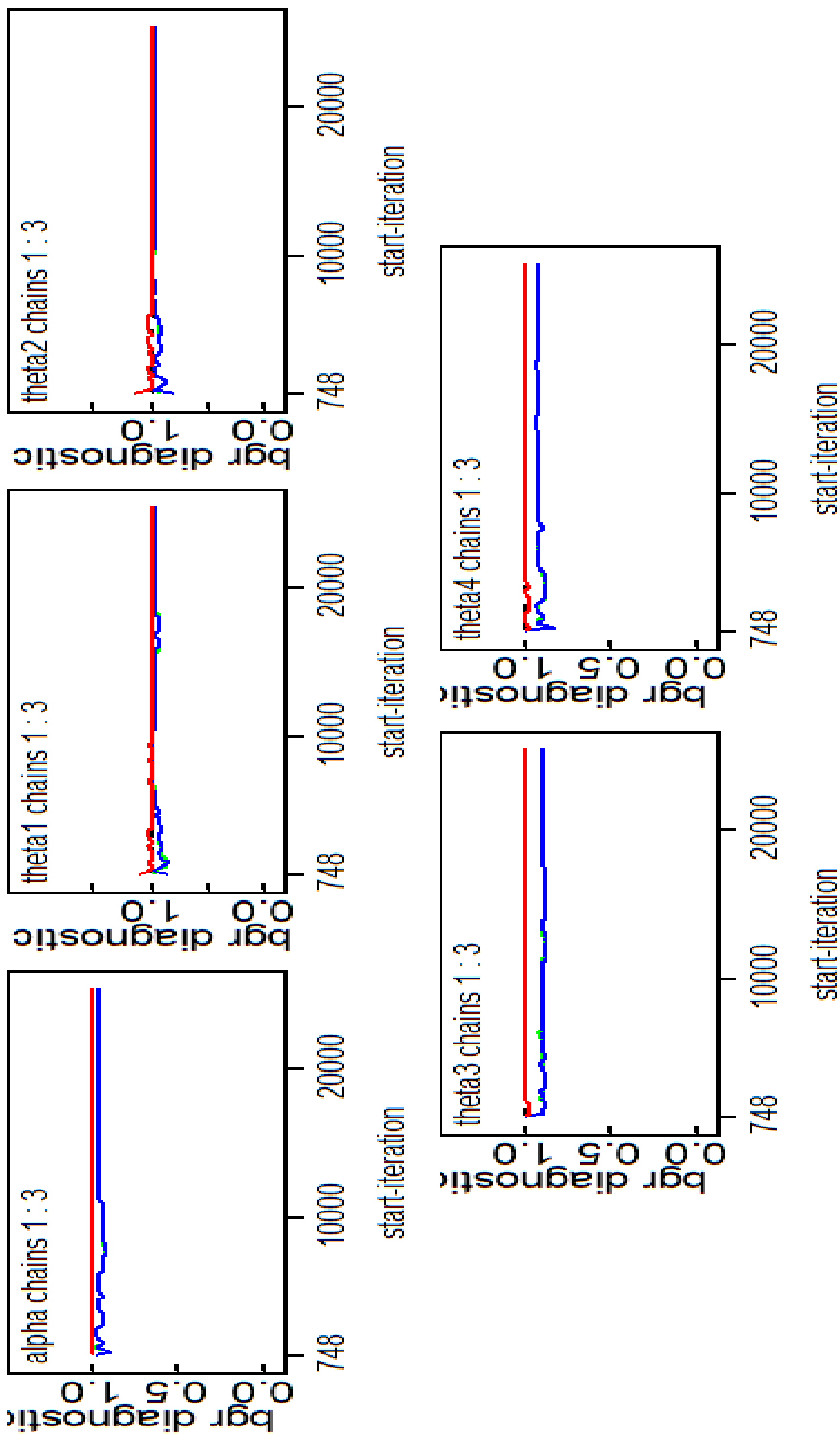

Figure A.19: BGR plots for the $GEBS_4$ model.

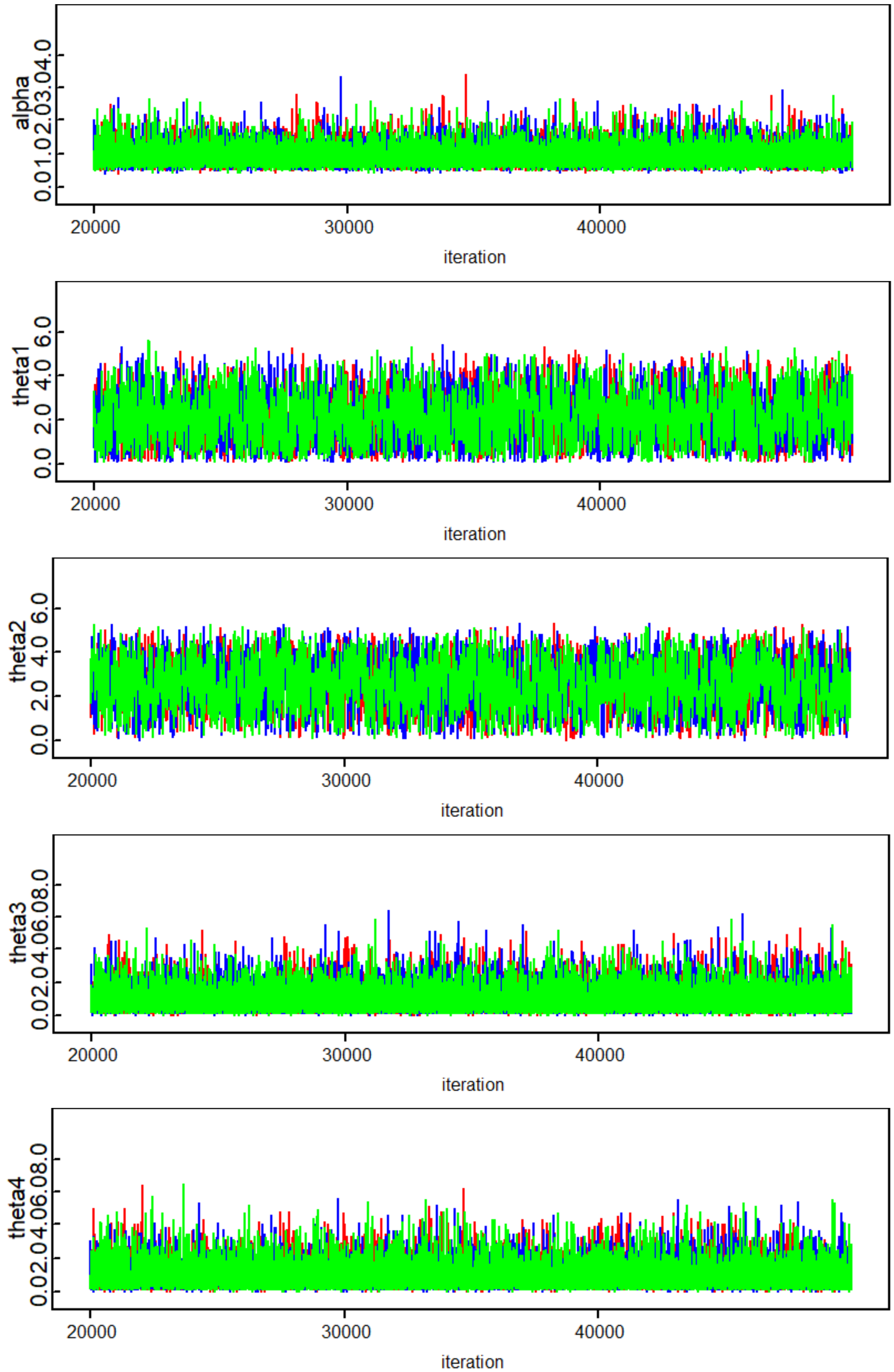

Figure A.20: Trace plots for the $GEBS_4$ model.

A.2.5 $GEBS_5$

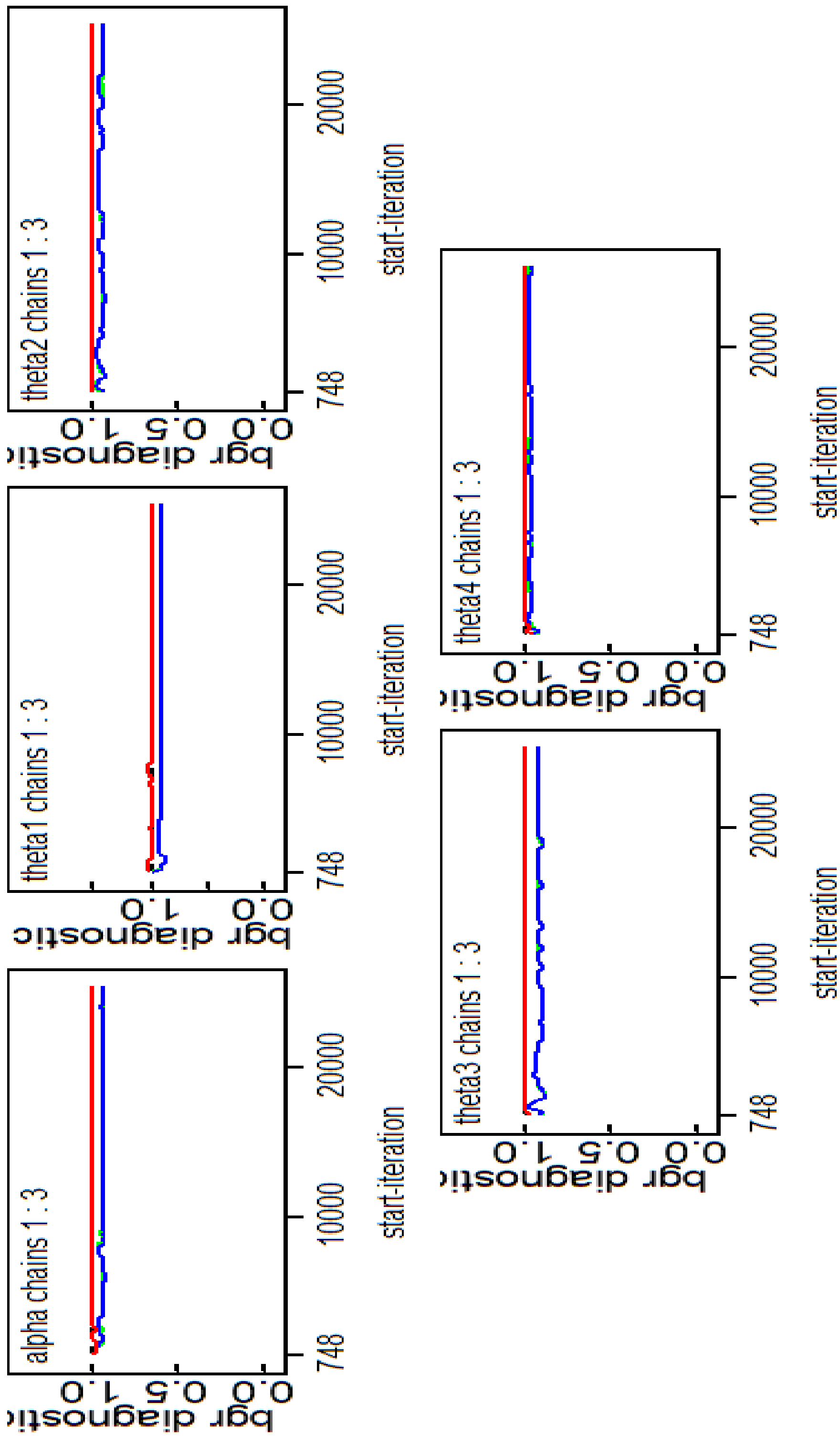

Figure A.21: BGR plots for the $GEBS_5$ model.

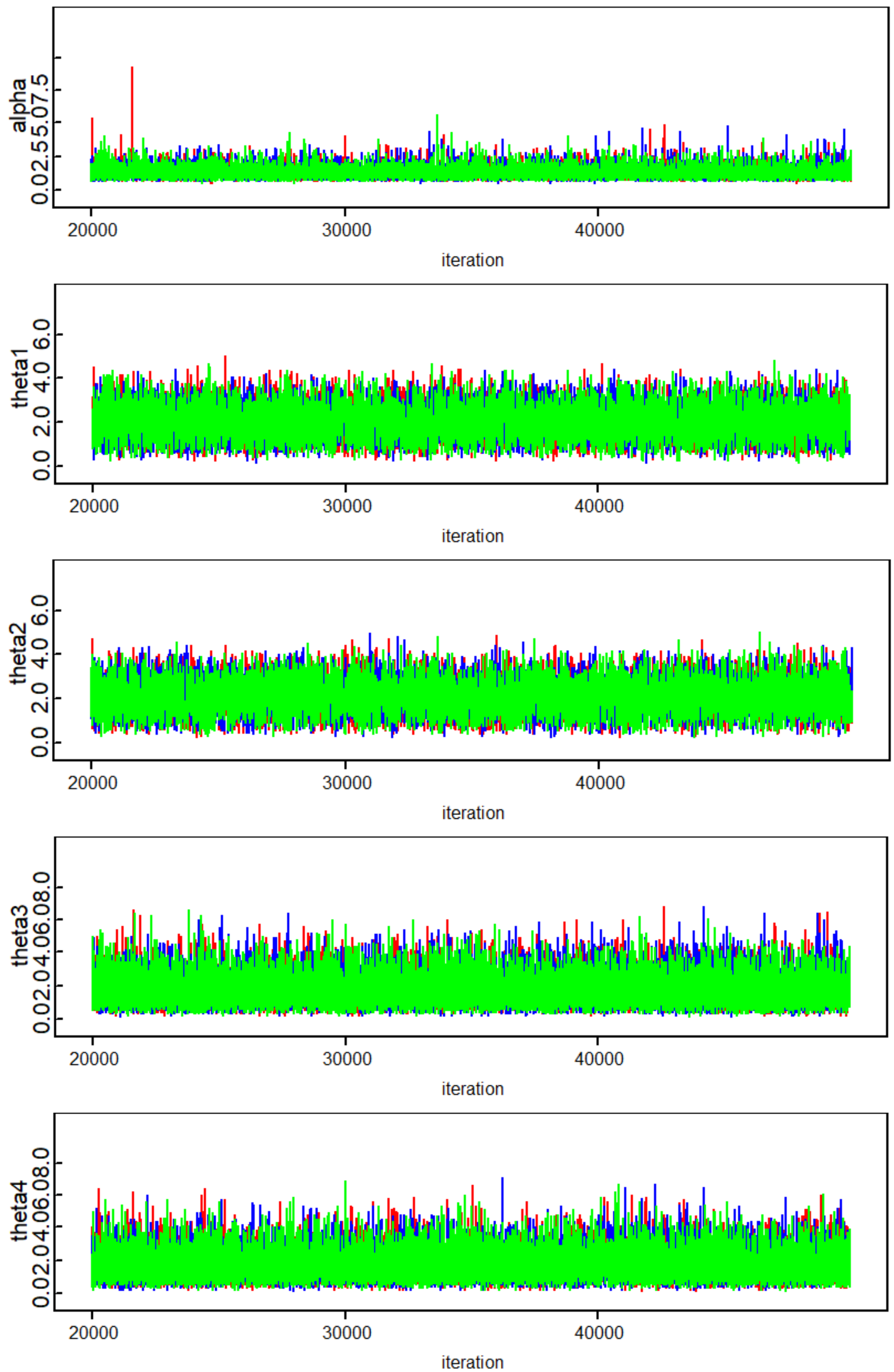

Figure A.22: Trace plots for the $GEBS_5$ model.

A.2.6 $GEBS_6$

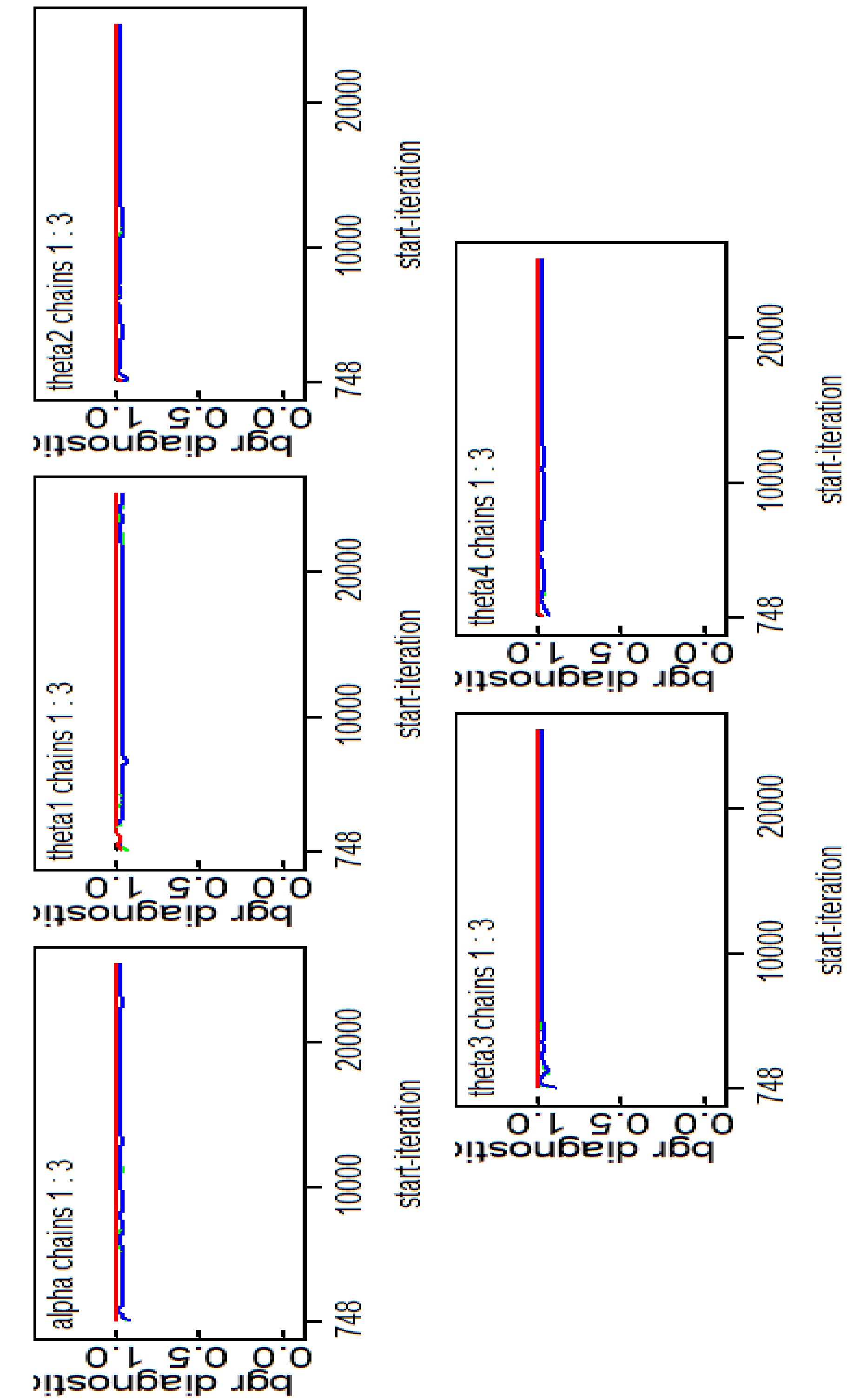

Figure A.23: BGR plots for the $GEBS_6$ model.

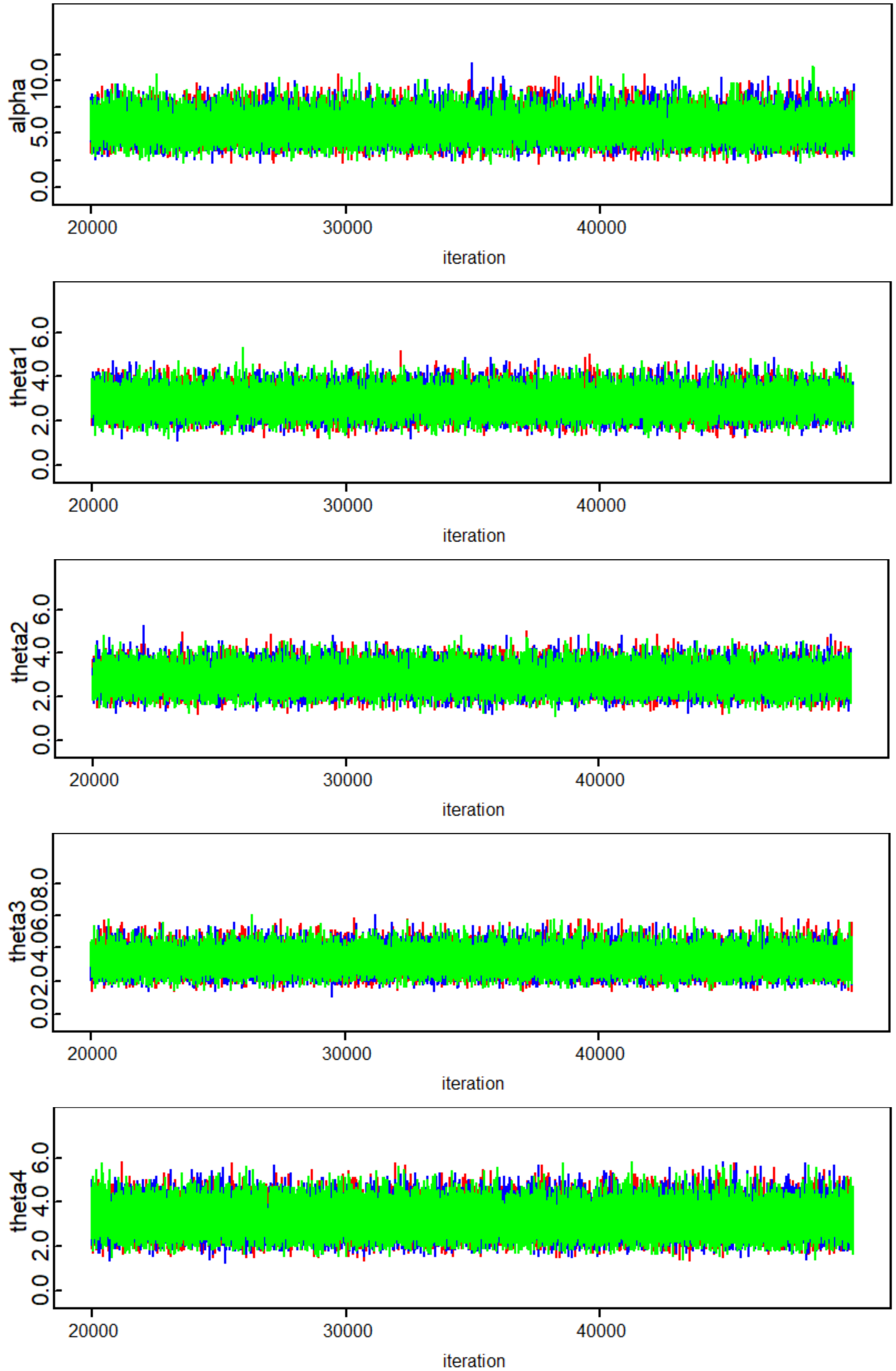

Figure A.24: Trace plots for the $GEBS_6$ model.

Appendix B: Additional Results

B.1 OpenBUGS® Code: MCMC Samples for GEW and GEBS Models

OpenBUGS® code for GEW model

```
model
{
for (i in 1:N)
{alpha[i] <- (Ti[i]/426)*exp(-theta1- theta2/(Ti[i]/426) -
theta3*Vi[i]/(-1.203972804) - theta4*(Vi[i]/(-1.203972804))/(Ti[i]/426))
X[i]~dweib(beta,alpha[i])
}
beta~dgamma(1,0.001)
theta1~dgamma(1,0.001)
theta2~dgamma(1,0.001)
theta3~dgamma(1,0.001)
theta4~dgamma(1,0.001)
}
Data
list(X=c(248, 456, 528, 731, 813, 164, 176, 289, 319, 340, 543, 92, 105, 184,
155, 219, 235),
N=17,
Ti=c(406, 406, 406, 406, 406, 416, 416, 416, 416, 416, 416, 426, 426, 426,
426, 426, 426),
Vi=c(-0.6931471806, -0.6931471806, -0.3566749439, -0.3566749439, -0.3566749439,
-0.6931471806, -0.6931471806, -0.6931471806, -0.3566749439, -0.3566749439,
-0.3566749439, -0.6931471806, -0.6931471806, -0.6931471806, -0.3566749439,
-0.3566749439, -0.3566749439))
Initials
```

```
list(beta=0.1, theta1=0.1, theta2=0.1, theta3=0.1, theta4=0.1)
```

OpenBUGS® code for GEBS model

```
model
{
for (i in 1:N)
{beta[i] <- (1/(Ti[i]/426))*exp(theta1 + theta2/(Ti[i]/426) +
theta3*Vi[i]/(-1.203972804) + theta4*(Vi[i]/(-1.203972804))/(Ti[i]/426))
X[i]~dbs(alpha,beta[i])
}
alpha~dgamma(1,0.001)
theta1~dgamma(1,0.001)
theta2~dgamma(1,0.001)
theta3~dgamma(1,0.001)
theta4~dgamma(1,0.001)   }
Data
list(X=c(248, 456, 528, 731, 813, 164, 176, 289, 319, 340, 543, 92, 105,
184, 155, 219, 235),
N=17,
Ti=c(406, 406, 406, 406, 406, 416, 416, 416, 416, 416, 416, 426, 426, 426,
426, 426, 426),
Vi=c(-0.6931471806, -0.6931471806, -0.3566749439, -0.3566749439, -0.3566749439,
-0.6931471806, -0.6931471806, -0.6931471806, -0.3566749439, -0.3566749439,
-0.3566749439, -0.6931471806, -0.6931471806, -0.6931471806, -0.3566749439,
-0.3566749439, -0.3566749439))
Initials
list(alpha=0.1, theta1=0.1, theta2=0.1, theta3=0.1, theta4=0.1)
```

B.2 R® Code: Predictive Reliability of GEW and GEBS Models

Code for GEW model

```
data = read.table("C:/...", sep="\t", header = T)
beta = data$beta
theta1 = data$theta1
theta2 = data$theta2
```

```
theta3 = data$theta3
theta4 = data$theta4
Tu = 350
Vu = -1.203972804
alpha = (Tu/426)*exp(-theta1 - theta2/(Tu/426) - theta3*(Vu/(-1.203972804))
- theta4*(Vu/(-1.203972804))/(Tu/426))
x = seq(1,10000,1)
result = vector("list")
for (i in 1:10000)
{
rel = exp(-alpha*(i^beta))
Pred_rel = mean(rel)
result[i]=Pred_rel
}
options(scipen = 5)
plot(x, result, xlab = "x", ylab = "Reliability")
write.table(cbind(result),file="C:/....txt")
```

Code for GEBS model

```
data = read.table("C:/...", sep="\t", header = T)
alpha = data$alpha
theta1 = data$theta1
theta2 = data$theta2
theta3 = data$theta3
theta4 = data$theta4
#load ExtraDist package
Tu = 350
Vu = -1.203972804
beta = (1/(Tu/426))*exp(theta1 + theta2/(Tu/426) + theta3*(Vu/(-1.203972804))
+ theta4*(Vu/(-1.203972804))/(Tu/426))
x = seq(1,10000,1)
result = vector("list")
for (i in 1:10000)
{
rel = 1 - pfatigue(i, alpha=alpha, beta=beta, mu=0, lower.tail=TRUE)
Pred_rel = mean(rel)
```

```
result[i]=Pred_rel
}
options(scipen = 5)
plot(x, result, xlab = "x", ylab = "Reliability")
write.table(cbind(result),file="C:/....txt")
```

B.3 R® Code: Simple Monte Carlo Estimation of Bayes Factors

Code for GEW model

```
beta = rgamma(200000, shape=1, rate=0.001)
theta1 = rgamma(200000, shape=1, rate=0.001)
theta2 = rgamma(200000, shape=1, rate=0.001)
theta3 = rgamma(200000, shape=1, rate=0.001)
theta4 = rgamma(200000, shape=1, rate=0.001)
X=c(521, 561, 575, 599, 609, 684, 709, 713, 345, 357, 439, 504, 115, 119, 150,
152, 153, 155, 156, 164, 199)
T=c(333, 333, 333, 333, 333, 333, 333, 333, 353, 353, 353, 353, 353, 353, 353,
353, 353, 353, 353, 353, 353)/353
V=c(-0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157,
-0.1053605157, -0.1053605157, -0.1053605157, -0.2231435513, -0.2231435513,
-0.2231435513, -0.2231435513, -0.1053605157, -0.1053605157, -0.1053605157,
-0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157,
-0.1053605157)/(-0.6931471806)
n=21
SumnT=8/T[1] + 4/T[9] + 9/T[13]
SumnV=8*V[1] + 4*V[9] + 9*V[13]
SumnVT=8*V[1]/T[1] + 4*V[9]/T[9] + 9*V[13]/T[13]
Like=0
Temp=c(rep(0,200000))
TempLike=c(rep(0,200000))
ret1=c(rep(0,200000))
ret2=c(rep(0,200000))
for (m in 1:200000)
{
TempLike[m]=n*log(beta[m]) - theta1[m]*n - theta2[m]*SumnT - theta3[m]*SumnV
- theta4[m]*SumnVT
```

```
C1=0
C2=0
for (i in 1:21)
{
TempCalc1=-T[i]*exp(-theta1[m] - theta2[m]/T[i] - theta3[m]*V[i]
- theta4[m]*V[i]/T[i])*(X[i]^beta[m])
TempCalc2=log(T[i]) + (beta[m]-1)*log(X[i])
C1=C1+TempCalc1
C2=C2+TempCalc2
}
ret1[m]=C1
ret2[m]=C2
Temp[m]=(TempLike[m]+ret1[m]+ret2[m])
Like=Like+exp(Temp[m])
}
Like
SME=Like/200000
SME
```

Code for GEBS model

```
beta = rgamma(200000, shape=1, rate=0.001)
theta1 = rgamma(200000, shape=1, rate=0.001)
theta2 = rgamma(200000, shape=1, rate=0.001)
theta3 = rgamma(200000, shape=1, rate=0.001)
theta4 = rgamma(200000, shape=1, rate=0.001)
X=c(521, 561, 575, 599, 609, 684, 709, 713, 345, 357, 439, 504, 115, 119, 150,
152, 153, 155, 156, 164, 199)
T=c(333, 333, 333, 333, 333, 333, 333, 333, 353, 353, 353, 353, 353, 353, 353,
353, 353, 353, 353, 353, 353)/353
V=c(-0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157,
-0.1053605157, -0.1053605157, -0.1053605157, -0.2231435513, -0.2231435513,
-0.2231435513, -0.2231435513, -0.1053605157, -0.1053605157, -0.1053605157,
-0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157,
-0.1053605157)/(-0.6931471806)
n=21
SumnT=8/T[1] + 4/T[9] + 9/T[13]
```

```
SumnV=8*V[1] + 4*V[9] + 9*V[13]
SumnVT=8*V[1]/T[1] + 4*V[9]/T[9] + 9*V[13]/T[13]
Like=0
Temp=c(rep(0,200000))
TempLike=c(rep(0,200000))
ret1=c(rep(0,200000))
ret2=c(rep(0,200000))
for (m in 1:200000)
{
TempLike[m]=((2*sqrt(2*pi)*alpha[m])^(-n))*exp(n/(alpha[m])^2)
C1=1
C2=0
for (i in 1:21)
{
TempCalc1=(X[i] + (1/T[i]) * exp(theta1[m] + theta2[m]/T[i] + theta3[m]*V[i]
+ theta4[m]*V[i]/T[i]))/(((X[i])^(3/2)) * sqrt((1/T[i]) * exp(theta1[m]
+ theta2[m]/T[i] + theta3[m]*V[i] + theta4[m]*V[i]/T[i])))
TempCalc2=(X[i] * T[i] * exp(-theta1[m] - theta2[m]/T[i] - theta3[m]*V[i]
- theta4[m]*V[i]/T[i])) + ((1/(X[i]*T[i])) * exp(theta1[m] + theta2[m]/T[i]
+ theta3[m]*V[i] + theta4[m]*V[i]/T[i]))
C1=C1*TempCalc1
C2=C2+TempCalc2
}
ret1[m]=C1
ret2[m]=exp(((-1)/(2*alpha[m]^2))*C2)
Temp[m]=(TempLike[m]*ret1[m]*ret2[m])
Like=Like+Temp[m]
}
Like
SME=Like/200000
SME
```

B.4 R® Code: Estimation of Posterior Bayes Factors

Code for GEW model

```
data = read.table("C:/...", sep="\t", header = TRUE)
```

```
beta = data$beta
theta1 = data$theta1
theta2 = data$theta2
theta3 = data$theta3
theta4 = data$theta4
X=c(521, 561, 575, 599, 609, 684, 709, 713, 345, 357, 439, 504, 115, 119, 150,
152, 153, 155, 156, 164, 199)
T=c(333, 333, 333, 333, 333, 333, 333, 333, 353, 353, 353, 353, 353, 353, 353,
353, 353, 353, 353, 353, 353)/353
V=c(-0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157,
-0.1053605157, -0.1053605157, -0.1053605157, -0.2231435513, -0.2231435513,
-0.2231435513, -0.2231435513, -0.1053605157, -0.1053605157, -0.1053605157,
-0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157,
-0.1053605157)/(-0.6931471806)
n=21
SumnT=8/T[1] + 4/T[9] + 9/T[13]
SumnV=8*V[1] + 4*V[9] + 9*V[13]
SumnVT=8*V[1]/T[1] + 4*V[9]/T[9] + 9*V[13]/T[13]
Like=0
Temp=c(rep(0,200000))
TempLike=c(rep(0,200000))
ret1=c(rep(0,200000))
ret2=c(rep(0,200000))
for (m in 1:200000)
{
TempLike[m]=n*log(beta[m]) - theta1[m]*n - theta2[m]*SumnT - theta3[m]*SumnV
- theta4[m]*SumnVT
C1=0
C2=0
for (i in 1:21)
{
TempCalc1=-T[i]*exp(-theta1[m] - theta2[m]/T[i] - theta3[m]*V[i]
- theta4[m]*V[i]/T[i])*(X[i]^beta[m])
TempCalc2=log(T[i]) + (beta[m]-1)*log(X[i])
C1=C1+TempCalc1
C2=C2+TempCalc2
}
```

```
ret1[m]=C1
ret2[m]=C2
Temp[m]=(TempLike[m]+ret1[m]+ret2[m])
Like=Like+exp(Temp[m])
}
Like
SME=Like/200000
SME
```

Code for GEBS model

```
data = read.table("C:/...", sep="\t", header = TRUE)
beta = data$beta
theta1 = data$theta1
theta2 = data$theta2
theta3 = data$theta3
theta4 = data$theta4
X=c(521, 561, 575, 599, 609, 684, 709, 713, 345, 357, 439, 504, 115, 119, 150,
152, 153, 155, 156, 164, 199)
T=c(333, 333, 333, 333, 333, 333, 333, 333, 353, 353, 353, 353, 353, 353, 353,
353, 353, 353, 353, 353, 353)/353
V=c(-0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157,
-0.1053605157, -0.1053605157, -0.1053605157, -0.2231435513, -0.2231435513,
-0.2231435513, -0.2231435513, -0.1053605157, -0.1053605157, -0.1053605157,
-0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157,
-0.1053605157)/(-0.6931471806)
n=21
SumnT=8/T[1] + 4/T[9] + 9/T[13]
SumnV=8*V[1] + 4*V[9] + 9*V[13]
SumnVT=8*V[1]/T[1] + 4*V[9]/T[9] + 9*V[13]/T[13]
Like=0
Temp=c(rep(0,200000))
TempLike=c(rep(0,200000))
ret1=c(rep(0,200000))
ret2=c(rep(0,200000))
for (m in 1:200000)
{
```

```
TempLike[m]=((2*sqrt(2*pi)*alpha[m])^(-n))*exp(n/(alpha[m])^2)
C1=1
C2=0
for (i in 1:21)
{
TempCalc1=(X[i] + (1/T[i]) * exp(theta1[m] + theta2[m]/T[i] + theta3[m]*V[i]
+ theta4[m]*V[i]/T[i]))/(((X[i])^(3/2)) * sqrt((1/T[i]) * exp(theta1[m]
+ theta2[m]/T[i] + theta3[m]*V[i] + theta4[m]*V[i]/T[i])))
TempCalc2=(X[i] * T[i] * exp(-theta1[m] - theta2[m]/T[i] - theta3[m]*V[i]
- theta4[m]*V[i]/T[i])) + ((1/(X[i]*T[i])) * exp(theta1[m] + theta2[m]/T[i]
+ theta3[m]*V[i] + theta4[m]*V[i]/T[i]))
C1=C1*TempCalc1
C2=C2+TempCalc2
}
ret1[m]=C1
ret2[m]=exp(((-1)/(2*alpha[m]^2))*C2)
Temp[m]=(TempLike[m]*ret1[m]*ret2[m])
Like=Like+Temp[m]
}
Like
SME=Like/200000
SME
```

B.5 R® Code: Laplace Approximation of Bayes Factors

Code for GEW model

```
CorrelData=c(1.000000, -0.032738, 0.827194, 0.001227, 0.029451,-0.032738,
1.000000, -0.564601, -0.054825, -0.068704,0.827194, -0.564601, 1.000000,
-0.043728, -0.045860,0.001227, -0.054825, -0.043728, 1.000000, -0.242574,
0.029451, -0.068704, -0.045860, -0.242574, 1.000000)
Correl=matrix(CorrelData, nrow=5, ncol=5)
det(Correl)
Sdev=c(1.8983, 3.2703, 2.1646, 2.5878, 0.4413)
SumSdev=0
for (i in 1:5)
```

```
{
SumSdev = SumSdev + log(Sdev[i])
}
SumSdev
beta=2.4652
theta1=1.9918
theta2=11.5327
theta3=2.5876
theta4=3.2155
X=c(521, 561, 575, 599, 609, 684, 709, 713, 345, 357, 439, 504, 115, 119, 150,
152, 153, 155, 156, 164, 199)
T=c(333, 333, 333, 333, 333, 333, 333, 333, 353, 353, 353, 353, 353, 353, 353,
353, 353, 353, 353, 353, 353)/353
V=c(-0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157,
-0.1053605157, -0.1053605157, -0.1053605157, -0.2231435513, -0.2231435513,
-0.2231435513, -0.2231435513, -0.1053605157, -0.1053605157, -0.1053605157,
-0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157,
-0.1053605157)/(-0.6931471806)
n=21
SumnT=8/T[1] + 4/T[9] + 9/T[13]
SumnV=8*V[1] + 4*V[9] + 9*V[13]
SumnVT=8*V[1]/T[1] + 4*V[9]/T[9] + 9*V[13]/T[13]
Like=0
C0=0
C1=0
FirstPart = n*log(beta) - theta1*n - theta2*SumnT - theta3*SumnV - theta4*SumnVT
for (i in 1:n)
{
TempCalc1 = -T[i]*exp(-theta1 - theta2/T[i] - theta3*V[i] - theta4*V[i]/T[i])
*(X[i]^beta)
TempCalc2 = log(T[i]) + (beta-1)*log(X[i])
C0 = C0 + TempCalc1
C1 = C1 + TempCalc2 }
Like = FirstPart + C0 + C1
Like
PriorAlpha = 1
PriorBeta = 0.001
```

```
LogPrior = 5*log((PriorBeta^PriorAlpha)/gamma(PriorAlpha)) + (PriorAlpha-1)
*(log(theta1)+log(theta2)+log(theta3)+log(theta4)+log(beta))
- PriorBeta*(theta1+theta2+theta3+theta4+beta)
LogLike = 0
LogLike = (5/2)*log(2*pi) + (1/2)*log(det(Correl)) + SumSdev + Like + LogPrior
MarginalLike = exp(LogLike)
MarginalLike
```

Code for GEBS model

```
CorrelData=c(1.000000, 0.280365, -0.330758, 0.151960, 0.106404,0.280365,
1.000000, -0.953070, -0.014423, -0.013875,-0.330758, -0.953070, 1.000000,
-0.120922, -0.167490,0.151960, -0.014423, -0.120922, 1.000000, -0.352438,
0.106404, -0.013875, -0.167490, -0.352438, 1.000000)
Correl=matrix(CorrelData, nrow=5, ncol=5)
det(Correl)
Sdev=c(1.0410, 1.0573, 1.3579, 1.5283, 0.1133)
SumSdev=0
for (i in 1:5)
{
SumSdev = SumSdev + log(Sdev[i])
}
SumSdev
alpha=0.5954
theta1=1.1009
theta2=3.8378
theta3=1.6984
theta4=2.1033
X=c(521, 561, 575, 599, 609, 684, 709, 713, 345, 357, 439, 504, 115, 119, 150,
152, 153, 155, 156, 164, 199)
T=c(333, 333, 333, 333, 333, 333, 333, 333, 353, 353, 353, 353, 353, 353, 353,
353, 353, 353, 353, 353, 353)/353
V=c(-0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157,
-0.1053605157, -0.1053605157, -0.1053605157, -0.2231435513, -0.2231435513,
-0.2231435513, -0.2231435513, -0.1053605157, -0.1053605157, -0.1053605157,
-0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157,
```

```
-0.1053605157)/(-0.6931471806)
n=21
SumnT=8/T[1] + 4/T[9] + 9/T[13]
SumnV=8*V[1] + 4*V[9] + 9*V[13]
SumnVT=8*V[1]/T[1] + 4*V[9]/T[9] + 9*V[13]/T[13]
Like=0
C0=0
C1=0
FirstPart = -n*log(2*sqrt(2*pi)*alpha) + n/(alpha^2)
for (i in 1:n)
{
TempCalc1 = log((X[i] + (1/T[i]) * exp(theta1 + theta2/T[i] + theta3*V[i]
+ theta4*V[i]/T[i]))/(((X[i])^(3/2)) * sqrt((1/T[i]) * exp(theta1 +
theta2/T[i] + theta3*V[i] + theta4*V[i]/T[i]))))
TempCalc2 = (X[i] * T[i] * exp(-theta1 - theta2/T[i] - theta3*V[i]
- theta4*V[i]/T[i])) + ((1/(X[i]*T[i])) * exp(theta1 + theta2/T[i]
+ theta3*V[i] + theta4*V[i]/T[i]))
C0 = C0 + TempCalc1
C1 = C1 + TempCalc2
}
Like = FirstPart + C0 - (1/(2*alpha^2))*C1
Like
PriorAlpha = 1
PriorBeta = 0.001
LogPrior = 5*log((PriorBeta^PriorAlpha)/gamma(PriorAlpha)) +
(PriorAlpha-1)*(log(theta1)+log(theta2)+log(theta3)+log(theta4)+log(alpha))
- PriorBeta*(theta1+theta2+theta3+theta4+alpha)
LogLike = 0
LogLike = (5/2)*log(2*pi) + (1/2)*log(det(Correl)) + SumSdev + Like
+ LogPrior
MarginalLike = exp(LogLike)
MarginalLike
```

B.6 R® Code: Harmonic Mean Estimator for Bayes Factors

Code for GEW model

```
data = read.table("C:/...", sep="\t", header = TRUE)
```

```
beta = data$beta
theta1 = data$theta1
theta2 = data$theta2
theta3 = data$theta3
theta4 = data$theta4
X=c(521, 561, 575, 599, 609, 684, 709, 713, 345, 357, 439, 504, 115, 119, 150,
152, 153, 155, 156, 164, 199)
T=c(333, 333, 333, 333, 333, 333, 333, 333, 353, 353, 353, 353, 353, 353, 353,
353, 353, 353, 353, 353, 353)/353
V=c(-0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157,
-0.1053605157, -0.1053605157, -0.1053605157, -0.2231435513, -0.2231435513,
-0.2231435513, -0.2231435513, -0.1053605157, -0.1053605157, -0.1053605157,
-0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157,
-0.1053605157)/(-0.6931471806)
n=21
SumnT=8/T[1] + 4/T[9] + 9/T[13]
SumnV=8*V[1] + 4*V[9] + 9*V[13]
SumnVT=8*V[1]/T[1] + 4*V[9]/T[9] + 9*V[13]/T[13]
Like=0
Temp=c(rep(0,200000))
TempLike=c(rep(0,200000))
ret1=c(rep(0,200000))
ret2=c(rep(0,200000))
for (m in 1:200000)
{
TempLike[m]=beta[m]^n * exp(-theta1[m]*n - theta2[m]*SumnT - theta3[m]*SumnV
- theta4[m]*SumnVT)
C1=0
C2=1
for (i in 1:21)
{
TempCalc1=-T[i]*exp(-theta1[m] - theta2[m]/T[i] - theta3[m]*V[i]
- theta4[m]*V[i]/T[i])*(X[i]^beta[m])
TempCalc2=T[i]*(X[i]^(beta[m]-1))
C1=C1+TempCalc1
C2=C2*TempCalc2
}
```

```
ret1[m]=exp(C1)
ret2[m]=C2
Temp[m]=1/(TempLike[m]*ret1[m]*ret2[m])
Like=Like+Temp[m]
}
Like
HME=1/(Like/200000)
HME
```

Code for GEBS model

```
data = read.table("C:/...", sep="\t", header = TRUE)
beta = data$beta
theta1 = data$theta1
theta2 = data$theta2
theta3 = data$theta3
theta4 = data$theta4
X=c(521, 561, 575, 599, 609, 684, 709, 713, 345, 357, 439, 504, 115, 119, 150,
152, 153, 155, 156, 164, 199)
T=c(333, 333, 333, 333, 333, 333, 333, 333, 353, 353, 353, 353, 353, 353, 353,
353, 353, 353, 353, 353, 353)/353
V=c(-0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157,
-0.1053605157, -0.1053605157, -0.1053605157, -0.2231435513, -0.2231435513,
-0.2231435513, -0.2231435513, -0.1053605157, -0.1053605157, -0.1053605157,
-0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157, -0.1053605157,
-0.1053605157)/(-0.6931471806)
n=21
SumnT=8/T[1] + 4/T[9] + 9/T[13]
SumnV=8*V[1] + 4*V[9] + 9*V[13]
SumnVT=8*V[1]/T[1] + 4*V[9]/T[9] + 9*V[13]/T[13]
Like=0
Temp=c(rep(0,200000))
TempLike=c(rep(0,200000))
ret1=c(rep(0,200000))
ret2=c(rep(0,200000))
for (m in 1:200000)
{
```

```
TempLike[m]=((2*sqrt(2*pi)*alpha[m])^(-n))*exp(n/(alpha[m])^2)
C1=1
C2=0
for (i in 1:21)
{
TempCalc1=(X[i] + (1/T[i]) * exp(theta1[m] + theta2[m]/T[i] + theta3[m]*V[i]
+ theta4[m]*V[i]/T[i]))/(((X[i])^(3/2)) * sqrt((1/T[i]) * exp(theta1[m]
+ theta2[m]/T[i] + theta3[m]*V[i] + theta4[m]*V[i]/T[i])))
TempCalc2=(X[i] * T[i] * exp(-theta1[m] - theta2[m]/T[i] - theta3[m]*V[i]
- theta4[m]*V[i]/T[i])) + ((1/(X[i]*T[i])) * exp(theta1[m] + theta2[m]/T[i]
+ theta3[m]*V[i] + theta4[m]*V[i]/T[i]))
C1=C1*TempCalc1
C2=C2+TempCalc2
}
ret1[m]=C1
ret2[m]=exp(((-1)/(2*alpha[m]^2))*C2)
Temp[m]=1/(TempLike[m]*ret1[m]*ret2[m])
Like=Like+Temp[m]
}
Like
HME=1/(Like/200000)
HME
```