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ON THE SOLUTIONS AND CONSERVATION
LAWS OF BOUSSINESQ SYSTEM OF
KdV-KdV TYPE, GENERALIZED COUPLED
KdV EQUATIONS AND COUPLED SYSTEM
OF ONE-LAYER SHALLOW WATER WAVES

by

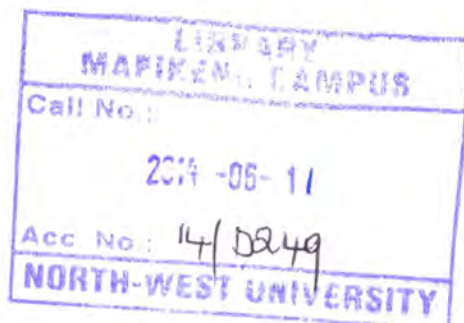
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Dissertation submitted for the degree of Master of Science in Applied
Mathematics in the Department of Mathematical Sciences in the
Faculty of Agriculture, Science and Technology at North-West
University, Mafikeng Campus

November 2013

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Declaration

I, DANIEL MPHO NKWANAZANA, student number 21398445, declare that this dissertation for the degree of Master of Science in Applied Mathematics at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other university, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

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This dissertation has been submitted with my approval as a University supervisor and would certify that the requirements for the applicable Master of Science degree rules and regulations have been fulfilled.

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Declaration of Publications

Details of contribution to publications that form part of this dissertation.

Chapter 3

D. M. Nkwanazana, B. Muatjetjeja, C. M. Khalique, Conservation Laws for a Generalized Coupled Korteweg-de Vries System, Mathematical Problems in Engineering, Volume 2013, Article ID 240797, 5 pages <http://dx.doi.org/10.1155/2013/240797>

Dedication

To my family and everyone who showed me support throughout my studies.

Acknowledgements

I would like to thank my supervisor, Professor C.M. Khalique and co-supervisor, Doctor B. Muatjetjeja for their guidance and encouragement in compiling this project. I also thank Mr A.R. Adem for his assistance.

Finally, I would also like to thank SANHARP, NRF and the North-West University, Mafikeng Campus for financial support.

Abstract

In this dissertation three coupled systems of partial differential equations, namely, third-order Boussinesq system of KdV-KdV type equations, third-order generalized coupled KdV system and first-order coupled system of one-layer shallow water waves are studied. Exact solutions for Boussinesq system of KdV-KdV type equations and a generalized coupled KdV equations are obtained using Lie group analysis in conjunction with the simplest equation method.

Furthermore, Noether theorem is employed to construct conservation laws for the three above mentioned coupled systems. Since these coupled systems are of third-order and first-order, they do not have a Lagrangian. Therefore we use the transformations $u = U_x$ and $v = V_x$ to increase their order by one. These new systems have Lagrangians and so the Noether approach can now be used to obtain conservation laws. Finally, the conservation laws are expressed in u and v variables. For each system, some local and infinitely many nonlocal conserved quantities are found.

List of Acronyms

PDEs:	Partial differential equations
ODEs:	Ordinary differential equations
KdV:	Korteweg-de Vries

Introduction

Most scientific problems that arise in mathematical physics are modeled by nonlinear partial differential equations (PDEs). A number of methods have been developed by various scientists for solving nonlinear PDEs. Some of the important techniques used in literature are homogeneous balance method [1], Jacobi elliptic function expansion method [2,3], Hirota's bilinear method [4], ansatz method [5,6] and variable separation approach [7].

The Lie group analysis [8–11], also known as the Lie symmetry method, was initiated by the Norwegian mathematician Sophus Lie (1842-1899) and is one of the useful techniques in finding solutions of (linear or nonlinear) differential equations. Lie symmetry method has a profound impact on all areas of mathematics, as well as physics, engineering and other mathematically based sciences. A symmetry is described generally as a change or a transformation that leaves an object unchanged. Recently, there have been considerable developments in the symmetry methods for differential equations as it is evident by the number of research papers, books and amount of new symbolic softwares devoted to the subject (see, for example, [8–18]).

In the study of differential equations, the concept of a conservation law, which is a mathematical formulation of the familiar physical laws of conservation of energy and conservation of momentum, plays an important role in the analysis of basic properties of the solutions. Conservation laws have been applied in the development and use of numerical methods [19,20]. Recently, conserved vectors associated with Lie point symmetries have been used to find exact solutions using double reduction method of some classical PDEs [21–24]. For variational problems, the Noether theorem

[25] provides an elegant way to construct conservation laws. The knowledge of a Lagrangian is important in finding Noether point symmetries and Noether conserved vectors. However, in the absence of a Lagrangian, there are other methods that can be applied to obtain the conserved vectors. See, for example, [26–33].

In this research we study three systems of PDEs, namely, Boussinesq system of KdV-KdV type equations, a generalized coupled KdV equations and coupled system of PDEs which describe one-layer shallow water waves.

Firstly, we study the Boussinesq system of KdV-KdV type equations [34]

$$\begin{aligned}u_t + v_x + v_{xxx} &= 0, \\v_t + u_x + u_{xxx} &= 0,\end{aligned}$$

where v is the elevation from the equilibrium and u describe the horizontal velocity of the fluid. Boussinesq system of KdV-KdV type equations describe the two-way propagation of small-amplitude gravity waves on the surface of water in a canal.

Secondly, we consider the coupled system

$$\begin{aligned}u_t + au_{xxx} + buu_x + cvv_x &= 0, \\v_t + dv_{xxx} + cuv_x + cu_xv &= 0,\end{aligned}$$

which is a special case of a generalized coupled KdV system given by [35]

$$\begin{aligned}u_t + au_{xxx} - buu_x + cvv_x &= 0, \\v_t + dv_{xxx} - evv_x + fu_xv &= 0,\end{aligned}$$

where a, b, c, d, e and f are real constants. The generalized coupled KdV system describes the interaction of two long waves, whose dispersion relations are different. These equations also models ion-acoustic waves and magnetohydrodynamic waves in plasmas, waves in elastic rods, mid-latitude, equatorial planetary waves and acoustic waves on a crystal lattice.

Thirdly, the coupled system of PDEs

$$\begin{aligned}v_t + vu_x + uv_x &= 0, \\u_t + uu_x + v_x &= 0,\end{aligned}$$

which describes one-layer shallow water waves [36] is studied. Here u is the horizontal component of the velocity and v is the amplitude. Shallow water equations are the simplest form of the equations of motion that can be used to describe the horizontal structure of an atmosphere. They describe the evolution of an incompressible fluid in response to gravitational and rotational accelerations [37].

The outline of this dissertation is as follows:

In Chapter one, the basic definitions and theorems concerning the one-parameter groups of transformations are presented.

In Chapter two, Lie group analysis is employed to find exact solutions of a Boussinesq system of KdV-KdV type equations. Thereafter, we derive conservation laws of the system using Noether theorem [25].

In Chapter three, exact solutions of a generalized coupled KdV equations are obtained with the aid of Lie symmetry method in conjunction with the simplest equation method [38–41]. Moreover, Noether theorem [25] is used to construct conservation laws of the system.

In Chapter four, the conservation laws of a coupled system describing one-layer shallow water waves are obtained using Noether theorem [25].

Finally, in Chapter five, a summary of the results of the dissertation are presented and future work will be discussed.

A bibliography is given at the end of this dissertation.

Chapter 1

Lie symmetry methods for partial differential equations

In this chapter we present basic Lie group theory and the algorithm for calculation of Lie point symmetries of partial differential equations. We also give some basic definitions and theorems concerning Noether point symmetries and conservation laws.

1.1 Introduction

Lie group analysis originated in the late nineteenth century by an outstanding mathematician Sophus Lie (1842-1899). He discovered that majority of the methods for solving differential equations could be explained and deduced simply by means of his theory which is based on the invariance of the differential equations under a continuous group of symmetries. The mathematical ideas of Lie's theory are presented in several books, e.g., P.J. Olver [8], G.W. Bluman and S. Kumei [9], Stephani [42] and Cantwell [43]. For more information on the definitions and results presented in this chapter the books mentioned above can be consulted.

1.2 Local continuous one-parameter Lie group

Let us take $x = (x^1, \dots, x^n)$ to be the independent variable with coordinates x^i and $u = (u^1, \dots, u^m)$ to be the dependent variable with coordinates u^α (n and m finite).

Definition 1.1 A set G of transformations

$$T_a : \bar{x}^i = f^i(x, u, a), \quad \bar{u}^\alpha = \phi^\alpha(x, u, a), \quad i = 1, \dots, n, \quad \alpha = 1, \dots, m, \quad (1.1)$$

where a is a real parameter which continuously takes values from a neighborhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$ and f^i, ϕ^α are differentiable functions, is called a *local continuous one-parameter Lie group of transformations* in the space of variables x and u if

- (i) For $T_a, T_b \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then $T_b T_a = T_c \in G$, $c = \phi(a, b) \in \mathcal{D}$ (Closure)
- (ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0 T_a = T_a T_0 = T_a$ (Identity), and
- (iii) For $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that $T_a T_{a^{-1}} = T_{a^{-1}} T_a = T_0$ (Inverse).

The associativity property follows from (i). The group property (i) can be written as

$$\begin{aligned} \bar{\bar{x}}^i &\equiv f^i(\bar{x}, \bar{u}, b) = f^i(x, u, \phi(a, b)), \\ \bar{\bar{u}}^\alpha &\equiv \phi^\alpha(\bar{x}, \bar{u}, b) = \phi^\alpha(x, u, \phi(a, b)) \end{aligned} \quad (1.2)$$

and the function ϕ is called the *group composition law*. A group parameter a is called *canonical* if $\phi(a, b) = a + b$.

Theorem 1.1 For any composition law $\phi(a, b)$, there exists the canonical parameter $\tilde{a}$ defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)}$$

where

$$w(s) = \left. \frac{\partial \phi(s, b)}{\partial b} \right|_{b=0}.$$

1.3 Prolongation formulas and Group generator

The derivatives of u with respect to x are defined as

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j D_i(u_i), \dots, \quad (1.3)$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots, \quad i = 1, \dots, n, \quad \alpha = 1, \dots, m \quad (1.4)$$

is the operator of total differentiation. The collection of all first derivatives u_i^α is denoted by $u_{(1)}$, i.e.,

$$u_{(1)} = \{u_i^\alpha\} \quad \alpha = 1, \dots, m, \quad i = 1, \dots, n.$$

Similarly

$$u_{(2)} = \{u_{ij}^\alpha\} \quad \alpha = 1, \dots, m, \quad i, j = 1, \dots, n$$

and $u_{(3)} = \{u_{ijk}^\alpha\}$ and likewise $u_{(4)}$ etc. Since $u_{ij}^\alpha = u_{ji}^\alpha$, $u_{(2)}$ contains only u_{ij}^α for $i \leq j$. In the same manner $u_{(3)}$ has only terms for $i \leq j \leq k$. There is natural ordering in $u_{(4)}, u_{(5)} \dots$.

In group analysis all variables $x, u, u_{(1)} \dots$ are considered functionally independent variables connected only by the differential relations (1.3). Thus the u_s^α are called differential variables and a p th-order partial differential equation is given as

$$E(x, u, u_{(1)}, \dots, u_{(p)}) = 0. \quad (1.5)$$

Prolonged or extended groups

If $z = (x, u)$, one-parameter group of transformations G is

$$\begin{aligned} \bar{x}^i &= f^i(x, u, a), \quad f^i|_{a=0} = x^i, \\ \bar{u}^\alpha &= \phi^\alpha(x, u, a), \quad \phi^\alpha|_{a=0} = u^\alpha. \end{aligned} \quad (1.6)$$

According to Lie's theory, the construction of the symmetry group G is equivalent to the determination of the corresponding *infinitesimal transformations*:

$$\bar{x}^i \approx x^i + a \xi^i(x, u), \quad \bar{u}^\alpha \approx u^\alpha + a \eta^\alpha(x, u) \quad (1.7)$$

obtained from (1.1) by expanding the functions f^i and ϕ^α into Taylor series in a about $a = 0$ and also taking into account the initial conditions

$$f^i|_{a=0} = x^i, \quad \phi^\alpha|_{a=0} = u^\alpha.$$

Thus, we have

$$\xi^i(x, u) = \left. \frac{\partial f^i}{\partial a} \right|_{a=0}, \quad \eta^\alpha(x, u) = \left. \frac{\partial \phi^\alpha}{\partial a} \right|_{a=0}. \quad (1.8)$$

One can now introduce the *symbol* of the infinitesimal transformations by writing (1.7) as

$$\bar{x}^i \approx (1 + a X)x, \quad \bar{u}^\alpha \approx (1 + a X)u,$$

where

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}. \quad (1.9)$$

The differential operator (1.9) is called the infinitesimal operator or generator of the group G .

Here we see how the derivatives are transformed.

The D_i transforms as

$$D_i = D_i(f^j) \bar{D}_j, \quad (1.10)$$

where $\bar{D}_j$ is the total differentiations in transformed variables $\bar{x}^i$. Therefore

$$\bar{u}_i^\alpha = \bar{D}_j(u^\alpha), \quad \bar{u}_{ij}^\alpha = \bar{D}_j(\bar{u}_i^\alpha) = \bar{D}_i(\bar{u}_j^\alpha), \dots,$$

and

$$\begin{aligned} D_i(\phi^\alpha) &= D_i(f^j) \bar{D}_j(\bar{u}^\alpha) \\ &= D_i(f^j) \bar{u}_j^\alpha. \end{aligned} \quad (1.11)$$

Hence

$$\left(\frac{\partial f^j}{\partial x^i} + u_i^\beta \frac{\partial f^j}{\partial u^\beta} \right) \bar{u}_j^\alpha = \frac{\partial \phi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \phi^\alpha}{\partial u^\beta}. \quad (1.12)$$

The quantities $\bar{u}_j^\alpha$ can be represented as functions of $x, u, u_{(i)}$, for small a , ie., (1.12) is locally invertible:

$$\bar{u}_i^\alpha = \psi_i^\alpha(x, u, u_{(1)}, a), \quad \psi_i^\alpha|_{a=0} = u_i^\alpha. \quad (1.13)$$

The transformations in $x, u, u_{(1)}$ space given by (1.6) and (1.13) form a one-parameter group (one can prove this but we do not consider the proof) called the first prolongation or just extension of the group G and denoted by $G^{[1]}$.

Let

$$\bar{u}_i^\alpha \approx u_i^\alpha + a\zeta_i^\alpha \quad (1.14)$$

to be the infinitesimal transformation of the first derivatives so that the infinitesimal transformation of the group $G^{[1]}$ is (1.7) and (1.14). Higher-order prolongations of G , viz. $G^{[2]}$, $G^{[3]}$ can be obtained by derivatives of (1.11).

Prolonged generators

Using (1.11) together with (1.7) and (1.14) we get

$$\begin{aligned} D_i(f^j)(\bar{u}_j^\alpha) &= D_i(\phi^\alpha) \\ D_i(x^j + a\xi^j)(u_j^\alpha + a\zeta_j^\alpha) &= D_i(u^\alpha + a\eta^\alpha) \\ (\delta_i^j + aD_i\xi^j)(h_j^\alpha + a\zeta_j^\alpha) &= h_i^\alpha + aD_i\eta^\alpha \\ u_i^\alpha + a\zeta_i^\alpha + au_j^\alpha D_i\xi^j &= u_i^\alpha + aD_i\eta^\alpha \\ \zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \quad (\text{sum on } j). \end{aligned} \quad (1.15)$$

This is called the first prolongation formula. Likewise, one can obtain the second prolongation, viz.,

$$\zeta_{ij}^\alpha = D_j(\eta_i^\alpha) - u_{ik}^\alpha D_j(\xi^k), \quad (\text{sum on } k). \quad (1.16)$$

By induction (recursively)

$$\zeta_{i_1, i_2, \dots, i_p}^\alpha = D_{i_p}(\zeta_{i_1, i_2, \dots, i_{p-1}}^\alpha) - u_{i_1, i_2, \dots, i_{p-1} j}^\alpha D_{i_p}(\xi^j), \quad (\text{sum on } j). \quad (1.17)$$

The first and higher prolongations of the group G form a group denoted by $G^{[1]}, \dots, G^{[p]}$.

The corresponding prolonged generators are

$$\begin{aligned} X^{[1]} &= X + \zeta_i^\alpha \frac{\partial}{\partial h_i^\alpha} \quad (\text{sum on } i, \alpha), \\ &\vdots \\ &\vdots \\ X^{[p]} &= X^{[p-1]} + \zeta_{i_1, \dots, i_p}^\alpha \frac{\partial}{\partial h_{i_1, \dots, i_p}^\alpha} \quad p \geq 1, \end{aligned}$$

where

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}.$$

1.4 Group admitted by a PDE

Definition 1.2 The vector field

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}, \quad (1.18)$$

is a *point symmetry* of the p th-order PDE (1.5), if

$$X^{[p]}(E) = 0 \quad (1.19)$$

whenever $E = 0$. This can also be written as

$$X^{[p]} E \Big|_{E=0} = 0, \quad (1.20)$$

where the symbol $|_{E=0}$ means evaluated on the equation $E = 0$.

Definition 1.3 Equation (1.19) is called the *determining equation* of (1.5) because it determines all the infinitesimal symmetries of (1.5).

Definition 1.4 (Symmetry group) A one-parameter group G of transformations (1.1) is called a symmetry group of equation (1.5) if (1.5) is form-invariant (has the same form) in the new variables $\bar{x}$ and $\bar{u}$, i.e.,

$$E(\bar{x}, \bar{u}, u_{(1)}^-, \dots, u_{(p)}^-) = 0, \quad (1.21)$$

where the function E is the same as in equation (1.5).

1.5 Group invariants

Definition 1.5 A function $F(x, u)$ is called an *invariant of the group of transformation* (1.1) if

$$F(\bar{x}, \bar{u}) \equiv F(f^i(x, u, a), \phi^\alpha(x, u, a)) = F(x, u), \quad (1.22)$$

identically in x, u and a .

Theorem 1.2 (Infinitesimal criterion of invariance) A necessary and sufficient condition for a function $F(x, u)$ to be an invariant is that

$$X F \equiv \xi^i(x, u) \frac{\partial F}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial F}{\partial u^\alpha} = 0. \quad (1.23)$$

It follows from the above theorem that every one-parameter group of point transformations (1.1) has $n - 1$ functionally independent invariants, which can be taken to be the left-hand side of any first integrals

$$J_1(x, u) = c_1, \dots, J_n(x, u) = c_n$$

of the characteristic equations

$$\frac{dx^1}{\xi^1(x, u)} = \dots = \frac{dx^n}{\xi^n(x, u)} = \frac{du^1}{\eta^1(x, u)} = \dots = \frac{du^n}{\eta^n(x, u)}.$$

Theorem 1.3 If the infinitesimal transformation (1.7) or its symbol X is given, then the corresponding one-parameter group G is obtained by solving the Lie equations

$$\frac{d\bar{x}^i}{da} = \xi^i(\bar{x}, \bar{u}), \quad \frac{d\bar{u}^\alpha}{da} = \eta^\alpha(\bar{x}, \bar{u}) \quad (1.24)$$

subject to the initial conditions

$$\bar{x}^i|_{a=0} = x, \quad \bar{u}^\alpha|_{a=0} = u.$$

1.6 Lie algebra

Let us consider two operators X_1 and X_2 defined by

$$X_1 = \xi_1^i(x, u) \frac{\partial}{\partial x^i} + \eta_1^\alpha(x, u) \frac{\partial}{\partial u^\alpha}$$

and

$$X_2 = \xi_2^i(x, u) \frac{\partial}{\partial x^i} + \eta_2^\alpha(x, u) \frac{\partial}{\partial u^\alpha}.$$

Definition 1.6 The *commutator* of X_1 and X_2 , written as $[X_1, X_2]$, is defined by $[X_1, X_2] = X_1(X_2) - X_2(X_1)$.

Definition 1.7 A Lie algebra is a vector space L (over the field of real numbers) of operators $X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}$ with the following property: If the operators

$$X_1 = \xi_1^i(x, u) \frac{\partial}{\partial x^i} + \eta_1^\alpha(x, u) \frac{\partial}{\partial u^\alpha}, \quad X_2 = \xi_2^i(x, u) \frac{\partial}{\partial x^i} + \eta_2^\alpha(x, u) \frac{\partial}{\partial u^\alpha}$$

are any elements of L , then their commutator

$$[X_1, X_2] = X_1(X_2) - X_2(X_1)$$

is also an element of L . It follows that the commutator is

1. Bilinear: for any $X, Y, Z \in L$ and $a, b \in \mathbb{R}$,

$$[aX + bY, Z] = a[X, Z] + b[Y, Z],$$

$$[X, aY + bZ] = a[X, Y] + b[X, Z];$$

2. Skew-symmetric: for any $X, Y \in L$,

$$[X, Y] = -[Y, X];$$

3. and satisfies the Jacobi identity: for any $X, Y, Z \in L$,

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

1.7 Fundamental relationship concerning the Noether theorem

In this section we briefly present the notation and pertinent results that will be used in this research. For details the reader is referred to [25, 44, 45]. Consider the system of k th order PDEs

$$E_\alpha(x, u, u_{(1)}, u_{(2)}, \dots, u_{(k)}) = 0, \quad \alpha = 1, 2, \dots, m, \quad (1.25)$$

if there exist a function $L(x, u, u_{(1)}, u_{(2)}, \dots, u_{(s)}) \in \mathcal{A}$ (space of differential functions), $s < k$ such that system (1.25) is equivalent to

$$\frac{\delta L}{\delta u^\alpha} = 0, \quad \alpha = 1, 2, \dots, m, \quad (1.26)$$

then L is called a Lagrangian of (1.25) and (1.26) are the corresponding Euler-Lagrange differential equations.

In (1.26), $\delta/\delta u^\alpha$ is the Euler-Lagrange operator defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \dots D_{i_s} \frac{\delta}{\delta u_{i_1 \dots i_s}^\alpha}, \quad \alpha = 1, \dots, m. \quad (1.27)$$

Definition 1.8 A Lie-Bäcklund operator X is a Noether symmetry generator associated with a Lagrangian L of (1.26) if there exist a vector $B = (B^1, \dots, B^n)$, $B^i \in \mathcal{A}$, such that

$$X(L) + LD_i(\xi^i) = D_i(B^i). \quad (1.28)$$

If in (1.28) $B^i = 0$, $i = 1, \dots, n$ then X is referred to as a strict Noether symmetry generator associated with Lagrangian $L \in \mathcal{A}$.

Definition 1.9 For each Noether symmetry generator X associated with a given Lagrangian L , there correspond a vector $T = (T^1, T^2, \dots, T^n)$, $T^i \in \mathcal{A}$, defined by

$$T^i = -B^i + N^i L, \quad i = 1, \dots, n, \quad (1.29)$$

which is a conserved vector for the Euler-Lagrange equations (1.26) and the Noether operator associated with X is

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s \geq 1} D_{i_1} \dots D_{i_s} (W^\alpha) \frac{\delta}{\delta u_{i_1 \dots i_s}^\alpha}, \quad i = 1, \dots, n, \quad (1.30)$$

in which the Euler-Lagrange operators with respect to derivatives of u^α are obtained from (1.27) by replacing u^α by the corresponding derivatives, e.g.,

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s \geq 1} (-1)^s D_{j_1} \dots D_{j_s} \frac{\delta}{\delta u_{ij_1 \dots j_s}^\alpha}, \quad i = 1, \dots, n, \quad \alpha = 1, \dots, m.$$

In (1.30), W^α is the Lie characteristic function given by

$$W^\alpha = \eta^\alpha - \xi^i u_j^\alpha, \quad \alpha = 1, \dots, m.$$

The vector (1.29) is a conserved vector for (1.25) if T^i satisfies

$$D_i T^i|_{(1.25)} = 0. \quad (1.31)$$

1.8 Conclusion

In this chapter we presented some basic definitions and results of the Lie group analysis including the algorithm to determine the Lie point symmetries of PDEs. We also briefly recalled the fundamental relations concerning Noether approach.

Chapter 2

Solutions and conservation laws of Boussinesq system of KdV-KdV type equations

In this chapter Lie group analysis is applied to find exact solution of Boussinesq system of KdV-KdV type equations [34]

$$\begin{aligned}u_t + v_x + v_{xxx} &= 0, \\v_t + u_x + u_{xxx} &= 0.\end{aligned}\tag{2.1}$$

Conservation laws of system (2.1) will also be constructed using Noether theorem.

The linearized coupled Boussinesq system of KdV-KdV type equations (2.1) was studied by Pazoto and Rosier [34] using classical semigroup theory to derive the exponential decay in the L^2 -norm. They designed a two-parameter family of feedback laws for which the solutions issuing from small data are globally defined and exponentially decreasing in the energy space.

2.1 Lie point symmetries of (2.1)

In this section we determine the Lie point symmetry of the coupled Boussinesq system of KdV-KdV type equations (2.1). The symmetry group of the coupled system (2.1) is generated by the vector field

$$X = \xi^1(t, x, u, v) \frac{\partial}{\partial t} + \xi^2(t, x, u, v) \frac{\partial}{\partial x} + \eta^1(t, x, u, v) \frac{\partial}{\partial u} + \eta^2(t, x, u, v) \frac{\partial}{\partial v}. \quad (2.2)$$

Applying the third prolongation $\text{pr}^{(3)}X$ [8] to (2.1) results in an overdetermined system of linear PDEs, namely

$$\xi_t^2 = 0, \quad (2.3)$$

$$\xi_u^2 = 0, \quad (2.4)$$

$$\xi_x^1 = 0, \quad (2.5)$$

$$\xi_v^2 = 0, \quad (2.6)$$

$$\xi_u^1 = 0. \quad (2.7)$$

$$\xi_v^1 = 0, \quad (2.8)$$

$$\eta_{uv}^2 = 0, \quad (2.9)$$

$$\eta_{uu}^1 = 0, \quad (2.10)$$

$$\eta_{vv}^2 = 0, \quad (2.11)$$

$$\eta_{uv}^1 = 0, \quad (2.12)$$

$$\eta_{vv}^1 = 0, \quad (2.13)$$

$$\eta_{xu}^2 = 0, \quad (2.14)$$

$$\eta_{uu}^2 = 0, \quad (2.15)$$

$$\eta_{xv}^1 = 0, \quad (2.16)$$

$$\eta_v^1 - \eta_u^2 = 0, \quad (2.17)$$

$$\eta_{xu}^1 - \xi_{xx}^2 = 0, \quad (2.18)$$

$$\eta_{xv}^2 - \xi_{xx}^2 = 0, \quad (2.19)$$

$$\eta_t^1 + \eta_{xxx}^2 + \eta_x^2 = 0, \quad (2.20)$$

$$-3\eta_{xxu}^1 + \xi_{xxx}^2 - 2\xi_x^2 = 0, \quad (2.21)$$

$$\eta_t^2 + \eta_{xxx}^1 + \eta_x^1 = 0, \quad (2.22)$$

$$-\eta_u^1 + \xi_t^1 + \eta_v^2 - 3\xi_x^2 = 0, \quad (2.23)$$

$$-\eta_u^1 - 3\eta_{xxu}^1 + \xi_x^2 + \eta_v^2 + \xi_{xxx}^2 - \xi_t^1 = 0, \quad (2.24)$$

$$-\eta_u^1 - \xi_{xxx}^2 + 3\eta_{xxv}^2 + \xi_t^1 + \eta_v^2 - \xi_x^2 = 0. \quad (2.25)$$

Solving the above system of linear PDEs for ξ^1 , ξ^2 , η^1 and η^2 with the aid of mathematical software, say Maple, we obtain:

$$\begin{aligned} \xi^1(t, x, u, v) &= C_4, \\ \xi^2(t, x, u, v) &= C_2, \\ \eta^1(t, x, u, v) &= C_1u + C_3v + F_1(t, x), \\ \eta^2(t, x, u, v) &= C_1v + C_3u + F_2(t, x), \end{aligned}$$

where $C_1, \dots, C_4$ are arbitrary constants and $F_1(t, x)$, $F_2(t, x)$ are arbitrary solutions of the Boussinesq system of KdV-KdV type equations (2.1). Thus, the Lie point symmetries of the coupled system (2.1) are spanned by the following vector fields:

$$\begin{aligned} X_1 &= u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= v \frac{\partial}{\partial u} + u \frac{\partial}{\partial v}, \\ X_4 &= \frac{\partial}{\partial t}, \\ X_{F_1} &= F_1(t, x) \frac{\partial}{\partial u}, \\ X_{F_2} &= F_2(t, x) \frac{\partial}{\partial v}, \end{aligned}$$

where $F_1(t, x)$ and $F_2(t, x)$ are arbitrary solutions of (2.1).

2.1.1 Symmetry reduction and exact solution of (2.1)

In this subsection we use symmetries obtained in the above section to reduce the coupled system (2.1) into system of ODEs. In order to find symmetry reductions and exact solutions, one has to solve the characteristic equation

$$\frac{dt}{\xi^1(t, x, u, v)} = \frac{dx}{\xi^2(t, x, u, v)} = \frac{du}{\eta^1(t, x, u, v)} = \frac{dv}{\eta^2(t, x, u, v)}$$

and then substitute the resulting expression into (2.1).

Firstly we take symmetry generator $kX_2 + X_4$ which is a linear combination of translational symmetries, where k is a constant. The characteristic equations for the invariants of the symmetry generator $kX_2 + X_4$ are

$$\frac{dt}{1} = \frac{dx}{k} = \frac{du}{0} = \frac{dv}{0}. \quad (2.26)$$

Solving (2.26) gives the following invariants

$$u(t, x) = E(\chi), \quad v(t, x) = F(\chi), \quad \chi = x - kt. \quad (2.27)$$

Using (2.27) the Boussinesq system of KdV-KdV type equations (2.1) reduces to the following system of ODEs:

$$F'''(\chi) + F'(\chi) - kE'(\chi) = 0, \quad (2.28a)$$

$$E'''(\chi) + E'(\chi) - kF'(\chi) = 0. \quad (2.28b)$$

Integrating (2.28b) and taking $F(\chi)$ as a subject of the equation we obtain

$$F(\chi) = \frac{1}{k}(E(\chi) + E''(\chi)) + C, \quad (2.29)$$

where C is an arbitrary constant of integration. Substituting (2.29) into (2.28a) gives the following fifth-order ODE

$$(1 - k^2)E'(\chi) + 2E'''(\chi) + E^{(5)}(\chi) = 0. \quad (2.30)$$

Solving (2.30) and substituting the results into (2.29) and reverting back to the original variables we obtain the following solutions of (2.1)

$$\begin{aligned} u(t, x) &= \frac{1}{A}(C_1 e^{A(x-kt)} - C_2 e^{-A(x-kt)}) + \frac{1}{B}(C_3 e^{B(x-kt)} - C_4 e^{-B(x-kt)}) + C_5, \\ v(t, x) &= \frac{1}{A}(-C_1 e^{A(x-kt)} + C_2 e^{-A(x-kt)}) + \frac{1}{B}(C_3 e^{B(x-kt)} - C_4 e^{-B(x-kt)}) + S, \end{aligned}$$

where $A = \sqrt{-k-1}$, $B = \sqrt{k-1}$, $S = \frac{C_5}{k} + C$ and $C_1, \dots, C_5$ are constants of integration.

Secondly we take symmetry generator $X_4 + kX_2 + X_1$, where k is a constant. The characteristic system for the invariants of the symmetry generator $X_4 + kX_2 + X_1$ are

$$\frac{dt}{1} = \frac{dx}{k} = \frac{du}{u} = \frac{dv}{v}. \quad (2.31)$$

The integration of equation (2.31) yields the following invariants:

$$u = e^t E(\chi), \quad v = e^t F(\chi), \quad \chi = x - kt. \quad (2.32)$$

Substituting (2.32) into system (2.1) we obtain the reduced system of ODEs:

$$\begin{aligned} e^t (E(\chi) - kE'(\chi) + F'(\chi) + F'''(\chi)) &= 0, \\ e^t (F(\chi) - kF'(\chi) + E'(\chi) + E'''(\chi)) &= 0. \end{aligned} \quad (2.33)$$

Since $e^t \neq 0$, we have

$$\begin{aligned} E(\chi) - kE'(\chi) + F'(\chi) + F'''(\chi) &= 0, \\ F(\chi) - kF'(\chi) + E'(\chi) + E'''(\chi) &= 0. \end{aligned} \quad (2.34)$$

Lastly, we take symmetry X_4 . The corresponding characteristic equations for the invariant of X_4 are

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{0} = \frac{dv}{0}. \quad (2.35)$$

Solving (2.35) gives the following invariants:

$$u(t, x) = E(\chi), \quad v(t, x) = F(\chi), \quad \chi = x. \quad (2.36)$$

Substituting (2.36) into system (2.1) we obtain the following reduced uncoupled system of ODEs:

$$\begin{aligned} F'(\chi) + F'''(\chi) &= 0, \\ E'(\chi) + E'''(\chi) &= 0. \end{aligned} \quad (2.37)$$

Solving (2.37) and returning back to the original variables we obtain the following group-invariant solutions of (2.1)

$$\begin{aligned} u(t, x) &= C_1 \sin x - C_2 \cos x + C_3, \\ v(t, x) &= C_4 \sin x - C_5 \cos x + C_6, \end{aligned} \quad (2.38)$$

where $C_1, \dots, C_6$ are arbitrary constants of integration.

2.2 Conservation laws for (2.1)

In this section we construct the conservation laws for the Boussinesq system of KdV-KdV type equations (2.1). We note that the coupled system (2.1) does not have a Lagrangian. In order to employ the Noether theorem we transform system (2.1) to fourth-order system using transformations $u = U_x$ and $v = V_x$, then the coupled system (2.1) becomes

$$\begin{aligned} U_{tx} + V_{xx} + V_{xxxx} &= 0, \\ V_{tx} + U_{xx} + U_{xxxx} &= 0. \end{aligned} \quad (2.39)$$

We note that coupled system (2.39) possess two Lagrangians given by

$$L = 2U_{xx}V_{xx} - 2U_xV_x - U_tU_x - V_tV_x \quad (2.40)$$

and

$$L = \frac{1}{2}(U_{xx}^2 + V_{xx}^2 - U_x^2 - V_x^2 - U_xV_t - U_tV_x). \quad (2.41)$$

It can be verified that the two second-order Lagrangians (2.40) and (2.41) satisfies the Euler-Lagrange equations. Thus

$$\frac{\delta L}{\delta U} = 0 \quad \text{and} \quad \frac{\delta L}{\delta V} = 0, \quad (2.42)$$

where $\delta/\delta U$ and $\delta/\delta V$ are defined by

$$\frac{\delta}{\delta U} = \frac{\partial}{\partial U} - D_t \frac{\partial}{\partial U_t} - D_x \frac{\partial}{\partial U_x} + D_t^2 \frac{\partial}{\partial U_{tt}} + D_x^2 \frac{\partial}{\partial U_{xx}} + D_x D_t \frac{\partial}{\partial U_{tx}} - \dots, \quad (2.43)$$

$$\frac{\delta}{\delta V} = \frac{\partial}{\partial V} - D_t \frac{\partial}{\partial V_t} - D_x \frac{\partial}{\partial V_x} + D_t^2 \frac{\partial}{\partial V_{tt}} + D_x^2 \frac{\partial}{\partial V_{xx}} + D_x D_t \frac{\partial}{\partial V_{tx}} - \dots, \quad (2.44)$$

and

$$D_t = \frac{\partial}{\partial t} + U_t \frac{\partial}{\partial U} + V_t \frac{\partial}{\partial V} + U_{tt} \frac{\partial}{\partial U_t} + V_{tt} \frac{\partial}{\partial V_t} + U_{tx} \frac{\partial}{\partial U_x} + V_{tx} \frac{\partial}{\partial V_x} + \dots,$$

$$D_x = \frac{\partial}{\partial x} + U_x \frac{\partial}{\partial U} + V_x \frac{\partial}{\partial V} + U_{xx} \frac{\partial}{\partial U_x} + V_{xx} \frac{\partial}{\partial V_x} + U_{tx} \frac{\partial}{\partial U_t} + V_{tx} \frac{\partial}{\partial V_t} + \dots.$$

We now show the calculations which verifies that the L given by (2.40) satisfies system (2.42).

$$\begin{aligned} \frac{\delta L}{\delta U} &= D_t(U_x) + D_x(U_t + 2V_x) + 2D_x^2(V_{xx}) \\ &= 2(U_{tx} + V_{xx} + V_{xxx}) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \frac{\delta L}{\delta V} &= D_t(V_x) + D_x(V_t + 2U_x) + 2D_x^2(U_{xx}) \\ &= 2(V_{tx} + U_{xx} + U_{xxx}) \\ &= 0. \end{aligned}$$

Hence L given by (2.40) is a Lagrangian for system (2.39). The L given by (2.41) can also be verified using the same procedure.

Consider the vector field

$$\begin{aligned} X &= \xi^1(t, x, U, V) \frac{\partial}{\partial t} + \xi^2(t, x, U, V) \frac{\partial}{\partial x} + \eta^1(t, x, U, V) \frac{\partial}{\partial U} \\ &\quad + \eta^2(t, x, U, V) \frac{\partial}{\partial V}, \end{aligned} \quad (2.45)$$

which have the second-order prolongation defined by

$$\begin{aligned} X^{[2]} &= \xi^1(t, x, U, V) \frac{\partial}{\partial t} + \xi^2(t, x, U, V) \frac{\partial}{\partial x} + \eta^1(t, x, U, V) \frac{\partial}{\partial U} \\ &\quad + \eta^2(t, x, U, V) \frac{\partial}{\partial V} + \zeta_t^1 \frac{\partial}{\partial U_t} + \zeta_t^2 \frac{\partial}{\partial V_t} + \zeta_x^1 \frac{\partial}{\partial U_x} + \zeta_x^2 \frac{\partial}{\partial V_x} + \dots, \end{aligned} \quad (2.46)$$

where

$$\zeta_t^1 = D_t(\eta^1) - U_t D_t(\xi^1) - U_x D_t(\xi^2), \quad (2.47)$$

$$\zeta_x^1 = D_x(\eta^1) - U_t D_x(\xi^1) - U_x D_x(\xi^2), \quad (2.48)$$

$$\zeta_t^2 = D_t(\eta^2) - V_t D_t(\xi^1) - V_x D_t(\xi^2), \quad (2.49)$$

$$\zeta_x^2 = D_x(\eta^2) - V_t D_x(\xi^1) - V_x D_x(\xi^2). \quad (2.50)$$

The Lie-Bäcklund operator X defined in (2.45) is a Noether operator corresponding to the Lagrangian L if it satisfies

$$X(L) + L[D_t(\xi^1) + D_x(\xi^2)] = D_t(B^1) + D_x(B^2), \quad (2.51)$$

where $B^1(t, x, U, V)$ and $B^2(t, x, U, V)$ are the gauge terms. The expansion of (2.51) together with Lagrangian (2.40) gives

$$\begin{aligned} & -U_x[\eta_t^1 + U_t \eta_U^1 + V_t \eta_V^1 - U_t \xi_t^1 - U_t^2 \xi_U^1 - U_t V_t \xi_V^1 - U_x \xi_t^2 - U_x U_t \xi_U^2 \\ & -U_x V_t \xi_V^2] - V_x[\eta_t^2 + U_t \eta_U^2 + V_t \eta_V^2 - V_t \xi_t^1 - U_t V_t \xi_U^1 - V_t^2 \xi_V^1 - V_x \xi_t^2 \\ & -U_t V_x \xi_U^2 - V_t V_x \xi_V^2] - (2V_x + U_t)[\eta_x^1 + U_x \eta_U^1 + V_x \eta_V^1 - U_t \xi_x^1 - U_t U_x \xi_U^1 \\ & -U_t V_x \xi_V^1 - U_x \xi_x^2 - U_x^2 \xi_U^2 - U_x V_x \xi_V^2] - (V_t + 2U_x)[\eta_x^2 + U_x \eta_U^2 + V_x \eta_V^2 \\ & -V_t \xi_x^1 - U_x V_t \xi_U^1 - V_t V_x \xi_V^1 - V_x \xi_x^2 - U_x V_x \xi_U^2 - V_x^2 \xi_V^2] + 2V_{xx}[D_x^2(\eta^1) \\ & -U_t D_x^2(\xi^1) - U_x D_x^2(\xi^2) - 2U_{tx}(\xi_x^1 + U_x \xi_U^1 + V_x \xi_V^1) \\ & -2U_{xx}(\xi_x^2 + U_x \xi_U^2 + V_x \xi_V^2)] + 2U_{xx}[D_x^2(\eta^2) - V_t D_x^2(\xi^1) - V_x D_x^2(\xi^2) \\ & -2V_{tx}(\xi_x^1 + U_x \xi_U^1 + V_x \xi_V^1) - 2V_{xx}(\xi_x^2 + U_x \xi_U^2 + V_x \xi_V^2)] \\ & + [2U_{xx}V_{xx} - 2U_x V_x - U_t U_x - V_t V_x][\xi_t^1 + U_t \xi_U^1 + V_t \xi_V^1 + \xi_x^2 + U_x \xi_U^2 \\ & + V_x \xi_V^2] = B_t^1 + U_t B_U^1 + V_t B_V^1 + B_x^2 + U_x B_U^2 + V_x B_V^2. \end{aligned} \quad (2.52)$$

Splitting (2.52) with respect to different combinations of derivatives of U and V results in an overdetermined system of linear PDEs:

$$\xi_U^1 = 0, \quad (2.53)$$

$$\xi_V^1 = 0, \quad (2.54)$$

$$\xi_t^1 = 0, \quad (2.55)$$

$$\xi_x^1 = 0, \quad (2.56)$$

$$\xi_U^2 = 0, \quad (2.57)$$

$$\xi_V^2 = 0, \quad (2.58)$$

$$\xi_x^2 = 0, \quad (2.59)$$

$$\xi_t^2 = 0, \quad (2.60)$$

$$\eta_U^1 = 0, \quad (2.61)$$

$$\eta_V^1 = 0, \quad (2.62)$$

$$\eta_U^2 = 0, \quad (2.63)$$

$$\eta_V^2 = 0. \quad (2.64)$$

$$\eta_{xx}^1 = 0, \quad (2.65)$$

$$\eta_{xx}^2 = 0, \quad (2.66)$$

$$B_U^1 = -\eta_x^1 \quad (2.67)$$

$$B_V^1 = -\eta_x^2 \quad (2.68)$$

$$B_U^2 = -\eta_t^1 - 2\eta_x^2 \quad (2.69)$$

$$B_V^2 = -\eta_t^2 - 2\eta_x^1 \quad (2.70)$$

$$B_t^1 + B_x^2 = 0. \quad (2.71)$$

We now solve the above system of linear PDEs for ξ^1 , ξ^2 , η^1 , η^2 , B^1 and B^2 .

From equations (2.53)-(2.56) we obtain

$$\xi^1(t, x, U, V) = A_1,$$

where A_1 is an arbitrary constant. Solving equations (2.57)-(2.60) we obtain

$$\xi^1(t, x, U, V) = A_2,$$

where A_2 is an arbitrary constant. Integrating (2.65) twice with respect to x gives

$$\eta^1(t, x, U, V) = c(t)x + d(t), \quad (2.72)$$

where $c(t)$ and $d(t)$ are arbitrary functions. Therefore, (2.61) and (2.62) are satisfied.

The integration of (2.66) twice with respect to x yields

$$\eta^2(t, x, U, V) = e(t)x + f(t), \quad (2.73)$$

where $e(t)$ and $f(t)$ are arbitrary functions. Thus, (2.63) and (2.64) are satisfied. Differentiating equations (2.72) and (2.73) with respect to x respectively and substituting the results into (2.67) and (2.68) we get

$$B_U^1 = -c(t), \quad (2.74)$$

$$B_V^1 = -e(t). \quad (2.75)$$

Integrating (2.74) with respect to U we obtain

$$B^1(t, x, U, V) = -c(t)U + K(t, x, V), \quad (2.76)$$

where $K(t, x, V)$ is an arbitrary function. Differentiating (2.76) with respect to V and substituting the results into (2.75) we obtain

$$K_V = -e(t). \quad (2.77)$$

The integration of (2.77) with respect to V gives

$$K = -e(t)V + H(t, x). \quad (2.78)$$

Substituting (2.78) into (2.76) yields

$$B^1(t, x, U, V) = -[c(t)U + e(t)V] + H(t, x), \quad (2.79)$$

where $H(t, x)$ is an arbitrary function. Solving equations (2.69) and (2.70) and use equations (2.72) and (2.73) we obtain

$$\begin{aligned} B^2(t, x, U, V) = & -[c'(t)x + d'(t)]U - 2e(t)U - [e'(t)x + f'(t)]V \\ & - 2c(t)V + P(t, x), \end{aligned} \quad (2.80)$$

where $P(t, x)$ is an arbitrary function.

Differentiating (2.79) and (2.80) with respect to t and x respectively and substituting the results into (2.71) we obtain

$$-2c'(t)U - 2e'(t)V + H_t + P_x = 0. \quad (2.81)$$

Splitting equation (2.81) with respect to U and V we get

$$c'(t) = 0, \quad (2.82)$$

$$e'(t) = 0, \quad (2.83)$$

$$H_t + P_x = 0. \quad (2.84)$$

Integrating equations (2.82) and (2.83) yields

$$c(t) = A_3, \quad e(t) = A_4, \quad (2.85)$$

where A_3 and A_4 are arbitrary constants. The substitution of (2.85) into (2.72), (2.73), (2.79) and (2.80) gives

$$\eta^1(t, x, U, V) = A_3x + d(t),$$

$$\eta^2(t, x, U, V) = A_4x + f(t),$$

$$B^1(t, x, U, V) = -(A_3U + A_4V) + H(t, x),$$

$$B^2(t, x, U, V) = -[d'(t)U + f'(t)V] - 2A_3V - 2A_4U + P(t, x).$$

Thus the general solutions of system (2.53)-(2.71) are

$$\begin{aligned} \xi^1 &= A_1, \quad \xi^2 = A_2, \quad \eta^1 = A_3x + d(t), \quad \eta^2 = A_4x + f(t), \\ B^1 &= -A_3U - A_4V + H(t, x), \\ B^2 &= -d'(t)U - f'(t)V - 2A_3V - 2A_4U + P(t, x), \\ H_t + P_x &= 0. \end{aligned} \quad (2.86)$$

We can choose $H(t, x) = P(t, x) = 0$ as they contribute to the trivial part of the conserved vectors. Hence the Noether symmetries and gauge functions are

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \quad B^1 = B^2 = 0, \\ X_2 &= \frac{\partial}{\partial x}, \quad B^1 = B^2 = 0, \\ X_3 &= x \frac{\partial}{\partial U}, \quad B^1 = -U, \quad B^2 = -2V, \\ X_4 &= x \frac{\partial}{\partial V}, \quad B^1 = -V, \quad B^2 = -2U, \\ X_{d(t)} &= d(t) \frac{\partial}{\partial U}, \quad B^1 = 0, \quad B^2 = -Ud'(t), \\ X_{f(t)} &= f(t) \frac{\partial}{\partial V}, \quad B^1 = 0, \quad B^2 = -Vf'(t). \end{aligned}$$

The above results will now be used to find the components of the conserved vectors.

We now recall the formula for the conserved vectors [25, 46], viz.,

$$\begin{aligned}
T^1 = & -B^1 + \xi^1 L + W^1 \left[\frac{\partial L}{\partial U_t} - D_t \frac{\partial L}{\partial U_{tt}} - D_x \frac{\partial L}{\partial U_{tx}} + \dots \right] \\
& + W^2 \left[\frac{\partial L}{\partial V_t} - D_t \frac{\partial L}{\partial V_{xt}} - D_x \frac{\partial L}{\partial V_{tt}} + \dots \right] \\
& + D_t(W^1) \frac{\partial L}{\partial U_{tt}} + D_t(W^2) \frac{\partial L}{\partial V_{tt}}, \tag{2.87}
\end{aligned}$$

$$\begin{aligned}
T^2 = & -B^2 + \xi^2 L + W^1 \left[\frac{\partial L}{\partial U_x} - D_t \frac{\partial L}{\partial U_{xt}} - D_x \frac{\partial L}{\partial U_{xx}} + \dots \right] \\
& + W^2 \left[\frac{\partial L}{\partial V_x} - D_t \frac{\partial L}{\partial V_{xt}} - D_x \frac{\partial L}{\partial V_{xx}} + \dots \right] \\
& + D_x(W^1) \frac{\partial L}{\partial U_{xx}} + D_x(W^2) \frac{\partial L}{\partial V_{xx}}, \tag{2.88}
\end{aligned}$$

where $W^1 = \eta^1 - U_t \xi^1 - U_x \xi^2$ and $W^2 = \eta^2 - V_t \xi^1 - V_x \xi^2$ are the Lie characteristic functions. Employing equations (2.87), (2.88) and X_1 we obtain the following independent conserved vector

$$\begin{aligned}
T_1^1 &= 2(U_{xx} V_{xx} - U_x V_x), \\
T_1^2 &= 2U_t(V_x + V_{xxx}) + U_t^2 + 2V_t(U_x + U_{xxx}) + V_t^2 - 2U_{tx} V_{xx} \\
&\quad - 2V_{tx} U_{xx}. \tag{2.89}
\end{aligned}$$

Using the inverse transformations $U = \int u dx$ and $V = \int v dx$ into (2.89) yields

$$\begin{aligned}
T_1^1 &= 2(u_x v_x - uv), \\
T_1^2 &= -2(u_x v_t + u_t v_x) + 2(u + u_{xx}) \int v_t dx + 2(v + v_{xx}) \int u_t dx + \\
&\quad \int v_t dx \int v_t dx + \int u_t dx \int u_t dx. \tag{2.90}
\end{aligned}$$

Likewise, for symmetries X_2 , X_3 , X_4 , $X_{d(t)}$ and $X_{f(t)}$ the following independent conserved vectors for system (2.1) are obtained:

$$\begin{aligned}
T_2^1 &= u^2 + v^2, \\
T_2^2 &= 2[uv_{xx} + vu_{xx} - u_x v_x + uv], \tag{2.91}
\end{aligned}$$

$$\begin{aligned}
T_3^1 &= -xu + \int u dx, \\
T_3^2 &= 2[v_x - x(v + v_{xx})] - x \int u_t dx + 2 \int v dx,
\end{aligned} \tag{2.92}$$

$$\begin{aligned}
T_4^1 &= -xv + \int v dx, \\
T_4^2 &= 2[u_x - x(u + u_{xx})] - x \int v_t dx + 2 \int u dx,
\end{aligned} \tag{2.93}$$

$$\begin{aligned}
T_{(d,f)}^1 &= -ud(t) - vf(t), \\
T_{(d,f)}^2 &= d'(t) \int u dx + f'(t) \int v dx - d(t)(2v + 2v_{xx} + \int u_t dx) \\
&\quad - f(t)(2u + 2u_{xx} + \int v_t dx).
\end{aligned} \tag{2.94}$$

We note that the conserved vector (2.91) is a local conserved vector and (2.90), (2.92), (2.93) are nonlocal conserved vectors for the system (2.1). We now find two special cases from the conserved vector (2.94) by setting $d(t) = 1$ and $f(t) = 0$, which yields a nonlocal conserved vector

$$\begin{aligned}
T_5^1 &= -u, \\
T_5^2 &= -2(v + v_{xx}) - \int u_t dx,
\end{aligned} \tag{2.95}$$

and by letting $d(t) = 0$ and $f(t) = 1$ we obtain a nonlocal conserved vector

$$\begin{aligned}
T_6^1 &= -v, \\
T_6^2 &= -2(u + u_{xx}) - \int v_t dx.
\end{aligned} \tag{2.96}$$

Here we note that for different values of the arbitrary function $d(t)$ and $f(t)$, we obtain infinitely many nonlocal conserved vectors.

We now follow the same procedure as above for Lagrangian (2.41) and obtain the values of $\xi^1, \xi^2, \eta^1, \eta^2, B^1$ and B^2 , viz.,

$$\begin{aligned}
\xi^1 &= A_1, \quad \xi^2 = A_2, \quad \eta^1 = A_3x + d(t), \quad \eta^2 = A_4x + f(t), \\
B^1 &= -\frac{1}{2}A_3V - \frac{1}{2}A_4U + H(t, x), \\
B^2 &= -\frac{1}{2}Vd'(t) - \frac{1}{2}Uf'(t) - A_3U - A_4V + P(t, x), \\
H_t + P_x &= 0.
\end{aligned} \tag{2.97}$$

Again we can set $H(t, x) = P(t, x) = 0$ as they contribute to the trivial parts of the conserved vectors. Therefore, we have the following Noether point symmetries and gauge terms:

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, & B^1 &= B^2 = 0, \\
X_2 &= \frac{\partial}{\partial x}, & B^1 &= B^2 = 0, \\
X_3 &= x \frac{\partial}{\partial U}, & B^1 &= -\frac{1}{2}V, & B^2 &= -U, \\
X_4 &= x \frac{\partial}{\partial V}, & B^1 &= -\frac{1}{2}U, & B^2 &= -V, \\
X_{d(t)} &= d(t) \frac{\partial}{\partial U}, & B^1 &= 0, & B^2 &= -\frac{1}{2}V d'(t), \\
X_{f(t)} &= f(t) \frac{\partial}{\partial V}, & B^1 &= 0, & B^2 &= -\frac{1}{2}U f'(t).
\end{aligned}$$

The associated independent conserved vectors for the coupled system (2.1) are:

$$\begin{aligned}
T_1^1 &= \frac{1}{2}(u_x^2 + v_x^2 - u^2 - v^2), \\
T_1^2 &= -u_x u_t - v_t v_x + (v + v_{xx}) \int v_t dx + (u + u_{xx}) \int u_t dx + \\
&\quad \int u_t dx \int v_t dx,
\end{aligned} \tag{2.98}$$

$$\begin{aligned}
T_2^1 &= uv, \\
T_2^2 &= uu_{xx} + vv_{xx} + \frac{1}{2}(u^2 + v^2 - u_x^2 - v_x^2),
\end{aligned} \tag{2.99}$$

$$\begin{aligned}
T_3^1 &= -\frac{1}{2}xv + \frac{1}{2} \int v dx, \\
T_3^2 &= u_x - x(u + u_{xx}) + \int u dx - \frac{1}{2}x \int v_t dx,
\end{aligned} \tag{2.100}$$

$$\begin{aligned}
T_4^1 &= -\frac{1}{2}xu + \frac{1}{2} \int u dx, \\
T_4^2 &= v_x - x(v + v_{xx}) + \int v dx - \frac{1}{2}x \int u_t dx,
\end{aligned} \tag{2.101}$$

$$\begin{aligned}
T_{(d,f)}^1 &= -\frac{1}{2}(vd(t) + uf(t)), \\
T_{(d,f)}^2 &= \frac{1}{2}d'(t) \int v dx + \frac{1}{2}f'(t) \int u dx - d(t)(u + u_{xx} + \frac{1}{2} \int v_t dx) \\
&\quad - f(t)(v + v_{xx} + \frac{1}{2} \int u_t dx).
\end{aligned} \tag{2.102}$$

Here we note that the conserved vector (2.99) is a local conserved vector and (2.98), (2.100), (2.101) are nonlocal conserved vectors for the system (2.1). Again we can

extract two particular cases from conserved vector (2.102) by taking $d(t) = 1$ and $f(t) = 0$ which gives a nonlocal conserved vector

$$\begin{aligned} T_5^1 &= -\frac{1}{2}v, \\ T_{(5)}^2 &= -u - u_{xx} - \frac{1}{2} \int v_t dx, \end{aligned} \quad (2.103)$$

and by letting $d(t) = 0$ and $f(t) = 1$ we get a nonlocal conserved vector

$$\begin{aligned} T_{(6)}^1 &= -\frac{1}{2}u, \\ T_{(6)}^2 &= -v - v_{xx} - \frac{1}{2} \int u_t dx. \end{aligned} \quad (2.104)$$

Again we note that for arbitrary values of $d(t)$ and $f(t)$ infinitely many nonlocal conservation laws exist for system (2.1).

2.3 Conclusion

In this chapter we studied the third-order coupled Boussinesq system of KdV-KdV type equations (2.1). The Lie group analysis was applied to find exact solutions for the coupled system (2.1). Thereafter, Noether theorem was used to determine the conservation laws for (2.1).

Chapter 3

Solutions and conservation laws for a generalized coupled KdV equations

The generalized coupled KdV system given by [35]

$$\begin{aligned}u_t + au_{xxx} - buu_x + cvv_x &= 0, \\v_t + dv_{xxx} - evv_x + fu_xv &= 0,\end{aligned}\tag{3.1}$$

where a, b, c, d, e and f are real constants, describes the interaction of two long waves, whose dispersion relations are different. For the case, when $f = 0$, soliton solutions have been obtained in [47, 48]. Many other special cases of (3.1) have been considered in literature and various methods have been used to find its exact solutions. See, for example, [49–56].

In this chapter we consider a special case of the generalized coupled KdV system, namely

$$\begin{aligned}u_t + au_{xxx} + buu_x + cvv_x &= 0, \\v_t + dv_{xxx} + cuv_x + cu_xv &= 0,\end{aligned}\tag{3.2}$$

and find exact solutions and conservation laws for (3.2). The conservation laws of system (3.2) for special values of the constants $a = d = -1$ and $b = c = -6$ were

derived in [57] using the multiplier approach.

In this chapter we use Lie symmetry method together with simplest equation method [38,39] to obtain exact solutions of the generalized coupled KdV equations (3.2). We also derive the conservation laws for the generalized coupled KdV equations (3.2) using Noether theorem [25]. This work has been published in [58].

3.1 Exact solutions of (3.2)

3.1.1 Lie point symmetries and one-dimensional optimal system of (3.2)

The symmetry group of the generalized coupled KdV equations (3.2) will be generated by the vector field of the form

$$X = \xi^1(t, x, u, v) \frac{\partial}{\partial t} + \xi^2(t, x, u, v) \frac{\partial}{\partial x} + \eta^1(t, x, u, v) \frac{\partial}{\partial u} + \eta^2(t, x, u, v) \frac{\partial}{\partial v}. \quad (3.3)$$

Applying the third-order prolongation $\text{pr}^{(3)}X$ to (3.2) and solving the resultant overdetermined system of PDEs we obtain the following three Lie point symmetries:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= 3t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - 2u \frac{\partial}{\partial u} - 2v \frac{\partial}{\partial v}, \\ X_3 &= \frac{\partial}{\partial x}. \end{aligned}$$

We now compute the commutation relations for the above Lie point symmetries. By definition of the Lie bracket, we have

$$[X_i, X_j] = X_i X_j - X_j X_i, \quad (3.4)$$

where the subscripts i and j take values from 1 to 3. Using (3.4), we calculate the commutator $[X_1, X_2]$, viz.,

$$\begin{aligned} [X_1, X_2] &= X_1 X_2 - X_2 X_1 \\ &= \frac{\partial}{\partial t} \left(3t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - 2u \frac{\partial}{\partial u} - 2v \frac{\partial}{\partial v} \right) - \left(3t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - 2u \frac{\partial}{\partial u} - 2v \frac{\partial}{\partial v} \right) \frac{\partial}{\partial t} \\ &= 3X_1. \end{aligned}$$

Likewise, one can obtain the commutation relations between other vector fields. In Table 1 below we present the commutator table of the Lie algebra of coupled system (3.2). Here the entry in row i and column j is represented by $[X_i, X_j]$:

Table 1.

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	$3X_1$	0
X_2	$-3X_1$	0	$-X_3$
X_3	0	X_3	0

Using Table 1 of commutators and the formula

$$Ad(\exp(\varepsilon X_i))X_j = X_j - \varepsilon[X_i, X_j] + \frac{\varepsilon^2}{2!}[X_i, [X_i, X_j]] + \cdots,$$

where i and j take values from 1 to 3, we find the adjoint representation of X_i . Here we will show the detailed calculations for the following adjoint representation:

$$\begin{aligned} Ad(\exp(\varepsilon X_1))X_2 &= X_2 - \varepsilon[X_1, X_2] + \frac{\varepsilon^2}{2}[X_1, [X_1, X_2]] + \cdots \\ &= X_2 - 3\varepsilon X_1 + \frac{\varepsilon^2}{2}[X_1, 3X_1] + \cdots \\ &= X_2 - 3\varepsilon X_1 \end{aligned}$$

and

$$\begin{aligned} Ad(\exp(\varepsilon X_2))X_3 &= X_3 - \varepsilon[X_2, X_3] + \frac{\varepsilon^2}{2}[X_2, [X_2, X_3]] + \cdots \\ &= X_3 + \varepsilon X_3 - \frac{\varepsilon^2}{2}[X_2, X_3] + \cdots \\ &= X_3 + \varepsilon X_3 + \frac{\varepsilon^2}{2}X_3 + \cdots \\ &= e^\varepsilon X_3. \end{aligned}$$

All the adjoint representations are summarized in Table 2.

Table 2.

Adj	X_1	X_2	X_3
X_1	X_1	$X_2 - 3\varepsilon X_1$	X_3
X_2	$\exp(3\varepsilon)X_1$	X_2	$\exp(\varepsilon)X_3$
X_3	X_1	$X_2 - \varepsilon X_3$	X_3

We now construct the one-dimensional optimal system.

Consider the general operator

$$X = \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 \quad (3.5)$$

where α_1, α_2 and α_3 are arbitrary constants. First we assume $\alpha_2 \neq 0$, for simplicity let $\alpha_2 = 1$. Therefore (3.5) becomes

$$X = \alpha_1 X_1 + X_2 + \alpha_3 X_3. \quad (3.6)$$

Referring to Table 2, we act on (3.6) by $Ad(\exp(\frac{1}{3}\varepsilon X_1))$ to eliminate $\alpha_1 X_1$ and obtain

$$X' = X_2 + \alpha_3 X_3.$$

We now eliminate $\alpha_3 X_3$ by acting on X' by $Ad(\exp(\alpha_3 \varepsilon X_3))$ and get

$$X'' = X_2.$$

Next we assume $\alpha_2 = 0$ with $\alpha_1 \neq 0$. Taking $\alpha_1 = 1$, operator (3.5) becomes

$$X = X_1 + \alpha_3 X_3. \quad (3.7)$$

Lastly, we act on (3.7) by $Ad(\exp(\alpha_2 X_2))$ and obtain

$$X' = e^{3\varepsilon} X_1 + \alpha_3 e^\varepsilon X_3. \quad (3.8)$$

The operator (3.8) is a scalar multiple of

$$X'' = X_1 + \alpha_3 X_3. \quad (3.9)$$

Depending on the sign of α_3 from (3.9), we can make the coefficient of X_3 to be ± 1 . Thus for $\alpha_3 > 0$: $X'' = X_1 + X_3$ and $\alpha_3 < 0$: $X'' = X_1 - X_3$. Hence the one-dimensional optimal system of subalgebras are

$$X_2, \quad X_1 + X_3, \quad X_1 - X_3.$$

3.1.2 Symmetry reductions for (3.2)

In this subsection we use the optimal system of one-dimensional subalgebras calculated above to obtain symmetry reductions.

First we consider the symmetry generator X_2 which gives rise to the group-invariant solutions

$$u(t, x) = t^{-2/3}E(\tau), \quad v(t, x) = t^{-2/3}F(\tau),$$

where $\tau = xt^{-1/3}$ is an invariant of the symmetry X_2 and the functions $E(\tau)$ and $F(\tau)$ satisfy the following system of ODEs:

$$\begin{aligned} 3aE'''(\tau) + 3bE(\tau)E'(\tau) + 3cF(\tau)F'(\tau) - \tau E'(\tau) - 2E(\tau) &= 0, \\ 3dF'''(\tau) + 3cE(\tau)F'(\tau) + 3cF(\tau)E'(\tau) - \tau F'(\tau) - 2F(\tau) &= 0. \end{aligned} \quad (3.10)$$

We now take symmetry generator $X_1 + kX_3$, where $k = \pm 1$. This symmetry generator gives rise to the group-invariant solutions

$$u(t, x) = E(\tau), \quad v(t, x) = F(\tau), \quad (3.11)$$

where $\tau = x - kt$ is an invariant of the symmetry generator $X_1 + kX_3$. Substituting (3.11) into (3.2) yields the following system of ODEs:

$$\begin{aligned} aE'''(\tau) + bE(\tau)E'(\tau) + cF(\tau)F'(\tau) - kE'(\tau) &= 0, \\ dF'''(\tau) + cE(\tau)F'(\tau) + cF(\tau)E'(\tau) - kF'(\tau) &= 0. \end{aligned} \quad (3.12)$$

3.1.3 Exact solutions of (3.2) using simplest equation method

We now use the simplest equation method which was introduced by Kudryashov [38, 39] and modified by Vitanov [40], to solve the ODE system (3.12). The simplest equations that will be used are the Bernoulli and Ricatti equations. We briefly recall the simplest equations method here. Let us consider the solutions of (3.12) in the form

$$E(\tau) = \sum_{i=0}^M A_i (H(\tau))^i, \quad F(\tau) = \sum_{i=0}^M B_i (H(\tau))^i, \quad (3.13)$$

where $H(\tau)$ satisfies the Bernoulli or Ricatti equations, M is a positive integer that can be determined by balancing procedure as in [40] and A_i and B_i ($i=0,1,\dots,M$), are parameters to be determined. We note that the Bernoulli and Ricatti equations are well-known nonlinear ODEs whose solutions can be expressed in terms of elementary functions. For the Bernoulli equation we have

$$H'(\tau) = qH(\tau) + pH^2(\tau), \quad (3.14)$$

where q and p are constants. As a result the solution of the Bernoulli equation is

$$H(\tau) = q \left\{ \frac{\cosh[q(\tau + C)] + \sinh[q(\tau + C)]}{1 - p \cosh[q(\tau + C)] - p \sinh[q(\tau + C)]} \right\}. \quad (3.15)$$

For the Ricatti equation

$$H'(\tau) = qH^2(\tau) + pH(\tau) + \gamma, \quad (3.16)$$

where q , p and γ are constants, we use the solutions

$$H(\tau) = -\frac{p}{2q} - \frac{\theta}{2q} \tanh \left[\frac{1}{2} \theta (\tau + C) \right] \quad (3.17)$$

and

$$H(\tau) = -\frac{p}{2q} - \frac{\theta}{2q} \tanh\left(\frac{1}{2}\theta\tau\right) + \frac{\operatorname{sech}\left(\frac{\theta\tau}{2}\right)}{C \cosh\left(\frac{\theta\tau}{2}\right) - \left(\frac{2q}{\theta}\right) \sinh\left(\frac{\theta\tau}{2}\right)}, \quad (3.18)$$

with $\theta^2 = p^2 - 4q\gamma > 0$ and C is a constant of integration.

Solutions of (3.2) using the Bernoulli equation as the simplest equation

The balancing procedure yields $M = 2$ so the solutions of the ODE system (3.12) are of the form

$$E(\tau) = A_0 + A_1 H + A_2 H^2, \quad F(\tau) = B_0 + B_1 H + B_2 H^2. \quad (3.19)$$

Substituting (3.19) into (3.12), and making use of the Bernoulli equation (3.14), and equating all coefficients of the functions H^i to zero, we obtain an algebraic system of equations in terms of A_0, A_1, A_2 and B_0, B_1, B_2 . These algebraic equations are

$$\begin{aligned} 4cA_2B_2p + 24dB_2p^3 &= 0, \\ 24aA_2p^3 + 2bA_2^2p + 2cB_2^2p &= 0, \\ bA_0A_1q - kA_1q + cB_0B_1q + aA_1q^3 &= 0, \\ cA_0B_1q - kB_1q + cB_0A_1q + dB_1q^3 &= 0, \end{aligned}$$

$$\begin{aligned}
54 d B_2 p^2 q + 3 c A_1 B_2 p + 4 c A_2 B_2 q + 3 c A_2 B_1 p + 6 d B_1 p^3 &= 0, \\
2 b A_2^2 q + 54 a A_2 p^2 q + 3 b A_1 A_2 p + 2 c B_2^2 q + 6 a A_1 p^3 + 3 c B_1 B_2 p &= 0, \\
3 c A_1 B_2 q + 12 d B_1 p^2 q + 2 c B_0 A_2 p - 2 k B_2 p + 2 c A_1 B_1 p + 2 c A_0 B_2 p + 38 d B_2 p q^2 \\
&+ 3 c A_2 B_1 q = 0, \\
2 c A_0 B_2 q - 2 k B_2 q + 2 c B_0 A_2 q + 8 d B_2 q^3 + 2 c A_1 B_1 q + 7 d B_1 p q^2 + c B_0 A_1 p - k B_1 p \\
&+ c A_0 B_1 p = 0, \\
2 c B_0 B_2 p - 2 k A_2 p + b A_1^2 p + 38 a A_2 p q^2 + c B_1^2 p + 3 b A_1 A_2 q + 3 c B_1 B_2 q + 12 a A_1 p^2 q \\
&+ 2 b A_0 A_2 p = 0, \\
-2 k A_2 q - k A_1 p + 2 b A_0 A_2 q + 8 a A_2 q^3 + 7 a A_1 p q^2 + b A_1^2 q + c B_0 B_1 p + 2 c B_0 B_2 q \\
&+ c B_1^2 q + b A_0 A_1 p = 0.
\end{aligned}$$

Solving the above system of algebraic equations with the aid of Maple, one obtains

$$\begin{aligned}
c &= b, \\
A_0 &= \frac{2k - dq^2}{2b}, \\
A_1 &= -\frac{6dpq}{b}, \\
A_2 &= -\frac{6dp^2}{b}, \\
B_0 &= \pm \frac{q^2 \sqrt{d(2a - d)}}{2b}, \\
B_1 &= \pm \frac{6qp \sqrt{d(2a - d)}}{b}, \\
B_2 &= \pm \frac{6p^2 \sqrt{d(2a - d)}}{b}.
\end{aligned}$$

Hence the solutions of (3.2) are given by

$$\begin{aligned}
u(t, x) &= A_0 + A_1 q \left\{ \frac{\cosh[q(\tau + C)] + \sinh[q(\tau + C)]}{1 - p \cosh[q(\tau + C)] - p \sinh[q(\tau + C)]} \right\} \\
&+ A_2 q^2 \left\{ \frac{\cosh[q(\tau + C)] + \sinh[q(\tau + C)]}{1 - p \cosh[q(\tau + C)] - p \sinh[q(\tau + C)]} \right\}^2, \quad (3.20)
\end{aligned}$$

$$\begin{aligned}
v(t, x) &= B_0 + B_1 q \left\{ \frac{\cosh[q(\tau + C)] + \sinh[q(\tau + C)]}{1 - p \cosh[q(\tau + C)] - p \sinh[q(\tau + C)]} \right\} \\
&+ B_2 q^2 \left\{ \frac{\cosh[q(\tau + C)] + \sinh[q(\tau + C)]}{1 - p \cosh[q(\tau + C)] - p \sinh[q(\tau + C)]} \right\}^2 \quad (3.21)
\end{aligned}$$

where $\tau = x - kt$ and C is a constant of integration.

Solutions of (3.2) using the Ricatti equation as the simplest equation

The balancing procedure yields $M = 2$ so the solutions of system (3.12) are of the form

$$E(\tau) = A_0 + A_1 H + A_2 H^2, \quad F(\tau) = B_0 + B_1 H + B_2 H^2. \quad (3.22)$$

Substituting (3.22) into (3.12) and making use of the Ricatti equation (3.16), we obtain an algebraic system of equations in terms of A_0, A_1, A_2, B_0, B_1 and B_2 by equating all coefficients of the functions H^i to zero. The corresponding algebraic system of equations are

$$\begin{aligned} 24 dB_2 q^3 + 4 c A_2 B_2 q &= 0, \\ 2 c B_2^2 q + 24 a A_2 q^3 + 2 b A_2^2 q &= 0, \\ 3 c A_2 B_1 q + 3 c A_1 B_2 q + 6 dB_1 q^3 + 4 c A_2 B_2 p + 54 dB_2 q^2 p &= 0, \\ 6 a A_2 p \gamma^2 + b A_0 A_1 \gamma - k A_1 \gamma + 2 a A_1 q \gamma^2 + c B_0 B_1 \gamma + a A_1 p^2 \gamma &= 0, \\ 3 b A_1 A_2 q + 2 b A_2^2 p + 3 c B_1 B_2 q + 6 a A_1 q^3 + 2 c B_2^2 p + 54 a A_2 q^2 p &= 0, \\ 6 dB_2 p \gamma^2 + c A_0 B_1 \gamma + c B_0 A_1 \gamma + 2 dB_1 q \gamma^2 - k B_1 \gamma + dB_1 p^2 \gamma &= 0, \\ 40 dB_2 q^2 \gamma + 2 c A_0 B_2 q + 3 c A_2 B_1 p + 12 dB_1 q^2 p + 38 dB_2 q p^2 + 2 c B_0 A_2 q + 4 c A_2 B_1 B_2 \gamma + 3 c A_1 B_2 p + 2 c A_1 B_1 q - 2 k B_2 q &= 0, \\ -k B_1 p + 2 c B_0 A_2 \gamma - 2 k B_2 \gamma + c A_0 B_1 p + dB_1 p^3 + c B_0 A_1 p + 2 c A_0 B_2 \gamma + 8 dB_1 q p \gamma + 2 c A_1 B_1 \gamma + 16 dB_2 q \gamma^2 + 14 dB_2 p^2 \gamma &= 0, \\ -k A_1 p + c B_0 B_1 p + b A_1^2 \gamma + a A_1 p^3 - 2 k A_2 \gamma + c B_1^2 \gamma + b A_0 A_1 p + 8 a A_1 q p \gamma + 2 b A_0 A_2 \gamma + 16 a A_2 q \gamma^2 + 2 c B_0 B_2 \gamma + 14 a A_2 p^2 \gamma &= 0, \\ 2 c B_0 B_2 q + 12 a A_1 q^2 p + 38 a A_2 q p^2 - 2 k A_2 q + 2 c B_2^2 \gamma + 2 b A_2^2 \gamma + 40 a A_2 q^2 \gamma + 2 b A_0 A_2 q + c B_1^2 q + 3 c B_1 B_2 p + 3 b A_1 A_2 p + b A_1^2 q &= 0, \\ 2 c B_0 A_2 p + 8 dB_1 q^2 \gamma + c A_0 B_1 q - k B_1 q + 3 c A_1 B_2 \gamma + 52 dB_2 q p \gamma - 2 k B_2 p + 3 c A_2 B_1 \gamma + 8 dB_2 p^3 + 2 c A_1 B_1 p + 2 c A_0 B_2 p + c B_0 A_1 q + 7 dB_1 q p^2 &= 0, \\ -2 k A_2 p - k A_1 q + b A_1^2 p + 3 b A_1 A_2 \gamma + 7 a A_1 q p^2 + c B_0 B_1 q + b A_0 A_1 q + 8 a A_2 p^3 + 3 c B_1 B_2 \gamma + 2 b A_0 A_2 p + 52 a A_2 q p \gamma + c B_1^2 p + 8 a A_1 q^2 \gamma + 2 c B_0 B_2 p &= 0. \end{aligned}$$

Solving the above equations yields

$$\begin{aligned}
c &= b, \\
A_0 &= \frac{-dp^2 - 8dq\gamma + 2k}{2b}, \\
A_1 &= -\frac{6dpq}{b}, \\
A_2 &= -\frac{6dq^2}{b}, \\
B_0 &= \pm \frac{1}{2} \frac{(p^2 + 8q\gamma)\sqrt{d(2a-d)}}{b}, \\
B_1 &= \pm \frac{6pq\sqrt{d(2a-d)}}{b}, \\
B_2 &= \pm \frac{6q^2\sqrt{d(2a-d)}}{b}.
\end{aligned}$$

As a result, we obtain the solutions of system (3.2) as

$$\begin{aligned}
u(t, x) &= A_0 + A_1 \left\{ -\frac{p}{2q} - \frac{\theta}{2q} \tanh \left[\frac{1}{2} \theta (\tau + C) \right] \right\} \\
&\quad + A_2 \left\{ -\frac{p}{2q} - \frac{\theta}{2q} \tanh \left[\frac{1}{2} \theta (\tau + C) \right] \right\}^2, \tag{3.23}
\end{aligned}$$

$$\begin{aligned}
v(t, x) &= B_0 + B_1 \left\{ -\frac{p}{2q} - \frac{\theta}{2q} \tanh \left[\frac{1}{2} \theta (\tau + C) \right] \right\} \\
&\quad + B_2 \left\{ -\frac{p}{2q} - \frac{\theta}{2q} \tanh \left[\frac{1}{2} \theta (\tau + C) \right] \right\}^2 \tag{3.24}
\end{aligned}$$

and

$$\begin{aligned}
u(t, x) &= A_0 + A_1 \left\{ -\frac{p}{2q} - \frac{\theta}{2q} \tanh \left(\frac{1}{2} \theta \tau \right) + \frac{\operatorname{sech}(\frac{\theta \tau}{2})}{C \cosh(\frac{\theta \tau}{2}) - (\frac{2q}{\theta}) \sinh(\frac{\theta \tau}{2})} \right\} \\
&\quad + A_2 \left\{ -\frac{p}{2q} - \frac{\theta}{2q} \tanh \left(\frac{1}{2} \theta \tau \right) + \frac{\operatorname{sech}(\frac{\theta \tau}{2})}{C \cosh(\frac{\theta \tau}{2}) - (\frac{2q}{\theta}) \sinh(\frac{\theta \tau}{2})} \right\}^2, \\
v(t, x) &= B_0 + B_1 \left\{ -\frac{p}{2q} - \frac{\theta}{2q} \tanh \left(\frac{1}{2} \theta \tau \right) + \frac{\operatorname{sech}(\frac{\theta \tau}{2})}{C \cosh(\frac{\theta \tau}{2}) - (\frac{2q}{\theta}) \sinh(\frac{\theta \tau}{2})} \right\} \\
&\quad + B_2 \left\{ -\frac{p}{2q} - \frac{\theta}{2q} \tanh \left(\frac{1}{2} \theta \tau \right) + \frac{\operatorname{sech}(\frac{\theta \tau}{2})}{C \cosh(\frac{\theta \tau}{2}) - (\frac{2q}{\theta}) \sinh(\frac{\theta \tau}{2})} \right\}^2, \tag{3.25}
\end{aligned}$$

where $\tau = x - kt$ and C is a constant of integration.

3.2 Conservation laws of (3.2)

In this section we derive the conservation laws for the generalized coupled KdV system (3.2). This system does not have a Lagrangian. In order to apply the Noether theorem we transform system (3.2) to a fourth-order system, using the transformations $u = U_x$ and $v = V_x$. Then system (3.2) becomes

$$\begin{aligned} U_{tx} + aU_{xxxx} + bU_x U_{xx} + cV_x V_{xx} &= 0, \\ V_{tx} + dV_{xxxx} + cU_x V_{xx} + cV_x U_{xx} &= 0. \end{aligned} \quad (3.26)$$

It can be verified that the second-order Lagrangian for system (3.26) is given by

$$L = \frac{1}{2} \left(aU_{xx}^2 + dV_{xx}^2 - \frac{1}{3}bU_x^3 - cU_x V_x^2 - U_x U_t - V_t V_x \right). \quad (3.27)$$

The insertion of L into the determining equation (2.51) yields

$$\begin{aligned} & -\frac{1}{2}U_x[\eta_t^1 + U_t\eta_U^1 + V_t\eta_V^1 - U_t\xi_t^1 - U_t^2\xi_U^1 - U_tV_t\xi_V^1 - U_x\xi_t^2 - U_xU_t\xi_U^2 \\ & - U_xV_t\xi_V^2] - \frac{1}{2}V_x[\eta_t^2 + U_t\eta_U^2 + V_t\eta_V^2 - V_t\xi_t^1 - U_tV_t\xi_U^1 - V_t^2\xi_V^1 - V_x\xi_t^2 \\ & - U_tV_x\xi_U^2 - V_tV_x\xi_V^2] - \frac{1}{2} \left(bU_x^2 + cV_x^2 + U_t \right) [\eta_x^1 + U_x\eta_U^1 + V_x\eta_V^1 - U_t\xi_x^1 \\ & - U_tU_x\xi_U^1 - U_tV_x\xi_V^1 - U_x\xi_x^2 - U_x^2\xi_U^2 - U_xV_x\xi_V^2] - \frac{1}{2} \left(cU_xV_x + V_t \right) \\ & [\eta_x^2 + U_x\eta_U^2 + V_x\eta_V^2 - V_t\xi_x^1 - U_xV_t\xi_U^1 - V_tV_x\xi_V^1 - V_x\xi_x^2 - U_xV_x\xi_U^2 - V_x^2\xi_V^2] \\ & + dV_{xx}[D_x^2(\eta^1) - U_tD_x^2(\xi^1) - U_xD_x^2(\xi^2) - 2U_{tx}(\xi_x^1 + U_x\xi_U^1 + V_x\xi_V^1) \\ & - 2U_{xx}(\xi_x^2 + U_x\xi_U^2 + V_x\xi_V^2)] + aU_{xx}[D_x^2(\eta^2) - V_tD_x^2(\xi^1) - V_xD_x^2(\xi^2) \\ & - 2V_{tx}(\xi_x^1 + U_x\xi_U^1 + V_x\xi_V^1) - 2V_{xx}(\xi_x^2 + U_x\xi_U^2 + V_x\xi_V^2)] \\ & + \frac{1}{2}(aU_{xx}^2 + dV_{xx}^2 - \frac{1}{3}bU_x^3 - cU_xV_x^2 - U_xU_t - V_tV_x)[\xi_t^1 + U_t\xi_U^1 + V_t\xi_V^1 + \xi_x^2 \\ & + U_x\xi_U^2 + V_x\xi_V^2] = B_t^1 + U_tB_U^1 + V_tB_V^1 + B_x^2 + U_xB_U^2 + V_xB_V^2. \end{aligned} \quad (3.28)$$

The separation of (3.28) with respect to different combinations of derivatives of U and V results in an overdetermined system of linear PDEs, namely

$$\xi_U^1 = 0, \quad (3.29)$$

$$\xi_V^1 = 0, \quad (3.30)$$

$$\xi_t^1 = 0, \quad (3.31)$$

$$\xi_x^1 = 0, \quad (3.32)$$

$$\xi_U^2 = 0, \quad (3.33)$$

$$\xi_V^2 = 0, \quad (3.34)$$

$$\xi_x^2 = 0, \quad (3.35)$$

$$\eta_U^1 = 0, \quad (3.36)$$

$$\eta_V^1 = 0, \quad (3.37)$$

$$\eta_U^2 = 0, \quad (3.38)$$

$$\eta_V^2 = 0, \quad (3.39)$$

$$\eta_x^2 = 0, \quad (3.40)$$

$$\eta_{xx}^1 = 0, \quad (3.41)$$

$$(b - c)\eta_x^1 = 0, \quad (3.42)$$

$$\xi_t^2 = (b + c)\eta_x^1, \quad (3.43)$$

$$B_U^1 = -\frac{1}{2}\eta_x^1, \quad (3.44)$$

$$B_V^1 = 0, \quad (3.45)$$

$$B_U^2 = -\frac{1}{2}\eta_t^1, \quad (3.46)$$

$$B_V^2 = -\frac{1}{2}\eta_t^2, \quad (3.47)$$

$$B_t^1 + B_x^2 = 0. \quad (3.48)$$

Solving this system of PDEs we arrive at the following two cases for which Noether symmetries exist:

Case 1. $b \neq c$.

In this case we obtain

$$\begin{aligned}
\xi^1 &= A_1, \\
\xi^2 &= A_2, \\
\eta^1 &= E(t), \\
\eta^2 &= F(t), \\
B^1 &= P(t, x), \\
B^2 &= -\frac{1}{2}U E'(t) - \frac{1}{2}V F'(t) + S(t, x), \\
P_t + S_x &= 0.
\end{aligned} \tag{3.49}$$

Here we can set $P(t, x) = S(t, x) = 0$ as they contribute to the trivial part of the conserved vectors. Thus, we obtain the following Noether symmetries and gauge functions:

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, & B^1 &= B^2 = 0, \\
X_2 &= \frac{\partial}{\partial x}, & B^1 &= B^2 = 0, \\
X_{E(t)} &= E(t) \frac{\partial}{\partial U}, & B^1 &= 0, & B^2 &= -\frac{1}{2}U E'(t), \\
X_{F(t)} &= F(t) \frac{\partial}{\partial V}, & B^1 &= 0, & B^2 &= -\frac{1}{2}V F'(t).
\end{aligned}$$

The above Noether operators and gauge functions are now used to find the components of the conserved vectors for the second-order Lagrangian. Using equations (2.87), (2.88) along with (3.49) and $u = U_x, v = V_x$, we find the following independent conserved vectors for the coupled system (3.2):

$$\begin{aligned}
T_1^1 &= \frac{1}{2}(au_x^2 + dv_x^2 - \frac{1}{3}bu^3 - cuv^2), \\
T_1^2 &= \frac{1}{2} \int u_t dx \int u_t dx + \frac{1}{2}(bu^2 + cv^2) \int u_t dx + au_{xx} \int u_t dx + \frac{1}{2} \int v_t dx \int v_t dx \\
&\quad + dv_{xx} \int v_t dx + cuv \int v_t dx - au_t u_x - dv_t v_x,
\end{aligned} \tag{3.50}$$

$$\begin{aligned}
T_2^1 &= \frac{1}{2}(u^2 + v^2), \\
T_2^2 &= auu_{xx} + dvv_{xx} - \frac{1}{2}au_x^2 - \frac{1}{2}dv_x^2 + \frac{1}{3}bu^3 + cuv^2,
\end{aligned} \tag{3.51}$$

$$\begin{aligned}
T_{(E,F)}^1 &= -\frac{1}{2}uE(t) - \frac{1}{2}vF(t), \\
T_{(E,F)}^2 &= \frac{1}{2}E'(t) \int u dx + \frac{1}{2}F'(t) \int v dx - \frac{1}{2}E(t) \int u_t dx - \frac{1}{2}F(t) \int v_t dx \\
&\quad - \frac{1}{2}(bu^2 + cv^2)E(t) - au_{xx}E(t) - dv_{xx}F(t) - cuvF(t). \tag{3.52}
\end{aligned}$$

The conserved vector (3.51) is a local conserved vector and (3.50) is a nonlocal conserved vector for the system (2.39). We can obtain two special cases from conserved vector (3.52) by choosing $E(t) = 1$ and $F(t) = 0$ which gives a nonlocal conserved vector

$$\begin{aligned}
T_3^1 &= -\frac{1}{2}u, \\
T_3^2 &= -\frac{1}{2}(bu^2 + cv^2) - au_{xx} - \frac{1}{2} \int u_t dx, \tag{3.53}
\end{aligned}$$

and by taking $E(t) = 0$ and $F(t) = 1$ we obtain a nonlocal conserved vector

$$\begin{aligned}
T_4^1 &= -\frac{1}{2}v, \\
T_4^2 &= -cuv - dv_{xx} - \frac{1}{2} \int v_t dx. \tag{3.54}
\end{aligned}$$

Case 2. $b = c$.

This case gives the following solutions:

$$\begin{aligned}
\xi^1 &= A_1, \\
\xi^2 &= cA_2t + A_3, \\
\eta^1 &= A_2x + E(t), \\
\eta^2 &= F(t), \\
B^1 &= -\frac{1}{2}A_2U + P(t, x), \\
B^2 &= -\frac{1}{2}UE'(t) - \frac{1}{2}VF'(t) + R(t, x), \\
P_t + R_x &= 0. \tag{3.55}
\end{aligned}$$

Again we can set $P = R = 0$ as they contribute to the trivial part of the conserved

vector. Therefore, the Noether symmetries and gauge functions are

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, & B^1 &= B^2 = 0, \\
X_2 &= ct \frac{\partial}{\partial x} + x \frac{\partial}{\partial U}, & B^1 &= -\frac{1}{2}U, \quad B^2 = 0, \\
X_3 &= \frac{\partial}{\partial x}, & B^1 &= B^2 = 0, \\
X_{E(t)} &= E(t) \frac{\partial}{\partial U}, & B^1 &= 0, \quad B^2 = -\frac{1}{2}U E'(t), \\
X_{F(t)} &= F(t) \frac{\partial}{\partial V}, & B^1 &= 0, \quad B^2 = -\frac{1}{2}V F'(t).
\end{aligned}$$

The associated conserved vectors for system (3.2), are

$$\begin{aligned}
T_1^1 &= \frac{1}{2}(au_x^2 + dv_x^2 - \frac{1}{3}bu^3 - cuv^2), \\
T_1^2 &= \frac{1}{2} \int u_t dx \int u_t dx + \frac{1}{2}(bu^2 + cv^2) \int u_t dx + au_{xx} \int u_t dx + \frac{1}{2} \int v_t dx \int v_t dx \\
&\quad + dv_{xx} \int v_t dx + cuv \int v_t dx - au_t u_x - dv_t v_x, \tag{3.56}
\end{aligned}$$

$$\begin{aligned}
T_2^1 &= \frac{1}{2}(-xu + ctu^2 + ctv^2 + \int u dx), \\
T_2^2 &= au_x + actu_{xx} + cdtv_{xx} + c^2tuv^2 - axu_{xx} + \frac{1}{3}cbtu^3 \\
&\quad - \frac{1}{2}(actu_x^2 + cdtv_x^2 + bxu^2 + cxv^2 + x \int u_t dx), \tag{3.57}
\end{aligned}$$

$$\begin{aligned}
T_3^1 &= \frac{1}{2}(u^2 + v^2), \\
T_3^2 &= auu_{xx} + dvv_{xx} - \frac{1}{2}au_x^2 - \frac{1}{2}dv_x^2 + \frac{1}{3}bu^3 + cuv^2, \tag{3.58}
\end{aligned}$$

$$\begin{aligned}
T_{(E,F)}^1 &= -\frac{1}{2}uE(t) - \frac{1}{2}vF(t), \\
T_{(E,F)}^2 &= \frac{1}{2}E'(t) \int u dx + \frac{1}{2}F'(t) \int v dx - \frac{1}{2}E(t) \int u_t dx - \frac{1}{2}F(t) \int v_t dx \\
&\quad - \frac{1}{2}(bu^2 + cv^2)E(t) - au_{xx}E(t) - (dv_{xx} + cuv)F(t). \tag{3.59}
\end{aligned}$$

The conserved vector (3.58) is a local conserved vector and (3.56) is a nonlocal conserved vector for the system (2.39). The conserved vector (3.59) for $E(t) = 1$ and $F(t) = 0$ gives

$$\begin{aligned}
T_3^1 &= -\frac{1}{2}u, \\
T_3^2 &= -\frac{1}{2}(bu^2 + cv^2) - au_{xx} - \frac{1}{2} \int u_t dx, \tag{3.60}
\end{aligned}$$

and for $E(t) = 0$ and $F(t) = 1$ gives

$$\begin{aligned} T_4^1 &= -\frac{1}{2}v, \\ T_4^2 &= -cuv - dv_{xx} - \frac{1}{2} \int v_t dx. \end{aligned} \quad (3.61)$$

We note that for arbitrary values of $E(t)$ and $F(t)$ infinitely many nonlocal conservation laws exist for system (3.2).

3.3 Conclusion

In this chapter we employed Lie symmetry analysis to study the generalized coupled KdV system (3.2). The Lie point symmetries obtained were then used to transform the coupled KdV system (3.2) to a coupled system of ordinary differential equations. Thereafter the simplest equation method was applied to construct exact solutions of (3.2). In addition conservation laws were obtained for (3.2) by using Noether theorem. The conservation laws obtained consisted of some local and infinite number of nonlocal conserved vectors.

Chapter 4

Conservation laws of coupled system describing one-layer shallow water waves

In this chapter we study coupled system of PDEs which describe one-layer shallow water waves [36] given by

$$\begin{aligned}v_t + vu_x + uv_x &= 0, \\u_t + uu_x + v_x &= 0,\end{aligned}\tag{4.1}$$

Sahim et al. [36] studied coupled system (4.1) using Lie group analysis. They obtained self-similarity solutions of the one-layer shallow water equations representing gravity currents and they showed that Lie group analysis is the generalization of the dimensional analysis for investigating the self-similarity solutions of (4.1). Conservation laws for system (4.1) were constructed in [59] using new conservation theorem [60]. Infinite local conservation laws and potential symmetries were obtained.

In this research a different approach is applied to construct conservation laws for coupled system (4.1).

4.1 Construction of conservation laws for (4.1)

In this section conservation laws for coupled system of PDEs which describe one-layer shallow water waves (4.1) are derived by utilizing Noether theorem. The use of Noether theorem requires the knowledge of the Lagrangian. This coupled system (4.1) does not have the Lagrangian. In order to apply Noether theorem we increase the first-order coupled system (4.1) to a second-order coupled system using $v = V_x$ and $u = U_x$. Now system (4.1) becomes

$$\begin{aligned} V_{tx} + V_x U_{xx} + U_x V_{xx} &= 0, \\ U_{tx} + U_x U_{xx} + V_{xx} &= 0, \end{aligned} \quad (4.2)$$

and the Lagrangian for system (4.2) is

$$L = -\frac{1}{2}(U_t V_x + U_x V_t + V_x U_x^2 + V_x^2). \quad (4.3)$$

We can easily use equation (2.42) to verify that equation (4.3) is indeed a Lagrangian for system (4.2). The substitution of Lagrangian (4.3) into the determining equation (2.51) gives

$$\begin{aligned} & -\frac{1}{2}V_x[\eta_t^1 + U_t\eta_U^1 + V_t\eta_V^1 - U_t\xi_t^1 - U_t^2\xi_U^1 - U_tV_t\xi_V^1 - U_x\xi_t^2 - U_xU_t\xi_U^2 - U_xV_t\xi_V^2] \\ & -\frac{1}{2}U_x[\eta_t^2 + U_t\eta_U^2 + V_t\eta_V^2 - V_t\xi_t^1 - U_tV_t\xi_U^1 - V_t^2\xi_V^1 - V_x\xi_t^2 - U_tV_x\xi_U^2 - V_tV_x\xi_V^2] \\ & -\left(\frac{1}{2}V_t + V_xU_x\right)[\eta_x^1 + U_x\eta_U^1 + V_x\eta_V^1 - U_t\xi_x^1 - U_tU_x\xi_U^1 - U_tV_x\xi_V^1 - U_x\xi_x^2 - U_x^2\xi_U^2 \\ & -U_xV_x\xi_V^2] - \left(\frac{1}{2}U_t + \frac{1}{2}U_x^2 + V_x\right)[\eta_x^2 + U_x\eta_U^2 + V_x\eta_V^2 - V_t\xi_x^1 - U_xV_t\xi_U^1 - V_tV_x\xi_V^1 \\ & -V_x\xi_x^2 - U_xV_x\xi_U^2 - V_x^2\xi_V^2] + \left[-\frac{1}{2}(U_tV_x + U_xV_t - V_xU_x^2 + V_x^2)\right] \\ & [\xi_t^1 + U_t\xi_U^1 + V_t\xi_V^1 + \xi_x^2 + U_x\xi_U^2 + V_x\xi_V^2] \\ & = B_t^1 + U_tB_U^1 + V_tB_V^1 + B_x^2 + U_xB_U^2 + V_xB_V^2. \end{aligned} \quad (4.4)$$

The separation of (4.4) with respect to different combinations of derivatives of V and U results in an overdetermined system of equations for $\xi^1, \xi^2, \eta^1, \eta^2, B^1$ and B^2 . The

solution of the system yields

$$\begin{aligned}
\xi^1 &= A_1 t + A_2, \\
\xi^2 &= \frac{3}{5} A_1 x + A_3 t + A_4, \\
\eta^1 &= \frac{1}{5} A_1 U + A_3 x + g(t), \\
\eta^2 &= \frac{1}{5} A_1 V + k(t), \\
B^1 &= -\frac{1}{2} A_3 V + \alpha(t, x), \\
B^2 &= -\frac{1}{2} U k'(t) - \frac{1}{2} V g'(t) + \beta(t, x), \\
\alpha_t + \beta_x &= 0,
\end{aligned} \tag{4.5}$$

where A_1, A_2, A_3 and A_4 are constants. We can let $\alpha(t, x) = \beta(t, x) = 0$ as they contribute to the trivial part of the conserved vectors. Then the Noether point symmetries and gauge functions are

$$\begin{aligned}
\Upsilon_1 &= t \frac{\partial}{\partial t} + \frac{3}{5} x \frac{\partial}{\partial x} + \frac{1}{5} U \frac{\partial}{\partial U} - \frac{1}{5} V \frac{\partial}{\partial V}, \quad B^1 = B^2 = 0, \\
\Upsilon_2 &= \frac{\partial}{\partial t}, \quad B^1 = B^2 = 0, \\
\Upsilon_3 &= t \frac{\partial}{\partial x} + x \frac{\partial}{\partial U}, \quad B^1 = -\frac{1}{2} V, \quad B^2 = 0, \\
\Upsilon_4 &= \frac{\partial}{\partial x}, \quad B^1 = B^2 = 0, \\
\Upsilon_{g(t)} &= g(t) \frac{\partial}{\partial U}, \quad B^1 = 0, \quad B^2 = -\frac{1}{2} V g'(t), \\
\Upsilon_{k(t)} &= k(t) \frac{\partial}{\partial V}, \quad B^1 = 0, \quad B^2 = -\frac{1}{2} U k'(t).
\end{aligned}$$

The formula for first-order conserved vectors are

$$T^1 = -B^1 + \xi^1 L + (\eta^1 - U_t \xi^1 - U_x \xi^2) \frac{\partial L}{\partial U_t} + (\eta^2 - V_t \xi^1 - V_x \xi^2) \frac{\partial L}{\partial V_t}, \tag{4.6}$$

$$T^2 = -B^2 + \xi^2 L + (\eta^1 - U_t \xi^1 - U_x \xi^2) \frac{\partial L}{\partial U_x} + (\eta^2 - V_t \xi^1 - V_x \xi^2) \frac{\partial L}{\partial V_x}. \tag{4.7}$$

We now use (4.6), (4.7) together with (4.5) and $u = U_x, v = V_x$ and obtain the

following independent conserved vectors:

$$\begin{aligned}
T_1^1 &= -\frac{1}{2}t(u^2v + v^2) + \frac{6}{10}xuv - \frac{1}{10}v \int u dx + \frac{1}{10}u \int v dx, \\
T_1^2 &= \frac{3}{10}xv^2 + \frac{3}{10}xu^2v + tuv \int u_t dx + \frac{1}{2}tu^2 \int v_t dx + \frac{1}{5}v \int v dx \\
&\quad + tv \int v_t dx + \frac{1}{10}u^2 \int v dx - \frac{1}{5}uv \int u dx + t \int u_t dx \int v_t dx + \\
&\quad \frac{1}{10} \int v dx \int u_t dx - \frac{1}{10} \int u dx \int v_t dx,
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
T_2^1 &= -\frac{1}{2}(u^2v + v^2), \\
T_2^2 &= uv \int u_t dx + v \int v_t dx + \int u_t dx \int v_t dx + \frac{1}{2}u^2 \int v_t dx,
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
T_3^1 &= tuv - \frac{1}{2}xv + \frac{1}{2} \int v dx, \\
T_3^2 &= tvu^2 - xuv + \frac{1}{2}tv^2 - \frac{1}{2}x \int v_t dx,
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
T_4^1 &= uv, \\
T_4^2 &= \frac{1}{2}v^2 + vu^2,
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
T_{(g,k)}^1 &= -\frac{1}{2}[vg(t) + uk(t)], \\
T_{(g,k)}^2 &= \frac{1}{2}k'(t) \int u dx + \frac{1}{2}g'(t) \int v dx - \frac{1}{2}g(t) \int v_t dx - \frac{1}{2}k(t) \int u_t dx \\
&\quad - \frac{1}{2}u^2k(t) - uv g(t) - vk(t).
\end{aligned} \tag{4.12}$$

Notice that the conserved vector (4.11) is a local conserved vector and (4.8), (4.9) and (4.10) are nonlocal conserved vectors for the system (4.1). The conserved vector (4.12) yields two special cases when we let $g(t) = 1$ and $k(t) = 0$ which gives a nonlocal conserved vector

$$\begin{aligned}
T_5^1 &= -\frac{1}{2}v, \\
T_5^2 &= -\frac{1}{2} \int v_t dx - uv,
\end{aligned} \tag{4.13}$$

and when we take $g(t) = 0$ and $k(t) = 1$ we obtain nonlocal conserved vector

$$\begin{aligned}
T_6^1 &= -\frac{1}{2}u, \\
T_6^2 &= -\frac{1}{2} \int u_t dx - \frac{1}{2}u^2 - v.
\end{aligned} \tag{4.14}$$

For the arbitrary values of $g(t)$ and $k(t)$ we note that infinitely many nonlocal conservation laws exist for system (4.1).

Note. The conservation laws obtained consist of infinitely many nonlocal conserved vector and one local conserved vector (4.11) which is also obtained in [59] using new conservation theorem.

4.2 Conclusion

In this chapter we used Noether theorem to study the first-order coupled system of PDEs which describe one-layer shallow water waves (4.1). We increased the order of the coupled system (4.1) to a second-order coupled system (4.2) which admitted a Lagrangian (4.3). The Noether theorem was then applied to find conservation laws in U and V variables. Finally, conservation laws for the first-order coupled system describing one-layer shallow water waves (4.1) were obtained by reverting back to the original variables u and v . The conservation laws for the first-order coupled system (4.1) consist of local and infinite number of nonlocal conserved vectors.

Chapter 5

Conclusion and Discussions

In this research project the Lie group method and Noether theorem were utilized to study two third-order and one first-order coupled systems of partial differential equations. We first recalled some important definitions and results from Lie group theory, and the fundamental relations concerning Noether theorem which were later used in the dissertation. The Noether theorem requires the knowledge of Lagrangian.

In Chapter two, we found exact solutions for the Boussinesq system of KdV-KdV type equations (2.1). The Lie group analysis was used to carry out the integration of this equations. Furthermore, we increased the third-order coupled system (2.1) to fourth-order coupled system (2.39) using $u = U_x$ and $v = V_x$. Thereafter we applied the Noether theorem to derive conservation laws in U and V variables. Finally, by reverting back to the original variables u and v we obtained the conservation laws for the third-order coupled system (2.1).

In Chapter three, Lie symmetry method together with simplest equation method was employed to find exact solutions for the generalized coupled KdV system (3.2). Moreover, the same procedure as in chapter two was followed to construct the conservation laws for system (3.2).

In Chapter four, a similar procedure as in chapter two was used to derive conservation laws for the coupled system of one-layer shallow water waves (4.1).

In future conservation laws of various nonlinear differential equations will be utilized to construct exact solutions.

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