



On the Cosmology of the $\text{Rh}=\text{ct}$ Universe

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Abstract

Decades of astronomical observations show that the universe is undergoing accelerated expansion. The standard model of cosmology is the closest to a standard theory of gravitation. Still, it fails to adequately describe the universe without the ad hoc introduction of dark energy and dark matter to late-time cosmology and inflation to early-universe cosmology. This failure of the standard model has certainly created dilemmas for cosmology and the wider astronomical community, and several alternative cosmological models are currently being considered. Most current work is limited to the study of background dynamics, so a brief exploration of the dynamical evolution of the universe is presented in the context of an alternative cosmological model known as the $R_h = ct$ model. This model is consistent with observations and has received particular attention in recent years. Like the Λ CDM model, the $R_h = ct$ universe is based on an FLRW cosmology with the cosmic fluid's total energy density ϵ and pressure p satisfying the condition of vanishing total active gravitational mass: $\epsilon + 3p = 0$. The field equations corresponding to this universe will be solved, and the results derived will be compared to those from the standard model and analysed to see if the model allows for structure formation in the universe.

Keywords: Cosmology; Dark energy; Gravitation; Instabilities; Perturbation theory; $R_h = ct$

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Chapter 1

Literature Review

In the last century, many models have been proposed to explain cosmic expansion. Although the literature explores a number of models, this work will critically examine research conducted on three major aspects which emerge repeatedly throughout the literature reviewed. These aspects are: inflation, structure formation and dark energy.

Most early models (for example, those by Einstein and Friedmann in 1931 [1], and Einstein and de Sitter in 1932 [2]) were concerned with matter-dominated universes. The most widely accepted model of the universe is the Λ CDM¹ model which invokes inflation [3] and dark energy [4, 5, 6] to explain the dynamics of the early-time and late-time universe, respectively. However, a recent proposal by Melia and Shevchuk [7] suggests a new methodology in which cosmic expansion may be described by a linearly expanding model, namely the $R_h = ct$ model. Their proposed model does not discount the earlier models but builds on them to include dark matter, radiation, and dark energy implicitly without invoking the mechanism of inflation.

The proposed model offers a simplified view of cosmic expansion. In particular, Melia and Shevchuk argue that this simplicity has the vanishing active gravitational mass as its basis. Despite criticism by Bilicki and Seikel [8] and Mitra [9], this view is supported by the numerous analyses that have been conducted to examine the extent to which the $R_h = ct$ model may be regarded as an accurate description of the universe (see references [10] and [11]). It can be argued that the $R_h = ct$ model is essentially a Λ CDM model with the additional constraint of vanishing active gravitational mass, which is why most of these analyses adequately fit late-time cosmological data. However, the model fails to explain the explicit roles of the different components of the cosmic fluid – particularly dark energy. This leaves a wide gap in this field of research. Studying the different components of the cosmic fluid in the context of the $R_h = ct$ model would be interesting from an academic standpoint, especially if it were to result in a simplified description of cosmic expansion.

For convenience, this work has been divided into the following chapters.

¹Discussed at length in Chapter 3

Chapter 2 provides an overview of the Λ CDM model and its shortcomings. Chapter 3 traces the mathematical development of the Λ CDM model within the framework of general relativity. Chapter 4 examines a proposed alternative to the Λ CDM model. Chapter 5 will provide a statistical analysis of the proposed alternative to the Λ CDM model. The results of the statistical analysis will be assessed against SNe Ia and OHD data. Chapter 6 explores the ways in which structures form in the universe by implementing the $1 + 3$ covariant spacetime splitting formalism and studying the resulting cosmological perturbations. The final chapter offers a few concluding remarks and possible future work.

Chapter 2

Introduction

The standard model of cosmology, also known as Lambda Cold Dark Matter (Λ CDM), is the simplest cosmological model that explains the evolution of an inflationary big bang universe in terms of expansion, matter and energy content [12]. Dark energy comprises about 70% of the total energy, followed by dark matter at 25%, and ordinary baryonic matter at 5%.

The name of the model represents its two principal ingredients:

- In the Einstein field equations, Λ , is the famous cosmological constant. It represents the "dark" energy density¹ that uniformly (albeit sparsely – roughly 10^{-27} kg/m³ [13]) fills empty space [14]. Einstein first introduced the constant to offset the net force of gravity on cosmological scales so that he could obtain a static universe. Decades later, Perlmutter [4, 6] and Riess [5] discovered the accelerated expansion of the universe, which clearly showed that, although Λ does not have the value needed for a static universe, it does seem to have a nonzero value, which impinges on the accelerated expansion of the universe.

According to the mass-energy principle proposed by general relativity, the physical quantity that causes matter to generate gravitational effects is the energy-momentum tensor, which contains both the mass density and pressure. Given that a substance's internal pressure plays a role in its gravitational attraction to other objects, dark energy in a Friedmann-Lemaître-Robertson-Walker (FLRW) universe would need to have a strong, constant negative pressure to account for the universe's observed expansion – regardless of its actual nature.

Even though it is undetectable in laboratory experiments, vacuum energy has frequently been cited as the main contributor to dark energy [15]. The problem (commonly referred to as the cosmological constant problem or "the worst fine-tuning problem in physics") is that the vacuum energy density predicted by general relativ-

¹"Dark" because many things about its origin and fundamental nature are not well understood; and "energy" because it usually takes energy to increase the expansion rate of the universe.

ity and the theoretical value of zero-point energy proposed by quantum field theory differ significantly – by up to 120 orders of magnitude [16, 17];

- CDM refers to cold (“collisionless”) dark matter. Compared to the "dark matter" part, the "cold" part is less certain. In 1933, Zwicky made the speculative "dark matter" hypothesis because it explains some otherwise unusual cosmological observations that appear to suggest the existence of "missing" mass. It is referred to as "dark" because it does not seem to interact with either ordinary baryonic matter and radiation (except weakly through gravity) or the electromagnetic field.

But why "cold" dark matter? Although it is known that massive neutrinos, which would make up hot dark matter, do exist, there are numerous competing theories regarding the composition and characteristics of dark matter. However, it is generally agreed that cosmological observations can be easily explained within Λ CDM by assuming that dark matter must be "cold." Meaning that the dark matter particles, whatever they may be, move in a non-relativistic manner and that structures form hierarchically through the slow accumulation of particles, their collapse due to self-gravity, and their eventual merging to form larger, more massive objects. In the hot dark matter paradigm, structure forms through fragmentation; the largest superclusters form first in flat pancake-like sheets before breaking up into smaller pieces like our galaxy, the Milky Way. The predictions of the hot dark matter theory are at odds with large-scale structure observations, whereas the cold dark matter paradigm is generally supported by observations.

Since it is hard to detect cold dark matter directly in Earth experiments, its distribution is deduced from its gravitational effects on other matter. Such effects occur in the context of cosmic microwave background anisotropies, galaxy formation and evolution, gravitational lensing, mass positions in galactic collisions, and the current structure of the observable universe—although, the first and possibly most well-known of these unusual effects relates to the rotation curves of galaxies.

Based on the observed matter, the laws of physics would suggest that the velocity of stars in galaxies (or galaxies in clusters) should scale as $1/\sqrt{r}$ – yet this is not the case. Consider a galaxy with an odd rotation. If that puzzle is solved, it is as if a halo of gravity surrounds it, accounting for the strangeness. But what is that? Does it qualify as matter at all? What else can possibly have gravity? Following half a century of unsuccessful dark matter particle searches, recent gravitational wave and James Webb Space Telescope (JWST) observations suggest that the most common explanation for dark matter is that it is made up of primordial black holes [18]. Other explanations of these observations may need more parameters than dark matter and rather drastic changes to the dynamics and behaviour of gravity.

While the standard model has endured numerous observational tests and developed into what is arguably the most thorough description of some significant cosmological observations, it still faces the challenges that come with a theory that was developed phenomenologically and heuristically. It is confronted with several contradictions and disagreeable

coincidences that suggest that our knowledge of the universe is incomplete. The horizon problem is one of these coincidences.

Consider an observer looking at an object $\mathcal{O}_1$ 10 billion light-years away in a single direction. At the same time, the distance of this object to another object $\mathcal{O}_2$ 10 billion light-years away from the observer, but in the opposite direction, is 20 billion light-years. Could information be exchanged between these objects? No, as the light from $\mathcal{O}_1$ has not yet reached $\mathcal{O}_2$ because either the time needed to transmit information from one object to the other would have to be longer than the age of the universe or the information would have to travel faster than light. In other words, parts of the universe are visible to us but invisible to each other: the two objects are causally disconnected and cannot have come into contact with one another physically. In a big bang without a mechanism (like cosmic inflation) that sets similar initial conditions, one would expect that their physical properties would differ and, more generally, that the universe would have different properties in causally disconnected regions.

Contrary to this expectation, galaxy surveys show that the observable universe is almost homogeneous and isotropic; as a result, it must have been causally connected for a sufficient amount of time for the early universe to equilibrate. This is the well-known horizon problem, first pointed out by Wolfgang Rindler in 1956.

The horizon problem only emerges for a subset of FLRW cosmologies (such as Λ CDM) that include an early phase of rapid deceleration. It is considered a significant shortcoming of the standard model, not because of an observational or internal inconsistency but rather because the universe appears to have needed some extremely precise initial conditions that are extremely unlikely given what is known about physics [19].

In its most direct manifestation, the horizon problem results from the standard model's failure to explain the observed statistical uniformity of the cosmic microwave background (CMB) radiation, which shows that, aside from fluctuations at the level of about 1 part in 10^5 observed in the Wilkinson microwave anisotropy probe (WMAP)'s measured relic signal, distant regions of the universe have roughly the same temperature [19, 20]. Regions on opposite sides of the sky lie beyond each other's horizons, yet their current temperatures are identical, even though they could never have been in thermal equilibrium. No physical process propagating at or below the speed of light could have causally connected the features of cosmological structure formation because they are constrained by the finite speed of light [21, 22, 23].

This attack on the laws of thermodynamics has become so blatant that the inflationary model of cosmology [3] was incorporated into the standard model to address it. It is believed that, in this picture, the very early universe experienced an inflationary spurt from roughly 10^{-35} to between 10^{-33} and 10^{-32} seconds after the big bang [24]. The spurt caused the universe to expand at a significantly faster rate (by a factor of about e^{60}) than would otherwise have been possible solely under the influence of radiation, matter, and dark energy, carrying causally connected regions beyond the horizon each would have had

if not for this brief acceleration. Thus, before this inflationary period, widely separated regions of the universe were in very close contact in the early universe, allowing them to exchange heat and acclimate to a common temperature.

Although inflation has successfully explained a large number of the universe's observed features, it remains controversial and presents several significant problems that underscore the need for further theoretical and observational developments to help understand the early universe [25]. For starters, there are many inflation models, so finding a self-consistent and comprehensive theory of inflation has proved difficult even after 40 years. Additionally, after a model is selected, one is then presented with the second problem: the theory calls for the introduction of extremely specific parameters (like the inflation potential, which, incidentally, does not correspond to any known physical field) and initial conditions of its own, which need to be fine-tuned to precise values so that the issue of fine-tuned initial conditions is unresolved. The previously mentioned has caused some to question whether inflation is the most plausible explanation for the early universe.

Therefore, there is great interest in improving the fundamental Λ CDM model or, if necessary, replacing it in the future to improve the comparison between theory and observation [26]. As a result, alternative cosmological models were proposed that provide more satisfactory explanations for the universe's accelerated expansion, structure formation and possible resolutions to the horizon problem, without referring to dark energy, dark matter, and cosmic inflation, respectively. How well these models fit observational data is still under investigation.

Chapter 3

Review of Λ CDM Cosmology

3.1 Einstein's Field Equations

Several writers undertook to explain the nature of cosmological expansion with varying degrees of success. In light of current knowledge, it is reasonable to say that an increased understanding of the foremost observational features of the universe is best developed in the context of general relativity [27].

Einstein's field equations (EFEs) of general relativity relate spacetime curvature to its source, which is mass and energy, in the same way that Maxwell's equations of electromagnetism relate the electromagnetic field to its source – charges and currents. They are a set of nonlinear partial differential equations that explain how matter curves spacetime and how spacetime directs matter's motion. Mathematically,

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4}T_{\alpha\beta}, \quad (3.1)$$

where the Einstein tensor $G_{\alpha\beta}$ depends on the metric and its first and second derivatives. It is entirely geometrical and reflects Einstein's profound realisation that gravity is a property of spacetime geometry. Λ is the cosmological constant; $R_{\alpha\beta}$ is the Ricci tensor that measures the degree to which a volume in curved space differs locally from a volume in ordinary Euclidean space [28]; R is the Ricci scalar that defines the intrinsic curvature at every point in spacetime; $g_{\alpha\beta}$ is the four-dimensional metric tensor used to measure the geometry of spacetime, and $T_{\alpha\beta}$ is the energy-momentum tensor that describes the associated mass-energy that causes the curvature of spacetime.

The EFEs are solved for the metric tensor, but because they are nonlinear, they are inordinately difficult to solve without making suitable assumptions.

3.2 FLRW Metric

One way to conceptualise the metric tensor in general relativity is as a mathematical object that describes how distances and angles are measured in a particular spacetime. The metric that describes the large-scale geometry of the entire universe can be derived from the two basic assumptions that underlie practically all modern cosmology:

1. the cosmological principle (which asserts that, on sufficiently large scales, the visible universe is spatially homogeneous¹ and isotropic²) and
2. uniform time-dependent cosmological expansion.

The only way to simultaneously preserve the homogeneity and isotropy of space and incorporate cosmological expansion is to introduce a time-dependent expansion factor $a(t)$ that completely describes the time evolution of the universe [29]. To investigate what kind of cosmological expansion is permitted by general relativity, it is necessary to determine what type of cosmological expansion solves the EFEs.

From a mathematical perspective, there are numerous geometries (such as toruses) that are spatially homogeneous and isotropic. However, this discussion will only cover simply-connected flat, closed (spherically-curved) and open (hyperbolically-curved) geometries with no holes in them.

The resulting diagonal 4-dimensional spacetime metrics that characterise these spatial geometries are such that uniform scale expansion can be incorporated into each of these metrics by multiplying each of the three spatial components by a single time-dependent scale factor, i.e.:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t)r^2 & 0 \\ 0 & 0 & 0 & a^2(t)r^2 \sin^2 \theta \end{pmatrix},$$

where

$$r = \begin{cases} \sin \chi, & \text{closed} \\ \chi, & \text{flat} \\ \sinh \chi, & \text{open} \end{cases};$$

in which case the r , θ and ϕ coordinates are called reduced circumference polar coordinates. These metrics are then freely inserted into the EFEs to obtain constraints on the scale factors in the form of differential equations. However, recall that the basis vectors are just the partial derivatives with respect to coordinate variables so to convert between the χ

¹looks the same from all points, has no spatial centre, and does not change under spatial translation

²looks the same in all spatial directions and does not change under spatial rotation

basis vector and an r basis vector, use the chain rule:

$$\begin{aligned}\frac{\partial}{\partial \chi} &\equiv \hat{e}_\chi = \frac{dr}{d\chi} \frac{\partial}{\partial r} \\ &= \frac{dr}{d\chi} \hat{e}_r,\end{aligned}$$

where $d_\chi r$ is the conversion term. For each of the three universes, the conversion term can be calculated exactly, so the metric component $g_{\chi\chi} \equiv \hat{e}_\chi \cdot \hat{e}_\chi$ is given by:

$$g_{\chi\chi} = \left(\frac{dr}{d\chi} \right)^2 g_{rr}.$$

Thus,

$$g_{rr} = \frac{a^2(t)}{1 - kr^2}, \quad (3.2)$$

where the curvature parameter $k \in \{1, 0, -1\}$ for a closed, flat, and open universe, respectively. The resulting metric is the standard +2 signature FLRW metric

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a^2(t)}{1-kr^2} & 0 & 0 \\ 0 & 0 & a^2(t)r^2 & 0 \\ 0 & 0 & 0 & a^2(t)r^2 \sin^2 \theta \end{pmatrix}, \quad (3.3)$$

which represents a curved four-dimensional spacetime with a non-zero Riemann curvature tensor [30]. This is the metric inserted into the EFEs and collapsed down to the Friedmann equations.

3.3 Christoffel Symbols

Christoffel symbols, also known as connection coefficients or the Levi-Civita connection [21], are essential for characterising the spacetime curvature and to offer a mathematical framework for studying how particles and light behave in gravitational fields. Their explicit calculation can be a complicated task since it requires calculating partial derivatives of the metric tensor's components with respect to the spacetime coordinates:

$$\Gamma_{\beta\gamma}^\alpha = \frac{1}{2} g^{\alpha\mu} (\partial_\gamma g_{\mu\beta} + \partial_\beta g_{\mu\gamma} - \partial_\mu g_{\beta\gamma}). \quad (3.4)$$

There are $4 \times 4 \times 4 = 64$ symbols for four-dimensional spacetime, but the symmetry of the Christoffel symbols in the lower indices reduces this number to 40 and guarantees the conservation of momentum and energy in the previously mentioned gravitational fields. Once obtained, they allow for an understanding of how vectors change as they move along curved paths, e.g. the Christoffel symbol $\Gamma_{\beta\gamma}^\alpha$ describes how the $\vec{e}_\beta$ basis vector changes with respect to x^γ along the $\vec{e}^\alpha$ direction.

An alternative approach is to derive the equations of motion along a geodesic³ using the Euler-Lagrange principle of least action, i.e.:

$$\frac{d}{d\sigma} \left(\frac{\partial L}{\partial \dot{x}^\alpha} \right) = \frac{\partial L}{\partial x^\alpha}, \quad (3.5)$$

³A geodesic is the path a free particle takes due to the curvature of spacetime.

where $\dot{x}^\alpha$ denotes the derivative of x^α with respect to a path parameter σ , and L is the Lagrangian for massive particles given by

$$L^2 \equiv \left(\frac{d\tau}{d\sigma} \right)^2 = -g_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma}$$

in terms of the proper time τ . Summation over Greek letters refers to a summation over all four spacetime indices $(x^0, x^i) = (ct, r, \theta, \phi)$, and a summation over Latin letters refers to a summation over only the spatial coordinates.

Written explicitly, the Lagrangian is

$$\begin{aligned} L^2 &= -g_{00} \left(\frac{dx^0}{d\sigma} \right)^2 - 2g_{0i} \left(\frac{dx^0}{d\sigma} \right) \left(\frac{dx^i}{d\sigma} \right) - g_{ij} \left(\frac{dx^i}{d\sigma} \right) \left(\frac{dx^j}{d\sigma} \right) \\ &= \left[\frac{d(ct)}{d\sigma} \right]^2 - a^2(t) \left[\frac{1}{1 - kr^2} \left(\frac{dr}{d\sigma} \right)^2 + r^2 \left(\frac{d\theta}{d\sigma} \right)^2 + r^2 \sin^2 \theta \left(\frac{d\phi}{d\sigma} \right)^2 \right] \end{aligned}$$

for a diagonal metric, where $g_{\alpha\beta} = 0$ for $\alpha \neq \beta$. The Euler-Lagrange equation predicts one equation of motion for each spacetime coordinate. The ct , r and θ coordinates yield

$$\frac{d^2(ct)}{d\tau^2} = -\Gamma_{ii}^0 \left(\frac{dx^i}{d\tau} \right)^2, \quad (3.6a)$$

$$\frac{d^2r}{d\tau^2} = -2\Gamma_{01}^1 \frac{d(ct)}{d\tau} \frac{dr}{d\tau} - \Gamma_{11}^1 \left(\frac{dr}{d\tau} \right)^2 - \Gamma_{22}^1 \left(\frac{d\theta}{d\tau} \right)^2 - \Gamma_{33}^1 \left(\frac{d\phi}{d\tau} \right)^2, \quad (3.6b)$$

$$\frac{d^2\theta}{d\tau^2} = -2\Gamma_{02}^2 \frac{d(ct)}{d\tau} \frac{d\theta}{d\tau} - 2\Gamma_{12}^2 \frac{dr}{d\tau} \frac{d\theta}{d\tau} - \Gamma_{33}^2 \left(\frac{d\phi}{d\tau} \right)^2. \quad (3.6c)$$

An important property of the Lagrangian is that conserved quantities can be deduced directly from it. According to Noether's theorem, if a non-zero Lagrangian does not depend on some cyclic coordinate $q_i \equiv \phi$, it follows from the Euler-Lagrange equations that the corresponding momentum is a conserved quantity:

$$\begin{aligned} \dot{p}_\phi &\equiv \frac{d}{d\tau} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi} \\ &= 0 \quad \Rightarrow \quad p_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = \text{constant}. \end{aligned}$$

Furthermore,

$$\frac{d^2\phi}{d\tau^2} = -2\Gamma_{03}^3 \frac{d(ct)}{d\tau} \frac{d\phi}{d\tau} - 2\Gamma_{13}^3 \frac{dr}{d\tau} \frac{d\phi}{d\tau} - 2\Gamma_{23}^3 \frac{d\theta}{d\tau} \frac{d\phi}{d\tau}, \quad (3.7)$$

where the coefficients of the Euler-Lagrange equations are essentially the Christoffel symbols:

$$\begin{aligned} \Gamma_{ii}^0 &= \frac{g_{ii}}{ac} \frac{da}{dt}, & \Gamma_{0i}^i &= \frac{1}{ac} \frac{da}{dt}, & \Gamma_{11}^1 &= \frac{kr}{1 - kr^2}, & \Gamma_{jj}^1 &= -\frac{g_{jj}}{a^2 r} (1 - kr^2), \\ \Gamma_{1j}^j &= \frac{1}{r}, & \Gamma_{33}^2 &= -\frac{1}{2} \sin 2\theta, & \Gamma_{23}^3 &= \cot \theta, \end{aligned}$$

for $i \in \{1, 2, 3\}$ and $j \in \{2, 3\}$. The results enable the calculation of the Ricci curvature tensor, which quantifies the local curvature of spacetime.

3.4 Ricci Tensor

The Ricci terms in the EFEs are mathematical objects used to study spacetime curvature. In particular, the physical meaning of the Ricci tensor is that it describes how the presence (and distribution) of mass and energy in a given region of spacetime affects the curvature of that region. Recall the Riemann tensor:

$$R_{\alpha\gamma\beta}^{\mu} = \partial_{\gamma}\Gamma_{\alpha\beta}^{\mu} - \partial_{\beta}\Gamma_{\alpha\gamma}^{\mu} + \Gamma_{\sigma\gamma}^{\mu}\Gamma_{\alpha\beta}^{\sigma} - \Gamma_{\sigma\beta}^{\mu}\Gamma_{\alpha\gamma}^{\sigma}.$$

The Ricci tensor is a rank-2 tensor whose components are just the Riemann tensor components summed over the upper and third lower index:

$$\begin{aligned} R_{\alpha\beta} &\equiv R_{\alpha\beta\mu}^{\mu} = \partial_{\mu}\Gamma_{\alpha\beta}^{\mu} - \partial_{\beta}\Gamma_{\alpha\mu}^{\mu} + \Gamma_{\sigma\mu}^{\mu}\Gamma_{\alpha\beta}^{\sigma} - \Gamma_{\sigma\beta}^{\mu}\Gamma_{\alpha\mu}^{\sigma} \\ &= \partial_0\Gamma_{\alpha\beta}^0 + \partial_i\Gamma_{\alpha\beta}^i - \partial_{\beta}\Gamma_{\alpha i}^i + 3\Gamma_{01}^1\Gamma_{\alpha\beta}^0 + \Gamma_{ij}^i\Gamma_{\alpha\beta}^j - \Gamma_{i\beta}^0\Gamma_{0\alpha}^i - \Gamma_{0\beta}^i\Gamma_{i\alpha}^0 - \Gamma_{j\beta}^i\Gamma_{i\alpha}^j. \end{aligned}$$

The Ricci tensor is diagonal; it is thus sufficient to calculate only the components $R_{\alpha\alpha}$:

$$R_{\alpha\alpha} = \partial_0\Gamma_{\alpha\alpha}^0 + \partial_i\Gamma_{\alpha\alpha}^i - \partial_{\alpha}\Gamma_{i\alpha}^i + 3\Gamma_{01}^1\Gamma_{\alpha\alpha}^0 + \Gamma_{ij}^i\Gamma_{\alpha\alpha}^j - 2\Gamma_{i\alpha}^0\Gamma_{0\alpha}^i - \Gamma_{j\alpha}^i\Gamma_{i\alpha}^j.$$

The components of the Ricci tensor become

$$\begin{aligned} \text{for } \alpha = 0 : \quad R_{00} &= -3\partial_0\Gamma_{01}^1 - \Gamma_{0j}^i\Gamma_{0i}^j \\ &= -3\partial_0\Gamma_{01}^1 - 3(\Gamma_{01}^1)^2 \\ &= -\frac{3\ddot{a}}{ac^2} \end{aligned}$$

$$\begin{aligned} \text{for } \alpha = i : \quad R_{ii} &= \partial_0\Gamma_{ii}^0 + \partial_i\Gamma_{jj}^i - \partial_j\Gamma_{ij}^i + 3\Gamma_{ii}^0\Gamma_{01}^1 + \Gamma_{ij}^i\Gamma_{kk}^j - 2\Gamma_{ij}^0\Gamma_{0j}^i - \Gamma_{jk}^i\Gamma_{ik}^j \\ &= \partial_0\Gamma_{ii}^0 + \partial_i(\Gamma_{22}^1 + \Gamma_{33}^1 - 2\Gamma_{12}^2) + \partial_2(\Gamma_{33}^2 - \Gamma_{23}^3) \\ &\quad + \Gamma_{ii}^0\Gamma_{01}^1 + \Gamma_{11}^1(\Gamma_{22}^1 + \Gamma_{33}^1 + 2\Gamma_{12}^2) - 2(\Gamma_{12}^2)^2 - (\Gamma_{33}^2 + \Gamma_{23}^3)\Gamma_{23}^3 \\ &= \frac{g_{ii}}{a^2c^2}(a\ddot{a} + 2\dot{a}^2 + 2kc^2). \end{aligned}$$

We use these results to calculate the Ricci scalar.

3.5 Ricci Scalar

Contracting the rank-2 Ricci tensor with the metric reduces the order by one and yields the Ricci scalar:

$$\begin{aligned} R &\equiv R_{\mu}^{\mu} = R_{\mu\nu}g^{\mu\nu} \\ &= R_{0\nu}g^{0\nu} + R_{i\nu}g^{i\nu} \\ &= R_{00}g^{00} + 2R_{0i}g^{0i} + R_{ij}g^{ij} \end{aligned}$$

The inverse metric g^{ij} is the metric such that $g_{ij}g^{ij} = \delta_{ii}$. It is easy to obtain for the FLRW metric since the inverse of a diagonal metric involves just taking the reciprocal of all the diagonal elements, i.e.

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1-kr^2}{a^2(t)} & 0 & 0 \\ 0 & 0 & \frac{1}{a^2(t)r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{a^2(t)r^2 \sin^2 \theta} \end{pmatrix}.$$

Substitute these results into the expression for the Ricci scalar to yield

$$\begin{aligned} R &= -R_{00} + \frac{g_{ii}g^{ii}}{a^2c^2} (a\ddot{a} + 2\dot{a}^2 + 2kc^2) \\ &= \frac{3a\ddot{a}}{a^2c^2} + \frac{3}{a^2c^2} (a\ddot{a} + 2\dot{a}^2 + 2kc^2) \\ &= \frac{6}{a^2c^2} (a\ddot{a} + \dot{a}^2 + kc^2). \end{aligned}$$

This scalar represents the curvature of spacetime at a specific point.

3.6 Energy-Momentum Tensor

The distribution of mass, momentum, and energy in the universe is defined by the energy-momentum (E-M) tensor. In simpler terms, it helps to keep track of the 4-momentum inside a 3-D box in spacetime. The component $T^{\alpha\beta}$ represents the flux of α -momentum through a unit box of constant β . Thus, the components of the E-M tensor are

- the energy density T^{00} ,
- energy flux T^{0i} ,
- momentum density T^{i0} and
- momentum flux T^{ij} , which can be further divided into
 - the pressure T^{ii} , when the momentum is perpendicular to a surface, and
 - the shear stress T^{ij} for $i \neq j$, when the momentum is parallel to the surface.

Given that a simultaneously spatially homogeneous and isotropic universe is considered a perfect fluid, the pressure p exerted by its particles is equal in all directions. In general relativity, the E-M tensor for a perfect fluid in curved spacetime is given by:

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2} \right) U^\mu U^\nu + pg^{\mu\nu},$$

where

$$\vec{U} = \gamma \begin{pmatrix} c & u^x & u^y & u^z \end{pmatrix}^T$$

and $\gamma = 1$ is the Lorentz factor for a particle at rest (i.e. $\vec{u}^i = 0$). The mass density ρ and pressure p are measured in a reference frame where the fluid is stationary in space but moves through time. The off-diagonal components disappear since shear stress and viscosity are absent in a perfect fluid.

It is possible to write the E-M tensor with mixed and lower indices:

$$\begin{aligned} T_\nu^\mu &\equiv T^{\mu\alpha} g_{\alpha\nu} = \left(\rho + \frac{p}{c^2}\right) U^\mu U^\alpha g_{\alpha\nu} + p g^{\mu\alpha} g_{\alpha\nu} \\ &= \left(\rho + \frac{p}{c^2}\right) U^\mu U_\nu + p \delta_\nu^\mu, \end{aligned}$$

where

$$U_\nu \equiv U^\beta g_{\nu\beta} = U^0 g_{0\nu} + U^i g_{\nu i}.$$

$\nu = 0$ sets $U_0 = -U^0$ and $\nu = j$ sets the spatial components $U_j = 0$, since $\vec{u}^i = 0$ for a perfect fluid. Thus, the mixed and lower index E-M tensors are given by

$$\begin{aligned} T_\nu^\mu &= -\left(\rho + \frac{p}{c^2}\right) U^\mu U^\nu + p \delta_\nu^\mu \\ \text{and } T_{\mu\nu} &\equiv T_\mu^\beta g_{\beta\nu} = -\left(\rho + \frac{p}{c^2}\right) U^\beta U^\mu g_{\beta\nu} + p \delta_\mu^\beta g_{\beta\nu} \\ &= -\left(\rho + \frac{p}{c^2}\right) U_\nu U^\mu + p g_{\mu\nu} \\ &= \left(\rho + \frac{p}{c^2}\right) U^\mu U^\nu + p g_{\mu\nu}, \end{aligned}$$

respectively.

Some concerns:

- If the E-M tensor $T_{\mu\nu}$ represents matter and radiation in the universe as a fluid at rest, in which frame is it at rest? In special and general relativity, there should not be an objective rest frame since all inertial frames are equivalent.
- The FLRW metric has the spatial scaling factor $a(t)$, but which reference frame does the t in $a(t)$ belong to? There should not be an objective t coordinate since every reference frame should have an individual t coordinate.

It is possible to address the concerns regarding the E-M tensor's rest frame and the scale factor's coordinate concurrently. It turns out that there is a rest frame for all the radiation and matter in the universe.

Consider observers looking up at the sky from different parts of the Earth. They will measure the CMB radiation coming at them from all directions. The CMB radiation will be redshifted in half of the sky and blueshifted in the other half. Since the Earth is travelling through an ocean of CMB radiation, the radiation will be more redshifted on the half of the sky that the Earth is travelling away from, and it will be more blueshifted on the half that the Earth is travelling toward.

Most reference frames for all the stars and planets in the universe will have the CMB radiation Doppler shifted in one direction or another. Still, there is one unique frame in

which the CMB radiation has no Doppler shift. The reference frame in which this occurs is called the **Cosmic Rest Frame (CRF)**. The 3-D spatial slices observed in this frame define the r, θ, ϕ coordinates in the FLRW metric. The direction orthogonal to these slices defines the time coordinate t used in the scale factor $a(t)$. The coordinates (ct, r, θ, ϕ) collectively describe events from the perspective of an observer falling freely with cosmic expansion [31]. To see this, one needs only to remember Einstein's demonstration that his theory of gravity, based on the equivalence principle, correctly reduces to Newton's law in the weak-field limit if

$$g_{00} = - \left(1 + \frac{2\Phi}{c^2} \right),$$

where Φ is the gravitational potential. Obviously if $g_{00} = -1$, $\Phi \stackrel{!}{=} 0$, and an observer in this frame sees no gravity [26, 32].

3.7 Friedmann Equations

Substituting the metric, Ricci scalar, Ricci tensor, and E-M tensor into the EFEs yields the Friedmann equations:

$$\begin{aligned} R_{00} - \frac{1}{2}Rg_{00} + \Lambda g_{00} &= -\frac{3\ddot{a}}{a^2c^2} + \frac{1}{2}\frac{6}{a^2c^2}(a\ddot{a} + \dot{a}^2 + kc^2) - \Lambda \\ &= \frac{3}{a^2c^2}(\dot{a}^2 + kc^2) - \Lambda \\ &\stackrel{!}{=} \frac{8\pi G}{c^2}\rho. \end{aligned}$$

Let $H = \dot{a}/a$ be the Hubble constant so that

$$\left(\frac{\dot{a}}{a} \right)^2 \equiv H^2 = \frac{8\pi G}{3}(\rho + \rho_\Lambda) - \frac{kc^2}{a^2}, \quad (3.8)$$

where the second term in parenthesis (i.e., $\rho_\Lambda \equiv \Lambda c^2/8\pi G$) is the constant mass density of the vacuum. This equation, which applies to the universe on large scales, describes the universe's rate of expansion. Small objects are not necessarily expanding as a result. The solar system, for instance, is entirely controlled by the Sun's gravitational force and has a constant physical size because local motions within it are dynamically dependent only on local densities.

Equipped with this equation, an expression for the scale factor $a(t)$ can be found as a function of t in various cosmological models. Even without knowing this dependence, significant inferences are effected regarding the general features of the expansion.

First, the parameter k in the first Friedmann equation is particularly interesting. It needs to be spatially independent since the other quantities in the equation are, and it also needs to be independent of time because energy is conserved for any given particle. As a result, k is merely a constant that is proportional to the total energy and independent of space and time. Any expanding universe has a unique value of k that remains constant throughout

its evolution. The value of k determines the form of the evolution, and it can, therefore, be immediately appreciated that:

- as long as ρ remains positive, it is only possible for the expansion of the universe to stop at some time t' if $k = 1$:

$$H^2 \equiv \frac{8\pi G}{3} \rho_T(t) - \frac{c^2}{a^2(t)} \begin{cases} > 0 & \text{for } t < t' \\ = 0 & \text{for } t = t' \\ < 0 & \text{for } t > t' \end{cases},$$

where ρ_T is the total mass density. After that, the universe will collapse;

- a negative k implies

$$H^2 \equiv \frac{8\pi G}{3} \rho_T(t) + \frac{c^2}{a^2(t)} > 0 \text{ for all } t,$$

and the expansion will continue forever;

- $k = 0$ implies the expansion will slow down but only stop at $t = \infty$ [33].

SUPPLEMENTARY:

Cosmological observations indicate that the universe is nearly flat, so $k \approx 0$ [13]. Nonetheless, given what is known about k , that means that the universe's total energy is zero, since every particle in the cosmos will satisfy the first Friedmann equation. Positive kinetic and negative potential energy were somehow separated but precisely balanced by the big bang.

Second, the universe being very nearly flat, means its density is very nearly its critical value⁴ [34]:

$$H_0^2 \equiv \frac{8\pi G}{3} (\rho + \rho_\Lambda) - \frac{kc^2}{a^2} \approx \frac{8\pi G}{3} \rho_c - \overset{0}{\cancel{\frac{kc^2}{a^2}}} \\ \rho_c = \frac{3H_0^2}{8\pi G},$$

where H_0 and ρ_c are the Hubble constant and critical density, respectively. The spatial component of the EFEs gives the second Friedmann equation:

$$\begin{aligned} G_{ii} &\equiv R_{ii} - \frac{1}{2} R g_{ii} + \Lambda g_{ii} = \frac{1}{a^2 c^2} (a\ddot{a} + 2\dot{a}^2 + 2kc^2) g_{ii} - \frac{3}{a^2 c^2} (a\ddot{a} + \dot{a}^2 + kc^2) g_{ii} + \Lambda g_{ii} \\ &= -\frac{1}{a^2 c^2} (2a\ddot{a} + \dot{a}^2 + kc^2) g_{ii} + \Lambda g_{ii} \\ &= -\frac{2}{c^2} \left(\frac{\ddot{a}}{a} + \frac{4\pi G}{3} \rho - \frac{\Lambda c^2}{3} \right) g_{ii} \\ &\stackrel{!}{=} \frac{8\pi G}{c^4} p g_{ii}. \end{aligned}$$

⁴the density necessary to obtain a flat universe

All the terms in the above expression have a common factor of g_{ii} ; thus, multiplying throughout by g^{ii} has the effect of multiplying each term by 3, which can be eliminated by subsequently dividing each term by 3, so that

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left(\rho + \frac{3p}{c^2} - 2\rho_\Lambda \right) \quad (3.9)$$

describes the universe's accelerated expansion. It is also possible to arrive at this result by taking the trace of the EFEs. In this situation, pressure increases the gravitational force, further slowing the expansion. Additionally, note that the constant k , crucial to the first Friedmann equation, is absent from this expression, conveying that the acceleration is not directly influenced by the universe's geometry.

According to [21], general relativity distinguishes between passive mass (the inertial mass that determines the acceleration an object experiences in response to curvature), and active mass (the total source of gravity) [22, 27]. In this case, the active gravitational mass density ρ_{act} of the matter and fields present is

$$\rho_{act} = \rho + \frac{3p}{c^2} - 2\rho_\Lambda.$$

Accordingly, if $\rho_{act} > 0$ (as is the case for all known forms of matter), then $\ddot{a} < 0$ and the universe undergoes a decelerating expansion; if $\rho_{act} < 0$, a positive cosmological constant will cause an accelerated expansion. Recent observations of distant type Ia supernovae indicate that the expansion of the universe is indeed expanding faster [8].

3.8 Fluid Equation

It is evident from the Friedmann equations that the universe is not static; the two Friedmann equations alone can thus not adequately describe the time evolution of the universe's scale factor without an additional equation to describe the time evolution of the density of material in the universe as it changes size. It can be obtained from

- i. differentiating the first Friedmann equation and making the necessary substitutions,
- ii. the first law of thermodynamics – assuming the universe expands adiabatically⁵, and
- iii. Noether's theorem. Energy is not **globally** conserved since the universe is not invariant under time-translation – that is, the universe's scale changes over time. The Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$ guarantees the corresponding **local** conservation of energy-momentum:

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= \partial_\mu T^{\mu\nu} + \Gamma_{\gamma\mu}^\mu T^{\gamma\nu} + \Gamma_{\gamma\mu}^\nu T^{\mu\gamma} \\ &= \partial_0 T^{0\nu} + \partial_i T^{i\nu} + 3\Gamma_{01}^1 T^{0\nu} + (\Gamma_{11}^1 + 2\Gamma_{12}^2) T^{1\nu} + \Gamma_{23}^3 T^{2\nu} + \Gamma_{ii}^\nu T^{ii}. \end{aligned}$$

⁵the symmetry imposed by the cosmological principle implies that there is no net heat flow through any surface, and the entropy is conserved (i.e. $dS = 0$)

$\nu = 0$ sets the energy conservation condition:

$$\begin{aligned}\nabla_\mu T^{0\mu} \stackrel{!}{=} 0 &\Rightarrow \partial_0 T^{00} + 3\Gamma_{01}^1 T^{00} + \Gamma_{ii}^0 T^{ii} = \partial_0 (\rho c^2) + \frac{3\dot{a}}{ac} (\rho c^2) + \frac{\dot{a} p g_{ii} g^{ii}}{ac} \\ &= c \left[\partial_t \rho + \frac{3\dot{a}}{a} \left(\rho + \frac{p}{c^2} \right) \right];\end{aligned}$$

$c \neq 0$ implies

$$\partial_t \rho + \frac{3\dot{a}}{a} \left(\rho + \frac{p}{c^2} \right) = 0.$$

For a spatially homogeneous density

$$\frac{d\rho}{dt} \equiv \frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho = \frac{\partial \rho}{\partial t}$$

so that

$$\frac{d\rho}{dt} + 3H \left(\rho + \frac{p}{c^2} \right) = 0. \quad (3.10)$$

$\nu \in \{1, 2, 3\}$ satisfies the momentum conservation condition trivially for an FLRW universe: $\nabla_\mu T^{\mu i} = 0$.

Eq. 3.10 is commonly known as the **fluid equation** and indicates that two terms contribute to the change in density as the universe expands. It is simple to understand that the first term in the brackets represents the dilution in the density due to the increased volume. The second term is the energy loss because the cosmic fluid works to increase the universe's volume. Since density and pressure are constant throughout a homogeneous universe, "pressure" does not refer to a pressure gradient that provides the force propelling expansion. The prevailing assumption in cosmology is that every density has a unique associated pressure.

In the standard model of cosmology, the viewpoint that the cosmological fluid's total energy content is composed of three components is typical: matter, radiation and dark (or vacuum) energy [35, 36]. The density and pressure of these components are related by an equation of state (EOS), the form of which depends on the properties of the universe's constituents and must be specified.

3.9 Equation of State

An equation of state originates in thermodynamics and chemistry, where variables that describe a system's state – such as pressure, volume, temperature, density, etc. – are related to one another in an equation. The most well-known equation of state is the ideal gas law:

$$pV = nRT,$$

which relates the pressure, volume, temperature and number of moles of gas. For a perfect fluid, the equation of state should relate the density ρ and the pressure p . Although various

equations of state could describe a perfect fluid, it is customary, in cosmological cases, to assume the linear form

$$p_i = w_i \rho_i c^2, \quad (3.11)$$

which solves the Friedmann equations exactly by setting the pressure $p_i(\rho)$ proportional to the energy density $\rho_i c^2$. The equation-of-state parameter w_i is a constant of proportionality, and i denotes the component [29, 37]. In more exotic cosmological models, w_i may depend on cosmic time t . If w_i is a constant, provided that the weak energy condition is fulfilled and the local sound speed $(dp/d\rho)^{1/2}$ is less than c , w_i must fall within the range $-1 \leq w_i \leq 1$ [34]. In general, EOS with smaller values of w are associated with faster universe expansion. As a result, hypothetical "phantom energy" would correspond to negative kinetic energy or negative mass and, therefore, an expanding universe in which, in the future, distances between particles will eventually become infinite. Such a universe would have an equation of state parameter $w < -1$ and cause a "big rip"; therefore, the idea of phantom energy is often disregarded.

A solution to the fluid equation in terms of the EOS parameter follows:

$$\begin{aligned} \frac{d\rho_i}{dt} &= -\frac{3\rho_i}{a} \frac{da}{dt} (1 + w_i) \\ \int \frac{1}{\rho_i} d\rho_i &= -3(1 + w_i) \int \frac{1}{a} da \\ \ln \rho_i &= -3(1 + w_i) \ln a + C. \end{aligned}$$

Assert that $a(t_0) = 1$ so that

$$\begin{aligned} \ln \rho_{i,0} = C &\quad \Rightarrow \quad \ln \rho_i = \ln \rho_{i,0} - 3(1 + w_i) \ln a \\ \rho_i &= \rho_{i,0} a^{-3(1+w_i)}, \end{aligned}$$

where t_0 is the current age of the universe and $\rho_{i,0}$ is the current value of the i -th component of the cosmic fluid.

- In general, non-relativistic matter in the universe is classified into two types: more exotic forms of matter made up of fundamental particles that are outside the standard model of particle physics, and the normal baryonic matter of everyday experience (such as protons and neutrons). Since the evolution of the scale factor with cosmic time t depends only on the total matter density, no distinction is made between different types of matter. A common assumption is that the matter particles (regardless of their form) have a thermal energy significantly lower than their rest mass energy [34], i.e., $k_B T \approx mv^2$ with $v \ll c$. Cosmologists sometimes refer to this as "dust" precisely to describe matter with negligible pressure. In this case, $w = 0$ [19, 33] and $\rho_m = \rho_{m,0} a^{-3}$, and this supports the density's expected behaviour that all the energy in pressureless matter is in the mass, and the density merely dilutes with the volume scale $a^3(t)$. This makes intuitive sense. If the amount of material is constant in space but the volume increases, the density will decrease proportionately.

- Photons and other highly relativistic species (like neutrinos) with very small or zero rest masses are naturally included in the term "radiation". Cosmic microwave background photons are evenly distributed throughout the universe and are the primary source of the universe's radiation energy density [34].

To describe radiation, recall the E-M tensor for pure electromagnetic radiation

$$T^{\mu\nu} = \frac{1}{\mu_0} \left(F^{\mu\alpha} F_{\alpha}^{\nu} - \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right),$$

where $F^{\mu\alpha}$ is the Faraday tensor given by

$$F^{\mu\alpha} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}.$$

The electromagnetic field has no Lorentz-invariant energy scale; thus, its E-M tensor must have a vanishing trace. Therefore, take the trace of the E-M tensor:

$$\begin{aligned} T_{\sigma}^{\mu} &= g_{\nu\sigma} T^{\mu\nu} \\ &= \frac{1}{\mu_0} \left(g_{\nu\sigma} F^{\mu\alpha} F_{\alpha}^{\nu} - \frac{1}{4} g^{\mu\nu} g_{\nu\sigma} F^{\alpha\beta} F_{\alpha\beta} \right) \\ &= \frac{1}{\mu_0} \left(F^{\mu\alpha} F_{\sigma\alpha} - \frac{1}{4} \delta_{\sigma}^{\mu} F^{\alpha\beta} F_{\alpha\beta} \right). \end{aligned}$$

Set $\sigma = \mu$ so that

$$\begin{aligned} T_{\mu}^{\mu} &= \frac{1}{\mu_0} \left(F^{\mu\alpha} F_{\mu\alpha} - \frac{1}{4} \delta_{\mu}^{\mu} F^{\alpha\beta} F_{\alpha\beta} \right) \\ &= \frac{1}{\mu_0} \left(F^{\mu\alpha} F_{\mu\alpha} - F^{\alpha\beta} F_{\alpha\beta} \right). \end{aligned}$$

α and μ are dummy indices, so relabel them as α and β to yield the expected result:

$$T_{\mu}^{\mu} = 0.$$

Recall the trace for a perfect fluid E-M tensor:

$$\begin{aligned} T_{\mu}^{\mu} &= - \left(\rho + \frac{p}{c^2} \right) + 4p \\ &= -\rho c^2 + 3p \\ &\stackrel{!}{=} 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} p &= \frac{1}{3} \rho c^2 \\ p &= w \rho c^2, \end{aligned}$$

where $w = 1/3$ [19, 33] and $\rho_r = \rho_{r,0} a^{-4}$. Radiation density is also diluted by the expansion of space and scales as $\rho \propto 1/a^3$. However, the photons' cosmological redshift adds an extra factor of $1/a$.

- It is assumed that the universe has an intrinsic non-zero cosmological constant Λ [34] in addition to matter and radiation. The modern interpretation of Λ is in terms

of the energy density of the vacuum (dark energy) with a strange equation of state $p = -\epsilon$, resulting in a fluid with a negative effective pressure [34], corresponding to $w = -1$ and constant mass density $\rho_\Lambda = \rho_{\Lambda,0}$. Moreover, unlike radiation or ordinary matter, dark energy is gravitationally repulsive. As a result, it is undiluted by the expansion of space and eventually surpasses the steadily declining densities of matter and radiation, leading to a late-time accelerated expansion [19, 33, 36]. Set $w = -1$ so that

$$\begin{aligned} T_{\mu\nu} &= \left(\rho - \frac{\rho c^2}{c^2} \right) U^\mu U^\nu - \rho c^2 g_{\mu\nu} \\ &= -\rho c^2 g_{\mu\nu}. \end{aligned}$$

The E-M tensor is simply a constant multiplied by the metric – which appears exactly like the cosmological constant term in the EFEs. Replace the cosmological constant term with an energy density term to yield

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \rho c^2 g_{\mu\nu} &= \frac{8\pi G}{c^4} T_{\mu\nu} \\ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} &= \frac{8\pi G}{c^4} T_{\mu\nu} + \rho c^2 g_{\mu\nu} \\ &= \frac{8\pi G}{c^4} T_{\mu\nu} + \tilde{T}_{\mu\nu}, \end{aligned}$$

where $\tilde{T}_{\mu\nu} = \rho c^2 g_{\mu\nu}$; because of this, the cosmological constant is deemed similar to an E-M tensor with negative pressure, which accelerates rather than slows down the universe's expansion. It is possible to consider other constituents, such as quintessence – a hypothetical form of dark energy with $-1 < w < -1/3$.

Substituting these results into the Friedmann equation yields

	w	Density	a(t)
Radiation	$w = 1/3$	$\rho_r = \rho_{r,0} a^{-4}$	$a(t) = (2H_0 t)^{1/2}$
Dust	$w = 0$	$\rho_m = \rho_{m,0} a^{-3}$	$a(t) = \left(\frac{3H_0 t}{2}\right)^{2/3}$
Dark Energy	$w = -1$	$\rho_\Lambda = \rho_{\Lambda,0}$	$a(t) \propto e^{H_0 t}$

Table 3.1: Scale factor for the matter, radiation and dark-energy-dominated eras

Even though the universe is a mixture of these three components [35], it still obeys the Friedmann equations in the same way as the individual densities would when combined into a single value of $\rho(t)$. Combining the above results, the total equivalent mass density can be expressed as the sum of the individual contributions

$$\begin{aligned} \rho(t) &= \rho_m(t) + \rho_r(t) + \rho_\Lambda(t) \\ &= \frac{\rho_{m,0}}{a^3(t)} + \frac{\rho_{r,0}}{a^4(t)} + \rho_{\Lambda,0}, \end{aligned}$$

where t is the cosmic time [19, 34, 35, 36]. Furthermore, suppose that these three components do not interact, e.g., lack the ability to exchange momentum or energy. It is not accurate to assume that before recombination, the matter and radiation densities scaled

as $\rho_m \propto a^{-3}$ and $\rho_r \propto a^{-4}$, respectively. Matter was created and annihilated, exchanging its density with that of the radiation field (and possibly other fields that may arise from extensions of the standard model of particle physics) [26]. However, it is reasonable to assume that interactions between matter and radiation are insignificant because, for most of the universe's history, matter and radiation evolved independently of one another [26].

In this model of the universe, as the universe evolves, so do the relative contributions of matter, radiation and dark energy to the total density. The universe started as a thermalised radiation plasma that was infinitely dense, i.e.,

$$\begin{aligned} \lim_{a \rightarrow 0} w &\equiv \lim_{a \rightarrow 0} \frac{\sum_i p_i}{\sum_i \rho_i} = \frac{1}{3} \lim_{a \rightarrow 0} \frac{\rho_{r,0} - 3\rho_{\Lambda,0}a^4}{\rho_{m,0}a + \rho_{r,0} + \rho_{\Lambda,0}a^4} \\ &= \frac{1}{3} \left(\frac{\rho_{r,0} - 3\rho_{\Lambda,0}a^4}{\rho_{m,0}a + \rho_{r,0} + \rho_{\Lambda,0}a^4} \right) \\ &= \frac{1}{3}. \end{aligned}$$

As the universe cools and matter takes over, the radiation density decreases. Dark energy, which is primarily responsible for the apparent acceleration currently observed [5], takes over the density at large times as a result of further expansion, causing the matter density to decrease as well, i.e.,

$$\begin{aligned} \lim_{a \rightarrow \infty} w &\equiv \lim_{a \rightarrow \infty} \frac{\sum_i p_i}{\sum_i \rho_i} = \frac{1}{3} \lim_{a \rightarrow \infty} \frac{\rho_{r,0} - 3\rho_{\Lambda,0}a^4}{\rho_{m,0}a + \rho_{r,0} + \rho_{\Lambda,0}a^4} \\ &= -4 \lim_{a \rightarrow \infty} \frac{\rho_{\Lambda,0}a^3}{\rho_{m,0} + 4\rho_{\Lambda,0}a^3} \\ &= - \lim_{a \rightarrow \infty} \frac{a^2}{a^2} \\ &= -1. \end{aligned}$$

Furthermore, the value of each density, and thus the total density, is determined at all other cosmic times t if the values of the equivalent mass densities $\rho_m(t')$, $\rho_r(t')$, and $\rho_{\Lambda}(t')$ are specified at a specific cosmic time t' . The previously mentioned values are sufficient to find the scale factor $a(t)$ at all cosmic times. The current cosmic time t_0 is most naturally assumed to be t' .

However, cosmological times cannot be directly observed, as all human observers are confined to Earth, and the timescale of life is much too short in relation to the age of the cosmos for anything to change significantly over a lifespan. Therefore, the best way to observe the universe's evolution is to observe the redshift of light caused by the universe's expansion.

3.10 Cosmological Redshift

When light from distant objects is measured, it produces a spectrum with spectral lines. The frequency and pattern of spectral lines expected from various elements and ions are

known from measurements made on Earth. Thus, conclusions are deduced from comparisons of the observed spectral lines with known spectra and the composition of the objects. The observed spectra are redshifted (the wavelengths are scaled compared to laboratory measurements) and are not the same as on Earth. If the wavelength of the emission line in the laboratory is λ and the observed wavelength is $\lambda_0 > \lambda$, then the line is said to be redshifted by a fraction z given by

$$z \equiv \frac{\lambda_0 - \lambda}{\lambda} = \frac{\lambda_0}{\lambda} - 1.$$

The redshift correlates with observables and is a natural byproduct of the Doppler effect. It is one of the few observational parameters determined from accurate measurements of the frequencies of spectral lines. In an expanding universe, the cosmological redshift can be written as a function of the monotonically increasing scale factor $a(t)$ via the wavelength as:

$$\begin{aligned} \frac{a(t_0) = 1}{a(t)} &= \frac{\lambda_0}{\lambda} \\ &= 1 + z. \end{aligned}$$

If the scale factor $a(t)$ increases or decreases with cosmic time t , i.e. the universe is expanding or contracting, then the photon is redshifted or blueshifted by an amount z , respectively.

Therefore, from the point of view of observations it is not necessary to proceed by the direct route of determining $a(t)$. To the observer, the evolution of the scale factor is directly characterised by the change in the Hubble parameter with redshift – $H(z)$. Therefore, it makes sense to rewrite the Friedmann equations using these parameters [34]

$$\begin{aligned} H^2 &\equiv \frac{8\pi G}{3} \sum_i \rho_i = \frac{H_0^2}{\rho_c} \left(\frac{\rho_{m,0}}{a^3} + \frac{\rho_{r,0}}{a^4} + \rho_{\Lambda,0} \right) \\ &= \frac{H_0^2}{\rho_c} \left[\rho_{m,0} (1+z)^3 + \rho_{r,0} (1+z)^4 + \rho_{\Lambda,0} \right] \\ &= H_0^2 \left[\Omega_{m,0} (1+z)^3 + \Omega_{r,0} (1+z)^4 + \Omega_{\Lambda,0} \right], \end{aligned}$$

where $\Omega_{i,0} \equiv \rho_{i,0}/\rho_c$ is the dimensionless fractional density parameter that satisfies the flatness condition $\sum_i \Omega_{i,0} = 1$. Similarly,

$$\begin{aligned} \frac{\ddot{a}}{a} &\equiv -\frac{4\pi G}{3} \left[\sum_{i=m,r} \left(\rho_i + \frac{3p_i}{c^2} \right) - 2\rho_{\Lambda} \right] = -\frac{4\pi G}{3} (\rho_m + 2\rho_r - 2\rho_{\Lambda}) \\ &= -\frac{H_0^2}{2\rho_c} \left(\frac{\rho_{m,0}}{a^3} + \frac{2\rho_{r,0}}{a^4} - 2\rho_{\Lambda,0} \right) \\ &= -\frac{H_0^2}{2} \left[\Omega_{m,0} (1+z)^3 + 2\Omega_{r,0} (1+z)^4 - 2\Omega_{\Lambda,0} \right]. \end{aligned}$$

Chapter 4

On the $R_h = ct$ Universe

4.1 Hubble Radius

When American astronomer Edwin Hubble combined measurements from multiple sources in 1929, he discovered that distant galaxies are moving away from each other at a velocity $\vec{v}$ proportional to the proper distance $\vec{x}$ between them, i.e. the famous Hubble's law

$$\vec{v} = H_0 \vec{x}, \quad (4.1)$$

where H_0 is the Hubble “constant” of proportionality [34]. As is known, this is proof that the universe is expanding. According to current calculations, a galaxy that is 1 Mpc away from Earth is moving away at roughly H_0 km/s; another, that is 2 Mpc away is moving at $2H_0$ km/s, and so on. The distance a photon must travel to reach the galaxy moving away from Earth at the speed of light is approximately 14 billion light-years since the speed of light is about 3×10^5 km/s. This means that the surface of a sphere with a radius of roughly 14 billion light-years centred on the Earth is thus moving away from Earth at the speed of light. It also means that larger spheres are moving away at arbitrary speeds greater than c and that Earth will never be able to receive the light emitted from these galaxies. Since galaxies outside the Hubble sphere are invisible to the human eye, this threshold distance is the Hubble radius [38, 39]. Sources outside the Hubble sphere are not currently observable, despite [40] suggesting otherwise. Much of this confusion stems from a misunderstanding of proper and comoving speeds. Proper speed has a limit, yet comoving speeds can surpass the speed of light. The erroneous conclusion that such "superluminal" recession speeds are incompatible with special relativity can result from failing to recognise this distinction [41].

The claim that these galaxies are moving away at arbitrary speeds greater than c is not entirely true; the expanding distance between Earth and the galaxies is because of the universe expanding rather than the galaxies' motion away from Earth. Consider drawing dots on an inflated balloon. The balloon's stretching causes the dots to get farther apart even though they remain stationary on its surface. It is, therefore, reasonable to say that

galaxies that travel faster than c exist, though only insofar as caused by the expansion of space. An observer would see all the grid points surrounding them recede on a uniform grid. Since any observer in the grid would see the same thing, homogeneity is fully respected, despite the observer's misconception that they are in the centre of the universe.

4.2 Gravitational Horizon

Before Wolfgang Rindler's distinction between event and particle horizons in 1956, the subject matter of cosmological horizons was unclear [38]. In cosmology, the term "horizon" now refers to either i) the proper distance particles could have travelled as regards an observer – since the big bang – or ii) the two-dimensional surface that forever separates causally connected spacetime events from those that are not [27]. A "toy" model based on a universe dominated by a single, stable, static, isotropic, perfect-fluid energy was used by [42] to discuss the possibility that a gravitational horizon might exist. The conclusion was that the energy would gravitationally affect an object over a maximum scale length, as implied by the Friedmann equations. Ironically, a limiting radius equal to the gravitational horizon appeared more than a century ago in de Sitter's solution to the Einstein equations. The choice of coordinates for which the gravitational horizon appears in his metric coefficients lost favour after the introduction of comoving coordinates in the 1920s.

In general relativity, a gravitational horizon is an imaginary surface beyond which all null geodesics – even those pointing towards the observer – actually move away from the observer. When matter is distributed within a spherical volume surrounded by near-vacuum, it is easy to understand how the mass curves the surrounding spacetime. A gravitational horizon appears for the mass when its radius is small enough for the escape velocity to equal or exceed the speed of light [27]. Karl Schwarzschild (1873–1916) formally derived the radius at which this occurs. The Schwarzschild metric that describes such a spacetime is time-independent so the gravitational horizon is actually an event horizon, and these terms are often used interchangeably.

The idea of a gravitational horizon is more complex to apply in cosmology than, for instance, for a compact star. To understand the emergence of a non-static gravitational horizon in cosmology, imagine an observer at the centre of a spherical cavity (with proper radius R_C), surrounded by a spherically symmetric mass (with proper radius $R_S < R_C$) [7]. The metric in the space between the mass and the cavity's edge, given by the Schwarzschild solution, describes the spacetime around a compact mass [7]. The relative acceleration between the observer and R_S is only due to the mass enclosed within this radius [26, 31].

The density within this region is a function of cosmic time t [7], so when the observer extends their perspective to progressively larger radii R_S , a threshold of enclosed mass, for which R_S becomes the gravitational horizon R_h , is achieved. Regardless of which specific

cosmological model one adopts, the gravitational horizon is expressed as:

$$R_h = \frac{2GM}{c^2},$$

where M is the proper mass (sometimes referred to as the Misner-Sharp mass [43]) contained within a sphere of proper radius R_h [26, 31], i.e.

$$M \equiv \rho V = \frac{4\pi\rho}{3} R_h^3.$$

The mass is not arbitrary since only the Misner-Sharp mass is defined this way. It is possible to identify other spherically symmetric mass distributions that reduce to the Misner-Sharp mass, such as the Hawking-Hayward and Brown-Yang mass.

From these results

$$R_h \equiv \frac{2GM}{c^2} = \frac{8\pi G}{3c^2} \rho R_h^3$$

$$R_h \left(1 - \frac{H^2 R_h^2}{c^2} \right) = 0.$$

$R_h = 0$ is a vacuum solution, i.e. it would lead to the conclusion that, since the density is proportional to $\dot{a}$ in the Friedmann equation, then $R_h = 0$ must correspond to a spacetime with $\dot{a} = 0$, i.e. the metric is static, which is incorrect. Unlike the Milne universe, this model is not empty, thus, the only non-empty physical solution is

$$R_h = \frac{c}{H}. \quad (4.2)$$

Unlike the Schwarzschild horizon, R_h , in this case, may depend on time; a sphere with this radius is not necessarily an event horizon, but may become one in the asymptotic future, depending on the properties of the cosmic fluid [27]. Therefore, at any cosmic time, R_h defines a gravitational horizon that separates null geodesics approaching, from those receding. Such a horizon is more advantageous than a stationary event horizon in dynamic situations like those that arise when gravitational waves are produced during black-hole merging events.

4.3 The $R_h = ct$ Model

The proper radius $x(t)$ express the changing distance that increases with the expansion of the universe rather than the comoving distance r between two points. This definition of $x(t)$ is a result of Weyl's postulate, which states that, for the cosmological principle to hold from one time-slice to the next, no two worldlines can ever cross after the big bang (apart from local peculiar motion that may exist on top of the averaged Hubble flow) [15, 27]. To meet this requirement, every proper distance in an FLRW cosmology must be expressible as the product of the expansion factor $a(t)$ and a comoving distance r . A galaxy a proper distance $\vec{x}(t)$ from an observer must, therefore, be receding at proper speed

$$x = a(t)r \Rightarrow \frac{dx}{dt} = \frac{da}{dt}r + a\frac{dr}{dt}$$

$$= Hx + a\frac{dr}{dt}.$$

However, the null geodesic condition ($ds = 0$) applied to the FLRW metric yields

$$c^2 dt^2 = \frac{a^2(t)}{1 - kr^2} dr^2 \equiv a^2(t) dr^2; \text{ for a flat universe}$$

$$\frac{dr}{dt} = \mp \frac{c}{a}.$$

In the asymptotic future $\dot{r} = 0$; this means there is no reason to assume that the comoving distance is constant at all other cosmic times $t \neq \infty$, and, consequently, that the relative comoving velocity $a\dot{r}$ is zero.

According to the Friedmann equations [31, 44] and the general definition of R_h as an implicit function of time, the gravitational horizon in all expanding homogeneous and isotropic universes must inevitably be a proper distance $R_h(t) = a(t) r_h$ and evolve in accordance with the following dynamical equation [45]:

$$\begin{aligned} \frac{dR_h}{dt} &\equiv \frac{da}{dt} r_h + a \frac{dr_h}{dt} = c \frac{d}{dt} \left(\frac{1}{H} \right) \\ a H r_h + a \frac{dr_h}{dt} &= -\frac{c}{H^2} \frac{dH}{dt} \\ H R_h + a \frac{dr_h}{dt} &= H R_h + \frac{1}{2} c (1 + 3w), \end{aligned}$$

where the EOS parameter w now relates the total pressure in terms of the total energy density. All comoving bodies at the horizon recede from the observer at velocity $v_h = H R_h$ and the horizon itself overtakes those comoving bodies at relative velocity

$$a \frac{dr_h}{dt} \equiv \frac{1}{2} c (1 + 3w) = \begin{cases} c/2 & \text{in a matter-dominated universe} \\ c & \text{in a radiation-dominated universe.} \end{cases}$$

It is important to note the following:

- R_h is constant only in a de Sitter universe with $w = -1$. $\dot{R}_h > 0$ for all other values of $w > -1$. For instance, in Λ CDM, where dark energy and dark matter coexist to dominate the universe, $\dot{R}_h > 0$. If dark energy were a true cosmological constant, the universe would eventually become de Sitter as the density of matter drops to zero, and R_h would asymptotically approach a constant value equal to the horizon in Λ CDM;
- Given what is known about the properties of R_h

$$\frac{dR_h}{dt} \stackrel{!}{=} c \Rightarrow a \frac{dr_h}{dt} = \frac{1}{2} c (1 + 3w) \stackrel{!}{=} 0,$$

which is only true if, and only if, r_h remains constant as the universe expands and $w = -1/3$ simultaneously and allows the proper speed $\dot{R}_h$ to be rewritten as:

$$\begin{aligned} \frac{dR_h}{dt} &= \frac{da}{dt} r_h + a \frac{dr_h}{dt} \overset{0}{\nearrow} \\ &= H R_h \\ &= c. \end{aligned}$$

This is the well-known Hubble law, according to which the recession speed reaches c at the Hubble radius $R_h(t)$. This proper distance also coincides with the gravitational horizon. The Hubble radius is not merely an empirical artefact of the expanding universe but a manifestation of the gravitational horizon since it effectively represents the radius at which a sphere centred on the observer contains enough mass energy for its surface to function as a static horizon [31]. In other words, the very existence of a Hubble radius is, therefore, due to the presence of the gravitational horizon, which may shift and eventually reveal hitherto unseen regions at greater proper distances. The proper size of the visible universe is determined by the furthest proper distance achieved by the null geodesics that reach the observer at time t_0 [26, 31, 27].

Furthermore,

$$R_h \equiv \frac{c}{H} = ar_h \quad \Rightarrow \quad \begin{aligned} \frac{c}{r_h} &= aH \\ &= \frac{da}{dt} \\ &= \text{constant}. \end{aligned}$$

A cursory examination of the Friedmann equations suggests that this is true if, and only if, the acceleration $\ddot{a}(t) = 0$, i.e. for either an empty universe in which $p = \rho = 0$ or one characterised by total EOS parameter $w = -1/3$ [36, 7]:

$$\frac{\ddot{a}}{a} \equiv -\frac{4\pi G}{3c^2} (\epsilon_T + 3p_T) \stackrel{!}{=} 0 \quad (4.3)$$

if

$$\begin{aligned} \epsilon_T + 3p_T &= \epsilon_T + 3w\epsilon_T \\ &= \epsilon_T (1 + 3w) \\ &= 0. \end{aligned}$$

Where the overall EOS of the constituents in the Λ CDM model must be empirically inferred, the total energy density ϵ_T and pressure p_T in the $R_h = ct$ model are related by the EOS parameter $w = -1/3$, resulting in the simplification of observable quantities, such as the redshift-dependent Hubble parameter and luminosity distance.

For a perfect cosmological fluid, the constraint $\epsilon + 3p = 0$ means that its active mass is zero: an elegant and meaningful physical property that results in zero acceleration or constant expansion for what is known as the non-empty "always coasting" model [46]. The absence of an early deceleration phase is perhaps its most alluring feature: the $R_h = ct$ universe does not require inflation to avoid the well-known horizon problem [26], whereas Λ CDM cannot survive without it.

For such a model, $\dot{a} = \text{constant}$ implies $a \propto t$ and

$$\begin{aligned} \frac{a(t=t_0)=1}{a(t)} &\equiv 1+z = \frac{t_0}{t} \\ a &= \frac{t}{t_0}, \end{aligned}$$

which leads directly to the expression for the Hubble parameter:

$$H \equiv \frac{1}{a} \frac{da}{dt} = \frac{t_0}{t} \left(\frac{1}{t_0} \right) \quad (4.4a)$$

$$= H_0 (1 + z). \quad (4.4b)$$

Substituting this result into the expression for the gravitational horizon yields

$$R_h \equiv \frac{c}{H} = ct.$$

Unlike the Λ CDM model, which may contain as many as four parameters, $H_0, \Omega_{m,0}, \Omega_{\Lambda,0}$ and $\Omega_{r,0}$, depending on how one parameterises the dark energy [11], the $R_h = ct$ model solely relies on the Hubble constant H_0 [36]. But what does all of this mean? Is the universe really this simple? The expansion dynamics in the standard model are inferred from the properties of the individual components: matter, radiation and dark energy. In a flat, expanding universe that adheres to Weyl's principle, each component evolves such that radiation dominates early and matter becomes more significant later. Late times are dominated by the nonzero contribution of dark energy.

In $R_h = ct$, the situation is very different. The universe, as we know it, has the same mixture of baryonic matter ρ_b , cold dark matter ρ_d , radiation ρ_r , and dark energy ρ_{Λ} , but it also has zero active mass. In many applications, particularly when determining observable quantities such as the luminosity distance and the redshift-dependent Hubble parameter $H(z)$ [47], the precise composition of the cosmic fluid is not necessary [19]. It makes no difference how all the varying components adjust their relative densities and balance each other through the various forces and particle-particle interactions experienced [26] to produce a total equation-of-state

$$\epsilon + 3p = 0$$

for which

$$\rho = \rho_c (1 + z)^2. \quad (4.5)$$

The evolution of each constituent should differ significantly from that the standard model due to these restrictions.

When the evolution of individual components is needed, several conservation laws and reasonable assumptions delimit their behaviour [46]:

$$\begin{aligned} \sum_i \rho_i &\equiv \rho_m + \rho_r + \rho_{\Lambda} = \rho_T \\ \sum_i p_i &\equiv \frac{1}{3}\rho_r + w_{\Lambda}\rho_{\Lambda} = -\frac{1}{3}\rho_T \quad \Rightarrow \quad \rho_{\Lambda} = -\frac{\rho_T}{3w_{\Lambda}} \left(1 + \frac{\rho_r}{\rho_T} \right). \end{aligned}$$

Substitute into the density equation to yield

$$\begin{aligned} \rho_m + \rho_r + \rho_{\Lambda} &\equiv \rho_m + \rho_r - \frac{\rho_T}{3w_{\Lambda}} \left(1 + \frac{\rho_r}{\rho_T} \right) = \rho_T \\ \frac{\rho_T}{3w_{\Lambda}} \left(1 + \frac{\rho_r}{\rho_T} \right) &= \rho_m + \rho_r - \rho_T \\ w_{\Lambda} &= \frac{\rho_r + \rho_T}{3(\rho_m + \rho_r - \rho_T)}. \end{aligned}$$

- for $a = 0$,

$$w_\Lambda \equiv \frac{a^2 + \Omega_r}{3(a\Omega_m + \Omega_r - a^2)} = \frac{\Omega_r}{3\Omega_r} = \frac{1}{3}.$$

- for $a=1$,

$$w_\Lambda = \frac{1 + \Omega_{r,0}}{3(\Omega_{m,0} + \Omega_{r,0} - 1)}.$$

The current temperature of the CMB radiation ($T_0 \approx 2.73 \text{ K}$) translates to a normalised radiation energy density $\Omega_r \approx 6 \times 10^{-5}$ [11]. Therefore,

$$w_\Lambda \approx -\frac{1}{3\Omega_{\Lambda,0}}.$$

w_Λ must be approximately equal to $-1/2$ in order to produce a partitioning of constituents in line with what is observed in the local universe. With $w_\Lambda \approx -1/2$,

$$\begin{aligned} \Omega_{\Lambda,0} &\equiv \frac{\rho_{\Lambda,0}}{\rho_c} \approx -\frac{1}{3w_\Lambda} \left(1 + \cancel{\Omega_{r,0}}\right)^0 \\ \Omega_{\Lambda,0} &= \frac{2}{3} \end{aligned}$$

and

$$\begin{aligned} \rho_{m,0} + \rho_{r,0} + \frac{2}{3}\rho_c &= \rho_c & \Rightarrow & & \rho_{m,0} &= \frac{1}{3}\rho_c - \rho_{r,0} \\ & & & & \Omega_{m,0} &\approx \frac{1}{3}. \end{aligned}$$

Even without taking into account the case for $a \rightarrow \infty$, it is clear that if the cosmic fluid is to maintain $1 + 3w = 0$, dark energy cannot be a cosmological constant. It needs to be dynamic, and it might have a particle origin that goes beyond the parameters of particle physics.

The Friedmann equation is re-written as

$$\begin{aligned} H^2 &\equiv \frac{1}{a^2} \left(\frac{da}{dt}\right)^2 = \frac{H_0^2}{a^4} (a\Omega_{m,0} + \Omega_{r,0} + a^4\Omega_{\Lambda,0}) \\ \frac{a^2}{a\Omega_{m,0} + \Omega_{r,0} + a^4\Omega_{\Lambda,0}} (da)^2 &= H_0^2 (dt)^2 \\ R_h(t_0) \int_0^1 \frac{a}{\sqrt{a\Omega_{m,0} + a^4(1 - \Omega_{m,0})}} da &= c \int_0^{t_0} dt. \end{aligned}$$

The proposed form of the gravitational (i.e. Hubble) radius leads to the equality $R_h(t_0) = ct_0$. Therefore, any cosmological model consistent with the cosmological principle and Weyl's postulate must satisfy the condition [7]:

$$\int_0^1 \frac{a}{\sqrt{a\Omega_{m,0} + a^4(1 - \Omega_{m,0})}} da = 1. \quad (4.6)$$

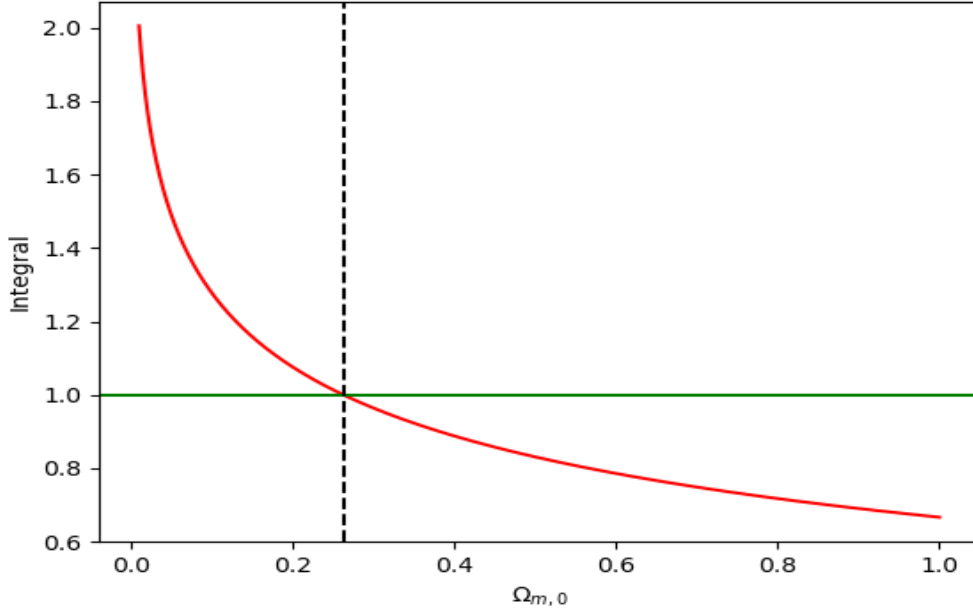


Figure 4.1: The integral as a function of $\Omega_{m,0}$ for the standard model (i.e. Λ CDM).

Not surprisingly, this integral is 1 when $\Omega_{m,0} \approx 0.263$. This inferred value of $\Omega_{m,0}$ does not come from fits to cosmological data but is consistent with the optimised parameters of the standard model. The fact that the universe comprises $\approx 26\%$ matter and $\approx 74\%$ dark energy is a mystery in Λ CDM [48]. In the $R_h = ct$ universe, it is clear why $\Omega_{m,0} \approx 0.263$. When one forces ρ to have the specific constituents ρ_m , ρ_r and ρ_Λ , only $\Omega_{m,0} \approx 0.263$ will allow the universe to evolve in a way that satisfies $R_h(t_0) = ct_0$ today. Despite not being immediately apparent, this constraint suggests that regardless of any prior deceleration or acceleration, the overall acceleration averaged over the universe's history must be zero [7], and can be seen directly from the FLRW equations, which indicate that

$$\frac{dR_h}{dt} = \frac{3}{2}c(1+w).$$

Assume that the gravitational horizon was much smaller in the distant past than it is today [7] and integrate the equation to get:

$$\begin{aligned} \int_0^{R_h(t_0)} dR_h &= \frac{3}{2}c \int_0^{t_0} (1+w) dt \\ R_h(t_0) &= \frac{3}{2}c \left[t_0 + \langle w \rangle \int_0^{t_0} dt \right] \\ &= \frac{3}{2}ct_0 (1 + \langle w \rangle). \end{aligned}$$

For $R_h(t_0)$ to be equal to ct_0 requires

$$\begin{aligned} \frac{3}{2}ct_0 (1 + \langle w \rangle) - ct_0 &\equiv \frac{1}{2}ct_0 (1 + 3\langle w \rangle) = 0 \\ ct_0 \neq 0 &\Rightarrow 1 + 3\langle w \rangle \stackrel{!}{=} 0 \\ \langle w \rangle &= -\frac{1}{3} \end{aligned}$$

to correspond to to an average acceleration of zero.

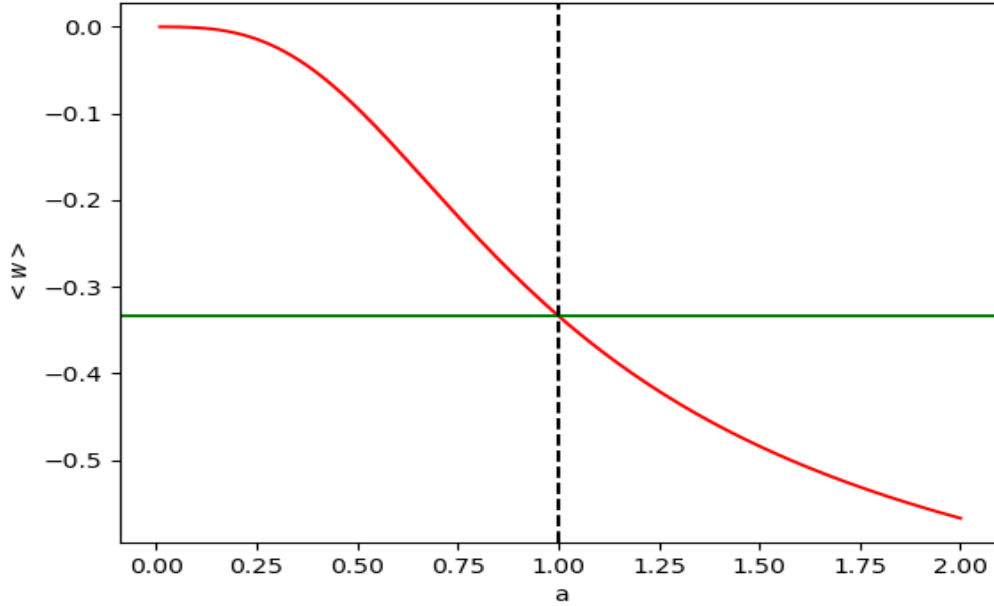


Figure 4.2: The average value of w as a function of the scale factor a for the standard model (i.e. Λ CDM).

Using the same optimised parameters, it appears that the time-averaged value $\langle w \rangle = -1/3$ at $a \approx 1$. The simplest way to simultaneously have the integral equal to 1 and satisfy the condition $\langle w \rangle = -1/3$ would be to have $w = -1/3$ for all cosmic time. However, this does not occur in Λ CDM. Instead, one must adjust the value of $\Omega_{m,0}$ to ensure the integral is equal to 1, which ensures that $\langle w \rangle = -1/3$ today, but neither w nor $\langle w \rangle$ is equal to $-1/3$ at any other time [7]. However, this is far from satisfactory because such an average could only appear once in the universe's history, and that would have to be right now – unless w was always equal to $-1/3$. Such an occurrence at present would be remarkable if this were not the case [26], and is still merely circumstantial evidence at best, but it does suggest that the condition $\langle w \rangle = -1/3$ restricts the optimisation of the Λ CDM parameters [26].

The standard model, therefore, predicts a moderately complex pattern of behaviour; despite this, the universe seems to have experienced just the right amount of deceleration at the beginning (during the radiation and matter-dominated epochs) followed by just the right amount of acceleration in recent times to leave it with an observed value of $R_h(t_0) = ct_0$ today. This is equivalent to the universe simply expanding at a constant rate all along [26]. The results call into question the conclusion that the universe underwent a period of deceleration and is currently undergoing a period of acceleration [49].

The impact of R_h on observations and interpretations of the data became quite evident after the optimisation of model parameters in the standard model [50, 51] revealed a gravitational horizon $R_h = ct_0$ within measurement error [52, 53]. This observed equality cannot be a mere coincidence [54]. In the context of Λ CDM, the equality $R_h = ct$ can

only occur once, making it an extremely unlikely event. The universe is spatially flat, meaning it will expand indefinitely and likely has an infinite timeline. The probability of the previously mentioned happening right now is essentially zero. There may be various physical explanations for such a constraint to exist. The most satisfactory, however, seems to be that, even though Λ CDM sometimes serves as a reasonable approximation to the 'real' universe in some instances, the more accurate representation of the universe is consistent with Weyl's principle, which requires that R_h must always be equal to ct and $a(t) \propto t$. In this case, the condition would be satisfied regardless of the time of measurements.

Chapter 5

Constraints from Cosmological Data

Simplicity is not the deciding factor for a cosmological model to be recognised as a proper description of the universe [8]. To determine whether $\epsilon + 3p = 0$ is maintained from one moment to the next, several comparative tests using Λ CDM and $R_h = ct$, along with a wide range of available observations, were performed, including relatively close type Ia supernovae, cosmic chronometers at intermediate redshifts, and CMB [26].

5.1 SNe Ia Data

The first indication that the universe is currently going through an accelerated expansion phase was provided by observations of rare (possibly a few per century in a Milky Way-sized galaxy) type Ia supernovae (SNe Ia) [8]. Although the specifics of what causes the explosion of SNe Ia are not well understood from theory, their observed properties are [55]. SNe Ia are viewed as approximate standard candles; they are so bright that they are detected at very high redshifts and have a nearly uniform intrinsic luminosity, which is well-calibrated by their light curves, independently of the redshift [14], and can, therefore, serve as probes to investigate the universe's expansion history.

5.1.1 Luminosity Distance

In the age of precision cosmology, cosmological models are constrained by observational data. One of the most significant and fundamental tests implemented is comparing the redshifts and distances of far-off objects with the predictions of specific models. In a generic curved spacetime, there is no preferred concept of distance between two objects [14]; several useful measures of distance [34, 56] are derived from observable quantities and used as the foundation for observational tests. For instance, in the case of type Ia supernovae, the quantity used is the luminosity distance [14], which is obtained by fitting a light curve to observed peak luminosities and then analysed as a function of redshift.

Consider a standard candle (with known absolute luminosity L – expressed in $W = J/s$) that emits photons isotropically and without loss of energy. At the luminosity distance D_L (often expressed in Mpc) from the object, only the surface brightness B_S , is observed; given by the inverse square distance law

$$B_S = \frac{L}{4\pi D_L^2}.$$

In a normal Euclidean universe, if the source is stationary at proper distance D_P , a telescope with area A will receive a fraction $A/4\pi D_P^2$ of the photons [34, 57]. In an expanding universe characterised by a scale factor $a(t)$, the object is not stationary, so the energy of the photons emitted at $t = t_e$ is redshifted by a factor $1/a_e = 1 + z$. Additionally, the arrival rate of the photons suffers time dilation and is reduced by an additional factor of $1 + z$. This phenomenon is referred to as the **energy effect** [57]. The apparent brightness B_A is then given by

$$B_A = \frac{L}{4\pi D_P^2 (1 + z)^2}$$

so that $B_A = B_S$ requires $D_L = D_P (1 + z)$, where

$$\begin{aligned} D_P &\equiv c \int_{t_e}^{t_0} \frac{1}{a} dt = c \int_{a(t_e)}^1 \frac{1}{a^2 H} da \\ &= c \int_{z_e}^0 \frac{(1 + z)^2}{H(z)} \frac{da}{dz} dz \\ &= c \int_0^{z_e} \frac{1}{H(z)} dz \end{aligned}$$

is the particle horizon¹ for geodesics diverging from the source at $t = t_e$. Of course, assuming spatial flatness, the luminosity distance is given by

$$D_L = \frac{c}{H_0} (1 + z) \int_0^z \frac{1}{\sqrt{\Omega_{m,0} (1 + z)^3 + 1 - \Omega_{m,0}}} dz \quad (5.1)$$

for the Λ CDM model. In contrast, the proper form of the luminosity distance used in the analysis of type Ia supernovae data in the context of the $R_h = ct$ model takes the following closed-form expression:

$$D_L = \frac{c}{H_0} (1 + z) \int_0^z \frac{1}{1 + z} dz \quad (5.2a)$$

$$= \frac{c}{H_0} (1 + z) \ln(1 + z). \quad (5.2b)$$

Here, the only parameter is the Hubble "constant."

Comparing these expressions with currently available data is used to verify the validity of the models. Both the Λ CDM and $R_h = ct$ models predict similar behaviour for the luminosity distance at low redshift:

$$\lim_{z \rightarrow 0} D_L^{\Lambda\text{CDM}} = \lim_{z \rightarrow 0} D_L^{R_h=ct} = 0.$$

¹Assuming that the universe has a finite age, an observer's particle horizon forms a spherical surface in space whose radius is the proper distance travelled by a photon of light emitted at the big bang and observed at cosmic time $t = t_0$ [38]

Constraints are, therefore, needed to differentiate between the two models at high enough redshifts. Indeed, one can obtain estimates of the values of H_0 and $\Omega_{m,0}$ in the Λ CDM model and H_0 in the $R_h = ct$ model by estimating the distance modulus of type Ia supernovae light curves [39, 58]:

$$\begin{aligned}\mu \equiv m - M &= 5 \log_{10} \left(\frac{D_L}{10 \text{ pc}} \right) \\ &= 5 \log_{10} \left(\frac{D_L \times 10^6}{10} \right) \\ &= 25 + 5 \log_{10} D_L,\end{aligned}$$

where m and M are the apparent and absolute magnitude of the supernovae, respectively [14].

5.2 Observational Hubble Data (OHD)

Since the discovery of the universe's expansion, the precise determination of the Hubble parameter $H(z)$ has become one of the main drivers of modern cosmology [59, 60] and is used to test the expansion rates predicted by different competing cosmological models [61] and to give basic information about the current acceleration. However, its measurement is exceedingly difficult, because many approaches (like SNe Ia analyses) concentrate on its local value $H(z = 0)$. So, while using SNe Ia data has advantages, it depends on the use of integrated distance, which inevitably introduces some prior assumptions about the underlying cosmological model [49] and the calibration of the distance ladder. When using a light-curve fitter to determine the distance moduli from raw data, a few nuisance parameters are fitted simultaneously along with the cosmological parameters. Hence, their distance moduli datasets are somewhat dependent on the model [8] and cannot be used for a comparative test between different expansion scenarios. Given the challenge of identifying model-independent data from SNe Ia observations, comparing the two models based solely on these measurements is complex [11]. The need for alternate methods for comparison and ultimate integration with other determinations for determining the Hubble parameter is, thus, supported.

What follows is an analysis based on observational Hubble parameter data collected from multiple SNe Ia independent sources and obtained using two methods: radial Baryonic Acoustic Oscillation (BAO) signals from galaxy clustering in redshift surveys and cosmic chronometers (CCs). Using these methods should ensure the robustness of this study's results. Any potential systematics in these two methods are expected to differ from those in SNe Ia analyses [8].

5.2.1 BAO Data

As standard rulers, BAO constraints provide a powerful tool for probing cosmic expansion history, mainly because CMB measurements are used to estimate the characteristic length scale in the early universe that these oscillations correspond to. For instance, a significant geometric relation used in this analysis is the angle subtended by a cosmic object of comoving length r_* located at redshift z :

$$\begin{aligned}\theta_*(z) &= \frac{r_*}{D_A(z)(1+z)} \\ &= \frac{r_*(1+z)}{D_L(z)}.\end{aligned}$$

$D_A(z)$ is the angular diameter distance, and $D_L(z)$ is the luminosity distance defined in the usual way. To determine r_* , use the angular BAO scales as measured by [62] and [63] from the analyses of the two-point angular correlation functions of 409 337 red galaxies in the interval $z \in [0.1140, 0.555]$ and quasars from the twelfth SDSS data release catalogue (DR12Q) in the interval $z \in [2.20, 2.25]$, respectively [64].

Evidently, in the BAO approach, the unique determination of $H(z)$ often necessitates the assumption of a specific cosmological model. Consequently, BAO data are useful when analysing a particular model, but careful calibration of the length scale for each case is necessary for a comparative analysis of multiple cosmologies [61]. Furthermore, this approach relies on the evolution of "standard rulers" with redshift instead of the local changes in cosmic time with z .

5.2.2 Cosmic Chronometers

The cosmic chronometer method is a new cosmological probe that provides a direct estimate of the Hubble parameter without assuming anything about the functional form of the expansion history or the universe's spatial geometry. Therefore, the constraints obtained using this approach can be applied to a wide range of cosmological models.

The main idea, introduced by [65], is based on the fact that in an FLRW universe, the scale factor is directly related to the redshift [66]. With this initial assumption, it is, therefore possible to directly express the Hubble parameter as

$$\begin{aligned}H(z) &\equiv \frac{1}{a} \frac{da}{dt} = (1+z) \frac{da}{dz} \frac{dz}{dt} \\ &= -\frac{1}{1+z} \frac{dz}{dt} \\ &\approx -\frac{1}{1+z} \frac{\Delta z}{\Delta t}.\end{aligned}$$

The local redshift-time derivative dz/dt represents the change in cosmic time with respect to redshift. Since redshift is a direct observable, the challenge then becomes one of finding a reliable estimate of Δz and Δt [66].

For a variety of reasons, it appears that the best cosmic chronometers are extremely massive ($M \geq 10^{11} M_{\odot}$) early-type elliptical galaxies that evolve on a timescale significantly longer than their age difference [65, 67, 68]. Based on empirical data, galaxies in the densest clusters are assumed to have formed about 90% of their stellar content in the redshift range $2 \lesssim z \lesssim 3$. Since then, they have evolved without any additional episodes of star formation. Thus, because of their identical evolution histories and suitability as cosmic chronometers, a copious amount of work is done to calculate the redshift-time derivative dz/dt and deduce the expansion rate $H(z)$ using their measured properties [61].

For instance, one of the most direct ways to determine the galaxy's age is by observing the $4000 \text{ \AA}$ break in its spectral continuum. For old stellar populations, metal absorption produces a discontinuity in the spectral continuum such that the amplitude of the break increases linearly with the age and metal abundance [11, 60, 69]. If the metallicity is known, then the age difference, Δt , between two nearby galaxies (separated by Δz), is proportional to the difference in their $4000 \text{ \AA}$ amplitudes [11, 61].

Of course, there is a variety of systematic sources of uncertainty that may hamper the accuracy of this type of analysis. These include:

- the degeneracy between the effects of age changes and those caused by a change in stellar metallicity or the star formation history;
- the potential bias in the choice of stellar population synthesis model used to calibrate the $4000 \text{ \AA}$ amplitude versus age relation;
- the potential progenitor bias in which high-redshift samples of early-type galaxies may not be statistically equivalent to those at low redshift.

Fortunately, extensive tests have shown that the $4000 \text{ \AA}$ feature primarily depends on the age and metallicity of the host galaxies [70]. Despite these limitations, one is nonetheless encouraged by not only the agreement between the results of several different approaches but also the fact that the inferred values of $H(z)$ are quasi-model-independent, which makes this a highly desirable and meaningful approach for fitting different cosmological models [11].

5.3 Model Selection and Theoretical Fits

Given that competing models tend to have different formulations and often feature different unmatched parameters, one must use model selection tools to distinguish between cosmological models and determine which (if any) is most supported by the data.

Let $\{(x_1, y_1), (x_2, y_2), \dots, (x_m, y_m)\}$ be a set of data points, and $f(x, \vec{\beta})$ be a differentiable model function that depends on $n \leq m$ unknown parameters $\vec{\beta} = (\beta_1, \beta_2, \dots, \beta_n)$

in addition to the independent variable x . In particular, the model function should be nonlinear with respect to the parameters in only a neighbourhood of its minimum value, where the truncated Taylor series is a good approximation. The goal then becomes to estimate the values of the model parameters $\vec{\beta}$, which would minimise the residual sum of squares [71]

$$RSS = \sum_i^m r_i^2 \quad (5.3)$$

over the model parameters.

$$r_i = y_i - f(x_i, \vec{\beta})$$

for $i = 1, 2, \dots, m$ is the i -th residual. Alternatively, because the uncertainties are known for both the SNe Ia and OHD data, the best-fit cosmology is determined by an iterative χ^2 -minimisation of the function

$$\chi^2 = \sum_{i=1}^m \left(\frac{r_i}{\sigma_i} \right)^2. \quad (5.4)$$

The corresponding reduced χ^2 goodness of fit to the data is then given by

$$\chi_{d.o.f}^2 = \frac{1}{m-n} \sum_{i=1}^m \left(\frac{r_i}{\sigma_i} \right)^2. \quad (5.5)$$

$\sigma_1, \sigma_2, \dots, \sigma_m$ are the normally distributed errors $y_1, y_2, \dots, y_m$ [10]. Generally speaking, the preferred model is the one whose $\chi_{d.o.f}^2$ value is closest to 1.

Since there are no direct ways to minimise Eq. 5.3, one must rely on iterative computational techniques. In other words, one starts with an initial guess of the solution β_n^0 of the solution β_n and then proceeds to generate a (hopefully convergent) sequence $\beta_n^1, \beta_n^2, \dots, \beta_n^k$ using a preassigned rule.

It is common in cosmology to employ the Akaike information criterion (AIC),

$$AIC = \chi^2 + 2n, \quad (5.6)$$

for the resulting fitted model, and is seen as an improved goodness of fit criterion that goes beyond the standard χ^2 criterion by accounting for the number of model parameters. It has a penalty term that favours models with fewer parameters and disfavors those with more, unless the latter offers a significantly better fit to the data, which reduces the likelihood of overfitting, which is preferable since adding more parameters in a model nearly always improves the goodness of fit.

Comparing the AIC values of two or more fitted models yields a consistent numerical measure of confidence that each model is the best, in addition to their relative ranks [11, 49]. These confidences are comparable to likelihoods or posterior probabilities in conventional statistical inferences.

If there are two or more competing models (i.e. $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_x$) that have been fitted to the data separately, the one with the lowest resulting AIC is viewed as the one that is most likely to be closest to the "truth" – that is, to the model $\mathcal{M}_*$ that produced the data.

The models can be ranked more qualitatively using the formula below. If AIC_α comes from model $\mathcal{M}_\alpha$,

$$\mathcal{P}(\mathcal{M}_\alpha) = \frac{e^{-AIC_\alpha/2}}{\sum_x e^{-AIC_x/2}} \quad (5.7)$$

is the likelihood of $\mathcal{M}_\alpha$ being the best choice [72]. In the case of a pair of models, the difference (or evidence) $\Delta AIC = AIC_2 - AIC_1$ gives a numerical measurement of how much $\mathcal{M}_1$ is preferred over $\mathcal{M}_2$. When utilising model comparison tools, the following qualitative interpretation is standard [49]. The evidence ΔAIC

- weakly favours $\mathcal{M}_1$ if $\Delta AIC \leq 2$;
- strongly favours $\mathcal{M}_1$ if it lies in the range $4 \leq \Delta AIC \leq 7$;
- very strongly favours $\mathcal{M}_1$ if $\Delta AIC \geq 10$ [73].

The outcome ΔAIC is used to estimate the percentage likelihood that, given the data, model $\mathcal{M}_1$ is statistically preferred over $\mathcal{M}_2$ according to the prescription

$$\mathcal{P}(\mathcal{M}_\alpha) \equiv \frac{e^{-AIC_1/2}}{e^{-AIC_1/2} + e^{-AIC_2/2}} = \frac{1}{1 + e^{-\Delta/2}} \quad (5.8)$$

when only two models are directly compared [19, 49].

Several alternatives to the AIC have been reviewed in the literature [11]. A more well-known alternative is the Bayesian information criterion (BIC). Contrary to its name, the BIC is not based on information theory but instead relies on an asymptotic ($m \rightarrow \infty$) approximation to the outcome of a traditional Bayesian inference procedure for deciding between competing models [49]. It is defined by

$$BIC = \chi^2 + n \ln m, \quad (5.9)$$

and very strongly suppresses overfitting if m is large. The use of BIC is particularly stimulated when the data points are independent and identically distributed (which is a reasonable assumption for SNe Ia luminosity-redshift data) [49]. Additionally, when the sample size is large, it has been shown that the BIC produces more reliable results, such as $n = 1048$, which lies deep within the asymptotic regime.

The evidence and likelihood are defined analogously for BIC [44]. It is clear that the logarithmic penalty term ($n \ln m$) exponentially penalises models with too many parameters because these are perceived to fit the noise. Some models, though, are favoured if they perform significantly better at fitting the data.

5.3.1 SNe Ia Data

As explained in Section 4.3, a Λ CDM scenario is considered in which two cosmological parameters simultaneously vary, the matter density $\Omega_{m,0}$ and the Hubble constant H_0 . Additionally, the case of $R_h = ct$, which has just one free parameter, H_0 , is considered.

Publicly available SNe Ia apparent magnitude measurements from the Pantheon compilation [74], which includes 1048 Hubble Space Telescope (HST), low- z , Panoramic Survey Telescope and Rapid Response System (Pan-STARRS1), Sloan Digital Sky Survey (SDSS) and Supernova Legacy Survey (SNLS) measurements within the range $0.01 < z \leq 2.26$ have been used to fit the observations.

The combined efforts of numerous groups have led to an impressive accumulation of SNe Ia events and an ever-improving quality of individual measurements [10]. More recently, [74] developed a framework for homogeneous analysis of such datasets, resulting in an evolving compilation that contains 1048 supernova detections and is referred to as the Pantheon sample.

The events constrain cosmological parameters by comparing their distance moduli with those predicted by the models across a range of redshifts. This exercise's main takeaway is that models with different expansion histories predict specific (and presumably distinguishable) distance vs. redshift relationships, which can be compared to the observations. The aforementioned cosmological parameter constraints were obtained by implementing a Metropolis-Hastings (MH) algorithm to perform a Markov chain Monte Carlo (MCMC) simulation using the previously mentioned data. The `curve_fit` and `EMCEE` Python modules were used to process the data and plot the figures. This method is currently the most popular but was modified by [56]. The primary results for the constraints within 4σ are displayed in Figure 5.1 below.

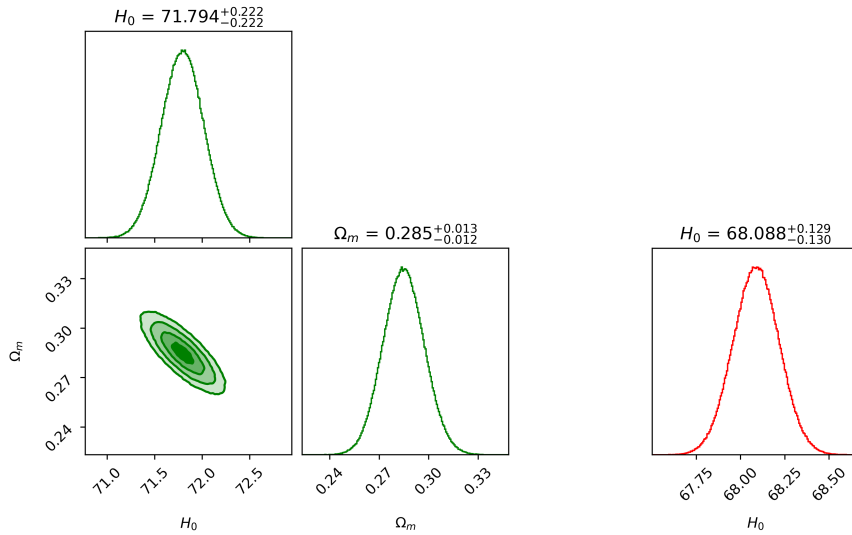


Figure 5.1: 4σ confidence regions of the parameters H_0 and $\Omega_{m,0}$ for the flat Λ CDM model (left panel) – along with their probability distributions – and H_0 for the $R_h = ct$ model (right panel).

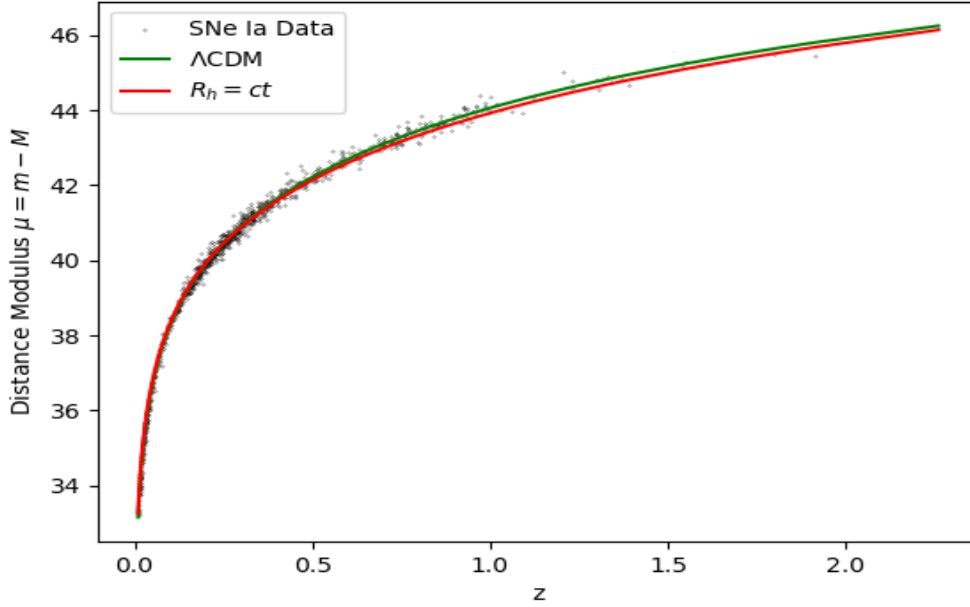


Figure 5.2: Side-by-side comparison of the best-fit Λ CDM and $R_h = ct$ models for the Pantheon compilation of 1048 SNe Ia events. Dark energy is assumed to be a constant. The Pantheon data is consistent with an $H_0 = 71.794 \pm 0.222$ km/s/Mpc, $\Omega_{m,0} = 0.285 \pm 0.013$ ($\Omega_{\Lambda,0} \equiv 1 - \Omega_{m,0} = 0.715 \pm 0.013$) Λ CDM cosmology and an $H_0 = 68.088 \pm 0.13$ km/s/Mpc $R_h = ct$ cosmology.

The most obvious aspect of the comparison is how closely the best-fit Λ CDM model mimics the $R_h = ct$ in the local universe. The difference between the two cosmologies becomes more noticeable at higher redshifts. Theoretically, one should be able to tell them apart by observing events in the universe's early expansion, which is hardly surprising, considering the two distance moduli depend differently on the assumed parameters. Despite their complexity, multiple parameters in Λ CDM must be adjusted to fit the data so that the universal expansion is nearly identical to what it would have been in the $R_h = ct$ universe [10]. In this way, the standard model functions as an empirical approximation to $R_h = ct$ despite having more free parameters but lacks the crucial ingredient: the equation of state $\epsilon + 3p = 0$ [49].

For $R_h = ct$, the inferred value of the Hubble "constant", $H_0 = 68.088 \pm 0.13$ km/s/Mpc, is more consistent with the CMB Planck optimisation than the locally measured value using Cepheid variables in the context of Λ CDM [59].

Combining numerous datasets into a single compilation has clear benefits: it treats all supernovae equally and uses the same light curve fitting. However, this process introduces a unique set of possible errors, such as the fact that the systematics may differ across the different datasets.

For these best-fit parameters, the universe's expansion changed from decelerating to accel-

erating at

$$\begin{aligned}
 \rho_m + 2\rho_r - 2\rho_\Lambda &\equiv \Omega_{m,0} (1+z)^3 + 2\Omega_{r,0} (1+z)^4 - 2\Omega_{\Lambda,0} = 0 \\
 \Omega_{m,0} (1+z)^3 &= 2(1 - \Omega_{m,0}) \\
 z &= \left[\frac{2(1 - \Omega_{m,0})}{\Omega_{m,0}} \right]^{1/3} - 1 \\
 &\approx 0.713 \pm 0.035.
 \end{aligned}$$

Equality between ρ_Λ and ρ_m occurred later at

$$\begin{aligned}
 \Omega_{m,0} (1+z)^3 &= \Omega_{\Lambda,0} \quad \Rightarrow \quad \Omega_{m,0} (1+z)^3 \approx 1 - \Omega_{\Lambda,0} \\
 z &= 0.360 \pm 0.028.
 \end{aligned}$$

A side-by-side comparison of the diagrams shows that the distance moduli are slightly different when the free parameters are optimised using different models. The Λ CDM and $R_h = ct$ models fit the SNe Ia data very well at relatively low redshifts. However, since these models formulate the distance modulus differently, a choice between the models must be made using formal model selection techniques [49].

The best-fit parameters for each model are shown to make comparisons simpler. For completeness, the AIC, BIC, ΔAIC and ΔBIC values are also shown in each case.

Model	χ^2	AIC	BIC	ΔAIC	ΔBIC
Λ CDM	1035.679	1039.679	1049.588	0	0
$R_h = ct$	1313.840	1315.840	1320.795	276.161	271.206

Table 5.1: The so-called corrected AIC (abbreviated AIC_c), which includes a small correction term meant to remove the bias when m is small is not utilised. The form of the term is only appropriate for datasets without explicit error bars [11].

With $1048 - 2 = 1046$ degrees of freedom, the $\chi^2_{d.o.f}$ for the best-fit Λ CDM solution is 0.990. In contrast, the optimal $R_h = ct$ fit has $1048 - 1 = 1047$ degrees of freedom and a corresponding $\chi^2_{d.o.f} = 1.255$. Generally, $\chi^2_{d.o.f}$ should be as close to one as possible. In this way, estimates and observations properly match the errors in the data. Statistically, the Λ CDM distance-redshift predictions are more likely than those of $R_h = ct$ to be closer to the actual cosmology.

According to the distance modulus's optimised fits, the evidences $\Delta AIC \equiv AIC_{R_h=ct} - AIC_{\Lambda CDM} \approx 276.161$ and $\Delta BIC \equiv BIC_{R_h=ct} - BIC_{\Lambda CDM} = 271.206$ indicate that the Λ CDM cosmology is a better match to the observations than $R_h = ct$.

However, remember that the AIC and BIC only indicate a model's quality compared to the other models, not its absolute quality. For instance, if all the candidate models fit poorly, AIC and BIC will not warn about that. Therefore, after choosing a model using AIC and/or BIC, it is generally a good idea to confirm the absolute quality of the model. Such confirmation usually involves checks of the models' residuals, r_i – see Figure 5.3.

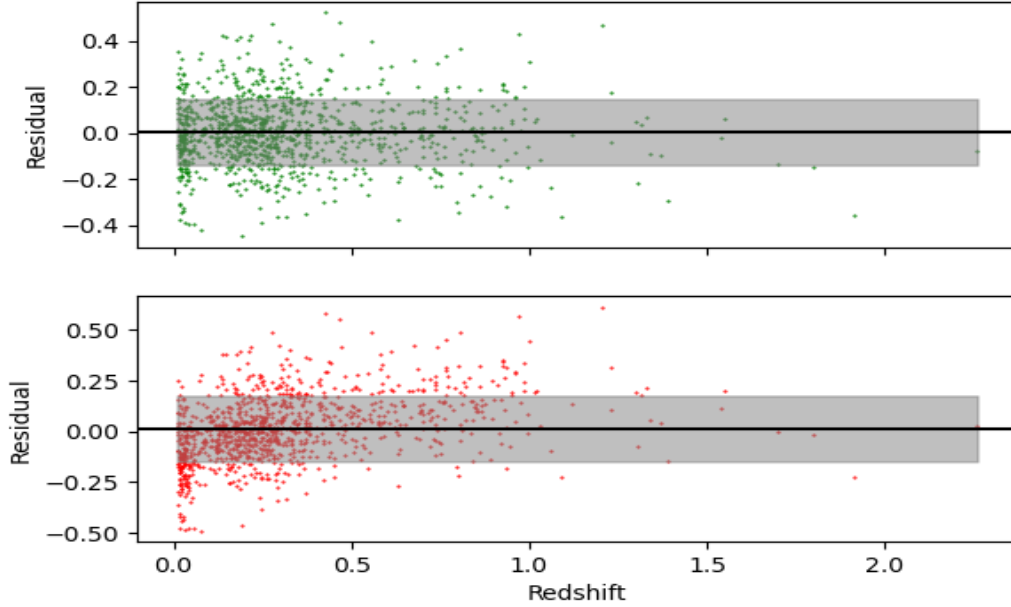


Figure 5.3: Side-by-side comparison of the SNe Ia residuals for the Pantheon sample in the standard (Λ CDM) (top) and $R_h = ct$ models (bottom). On average, the Λ CDM model overestimates the SNe Ia data with average residual $\bar{r}^\Lambda = 0.00455$ and standard deviation $\sigma_r^\Lambda = 0.143$, showing that this is a very tight fit. In contrast, the average residual and standard deviation are $\bar{r}^R = 0.0133$ and $\sigma_r^R = 0.162$ for the $R_h = ct$ model. As a result, only 293 points lie more than 1σ away from $\bar{r}^\Lambda$ in Λ CDM compared to 314 points in $R_h = ct$.

When referring to the luminosity distance, $R_h = ct$ raises doubts about the validity of introducing parameters such as the deceleration parameter $q(t)$ into the analyses. However, by challenging this parameter, further constraints are imposed on the validity of the $R_h = ct$ model. $q(t)$ has been used extensively for many decades as a direct probe of cosmic expansion and has a simple interpretation [8].

If a photon is emitted from a nearby galaxy at cosmic time $t = t_0 - \delta t$ (where t_0 is the time when the photon is received and $|\delta t| \ll |t_0|$), expanding the scale factor as a Taylor series about t_0 results in:

$$\begin{aligned}
 a(t) &\equiv a(t_0 + \delta t) = a(t_0) + \left. \frac{da}{dt} \right|_{t=t_0} \delta t + \frac{1}{2} \left. \frac{d^2 a}{dt^2} \right|_{t=t_0} (\delta t)^2 + \dots \\
 &= a(t_0) \left[1 + \frac{1}{a(t_0)} \left. \frac{da}{dt} \right|_{t=t_0} \delta t + \frac{1}{2a(t_0)} \left. \frac{d^2 a}{dt^2} \right|_{t=t_0} (\delta t)^2 + \dots \right] \\
 &= a(t_0) \left[1 + H(t_0) \delta t - \frac{1}{2} q(t_0) H^2(t_0) (\delta t)^2 + \dots \right].
 \end{aligned}$$

The deceleration parameter is generally defined as

$$q(t) = -\frac{\ddot{a}}{aH^2} \quad (5.10a)$$

$$= \frac{4\pi G(\rho_m + 2\rho_r - 2\rho_\Lambda)}{8\pi G a(\rho_m + \rho_r + \rho_\Lambda)} \quad (5.10b)$$

$$= \frac{a\Omega_{m,0} + 2\Omega_{r,0} - 2a^4\Omega_{\Lambda,0}}{2a(a\Omega_{m,0} + \Omega_{r,0} + a^4\Omega_{\Lambda,0})}. \quad (5.10c)$$

According to this definition, q is negative if the universal expansion accelerates. Furthermore, it is not constant in time [8]. In the $R_h = ct$ model, however, $q = 0$ at all times. According to the SNe Ia analysis, the deceleration parameter's predicted current value

$$\begin{aligned} q(t_0) \equiv q_0 &\approx \frac{\Omega_{m,0}}{2} - \Omega_{\Lambda,0} \\ &= \frac{3}{2}\Omega_{m,0} - 1 \\ &= -0.573 \pm 0.019 \end{aligned}$$

is significantly less than zero. The conclusion is that the suggested model is incompatible with the various analyses (particularly those that include more recent observations of SNe Ia at redshifts greater than 1) that point towards a deceleration parameter that varies over time and transitions from past deceleration to current acceleration [8].

5.3.2 OHD

A compilation of 57 Hubble parameter measurements (provided in Table A.2) within the range $0.07 < z < 2.36$ have been used to constrain the models. The compilation includes 31 data points derived from cosmic chronometers and 26 data points from BAO provided by the Sloan Digital Sky Survey (SDSS), Data Release 9&11 (*DR9* and *DR11*).

In this work, we consider two flat cosmological models:

1. Λ CDM – with matter density parameter $\Omega_{m,0}$. For this model,

$$\begin{aligned} H &= H_0 \sqrt{\Omega_{m,0}(1+z)^3 + \cancel{\Omega_{r,0}(1+z)^4} + \Omega_{\Lambda,0}} \\ &= H_0 \sqrt{\Omega_{m,0}(1+z)^3 + 1 - \Omega_{m,0}}. \end{aligned}$$

Ignore the negligible contribution from radiation in the local universe and

2. $R_h = ct$. In this model, $H(z)$ is given by

$$H = H_0(1+z).$$

The fitting is a non-linear regression in the case of Λ CDM and linear for $R_h = ct$, as is evident from the equations above [11]. The results of the constraints recorded within 4σ are depicted in Figure 5.4 below.

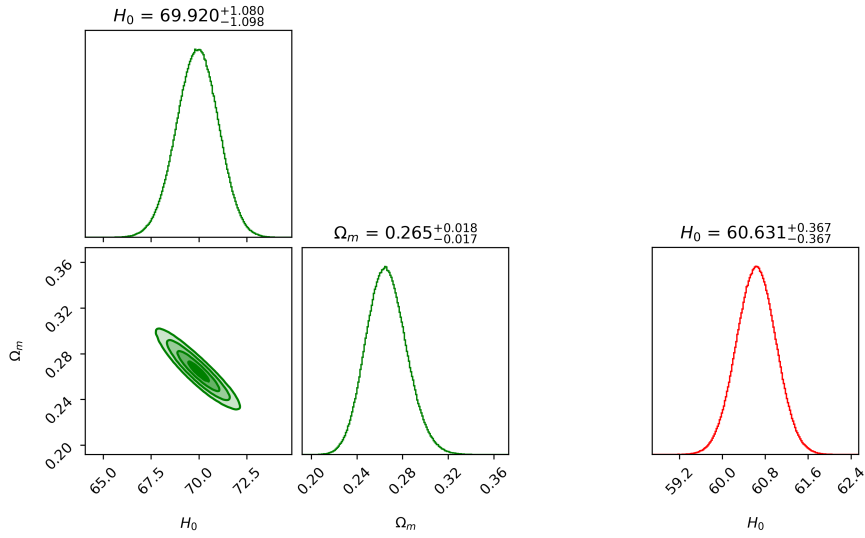


Figure 5.4: 4σ confidence regions of the parameters H_0 and $\Omega_{m,0}$ for the flat Λ CDM model (left panel) along with their probability distribution, and H_0 for the $R_h = ct$ model (right panel)

The resulting best-fit Λ CDM value of $H_0 = 69.920 \pm 1.089$ km/s/Mpc not only falls halfway within the range defined by the current Hubble tension but is also in agreement with the value derived from a calibration of the tip of the red giant branch (TRGB) applied to SNe Ia [55].

The universe's expansion changed from decelerating to accelerating at $z \approx 0.778 \pm 0.051$, and matter-dark energy equality was reached much much later at $z \approx 0.411 \pm 0.041$, using these best-fit values. From the figure, it is apparent how sensitive each models is to the choice of H_0 . For example, note that in the case of $R_h = ct$, which only has one free parameter, the optimal value of H_0 is extremely constrained. A 1σ standard deviation could be produced with just a change of ± 0.369 km/s/Mpc. However, in Λ CDM, which has two free parameters – and, thus, more flexibility – the same 1σ standard deviation necessitates a change in H_0 of roughly ± 1.089 km/s/Mpc from this optimised value.

The fact that the optimised value of H_0 in $R_h = ct$ is significantly smaller than in the standard model is worth mentioning. According to [39], the age of a flat universe is given by

$$t_0 \equiv \int_0^{t_0} dt = \int_0^1 \frac{1}{aH} da \quad (5.11a)$$

$$= - \int_\infty^0 \frac{1}{H(1+z)} dz. \quad (5.11b)$$

For Λ CDM and $R_h = ct$,

$$t_0 = \frac{1}{H_0} \int_0^\infty \left[\Omega_{m,0} (1+z)^5 + (1 - \Omega_{m,0}) (1+z)^2 \right]^{-1/2} dz$$

and

$$t_0 = \frac{1}{H_0},$$

respectively.

The best-fit OHD values yield $t_0 = 13.962 \pm 0.323$ and $t_0 = 16.127 \pm 0.098$ billion years for Λ CDM and $R_h = ct$, respectively. The Λ CDM value is no surprise since that is the age expected and possibly even quoted in cosmology texts. What is surprising is the age calculated from $R_h = ct$ because it points to a universe significantly older than Λ CDM predicts. The previously mentioned is not necessarily an issue in and of itself because $R_h = ct$ is a proposed alternative to Λ CDM; there is little likelihood its parameter values would precisely match those of Λ CDM. The following, however, is problematic.

The most accurate determination of H_0 comes from the CMB's anisotropies [13]. The OHD-measured value of H_0 in $R_h = ct$ is approximately 7.03 ± 0.559 km/s/Mpc less than the value derived from the Planck measurement; it would, however, be intriguing to explore what kind of new physics might resolve this disparity. One explanation could be that the CMB value of H_0 in and of itself is model-dependent. Additionally, the temperature fluctuation spectrum has not been analysed using $R_h = ct$ [72]. It is too early to determine whether the CC and CMB values of H_0 are consistent with one another in the $R_h = ct$ model, as the Hubble constant is only optimised for Λ CDM.

It is important to note that although some local measurements of H_0 do point to a value lower than $H_0 = 69.920 \pm 1.089^2$, most local measurements suggest a higher value. The main question for $R_h = ct$ is how the CC measurements will compare with those based on model fits to the CMB temperature anisotropies [72].

The best-fit curves of the two cosmologies are depicted in Figure 5.5, superimposed on the data. As shown in Figure 5.5, the two cosmologies are in excellent agreement at relatively low redshifts, and they diverge when probing further back in the universe's expansion history. In particular

$$H_{0\Lambda}^2 \left[\Omega_{m,0} (1+z)^3 + 1 - \Omega_{m,0} \right] = H_{0R}^2 (1+z)^2 \quad \Rightarrow \quad az^3 + bz^2 + cz + d = 0,$$

where

$$\begin{aligned} a &= H_{0\Lambda}^2 \Omega_{m,0} \\ b &= 3H_{0\Lambda}^2 \Omega_{m,0} - H_{0R}^2 \\ c &= 3H_{0\Lambda}^2 \Omega_{m,0} - 2H_{0R}^2 \\ d &= H_{0\Lambda}^2 - H_{0R}^2. \end{aligned}$$

$H_{0\Lambda}$ and H_{0R} are the best-fit values of the Hubble constant for Λ CDM and $R_h = ct$, respectively.

²For example, Tammann and Reindl (2013) used red-giant branch stars in the haloes of local galaxies to calculate the SNe Ia luminosity and inferred $H_0 = 64.0 \pm 1.8$ km/s/Mpc

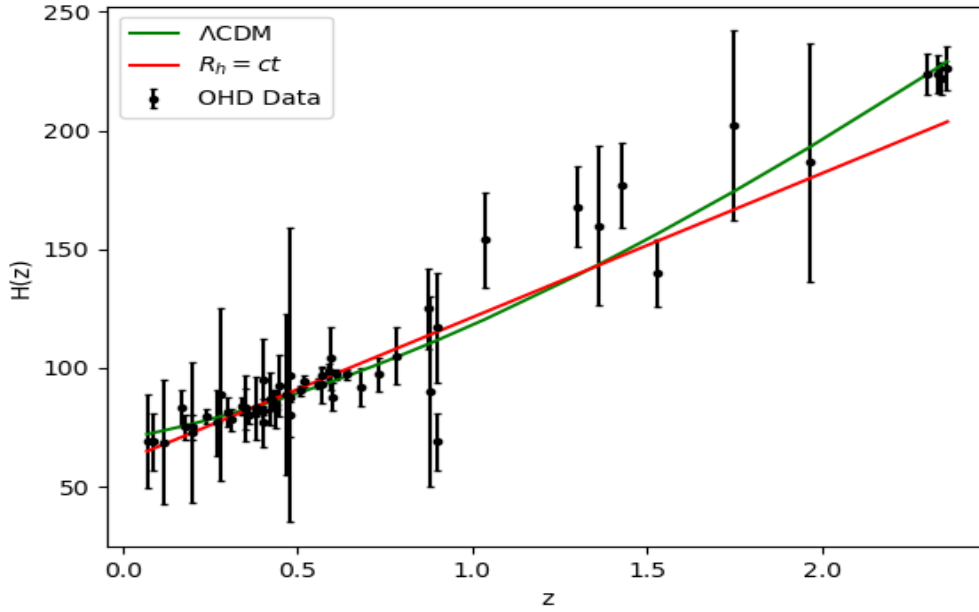


Figure 5.5: Side-by-side comparison of the best-fit Λ CDM and $R_h = ct$ models for the OHD compilation of 57 measurements. The data is consistent with an $H_0 = 69.920 \pm 1.089$ km/s/Mpc and $\Omega_{m,0} = 0.265 \pm 0.0175$ ($\Omega_{\Lambda,0} \equiv 1 - \Omega_{m,0} = 0.735 \pm 0.0175$) Λ CDM cosmology and an $H_0 = 60.588 \pm 0.369$ km/s/Mpc $R_h = ct$ cosmology.

There are, of course, a maximum of three independent solutions for z in this equation, i.e. $z = -1.87$, $z = 0.382$ and $z = 1.34$. The first solution is not physical since, at infinite time, the redshift approaches -1 . The second solution falls within 1σ of the Λ CDM redshift at matter-dark energy equality. The Hubble parameter is highly dependent on the universe's composition. Thus far, no explicit mechanism clarifies the time dependence of the individual components in $R_h = ct$. It is possible that the equality of these two cosmologies, in this particular instance, reveals some intriguing physics regarding the makeup of the universe.

In fact, this may very well be the case for the third solution as well. $z = 1.34$ is deep in the deceleration epoch, much earlier than the expected redshift at the time when the universe's expansion changed from deceleration to acceleration. It is unclear what it physically corresponds to in the context of Λ CDM or $R_h = ct$, which warrants a complete analysis of its own as such an analysis is outside the purview of this work.

The model selection tools are given in Table 5.2 below.

Model	χ^2	AIC	BIC	ΔAIC	ΔBIC
Λ CDM	44.31	48.311	52.432	0	0
$R_h = ct$	78.226	80.226	82.287	31.916	29.855

Table 5.2: Model selection tools for Λ CDM and $R_h = ct$ using the data in Table A.2.

With $57 - 2 = 55$ degrees of freedom, the $\chi^2_{d.o.f}$ for the best-fit Λ CDM solution is 0.791, and the large error bars do, in fact, support this value of $\chi^2_{d.o.f}$. The model fits the data poorly due to the errors being overestimated. In contrast, the $R_h = ct$ fit has $57 - 1 = 56$ degrees of freedom and a corresponding $\chi^2_{d.o.f} = 1.372$. Even on the surface, this would indicate a poor model fit, but the value of $\chi^2_{d.o.f}$ points to the underestimation of the errors in the data. Both models appear to deviate significantly from the optimum value of $\chi^2_{d.o.f} = 1$; it would, thus, be premature to rule out either model without considering the AIC, BIC and related evidences.

The probability that $R_h = ct$ is correct is 1.17×10^{-7} , as can be reasonably seen from $\Delta AIC \equiv AIC_{R_h=ct} - AIC_{\Lambda CDM} = 31.916$. Using the BIC leads to the same conclusion, i.e. the OHD data very strongly favours the Λ CDM model. There is only a 3.29×10^{-7} probability of $R_h = ct$ being correct: $\Delta BIC \equiv BIC_{R_h=ct} - BIC_{\Lambda CDM} = 29.855$.

At this stage, Λ CDM has convincingly shown that it provides a more accurate description of the universe according to the OHD data. Nevertheless, the residuals are displayed below:

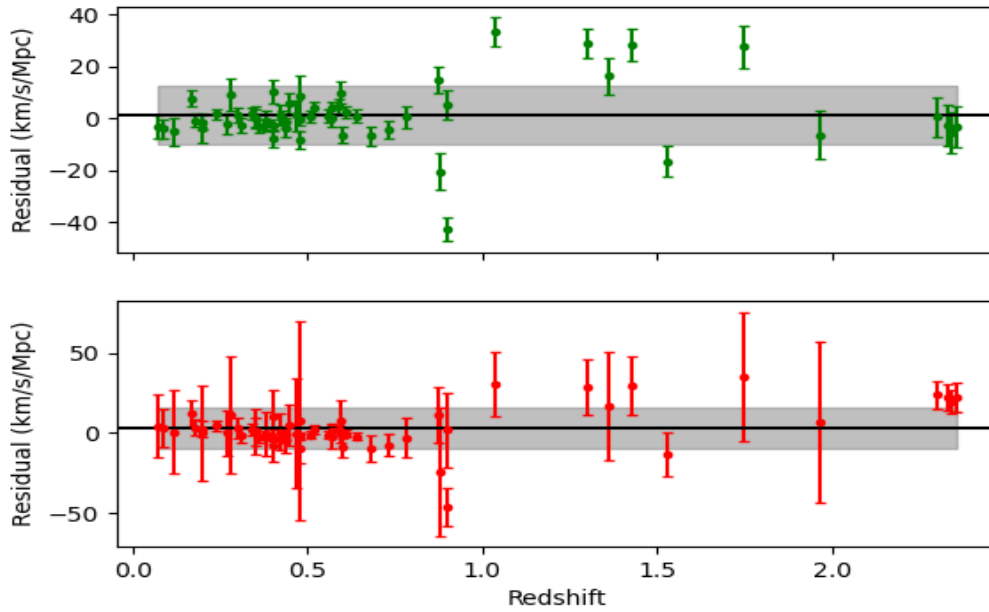


Figure 5.6: Side-by-side comparison of the residuals for the $H(z)$ data in the standard (Λ CDM) (top) and $R_h = ct$ models (bottom). The average residuals and standard deviations are $\mu_{\Lambda} = 1.19$ and $\sigma_{\Lambda} = 11.32$ for the standard model. As with the SNe Ia data, the $R_h = ct$ model is less precise and less accurate for $\mu_{R_h} = 3.05$ and $\sigma_{R_h} = 12.92$.

SNe Ia data assumes a Λ CDM cosmology, and since $R_h = ct$ is essentially a more constrained Λ CDM model, both models fit the data very well. When quasi-model-independent data (like OHD) are used, Λ CDM tends to fit much better than $R_h = ct$ as it reproduces theoretically consistent results: the best-fit value of $\Omega_{m,0} = 0.265 \pm 0.018$ agrees very well with the theoretical value obtained from the integral in Eq. 4.6. From this result alone,

it might be easy to conclude that Λ CDM is unquestionably the superior model. The extremely large error bars must be considered as they cast doubt on the confidence in the accuracy of the MCMC fit.

Attempting to identify why Λ CDM produces deceleration and acceleration phases reminds one of trying to fit a straight line with a low-order polynomial. It is always possible to make the ends meet, but there will inevitably be "wiggles" in between. The early deceleration and current acceleration indicated by Λ CDM appear to be two such "wiggles", whereas $R_h = ct$ fits the straight line [49].

Examining the consequences of the $R_h = ct$ model based on these results leads to the conclusion that it is strongly disfavoured by observations.

Chapter 6

Cosmological Perturbations

Over the past few decades, various surveys have measured numerous galaxies' redshifts and used them to plot their three-dimensional distribution in the universe. The conclusion is that, on the largest scales, it approaches a homogeneous distribution per the underlying assumptions of FLRW models. There is unmistakable proof of inhomogeneity as the universe is observed on a small scale [24]. Density fluctuations (overdense regions like clusters and underdense regions like voids) become more pronounced on smaller smoothing scales. This section addresses deriving the necessary mathematical formalism for structure formation in the two cosmological models.

The origin and evolution of large-scale structures in the universe is one of the key issues in cosmology. Therefore, the condition under which a cosmological model can be regarded as valid, or at the very least a good approximation to reality, is that it offers a plausible explanation for how the observed structure formed in the universe. While many different structure formation models have been considered, the basic idea that has endured since Newton's time is gravitational instability. The large-scale structure revealed by the distribution of galaxies is assumed to have arisen via a process of gravitational instability through the growth of density fluctuations that originated in the early universe. The initial fluctuations were either primordial¹ or spontaneous². The fluctuations were amplified by the exponential inflation of the universe and evolved linearly for a considerable amount of time [75] so that, when averaged over a few hundred Mpc, the inhomogeneities remain small. Because small perturbations imply weak gravitational fields, much of the fundamental physics is derived from the usual non-relativistic Newtonian linear perturbation theory, where the governing equations are considerably simplified.

¹fed into the initial conditions of the classical universe

²arose at a very early epoch through some physical process like quantum fluctuations

6.1 Newtonian Perturbation Theory

In this regime, volumes of the universe on sub-Hubble scales are considered, where $|\vec{x}| \ll cH^{-1}$ and $|\vec{v}| \ll c$. It is assumed that the effects of spacetime curvature are negligible, and relativistic corrections are disregarded.

Consider the case of non-relativistic matter, which is more familiar and explained using the equations of self-gravitating fluid dynamics (assuming negligible radiative effects):

- the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \quad (6.1)$$

- the Euler equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{\nabla P}{\rho} - \nabla \phi, \quad (6.2)$$

and

- the Poisson equation

$$\nabla^2 \phi = 4\pi G \rho \quad (6.3)$$

which relates the Newtonian gravitational potential ϕ to the density ρ .

Current observations suggest a universe which is very close to an FLRW universe, meaning that the analysis is limited to situations wherein the real universe does not differ significantly from the background. Mathematically, this is achieved by treating quantities that do not vanish in the background as zeroth-order and keeping only terms which are first-order in perturbations [76]:

$$\begin{aligned} \phi &= \phi_0 + \delta\phi \\ \rho(t, \vec{x} = a\vec{r}) &= \rho_0(t) + \delta\rho(t, \vec{x} = a\vec{r}) \\ p(\rho_0 + \delta\rho) &= p_0(t) + \left. \frac{dp}{d\rho} \right|_{\rho=\rho_0} (\rho - \rho_0) \\ &= p_0 + c_s^2 \delta\rho \\ \vec{v} &= \vec{v}_0 + \delta\vec{v} \end{aligned}$$

a_0 denotes the zeroth-order (i.e. homogeneous and isotropic) solution, and $|\delta a| \ll |a_0|$ denotes the small first-order perturbations about the background solutions [44]. Pressure perturbations in a perfect fluid generate sound waves such that the closure relation $c_s^2 = w c^2$ is the effective squared sound speed. The sound speed lies within the range

$$0 \leq c_s^2 \leq 1$$

to preserve matter stability and causality. Matter is unstable for $c_s^2 < 0$ and causality breaks for $c_s^2 > 1$ [21].

In deriving the dynamical equations for the evolution of the perturbations, one must carefully distinguish between the different roles played by the zeroth-order quantities p_0 and ρ_0 and the first-order quantities δp and $\delta \rho$ [44]. The former quantities determine the evolution of the Hubble parameter $H(t)$, and the local growth is affected by fluctuations.

The continuity equation to linear order is given by

$$\begin{aligned} \frac{\partial \rho_0}{\partial t} + \frac{\partial \delta \rho}{\partial t} + \nabla \cdot (\rho_0 \vec{v}_0 + \rho_0 \delta \vec{v} + \delta \rho \vec{v}_0) &= \frac{\partial \delta \rho}{\partial t} + \nabla \cdot (\rho_0 \delta \vec{v} + \delta \rho \vec{v}_0) \\ &= \frac{d \delta \rho}{dt} + \rho_0 \nabla \cdot \delta \vec{v} + \delta \rho \nabla \cdot \vec{v}_0, \end{aligned}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}_0 \cdot \nabla$$

is the total time derivative of an observer moving with the unperturbed expansion. The rate of change of a quantity at position $\vec{x}$ is given by the explicit change and the change brought on by the fluid's movement at that position. Define the normalised scalar density perturbation as

$$\delta \equiv \frac{\rho(t, \vec{x}) - \rho_0(t)}{\rho_0(t)} = \frac{\delta \rho}{\rho_0}, \quad (6.4)$$

where $\vec{x}$ is the proper distance so that

$$\rho_0 \frac{d \delta}{dt} = \frac{d \delta \rho}{dt} + \delta \rho \nabla \cdot \vec{v}_0 \Rightarrow \rho_0 \left(\frac{d \delta}{dt} + \nabla \cdot \delta \vec{v} \right) \stackrel{!}{=} 0$$

if, and only if,

$$\frac{d \delta}{dt} = -\nabla \cdot \delta \vec{v}$$

for a non-empty universe with $\rho_0 \neq 0$.

Thus far, proper length units have been used, and the Hubble expansion is explicitly present through $\vec{v}_0 \equiv H_0 \vec{x}$. Alternatively, use Lagrangian coordinates that include the property that the Lagrangian position of a particle or fluid element does not change with time. One kind of Lagrangian coordinates is the Hubble comoving coordinates, which label observers who follow the Hubble expansion in an unperturbed universe. In these coordinates,

$$\begin{aligned} \nabla &\equiv \frac{\partial}{\partial x^i} \hat{e}_i = \frac{\partial}{\partial (a r^i)} \hat{e}_i \\ &= \frac{1}{a} \nabla_r. \end{aligned}$$

Furthermore, let

$$\delta \vec{v} = a \vec{u}$$

be the peculiar velocity that represents a small deviation from a pure Hubble flow so that the corresponding second-order differential equation for the density perturbation is given by

$$\frac{d^2 \delta}{dt^2} = -\nabla_r \cdot \frac{d \vec{u}}{dt}. \quad (6.5)$$

The Euler equation gives

$$\frac{d \delta \vec{v}}{dt} + (\delta \vec{v} \cdot \nabla) \vec{v}_0 = -\frac{\nabla \delta P}{\rho_0 + \delta \rho} - \nabla \delta \phi.$$

The denominator can be approximated to first order using the Binomial theorem:

$$(\rho_0 + \delta\rho)^{-1} \equiv \left[\rho_0 \left(1 + \frac{\delta\rho}{\rho_0} \right) \right]^{-1} = \frac{1}{\rho_0} \left(1 - \frac{\delta\rho}{\rho_0} + \dots \right) \approx \frac{1}{\rho_0}.$$

Thus

$$\begin{aligned} \frac{d\vec{u}}{dt} &= -H\vec{u} - H(\vec{u} \cdot \nabla_r) \vec{r} - \frac{c_s^2 \nabla_r \delta}{a^2} - \frac{1}{a^2} \nabla_r \delta \phi \\ \Rightarrow \frac{d^2 \delta}{dt^2} &= H \nabla_r \cdot \vec{u} + H \nabla_r \cdot [(\vec{u} \cdot \nabla_r) \vec{r}] + \frac{c_s^2 \nabla_r^2 \delta}{a^2} + \frac{1}{a^2} \nabla_r^2 \delta \phi. \end{aligned}$$

SUPPLEMENTARY:

$$\begin{aligned} (\vec{u} \cdot \nabla_r) \vec{r} &\equiv u_i \frac{\partial r_j}{\partial r_i} \hat{e}_j = u_j \hat{e}_j \Rightarrow \nabla_r \cdot [(\vec{u} \cdot \nabla_r) \vec{r}] = \frac{\partial u_i}{\partial r_j} \hat{e}_i \cdot \hat{e}_j \\ &= \frac{\partial u_i}{\partial r_i} \\ &= \nabla_r \cdot \vec{u} \end{aligned}$$

and

$$\begin{aligned} \nabla^2 \phi &\equiv \nabla^2 (\phi_0 + \delta\phi) = \nabla^2 \phi_0 + \nabla^2 \delta\phi \\ &= \frac{1}{a^2} \nabla_r^2 \phi_0 + \frac{1}{a^2} \nabla_r^2 \delta\phi \\ &\stackrel{!}{=} 4\pi G (\rho_0 + \delta\rho) \Rightarrow \nabla_r^2 \delta\phi = 4\pi G \rho_0 a^2 \delta. \end{aligned}$$

A simple master equation is obtained by combining the continuity, Euler and Poisson equations:

$$\frac{d^2 \delta}{dt^2} + 2H \frac{d\delta}{dt} = \frac{c_s^2 \nabla_r^2 \delta}{a^2} + 4\pi G \rho_0 \delta. \quad (6.6)$$

The Hubble drag term ($2H\dot{\delta}$) represents an effective friction (damping) force which has the effect that, while self-gravity amplifies density perturbations in the cosmic fluid, cosmic expansion suppresses them [14, 44]; the Laplacian term represents the restoring force, and the last term is the source forcing term [21]. As per the Lichnerowicz-Obata theorem, the Laplacian in a flat, (nearly) FLRW spacetime is simply the Laplace-Betrami operator,

$$\nabla_r^2 \delta = -\lambda \delta, \quad (6.7)$$

where $\lambda \equiv k^2$ is the eigenvalue of eigenfunction δ . Therefore,

$$\frac{d^2 \delta}{dt^2} + 2H \frac{d\delta}{dt} = \left(4\pi G \rho_0 - \frac{c_s^2 k^2}{a^2} \right) \delta, \quad (6.8)$$

where $\vec{k}$ is the comoving wavenumber vector.

To describe the distribution of matter in the universe at a given time and its subsequent evolution, one could attempt to split it into multiple volumes, each of which evolves independently at first. However, the independence would no longer hold as the gravitational

forces between the volumes become too great. It is, therefore, not a good idea to think of a perturbation as a sum of spatial components. The perturbation should preferably be viewed as a superposition of plane waves that evolve independently [24]; this means that the distribution of matter can be represented in Fourier space as a sum of independent components (referred to as "modes"), each with a wave vector $\vec{k}$. If, as is customary, a continuous density field $\rho(\vec{r}, t)$ is assumed at a point $\vec{r}$ with respect to some arbitrary origin, the density perturbation field $\delta(\vec{r}, t)$ can be expressed as a wave-like decomposition [7]:

$$\begin{aligned}\delta(\vec{r}, t) &= \sum_{\vec{k}} \delta_{\vec{k}} e^{i\vec{k} \cdot \vec{r}} \\ &= \sum_{\vec{k}} \delta_{-\vec{k}} e^{-i\vec{k} \cdot \vec{r}}.\end{aligned}$$

Note that $\delta_{\vec{k}}^* = \delta_{-\vec{k}}$ for a real $\delta(\vec{r})$.

It is often easier to deal with the complex Fourier coefficients $\delta_{\vec{k}} = \delta_{\vec{k}}(t)$ given by

$$\delta_{\vec{k}} = \frac{1}{V} \int_V \delta(\vec{r}) e^{-i\vec{k} \cdot \vec{r}} d^3r.$$

V is a finite (but large) volume over which the universe is assumed to be periodic and is regarded as a fair sample of the universe. In this case, Eq. 6.8 can be re-written as

$$\frac{d^2\delta_k}{dt^2} + 2H \frac{d\delta_k}{dt} = \left(4\pi G\rho_0 - \frac{c_s^2 k^2}{a^2}\right) \delta_k.$$

The right-hand side defines a characteristic proper length:

$$4\pi G\rho_0 = \frac{c_s^2 k_J^2}{a^2} \quad \Rightarrow \quad k_J^2 \equiv \frac{4\pi^2 a^2}{\lambda_J^2} = \frac{4\pi G\rho_0 a^2}{c_s^2} \quad (6.9a)$$

$$\lambda_J = c_s \sqrt{\frac{\pi}{G\rho_0}}. \quad (6.9b)$$

λ_J critical length known as the Jeans length³. Physically, the existence of this critical length scale can be understood by considering the balance between gravity and pressure:

- $4\pi G\rho_0 \ll \frac{4\pi^2 c_s^2}{\lambda^2}$ implies $\lambda \ll \lambda_J$.

Pressure tends to oppose gravity effectively for wavelengths shorter than the Jeans length. The gravity term can be disregarded, and the equation for δ_k is the same as that for a damped oscillator;

- $4\pi G\rho_0 \gg \frac{4\pi^2 c_s^2}{\lambda^2}$ implies $\lambda \gg \lambda_J$.

On large scales, when the stabilising influence of pressure is significantly smaller than the effect of self-gravity, exclude the pressure term, and matter behaves like a pressureless fluid. Only unstable perturbations with linear dimensions larger than the Jeans length will grow in time [77].

³An alternative definition for the Jeans length is the scale at which sound waves can cross an object in about the time for gravitational collapse

There must, therefore, be some intermediate scale at which the transition from being pressure-dominated to being gravity-dominated occurs. The c_s^2 term is of particular interest:

$$\begin{aligned} c_s^2 &\equiv \frac{dp}{d\rho} = \frac{dp}{da} \frac{da}{d\rho} \\ &= -\frac{4\rho_r c^2}{3a} \left(-\frac{a}{3\rho_m + 4\rho_r} \right) \\ &= \frac{4\Omega_{r,0} c^2}{3(3a\Omega_{m,0} + 4\Omega_{r,0})}. \end{aligned}$$

Therefore,

- for $a \rightarrow 0$, $3c_s^2 \rightarrow c^2$ during the radiation-dominated era;
- for $a = 1$, $c_s^2 = 0$ in the current epoch when the universe is dominated by matter and dark energy;
- for $a \rightarrow \infty$, $c_s^2 \rightarrow 0$ during the dark energy-dominated epoch.

Thus, it follows that if the growth of structure has been dominated by cold dark matter ($\Omega_m \approx \Omega_d \approx 1$) for most of the universe's history, set $c_s = 0$ and conclude that perturbations on all scales are unstable. For collisionless dark matter, this description is a fairly good approximation. In such a case, the standard equation is:

$$\frac{d^2 \delta_m}{dt^2} + 2H \frac{d\delta_m}{dt} = \frac{3}{2} H^2 \delta_m; \quad (6.10)$$

which is solved by taking the ansatz $\delta \propto t^n$ and yields the equation

$$\frac{n(n-1)}{t^2} \delta_m + \frac{4n}{3t^2} \delta_m - \frac{2}{3t^2} \delta_m = 0;$$

which has two non-trivial, independent solutions if it satisfies

$$3n^2 + n - 2 = 0.$$

The solutions to this equation determine the fate of linear matter perturbations over time due to a combination of competing effects: universal expansion, pressure and the tendency for the fluctuations' self-gravity to cause overdense regions to collapse. $n = 2/3$ and $n = -1$ are associated with growing and decaying modes $\delta_+ \propto t^{2/3} \propto a$ and $\delta_- \propto t^{-1}$, respectively.

The Hubble parameter $H(t)$ obeys

$$\begin{aligned} 2H \frac{dH}{dt} &= \frac{8\pi G}{3} \frac{d\rho_m}{dt} + 2H \left(\frac{8\pi G}{3} \rho_m - H^2 \right) \\ \frac{dH}{dt} &= -\frac{4\pi G}{3} \rho_m - H^2 \\ \frac{d^2 H}{dt^2} + 2H \frac{dH}{dt} &= 4\pi G H \rho_m. \end{aligned}$$

Both δ and $H(t)$ obey the same equation, and since $H(t)$ decreases with t , $\delta_- \propto t^{-1} \propto H(t)$ gives the decaying mode, which is inconsequential because those modes disappear. The growing mode is of particular interest:

$$\delta_+ \propto t^{2/3} \propto a \Rightarrow \begin{cases} \delta(a) = a\delta(a=1) \\ \delta(t) = \delta(t=t_0) \left(\frac{t}{t_0}\right)^{2/3}. \end{cases}$$

Choose $t = t_0$ so that $\delta(\vec{r}, t = t_0)$ is sufficiently small.

The growth of perturbations in an expanding cold dark-matter-dominated universe is not exponential with time but a power law and, in particular, linear with the scale factor and independent of the wavelength. Assuming that the density fluctuations appeared at some very early time, then it is convenient to express the evolution of $\delta(a)$ in terms of the current value $\delta_0 \equiv \delta(a=1)$, and the linear growth factor $D_+(a)$:

$$\delta(a) = \frac{\delta(a=1) D_+(a)}{D_+(a=1)}, \quad (6.11)$$

where according to [78], $D_+(a)$ normalised to $D_+(a=1) = 1$ satisfies

$$D_+(a) \propto H \int_0^a \frac{1}{a^3 H^3} da$$

for a model with arbitrary Ω_Λ and Ω_m .

Up to this point, it was implicitly assumed that the universe is dominated by matter (insofar as perturbations are concerned). However, the initial density perturbations arose during the radiation-dominated era, where $3c_s^2 = c^2$. Theoretically, the correct treatment of perturbation growth during this era requires a relativistic treatment of the governing equations.

6.2 Relativistic Perturbation Theory

In general relativity, the density and pressure contribute to the "active" mass. For a more unified description of matter on all scales, especially considering the curvature and spacetime expansion and when the pressure is a significant fraction of the density, general relativity is not only more beneficial but necessary [44]. There are currently two distinct approaches to the general relativistic perturbation theory:

- the standard metric perturbation theory [79] and
- Gauge-Invariant⁴ Covariant⁵ (GIC) perturbation theory [75].

In the standard approach, Lifshitz started exactly from the metric tensor $g_{\alpha\beta}$. Provided $\bar{g}_{\alpha\beta}$ is the background metric in the standard system of coordinates, then $g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \delta g_{\alpha\beta}$

⁴variables take the same value in all coordinate systems

⁵equations take the same form in all coordinate systems

defines small variations of the metric tensor in these coordinates. The perturbed metric is written in the form

$$\begin{aligned} ds^2 &= (\bar{g}_{\alpha\beta} + \delta g_{\alpha\beta}) dx^\alpha dx^\beta \\ &= -(1 - \delta g_{00}) (dx^0)^2 + 2\delta g_{0i} dx^0 dx^i + (\bar{g}_{ij} + \delta g_{ij}) dx^i dx^j, \end{aligned}$$

where $|\bar{g}_{\alpha\beta}| \gg |\delta g_{\alpha\beta}|$ [75]; this has a drawback that, in addition to being a linear theory that does not account for nonlinear effects [21], the metric tensor itself is not a physically significant quantity. In other words, one cannot measure it directly, but only its second derivatives. As a result, it is not always obvious what the physical interpretation of a given metric perturbation is. It need not have any physical meaning at all, it must merely correspond to a coordinate transformation. Instead, it seems preferable to employ a method that considers small variations of more physically significant quantities [30].

6.2.1 1 + 3 Gauge-Invariant Covariant Formalism

General relativity is a covariant theory whose physical interpretation usually requires splitting tensors into parallel (time) and orthogonal (space) components. Spacetime is described via covariantly defined variables with respect to a partial frame formalism in the GIC perturbation theory. Prevailing examples such as the 1 + 3 or 1 + 1 + 2 (to indicate the number of dimensions in each slice) formalism assume a spacetime with a congruence of timelike curves [80]. This approach differs from the metric approach in that it can account for nonlinearities [21]. Furthermore, this approach produces differential equations of exactly the right order required to describe the true physical degrees of freedom, and, as such, no non-physical gauge modes exist.

Using the expanded metric about a homogeneous and isotropic background

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}, \quad (6.12)$$

and considering a perfect fluid flow in curved spacetime, there will always be a preferred family of fundamental worldlines representing the average motion of fundamental observers in the universe [21]. Let the average four-velocity vector tangent to these worldlines be:

$$u^\mu = \frac{dx^\mu}{d\tau}; \quad (6.13)$$

regarded as the four-velocity of a fundamental observer. τ is the proper time measured along the worldlines. At every point P of the spacetime, a subspace H_P of the tangent space T_P at P exists, which is orthogonal to u^μ [81]. For any u^μ , there exist two uniquely defined projection operators⁶:

⁶in $c = 1$ units

- $U_\nu^\mu = -u^\mu u_\nu$ projects along the four-velocity u^μ such that

$$\begin{aligned}
U_\gamma^\mu U_\nu^\gamma &= (-u^\mu u_\gamma)(-u^\gamma u_\nu) & U_\mu^\mu &= -u^\mu u_\mu & U_{\mu\nu} u^\nu &= (U_\mu^\gamma g_{\gamma\nu}) u^\nu \\
&= -u^\mu u_\nu & &= 1 & &= -(u^\gamma u_\mu) g_{\gamma\nu} u^\nu \\
&= U_\nu^\mu & & & &= -u_\nu u_\mu u^\nu \\
& & & & &= u_\mu;
\end{aligned}$$

- $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ determines the orthogonal metric properties of the instantaneous rest spaces of observers moving with four-velocity u^μ [21, 82]. This tensor is the metric in the subspace H_P of T_P and has properties

$$\begin{aligned}
h_\gamma^\mu h_\nu^\gamma &= h_{\alpha\gamma} h^{\alpha\mu} h_\nu^\gamma & h_\mu^\mu &= g_\mu^\mu + u_\mu u^\mu & h_{\mu\nu} u^\nu &= (g_{\mu\nu} + u_\mu u_\nu) u^\nu \\
&= \delta_\gamma^\mu h_\nu^\gamma & &= \delta_\mu^\mu - 1 & &= u_\mu - u_\mu \\
&= h_\nu^\mu & &= 3 & &= 0.
\end{aligned}$$

In this way, any affine parameters on the worldlines associated with u^α may be chosen to represent "time", and the tensor $h_{\mu\nu}$ determines the geometry of the instantaneous rest spaces of the observers [83]. Similarly, if isotropy is assumed, the matter energy-momentum tensor $T_{\mu\nu}$ of a perfect fluid can be decomposed locally using $h_{\mu\nu}$ and u^μ :

$$\begin{aligned}
T_{\mu\nu} &\equiv (\rho + p) u_\mu u_\nu + p g_{\mu\nu} = \rho u_\mu u_\nu + p u_\mu u_\nu + p (h_{\mu\nu} - u_\mu u_\nu) \\
&= \rho u_\mu u_\nu + p h_{\mu\nu},
\end{aligned}$$

where $T_{\mu\nu} u^\mu u^\nu = \rho$ and $T_{\mu\nu} h^{\mu\nu} = 3p$ define the relativistic energy density and isotropic pressure, respectively [21]. Furthermore, since it is a time-space formalism, two derivatives are defined for any tensor $T_{\mu\nu}^{\alpha\beta}$ [21]:

- the proper time derivative, as measured by a fundamental observer, is simply the covariant derivative along u^μ

$$\dot{T}_{\mu\nu}^{\alpha\beta} = u^\sigma \nabla_\sigma T_{\mu\nu}^{\alpha\beta} \quad (6.14)$$

- the fully-orthogonal covariant derivative D_σ in the local rest frame of the observer:

$$D_\sigma T_{\mu\nu}^{\alpha\beta} = h_\gamma^\alpha h_\lambda^\beta h_\mu^\eta h_\nu^\epsilon h_\sigma^\rho \nabla_\rho T_{\eta\epsilon}^{\gamma\lambda}. \quad (6.15)$$

The operator D_σ is a useful tool to avoid a plethora of indices.

For example, the covariant spatial derivative of the four-velocity is given by

$$\begin{aligned}
D_\mu u_\nu &\equiv h_\mu^\sigma \nabla_\sigma u_\nu = \left(\delta_\mu^\sigma + \frac{u_\mu u^\sigma}{c^2} \right) \nabla_\sigma u_\nu \\
&= \nabla_\mu u_\nu + \frac{a_\nu u_\mu}{c^2} \\
&= \sigma_{\mu\nu} + \omega_{\mu\nu} + H h_{\mu\nu} + \frac{a_\nu u_\mu}{c^2}.
\end{aligned}$$

$a_\nu = \dot{u}_\nu$ is the relativistic acceleration vector, which explains the effects of non-gravitational forces (such as pressure) acting on the observer [21, 82].

In general, every rank-2 tensor can be decomposed into its irreducible parts [82], namely a symmetric and anti-symmetric pair:

$$\nabla_\mu u_\nu \equiv \frac{1}{2}(\nabla_\mu u_\nu + \nabla_\nu u_\mu) + \frac{1}{2}(\nabla_\mu u_\nu - \nabla_\nu u_\mu) = u_{(\nu;\mu)} + u_{[\nu;\mu]}.$$

The symmetric and anti-symmetric parts of $\nabla_\mu u_\nu$ are the shear $\sigma_{\mu\nu}$ and vorticity $\omega_{\mu\nu}$, respectively [75]:

$$\begin{aligned}\sigma_{\mu\nu} &= h_\nu^\sigma h_\mu^\lambda u_{(\sigma;\lambda)} - H h_{\mu\nu} \\ \omega_{\mu\nu} &= h_\nu^\sigma h_\mu^\lambda u_{[\sigma;\lambda]},\end{aligned}$$

and they satisfy the conditions $\sigma_{\mu\nu}u^\nu = \sigma_\sigma^\sigma = 0$ and $\omega_{\mu\nu}u^\nu = 0$. The shear describes the rate of distortion of the observers' flow [21], and the vorticity describes the rotation of the observers relative to a non-rotating frame.

The quantities that characterise the variation of zeroth-order variables (pressure p_0 and mass density ρ_0) are typically non-zero and are, therefore, not gauge-invariant. However, [75] introduced gauge-invariant orthogonal spatial gradients.

- To characterise the density perturbation, the natural variable is $\chi_\mu = D_\mu \rho$; this vector represents any spatial variation in density ρ (such as clusters or voids) and is, at least in theory, measurable. However, a more suitable quantity to describe the density perturbation is the comoving fractional density gradient

$$\mathcal{D}_\mu = \frac{a D_\mu \rho}{\rho}, \quad (6.16)$$

where the ratio $\frac{D_\mu \rho}{\rho}$ allows one to evaluate the magnitude of the perturbations relative to the background density and the presence of the scale factor guarantees that it is comoving [84].

- Another important quantity pertinent to density perturbation evolution in general relativity is the comoving expansion gradient:

$$\mathcal{Z}_\mu = a D_\mu \theta, \quad (6.17)$$

where D_μ is the orthogonal spatial derivative and $\theta \equiv 3H$ represents the rate of volume expansion of the u^α worldlines [84].

In this approach, rather than concentrating on $\delta(\vec{r}, t)$, the gauge-invariant quantities allow for the extraction of information regarding the thermodynamics associated with the fluid. The benefit of using these variables lies in their capacity to offer a treatment of cosmology that is both mathematically sound and physically significant, especially to perturbation theory. Their evolution and constraint equations are equivalent to the EFEs and represent the complete evolution of a cosmological model [84].

One can derive the evolution and constraint equations for $\mathcal{D}_\mu$. From the fluid equation,

$$\begin{aligned}\dot{\rho} + 3H \left(\rho + \frac{p}{c^2} \right) &= 0 \quad \Rightarrow \quad D_\mu \dot{\rho} + 3H(1+w) D_\mu \rho + 3\rho(1+w) D_\mu H = 0 \\ h_\mu^\nu \nabla_\nu (u^\sigma \nabla_\sigma \rho) + \frac{3H\rho}{a} (1+w) \mathcal{D}_\mu + \frac{\rho}{a} (1+w) \mathcal{Z}_\mu &= 0\end{aligned}$$

$$h_\mu^\nu (\nabla_\nu u^\sigma) (\nabla_\sigma \rho) + h_\mu^\nu u^\sigma \nabla_\sigma (\nabla_\nu \rho) + \frac{3H\rho}{a} (1+w) \mathcal{D}_\mu + \frac{\rho}{a} (1+w) \mathcal{Z}_\mu = 0.$$

For the FLRW background $\sigma_{\mu\nu} = \omega_{\mu\nu} = 0$, thus,

$$\begin{aligned} H h_\mu^\sigma \nabla_\sigma \rho + h_\mu^\lambda u^\sigma \nabla_\sigma (h_\lambda^\nu \nabla_\nu \rho) - h_\mu^\lambda u^\sigma (\nabla_\sigma h_\lambda^\nu) (\nabla_\nu \rho) + \frac{3H\rho}{a} (1+w) \mathcal{D}_\mu + \frac{\rho}{a} (1+w) \mathcal{Z}_\mu &= 0 \\ H D_\mu \rho + h_\mu^\lambda (D_\lambda \rho)^\cdot - \frac{a_\mu \dot{\rho}}{c^2} + \frac{3H\rho}{a} (1+w) \mathcal{D}_\mu + \frac{\rho}{a} (1+w) \mathcal{Z}_\mu &= 0. \end{aligned}$$

The acceleration a_μ is given by the momentum conservation equation:

$$a_\mu = -\frac{wc^2}{1+w} \frac{D_\mu \rho}{\rho}.$$

Substitute this expression into the perturbation equation

$$\begin{aligned} \frac{4H\rho}{a} \mathcal{D}_\mu + h_\mu^\lambda (D_\lambda \rho)^\cdot + \frac{\rho}{a} (1+w) \mathcal{Z}_\mu &= 0 \\ \frac{4H\rho}{a} \mathcal{D}_\mu + \frac{\rho}{a} h_\mu^\lambda \left[\dot{\mathcal{D}}_\lambda - (4+3w) H \mathcal{D}_\lambda \right] + \frac{\rho}{a} (1+w) \mathcal{Z}_\mu &= 0 \\ \frac{\rho}{a} \left[\dot{\mathcal{D}}_\mu - 3Hw \mathcal{D}_\mu + (1+w) \mathcal{Z}_\mu \right] &= 0. \end{aligned}$$

$\rho/a = 0$ is true if the universe is of infinite size – in which case, there is no structure formation. Therefore, the linearised propagation equation for $\mathcal{D}_\mu$ is given by the coupled differential equation

$$\dot{\mathcal{D}}_\mu = 3Hw \mathcal{D}_\mu - (1+w) \mathcal{Z}_\mu. \quad (6.18)$$

The above equation can be reduced to a linearly independent second-order equation by deriving the equation for $\mathcal{Z}_\mu$ in the same way from the Raychaudri-Ehlers equation.

The Raychaudri-Ehlers equation (given here for the FLRW background) [21]

$$\dot{H} = \frac{1}{3} \nabla^\mu a_\mu - H^2 - \frac{4\pi G}{3} \left(\rho + \frac{3p}{c^2} \right) \quad (6.19a)$$

$$= -\frac{wc^2}{3a(1+w)} D^\nu \mathcal{D}_\nu - H^2 - \frac{4\pi G}{3} \left(\rho + \frac{3p}{c^2} \right) \quad (6.19b)$$

is a fundamental result describing the evolution of H along the fluid flow lines:

$$\begin{aligned} D_\mu \dot{H} &= -\frac{wc^2}{3a(1+w)} D_\mu D^\nu \mathcal{D}_\nu - 2H D_\mu H - \frac{4\pi G}{3a} \rho (1+3w) \mathcal{D}_\mu \\ \frac{H \mathcal{Z}_\mu}{3a} + h_\mu^\lambda (D_\lambda H)^\cdot - \frac{a_\mu \dot{H}}{c^2} &= -\frac{wc^2}{3a(1+w)} D_\mu D^\nu \mathcal{D}_\nu - \frac{2H \mathcal{Z}_\mu}{3a} - \frac{4\pi G}{3a} \rho (1+3w) \mathcal{D}_\mu \\ \dot{\mathcal{Z}}_\mu &= -\frac{wc^2}{1+w} D_\mu D^\nu \mathcal{D}_\nu - 2H \mathcal{Z}_\mu + \frac{3wH^2}{1+w} \mathcal{D}_\mu - \frac{4\pi G}{1+w} \rho (1+3w) \mathcal{D}_\mu. \end{aligned}$$

The dynamics of the variable $\mathcal{D}_\mu$ is given by the linear second-order differential equation

$$\begin{aligned} \dot{\mathcal{D}}_\mu = 3Hw \mathcal{D}_\mu - (1+w) \mathcal{Z}_\mu &\Rightarrow \ddot{\mathcal{D}}_\mu = 3\dot{H}w \mathcal{D}_\mu + 3Hw \dot{\mathcal{D}}_\mu - (1+w) \dot{\mathcal{Z}}_\mu \\ &= 3w \left[-\frac{wc^2}{3a(1+w)} D^\nu \mathcal{D}_\nu - H^2 - \frac{4\pi G}{3} \rho (1+3w) \right] \mathcal{D}_\mu \\ &\quad + 3Hw \dot{\mathcal{D}}_\mu + wc^2 D_\mu D^\nu \mathcal{D}_\nu + 2H (1+w) \mathcal{Z}_\mu \\ &\quad - 3wH^2 \mathcal{D}_\mu + 4\pi G \rho (1+3w) \mathcal{D}_\mu \\ &= - (2-3w) H \dot{\mathcal{D}}_\mu + 4\pi G \rho (1-w) (1+3w) \mathcal{D}_\mu \\ &\quad + wc^2 D_\mu D^\nu \mathcal{D}_\nu. \end{aligned}$$

During the linearisation of an equation, first-order quantity products are discarded because they become second-order, but products like $\theta \mathcal{D}_\mu$ are kept. Furthermore,

$$\begin{aligned} D_\mu D^\nu \mathcal{D}_\nu &= \delta_\mu^\nu \nabla^2 \mathcal{D}_\nu \\ &= \nabla^2 \mathcal{D}_\mu \\ &= -\frac{k^2}{a^2} \mathcal{D}_\mu \end{aligned}$$

from the Helmholtz equation [75]. Therefore, the evolution equation for the comoving spatial density gradient is given by

$$\ddot{\mathcal{D}}_\mu = -(2 - 3w) H \dot{\mathcal{D}}_\mu + 4\pi G \rho (1 - w) (1 + 3w) \mathcal{D}_\mu - \frac{w c^2 k^2}{a^2} \mathcal{D}_\mu. \quad (6.20)$$

There are two parts to the quantity $\mathcal{D}_\mu$: a scalar and a vector. Structure formation on cosmological scales is assumed to be due to the scalar part whose magnitude

$$\mathcal{D} \equiv (\mathcal{D}_\mu \mathcal{D}^\mu)^{\frac{1}{2}}$$

is a real, covariant and gauge-invariant variable that corresponds closest to $\delta(\vec{r}, t)$. It is easy to show that $\delta(\vec{r}, t)$ is the scalar harmonic component of $\mathcal{D}_\mu$ [84]. Nonetheless, Eq. 6.20 may be recast as

$$\ddot{\delta} + (2 - 3w) H \dot{\delta} = \left[4\pi G \rho (1 - w) (1 + 3w) - \frac{w c^2 k^2}{a^2} \right] \delta; \quad (6.21)$$

this equation has the advantage of making comparisons to the Newtonian approach easier [21]. The extra simplicity gained by using the Newtonian method (apart from giving formulae that are easier to understand) allows one to extend the investigation to a broader class of models, specifically those representing the steady-state universe [77].

The Jeans length is given by:

$$\lambda_J = \begin{cases} 0, & \text{for a matter-dominated universe (as before)} \\ \pi R_h \sqrt{\frac{2}{3}}, & \text{for a radiation-dominated universe.} \end{cases} \quad (6.22)$$

The Jeans length in the radiation-dominated regime is of the order of the Hubble radius, R_h . Does this mean that perturbations on scales comparable to the size of the observable universe are suppressed during the radiation-dominated era?

When considering scales $\lambda \ll R_h$, any perturbations in the radiation component can be disregarded. In this limit, the equation describing the growth of nonrelativistic matter perturbations before decoupling is

$$\frac{d^2 \delta}{dt^2} + 2H \frac{d\delta}{dt} = 4\pi G \rho_m \delta, \quad (6.23)$$

where

$$H^2 = \frac{8\pi G}{3} (\rho_m + \rho_r).$$

SUPPLEMENTARY:

$$\begin{aligned}
\frac{d}{dt} &= \frac{da}{dt} \frac{d}{da} \\
&= aH \frac{d}{da} \quad \Rightarrow \quad \frac{d^2}{dt^2} = aH \frac{d}{da} \left(aH \frac{d}{da} \right) \\
&= aH \left(H \frac{d}{da} + a \frac{dH}{da} \frac{d}{da} + aH \frac{d^2}{da^2} \right) \\
&= aH^2 \frac{d}{da} + a^2 H \frac{dH}{da} \frac{d}{da} + a^2 H^2 \frac{d^2}{da^2}
\end{aligned}$$

Substituting the expressions above into Eq. 6.23:

$$\begin{aligned}
a^2 H^2 \frac{d^2 \delta}{da^2} + a^2 H^2 \left(\frac{3}{a} + \frac{1}{H} \frac{dH}{da} \right) \frac{d\delta}{da} &= 4\pi G \rho_m \delta \\
\frac{d^2 \delta}{da^2} + \frac{3\rho_m + 2\rho_r}{2a(\rho_m + \rho_r)} \frac{d\delta}{da} &= \frac{3\rho_m}{2a^2(\rho_m + \rho_r)} \delta \\
\frac{d^2 \delta}{da^2} + \frac{2 + 3a}{2a(1 + a)} \frac{d\delta}{da} &= \frac{3}{2a(1 + a)} \delta.
\end{aligned}$$

Let

$$\frac{\rho_m}{\rho_r} \equiv \frac{a}{a_{eq}},$$

where a_{eq} is the scale factor at matter-radiation equality. The growing mode solution is

$$\delta = C \left(1 + \frac{3}{2}a \right), \quad (6.24)$$

for which $\delta \rightarrow \text{constant}$ for $a \ll a_{eq}$ (during the radiation-dominated era) and, $\delta \propto a$ at later times. Dark matter does not couple directly to radiation and, hence, is unaffected by the radiation pressure. However, the growth of perturbations on scales $\lambda \ll R_h$ is nonetheless suppressed. Consequently, before matter-radiation equality, small-scale dark matter perturbations are prevented from growing. Physically, this effect is understood as follows: on scales smaller than λ_J , the growth rate of the perturbations (driven by ρ_m) is much slower than the universe's expansion rate (driven by ρ_r), and thus δ is "frozen" at a roughly constant value for as long as $\rho_m \ll \rho_r$. If scales larger than λ_J are considered, both ρ_m and ρ_r contribute to the growth of the perturbations in such a way that the governing equation for the growth of density perturbations is given by

$$\frac{d^2 \delta_m}{dt^2} + 2H \frac{d\delta_m}{dt} - 4\pi G \rho_m \delta_m = 0, \quad (6.25)$$

when Ω_m drops appreciably below 1. Here, δ_m only depends on matter and $\rho_0 = \rho_m$. The argument is that although perturbations include radiation and dark and baryonic matter, neutrino masses are so tiny that they barely affect structure formation. After decoupling, photons stream freely and form a smooth, homogeneous background. Since dark matter is assumed to be collisionless and non-interacting, its perturbations grow due to self-gravity alone – unaffected by radiation pressure. In contrast, baryons could not form bound objects in the early universe because their growth was suppressed by the radiation to which they

were coupled. As they decoupled from radiation, they accreted into the potential wells created by dark matter to form the large-scale structure currently observed [85]. The conventional approach is to presume that dark energy also forms a smooth background [21] and does not significantly contribute to the internal evolution of matter perturbations.

From the definition of the Jeans length

$$\lambda_J = \pi R_h \sqrt{\frac{2}{3}}$$

for a dark energy-dominated universe. Essentially, the effect of radiation and dark energy is to define the cosmic time scale and change the relationship between the matter density and the expansion rate from what it would be in a matter-dominated universe, which, in turn, influences the growth of large-scale structure [14, 86, 87]. Consequently, the differential equation is more difficult to solve for the standard model.

Conversely, for the $R_h = ct$ universe, the dynamical equation for δ_k is

$$\frac{d^2 \delta_m^k}{dt^2} + 3H \frac{d\delta_m^k}{dt} - \frac{c^2 k^2}{3a^2} \delta_m^k = 0 \quad (6.26a)$$

$$\frac{d^2 \delta_m^k}{dt^2} + \frac{3}{t} \frac{d\delta_m^k}{dt} - \frac{\Delta_k^2}{3t^2} \delta_m^k = 0, \quad (6.26b)$$

where

$$\Delta_k = \frac{2\pi R_h}{\lambda}.$$

Both the fluctuation scale λ and the gravitational horizon R_h depend on t in the same manner. Δ_k is, therefore, essentially a constant of proportionality, meaning that the perturbations in this cosmology grow in amplitude while their wavelength remains a fixed fraction of the Hubble radius [44].

It is imperative to highlight a few fundamental aspects of this equation. First, it is unclear how the large-scale structure would emerge within this model [8]. The active mass in this universe vanishes, and the gravitational term that usually appears in the standard model is absent. But this does not prevent δ_k from growing. Because $p < 0$, the (usually dissipative) pressure term becomes a source of gravity and an agent of perturbation growth [7].

Second, the standard model predicts different distributions at different scales, in part because of the influence of the Jeans length. However, the $R_h = ct$ universe does not have such a transition region [7]. In its place is the gravitational radius.

A simple solution to Eq. 6.26 is the power law

$$\delta_m^k \propto t^n,$$

where, evidently

$$\begin{aligned} \frac{1}{3t^2} (3n^2 + 6n - \Delta_k^2) \delta &= 0 \\ \delta \neq 0 \text{ and } t \neq \infty \text{ simultaneously} &\Rightarrow 3n^2 + 6n - \Delta_k^2 = 0 \end{aligned}$$

whenever

$$n = -1 \pm \sqrt{1 + \frac{\Delta_k^2}{3}}.$$

Unlike, Λ CDM, where δ_m is independent of k (or rather λ), the perturbation in this case is highly dependent on whether λ is less than or greater than R_h :

- If $\lambda \ll R_h$, $\Delta_k \gg 1$ and

$$\begin{aligned} n \rightarrow -1 \pm \left[\frac{\Delta_k^2}{3} \left(1 + \frac{3}{\Delta_k^2} \right) \right]^{1/2} &= -1 \pm \frac{\Delta_k}{\sqrt{3}} \left(1 + \frac{3}{2\Delta_k^2} + \dots \right) \\ &= -1 \pm \frac{\Delta_k}{\sqrt{3}} \pm \frac{\sqrt{3}}{2\Delta_k} \pm \dots \\ &\approx \pm \frac{\Delta_k}{\sqrt{3}}, \end{aligned}$$

which corresponds to

$$\delta_m^k = At^{n_+} + Bt^{n_-},$$

where $\sqrt{3}(n_+, n_-) = (\Delta_k, -\Delta_k)$ and A and B are the amplitudes of the growing or decaying modes, respectively.

- If $\lambda \gg R_h$, $\Delta_k \ll 1$ and

$$\begin{aligned} n \rightarrow -1 \pm \left(1 + \frac{\Delta_k^2}{6} + \dots \right) &= -1 \pm 1 \pm \frac{\Delta_k^2}{6} \pm \dots \\ &\approx -1 \pm 1, \end{aligned}$$

which corresponds to

$$\delta_m^k = C + \frac{D}{t^2},$$

where $C = \text{constant}$. For large fluctuations, the dominant mode does not even grow.

In general, however, the solution to the differential equation is given by

$$\delta_m^k(t) = At^{-1+S} + Bt^{-1-S}, \quad (6.27)$$

where

$$S = \sqrt{1 + \frac{c^2 k^2}{3H_0^2}}.$$

Conservation of mass enclosed in V requires

$$\begin{aligned} \delta_{\vec{k}=0} &= \frac{1}{V} \int_V \delta(\vec{r}) d^3x \\ &= \langle \delta(\vec{r}) \rangle \\ &\stackrel{!}{=} 0 \end{aligned}$$

for a homogeneous and isotropic universe, i.e.

$$\delta_m^{k=0} \equiv A + \frac{B}{t^2} \stackrel{!}{=} 0 \quad \Rightarrow \quad \delta_m^k(t) = At^{-1+S} - At^{1-S}.$$

Due to gravitational instability, the deviation in the CMB temperature is of the order of 10^{-5} :

$$\delta_m^k(t_0) \equiv At_0^{-1+S} - At_0^{1-S} \approx 10^{-5} \quad \Rightarrow \quad A \approx \frac{10^{-5}}{t_0^{-1+S} - t_0^{1-S}}.$$

A guide to the physical interpretation of this solution is provided by the continuity equation

$$\frac{d\delta}{dt} = -\nabla_r \cdot \vec{u},$$

which confirms that the peculiar velocities induced by linear density perturbations result in what are referred to as vorticity modes for which $\nabla_r \cdot \vec{u} = 0$, and δ is covariantly conserved [21], i.e. $\dot{\delta} \stackrel{!}{=} 0$. However, $\dot{\delta} \neq 0$ for any nonzero δ , so these modes are, thus, not associated with any perturbation to the density field. Moreover, the condition for a perturbation to be a growing mode is that the velocity divergence correctly reinforces the gravitational growth:

$$\begin{aligned} \frac{d\delta}{dt} &\equiv -\nabla_r \cdot \vec{u} = \frac{\delta}{\delta} \frac{d\delta}{dt} \\ &= \left(\frac{a}{\delta}\right) H\delta(\vec{r}, t) \frac{d\delta}{da} \\ &= H\delta(\vec{r}, t) f(\Omega_m), \end{aligned}$$

where

$$f(\Omega_m) = \frac{d \ln \delta}{d \ln a} \quad (6.28)$$

is the growth rate representing a measure of the matter clustering and depends rather sensitively on the underlying dynamical expansion rate of the universe. To this end, measuring the growth factor is a promising method to test cosmological models.

In theory, it is possible to indirectly measure $f(\Omega_m)$ by measuring the peculiar velocity field $\vec{u}(\vec{r}, t)$ and the density field $\delta(\vec{r}, t)$. This method served as the foundation for the POTENT program (a popular technique for determining Ω_m using the galaxy two-point correlation function) in the 1990s [88]. However, strictly speaking, the galaxy distribution $\delta_g(\vec{r})$ used measures $\delta_g(\vec{r}, t) = b\delta(\vec{r}, t)$ – where b is a non-trivial linear bias factor of the cosmological tracer. Over the last few decades, the growth rates were measured using various techniques focused on the infall motion associated with condensing regions. The evolution of galaxies manifests these motions, imprinting the changing growth rate as the universe evolves [89, 90, 91]. The galaxy distribution is, therefore, not always the same as the matter distribution, and the relation between them is quantified by the bias factor. Therefore, what the peculiar velocity surveys measure is a parameter $\beta \approx f(\Omega_m)/b$ [92]. Furthermore, unfortunately, the bias factor varies between populations of galaxies, so measurements from different surveys are arduous to combine and compare with theoretical predictions [44].

A more popular statistical method for measuring the growth rate is based on measuring peculiar velocities from Redshift Space Distortions (RSDs)⁷ in galaxy redshift surveys, as

⁷Redshift Space Distortions are an effect in observational cosmology where the spatial distribution of galaxies appears "squashed" and distorted when the positions are plotted as a function of redshift rather than as a function of their distance

proposed by [93]. The overall effect is due to the peculiar velocities of galaxies, which cause a Doppler shift in addition to the redshift caused by the cosmological expansion [93]. Specifically, maps created using distances inferred from redshifts in spectroscopic galaxy surveys show that the galaxy distributions are anisotropic because the redshifts contain components from both the smooth Hubble flow and the peculiar velocities of the infalling matter [44]. Provided one can consistently distinguish between these two contributions to the redshift, one can extract a history of the structure build-up.

6.3 Evolution of the Power Spectrum

Although the mean value of the perturbation across the statistical ensemble is identically zero by definition, its positive-definite mean squared value (i.e. variance) is not [24].

SUPPLEMENTARY:

$$\delta(\vec{r}) = \sum_{\vec{k}} \delta_{\vec{k}} e^{i\vec{k} \cdot \vec{r}} \quad \Rightarrow \quad |\delta(\vec{r})|^2 \equiv \delta^2(\vec{r}) = \sum_{\vec{k}'} \sum_{\vec{k}} \delta_{\vec{k}'}^* \delta_{\vec{k}} e^{-i(\vec{k}' - \vec{k}) \cdot \vec{r}}$$

By the periodic boundary conditions, all the cross-terms with $\vec{k}' \neq \vec{k}$ average to zero. The double sum reduces to the integral

$$\delta^2(\vec{r}) = \frac{V}{(2\pi)^3} \int |\delta_{\vec{k}}|^2 d^3k$$

The variance

$$\sigma^2 \equiv \langle \delta^2(\vec{r}) \rangle - \langle \delta(\vec{r}) \rangle^2 = \frac{1}{(2\pi)^3} \int \mathcal{P}(\vec{k}) d^3k \quad (6.29a)$$

$$= \frac{1}{2\pi^2} \int \mathcal{P}(k) k^2 dk \quad (6.29b)$$

characterises the average fluctuation amplitude of the density field at a given scale and may be quantified by the quantity $\mathcal{P}(\vec{k}) = \langle |\delta_{\vec{k}}|^2 \rangle$, the power spectral density function (or simply the power spectrum) of the field $\delta(\vec{r})$. According to the Wiener-Khinchin theorem, the power spectrum, as defined, contains both scalar and tensor perturbation modes. In the case of an isotropic universe, however, only the direction-averaged power spectrum $\mathcal{P}(k) \equiv \mathcal{P}(|\vec{k}|) = \langle \delta_k^2 \rangle$ is of interest. $\mathcal{P}(k) \rightarrow 0$ as $k \rightarrow 0$ if homogeneity is to be recovered on large scales [24].

The power spectrum is a fundamental quantity in cosmology, and its determination is very important, but what is known about its functional form? It is usual to assume that, over a small range of k , the scalar component of the power spectrum in inflationary models (Λ CDM, for example) can be approximated by a power law,

$$\mathcal{P}_s(k) \propto k^{n_s},$$

where the exponent n_s is called the scalar spectral index [7]. It describes how the density fluctuations vary between large and small scales. Some models predict that the power

spectrum of matter fluctuations is almost scale invariant as the fluctuations cross the Hubble radius, such as the Harrison-Zel'dovich spectrum, for which $n_s = 1$.

Unfortunately, the power spectrum (or specifically, the variance) is insufficiently defined because the fluctuation field is smoothed to some resolution, L .

A common concept in the manipulation of cosmological density fields is that of filtering. Interest is often drawn towards the properties of the density field after it has been filtered and smoothed by convolution with a window function $\mathcal{W}_L(r)$ of characteristic radius L . A common choice for 3-D filter function $\mathcal{W}_L(r)$ is the Gaussian

$$\mathcal{W}_L(r) = \frac{1}{(2\pi)^{3/2} L^3} e^{-r^2/2L^2}.$$

Interest is also often focused on the variance of the smoothed field for use as a convenient statistic, not the convolved field:

$$\sigma_L^2 = \frac{1}{2\pi^2} \int_0^\infty \mathcal{P}(k) k^2 \tilde{\mathcal{W}}^2(kL) dk,$$

where

$$\tilde{\mathcal{W}}(kL) = e^{-(kL)^2/2}$$

is the Fourier transform of the window function [94].

A commonly used parameter is the variance in spherical volumes with a radius $L = 8$ Mpc/h ($h = H/100$) – roughly corresponding to the scale of massive galaxy clusters. Putting all these together, attained the final expression for σ_L^2 :

$$\sigma_8^2 = \frac{1}{2\pi^2} \int_0^\infty \mathcal{P}(k) k^2 e^{-64k^2} dk \quad (6.30a)$$

$$= \sigma_8^2(z=0) D^2(z). \quad (6.30b)$$

Here, the measured value of $\sigma_8(z)$ is known from the galaxy distribution to within a bias factor b . The probative power of f and σ_8 , individually, is mitigated by the uncertainty in the inferred bias [44]. Each, however, has a dependence on b that is the inverse of the other, so the product

$$f\sigma_8(z) \equiv -\frac{1+z}{\delta} \sigma_{8,0} D(z) \frac{d\delta}{dz} = -\frac{1+z}{\delta_0 D(z)} \sigma_{8,0} D(z) \frac{d}{dz} [\delta_0 D(z)] \quad (6.31a)$$

$$= -\sigma_{8,0} (1+z) \frac{dD}{dz} \quad (6.31b)$$

may be a much more suitable measure of the structural evolution [44, 95]. Measurements of the growth rate are now commonly reported using the quantity $f\sigma_8(z)$, and these are the data employed for the analysis.

6.4 Constraints from RSD Data

The literature reports diverse compilations of measurements of the growth rate of cosmic structures [96]. The criteria for selecting the data, shown in Table A.1, are:

- i. Data obtained from uncorrelated redshift bins when the same cosmological tracer was analysed;
- ii. Data from possibly correlated redshift bins when different tracers were analysed;
- iii. In Table A.1, only direct measurements of $f\sigma_8(z)$ were considered;
- iv. The latest measurements were considered when the same survey collaboration performed two or more measurements corresponding to diverse data releases.

The use of RSD, however, is practical before nonlinear effects start to emerge. In these cases, measurements provide information on the matter overdensity as well as the peculiar velocities of galaxies. The problem is typically reduced to solving a first-order differential equation in the linear regime [44].

For Λ CDM,

$$\frac{d^2 D}{dt^2} + 2H \frac{dD}{dt} - 4\pi G \rho_m D = 0$$

but

$$\begin{aligned} \frac{dD}{dt} &= \frac{da}{dt} \frac{dz}{da} \frac{dD}{dz} \\ &= -H(1+z) \frac{dD}{dz} \quad \Rightarrow \quad \frac{d^2 D}{dt^2} = H(1+z) \frac{d}{dz} \left[H(1+z) \frac{dD}{dz} \right] \\ &= H(1+z)^2 \frac{dH}{dz} \frac{dD}{dz} + H^2(1+z) \frac{dD}{dz} + H^2(1+z)^2 \frac{d^2 D}{dz^2}, \end{aligned}$$

which yields

$$\frac{d^2 D}{dt^2} + 2H \frac{dD}{dt} - 4\pi G \rho_m D = H^2(1+z)^2 \frac{d^2 D}{dz^2} + H(1+z)^2 \frac{dH}{dz} \frac{dD}{dz} - H^2(1+z) \frac{dD}{dz} - 4\pi G \rho_m D.$$

SUPPLEMENTARY:

$$\begin{aligned} f\sigma_8(z) \equiv -\sigma_{8,0}(1+z) \frac{dD}{dz} \quad \Rightarrow \quad \frac{d}{dz} [f\sigma_8(z)] &= -\sigma_{8,0} \frac{d}{dz} \left[(1+z) \frac{dD}{dz} \right] \\ &= -\sigma_{8,0} \frac{dD}{dz} - \sigma_{8,0}(1+z) \frac{d^2 D}{dz^2} \\ &= \frac{f\sigma_8(z)}{1+z} - \sigma_{8,0}(1+z) \frac{d^2 D}{dz^2} \end{aligned}$$

Substituting these results into the second-order perturbation equation yields

$$\frac{d}{dz} [f\sigma_8(z)] + \left(\frac{1}{H} \frac{dH}{dz} - \frac{2}{1+z} \right) f\sigma_8(z) + \frac{3\Omega_{m,0}(1+z)^2}{2E^2} \sigma_8(z) = 0, \quad (6.32)$$

where

$$\begin{aligned} E^2 &\equiv \left(\frac{H}{H_0} \right)^2 = \Omega_{m,0}(1+z)^3 + \cancel{\Omega_{r,0}}^0 (1+z)^4 + \Omega_{\Lambda,0} \\ &\approx \Omega_{m,0}(1+z)^3 + 1 - \Omega_{m,0}. \end{aligned}$$

In recent years, perturbation theory has matured to the point where the growth rate predicted by Λ CDM can be extensively compared to direct measurements and other models' predictions - in this case, the $R_h = ct$ model whose corresponding dynamic equation is:

$$\frac{d}{dz} [f\sigma_8(z)] - \frac{2}{1+z} f\sigma_8(z) + \frac{c^2 k^2}{3H_0^2(1+z)} \sigma_{8,0} = 0. \quad (6.33)$$

In this analysis, the values of H_0 and σ_8 must be consistently presented with another variable - k .

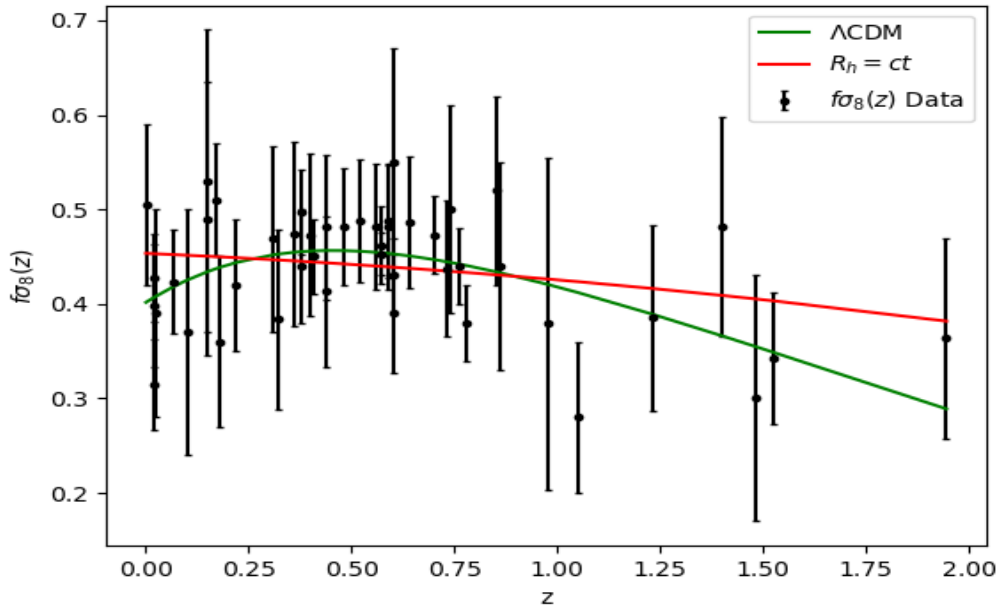


Figure 6.1: Plots of $f\sigma_8(z)$ for Λ CDM and $R_h = ct$ superimposed on the measured values from the sample in Table A.1. When evaluated at the present time, $f\sigma_8(z=0) = 0.402 \pm 0.023$ and $f\sigma_8(z=0) = 0.453 \pm 0.023$ for Λ CDM and $R_h = ct$, are found, respectively, at the 1σ confidence level. The best-fit values are $\Omega_m = 0.433 \pm 0.127$ and $\sigma_8 = 0.411 \pm 0.117$ for Λ CDM and $H_0 = 21.589$ km/s/Mpc, $k = -2.58 \times 10^{-4}$ /Mpc and $\sigma_8 = 0.217$ for $R_h = ct$.

The value of Ω_m indicates some inconsistency between the OHD, SNe Ia and $f\sigma_8(z)$ data. The $f\sigma_8(z)$ measurements seem to be finding a higher value of $\Omega_{m,0}$ than is expected. $\Omega_m = 0.433 \pm 0.127$ implies a dark energy-dominated universe with a significant amount of matter, and potentially overestimates the amount of structure in the universe. Standard estimates of σ_8 are within the range $0.7 \leq \sigma_8 \leq 1.1$, therefore, $\sigma_8 = 0.411 \pm 0.117$ significantly underestimates the expected value. Two explanations are possible: Λ CDM is either a poor model or does an excellent job fitting poor data. The second explanation seems more valid, given the order of the error bars in the raw data. As with the OHD results, such errors call into question the validity of the best-fit values since several other fits could adequately fit the data within the margin of error, albeit less optimally. Even the 1σ confidence levels in the best-fit values hint at this conclusion: the errors in the best-fit values are of the order of the values themselves.

This explanation, however, does not hold in the case of $R_h = ct$. All the best-fit values, except perhaps $f\sigma_8(z=0)$, do not seem to make physical sense.

For one, the value of H_0 obtained in this analysis is much smaller than that obtained using SNe Ia and OHD measurements. Therefore, while the additional variables offer added flexibility, H_0 evaluated in this manner is inconsistent with the previously optimised values of the Hubble constant.

Note that k has a negative value and σ_8 dramatically underestimates the expected value. A negative wavenumber may have a place in number theory, but as far as cosmology is concerned, it is impossible to derive a physical meaning from a negative wavenumber.

The proposed model is rejected from a mathematical viewpoint because in $R_h = ct$, $dD/dz = 0$: that is,

$$\begin{aligned} D(z) \equiv - (1+z) \int_{\infty}^z \frac{1}{(1+z)^2} dz = 1 & \quad \Rightarrow \quad f(z) = - \frac{(1+z)}{D} \frac{dD}{dz} \\ & = 0 \\ & = f\sigma_8(z). \end{aligned}$$

$R_h = ct$ does not allow for the growth of perturbations and predicts no structure formation. It is possible to obtain plots and figures (see above), but doing so requires unphysical parameter values, which may lead to incorrect conclusions regarding the structure formation in the universe. To see this, consider the optimisation values:

Model	χ^2	AIC	BIC	ΔAIC	ΔBIC
Λ CDM	22.032	28.032	33.518	0	0
$R_h = ct$	29.728	37.728	45.042	9.696	11.525

Table 6.1: The results were obtained by letting $f\sigma_8(z=0)$ be a free parameter. The initial conditions vary between datasets, and no assumptions were made about them in this case.

Λ CDM, in this case, has 3 free parameters ($f\sigma_{8,0}$, Ω_m and σ_8) while $R_h = ct$ has 4 ($f\sigma_{8,0}$, H_0 , k and σ_8). Using these values, $\chi^2_{d.o.f} = 0.512$ and $\chi^2_{d.o.f} = 0.708$ for Λ CDM and $R_h = ct$, respectively. Both models fit the data poorly due to the overestimated errors (as expected), but the $R_h = ct$ value is closer to 1. Supposing nothing else is considered, the reduced χ^2 makes a compelling argument supporting $R_h = ct$. There are several other statistical tools to consider before making any conclusions about any specific model, i.e. the evidence and probabilities.

From the AIC values, there is a $P(R_h = ct) = 0.00778$ probability that $R_h = ct$ is the correct model. Similarly, from the BIC values, there is a $P(R_h = ct) = 0.00313$ probability that it is a more accurate description of the universe. Furthermore, even a simple inspection by eye would suggest that the measurements, as shown in Figure 6.1, can statistically distinguish between the functional form of $f\sigma_8(z)$ in Λ CDM and $R_h = ct$ and is borne out by Figure 6.2, which compares the residuals in the two cosmologies.

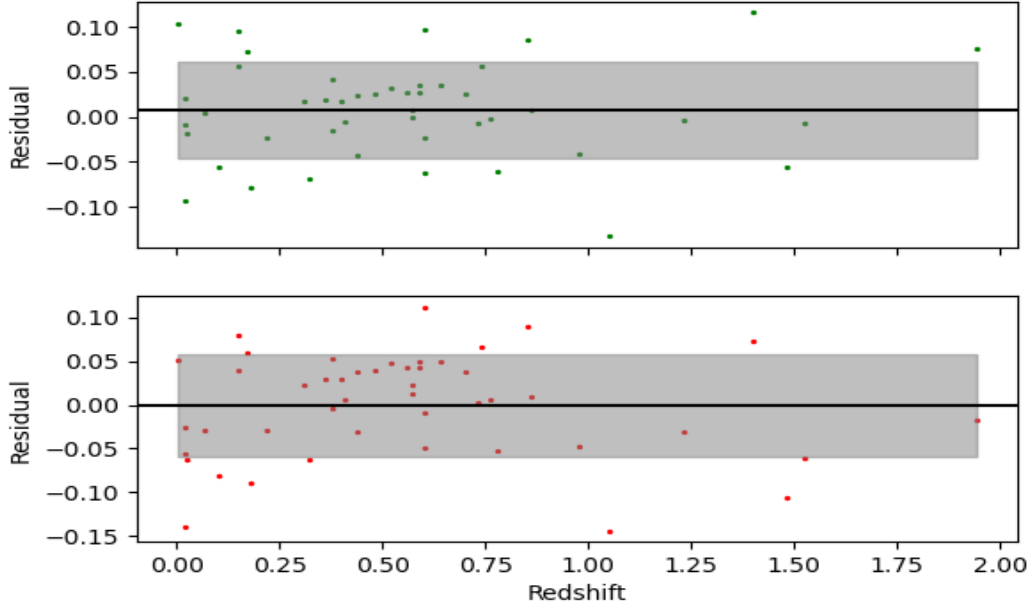


Figure 6.2: Side-by-side comparison of the residuals for the $f\sigma_8(z)$ data in the standard (Λ CDM) (top) and $R_h = ct$ models (bottom). The average residuals and standard deviations are $\mu_\Lambda = 0.00702$ and $\sigma_\Lambda = 0.0534$ for the standard model. The average residual and standard deviation are $\mu_{R_h} = -0.000499$ and $\sigma_{R_h} = 0.0584$.

The residuals indicate that the predictions made by Λ CDM are less accurate than those made by $R_h = ct$ while simultaneously being more precise.

After exhausting all statistical and mathematical arguments against the $R_h = ct$ model, it is concluded that the growth rate data strongly favours the Λ CDM model. While the $R_h = ct$ universe offers an alternative perspective on structure formation, it requires a novel and unphysical evolution of matter, which makes it less viable as an alternative to the established Λ CDM model [97]. It offers an accurate description of the local universe but fails to accurately explain phenomena like structure formation, which started in the early universe.

Chapter 7

Concluding Remarks and Future Outlook

This study presents an alternative to the standard model of cosmology. Basic background information about the mathematical foundation of general relativity, its historical evolution as a tool for studying standard cosmology, and its shortcomings within the framework of the standard model was presented in Chapter 3. Based on these shortcomings, the need to consider other models was motivated.

Chapter 4 examined an alternative, the $R_h = ct$ model, for a flat cosmology only. New observations, ongoing research and potential modifications could result in refinements or entirely new frameworks for understanding the universe. For instance, in Section 4.3, scalar fields evolve with time to ensure that they present a possible dynamic explanation for dark energy.

Chapter 5 considered SNe Ia and OHD constraints. With regard to SNe Ia data, Λ CDM currently provides a good fit. It may very well be that it does so because it has several free parameters. That is not to say that $R_h = ct$ necessarily provides a poor fit. It is just that that value of the Hubble constant in this model is more consistent with the value predicted by Λ CDM for CMB optimisation than with the locally measured Cepheid value. The value obtained using OHD was significantly less than the previously mentioned CMB value.

Chapter 6 provided an overview of the $1 + 3$ covariant spacetime splitting formalism as it applies to Λ CDM and $R_h = ct$. Various approaches to the theory of cosmological perturbations were studied; the linear evolution of scalar cosmological perturbations was examined, and the models were constrained by comparing the theoretical growth rates predicted by the models with $f\sigma_8(z)$ data once the equations were obtained. The analysis in this work used three very different measurements, and the fact that the optimised results are not self-consistent provides a stronger indication of the suitability and feasibility of the $R_h = ct$ cosmological model. With several observational probes on the horizon,

more accurate measurements of the main cosmological parameters should allow for tighter constraints of different cosmological models.

Overall, this study has shown that there are many other implications of the $R_h = ct$ constraint, such as those pertaining to Big Bang Nucleosynthesis (BBN) and baryogenesis, both of which would have been affected in terms of when they would have occurred given the physical conditions prevalent at the time. Although it remains beyond the scope of this work to thoroughly examine these processes, a thorough account is essential before a comprehensive assessment of the viability of the proposed model is possible.

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Appendix A

Appendix

Redshift z	$f\sigma_8$	Survey	Reference
0.001	0.505 ± 0.085	2MTF	[98]
0.02	0.314 ± 0.048	2MASS	[92]
0.02	0.398 ± 0.065	SnIa+IRAS	[99]
0.02	0.428 ± 0.0465	6dFGS+SnIa	[100]
0.025	0.39 ± 0.11	6dFGS	[101]
0.067	0.423 ± 0.055	6dFGS	[101]
0.10	0.37 ± 0.13	SDSS-veloc	[102]
0.15	0.490 ± 0.145	SDSS-MGS	[51]
0.15	0.53 ± 0.16	SDSS-IVS	[103]
0.17	0.510 ± 0.060	2dFGRS	[91]
0.18	0.360 ± 0.090	GAMA	[104]
0.22	0.42 ± 0.07	WiggleZ	[105]
0.31	0.469 ± 0.098	BOSS DR12	[106]
0.32	0.384 ± 0.095	BOSS-LOWZ	[107]
0.36	0.474 ± 0.097	BOSS DR12	[106]
0.38	0.440 ± 0.060	GAMA	[104]
0.38	0.497 ± 0.045	SDSS-IV	[103]
0.40	0.473 ± 0.086	BOSS DR12	[106]
0.41	0.45 ± 0.04	WiggleZ	[105]
0.44	0.413 ± 0.080	WiggleZ	[105]
0.44	0.481 ± 0.076	BOSS DR12	[106]
0.48	0.482 ± 0.062	BOSS DR12	[106]
0.52	0.488 ± 0.065	BOSS DR12	[106]
0.56	0.482 ± 0.067	BOSS DR12	[106]
0.57	0.453 ± 0.022	CMASS-BOSS	[108]

Redshift z	$f\sigma_8$	Survey	Reference
0.57	0.462 ± 0.041	BOSS-CMASS	[99]
0.59	0.481 ± 0.066	BOSS DR12	[106]
0.59	0.488 ± 0.060	SDSS-CMASS	[109]
0.60	0.390 ± 0.063	WiggleZ	[105]
0.60	0.43 ± 0.04	WiggleZ	[105]
0.60	0.550 ± 0.120	Vipers PDR-2	[110, 111]
0.64	0.486 ± 0.070	BOSS DR12	[106]
0.70	0.473 ± 0.041	SDSS-IV	[103]
0.73	0.437 ± 0.072	WiggleZ	[105]
0.74	0.50 ± 0.11	SDSS-IV	[103]
0.76	0.440 ± 0.040	VIPERS v7	[112]
0.78	0.38 ± 0.04	WiggleZ	[105]
0.85	0.52 ± 0.10	SDSS-IV	[103]
0.86	0.440 ± 0.110	Vipers PDR-2s	[110, 111]
0.978	0.379 ± 0.176	SDSS-IV	[103]
1.05	0.280 ± 0.080	VIPERS v7	[112]
1.23	0.385 ± 0.099	SDSS-IV	[113]
1.40	0.482 ± 0.116	FastSound	[114]
1.48	0.30 ± 0.13	SDSS-IV	[103]
1.526	0.342 ± 0.070	SDSS-IV	[113]
1.944	0.364 ± 0.106	SDSS-IV	[103]

Table A.1: Data compilation of 46 $f\sigma_8(z)$ measurements from various redshift surveys. The clumping of matter is being studied in the critical redshift range $z < 2$ where the differences in the function $f\sigma_8(z)$ between Λ CDM and $R_h = ct$ are most pronounced.

Redshift z	H_{obs}	σ_H	Reference	Redshift z	H_{obs}	σ_H	Reference
0.07	69	19.6	[115]	0.51	90.4	1.9	[70]
0.12	68.6	26.2	[115]	0.52	94.35	2.64	[119]
0.17	83	8	[116]	0.56	93.34	2.3	[121]
0.1791	75	4	[117]	0.57	87.6	7.8	[125]
0.1993	75	5	[117]	0.57	96.8	3.4	[126]
0.2	72.9	29.6	[118]	0.59	98.48	3.18	[121]
0.24	79.69	2.99	[119]	0.593	104	13	[117]
0.27	77	14	[116]	0.6	87.9	6.1	[127]
0.28	88.8	36.6	[70]	0.61	97.3	2.1	[123]
0.3	81.7	6.22	[120]	0.64	97.3	2.1	[121]
0.31	78.18	4.74	[121]	0.6797	92	8	[70]
0.34	83.8	3.66	[119]	0.7182	105	12	[117]
0.35	82.7	9.1	[122]	0.73	97.3	7	[122]
0.3519	83	14	[117]	0.8754	124	17	[117]
0.36	79.94	3.38	[121]	0.88	90	40	[115]
0.38	81.5	1.9	[123]	0.9	69	12	[116]
0.3802	83	13.5	[70]	0.9	117	23	[116]
0.4	95	17	[116]	1.037	154	20	[116]
0.4	82.04	2.03	[121]	1.3	168	17	[116]
0.4004	77	10.2	[70]	1.363	160	33.6	[128]
0.4247	87.1	11.2	[70]	1.43	177	17	[116]
0.43	86.45	3.97	[124]	1.53	140	14	[116]
0.44	82.6	7.8	[121]	1.75	202	40	[116]
0.44	84.81	1.83	[121]	1.965	186.5	54	[128]
0.4497	92.8	12.9	[70]	2.3	224	8.6	[129]
0.47	89	34	[124]	2.33	224	8	[130]
0.4783	80	90	[70]	2.34	222	8.5	[131]
0.48	97	62	[115]	2.36	226	9.3	[132]
0.48	87.79	2.03	[121]				

Table A.2: The behaviour of the Hubble parameter $H(z)$ with redshift