Goodness-of-fit tests based on new characterizations of the exponential distribution.

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Summary

The exponential density is probably one of the most widely used distributions in practice. Due to its importance, many goodness-of-fit tests for exponentiality have been proposed in the literature.

The objectives of this research are as follow:

- to study the importance of the exponential distribution in practical problems,
- to investigate alternative classes of distributions to the exponential distribution,
- to present an overview of existing characterizations of the exponential distribution,
- to evaluate existing goodness-of-fit tests for exponentiality,
- to develop new goodness-of-fit tests for exponentiality, and
- to compare the proposed goodness-of-fit tests to existing tests by means of relative efficiencies and simulation studies of the power of the tests.

To achieve these objectives, we begin with a brief discussion of the exponential distribution and other parametric families of life distributions, followed by a summary of six well-known nonparametric classes of alternative distributions.

A comprehensive literature study of existing characterizations of the exponential distribution and existing goodness-of fit tests for exponentiality are presented.

We then propose and prove two new characterizations of the exponential distribution in the class of NBUE life distributions based on properties of order statistics. These characterizations are used to develop a new class of goodness-of-fit tests for exponentiality. The tests are shown to be consistent and the limiting distributions under the null and alternative hypotheses are derived.

We show that the new class of test statistics includes two statistics which are equivalent to the well-known Gini test statistic (Gail and Gastwirth 1978a) and the coefficient of variation test statistic (Borges, Proschan and Rodrigues 1984).

The newly proposed tests are compared to existing goodness-of-fit tests by means of Pitman and approximate Bahadur relative efficiencies. Monte Carlo studies are conducted to compare the various tests with regard to power for small and moderate sample sizes against a wide range of alternative distributions.

We recommend three members of the class of test statistics as being very effective testing procedures for exponentiality.

In conclusion, practical examples based on real-life data are presented.

Opsomming

Titel: Passingstoetse gebaseer op nuwe karakteriserings van die eksponensiële verdeling.

Die eksponensiële verdeling is sekerlik een van die mees algemeen gebruikte verdelings in die praktyk. As gevolg van sy belangrikheid is daar baie passingstoetse vir eksponensialiteit in die literatuur voorgestel.

Die doelwitte van hierdie navorsing is as volg:

- om die belangrikheid van die eksponensiële verdeling in praktiese probleme te ondersoek,
- om alternatiewe klasse van verdelings tot die eksponensiële verdeling te bestudeer,
- om 'n oorsig te bied van bestaande karakteriserings van die eksponensiële verdeling,
- om bestaande passingstoetse vir eksponensialiteit te evalueer,
- om nuwe passingstoetse vir eksponensialiteit te ontwikkel, en
- om die voorgestelde passingstoetse met bestaande toetse te vergelyk op grond van relatiewe doeltreffendhede en simulasiestudies ten opsigte van die onderskeidingsvermoë van die toetse.

Om hierdie doelwitte te bereik, begin ons met 'n kort bespreking van die eksponensiële verdeling en ander parametriese verdelings, gevolg deur 'n opsomming van ses bekende nie-parametriese klasse van alternatiewe verdelings.

'n Uitgebreide literatuurstudie van bestaande karakteriserings van die eksponensiële verdeling en bestaande passingstoetse vir eksponensialiteit word gegee.

Twee nuwe karakteriserings van die eksponensiële verdeling in die klas van NBUE-verdelings, gebaseer op eienskappe van orde-statistieke, word bewys. Hierdie karakteriserings word gebruik om 'n klas van nuwe passingstoetse vir eksponensialiteit voor te stel. Dit word bewys dat hierdie toetse konsekwent is en die limietverdelings onder die nul- en alternatiewe hipoteses word afgelei.

Daar word aangetoon dat die nuwe klas van passingstoetse twee toetsstatistieke insluit wat ekwivalent is aan die bekende Gini toetsstatistiek (Gail and Gastwirth 1978a) en die koëffisiënt van variasie toetsstatistiek (Borges et al. 1984).

Die nuwe toetse word vergelyk met bestaande passingstoetse deur middel van Pitman en benaderde Bahadur doeltreffendhede. Monte-Carlo studies is uitgevoer om die verskillende toetse te vergelyk ten opsigte van onderskeidingsvermoë vir klein en matige steekproefgroottes teen 'n wye verskeidenheid van alternatiewe verdelings.

Ons beveel drie lede van die klas van toetsstatistieke aan as baie effektiewe toetsingsprosedures vir eksponensialiteit.

Ten slotte word praktiese voorbeelde gebaseer op werklike data bespreek.

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Contents

Chapter 1

Introduction: Reliability theory and the exponential distribution

1.1 Introduction

The exponential density is probably one of the most widely used distributions in practice. Important application areas of the exponential distribution include survival analysis and reliability theory.

The demand for accuracy and reliability of electronic equipment is continuously increasing, providing a reason for the development of the theory of reliability. It involves the problem of studying the failure times of items and components and calculating the reliability of instruments in an effort to increase efficiency and reduce operational costs.

Due to the importance of the exponential distribution in statistical analysis, many goodness-of-fit tests for exponentiality have been proposed in the literature. Spurrier (1984), Ascher (1990) and more recently Henze and Meintanis (2005) discuss and compare a wide selection of classical and recent tests for exponentiality.

The objectives of this research are to study the importance of the exponential distribution in practical problems, to investigate alternative classes of distributions to the exponential distribution, to evaluate existing goodness-of-fit tests for exponentiality, to develop new goodness-of-fit tests for exponentiality theoretically and to compare the proposed goodness-of-fit tests to existing tests by means of simulation studies on the power and relative efficiency of the tests.

In Chapter 1 a short introduction to reliability theory is given, followed by a discussion on general characteristics of life distributions. The chapter is concluded with definitions and properties of the exponential distribution as well as other parametric families of life distributions (e.g. the gamma, Weibull and Makeham distributions), which can be used as alternatives when testing for exponentiality.

Since it is usually difficult to determine which specific parametric family of densities is appropriate to use as alternative, nonparametric classes of life distributions are more often considered as alternatives. Well-known classes are the increasing failure rate (IFR), the increasing failure rate average (IFRA), the new better than used (NBU), the decreasing mean residual life (DMRL) and the new better than used in expectation (NBUE) classes, together with their respective dual classes (Hollander and Proschan (1975)). These classes and their properties are discussed in Chapter 2.

Chapters 3 and 4 present an overview of existing characterizations of the exponential distribution and goodness-of-fit tests for exponentiality.

We propose new characterizations of the exponential distribution in the class of NBUE life distributions based on properties of order statistics in Chapter 5.These characterizations are used to develop new goodness-of-fit tests for exponentiality, of which properties will be discussed. Simulation results are also presented.

1.2 Reliability theory, life distributions and the concept of aging

Since the 1940's and 1950's, the theory of reliability was developed due to a steadily increasing demand for accuracy and reliability of electronic equipment. It began with the problem of calculating the reliability of instruments and the development of measures to increase efficiency and reduce operational costs in connection with the unreliability of electro-vacuum instruments used in aviation (Azlarov and Volodin 1986, p. 1).

The term reliability is used in general to express a certain degree of assurance that a component or a system will operate successfully in a specified environment during a certain time period. In short, reliability deals with the study of the proper functioning of components and systems (Lawless 1982, p. 20).

However, if a component or system fails, this does not necessarily imply that it is unreliable - every piece of mechanical or electronic equipment fails eventually. The question to address is how frequently failures occur in specified time periods.

A key problem in reliability theory is to determine the reliability of a complex system from the knowledge of the reliability of its components. When aiming at the improvement of system reliability, the relative importance of each component to the reliability of the system needs to be determined. The reason is twofold: it allows the analyst to determine which components merit the most additional research and development to improve overall system reliability at minimum cost or effort, and it may suggest the most efficient way to diagnose system failure by generating a repair "check-list".

Consider a system consisting of n components. Each component (and the system itself) operates until it fails at some time. Assume that no repair is performed. To indicate the state of the i-th component at time t, define

$$y_i(t) = \begin{cases} 1, & \text{if component } i \text{ is functioning at time } t \\ 0, & \text{if component } i \text{ has failed at time } t \end{cases}$$

for $i = 1, \ldots, n$. Similarly, the binary variable ϖ indicates the state of the system at time t:

$$\varpi(t) = \begin{cases} 1, & \text{if the system is functioning at time } t \\ 0, & \text{if the system has failed at time } t \end{cases}$$

The system can have different structures, for example:

1. A series structure functions if and only if each component functions.
2. A parallel structure functions if and only if at least one component functions.
3. A k-out-of-n structure functions if and only if at least k of the n components function, for $1 \leq k \leq n$.

Clearly, the state of the system is determined completely by the state of its components, so

$$\varpi(t) = \varpi(\mathbf{y}(t)), \text{ where } \mathbf{y}(t) = (y_1(t), \ldots, y_n(t)).$$

The function $\varpi(\mathbf{y}(t))$ is called the structure function of the system. Barlow and Campo (1975) defined a **coherent system** as a system of components for which its structure function $\varpi(t)$ is increasing and each component is relevant (i.e. for each component i, $\varpi(t)$ is not constant in $y_i(t)$).

Now, suppose the state $Y_i(t)$ of the i-th component at time t is random with

$$P[Y_i(t) = 1] = p_i(t) = E[Y_i(t)].$$

The probability that component i functions at time t, $p_i(t)$, is referred to as the *reliability* of component i. Similarly, the reliability of the system at time t is given by

$$P[\varpi(\mathbf{Y}(t)) = 1] = \mathrm{h}(t) = E[\varpi(\mathbf{Y}(t))].$$

When discussing statistical problems in reliability, two main types of situations are considered (Lawless 1982, p. 20):

- those where the emphasis is on the lifetime (or failure-free operating time) of a system or component, and
- those where the emphasis is on broader aspects of a system's performance, the possibility of repeated failure and repair, or of varying levels of performance being allowed for.

In the second case, statistical methods related to stochastic processes such as renewal and Markov processes are important. The first case involves statistical methods related to the modeling and estimation of lifetime distributions, which leads us to the study of *life distributions* and their properties.

1.2.1 General characteristics of life distributions

Consider a nonnegative continuous random variable X, with distribution F, representing a lifetime of some component or system. Suppose a random sample $X_1, \ldots, X_n$ of lifetimes from F is observed and let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denote the order statistics.

The distribution function of the lifetime X is the probability that the lifetime does not exceed t, i.e.,

$$F(t) := P(X \leq t),\ 0 < t < \infty.$$

A life distribution F is a distribution satisfying $F(t) = 0$ for $t < 0$.

The survival function (or reliability function) of a component/system having distribution F is:

$$\bar{F}(t) := 1 - F(t) = P(X > t),\ 0 < t < \infty.$$

This is the probability that the lifetime of the component/system will exceed t.

The conditional probability of failure during the next interval of duration x of a unit of age t is:

$$F(x|t) := P(X < t + x|X > t) = \frac{F(t+x) - F(t)}{\bar{F}(t)}. \tag{1.1}$$

Similarly, the corresponding conditional survival probability of a unit of age t is

$$\bar{F}(x|t) := 1 - F(x|t) = \frac{\bar{F}(t+x)}{\bar{F}(t)}. \tag{1.2}$$

From (1.1) the failure rate (hazard rate) function at time t is obtained:

$$r(t) := \lim_{x \to 0} \frac{1}{x} \frac{F(t+x) - F(t)}{\bar{F}(t)},$$

i.e., if F is absolutely continuous,

$$r(t) = \frac{F'(t)}{\bar{F}(t)} = \frac{f(t)}{\bar{F}(t)}. \tag{1.3}$$

The cumulative failure rate,

$$R(t) := \int_0^t r(u)du, \tag{1.4}$$

is also referred to as the hazard function. The survival function can be expressed in terms of the hazard function as follows:

$$\bar{F}(t) = e^{-R(t)}, \tag{1.5}$$

which gives

$$R(t) = -\log \bar{F}(t). \tag{1.6}$$

The average failure rate is defined as

$$q(t) := \frac{1}{t}\int_0^t r(u)du = \frac{1}{t}R(t) = -\frac{1}{t}\log \bar{F}(t). \tag{1.7}$$

The mean time to failure (expected lifetime) is the average length of time until failure,

$$\mu := E[X] = \int_0^\infty u dF(u) = \int_0^\infty \bar{F}(u)du,$$

while the mean residual life function at time t is defined as

$$\varepsilon_F(t) := \frac{\int_t^\infty \bar{F}(u)du}{\bar{F}(t)}. \tag{1.8}$$

Note that $\mu = \varepsilon_F(0)$. The i-th spacing, D_i', between the order statistics is defined as

$$D_i' := X_{i:n} - X_{(i-1):n},\ i = 1, 2, \ldots, n, \tag{1.9}$$

where $X_{0:n} \equiv 0$. The normalized spacings are defined as

$$D_i := (n-i+1)\left(X_{i:n} - X_{(i-1):n}\right),\ i = 1, 2, \ldots, n. \tag{1.10}$$

1.2.2 Total time on test

An important concept in reliability and life testing is the **total time on test (TTT)** concept. It features in many goodness-of-fit tests for exponentiality, as discussed in Chapter 4.

The TTT-transform of a life distribution F is defined as

$$H_F^{-1}(t) := \int_0^{F^{-1}(t)} \bar{F}(u)du,\ 0 \le t \le 1, \tag{1.11}$$

where $F^{-1}(t) = \inf\{x : F(x) \geq t\}$.

There is a one-to-one correspondence between life distributions and their TTT-transforms. Further, H_F^{-1} is continuous if and only if F is strictly increasing for $0 \leq x \leq F^{-1}(1)$ and H_F^{-1} is strictly increasing if and only if F is continuous (Bergman 1979).

Since the mean of F is given by

$$\mu = H_F^{-1}(1) = \int_0^\infty \bar{F}(u)du,$$

the transform

$$\varphi_F(t) := \frac{H_F^{-1}(t)}{H_F^{-1}(1)} = \frac{H_F^{-1}(t)}{\mu}, \ 0 \leq t \leq 1, \tag{1.12}$$

is scale invariant and is called the scaled TTT-transform.

A natural estimator of the scaled TTT-transform defined in (1.12) is the empirical scaled TTT-transform defined as

$$\varphi_n(t) := \frac{H_n^{-1}(t)}{H_n^{-1}(1)} = \frac{H_n^{-1}(t)}{\bar{X}}, \ 0 \leq t \leq 1, \tag{1.13}$$

where $H_n^{-1}(t) = \int_0^{F_n^{-1}(t)} \bar{F}_n(u)du, \ 0 \leq t \leq 1$.

Suppose n independent components with life distribution F are put on test at the same time. Let $X_{1:n} < \ldots < X_{n:n}$ denote their ordered failure times. At time $X_{i:n}$, the total time that the n items have spent on test is

$$\begin{aligned} T_i &= nX_{1:n} + (n-1)(X_{2:n} - X_{1:n}) + \ldots + (n-i+1)(X_{i:n} - X_{(i-1):n}) \\ &= \sum_{j=1}^{i}(n-j+1)(X_{j:n} - X_{(j-1):n}) \\ &= \sum_{j=1}^{i} D_j, \end{aligned} \tag{1.14}$$

where D_j are the normalized spacings in (1.10). $T_0 = 0$ and $T_n = \sum_{j=1}^n D_j = \sum_{j=1}^n X_{j:n}$. Thus, T_i is called the total time on test at $X_{i:n}$.

Define

$$W_i := \frac{\sum_{j=1}^i D_j}{\sum_{j=1}^n D_j} = \frac{T_i}{T_n}, \quad i = 0, 1, \ldots, n, \tag{1.15}$$

where T_i is the total time on test at $X_{i:n}$ defined in (1.14). Calculations show that

$$\varphi_n\left(\frac{i}{n}\right) = W_i, \quad i = 0, 1, \ldots, n. \tag{1.16}$$

(Basu and Ebrahimi 1985).

1.3 The exponential distribution: Definitions and properties

A random variable X has an exponential distribution with rate θ, where θ is a positive scale parameter, if X has distribution function

$$F(t) = 1 - e^{-\theta t}, \ t \geq 0.$$

The probability density function of X is given by

$$f(t) = \theta e^{-\theta t}, \ t \geq 0,$$

and the survival function is

$$\bar{F}(t) = e^{-\theta t}, \ t \geq 0.$$

The moment generating function of X is $M(t) = \theta/(\theta - t)$, which gives

$$\mu = E[X] = \frac{1}{\theta} \quad \text{and} \quad \sigma^2 = Var[X] = \frac{1}{\theta^2}.$$

It is easy to see that the coefficient of variation of the exponential distribution is 1. In fact, this is a characterization of the exponential distribution which will be discussed in Chapter 3, page 42.

In general, for $r > -1$, the r-th moment of the exponential distribution is given by

$$\mu_r := E[X^r] = \int_0^\infty x^r \theta e^{-\theta x} dx = \frac{1}{\theta^r} \int_0^\infty y^r e^{-y} dy,$$

so that

$$\mu_r = \frac{1}{\theta^r} \Gamma(r+1).$$

In particular, for integer values $n = 1, 2, \ldots$

$$\mu_n = \frac{n!}{\theta^n}.$$

The most well-known property of the exponential distribution is the lack-of-memory property:

$$\bar{F}(t+x) = \bar{F}(t)\bar{F}(x), \ \forall \ t, x \geq 0. \tag{1.17}$$

If X denotes the lifetime of a certain component, then this property means that the probability that the component survives for at least $t + x$ time units, given that it has already survived for t time units, is the same as the probability that the component survives for at least x time units, i.e.,

$$P(X > t + x | X > t) = P(X > x) \text{ for all } t, x \geq 0. \tag{1.18}$$

It is easy to show that if X is exponential, then it has this property. It can also be shown that the solution of (1.17) is of the form

$$\bar{F}(t) = e^{-\theta t}, \ \theta > 0, \ t \geq 0,$$

which is the exponential survival function. Thus, property (1.17) characterizes the exponential distribution.

In reliability theory, the lifetimes of components are often assumed to be exponentially distributed and therefore exhibit the lack-of-memory property. This may hold if the components under consideration do not have any moving parts, e.g. fuses and air monitors. In these cases, failure is caused by random shocks and not by wear, and the exponential assumption is equivalent to the assumption that these shocks occur according to a Poisson process.

This has important practical and theoretical consequences. If it is assumed that lifetimes are exponentially distributed, then it follows that:

1. Since a used component is as good as new, there is no advantage in following a policy of planned replacement of used components known to be still functioning;

2. In statistical estimation of mean life, percentiles, reliability, etc., data may be collected consisting only of the number of hours of observed life and of the number of observed failures; the ages of components under observation are irrelevant.

The lack-of-memory property of the exponential distribution is further illustrated by the fact that the failure rate function is constant, which is easy to prove:

$$r(t) = \frac{f(t)}{\bar{F}(t)} = \frac{\theta e^{-\theta t}}{e^{-\theta t}} = \theta. \tag{1.19}$$

Conversely, a distribution with constant failure rate has the exponential survival function

$$\bar{F}(t) = e^{-\theta t}, \ \theta > 0, \ t \geq 0.$$

Therefore, the failure rate $r(t)$ of a distribution F is constant if and only if F is exponential.

Azlarov and Volodin (1986) stated that if F is a distribution function with a monotone failure rate, the relation

$$f(0) \equiv r(0) = \theta = 1/\mu$$

holds if and only if F is exponential with mean μ.

The scaled TTT-transform defined in (1.12) of the exponential distribution is easily determined as $\varphi_F(t) = t,\ 0 \leq t \leq 1$.

Another property of the exponential distribution is that if $X_1, X_2, \ldots, X_n$ are independent identically distributed (i.i.d.) exponential random variables with mean $1/\theta$, then the sum of these random variables, $X_1 + X_2 + \cdots + X_n$, has a gamma distribution with parameters n and θ.

It is also easy to calculate the probability that one exponential random variable is smaller than another one. That is, suppose X_1 and X_2 are independent exponential random variables with means $1/\theta_1$ and $1/\theta_2$ respectively, then $P[X_1 < X_2] = \frac{\theta_1}{\theta_1+\theta_2}$.

Further, if $X_1, X_2, \ldots, X_n$ are independent exponential random variables and X_j has mean $1/\theta_j,\ j = 1, \ldots, n$, then the minimum of the X_i's are exponentially distributed with mean $1/\sum_j \theta_j$.

Barlow and Campo (1975) proved the following properties of the spacings $D_1', D_2', \ldots, D_n'$ defined in (1.9) from the exponential distribution with mean $1/\theta$:

1. $P[D_k' \leq t] = 1 - e^{-(n-k+1)\theta t},\ k = 1, \ldots, n.$
2. $D_1', D_2', \ldots, D_n'$ are mutually independent.
3. $E[D_k'] = \frac{1}{(n-k+1)\theta}$, and $Var[D_k'] = \frac{1}{[(n-k+1)\theta]^2},\ k = 1, \ldots, n.$
4. The normalized spacings, D_i, defined in (1.10), from the exponential distribution are independently distributed with common exponential distribution $F(t) = 1 - e^{-\theta t}$.

Since $X_{k:n} = D_1' + D_2' + \cdots + D_k'$, the expression for $E[D_k']$ can be used to derive that the exponential order statistics have expected value

$$E[X_{k:n}] = \frac{1}{\theta}\left(\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{n-k+1}\right) \tag{1.20}$$

and variance

$$Var[X_{k:n}] = \frac{1}{\theta^2}\left(\frac{1}{n^2} + \frac{1}{(n-1)^2} + \cdots + \frac{1}{(n-k+1)^2}\right)$$

for $k = 1, \ldots, n$.

1.4 Other parametric families of life distributions

In this section some other parametric families of life distributions that appear in reliability applications are discussed. These distributions are often used as alternatives when testing for exponentiality. For detailed descriptions of these distributions see e.g. Cox and Lewis (1966), Barlow and Campo (1975) and Zacks (1992).

In Figure 1 the distribution function, density function and failure rate function for the most important of these distributions are illustrated, for different parameter values.

1.4.1 Erlang, Rayleigh, Chi-square and gamma distributions

Consider a system which consists of k similar units, connected sequentially. When unit 1 fails, unit 2 starts operating automatically, and so on, until all k units fail. Suppose further that each one of the units has an exponential life distribution with mean $1/\theta$ and that the units operate independently. Then the life length of the device, X_D, is the sum of the k life lengths $X_1, \ldots, X_k$ of its component units. That is, $X_D = X_1 + \cdots + X_k$.

The distribution of X_D is called the **Erlang** distribution, which is a special case of the gamma distribution. The probability density function of the Erlang distribution is

$$f_E(t) = \frac{\theta^k t^{k-1}}{(k-1)!} e^{-\theta t},\ 0 \leq t < \infty.$$

θ is a scale parameter, $0 < \theta < \infty$ and k is a shape parameter, $k = 1, 2, \ldots$. Note that the exponential distribution is a special case of the Erlang for $k = 1$.

The mean of an Erlang (k, θ) life distribution is $\mu_E = k/\theta$ and the variance is $\sigma_E^2 = k/(\theta^2)$.

The failure rate function of an Erlang (k, θ) life distribution is

$$r_E(t) = \frac{p(k-1, t\theta)}{\frac{1}{\theta} F(k-1, t\theta)},$$

where $p(j, \lambda) = e^{-\lambda} \frac{\lambda^j}{j!},\ j = 1, 2, \ldots$ and $F(j, \lambda) = \sum_{i=0}^{j} p(i, \lambda)$ are the probability mass function and the cumulative distribution functions of a Poisson random variable.

By considering any positive real number ν as shape parameter, the Erlang family of life distributions is extended to the **gamma** family with probability density function

$$f_G(t) = \frac{\theta^\nu}{\Gamma(\nu)} t^{\nu-1} e^{-\theta t},\ t > 0, \tag{1.21}$$

and distribution function

$$F_G(t) = \frac{1}{\Gamma(\nu)} \int_0^t x^{\nu-1} e^{-\theta x} dx,\ t > 0,$$

where the gamma function $\Gamma(\alpha), \alpha > 0$ is defined as $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$.

For $\nu = 1$, this is the exponential distribution. For ν greater than one, $f_G(t)$ is zero at the origin and has a single maximum at $t = \theta(\nu - 1)/(\nu^2)$, the failure rate function increasing monotonically from zero to θ as t goes from zero to infinity. When $0 < \nu < 1$, $f_G(t)$ is infinite at the origin and the failure rate decreases monotonically from infinity to θ (Cox and Lewis 1966, p. 136).

An important characterization of the gamma distribution is the following: X_1 and X_2 are independent gamma variables, if and only if the random variables $X_1/(X_1 + X_2)$ and $X_1 + X_2$ are independent.

A variation of the Erlang distribution is obtained when the shape parameter k is allowed to assume values which are multiples of $\frac{1}{2}$, i.e. if $k = m/2,\ m = 1, 2, \ldots$ and the scale parameter θ is fixed at $\theta = 1/2$.

The distributions obtained in this way are called **Chi-square distributions**, denoted by $\chi^2(m)$, where m is a shape parameter (degrees of freedom). The probability density function of $\chi^2(m)$ is

$$f_{\chi^2}(t) = \frac{1}{2^{m/2}\Gamma(m/2)} t^{m/2-1}e^{-t/2},\ t \geq 0.$$

The mean of $\chi^2(m)$ is $\mu_{\chi^2} = m$ and the variance is $\sigma^2_{\chi^2} = 2m$.

If X has the Erlang distribution with parameters $\theta = 1/2$ and $k = 1$, then the distribution of $Y = (X)^{1/2}$ is called the **Rayleigh** distribution. The probability density function of the Rayleigh distribution is given by $f_R(t) = te^{-t^2/2},\ t > 0$.

1.4.2 Weibull distributions

The Weibull family of life distributions has been found to provide good models in many empirical studies. Barlow and Campo (1975) mentioned the following areas where it has been used: fatigue failure, vacuum tube failure and ball bearing failure. The probability density function of a **Weibull** (ν, θ) distribution is

$$f_W(t) = \nu\theta^\nu t^{\nu-1}\exp(-(\theta t)^\nu),\ t \geq 0, \tag{1.22}$$

where ν is a shape parameter and θ is a scale parameter.

Note that if $\nu = 1$, the Weibull distribution reduces to the exponential distribution. For ν greater than one, $f_W(t)$ is zero at the origin, while for $0 < \nu < 1$, $f_W(t)$ is infinite at the origin.

The distribution function is

$$F_W(t) = 1 - \exp(-(\theta t)^\nu), \ t \geq 0. \tag{1.23}$$

The mean and variance of a Weibull (ν, θ) distribution is

$$\mu_W = \frac{1}{\theta}\,\Gamma\left(1+\frac{1}{\nu}\right) \quad \text{and} \quad \sigma_W^2 = \frac{1}{\theta^2}\left[\Gamma\left(1+\frac{2}{\nu}\right) - \Gamma^2\left(1+\frac{1}{\nu}\right)\right].$$

An important property of the Weibull distribution is that the minimum of n i.i.d. Weibull (ν, θ) random variables has the Weibull $(\nu, \frac{1}{\theta n^{1/\nu}})$ distribution. That is, if n independent devices start to operate at the same time, and if the life distributions of these devices are the same Weibull (ν, θ) , then the time until the first failure has the Weibull $(\nu, \frac{1}{\theta n^{1/\nu}})$ distribution (Zacks 1992, pp. 25-26). This property makes the Weibull family an attractive one for modelling the reliability of systems of similar components connected in series, or for mechanical systems where the "weakest link" model is appropriate.

The failure rate of the Weibull distribution is

$$r_W(t) = \nu\theta^\nu t^{\nu-1}, \ t > 0. \tag{1.24}$$

Thus, if $\nu = 1$, then $r_W(t)$ is constant, which corresponds to the exponential distribution. For $\nu \geq 1$ the Weibull distribution has increasing failure rate, and for $0 < \nu \leq 1$ the Weibull distribution has decreasing failure rate.

The following relationship exists between random variables with Weibull and exponential distributions: If X is Weibull (ν, θ), then X^ν is exponentially distributed with parameter θ^ν (Cox and Lewis 1966, p. 139).

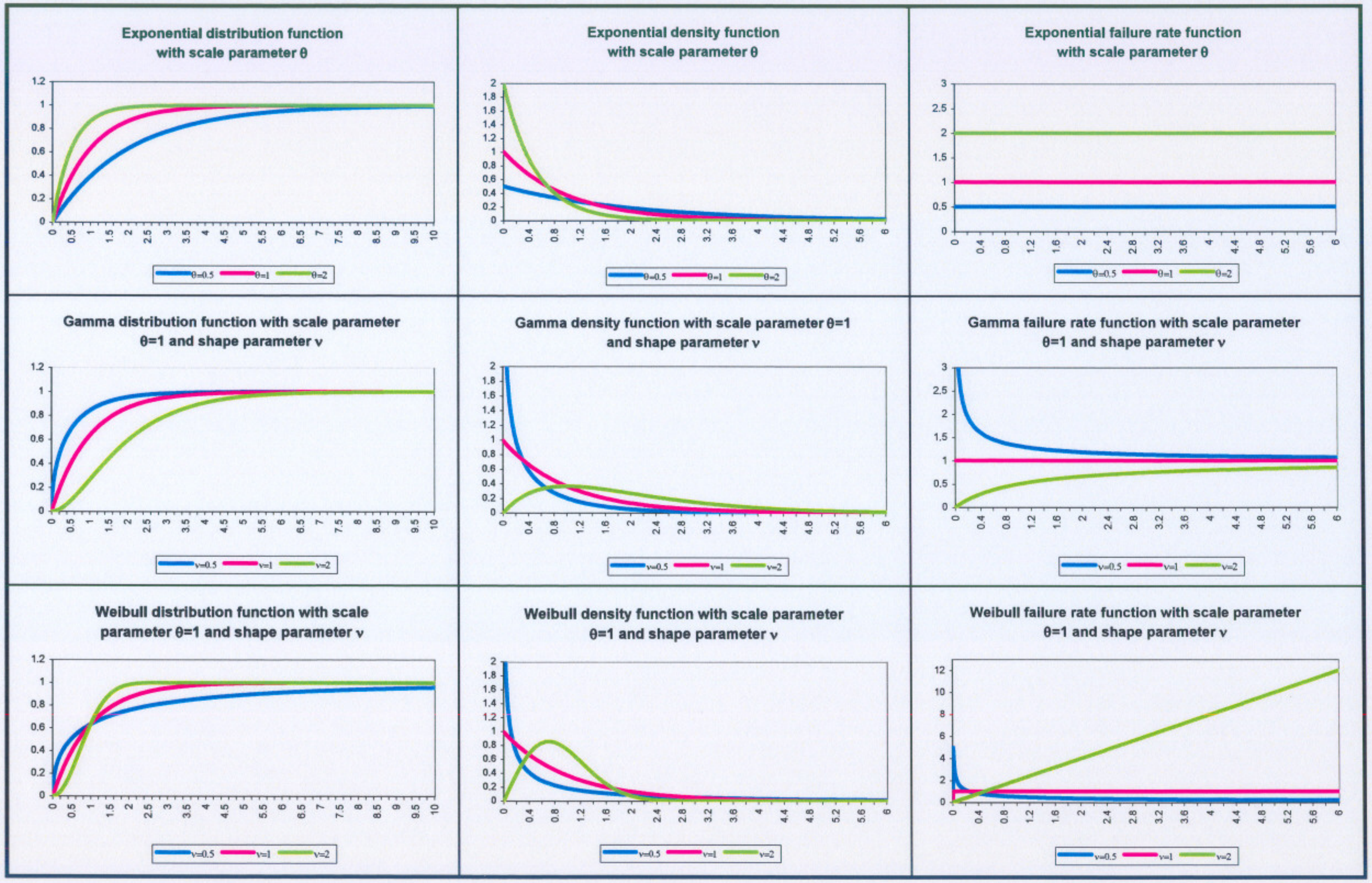

Figure 1: Distribution, density and failure rate functions of the exponential, gamma and Weibull distributions

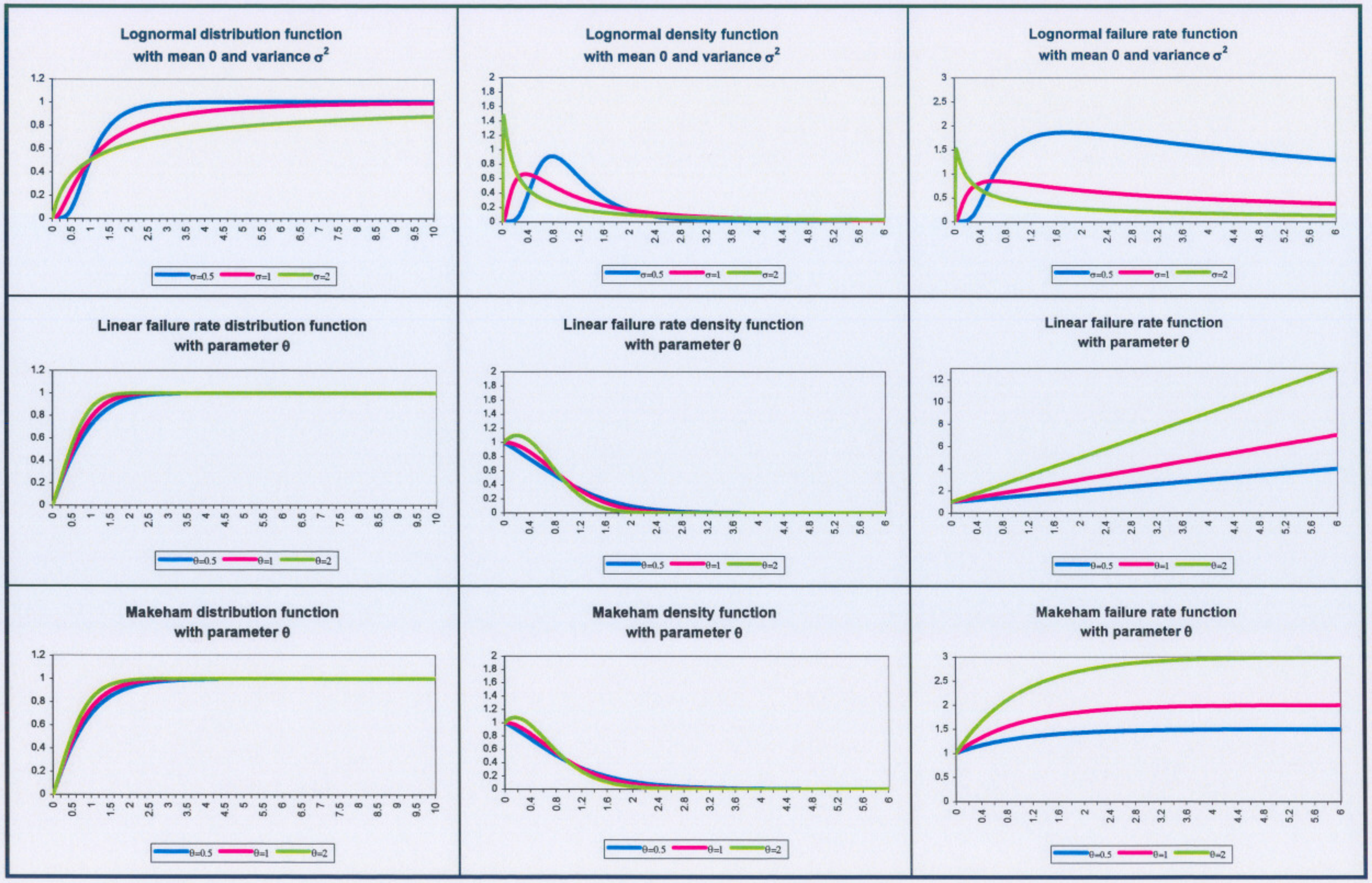

Figure 1 (contd.): Distribution, density and failure rate functions of the lognormal, linear failure rate and Makeham distributions

1.4.3 Normal, truncated normal and lognormal distributions

A random variable has a **normal** distribution if its probability density function is

$$f_N(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right), \quad -\infty < x < \infty.$$

The location parameter μ is the mean, and the scale parameter σ is the standard deviation.

A truncated version of the normal distribution with parameters μ, σ and a has probability density function

$$f_{TN}(x) = \begin{cases} \left[\sigma\sqrt{2\pi}\left(1-\Phi\left(\frac{a-\mu}{\sigma}\right)\right)\right]^{-1} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right], & x \geq a \\ 0, & x < a \end{cases} \tag{1.25}$$

where $\Phi(z)$ is the distribution function of a standard normal random variable.

If $a > 0$, the **truncated normal** distribution can be used as a model for a life distribution. The introduction of $\left(1-\Phi\left(\frac{a-\mu}{\sigma}\right)\right)^{-1}$ in (1.25) ensures that $\int_0^\infty f(x)dx = 1$, so that $f_{TN}(x)$ is the density of a (non-negative) life length. The truncated normal distribution has been used to model e.g. the distribution of material strength (Zacks 1992, p. 30).

The failure rate function is

$$r_{TN}(t) = \left[\sigma\sqrt{2\pi}\left(1-\Phi\left(\frac{t-\mu}{\sigma}\right)\right)\right]^{-1} \exp\left[-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right], \quad t \geq a.$$

A random variable X has a **lognormal** distribution with parameters μ and σ^2 if $Y = \ln(X)$ has the normal distribution. The probability density function of X is

$$f_{LN}(x) = \frac{1}{\sigma x\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(\ln x - \mu)^2\right), \quad 0 < x < \infty. \tag{1.26}$$

It is clear from Figure 1 that the lognormal distribution is highly skewed to the right, and therefore has been widely applied to model the distribution of e.g. material strength, air and water pollution, and other phenomena with highly skewed distributions (Zacks 1992, p. 33).

The mean and the variance of a lognormal random variable X is given by

$$\mu_{LN} = \exp(\mu + \sigma^2/2)$$

and

$$\sigma_{LN}^2 = \exp\left(2(\mu+\sigma^2/2)\right)\left(e^{\sigma^2}-1\right).$$

1.4.4 Linear failure rate, Makeham, Pareto and Beta distributions

The **linear failure rate** distribution has distribution function

$$F_{LFR}(x) = 1 - e^{-(x+\frac{\theta}{2}x^2)}, \quad x > 0, \quad \theta \geq 0. \tag{1.27}$$

For $\theta = 0$ this is the exponential distribution. The failure rate function of the linear failure rate distribution is given by

$$r_{LFR}(x) = 1 + \theta x, \tag{1.28}$$

which is a linear function in x.

The **Makeham** distribution has been widely applied in actuarial science and has distribution function

$$F_M(x) = 1 - e^{-[\lambda x + \frac{\gamma}{\kappa}(e^{\kappa x} - 1)]}, \quad x \geq 0, \quad \kappa \neq 0, \tag{1.29}$$

which is often written in the form

$$F_M(x) = 1 - e^{-[x+\theta(x+e^{-x}-1)]}, \quad \theta \geq 0. \tag{1.30}$$

For $\theta = 0$ (or $\lambda = 1$ and $\gamma = 0$), this is the exponential distribution. The failure rate function of the Makeham distribution is given by

$$r_M(x) = \lambda + \gamma e^{\kappa x},$$

or in alternative form

$$r_M(x) = 1 + \theta(1 - e^{-x}). \tag{1.31}$$

The distribution function of the **Pareto** distribution is

$$F_P(x) = 1 - \lambda^{1/\theta} (\lambda + x)^{-1/\theta}, \quad x \geq 0, \quad \theta, \lambda > 0, \tag{1.32}$$

with density function

$$f_P(x) = \frac{1}{\theta} \lambda^{1/\theta} (\lambda + x)^{-(1+\theta)/\theta}. \tag{1.33}$$

Consider the case where $\lambda = 1/\theta$, then:

$$F_P(x) = 1 - (1 + \theta x)^{-1/\theta},$$

so that the failure rate function of a Pareto random variable is

$$r_P(x) = \frac{1}{1 + \theta x}. \tag{1.34}$$

The density function of a **beta** distribution with positive parameters α and β, is given by

$$f_B(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1. \tag{1.35}$$

Provided that $\alpha > 1$ and $\beta > 1$, the density function is unimodal and falls to zero at the end points 0 and 1. If α and β have similar values, the density function is roughly symmetrical. When $\beta = 1$, the failure rate function of the beta distribution is equal to

$$r_{B1}(x) = \frac{\alpha x^{\alpha-1}}{1-x^{\alpha}}, \tag{1.36}$$

and when $\alpha = 1$ it is equal to

$$r_{B2}(x) = \frac{\beta}{1-x}. \tag{1.37}$$

1.5 Summary of mathematical notation

In this section a summary of the mathematical notation introduced in this chapter is presented, to serve as a quick reference for the reader while reading the rest of the dissertation.

- $X_1, X_2, \ldots, X_n$ are independent copies of a nonnegative random variable X with distribution F.
- $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denotes the order statistics of $X_1, X_2, \ldots, X_n$.
- $F(t) := P(X \leq t),\ t \geq 0$ is the distribution function.
- $\bar{F}(t) := 1 - F(t),\ t \geq 0$ is the survival function.
- $r(t) := \frac{f(t)}{\bar{F}(t)},\ t \geq 0$ is the failure rate function.
- $R(t) := \int_0^t r(u)du,\ t \geq 0$ is the cumulative failure rate function.
- $q(t) := \frac{1}{t}\int_0^t r(u)du,\ t \geq 0$ is the average failure rate function.
- $\mu := E(X) = \int_0^\infty \bar{F}(u)du$ is the mean of X.
- $\mu_s := E(X^s) = \int_0^\infty u^s dF(u),\ s = 2, 3, \ldots$ is the s-th moment of X.
- $\varepsilon_F(t) := \frac{\int_t^\infty \bar{F}(u)du}{\bar{F}(t)},\ t \geq 0$ is the mean residual life function.
- $D_i' := X_{i:n} - X_{(i-1):n},\ i = 1, 2, \ldots, n$ is the i-th spacing between the order statistics.
- $D_i := (n-i+1)\left(X_{i:n} - X_{(i-1):n}\right),\ i = 1, 2, \ldots, n$ is the normalized spacings.
- $H_F^{-1}(t) := \int_0^{F^{-1}(t)} \bar{F}(u)du,\ 0 \leq t \leq 1$ is the total time on test (TTT) transform of F.
- $\varphi_F(t) := \frac{H_F^{-1}(t)}{\mu},\ 0 \leq t \leq 1$ is the scaled TTT-transform.

Chapter 2

Alternative classes of distributions

2.1 Introduction

In Section 1.4 several parametric families of distributions were discussed. These distributions can be used as alternatives when testing for exponentiality. However, it is usually difficult to determine which specific parametric family of densities is appropriate to use as alternative.

For this reason, nonparametric classes of distributions are more often considered as alternatives. These classes of distributions are defined in terms of the monotonicity properties of the failure rate, defined in (1.3), the average failure rate, defined in (1.7), and the mean residual life function, defined in (1.8). They arise naturally from physical considerations like aging and wear.

Well-known classes are the increasing failure rate (IFR), the increasing failure rate average (IFRA), the new better than used (NBU), the decreasing mean residual life (DMRL), the new better than used in expectation (NBUE) and the harmonic new better than used in expectation (HNBUE) classes. These classes, together with their respective dual classes, are defined and discussed in Sections 2.2 to 2.4. From these discussions it will be clear that these classes of life distributions play a central role in the application of reliability theory.

2.2 IFR and IFRA classes of life distributions

Consider a component that does not age stochastically, i.e. its survival function over an additional period of duration x is the same regardless of its present age t. Thus,

$$\bar{F}(x|t) = \bar{F}(x), \ \forall \ t, x \geq 0,$$

or equivalently,

$$\bar{F}(t+x) = \bar{F}(t)\bar{F}(x), \ \forall \ t, x \geq 0.$$

This is the memoryless property (1.17) of the exponential distribution.

Suppose now that the component ages adversely in the sense that the conditional survival probability given in (1.2) is a decreasing function of age, i.e., $\bar{F}(x|t)$ is *decreasing* in $-\infty < t < \infty$ for each $x \geq 0$. This leads to a class of distributions corresponding to adverse aging, known as the increasing failure rate (IFR) class. The failure times of items with moving parts are usually modelled to have an increasing failure rate distribution, since friction would increase the rate of failure. Examples include rubber tyres, human beings after some initial period, and many mechanical parts which gradually wear out (Block and Savits 1981).

Formally, the increasing failure rate class is defined as follows:

Definition 2.1 *A distribution F is IFR if $\frac{\bar{F}(t+x)}{\bar{F}(t)}$ is decreasing in t for $x > 0$.*

The first systematic treatment of the IFR class appeared in Barlow, Marshall and Proschan (1975).

If F is absolutely continuous and has density f, then F is IFR if the failure rate, $r(t)$, defined in (1.3), is increasing for $0 \leq t < \infty$. Alternatively, F is IFR if the cumulative failure rate, $R(t) = -\log \bar{F}(t)$, defined in (1.6), is convex.

In analogy with IFR distributions there exists decreasing failure rate (DFR) distributions, when the component has *increasing* conditional survival probability as a function of age. This type of behaviour is often encountered in the initial phase of a lifetime where work hardening or debugging takes place. Examples include certain metals, certain complex systems like motorcars, and human beings in childhood (Block and Savits 1981).

The dual class is thus defined by reversing the direction of monotonicity, to describe situations where lifetimes of items improve with age. Therefore, F is in the DFR class if the failure rate, $r(t)$ is decreasing for $0 \leq t < \infty$.

An interesting way in which DFR lifetimes can occur is as *mixtures* of exponential lifetimes. Block and Savits (1981) discussed an example given by Proschan in 1963, where it was shown that the times of successive failures of air conditioning systems of a fleet of jet airplanes is DFR, even though the failure times of the individual air conditioning units are exponential.

The boundary member of both these classes are the exponential distribution, which models lifetimes that neither improve nor decline with age.

In terms of the scaled TTT-transform given in (1.12), Klefsjö (1982b) stated the following theorem:

Theorem 2.1 *A life distribution F is IFR (DFR) if and only if the scaled TTT-transform* $\varphi_F(t)$ *is concave (convex) for* $0 \leq t \leq 1$.

Langberg, Leon and Proschan (1980) considered the following characterizations of the IFR class of life distributions:

Theorem 2.2 *Let F be a life distribution and* D_i *be the normalized spacings (1.10).*

1. *F is IFR (DFR) if and only if F has a finite mean and* $E[D_k]$ *is decreasing (increasing) in k* $(k = 2, \ldots, n)$ *for infinitely many n.*

2. *Let F have a finite mean. Then F is IFR (DFR) if and only if for infinitely many* $n \geq N$ *and some l* $(1 \leq l \leq N)$, $E[\sum_{i=k}^{k+l} D_i]$ *is decreasing (increasing) in k* $(1 \leq k \leq n - l)$.

3. *Let F be continuous. Then F is IFR(DFR) if and only if for some fixed n and m,* $(2 \leq m + 1 \leq n)$ *and all* $u \geq 0$,

$$P(X_{(m+1):n} - X_{m:n} > u | X_{m:n} = x)$$

is decreasing (increasing) in x.

2.2.1 IFRA distributions and coherent systems

Barlow and Campo (1975, p. 83) argued that it would seem reasonable to suppose that if each component of a coherent system has an IFR distribution, then the system itself would also have an IFR distribution, but they could prove with a counterexample that this is not necessarily true. However, the failure rate of the system was still increasing *on average.*

Birnbaum, Esary and Marshall (1966) introduced the increasing failure rate average (IFRA) class of life distributions as the class of life distributions of coherent systems of IFR components:

Definition 2.2 *The distribution F is IFRA if* $-\frac{1}{t}\log \bar{F}(t)$ *is increasing in* $t > 0$.

Similarly, F is in the decreasing failure rate average (DFRA) class if $-\frac{1}{t}\log \bar{F}(t)$ is *decreasing* in $t > 0$.

Recall from (1.7) that $-\frac{1}{t}\log \bar{F}(t)$ is the average failure rate of the distribution.

Following Barlow and Campo (1975, p. 84) it is easy to see that an IFRA distribution F is characterised by $\bar{F}^{1/t}(t)$ decreasing on $[0, \infty)$. It is then obvious that F is IFRA if and only if

$$\bar{F}(\alpha t) \geq \bar{F}^{\alpha}(t) \qquad (2.1)$$

for all $0 < \alpha < 1$ and $t \geq 0$.

Similarly, a DFRA distribution F is characterised by $\bar{F}^{1/t}(t)$ *increasing* on $0 \leq t < \infty$, so that F is DFRA if and only if $\bar{F}(\alpha t) \leq \bar{F}^{\alpha}(t)$ for all $0 < \alpha < 1$ and $t \geq 0$.

Deshpande (1983) developed a test for exponentiality against IFRA alternatives based on (2.1) - the test statistic is given in (4.103).

Barlow and Campo (1975, p. 85) proved that a coherent system of independent IFRA components itself has an IFRA distribution (i.e. the IFRA class is closed under the formation of coherent systems). As two special cases, it follows that a coherent system of independent IFR components has an IFRA distribution, and a coherent system of independent exponential components is also IFRA. Thus, IFRA distributions arise naturally when coherent systems of independent IFR distributions are formed. Further, the IFRA class of distributions is the *smallest* class containing the exponential distributions which is closed under formation of coherent systems (Barlow and Campo 1975, pp. 86-69).

Klefsjö (1982b) studied properties of the IFRA class of distributions in terms of the scaled TTT-transform defined in (1.12). He stated the following theorem:

Theorem 2.3 *If F is a life distribution which is IFRA (DFRA), then $\varphi_F(t)/t$ is decreasing (increasing) for $0 < t < 1$.*

However, Klefsjö (1982b) referred to an F from Barlow (1979) for which $\varphi_F(t)/t$ is decreasing, but which is not IFRA.

This section is concluded with two basic properties of IFRA distributions from Barlow and Campo (1975, p. 89). But first the concept of a star-shaped function is defined:

Definition 2.3 *A function $h(x)$ defined on $[0, \infty)$ such that $h(x)/x$ is increasing on $[0, \infty)$ is called a star-shaped function.*

Alternatively, $h(x)$ is called star-shaped if $h(\alpha x) \leq \alpha h(x)$ for $0 \leq \alpha \leq 1$, $x \geq 0$ (Hollander and Proschan 1984, p. 618).

From Definition 2.2 and the definition of the hazard function (1.4), it then follows that if F is an IFRA distribution, then its hazard function, $R(x)$, is a star-shaped function.

Further, a distribution F is IFRA (DFRA) if and only if for each $\theta > 0$, $\bar{F}(x) - e^{-\theta x}$ has at most one change of sign, and if one change of sign actually occurs, it occurs from + to − (− to +).

Now, the first derivative of the hazard function $R(x)$, is just the failure rate function (1.3), which is increasing if F is IFR. Therefore $R(x)$ is convex. Further, a convex function passing through the origin is star-shaped and following Barlow and Campo (1975, p. 90), the IFR distribution F is IFRA. Thus, IFR $\subset$ IFRA.

2.2.2 Preservation of IFR and IFRA classes under reliability operations

In Section 2.2.1 it was stated that the IFRA class of distributions is closed under the formation of coherent systems. Two other reliability operations are considered in this section:

Addition of life lengths – The addition of life lengths is an important reliability operation in the study of maintenance policies. For example, when a failed component is replaced by a new component, the total life accumulated is obtained by the addition of the two life lengths. The question is whether the sum of the life lengths is in the same class of distributions as the life lengths of the respective components.

Mixture of distributions – Mixtures of distributions arise naturally in a number of reliability situations. For example, suppose a manufacturer produces 60% of a certain product in Assembly Line 1 and 40% in Assembly Line 2. Because of differences in machines, personnel, etc., the life length of a component produced in Assembly Line 1 has distribution F_1, while the life length of a component produced in Assembly Line 2 has distribution $F_2 \neq F_1$. After production, components from both assembly lines flow into a common shipping room, so that outgoing lots consist of a random mixture of the output of the two lines. It is clear that a component selected at random from a lot would have a life distribution $F = 0.6F_1 + 0.4F_2$, which is a mixture of the two underlying distributions (Barlow and Campo 1975, p. 101).

We present a summary of the results concerning preservation of the IFR and IFRA classes under these reliability operations in Table 2.1. For a detailed discussion, see Barlow and Campo (1975, pp. 98-104).

Table 2.1: Preservation of IFR and IFRA classes.

Class of life distributions	**Formation of coherent systems**	**Addition of life lengths**	**Mixture of distributions**
IFR	Not preserved	Preserved	Not preserved
IFRA	Preserved	No proof	Not preserved
DFR	Not preserved	Not preserved	Preserved
DFRA	Not preserved	Not preserved	Preserved

2.2.3 Distributions in the IFR and IFRA classes

Table 2.2 indicates for each of the parametric distributions from Section 1.4, for which value(s) of the parameter(s) the specific distribution is a member of the IFR(DFR) class (and therefore also a member of the IFRA(DFRA) class).

Table 2.2: Parametric distributions in the IFR(DFR) classes of life distributions.

Distribution	$f(x)$	$F(x)$	$r(x)$	Exponential	IFR	DFR
Linear failure rate		1.27	1.28	$\alpha = 0$	$\alpha > 0$	
Weibull	1.22	1.23	1.24	$\nu = 1$	$\nu \geq 1$	$0 < \nu \leq 1$
Gamma	1.21			$\nu = 1$	$\nu \geq 1$	$0 < \nu \leq 1$
Makeham		1.30	1.31	$\theta = 0$	$\theta > 0$	$\theta < 0$
Pareto		1.33	1.34	$\theta \to 0$	Not IFR	$\theta > 0$
Beta	1.35		1.36; 1.37		$\alpha \geq 1$, $\beta = 1$ or $\alpha = 1$ Not IFR $(0 < \alpha < 1)$)	Not DFR $(0 < \alpha < 1)$
Truncated normal	1.25				$a > 0$	Not DFR
Lognormal	1.26				Not IFR	Not DFR

Borges et al. (1984) used the IFR distribution

$$F_{u,\theta}(x) = \begin{cases} 1 - e^{-a_1 x}, & 0 \leq x \leq x_0 \\ 1 - e^{-a_1 x_0 - a_2(x - x_0)}, & x_0 \leq x < \infty \end{cases}$$

where $x_0 = \frac{-u \ln(1-u+\theta)}{u-\theta}$, $a_1 = 1 - \frac{\theta}{u}$, $a_2 = 1 + \frac{\theta}{1-u}$, and $0 \leq \theta \leq u$, for a fixed number u such that $0 < u < 1$. For $\theta = 0$, this is the exponential distribution.

Klefsjö (1983a) considered the life distribution

$$F(t) = \left(1 - e^{-3t}\right)\left(1 - e^{-7t}\right), \quad t \geq 0,$$

which is IFRA but not IFR. In general, the distribution

$$F(t) = \left(1 - e^{-ct}\right)\left(1 - e^{-dt}\right), \quad t \geq 0,$$

is IFRA but not IFR, provided that $c \neq d$.

2.3 NBU, DMRL and NBUE classes of life distributions

In this section we discuss classes of life distributions that are especially applicable in maintenance theory, namely the new better than used (NBU) and the new better than used in expectation (NBUE) classes. Their fundamental roles in maintenance analysis were shown by Marshall and Proschan (1972).

2.3.1 Classes of distributions applicable in replacement

The new better than used (NBU) class is defined as follows:

Definition 2.4 *The distribution F is NBU if*

$$\bar{F}(t+x) \leq \bar{F}(t)\bar{F}(x) \qquad \forall \quad t, x \geq 0. \tag{2.2}$$

This is equivalent to stating that $\bar{F}(x)$, the probability that a new unit will survive to age x, is greater than $\bar{F}(t+x)/\bar{F}(t)$, the probability that a unit of age t will survive an additional time x.

Similarly, the distribution F is new worse than used (NWU) if

$$\bar{F}(t+x) \geq \bar{F}(t)\bar{F}(x) \qquad \forall\, t, x \geq 0.$$

The boundary member of each of these classes is the exponential distribution. This is clear from the fact that when equality holds in (2.2), the memoryless property (1.17) of the exponential distribution is obtained.

Langberg et al. (1980) considered the following characterizations of the NBU class of life distributions:

Theorem 2.4 *Let F be a continuous life distribution.*

1. *Then F is NBU(NWU) if and only if*

$$P(X_{1:n-m} > u) \geq (\leq) P(X_{m+1:n} - X_{m:n} > u | X_{m:n} = x)$$

for some fixed n and m, $(1 \leq m \leq n)$ and all $u \geq 0$ and $x \geq 0$.

2. *Let F have a finite mean. Then F is NBU(NWU) if and only if, for every $t, s \in (0,1)$,*

$$E\left(X_{[nt]+[n(1-t)s]:n} - X_{[nt]:n}\right) \leq_{a.s.} (\geq_{a.s.}) E\left(X_{[n(1-t)s]:(n-[nt])}\right)$$

for infinitely many n.

In each case equality holds if and only if F is exponential.

Hollander and Proschan (1984, p. 618) gave the following characterization of NBU distributions: If a life distribution F has cumulative failure rate $R(x)$ given in (1.4), then F is NBU if and only if $R(x)$ is superadditive, where a superadditive function is defined as follows:

Definition 2.5 *A function $h(x) \geq 0$ defined on $[0, \infty)$ is superadditive if and only if*

$$h(x+y) \geq h(x) + h(y)$$

for all $x, y \geq 0$.

In contrast with the IFR and IFRA classes, it is interesting to note that no relationship seems to be known between the NBU class of life distributions and the scaled TTT-transform $\varphi_F(t)$ (Klefsjö 1982b).

The decreasing mean residual life class (DMRL) is defined as follows:

Definition 2.6 *The distribution F is DMRL if, for all $0 \leq s \leq t$,*

$$\varepsilon_F(s) \geq \varepsilon_F(t), \tag{2.3}$$

where $\varepsilon_F(s)$ defined in (1.8) is the mean residual life at time s.

That is, the mean of a component's remaining life, given that it has survived to time s, is no less than the mean of the component's remaining life given that it has survived to time t.

The DMRL class is especially important in medical research, where the mean residual life of a patient is often used as a measure of effectiveness of treatment. In general, the mean residual life is also an important measure in demography, life insurance, and comparison of diseases (Hollander and Proschan 1984).

The new better than used in expectation class (NBUE) is defined as follows:

Definition 2.7 *The distribution F is NBUE if*

$$\mu = \varepsilon_F(0) \geq \varepsilon_F(t) \qquad \text{for all} \quad t \geq 0. \tag{2.4}$$

This implies that a used component of age t has smaller mean residual life than a new component.

Similarly, the distribution F is new worse than used in expectation (NWUE) if $\varepsilon_F(0) \leq \varepsilon_F(t)$ for all $t \geq 0$.

It is easy to prove that the NBUE property can also be written as:

$$F \ is \ NBUE \quad \Leftrightarrow \quad \int_0^t \bar{F}(u)du \geq \mu F(t) \ for \ t \geq 0.$$

(Barlow and Campo 1975, p. 151, ex. 3).

The different classes of life distributions discussed up to now are related in the following manner:

$$\text{IFR} \subset \text{IFRA} \subset \text{NBU} \subset \text{NBUE}$$
$$\text{and IFR} \subset \text{DMRL} \subset \text{NBUE},$$

as well as

$$\text{DFR} \subset \text{DFRA} \subset \text{NWU} \subset \text{NWUE}$$
$$\text{and DFR} \subset \text{IMRL} \subset \text{NWUE}.$$

For detailed proofs of these relations the reader is referred to Bryson and Siddiqui (1969).

Langberg et al. (1980) considered the following characterizations of the NBUE class of life distributions:

Theorem 2.5 *Let F be a continuous life distribution with finite mean.*

1. *F is NBUE(NWUE) if and only if, for every $t \in (0, 1)$,*

$$\frac{1}{n-[nt]} \sum_{k=1}^{n-[nt]} E\left(X_{[nt]+k:n} - X_{[nt]:n} | X_{[nt]:n}\right) \leq_{a.s.} (\geq_{a.s.}) E\left(X_i\right)$$

   for infinitely many n.

2. *F is NBUE(NWUE) if and only if,*

$$E\left(X_{n:n} - X_{(n-1):n}\right) \leq (\geq) E\left(X_1\right)$$

   for some fixed $n \geq 2$.

In each case equality holds if and only if F is exponential.

For $n = 2$, the second characterization in Theorem 2.5 gives

$$E\left(X_{2:2} - X_{1:2}\right) \leq E\left(X_1\right)$$

and equality holds if and only if F is exponential.

Klefsjö (1982b) gave the following characterizations of the DMRL (IMRL) and NBUE (NWUE) classes in terms of the scaled TTT-transform (1.12):

- A life distribution F is DMRL (IMRL) if and only if $Q(t) = \left(1 - \varphi_F(t)\right)/(1 - t)$ is decreasing (increasing) for $0 \leq t < 1$.
- A life distribution F is NBUE (NWUE) if and only if $\varphi_F(t) \geq (\leq) t$ for $0 \leq t \leq 1$.

The NBUE characterization was first noted by Bergman (1979) and was used in connection with replacement policies.

2.3.2 Preservation of NBU and NBUE classes under reliability operations

In Section 2.2.2 the preservation of the IFR (DFR) and IFRA (DFRA) classes under certain reliability operations (formation of coherent systems, addition of life lengths and mixture of distributions) were discussed.

The results from Barlow and Campo (1975, pp. 182-187) for the NBU (NWU) and NBUE (NWUE) classes are summarised in Table 2.3.

Table 2.3: Preservation of NBU and NBUE classes.

Class of life distributions	**Formation of coherent systems**	**Addition of life lengths**	**Mixture of distributions**
NBU	Preserved	Preserved	Not preserved
NBUE	Not preserved	Preserved	Not preserved
NWU	Not preserved	Not preserved	Not preserved
NWUE	Not preserved	Not preserved	Not preserved

Note that the proof for mixtures of NWUE distributions was only recently presented independently by Bondesson and Mehrotra (Klefsjö 1982a).

2.3.3 Distributions in the NBU and NBUE classes

All the distributions in Section 2.2.3 are IFR and/or IFRA, and thus also NBU and NBUE (refer to p. 25).

Hollander and Proschan (1972) considered the following class of NBU alternatives: Let $\mathcal{F}_{a,b}$ denote the class of distributions with support $[a, b]$ where $b < 2a$. The class $\mathcal{F}_{a,b}$ contains distributions which are NBU but not IFR. Koul (1977) observed that $\mathcal{F}_{a,b}$ is at an extreme of the NBU class, since an F in $\mathcal{F}_{a,b}$ is an NBU which is as far away from being an exponential as possible.

Koul (1978) also constructed a family of life distributions which are NBUE, but not IFR, IFRA or NBU, and which are at a fixed distance Δ from the exponential distribution:

Let $0 < \Delta < 1$ be a given number. Choose $\Delta \leq u$ such that $u - H(u) = \Delta$, where H is a distribution on $[0, 1]$ which is used to construct the NBUE distribution. Choose $0 < p, q < 1$ and $u < v < w < 1$. Then there are positive numbers a_1, a_2, a_3 and a_4 such that

$$a_1 = (u - \Delta)/u; \quad (v - u)a_2 = v - u + \Delta(1 - p);$$

$$(w-v)a_3 = w - v + \Delta(p-q); \quad (1-w)a_4 = 1 - w + \Delta q.$$

Define $b_1 = a_1 u$, $b_2 = a_1 u + a_2(v-u)$, $b_3 = a_1 u + a_2(v-u) + a_3(w-v)$. Then the distribution defined as

$$\begin{aligned}
\bar{F}(x) &= e^{-a_1 x}, \quad 0 \le x \le -\ln(1-b_1) \\
&= e^{-a_2 x}, \quad -\ln(1-b_1) \le x \le -\ln(1-b_2) \\
&= e^{-a_3 x}, \quad -\ln(1-b_2) \le x \le -\ln(1-b_3) \\
&= e^{-a_4 x}, \quad -\ln(1-b_3) \le x < \infty. \qquad (2.5)
\end{aligned}$$

is neither IFRA nor NBU, but NBUE, since $a_2 > \max(a_1, a_3)$.

Klefsjö (1982b) gave the following distribution from Barlow (1979) as an example of a distribution that is NBUE, but not NBU:

$$F(x) = \begin{cases} 0, & 0 \le x < \frac{1}{2} \\ 1 - e^{-(c+x)}, & x \ge \frac{1}{2} \end{cases},$$

where $c = \ln 2 - \frac{1}{2}$.

2.3.4 Generalisations of the NBU and NBUE classes

The NBU (NWU) and NBUE (NWUE) classes of life distributions are restricted to one type of application, since they represent positive (negative) aging throughout the life span of an experimental component. In many situations there exists a particular age t_0 at (from) which deterioration sets in. For example, the performance of an airplane engine may deteriorate after t_0 hours of flight, making repair of the aircraft after t_0 hours essential.

For this reason various classes of life distributions have been created to describe such situations, e.g. the "NBU(NBUE) with respect to the set $[t_0, \infty)$"-class, the "NBU(NBUE) of age t_0"-class and the "NBU of order kt_0"-class, $k = 1, 2, \ldots$. For definitions and discussions see e.g. Bergman (1979), Hollander, Park and Proschan (1986), Ebrahimi and Habibullah (1990), and Reneau, Samaniego and Boyles (1991).

2.4 HNBUE class of life distributions

The harmonic new better than used in expectation (HNBUE) class of life distributions was introduced by Rolski (1975). He considered the mean residual life function (1.8) of a distribution and investigated relationships between various kinds of classes of distribution functions based on $\varepsilon_F(t)$.

The definition of the HNBUE class is based on the following order relation in the set of distribution functions which is used in reliability theory:

$$F <_C G \Longleftrightarrow \int_0^\infty x dG(x) < \infty \quad and \quad \int_x^\infty \bar{F}(t)dt \leq \int_x^\infty \bar{G}(t)dt, \quad x > 0.$$

Further, $F <_C G$ if and only if for every increasing convex function h,

$$\int_0^\infty h dF \leq \int_0^\infty h dG,$$

provided the integrals exist.

Rolski (1975) gave the following characterization of the class $\{F : F <_C M_\mu\}$, where $M_\mu = 1 - e^{-x/\mu}$, $x > 0$:

The relation $F <_C M_\mu$ holds if and only if

$$\frac{1}{\frac{1}{x}\int_0^x \frac{1}{\varepsilon_F(t)}dt} \leq \mu \qquad \forall x > 0. \tag{2.6}$$

This inequality means that for every $x > 0$ the integral harmonic mean of ε_F in the interval $(0, x)$ is less than the mean of F. Thus Rolski called the class of distribution functions $\{F : F <_C M_\mu\}$ the HNBUE class.

The different classes of life distributions are now related as follows:

IFR $\subset$ IFRA $\subset$ NBU $\subset$ NBUE $\subset$ HNBUE

(see Klefsjö (1982a).)

Klefsjö (1981) stated that F is HNBUE if

$$\int_t^\infty \bar{F}(x)dx \leq \mu e^{-t/\mu}, \qquad t \geq 0. \tag{2.7}$$

F is HNWUE if the reversed inequality is true and a life distribution which is both HNBUE and HNWUE is an exponential distribution.

Several authors have studied properties of the HNBUE class of distributions. These include Klefsjö (1981), Klefsjö (1982a), Klefsjö (1983a), Basu and Ebrahimi (1984), and Basu and Ebrahimi (1985).

Klefsjö (1982b) proved the following theorem based on the scaled TTT-transform (1.12):

Theorem 2.6 *A strictly increasing life distribution F is HNBUE if and only if*

$$\varphi_F(t) \;\geq\; (\leq)\; 1 - e^{-F^{-1}(t)/\mu}, \quad 0 \leq t \leq 1.$$

However, he showed that this does not hold for life distributions in general, by giving an example of a life distribution that is not HNBUE, but for which this inequality holds.
Klefsjö (1982a) presented further properties of the HNBUE class in terms of the equilibrium distribution, which is defined as follows:

Definition 2.8 *The equilibrium distribution of F, denoted by T_F, is defined as*

$$T_F(t) := \frac{1}{\mu}\int_0^t \bar{F}(x)dx, \quad t \geq 0. \tag{2.8}$$

Then the HNBUE property can be written as

$$\bar{T_F}(t) \leq \bar{G}(t), \quad t \geq 0,$$

where $\bar{G}(t) = e^{-t/\mu}$, $t \geq 0$.

2.4.1 Preservation of HNBUE properties under reliability operations

Table 2.4 provides a summary from Klefsjö (1982a) regarding the preservation of the HNBUE and HNWUE classes under reliability operations.

Table 2.4: Preservation of HNBUE class.

Class of life distributions	**Formation of coherent systems**	**Addition of life lengths**	**Mixture of distributions**
HNBUE	Not preserved	Preserved	Not preserved
HNWUE	Not preserved	Not preserved	Preserved

Note that a mixture of HNWUE life distributions is HNWUE, while the NWUE class is not closed under mixtures, as was mentioned in Section 2.3.2. Klefsjö (1982a) also mentioned that a mixture of HNBUE life distributions, *all of which have the same mean*, is HNBUE.

2.4.2 HNBUE distributions of higher order

Basu and Ebrahimi (1984) proposed a more general class of life distributions which they called the *k-order harmonic new better than used in expectation (k-HNBUE) class.* The motivation for proposing this class was twofold: Firstly, in testing for exponentiality there was now a larger class of available alternatives, since it was proved that the k-HNBUE class is the largest available class of distributions with the aging property. Secondly, analytical properties for as large a class as possible comprising of all known standard classes of distributions with aging properties could now be developed.
The k-HNBUE class is defined as follows:

Definition 2.9 *The non-negative random variable X is said to have a k-order harmonic new better than used in expectation distribution if*

$$\frac{1}{\frac{1}{x}\int_0^x \frac{1}{\{\varepsilon_F(t)\}}^k dt} \leq \mu^k \qquad \forall x > 0,$$

where $\varepsilon_F(s)$ is the mean residual life of a device at time s and $k \geq 1$.

The k-HNWUE is the dual class for which the reversed inequality holds. It is clear that if F is exponential, then F is both k-HNBUE and k-HNWUE.

For $k = 1$ this definition is equivalent to the definition of HNBUE given by Rolski (1975). Note that as k increases, the k-HNBUE class increases. Therefore HNBUE is the smallest class among the k-HNBUE classes. Conversely, by increasing k, the k-HNWUE class decreases, so that the HNWUE class is the largest among the k-HNWUE classes.

Basu and Ebrahimi (1984) stated and proved various closure properties of the k-HNBUE (k-HNWUE) class.

2.5 Multivariate classes of life distributions

The concepts of univariate aging and classes of life distributions as discussed here have also been extended to the multivariate case. A survey is presented in Block and Savits (1981). Basu, Ebrahimi and Klefsjö (1983) introduced and studied some multivariate versions of the HNBUE class and the reader is referred to the references in Basu et al. (1983) for literature on the multivariate versions of other classes of life distributions.

Chapter 3

Characterizations of the exponential distribution

3.1 Introduction

In this chapter we present a summary of characterizations of the exponential distribution as they appear in the literature from the early 1960's. Many of these characterizations have been used to construct goodness-of-fit tests for exponentiality, which are discussed in Chapter 4. The characterizations are divided into the following categories:

- Characterizations based on the lack-of-memory property;
- Characterizations based on identically distributed random variables;
- Characterizations based on the independence of random variables;
- Characterizations based on uniform random variables and normalized spacings;
- Characterizations based on moments of order statistics;
- Characterizations based on moment inequalities;
- Characterizations based on the mean residual life function;
- Other characterizations.

3.2 Characterizations based on the lack-of-memory property

The lack-of-memory property (1.17) of the exponential distribution was defined and discussed in Section 1.3, p. 7. We now formally present this as our first characterization of the exponential distribution:

Theorem 3.1 *If X is a non-negative random variable from an unknown continuous probability distribution with distribution function F, then*

$$P(X > t + x | X > t) = P(X > x) \text{ for all } t, x \geq 0 \tag{3.1}$$

if and only if F is exponential.

Azlarov and Volodin (1986, pp. 9-10) discussed several generalizations of Theorem 3.1.

Angus (1982) presented the following characterization of the exponential distribution which can be obtained from the lack-of-memory property by letting $t = x$ in (1.17):

Theorem 3.2 *Let X be a random variable with right-continuous distribution function F, satisfying $F(0-) = 0$ and*

$$\lim_{h \to 0^+} \frac{F(h) - F(0)}{h} = a \in [0, \infty].$$

Then $\bar{F}(2x) = \bar{F}^2(x)$ $\forall x \geq 0$ if and only if either $\bar{F}(x) = e^{-\theta x}$, $x \geq 0$ for some $\theta \in (0, \infty)$ or $P(X = 0) = 1$.

Theorem 3.2 is actually an order statistic characterization, since $1 - \bar{F}^2(x)$ is the cumulative distribution function of $X_{1:2}$ (where X_1, X_2 are i.i.d. F), and $1 - \bar{F}(2x)$ is the cumulative distribution function of $X_1/2$.

Restated in this context, Theorem 3.2 becomes:

Theorem 3.3 *For right-continuous F, $2X_{1:2}$ has the same distribution as X_1 if and only if $\bar{F}(x) = e^{-\theta x}$, $x \geq 0$ for some $\theta \in (0, \infty)$ or $P(X = 0) = 1$.*

(Also see the discussion on Theorem 3.5, p. 34).

Angus (1982) based three goodness-of-fit tests for exponentiality on Theorem 3.2: (4.16), (4.17) and (4.18) in Section 4.2.3, p. 52.

3.3 Characterizations based on identically distributed random variables

Puri and Rubin (1970) considered characterizations of distributions based on the absolute difference of two i.i.d. random variables, X_1 and X_2, whose common distribution is the same as that of a random variable X. The question they addressed was to characterise all possible distributions of X for which the distribution of $|X_1 - X_2|$ and X is identical.

They considered three different cases: the distribution of X is either discrete, or absolutely continuous, or singular. For the absolutely continuous case they proved the following theorem:

Theorem 3.4 *Let X_1 and X_2 be two independent copies of a random variable X with probability density function $f(x)$. Then X and $|X_1 - X_2|$ have the same distribution if and only if, for some $\theta > 0$,*

$$f(x) = \begin{cases} \theta e^{-\theta x} & x \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

Theorem 3.4 was used by Nikitin and Tchirina (1996) to explain the high efficiency properties of the Gini test defined in (4.65).

Desu (1971) gave the following characterization of the exponential distribution:

Theorem 3.5 *If F is a non-degenerate distribution function, then for each positive integer n, $nX_{1:n}$ and X are identically distributed if and only if $F(x) = 1 - e^{-\theta x}$, $x \geq 0$.*

Gupta (1973) proved that under the conditions that F is non-degenerate, with a continuous derivative at 0 and $F(0) = 0$, the identity $nX_{1:n} \stackrel{d}{=} X$ for *one* specific $n \geq 2$ characterises F to be exponential.

Gather (1989) proved the following theorem:

Theorem 3.6 *Let F be a continuous distribution function and let $F(x)$ be strictly increasing for all $x > 0$. Then F is exponential if and only if*

$$X_{j:n} - X_{i:n} \stackrel{d}{=} X_{(j-i):(n-i)}, \quad 1 \leq i < j \leq n \tag{3.2}$$

holds true for a sample from F with two distinct values j_1 and j_2 of j and some i, n, $1 \leq i < j_1 < j_2 \leq n$, $n \geq 3$.

This result was first proven by Ahsanullah (1975), where an implicit assumption was made in the proof that an absolutely continuous F is NBU or NWU.

In the special case that (3.2) holds for the one value $j = i + 1$, for some $1 \leq i \leq n$, $n \geq 2$, this characterises the exponential distribution in the class of all continuous distributions without further assumptions (Rossberg 1972b).

If F is absolutely continuous and IFR, and (3.2) holds for one arbitrary j, i, n, $1 \leq i < j \leq n$, then it is a characteristic property of the exponential distribution (Ahsanullah 1984).

Yeo and Milne (1989) proved the following theorem, which contains a result of Kotz and Steutel (1988):

Theorem 3.7 *Suppose that X_1 and X_2 are i.i.d. non-negative random variables which are independent of another random variable U, and that $Z = U(X_1 + X_2)$.*
Then any two of the following three conditions imply the third:

1. *Z has the same distribution as X_1 and X_2;*

2. *U has a uniform distribution on $(0, 1)$;*

3. *Z is exponentially distributed.*

The theorem also holds if the first condition is replaced by

1. *X_1 and X_2 are each exponentially distributed.*

Yeo and Milne discussed multivariate extensions of Theorem 3.7, while Alamatsaz (1993) interpreted a general characterization concerning α-unimodal distributions to yield several characterizations of exponential and gamma distributions which also contain Theorem 3.7.

3.4 Characterizations based on the independence of random variables

The previous section dealt with characterizations based on X_1 and $|X_1 - X_2|$ being identically distributed, where X_1, X_2 are i.i.d. random variables. Ferguson (1964), Basu (1965) and Crawford (1966) considered the problem of characterising distributions with the property that $X_{1:2}$ and $|X_1 - X_2|$ are *independent.*
Ferguson (1964) proved the following theorem:

Theorem 3.8 *Suppose that the random variables X_1 and X_2 are independent and have absolutely continuous distributions. Then, in order that $X_{1:2}$ and $V = |X_1 - X_2|$ be independent, it is necessary and sufficient that both X_1 and X_2 have exponential distributions with the same location parameter.*

Basu (1965) proved the same result (assuming $F(0) = 0$), while Crawford (1966) proved the theorem without the assumption of absolute continuity.

Govindarajulu (1966) proved the following two theorems, also based on the independence of random variables:

Theorem 3.9 *The vector of random variables* $(U_2, U_3, \ldots, U_n)$ *and* U_1 *are stochastically independent if and only if* $F(x) = 1 - \exp(-\theta x)$ *for some* $\theta > 0$ *where* $U_1 = X_{1:n}$ *and* $U_i = X_{i:n} - X_{1:n}$, $i = 2, 3, \ldots, n$.

Theorem 3.10 *The random variables* $X_{(j+1):n} - X_{j:n}$, $1 \leq j \leq n-1$ *and* $X_{i:n}$, $i \leq j$ *are stochastically independent if and only if* $F(x) = 1 - \exp(-\theta x)$ *for some* $\theta > 0$.

When $n = 2$, both Theorem 3.9 and Theorem 3.10 state exactly the result of Theorem 3.8, also without the assumption of absolute continuity.

For the remainder of this section, define $Z := n^{-1} \sum_{i=1}^{n} (X_{i:n} - X_{1:n})$. The following corollary follows from Theorem 3.9:

Corollary 3.1 Z and $X_{1:n}$ are stochastically independent if and only if $F(x) = 1 - \exp(-\theta x)$ for some $\theta > 0$.

The result of Corollary 3.1 was proved by Tanis (1964) for the absolutely continuous case. This result is also a special case of a much more general theorem developed later by Rossberg (1972a).

Dallas (1973) showed that, among all continuous distributions having *finite first moment*, the exponential distribution is the only distribution satisfying $E[Z|X_{1:n}] = E(Z)$ almost surely. This characterization is weaker than Corollary 3.1 using the independence of Z and $X_{1:n}$. Generalisations of Dallas's result were studied by Wang and Srivastava (1980) amongst others.

Swanepoel (1991) presented a characterization of the exponential distribution based on the covariance of $X_{1:n}$ and Z by defining a class of continuous distributions $\mathcal{G}$ consisting of all continuous distribution functions F such that $\int_{-\infty}^{\infty} x^2 dF(x) < \infty$ and $\varepsilon_F(x) := \frac{\int_x^\infty (1-F(y))dy}{1-F(x)}$ is a monotone function of x:

Theorem 3.11 *The following two conditions are equivalent:*

1. *F is exponential.*
2. $\mathrm{Cov}\,(X_{1:n}, Z) = 0$ *and* $F \in \mathcal{G}$.

3.5 Characterizations based on uniformly distributed random variables and normalized spacings

Patil and Seshadri (1964) derived a characterization of the exponential distribution for the case when the conditional distribution of X_1 given $X_1 + X_2$ is known to be uniform.

Theorem 3.12 *If the conditional distribution of X_1 given $X_1 + X_2 = Z$ is the uniform density over $(0, Z)$, where X_1 and X_2 are independent non-negative random variables, then both X_1 and X_2 have the exponential distribution with the same scale parameter.*

Wang and Chang (1977) proposed the transformation

$$\tilde{Z}_i = \left[\frac{\sum_{j=1}^{i} X_j}{\sum_{j=1}^{i+1} X_j}\right]^i, \; i = 1, \ldots, n-1. \tag{3.3}$$

Under one parameter exponentiality, $\tilde{Z}_1, \ldots, \tilde{Z}_{n-1}$ are i.i.d. uniform $(0,1)$ random variables, which is a characterization of the exponential distribution for $n \geq 3$. A goodness-of-fit test for exponentiality based on this characterization is defined in (4.35).

Another commonly used transformation in the development of goodness-of-fit tests for exponentiality is the transformation to normalized spacings D_i, as defined in (1.10). Goodness-of-fit tests for exponentiality based on normalized spacings are discussed in Section 4.2.5.

Under one parameter exponentiality the normalized spacings are i.i.d. as one parameter exponential random variables with the same parameter as the original sample. This property was first proved by Epstein and Sobel (1954), while Seshadri, Csörgo and Stephens (1969) proved the following theorem which states that the exponential distribution is the only non-negative distribution having this property:

Theorem 3.13 *Let $X_1, \ldots, X_n$ be a random sample from an unknown non-negative continuous distribution F. The normalized spacings D_i are exponentially distributed with density $f(x) = \theta e^{-\theta x}$ if and only if the X_i are exponentially distributed with density $f(x) = \theta e^{-\theta x}$.*

The tests (4.44) and (4.45) developed by Jammalamadaka and Taufer (2003) are based on Theorem 3.13.

Theorems 3.14 to 3.16 by **Ahsanullah (1976), Ahsanullah (1977)** and **Ahsanullah (1978)** state characterizations of the exponential distribution that require the identical distributions of D_i and D_1, D_i and X, and D_i and D_j, respectively. Note that all three these characterizations make assumptions regarding the class of distributions in which the characterization holds:

Theorem 3.14 *Let X be a non-negative random variable with absolutely continuous, strictly increasing distribution function $F(x)$ for all $x > 0$, and $F(x) < 1$ for all x. Then the following properties are equivalent:*

1. *X has an exponential distribution.*
2. *X has a monotone hazard rate and for one i and one n with $2 \leq i \leq n$, the statistics D_i and $D_1 = nX_{1:n}$ are identically distributed.*

For the case $i = n = 2$, this becomes

$$D_2 = X_{2:2} - X_{1:2} \stackrel{d}{=} 2X_{1:2}, \tag{3.4}$$

where $\stackrel{d}{=}$ indicates equal distributions.

Theorem 3.15 *Let X be a non-negative random variable with absolutely continuous distribution function $F(x)$ that is strictly increasing on $[0, \infty)$. Then the following properties are equivalent:*

1. *X has an exponential distribution.*
2. *For some i and n with $1 \leq i \leq n$, the statistics D_i and X are identically distributed and F is either NBU or NWU.*

Note that if $i = 1$, then

$$D_1 = nX_{1:n} \stackrel{d}{=} X. \tag{3.5}$$

The reader is referred to Theorem 3.5 and the subsequent result from Gupta (1973) on p. 34. Further, if $i = n = 2$, then $D_2 = X_{2:2} - X_{1:2} \stackrel{d}{=} X$. This is similar to the result in Theorem 3.4, p. 34 from Puri and Rubin (1970).

Theorem 3.16 *Let X be a non-negative random variable with absolutely continuous distribution function $F(x)$ that is strictly increasing on $[0, \infty)$. Then the following properties are equivalent:*

1. *X has an exponential distribution.*
2. *For some i, j and n with $1 \leq i < j < n$, the statistics D_i and D_j are identically distributed and F is either IFR or DFR.*

3.6 Characterizations based on moments of order statistics

The expected value of the k-th order statistic from an exponential distribution with mean $1/\theta$ is given in (1.20) as

$$E[X_{k:n}] = \frac{1}{\theta}\left(\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{n-k+1}\right).$$

This property was used by Patwardhan (1988) to develop a goodness-of-fit test (4.56).

The following recurrence relation for the moments of order statistics also appears in the literature often:

$$(n-k)E[X_{k:n}^{(j)}] + kE[X_{k+1:n}^{(j)}] = nE[X_{k:n-1}^{(j)}], \tag{3.6}$$

for $k = 1, 2, \ldots, n = k+1, k+2, \ldots$ and $j = 1, 2, \ldots$ (see e.g. Govindarajulu (1975), Azlarov and Volodin (1986, p. 28)).

Based on (3.6), Govindarajulu (1975) proved many characterizations of the exponential distribution in terms of the moments, variance and covariance of order statistics. His results are presented in Theorems 3.17 to 3.19:

Theorem 3.17 *1. For* $i = 0, 1, \ldots$

$$E[X_{(i+1):n}^2] - E[X_{i:n}^2] = 2(\theta(n-i))^{-1}E[X_{(i+1):n}],$$

$n = i+1, i+2, \ldots$ *if and only if* $F(x) = 1 - e^{-\theta x}$.

2. For $i = 1, 2, \ldots$

$$n\left(E[X_{i:n}^2] - E[X_{(i-1):(n-1)}]\right) = \frac{2}{\theta}E[X_{i:n}],$$

$n = i, i+1, \ldots$ *if and only if* $F(x) = 1 - e^{-\theta x}$.

Theorem 3.18 *1. For* $i = 0, 1, \ldots$

$$Var(X_{(i+1):n}) - Var(X_{i:n}) = \left(E[X_{(i+1):n}] - E[X_{i:n}]\right)^2,$$

$n = i+1, i+2, \ldots$ *if and only if* $F(x) = 1 - e^{-\theta x}$.

2. For $i = 1, 2, \ldots$

$$Var(X_{i:n}) - Var(X_{i-1:n-1}) = \left(E[X_{i:n}] - E[X_{i-1:n-1}]\right)^2,$$

$n = i, i+1, \ldots$ *if and only if* $F(x) = 1 - e^{-\theta x}$.

Theorem 3.19 *1. For $i = 1, 2, \ldots$*

$$Var(X_{i:n}) - Cov(X_{i:n}, X_{i+1:n}) = 0,$$

$n = i, i+1, \ldots$ if and only if $F(x) = 1 - e^{-\theta x}$.

2. For $i = 0, 1, \ldots$

$$Var(X_{i:n}) - (n-i)^{-1} \sum_{j=i+1}^{n} Cov(X_{i:n}, X_{j:n}) = 0,$$

$n = i+1, i+2, \ldots$ if and only if $F(x) = 1 - e^{-\theta x}$.

3. For $i = 0, 1, \ldots$ and $k = i, i+1, \ldots$

$$Cov(X_{i:n}, X_{k:n}) = Cov(X_{i:n}, X_{k+1:n}),$$

$n = k, k+1, \ldots$ if and only if $F(x) = 1 - e^{-\theta x}$.

4. Let $E[X_1] = 1/\theta$. For $i = 1, 2, \ldots$

$$\sum_{j=1}^{n} Cov(X_{i:n}, X_{j:n}) = \frac{1}{\theta} E(X_{i:n}),$$

$n = i, i+1, \ldots$ if and only if $F(x) = 1 - e^{-\theta x}$.

Characterizations of the exponential distribution based on moments and product moments of order statistics and defined in terms of a sequence of integers $\{n_j\}_{j=1}^{\infty}$ satisfying $0 < n_1 < n_2 < \ldots$ and $\sum_{j=1}^{\infty} \frac{1}{n_j} = \infty$ are discussed by Galambos and Kotz (1978, pp. 55-57), Lin (1988) and Lin (1989).

3.7 Characterizations based on moment inequalities

Ahmad (2001) derived several moment inequalities for the IFR, NBU, NBUE and HNBUE classes of life distributions, which lead to characterizations of the exponential distribution. These moment inequalities and characterizations are presented in this section.

Theorem 3.20 *If F is IFR, then for all integers $r_1, \ldots, r_k \geq 0, k \geq 2$,*

$$2^{C(\underline{r};k)} \prod_{i=1}^{k} (r_i + 1)! E[X_{1:2}^{(\sum_{i=1}^{k} r_i + k)}] \geq \left(\sum_{i=1}^{k} r_i + k \right)! \prod_{i=1}^{k} E[X^{r_i+1}]$$

where $C(\underline{r}; k) = (k+2)(k-1)/2 + \sum_{i=1}^{k} i r_i - r_k$.

Note that if $r_1 = \cdots = r_k = 0$, then for all $k \geq 2$,

$$2^{(k+2)(k-1)/2} E[X_{1:2}^k] \geq k! E[X]^k.$$

Note also that if $k = 2$ and $r_1 = r_2 = r$, a characterization of the exponential distribution follows:

Corollary 3.2 If F is IFR then

$$E[X_{1:2}^{(2r+2)}] \geq \binom{2r+2}{r+1} \left(\frac{1}{2}\right)^{2r+2} (E[X^{(r+1)}])^2 \tag{3.7}$$

and equality holds if and only if F is exponential.

Ahmad (2001) used (3.7) to define a measure of deviation in developing a goodness-of-fit test for exponentiality against IFR alternatives (4.110).

Theorem 3.21 *If F is NBU, and if $r_1, \ldots, r_k \geq 0, k \geq 2$ are integers, then*

$$\left(\sum_{i=1}^{k} r_i + k\right)! \prod_{i=1}^{k} E[X^{(r_i+1)}] \geq \prod_{i=1}^{k} (r_i + 1)! \mu_{(\sum_{i=1}^{k} r_i + k)}.$$

If $r_1 = \cdots = r_k = 0$, then for all $k \geq 2$, the following is a characterization of the exponential distribution:

Corollary 3.3 If F is NBU then

$$(E[X])^k \geq E[X^k]/k!. \tag{3.8}$$

and equality holds if and only if F is exponential.

Theorem 3.22 *If F is NBUE, then for all integers $r \geq 0$,*

$$\mu E[X^{(r+1)}] \geq E[X^{(r+2)}]/(r+2). \tag{3.9}$$

The last theorem of Ahmad (2001) is:

Theorem 3.23 *If F is HNBUE, then for all integers $r \geq 0$,*

$$(E[X])^{r+2} \geq E[X^{(r+2)}]/(r+2)! \tag{3.10}$$

In both Theorems 3.22 and 3.23 equality holds if and only if F is exponential.

Ahmad (2001) used (3.8) and (3.9) to define measures of deviation in developing goodness-of-fit tests for exponentiality against NBU and NBUE alternatives (see (4.119) and (4.137)).

Note that when $k = 2$ in (3.8) and $r = 0$ in (3.9) and (3.10) respectively, all three expressions give

$$2(E[X])^2 \geq E[X^2]. \tag{3.11}$$

For a distribution F with a monotone failure rate density (i.e. IFR or DFR), Azlarov and Volodin (1986, p.) stated that the relation

$$2(E[X])^2 = E[X^2]. \tag{3.12}$$

holds if and only if F is exponential. Since equality holds in (3.11) if and only if F is exponential, the relation (3.12) is a characterization of the exponential distribution in the larger HNBUE class of distributions.

Using the well-known relation $\sigma^2 := Var(X) = E[X^2] - (E[X])^2 = E[X^2] - \mu^2$, (3.11) can be written as $\mu^2 \geq \sigma^2$. Thus, for all F in the HNBUE class of life distributions we have that $\mu^2 \geq \sigma^2$ and equality holds if and only if F is exponential. This of course implies that for all $F \in$ HNBUE, $\sigma/\mu \leq 1$.

Borges et al. (1984) have already proven that if F is NBUE, a necessary and sufficient condition in order that F is exponential, is that $CV(F) = 1$, where $CV(F) = \sigma/\mu$ denotes the coefficient of variation of F. They used this property to base a goodness-of-fit test for exponentiality on the sample coefficient of variation (p. 92).

The characterization regarding the exponential coefficient of variation can also be derived from bounds on the *moments* of distributions from different classes of life distributions, assuming the mean is known (Barlow and Campo 1975, p. 118):

Let F be IFRA (DFRA) with known mean μ. Bounds for the r-th moment are given by:

$$E[X^r] \geq (\leq)\Gamma(r+1)\mu^r, \quad 0 < r \leq 1; \tag{3.13}$$

$$E[X^r] \leq (\geq)\Gamma(r+1)\mu^r, \quad 1 \leq r < \infty. \tag{3.14}$$

For $r = 2$, if F is IFRA, then (3.14) becomes (3.11).

For $F \in$ NBU or NBUE, the following inequalities then hold:

- If F is NBU (NWU) then $\frac{E[X^{r+s}]}{\Gamma(r+s+1)} \leq (\geq) \frac{E[X^r]}{\Gamma(r+1)} \frac{E[X^s]}{\Gamma(s+1)}$ for all $r, s \geq 0$.

- If F is NBUE (NWUE) then $\frac{E[X^{r+1}]}{\Gamma(r+2)} \leq (\geq) \frac{E[X^r]}{\Gamma(r+1)} E[X]$ for all $r > 0$.

Note that with $r = 1$, if F is NBUE (NWUE), then $\sigma/\mu \leq (\geq) 1$.

Another result containing the coefficient of variation characterization was proved by Basu and Ebrahimi (1984):

Lemma 3.1 *Let F be HNBUE with mean μ. In order that F is exponential it is necessary and sufficient that*

$$\int_0^\infty x^r dF(x) = \Gamma(r+1)\mu^r$$

holds for some positive $r \neq 1$.

Taking $r = 2$, then $F \in$ HNBUE is exponential if and only if its coefficient of variation 1.

Al-Ruzaiza, Hendi and Abu-Youssef (2003) followed the same approach as Ahmad (2001) to derive moment inequalities for the class of HNBUET (harmonic new better than used in expectation upper tail) distributions. **Ahmad and Mugdadi (2004)** proved moment inequalities for three classes of life distributions that were not considered by Ahmad (2001). These are the IFRA, NBUC (new better than used in convex ordering) and DMRL classes.

Theorem 3.24 *If F is IFRA, then for all integers $r \geq 0$ and any $0 < \alpha < 1$,*

$$E\left(X_1^{r+1}\right) \geq E\left\{\min\left(\frac{X_1}{\alpha}, \frac{X_2}{1-\alpha}\right)^{r+1}\right\},$$

where X_1 and X_2 are two non-negative independent copies of random variables with distribution F.

From Theorem 3.24, Ahmad and Mugdadi (2004) stated the following result:

Corollary 3.4 Let $r = 0$, then $\mu \geq E\{\min(X_1/\alpha, X_2/(1-\alpha))\}$.

For the NBUC class, Ahmad and Mugdadi (2004) proved the following theorem:

Theorem 3.25 *If F is NBUC, then for all integers r and $s \geq 0$,*

$$(r+2)!(s+1)!E[X^{r+s+3}] \leq (r+s+3)!E[X^{r+2}]E[X^{s+1}].$$

From Theorem 3.25 Ahmad and Mugdadi (2004) stated the following three results:

Corollary 3.5 Let $r = 0$, then $2\left((s+1)!E[X^{s+3}]\right) \leq (s+3)!E[X^2]E[X^{s+1}]$.

Corollary 3.6 Let $s = 0$, then $(r+2)!E[X^{r+3}] \leq (r+3)!E[X]E[X^{r+2}]$.

Corollary 3.7 Let $r = s = 0$, then $E[X^3] \leq 3E[X]E[X^2]$.

The last theorem of Ahmad and Mugdadi (2004) is for the DMRL class:

Theorem 3.26 *If F is DMRL, then for all integers $r \geq 0$,*

$$(r+1)E[X_1(X_{1:2})^r] \geq (r+2)E[X_{1:2}^{r+1}].$$

Corollary 3.8 Let $r = 0$, then $\mu \geq 2E[X_{1:2}]$.

3.8 Characterizations based on the mean residual life function

Consider the random life length X of a system or component. Two different kinds of residual lifetimes are considered in the literature (Lin 2003):

1. the truncated life from below, $(X-x)_+ = \max\{X-x, 0\}$;

2. the remaining life at age t, $X_t = (X-t)|X > t$;

Baringhaus and Henze (2000) stated the following characterization, which they used to develop two goodness-of-fit tests ((4.57) and (4.58)):

Theorem 3.27 *If X has finite positive mean, the distribution of X is exponential if and only if the mean residual life function is constant, i.e.*

$$\varepsilon_F(t) := E(X-t|X \geq t) = E(X), \ \forall \ t > 0.$$

Theorem 3.27 is contained in the following more general theorem from Azlarov and Volodin (1986, p. 36):

Theorem 3.28 *Assume that X is a non-degenerate,non-negative random variable and that $E[X^\alpha] < \infty$ for some $\alpha > 0$. Then*

$$E[(X-t)^\alpha|X \geq t] = E(X^\alpha)$$

holds for all $t \geq 0$ if and only if X is exponential.

Characterizations of the exponential distribution based on the truncated life from below were considered by Chong (1977) and Lin (2003).
We present only one result and one remark from Lin (2003):

Theorem 3.29 *Let X be a non-negative random variable with finite moments of all orders, and let $E(X) > 0$. Then the relation*

$$\frac{E(X^m)}{m!}\left(\frac{mE(X^{m-1})}{E(X^m)}\right)^m = 1, \qquad m = 2, 3, 4, \ldots,$$

holds if and only if X is exponentially distributed.

Similar to the relation in (2.1), Lin (2003) considered the following question:
"If r is a positive constant not equal to one and if X is a positive random variable with distribution F satisfying

$$\bar{F}^r(t) = \bar{F}(rt) \text{ for } t \geq 0, \tag{3.15}$$

is it true that F is exponential?"
Lin (2003) showed that the answer to this question is in general negative. However, if relation (3.15) holds for some positive $r \neq 1$ and if $F'(0^+) \equiv \lim_{x\to 0^+} F(x)/x > 0$, then F is exponential.

3.9 Other characterizations

We conclude this chapter with a discussion on characterizations of the exponential distribution based on the Laplace transform, the equilibrium distribution, the entropy and the characteristic function of a random variable. For characterizations of the exponential distribution based on record values the reader is referred to Ahsanullah (1978), Ahsanullah (1979), Azlarov and Volodin (1986, p. 53), Too and Lin (1989) and Balakrishnan and Stepanov (2004).

For characterizations using the geometric distribution or geometric compounding, see e.g. Arnold (1973), Azlarov and Volodin (1986, p. 79-81), Lin and Hu (2001) and Hu and Lin (2003).

Baringhaus and Henze (1991) stated the following characterization on which they also based a goodness-of-fit test:

Theorem 3.30 *If the distribution of X is exponential with parameter $\theta > 0$, then the Laplace transform of X is*

$$\psi(t) := E[e^{-tX}] = \frac{\theta}{\theta + t}, \qquad t \geq 0$$

and thus satisfies the differential equation

$$(\theta + t)\psi'(t) + \psi(t) = 0, \ \forall\ t \geq 0$$

subject to the boundary condition $\psi(0) = 1$.

This is a characterization of the exponential distribution, since the distribution of a non-negative random variable is determined by its Laplace transform.

Recall that the equilibrium distribution of a distribution F is $T_F = \frac{1}{\mu}\int_0^t \bar{F}(x)dx$, defined in (2.8).

Now, for T_F, since $\varepsilon_F(0) = \mu$,

$$\bar{T_F}(t) := 1 - T_F(t) = \frac{1}{\varepsilon_F(0)}\int_t^\infty \bar{F}(x)dx.$$

From this it follows that

$$\bar{T_F}(t)/\bar{F}(t) = \varepsilon_F(t)/\varepsilon_F(0).$$

Recall from Section 2.3.1, p. 25 that F is NBUE if and only if $\varepsilon_F(t) \leq \varepsilon_F(0)$ with strict inequality for at least one t; thus the following theorem holds:

Theorem 3.31 *Let T_F be the equilibrium distribution corresponding to F, then $\bar{T_F}(t) \leq \bar{F}(t)$ if and only if F is NBUE, and $\bar{T_F}(t) = \bar{F}(t)$ if and only if F is exponential.*

Kanjo (1993) used Theorem 3.31 to develop a test for exponentiality against NBUE alternatives defined in (4.135).

Grzegorzewski and Wieczorkowski (1999) used the following characterization in developing a goodness-of-fit test for exponentiality:

Theorem 3.32 *If X is a random variable with $P(X > 0) = 1$ and its mean $E(X) = \mu$ is given, then*

$$H(f) \leq 1 + \log \mu,$$

where $H(f) := -\int_{-\infty}^{\infty} f(x) \log f(x) dx$ is the entropy of X. And among all random variables with densities concentrated on $(0, +\infty)$ the exponential distribution with mean μ maximises $H(f)$ to $1 + \log \mu$.

Henze and Meintanis (2002a) proposed a new class of consistent tests for exponentiality defined in (4.77) based on the characteristic function

$$\phi(t) := E[e^{itX}] = \int_{-\infty}^{\infty} e^{itx} dF(x) = C(t) + iS(t), \qquad i = \sqrt{-1}, \tag{3.16}$$

of a non-negative random variable X, where $C(t) = E[cos(tX)]$ denotes the real part and $S(t) = E[sin(tX)]$ denotes the imaginary part of $\phi(t)$. They used the following characterization of the exponential distribution, which was also proved by Meintanis and Iliopoulos (2003):

Theorem 3.33 *Among all distributions of continuous non-negative random variables that possess a continuously differentiable density with finite limit as $x \to 0^+$ and absolutely integrable derivatives, the exponential distribution with mean $1/\theta$ is the only one that satisfies the equation*

$$S(t) - \frac{1}{\theta} tC(t) = 0, \quad t \in \mathbb{R}.$$

Another characterization of the exponential distribution proved by Meintanis and Iliopoulos (2003) and used by Henze and Meintanis (2005) to develop a goodness-of-fit test given in (4.80) is presented in the following theorem:

Theorem 3.34 *Let $|\phi(t)| = \sqrt{C(t)^2 + S(t)^2}$ be the modulus of the characteristic function of X. Then the distribution of X is exponential if and only if $\phi(t)$ satisfies*

$$|\phi(t)|^2 = C(t), \quad t \in \mathbb{R}.$$

Chapter 4

A discussion of existing goodness-of-fit tests for exponentiality

4.1 Introduction

Consider a non-negative continuous random variable X with distribution F, representing the lifetime of some component or system. Suppose a random sample $X_1, \ldots, X_n$ of lifetimes from F is observed and let $X_{1:n}, \ldots, X_{n:n}$ denote the order statistics. Considerable attention has been given in literature to the problem of testing the hypothesis that $X_1, \ldots, X_n$ are independent identically distributed (i.i.d.) exponential random variables.

The composite null hypothesis to be tested is

$$H_0 : F(x) = 1 - e^{-\theta x}, \qquad x > 0, \tag{4.1}$$

with $\theta > 0$.

The hypothesis can also be generalised in terms of a two parameter exponential distribution, i.e.,

$$H_0 : F(x) = 1 - e^{-\theta(x-\gamma)},\ x > \gamma,\ \theta > 0.$$

Spurrier (1984) discussed a transformation which allows one to use many tests designed for testing one parameter exponentiality to test for two parameter exponentiality.

In this chapter we summarize and discuss existing goodness-of-fit tests for exponentiality, many of them based on the characterizations presented in Chapter 3. Omnibus tests are discussed in Section 4.2, while tests for exponentiality against the different nonparametric classes of distributions defined in Chapter 2 are discussed in Section 4.3 to Section 4.7.

4.2 Omnibus tests for exponentiality

4.2.1 Introduction

Tests for exponentiality which aim at detecting *all* distributional departures from exponentiality are of great importance, since the alternatives to the exponential distribution are often not known in practice. These types of tests are called omnibus tests.

Let $F_0(x) = 1 - e^{-\theta x}$, $\theta > 0$. The omnibus test considers the hypothesis

$$H_0 : F(x) = F_0(x) \tag{4.2}$$

against one of three different alternatives:

$$H_1^{(a)} : F(x) \neq F_0(x) \tag{4.3}$$

$$H_1^{(b)} : F(x) \geq F_0(x) \quad \text{and} \quad F(x) \neq F_0(x) \quad \text{for some } x \tag{4.4}$$

$$H_1^{(c)} : F(x) \leq F_0(x) \quad \text{and} \quad F(x) \neq F_0(x) \quad \text{for some } x \tag{4.5}$$

The many omnibus tests for exponentiality available in literature are discussed under the following categories:

- Omnibus tests based on the empirical distribution function;
- Omnibus tests based on the lack-of-memory property;
- Omnibus tests based on transformations to uniformity;
- Omnibus tests based on normalized spacings;
- Omnibus tests based on order statistics;
- Omnibus tests based on the mean residual life function;
- Omnibus tests based on the sample Lorenz curve and Gini statistic;
- Omnibus tests based on the sample entropy, sample redundancy and other information theoretic measures;
- Omnibus tests based on the empirical Laplace transform and characteristic function.

4.2.2 Omnibus tests based on the empirical distribution function

One of the earliest approaches in defining goodness-of-fit tests was to use statistics based on the distance between a completely specified distribution function $F_0(x)$ and the empirical distribution function

$$F_n(x) := \frac{1}{n}\sum_{i=1}^{n} I\left(X_i \leq x\right). \tag{4.6}$$

These tests have the property of being *distribution-free*, in the sense that under the null hypothesis the distribution of the test statistic does not depend on any unknown parameters. Tests of this form include the well-known Kolmogorov-Smirnov statistic, the Cramér-von Mises statistic, and the Anderson-Darling statistic.

Kolmogorov-Smirnov statistics

The one- and two-sided Kolmogorov-Smirnov statistics are

$$D_n^+ = \sup_{-\infty\leq x\leq\infty} \{F_n(x) - F_0(x)\} = \max_{1\leq x\leq n} \left\{\frac{i}{n} - F_0(X_{i:n})\right\}, \tag{4.7}$$

$$D_n^- = \sup_{-\infty\leq x\leq\infty} \{F_0(x) - F_n(x)\} = \max_{1\leq x\leq n} \left\{F_0(X_{i:n}) - \frac{i-1}{n}\right\}, \tag{4.8}$$

$$D_n = \sup_{-\infty\leq x\leq\infty} |F_n(x) - F_0(x)| = \max\left\{D_n^+, D_n^-\right\}. \tag{4.9}$$

Large values of D_n lead to rejection of H_0.

Darling (1957) considered the distributions of these statistics, which are not asymptotically normal. Tables of percentiles of D_n for various values of n are given by e.g. Birnbaum (1952) and Miller (1956).

The Kolmogorov-Smirnov statistics ((4.7)-(4.9)) can however only be used to test whether a set of observations are from some completely specified continuous distribution. **Lilliefors (1969)** presented a procedure with a table of critical values for the Kolmogorov-Smirnov statistic when testing that a set of observations is from an exponential distribution with the population mean *unspecified*. According to Durbin (1975), much more extensive Monte Carlo experiments supplemented by sophisticated smoothing and other devices were carried out by M.A. Stephens and the resulting critical values were presented in Table 54 of Pearson and Hartley (1972).

Other authors who considered Kolmogorov-Smirnov type statistics were Srinavasan (1970), Finkelstein and Schafer (1971) and Stephens (1974).

Cramér-von-Mises statistic

The Cramér-von-Mises test statistic is defined as

$$\omega_n^2 := n \int_{-\infty}^{\infty} \{F_n(x) - F_0(x)\}^2 \, dF_0(x),$$

with its computational formula given by

$$\omega_n^2 = \frac{1}{12n} + \sum_{i=1}^{n} \left\{ F_0(X_{i:n}) - \frac{(2i-1)}{2n} \right\}^2 . \tag{4.10}$$

This statistic gives a consistent test of the two-sided hypothesis given in (4.3). The asymptotic null distribution is non-normal and was discussed by Anderson and Darling (1952).

Van Soest (1969) used the Cramér-von-Mises statistic

$$C_n = \frac{1}{12n} + \sum_{i=1}^{n} \left\{ F^*(X_{i:n}) - \frac{(2i-1)}{2n} \right\}^2 , \tag{4.11}$$

where $F^*(x) = 1 - e^{-x/\bar{X}}$.

Exponentiality is rejected if C_n is significantly large. Tables of critical values were given. Power comparisons showed that the test based on C_n is better than the tests WE_0 given in (4.51) and WE given in (4.52) (Hahn and Shapiro 1967) and two tests discussed by Epstein (1960).

For unspecified or only vaguely specified alternatives the tests based on D_n defined in (4.9) and ω_n^2 defined in (4.10) will on the average have good power (Cox and Lewis 1966). However, for very specific alternatives to the null hypothesis it will often be possible to derive more powerful tests than these two tests.

Anderson-Darling statistic

Cox and Lewis (1966) argued that the tests based on D_n given in (4.9) and ω_n^2 given in (4.10) are most sensitive to departures in the middle of the range of values. To prevent this, D_n or the kernel of the integral in ω_n^2 can be multiplied by a non-negative weight function $\kappa\{F_0(x)\}$:

$$W_n^2 := n \int_{-\infty}^{\infty} [F_n(x) - F_0(x)]^2 \, \kappa\,[F_0(x)] \, dF_0(x). \tag{4.12}$$

The most well-known weight function $\kappa(t) = \frac{1}{[t(1-t)]}$ was proposed by **Anderson and Darling (1952)** .

Integrated empirical distribution function

The distribution of a positive random variable X, with $E(X) < \infty$, is uniquely determined by its integrated distribution function (idf), defined as

$$\xi(t) := \int_t^\infty (1 - F(x))\,dx. \tag{4.13}$$

An omnibus test for the distribution may therefore be based on an L_2-difference between the idf and its empirical counterpart. **Klar (2001)** studied tests based on the idf for both the exponential and the normal distributions.

The integrated distribution function of the exponential distribution with mean $1/\theta$ is given by

$$\xi(t) = \frac{1}{\theta}e^{-\theta t},$$

with its empirical counterpart given by

$$\xi_n(t) := \int_t^\infty (1 - F_n(x))\,dx = \frac{1}{n}\sum_{j=1}^n (X_j - t)I(X_j > t). \tag{4.14}$$

Let T_n be the L_2-distance measure between $\xi_n(t)$, given in (4.14), and $\xi(t)$, then we have that

$$T_n = \frac{n}{2} - 2\sum_{i=1}^n e^{-Y_i} - (3n)^{-1}\sum_{i=1}^n (n-i+1)Y_{i:n}^3 + n^{-1}\sum_{i<j} Y_{i:n}^2 Y_{j:n},$$

where $Y_j = X_j/\bar{X}_n$.

T_n is scale-invariant and consistent against each alternative distribution with finite positive expectation. Klar (2001) also derived the asymptotic null distribution of T_n.

Adding a weight function e^{-at}, with $a > 0$, to his first test statistic, Klar (2001) obtained the modified test statistic which can be written as

$$\begin{aligned} T_{n,a} &= \frac{2(3a+2n)}{(2+a)(1+a)^2} - 2a^3\sum_{j=1}^n \frac{\exp(-(1+a)Y_j)}{(1+a)^2} - \frac{2}{n}\sum_{j=1}^n \exp(-aY_j) \\ &\quad + \frac{2}{n}\sum_{j<k}[a(Y_{k:n} - Y_{j:n}) - 2]\exp(-aY_{j:n}). \end{aligned} \tag{4.15}$$

H_0 is rejected for large values of $T_{n,a}$ and a test based on $T_{n,a}$ is consistent against all alternative distributions with finite positive expectation.

Power calculations showed that the power of the test depends heavily on a. For different values of a, there exists alternatives for which the test is most powerful. Klar (2001) therefore considered combinations of two or more test statistics T_{n,a_j} and rejected H_0 if at least one of the tests based on T_{n,a_j} rejects the hypothesis. Klar (2001) recommended rejecting H_0 if at least one of the tests based on $T_{n,1}$ and $T_{n,10}$ rejects H_0. The performance of this test was similar to that of the Cramér-von-Mises test given in (4.10) and the Anderson-Darling test given in (4.12).

Klar (2001) stated that the power of the idf test could be improved by an adaptive choice of the weight function.

4.2.3 Omnibus tests based on the lack-of-memory property

Angus (1982) developed goodness-of-fit tests for exponentiality based on the order statistic characterization presented in Theorem 3.2, p. 33 which is closely related to the lack-of-memory property (3.1). Recall that $\bar{F}(2x) = \bar{F}^2(x)\ \forall x \geq 0$ if and only if F is exponential.

Based on this, Angus (1982) defined the following three measures of deviation:

$$T_1(F) := \sup_{x \geq 0} \left| \bar{F}(2x) - \bar{F}^2(x) \right|, \tag{4.16}$$

$$T_2(F) := \int_0^\infty \left[\bar{F}(2x) - \bar{F}^2(x) \right] dF(x), \tag{4.17}$$

and

$$T_3(F) := \int_0^\infty \left[\bar{F}(2x) - \bar{F}^2(x) \right] \mu^{-1} e^{-x/\mu} dF(x). \tag{4.18}$$

By replacing F with the empirical distribution function F_n, three distribution-free tests for exponentiality were obtained, denoted by $T_1(F_n)$, $T_2(F_n)$ and $T_3(F_n)$ respectively. The power of these tests was compared to the power of the following tests:

- Gail and Gastwirth's (1978a) Lorenz curve test given in (4.61),
- Durbin's (1961) modified Kolmogorov-Smirnov test given in (4.32),
- Proschan and Pyke's (1967) IFR test given in (4.86),
- Lilliefors's (1969) adapted Kolmogorov-Smirnov test (p. 49),
- Van Soest's (1969) adapted Cramér-von-Mises test given in (4.11),
- Moran's (1951) test given in (4.46),
- The F-tests of Gnedenko given in (4.40), Harris given in (4.41) and Lin and Mudholkar given in (4.42).

Angus's new tests compared well with all these tests, and in some cases outperformed several of the tests. The new tests have the advantage of not having parameters such as degrees of freedom with the F-tests and p in the Lorenz curve-test. The tests based on $T_2(F_n)$ and $T_3(F_n)$ seemed to be the best of the three tests, with the test based on $T_3(F_n)$ slightly superior. However, the test based on $T_2(F_n)$ is computationally the simplest. The asymptotic null distribution and the Bahadur asymptotic efficiency of $T_1(F_n)$ were considered by Nikitin (1996).

Ahmad and Alwasel (1999) also proposed a test for exponentiality based on Theorem 3.2. They defined a measure of deviation similar to (4.17), namely

$$\Delta_2(F) := \int_0^\infty \left[\bar{F}(2x) - \bar{F}^2(x)\right]^2 dF(x), \tag{4.19}$$

and used the empirical distribution function to obtain the test statistic

$$\Delta_2(F_n) = \frac{1}{n}\sum_{i=1}^{n}\left[\bar{F}_n\left(2X_{i:n} - \left(\frac{n-i}{n}\right)^2\right)\right]^2. \tag{4.20}$$

Although this is a distribution-free and scale invariant statistic, its limiting null distribution is not normal. Therefore Ahmad and Alwasel (1999) proposed another estimate of (4.19) based on the following perturbed estimate of $F(x)$:

$$F_{n,\gamma}(x) = \frac{1}{n}\sum_{i=1}^{n} C_{i,n}(\gamma) I(x \leq X_i) \tag{4.21}$$

where $\{C_{i,n}(\gamma)\}_{i=1}^{n}$, $n \geq 1$ is a triangular array of real numbers depending on a parameter $0 < \gamma \leq 1$ and satisfying $\frac{1}{n}\sum_{i=1}^{n} C_{i,n}(\gamma) \to 1$ and $\frac{1}{n}\sum_{i=1}^{n} C_{i,n}^2(\gamma) \to C^2(\gamma) > 1$ as $n \to \infty$ for all $0 < \gamma \leq 1$.

With the equation $\bar{F}(2x) = \bar{F}^2(x)$, $\forall x \geq 0$, extended to $\bar{F}(rx) = \bar{F}^r(x)$, $\forall x \geq 0$, for any integer $r \geq 2$, the measure of deviation becomes

$$\Delta_r(F) = \int_0^\infty \left[\bar{F}(rx) - \bar{F}^r(x)\right]^2 dF(x),\ r \geq 2. \tag{4.22}$$

Ahmad and Alwasel (1999) estimated $\Delta_r(F)$ by

$$\hat{\Delta}_r(F_{n,\gamma}) = \int_0^\infty \bar{F}_n^2(rx) dF_n(x) - 2\int_0^\infty \bar{F}_{n,\gamma}(rx)\bar{F}_{n,\gamma}^r(x) dF_{n,\gamma}(x) + \frac{1}{2r+1}, \tag{4.23}$$

which, for $r = 2$, can be written as

$$\begin{aligned}\hat{\Delta}_2(F_{n,\gamma}) &= \int_0^\infty \bar{F}_n^2(2x) dF_n(x) - 2\int_0^\infty \bar{F}_{n,\gamma}(2x)\bar{F}_{n,\gamma}^2(x) dF_{n,\gamma}(x) + \frac{1}{5} \\ &= n^{-3}\sum_j\sum_k\sum_l I(X_j > 2X_l) I(X_k > 2X_l) \\ &\quad -2n^{-4}\sum_i\sum_j\sum_k\sum_l C_{i,n}(\gamma)C_{j,n}(\gamma)C_{k,n}(\gamma)C_{l,n}(\gamma) \\ &\quad \times I(X_j > X_i) I(X_k > X_i) I(X_l > 2X_i) + \frac{1}{5}.\end{aligned} \tag{4.24}$$

$\hat{\Delta}_2(F_{n,\gamma})$ is asymptotically normal under both the null and the alternative hypotheses.

A simple choice for $C_{i,n}(\gamma)$ is to take it equal to $1+\gamma$ if i is odd, and $1-\gamma$ if i is even. Ahmad and Alwasel found that values of $\gamma \in [0.3, 0.8]$ is highly satisfactory for most usable levels of significance.

Comparing the power of this test to the powers of the tests considered by Ebrahimi, Habibullah and Soofi (1992) (p. 66), Ahmad and Alwasel found that their test is more powerful than most others for gamma and log-normal alternatives. For Weibull alternatives, their test is better than most tests reported by Ebrahimi et al. (1992). Since the test by Ahmad and Alwasel has limiting normal distributions under the null and the alternative hypotheses, they believed their test to be better.

Alwasel (2001) generalized the work of Ahmad and Alwasel (1999) by considering the lack-of-memory property (3.1) in its full form. Thus, he considered the measure of deviation

$$\Delta(F) := \int_0^\infty \int_0^\infty \left[\bar{F}(x+y) - \bar{F}(x)\bar{F}(y)\right]^2 dF(x)dF(y), \tag{4.25}$$

which is similar to the distance measure used by Hollander and Proschan (1972) in developing a test for exponentiality against NBU alternatives (Section 2.3.1).

In order to obtain asymptotic normality under both the null and the alternative hypothesis, Alwasel considered the same perturbed estimate of $F(x)$ given in (4.21) as was done by Ahmad and Alwasel (1999), instead of using the ordinary empirical distribution function. Thus, (4.25) was estimated by

$$\begin{aligned} \hat{\Delta}(F_{n,\gamma}) &= \int_0^\infty \int_0^\infty \bar{F}_n^2(x+y)dF_n(x)dF_n(y) - 2\int_0^\infty \int_0^\infty \bar{F}_{n,\gamma}(x+y)\bar{F}_{n,\gamma}(x)\bar{F}_{n,\gamma}(y)dF_{n,\gamma}(y) + \frac{1}{9} \\ &= n^{-4}\sum_j\sum_k\sum_l\sum_m I(X_j > X_l + X_m)I(X_k > X_l + X_m) \\ &\quad -2n^{-5}\sum_i\sum_j\sum_k\sum_l\sum_m C_{i,n}(\gamma)C_{j,n}(\gamma)C_{k,n}(\gamma)C_{l,n}(\gamma)C_{m,n}(\gamma) \\ &\quad \times I(X_i > X_l)I(X_j > X_m)I(X_k > X_l + X_m) + \frac{1}{9}. \end{aligned} \tag{4.26}$$

The same recommendations regarding the choice of $C_{i,n}(\gamma)$ and γ were made as in Ahmad and Alwasel (1999). Monte Carlo methods were used to obtain critical values for various values of γ. The power of the new test based on (4.26) was found to be much higher than the power of the test by Ebrahimi et al. (1992) (p. 66) and as good as the power of the test by Ahmad and Alwasel (1999) given in (4.24) in most cases considered and slightly better in other cases.

4.2.4 Omnibus tests based on transformations to uniformity

The well-known probability integral transformation states that if the random variable X has distribution $F(x)$, then the random variable $U = F(X)$ is uniformly distributed over $(0, 1)$. The problem of testing for exponentiality, i.e. $H_0 : F = F_0$, where F_0 is a standard exponential distribution function, can thus be reduced to one of testing for uniformity.

Let $G_n(u)$ denote the empirical distribution derived from $U_{i:n} = F_0(X_{i:n})$, $i = 1, \ldots, n$. The Kolmogorov-Smirnov, Cramér-von-Mises and Anderson-Darling statistics discussed in Section 4.2.2 then become

$$D_n^+ = \sup_{0\leq u\leq 1} \{G_n(u) - u\} = \max_{1\leq i\leq n} \left\{\frac{i}{n} - U_{i:n})\right\}, \tag{4.27}$$

$$D_n^- = \sup_{0\leq u\leq 1} \{u - G_n(u)\} = \max_{1\leq i\leq n} \left\{U_{i:n} - \frac{i}{n})\right\}, \tag{4.28}$$

$$\omega_n^2 = n \int_0^1 \{G_n(u) - u\}^2 \, du, \tag{4.29}$$

and

$$\begin{aligned} W_n^2 &= n \int_0^1 [G_n(u) - u]^2 \frac{1}{u(1-u)} du \\ &= -n - \frac{1}{n} \sum_{i=1}^{n} [(2i-1) \log U_{i:n} + \{2(n-i)+1\} \log(1 - U_{i:n})] \,. \end{aligned} \tag{4.30}$$

Durbin (1961) considered the following further transformation of the order statistics $U_{i:n}$:

$$U'_{i:n} = \sum_{r=1}^{i} g_r = C_{1:n} + \ldots + C_{(i-1):n} + (n+2-i)C_{i:n}, \; i = 1, \ldots, n, \tag{4.31}$$

where $C_1 = U_{1:n}$, $C_i = U_{i:n} - U_{(i-1):n}$, $C_{n+1} = 1 - U_{n:n}$ and $g_i = \left(C_{i:n} - C_{(i-1):n}\right)$.
His modified Kolmogorov test statistic

$$K_m = \max_{r=1,\ldots,n} \left(\frac{r}{n} - U'_{r:n}\right), \tag{4.32}$$

which is to be used as a one-sided test, proved to be more powerful than the unmodified Kolmogorov test.

For another test for exponentiality based on the probability integral transformation, the reader is referred to **Gombay and Horvath (1992)** .

Other transformations to uniformity

Seshadri et al. (1969) considered two techniques of first transforming the observations and then testing for exponentiality. The first transformation is called the J transformation:

$$Z_i = \frac{\sum_{j=1}^{i} X_j}{\sum_{j=1}^{n} X_j}, \quad i = 1, \ldots, n-1. \tag{4.33}$$

The Z_i are uniformly distributed on $(0,1)$ if and only if $X_1, \ldots, X_n$ are exponential. The Kolmogorov-Smirnov test in (4.27) and Cramér-von-Mises test in (4.29) can then be used to test for uniformity.

The second transformation is called the K transformation:

$$W_i = \frac{\sum_{j=1}^{i} D_j}{\sum_{j=1}^{n} D_j}, \; i = 1, \ldots, n-1, \tag{4.34}$$

where D_j are the normalized spacings defined in (1.10). Note that W_i was already introduced in (1.15) in connection with the total time on test concept.

Like Z_i, the W_i are also uniformly distributed on $(0,1)$ if and only if $X_1, \ldots, X_n$ are exponential. Note that $\sum_{j=1}^{n} X_j = \sum_{j=1}^{n} D_j$, so that the only difference between (4.33) and (4.34) is in the numerators.

These two transformations were also discussed by Epstein (1960) amongst others. It was found that the K transformation should always be preferred to the J transformation, since it will generally give greater power.

Based on the transformation (3.3) and the subsequent characterization of the exponential distribution discussed in Section 3.3, **Wang and Chang (1977)** proposed the statistic

$$\chi_z^2 = -2 \sum_{j=1}^{n-1} \log\left(g(\tilde{Z}_j)\right), \tag{4.35}$$

where

$$g(z_j) = \begin{cases} 2z_j, & 0 \leq z_j \leq \frac{1}{2} \\ 2(1-z_j), & \frac{1}{2} < z_j \leq 1. \end{cases}$$

Under one parameter exponentiality the statistic has a chi-square distribution with $2(n-1)$ degrees of freedom. A Monte Carlo power study illustrated that a two sided test based on χ_z^2 generally outperforms the Kolmogorov-Smirnov-Lilliefors test (p. 49) for Weibull and gamma alternatives.

4.2.5 Omnibus tests based on normalized spacings

Transforming the data before testing for exponentiality can have many purposes, e.g. increasing the power of a test for certain alternatives. Following the probability integral transformation discussed in the previous section, one of the most commonly used transformations is the transformation to normalized spacings D_i, defined in (1.10.)

The first test based on normalized spacings is attributed to **Epstein (1960)** and is a direct consequence of Bartlett's test for homogeneity of variances. With D_i representing the good time between failures of k observations, the test statistic is defined as

$$l_k = \frac{2k\left\{\ln \frac{1}{k}\sum_{i=1}^{k} D_i - \frac{1}{k}\sum_{i=1}^{k} \ln D_i\right\}}{1+\frac{(k+1)}{6k}}, \tag{4.36}$$

which is approximately distributed under the null hypothesis as a chi-square variable with $(k-1)$ degrees of freedom.

When the ordered failure times are considered as k groups of size r, the j-th group has a grouped good time between failures of

$$V_j = \sum_{i=(j-1)r+1}^{jr} D_i, \tag{4.37}$$

and the test statistic, also due to **Epstein (1960)** is

$$\frac{2rk\left\{\ln \frac{1}{k}\sum_{i=1}^{k} V_i - \frac{1}{k}\sum_{i=1}^{k} \ln V_i\right\}}{1+\frac{(k+1)}{6rk}}. \tag{4.38}$$

Hartley's *F* max-test (Ascher 1990) also resulted from a test for homogeneity of variances. The failure times are grouped in the same k groups of size r, with V_i defined in (4.37). Then the test statistic is

$$U = \frac{\max_{1\le i\le k} V_i}{\min_{1\le i\le k} V_i}, \tag{4.39}$$

which is distributed under the null hypothesis as the F-distribution with $2r$ and k degrees of freedom.

Another grouped failure time test was developed by Gnedenko in 1960. Let $\sum_{i=1}^{r}$ and $\sum_{i=r+1}^{n}$ denote the sums of the first r and the last $n-r$ spacings respectively. Then **Gnedenko's *F*-test** (Lin and Mudholkar 1980) is based on the ratio

$$Q = \frac{\sum_{i=1}^{r} D_i/r}{\sum_{i=r+1}^{n} D_i/(n-r)}, \tag{4.40}$$

which has an F-distribution with $2r$ and $2(n-r)$ degrees of freedom under the null hypothesis. Fercho and Ringer (1972) found that Gnedenko's test in (4.40) is superior to Epstein's test given in (4.36) and Hartley's Fmax-test given in (4.39). They recommended using $r = n/2$.

Harris (1976) proposed a two-tailed version of Q by considering the sum of the $(n-2r)$ middle spacings, namely

$$Q' = \frac{\left(\sum_{i=1}^{r} D_i + \sum_{i=n-r+1}^{n} D_i\right)/(2r)}{\sum_{i=r+1}^{n-r} D_i/(n-2r)}, \tag{4.41}$$

which has an F-distribution with $4r$ and $2(n-2r)$ degrees of freedom under the null hypothesis. Lin and Mudholkar (1980) found that a test based on Q' is powerful against the lognormal distribution (which has a U-shaped hazard) but inferior for monotone hazards. Harris (1976) recommended using $r = n/4$.

Lin and Mudholkar (1980) proposed an alternative method by separately comparing the upper and lower sums of spacings to the middle sum. They defined

$$F_l = \frac{\sum_{i=1}^{r} D_i/r}{\sum_{i=r+1}^{n-r} D_i/(n-2r)} \quad \text{and} \quad F_u = \frac{\sum_{i=n-r+1}^{n} D_i/r}{\sum_{i=r+1}^{n-r} D_i/(n-2r)}. \tag{4.42}$$

F_l and F_u jointly follow a bivariate F-distribution under the null hypothesis. Lin and Mudholkar (1980) proposed accepting the null hypothesis if neither of the two statistics is too large or too small, i.e. rejecting exponentiality if either F_l or F_u is not within some interval which is determined by using a theorem from Hewett and Bulgren (1971). They recommended using $r = n/10$.

For n independent items put on test at the same time, the total time on test at $X_{i:n}$ was defined in (1.14), with the K-transformation W_i in (4.34) being discussed in Section 4.2.4, p. 56. The total time on test statistic (TTT-statistic), based on W_i, is defined as

$$V = \sum_{i=1}^{n-1} W_i. \tag{4.43}$$

Under the null hypothesis, V is distributed as the sum of uniform variables, which is very close to the normal distribution (Doksum and Yandell 1984, p. 590).

Recall from Theorem 3.13, p. 37, that the normalized spacings from the exponential distribution are i.i.d. exponential random variables with the same parameter as the original sample. Now, let $G_n(t)$ denote the empirical distribution function of the normalized spacings. Clearly, under the hypothesis of exponentiality, $F_n(t)$ and $G_n(t)$ should be close. **Jammalamadaka and Taufer (2003)** constructed new goodness-of-fit tests for exponentiality by measuring the distance between $F_n(t)$ and $G_n(t)$, using the classical Kolmogorov-Smirnov and Cramér-von-Mises type distances. Their test statistics are

$$T_{1,n} = \sqrt{\frac{n}{2}} \sup_{0 \le t < \infty} |F_n(t) - G_n(t)| \tag{4.44}$$

and

$$T_{2,n} = \frac{n}{2\bar{X}} \int (F_n(t) - G_n(t))^2 \, e^{-t/\bar{X}} dt. \tag{4.45}$$

The null hypothesis is rejected for large values of the statistics. These tests are consistent and scale-free.

Comparing results on Monte Carlo power calculations, the tests based on $T_{1,n}$ and $T_{2,n}$ seemed to compare well with the Kolmogorov-Smirnov statistic in (4.9), the Cramér-von-Mises statistic in (4.10), Gail and Gastwirth's Gini statistic in (4.65), Angus' tests in (4.16) and (4.17) based on the memoryless-property, Baringhaus and Henze's tests in (4.57) and (4.58) based on a characterization via the mean residual life, and the test of Ebrahimi et al. (1992) (p. 66) based on the sample entropy. Jammalamadaka and Taufer (2003) found that the statistics based on mean residual life and entropy show good power for most alternatives.

4.2.6 Omnibus tests based on order statistics

Bartholomew (1957) calculated the asymptotic relative efficiencies and compared the power of a test introduced by Moran (1951),

$$M(n-1) = -2\sum_{i=1}^{n} \log\left(\frac{X_i}{\bar{X}}\right) = -2\left\{\sum_{i=1}^{n} \log(X_i)\right\} + 2n\log\bar{X}, \tag{4.46}$$

and two other tests for exponentiality:

$$S = \frac{\sum_{i=1}^{n} X_i^2}{(\sum_{i=1}^{n} X_i)^2} = \sum_{i=1}^{n} \left(\frac{X_i}{n\bar{X}}\right)^2 \tag{4.47}$$

and

$$w = \sum_{i=1}^{n} \frac{|X_i - \bar{X}|}{(2n\bar{X})}. \tag{4.48}$$

The statistic S was proposed by Greenwood (1946) and w was proposed by Sherman (1950).

Bartholomew (1957) pointed out that the form of the alternative has a pronounced effect on the relative power of the tests. For certain alternatives $M(n-1)$ is the best test to use, while for other alternatives S is the best to use. Under circumstances where no information regarding the alternative distribution is known, he recommended w, because the maximum loss in power which might be incurred by a wrong choice of test, is minimised.

The main drawback of the test based on (4.46) is its sensitivity to recording errors in small observations (Cox and Lewis 1966, Bartholomew 1957). However, Moran did show that his test is asymptotically most powerful against gamma alternatives (see also Shorack (1972)).

Many of the distribution-free statistics discussed previously involve a great deal of computation, especially when using the transformed data as proposed by Durbin (1961) in Section 4.2.4, p. 55. **Lewis (1965)** proposed a modified mean test,which is based on a statistic that is much simpler to calculate, given by

$$S' = 2n - 2\sum_{i=1}^{n}\left(\frac{iX_{i:n}}{n\bar{X}}\right). \tag{4.49}$$

Note that if $U_k = \sum_{j=1}^{k} D_j / \sum_{j=1}^{n} D_j$, then S' can be written in terms of the normalized spacings defined in (1.10) as follows:

$$S' = \sum_{i=1}^{n-1} U_i,$$

(Henze and Meintanis 2005). Thus, U_k is equal to W_i defined in (1.15), and thus S' is equivalent to the total time on test statistic given in (4.43).

When testing exponentiality against gamma alternatives, the asymptotic relative efficiency of a test based on S' with respect to Moran's asymptotically most powerful test in (4.46) can be calculated. Combining this with the results of Bartholomew (1957), it follows that a test based on S' is not as good as a test based on Moran's test, but that it is slightly better than the test of Sherman (1950) defined in (4.48).

Jackson (1967) proposed a test statistic based on a direct comparison between the ordered observations and the corresponding expected values of the order statistics. The statistic is

$$T_n = \frac{\sum_{i=1}^{n}\sum_{j=1}^{i}(n-j+1)^{-1}X_{i:n}}{\sum_{i=1}^{n} X_{i:n}}. \tag{4.50}$$

An empirical comparison of the powers of T_n and S' in (4.49) indicated that T_n is better than S' for n up to about 35. Also taking into account the previously mentioned disadvantage of the statistic $M(n-1)$, Jackson recommended the use of T_n.

Hahn and Shapiro (1967, pp. 298-300) discussed two two-sided tests for exponentiality:

$$WE_0 = \frac{\sum_{i=1}^{n}\left(X_{i:n}-\bar{X}\right)^2}{\left(\sum_{i=1}^{n} X_{i:n}\right)^2} = \frac{\sum_{i=1}^{n} X_{i:n}^2 - \frac{1}{n}\left(\sum_{i=1}^{n} X_{i:n}\right)^2}{\left(\sum_{i=1}^{n} X_{i:n}\right)^2} \tag{4.51}$$

and

$$WE = \frac{\left(\bar{X} - X_{1:n}\right)^2}{\sum_{i=1}^{n}\left(X_{i:n}-\bar{X}\right)^2} = \frac{\left(\sum_{i=1}^{n} X_{i:n}/n - X_{1:n}\right)^2}{\sum_{i=1}^{n} X_{i:n}^2 - \frac{1}{n}\left(\sum_{i=1}^{n} X_{i:n}\right)^2}. \tag{4.52}$$

Tables of critical values are given by Hahn and Shapiro (1967, pp.334-335).

Shapiro and Wilk (1972) developed test procedures for exponentiality based on the same principles that they used in defining and extending test statistics for normality in the 1960's.

The statistic is defined as the ratio of the square of an appropriate linear combination of the sample order statistics to the usual sample variance:

$$W = \frac{n\left(\bar{X} - X_{1:n}\right)^2}{(n-1)\sum_{i=1}^{n}\left(X_{i:n} - \bar{X}\right)^2}. \tag{4.53}$$

As an omnibus procedure, W is to be used as a two-tailed statistic. However, if one can specify the class of alternatives, one may improve sensitivity by employing one or the other tail.

W is scale and origin invariant. The null distribution was studied by empirical sampling and Shapiro and Wilk presented tables of upper and lower tail percentile points for sample sizes from 3 to 100.

Sarkadi (1975) presented the analog of a test for normality developed by Shapiro and Francia in 1972 for testing two parameter exponentiality and showed that the test is consistent for omnibus alternatives. The test rejects exponentiality for small values of

$$W'' = \frac{\left[\sum_{i=1}^{n}\sum_{j=1}^{i} X_{i:n}/(n-j+1) - \sum_{i=1}^{n} X_{1:n}\right]^2}{\sum_{i=1}^{n}\left(X_{i:n} - \bar{X}\right)^2}. \tag{4.54}$$

Cox and Oakes (1984, pp. 43-45) introduced the score test

$$CO_n = n + \sum_{j=1}^{n} \log X_{j:n} - n\frac{\sum_{j=1}^{n} X_{j:n} \log X_{j:n}}{\sum_{j=1}^{n} X_{j:n}}, \tag{4.55}$$

which was based on a first partial derivative of the log likelihood function and which they claimed to be a useful test against alternative hypotheses which specify monotone hazard functions.

The statistic in (4.55) can also be written in simpler form as

$$CO_n = n + \sum_{j=1}^{n} (1 - Y_j)\log(Y_j),$$

where $Y_j = X_j/\bar{X}$ (Henze and Meintanis 2005).

Ascher (1990) found that the score test of Cox and Oakes appeared to be the best of all the procedures he compared. The test also did well in rejecting exponentiality for alternative distributions which are nearly exponential in shape. It is easy to calculate and can also accommodate censored data.

Patwardhan (1988) used the idea of measuring the departure of data by a plot of observed observations against their expectations to develop tests for exponentiality. He used the well known expression in (1.20) regarding the expected value of exponential order statistics from Section 3.6, p. 9.

Let $\delta_k := \frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{n-k+1}$ in (1.20), so that $E[X_{k:n}] = \frac{1}{\theta}\delta_k$.
Hence, a plot of $Z_k = X_{k:n}/\bar{X}$'s versus δ_k's should be a straight line with slope 1. The proposed statistic is written in simplified form in terms of the normalized spacings as

$$Q_1 = n(n+1)\frac{\sum_{i=1}^n D_i^2}{\left(\sum_{i=1}^n D_i\right)^2}. \tag{4.56}$$

Patwardhan also developed similar tests for the two parameter exponential distribution and the one parameter exponential distribution with right censoring.

4.2.7 Omnibus tests based on the mean residual life function

Using Theorem 3.27, **Baringhaus and Henze (2000)** proposed two new omnibus tests for exponentiality, taking a different approach based on the mean residual life function.
For large n, the random variables $Y_1, \ldots, Y_n$, with $Y_j = X_j/\bar{X}$, behave approximately like n independent exponential variables with mean 1. This motivated Baringhaus and Henze's proposal of the Kolmogorov-Smirnov type statistic

$$L_n = \sqrt{n}\sup_{z\geq 0}\left|\frac{1}{n}\sum_{k=1}^n \min(Y_k, z) - \frac{1}{n}\sum_{k=1}^n I(Y_k \leq z)\right| \tag{4.57}$$

and the Cramér-von-Mises type statistic

$$G_n = n\int_0^\infty \left(\frac{1}{n}\sum_{k=1}^n \min(Y_k, z) - \frac{1}{n}\sum_{k=1}^n I(Y_k \leq z)\right)^2 e^{-z}dz \tag{4.58}$$

as tests for exponentiality.
From Henze and Meintanis (2005), it follows that L_n can also be written as

$$L_n = \sqrt{n}\max\left(L_n^+, L_n^-\right),$$

where

$$L_n^+ = \max_{j=0,1,\ldots,n-1}\left[\frac{1}{n}\left(Y_{1:n} + \cdots + Y_{n:n}\right) + Y_{(j+1):n}\left(1 - \frac{j}{n}\right) - \frac{j}{n}\right],$$

and

$$L_n^- = \max_{j=0,1,\ldots,n-1}\left[\frac{j}{n} - \frac{1}{n}\left(Y_{1:n} + \cdots + Y_{n:n}\right) - Y_{j:n}\left(1 - \frac{j}{n}\right)\right],$$

while G_n can be written as

$$G_n = \frac{1}{n}\sum_{j,k=1}^n\left[2 - 3e^{-\min(Y_j,Y_k)} - 2\min(Y_j, Y_k)\left(e^{-Y_j} + e^{-Y_k}\right) + 2e^{-\max(Y_j,Y_k)}\right].$$

H_0 is rejected for large values of L_n or G_n.
The limiting null distributions of the test statistics are the same as the limiting null distributions of the classical Kolmogorov-Smirnov and Cramér-von-Mises test statistics (Section 4.2.2) when testing for uniformity in the unit interval. The tests based on L_n or G_n are consistent against each fixed alternative distribution having positive, possibly infinite, mean.

4.2.8 Omnibus tests based on the sample Lorenz curve and Gini statistic

Gail and Gastwirth (1978b) used the scale-free character of the Lorenz curve, which economists use to measure income inequality, to develop a goodness-of-fit test for exponentiality. Let μ be the mean of the underlying population with distribution F which is strictly increasing on its support. The population Lorenz curve is defined as

$$\lambda(p) := \mu^{-1} \int_0^{F^{-1}(p)} x dF(x) = \mu^{-1} \int_0^p F^{-1}(t) dt, \; 0 \leq p \leq 1. \tag{4.59}$$

If $F(x) = 1 - e^{-\theta x}, \; x \geq 0$, then

$$\lambda(p) = \theta \int_0^p \left[-\theta^{-1} \log(1-t) \right] dt = p + (1-p)\log(1-p). \tag{4.60}$$

Let $r = [np]$ denote the greatest integer less than or equal to np, then the sample Lorenz curve is defined as

$$L_n(p) := \frac{\sum_{i=1}^{r=[np]} X_{i:n}}{\sum_{i=1}^{n} X_{i:n}}, \tag{4.61}$$

where $0 < p < 1$.

The scale-free test based on $L_n(p)$ was proven to be consistent and $p = 0.5$ was recommended. The power of $L_n(0.5)$ was compared to the powers of the following tests for exponentiality: The Kolmogorov-Smirnov tests D_n in (4.9), D_n^+ in (4.7) and D_n^- in (4.8), Proschan and Pyke's (1967) IFR test in (4.86), Lilliefors' (1969) two-sided Kolmogorov-Smirnov test (Section 4.2.2, p. 49), Durbin's (1975) modified Kolmogorov-Smirnov test in (4.32), Shapiro and Wilk's (1972) test in (4.53), and Moran's (1951) test in (4.46).

The $L_n(0.5)$-test compared favourably with the Durbin Kolmogorov-Smirnov test against all alternatives except the Pareto. The Moran statistic is more powerful than $L_n(0.5)$ against gamma, Weibull and shifted exponential alternatives, and less powerful against uniform, Pareto and shifted Pareto alternatives; however, the differences are not great.

The Lorenz test substantially outperformed the IFR test against all alternatives, and it outperformed D_n, D_n^+ and D_n^- against all but uniform, Pareto and shifted Pareto alternatives. The Lorenz test also outperformed the Shapiro-Wilk test against all but uniform and shifted Pareto alternatives.

Gail and Gastwirth (1978a) developed another idea based on the sample Lorenz curve by studying the Gini and Pietra statistics, based on the Gini index and the Pietra ratio. The Gini index was introduced by Gini (1921) and is defined to be twice the area between the equiangular line and the population Lorenz curve. The Pietra ratio is defined to be twice the area of the largest triangle which can be inscribed in the area bounded by the equiangular line and the population Lorenz curve.

The Gini statistic is:

$$G_n = \frac{\sum\sum |X_i - X_j|}{2n(n-1)\bar{X}}, \tag{4.62}$$

and the Pietra statistic is:

$$P_n = \frac{\sum_{i=1}^{n} |X_i - \bar{X}|}{2n\bar{X}}. \tag{4.63}$$

They also considered the scale-free statistic

$$R_n = \bar{X}\left\{\frac{(n-1)}{\sum_{i=1}^{n}(X_i - \bar{X})^2}\right\}^{\frac{1}{2}}, \tag{4.64}$$

which is similar to the test of Shapiro and Wilk (1972) defined in (4.53).

Note that the Gini statistic in (4.65) can also be written as

$$\begin{aligned} G_n &= \frac{\sum_{i=1}^{n-1} iD_{i+1}}{(n-1)\sum_{i=1}^{n} D_i} \\ &= \frac{\sum_{i=1}^{n-1} i(n-i)\left(X_{(i+1):n} - X_{i:n}\right)}{(n-1)\sum_{i=1}^{n} X_{i:n}}. \end{aligned} \tag{4.65}$$

Monte Carlo simulations showed that G_{20} had greater power than P_{20}, $L_{20}(0.5)$ in (4.61) and R_{20} against most alternatives. The Gini statistic shares many advantages of $L_n(0.5)$ including ease of computation, the scale-free property, availability of exact distribution theory and robustness to truncation and rounding measurement error.

Gail and Gastwirth (1978a) noted a surprising connection between the Gini statistic and the total time on test statistic, V defined in (4.43), namely that $G_n = 1 - (n-1)^{-1}V$.

In a paper by Chandra and Singpurwalla (1981), some interesting relationships between the sample Lorenz curve, the Gini statistic and the cumulative total time on test statistic $(n-1)^{-1}V$ are proven.

Spurrier (1984) noted that the test based on G_n is not consistent for the beta distribution with parameters 0.5 and 1, and Henze and Meintanis (2005) also gave an example to illustrate that the Gini-test is not universally consistent.

Nikitin and Tchirina (1996) calculated a general expression for the local Bahadur efficiency of the two-sided test based on G_n for a very broad class of alternatives. They found that the

Gini test is locally asymptotically optimal in the Bahadur sense for the alternative with the Makeham density. They also considered a test based on large values of the two-sided statistic

$$G_n^* = |G_n - 1/2| .$$

They related the Gini statistic in (4.65) with the characterization of the exponential distribution presented by Puri and Rubin (1970) (Theorem 3.4, p. 34) which explains the high efficiency properties of the Gini test.

They also proposed the more general Gini statistic of order $r > 0$, defined as

$$G_n^{(r)} = \frac{1}{\bar{X}} \left[\frac{1}{n(n-1)} \sum_{i \neq j} |X_i - X_j|^r \right]^{1/r} \tag{4.66}$$

as a statistic for testing exponentiality. However, the properties of the generalised Gini statistic for $r > 2$ are still unknown.

4.2.9 Omnibus tests based on the sample entropy, the sample redundancy and other information theoretic measures

The concept of entropy was introduced by C.E. Shannon in 1948 and is one of the fundamental notions of information theory, communication, pattern recognition, etc. For a continuous random variable X with density function f and mean μ, Shannon's entropy is defined as

$$H(X) \equiv H(f) := -\int_{-\infty}^{\infty} f(x) \log f(x) dx, \tag{4.67}$$

while the redundancy of X is defined as

$$R(X) := E\left(\frac{X}{\mu} \log \frac{X}{\mu}\right). \tag{4.68}$$

Now, the sample entropy of $Y_1, \ldots, Y_n$, where $Y_i = X_i / \sum_{j=1}^n X_j$, is defined as

$$H_n(Y) := -\sum_{i=1}^n Y_i \log Y_i. \tag{4.69}$$

Note that $H_n(Y)$ attains its maximum value $\log n$ when $Y_i = 1/n \; \forall \; i$. The difference between $H_n(Y)$ and its maximum value is called the sample redundancy R_n, i.e.

$$R_n := \log n - H_n(Y), \tag{4.70}$$

which can be written as

$$R_n = \frac{1}{n\bar{X}} \sum_{i=1}^n X_i \log X_i - \log \bar{X}. \tag{4.71}$$

Chandra, de Wet and Singpurwalla (1982) proposed the sample redundancy as a scale-free test for exponentiality and presented a table of quantiles. They concluded that a test for exponentiality based on the sample redundancy defined in (4.71) performs as well as the test based on the Gini index in (4.65) for gamma, Weibull, uniform and Pareto alternatives. Due to ease of computation and the slight advantage of power, they recommended the test based on redundancy over other tests available at that time.

For more tests based on entropy, redundancy and other information theoretic measures (e.g. Kullback-Leibler information) the reader is referred to Ebrahimi et al. (1992), Grzegorzewski and Wieczorkowski (1999), Taufer (2002) and Choi, Kim and Song (2004).

4.2.10 Omnibus tests based on the empirical Laplace transform and the characteristic function

Recall that the Laplace transform of a random variable X is defined as $\psi(t) = E[e^{-tX}]$. **Baringhaus and Henze (1991)** based their test for exponentiality on the characterization regarding the Laplace transform presented in Theorem 3.30, p. 45. Their ideas were extended by Baringhaus and Henze (1992), while Henze (1993) proposed another test statistic based on the empirical Laplace transform, defined as

$$\psi_n(t) = n^{-1}\sum_{j=1}^{n} e^{-tX_j}. \tag{4.72}$$

Connections between these tests were studied by Baringhaus, Gurtler and Henze (2000).

Henze and Meintanis (2002b) used the Laplace transform of the unit exponential distribution, $1/(1+t)$, to define the statistic

$$L_{n,a} = n\int_0^\infty \left((1+t)\psi_n(t) - 1\right)^2 e^{-at}dt.$$

This can be written as

$$L_{n,a} = n\int_0^\infty \left(\psi_n(t) - \frac{1}{1+t}\right)^2 (1+t)^2 e^{-at}dt, \tag{4.73}$$

which differs from the statistic of Henze (1993) only by a weight function, which is $(1+t)^2e^{-at}$ instead of e^{-at}. This leads to the more general statistic

$$W_{n,a} = n\int_0^\infty \left((1+t)\psi_n(t) - 1\right)^2 w(t)e^{-at}dt, \tag{4.74}$$

where $w(t) = O(t^k)$ as $t \to \infty$.

The test based on $W_{n,a}$ is consistent against any alternative distribution not degenerate at zero, provided that $w(t) > 0$ for each t.

The test statistics $L_{n,0.75}$ and $L_{n,1.0}$ were suggested as computationally simple, yet powerful, omnibus tests for exponentiality.

The characteristic function of a non-negative random variable X was defined in Section 3.9 as

$$\phi(t) := E[e^{itX}] = C(t) + iS(t), \tag{4.75}$$

with real part $C(t) = E[\cos(tX)]$ and imaginary part $S(t) = E[\sin(tX)]$. The empirical characteristic function of the scaled data $Y_1, \dots, Y_n$, where $Y_j = X_j/\bar{X}$, is defined as

$$\phi_n(t) := \frac{1}{n}\sum_{j=1}^{n} \exp(itY_j) = C_n(t) + iS_n(t), \tag{4.76}$$

where $C_n(t) = n^{-1}\sum_{j=1}^{n}\cos(tY_j)$ and $S_n(t) = n^{-1}\sum_{j=1}^{n}\sin(tY_j)$ are the real and the imaginary parts of $\phi_n(t)$ respectively.

The empirical characteristic function has been used widely in goodness-of-fit tests for symmetric random variables (see e.g. Csörgö and Heathcote (1987)) as well as in testing for independence of random variables (see e.g. Csörgö (1985)).

Henze and Meintanis (2002a) proposed a new class of consistent omnibus tests for exponentiality based on the characterization of the exponential distribution in terms of the characteristic function stated in Theorem 3.33, p. 46.

Their test rejects exponentiality for large values of the statistic

$$W_n = n\int_0^{\infty} [S_n(t) - tC_n(t)]^2\, w(t)dt, \tag{4.77}$$

where $w(\cdot)$ denotes a non-negative weight function satisfying $\int_0^{\infty} t^2 w(t)dt < \infty$.

A test based on W_n is consistent against all alternatives.

Henze and Meintanis (2002a) considered two weight functions, $w_1(t) = \exp(-at)$ and $w_2(t) = \exp(-at^2)$ (where a is a positive parameter), for which the integral in the equation of W_n can be written in a simple closed form. These weight functions produce the following two statistics:

$$\begin{aligned} T_{n,a} \;=\; & \frac{a}{2n}\sum_{j,k=1}^{n}\left[\frac{1}{a^2+(Y_j-Y_k)^2} - \frac{1}{a^2+(Y_j+Y_k)^2} - \frac{4(Y_j+Y_k)}{(a^2+(Y_j+Y_k)^2)^2}\right. \\ & \left. + \frac{2a^2-6(Y_j-Y_k)^2}{(a^2+(Y_j-Y_k)^2)^3} + \frac{2a^2-6(Y_j+Y_k)^2}{(a^2+(Y_j+Y_k)^2)^3}\right] \end{aligned} \tag{4.78}$$

and

$$
\begin{aligned}
D_{n,a} &= \frac{\sqrt{\pi}}{4n\sqrt{a}} \sum_{j,k=1}^{n} \left[\left(1 + \frac{2a - (Y_j - Y_k)^2}{4a^2}\right) \exp\left(\frac{-(Y_j - Y_k)^2}{4a}\right) \right. \\
&\quad \left. + \left(\frac{2a - (Y_j + Y_k)^2}{4a^2} - \frac{Y_j + Y_k}{a} - 1\right) \exp\left(\frac{-(Y_j + Y_k)^2}{4a}\right) \right]. \qquad (4.79)
\end{aligned}
$$

The parameter a offers some flexibility with respect to the power of the tests based on (4.78) and (4.79). A small value of a means that the weight function decays slowly, whereas choosing a larger value of a speeds up the rate at which the weight functions decay. Baringhaus and Henze (1991) noted that a small value of the weight parameter should render the tests more powerful against alternatives having a point mass or an infinite density at $x = 0$, while a large value of a should result in powerful tests against distributions with markedly different tail behaviour compared to the exponential distribution.

The power of the tests based on (4.78) and (4.79) (for $a \in \{0.5, 0.75, 1.0, 1.5, 2.5\}$) were compared to the power of the following tests for exponentiality:

- The tests of Baringhaus and Henze (1991) and Henze (1993) (p. 66) based on the empirical Laplace transform;
- The test given in (4.108) of Epps and Pulley (1986) against IFR alternatives based on the empirical characteristic function;
- The Cramér-von-Mises test in (4.10);
- The Kolmogorov-Smirnov and Cramér-von-Mises type statistics in (4.57) and (4.58) of Baringhaus and Henze (2000).

The main conclusions were as follows:

- Against Weibull and gamma alternatives the tests based on (4.78) and (4.79) are inferior to the tests of Baringhaus and Henze (1991) and Henze (1993).
- The test based on (4.78) outperformed all its competitors for a small value of a under the lognormal alternative with parameter $\theta = 0.8$, while a larger value of a is required to make this test comparable to the other test procedures under the lognormal distribution with $\theta = 1.5$;
- For $a \geq 1$, the test based on $D_{n,a}$ is the most powerful;

- Under a power distribution with parameter $\theta = 1.0$ all tests performs satisfactorily, with $D_{n,a}$ being the most powerful. And $D_{n,a}$ is also the most powerful test against modified extreme value alternatives.

Henze and Meintanis (2005) also proposed a test for exponentiality based on the empirical characteristic function, $\phi_n(t)$ defined in (4.76). They used Theorem 3.34, p. 46 by Meintanis and Iliopoulos (2003) to define the statistic

$$T_n = n \int_0^{\infty} \left(|\phi_n(t)|^2 - C_n(t)\right)^2 w(t)dt, \tag{4.80}$$

where $w(\cdot)$ is a nonnegative weight function. The limiting null distribution of T_n is given, under the condition $\int_0^{\infty} t^4 w(t)dt < \infty$.

For the weight functions $w_1(t)$ and $w_2(t)$ (p. 67) the resulting statistics are:

$$\begin{aligned} T_{n,a}^{(1)} &= \frac{a}{n} \sum_{j,k=1}^{n} \left[\frac{1}{a^2 + Y_{jk-}^2} + \frac{1}{a^2 + Y_{jk+}}\right] \\ &-\frac{2a}{n^2} \sum_{j,k=1}^{n} \sum_{l=1}^{n} \left[\frac{1}{a^2 + [Y_{jk-} - Y_l]^2} + \frac{1}{a^2 + [Y_{jk-} + Y_l]^2}\right] \\ &+\frac{a}{n^3} \sum_{j,k=1}^{n} \sum_{l,m=1}^{n} \left[\frac{1}{a^2 + [Y_{jk-} - Y_{lm-}]^2} + \frac{1}{a^2 + [Y_{jk-} + Y_{lm-}]^2}\right] \end{aligned} \tag{4.81}$$

and

$$\begin{aligned} T_{n,a}^{(2)} &= \frac{1}{2n}\sqrt{\frac{\pi}{a}} \sum_{j,k=1}^{n} \left[\exp\left(-\frac{Y_{jk-}^2}{4a}\right) + \exp\left(-\frac{Y_{jk+}^2}{4a}\right)\right] \\ &-\frac{1}{n^2}\sqrt{\frac{\pi}{a}} \sum_{j,k=1}^{n} \sum_{l=1}^{n} \left[\exp\left(-\frac{[Y_{jk-} - Y_l]^2}{4a}\right) + \exp\left(-\frac{[Y_{jk-} + Y_l]^2}{4a}\right)\right] \\ &+\frac{1}{2n^3}\sqrt{\frac{\pi}{a}} \sum_{j,k=1}^{n} \sum_{l,m=1}^{n} \left[\exp\left(-\frac{[Y_{jk-} - Y_{lm-}]^2}{4a}\right) + \exp\left(-\frac{[Y_{jk-} + Y_{lm-}]^2}{4a}\right)\right] \end{aligned} \tag{4.82}$$

where $Y_{jk-} = Y_j - Y_k$, $Y_{jk+} = Y_j + Y_k$ and $Y_j = X_j/\bar{X}$.

A test for exponentiality against IFR alternatives based on the empirical characteristic function was proposed by Epps and Pulley (1986) and is discussed in Section 4.3.

4.3 Goodness-of-fit tests for exponentiality against IFR or IFRA alternatives

In this section, the null hypothesis is still

$$H_0 : F(x) = 1 - e^{-\theta x}, \tag{4.83}$$

with $\theta > 0$ unspecified, while the alternative hypothesis is either

$$H_1^{(1)} : \; F \text{ is IFR, but not exponential} \tag{4.84}$$

or

$$H_1^{(2)} : \; F \text{ is IFRA, but not exponential.} \tag{4.85}$$

The IFR and IFRA classes of life distributions were defined and discussed in Section 2.2 of Chapter 2. Recall that:

- A distribution F is IFR if $\bar{F}(t+x)/\bar{F}(t)$ is decreasing in t for $x > 0$. Or, if F has density f, then F is IFR if the failure rate $r(t) = f(t)/\bar{F}(t)$ is increasing for $0 \leq t < \infty$; and
- The distribution F is IFRA if $-(1/t)\log \bar{F}(t)$ is increasing in $t > 0$. Or, if F has density f, then F is IFRA if the average failure rate $q(t) = \frac{1}{t}\int_0^t r(u)du$ is increasing for $0 \leq t < \infty$.

From Section 2.2 it was clear that the IFRA class plays an especially important role in reliability theory, since it is the smallest class of probability distributions containing the exponential distribution that is closed under the formation of coherent systems.

The literature contains many tests for exponentiality against IFR and IFRA alternatives. These tests are summarised and discussed here in chronological order. One-sided tests for IFR or IFRA alternatives were developed by Proschan and Pyke (1967), Barlow (1968), Bickel and Doksum (1969), Ahmad (1975), Barlow and Campo (1975), Deshpande (1983), Klefsjö (1983a), Kochar (1985) and Ahmad (2001), while two-sided tests for IFR/DFR or IFRA/DFRA alternatives were developed by Lee, Locke and Spurrier (1980), Epps and Pulley (1986) and Link (1989)

Proschan and Pyke (1967) developed the first test statistic for exponentiality against IFR alternatives, based on the ranks of the normalized spacings $D_1, \ldots, D_n$, defined in (1.10). The test statistic is

$$V_n = \sum_{\substack{i,j=1 \\ i<j}}^{n} V_{i,j}, \tag{4.86}$$

where

$$V_{i,j} = \begin{cases} 1 & \textit{if } D_i \geq D_j,\ i,j = 1,2,\ldots,n \\ 0 & \textit{otherwise} \end{cases}$$

The null hypothesis is rejected in favour of $H_1^{(1)}$ for large values of V_n. The distribution of V_n is asymptotically normal under the null hypothesis.

Soon afterwards, **Barlow (1968)** developed likelihood ratio tests for testing exponentiality against IFR and IFRA alternatives. Based on a generalisation of the maximum likelihood estimate concept, he defined the following likelihood ratio statistic for testing exponentiality against IFR distributions:

$$\Lambda_n^*(X) = \left((n-1)/\sum_{i=1}^{n} X_i\right)^{n-1} \prod_{i=1}^{n-1} (n-i)(X_{(i+1):n} - X_{i:n}).$$

The test ϕ^* rejects exponentiality in favour of $H_1^{(1)}$ when $\Lambda_n^*(X)$ is small. This statistic is essentially the same as a conditional maximum likelihood ratio test developed in 1966 by M. Boswell, who also derived the asymptotic distribution. Barlow (1968) presented a table of critical values for small samples, obtained by using Monte Carlo methods.

A statistic related to $\Lambda_n^*(X)$ and which rejects exponentiality in favour of $H_1^{(1)}$ for sufficiently large values, was given by Barlow (1968) as

$$\Lambda_n^{**}(X) = \left(n/\sum_{i=1}^{n} X_{i:n}\right) \prod_{i=1}^{n} (n-i+1)(X_{i:n} - X_{(i-1):n}).$$

This test is related to (4.46), the test of Moran (1951). Further, note that

$$W = -2\log \Lambda_n^{**}(X)/[1 + (n+1)6n]$$

is exactly the test of Epstein (1960).

For testing against IFRA alternatives, Barlow (1968) considered:

$$\Lambda_n(X) = n^n \prod_{i=1}^{n} \left[1 - X_{i:n}/\sum_{i=1}^{n} X_{j:n}\right]^{\sum_{i=1}^{n} X_{j:n}/X_{i:n} - 1} \left[X_{i:n}/\sum_{i=1}^{n} X_{j:n}\right]. \tag{4.87}$$

The test ϕ^{**} rejects exponentiality in favour of $H_1^{(2)}$ when $\Lambda_n(X)$ is large. Although the test ϕ^{**} is not a likelihood ratio test, it is based on $\Lambda_n(X)$, which is essentially the maximum likelihood under the IFRA assumption. This may be the reason that ϕ^{**} performs better than the test ϕ^*.

Power comparisons indicated that ϕ^{**} has slightly greater power than the test based on the TTT-statistic (4.43). The test ϕ^* is distinctly inferior to the latter two tests, while the test V_n defined in (4.86) was seen to be distinctly inferior to ϕ^*.

Bickel and Doksum (1969) considered five rank tests of the form

$$W = n^{-1/2} \sum_{i=1}^{n} (c_n(i) - \bar{c}) J_n(R_i/(n+1)),$$

where R_i denotes the ranks of the normalized spacings D_i defined in (1.10), $-c_n(i)$ and $J_n(i)$ are nondecreasing functions in $i = 1, 2, \ldots$, and $\bar{c} = n^{-1} \sum_{i=1}^{n} c_n(i)$.

With the weights $c_n(i)$ of the form $c_n(i) = c\left(i(n+1)^{-1}\right)$ for some function c on $(0, 1)$, these test statistics are:

$$\begin{aligned}
W_0 &= \sum_{i=1}^{n} -(i/(n+1))\,(R_i/(n+1))\,; \\
W_1 &= \sum_{i=1}^{n} -(i/(n+1))\,[-\log(1 - R_i/(n+1))]\,; && (4.88) \\
W_2 &= \sum_{i=1}^{n} [-\log(1 - i/(n+1))]\,[-\log(1 - R_i/(n+1))]\,; && (4.89) \\
W_3 &= \sum_{i=1}^{n} -\{\log[-\log(1 - i/(n+1))]\}\,[-\log(1 - R_i/(n+1))]\,; \\
W_4 &= \sum_{i=1}^{n} g\,(i/(n+1))\,[-\log(1 - R_i/(n+1))]\,,
\end{aligned}$$

(4.90)

with $g(t) = (1-t)^{-1} \int_{-\log(1-t)}^{\infty} x^{-1} e^{-x} dx$.

They also proposed four statistics which are linear in the $D_i's$:

$$\begin{aligned}
S_1 &= \sum_{i=1}^{n} -(i/(n+1))\,D_i; && (4.91) \\
S_2 &= \sum_{i=1}^{n} [\log(1 - i/(n+1))]\,D_i; && (4.92) \\
S_3 &= \sum_{i=1}^{n} -\{\log[-\log(1 - i/(n+1))]\}\,D_i; && (4.93) \\
S_4 &= \sum_{i=1}^{n} g\,(i/(n+1))\,[-\log(1 - D_i/(n+1))]\,.
\end{aligned}$$

Large values of the statistics are significant for rejecting H_0 in favour of $H_1^{(1)}$.

W_0 is asymptotically equivalent to the statistic V_n (4.86) of Proschan and Pyke (1967), and is uniformly improved by W_1 asymptotically.

In general, the rank tests were uniformly less powerful than the corresponding studentised linear spacings tests, $S_i^* = S_i / \sum_{j=1}^{n} D_j,\ i = 1, 2, 3, 4$.

On the basis of Monte Carlo power studies and asymptotic efficiency studies, the statistics $S_1^* = S_1/\sum_{j=1}^n D_j$, with S_1 defined in (4.91), and $S_3^* = S_3/\sum_{j=1}^n D_j$, with S_3 defined in (4.93), generally performed the best. It was also seen that the power of S_3^* is about the same as Barlow's (1968) best test ϕ^{**}, the likelihood ratio test for IFRA alternatives based on (4.87).

Barlow and Doksum (1972) considered the class $\mathcal{F}$ of absolutely continuous distributions functions F such that $F(0) = 0$ with positive and right (or left) continuous density f on the interval where $0 < F < 1$. Then, for F and $G \in \mathcal{F}$, with respective densities f and g, they introduced the transformation

$$H_F^{-1}(t) = \int_0^{F^{-1}(t)} g\left[G^{-1}(F(u))\right] du, \; 0 \le t \le 1,$$

which was estimated by substituting the empirical distribution function F_n for F. Then

$$H_{F_n}^{-1}(\frac{i}{n}) \equiv H_n^{-1}\left(\frac{i}{n}\right) = n^{-1}\sum_{j=1}^{i} D_j.$$

That is, H_n^{-1} is just n^{-1} times T_i, the total time on test until the i-th observation (1.14).

To make the test scale invariant, Barlow and Doksum (1972) used the scaled total time on test

$$W_{i:n} = H_n^{-1}(i/n)/H_n^{-1}(1) = H_n^{-1}(i/n)/\bar{X}, \tag{4.94}$$

which is just the empirical scaled TTT-transform (1.16).

Barlow and Doksum (1972) considered three classes of test statistics:

1. General scores statistics of the form

   $$T_n(J) = n^{-1}\sum_{i=1}^{n} J[W_{i:n}] \tag{4.95}$$

   where J is an increasing function on $[0, 1]$.

   The test corresponding to $J(x) = x$ is called the uniform scores test and $n^{-1}\sum_{i=1}^n W_{i:n}$ is the cumulative TTT-statistic (4.43).

   Other general scores tests are Fisher's test with $J(x) = \log x$, the Pearson or exponential scores test with $J(x) = -\log(1-x)$, and the normal scores test with $J(x) = \Phi^{-1}(t)$, where $\Phi(t)$ is the standard normal distribution function.

   Barlow and Doksum (1972) showed that the statistics $T_n(J)$ are asymptotically equivalent to the classes of statistics considered by Bickel and Doksum (1969) (p. 72). Consequently, for a given parametric family of IFR alternatives, it is possible to find a $J = J_{F_\theta}$ such that the test that rejects exponentiality for large values of $T_n(J)$ is asymptotically most powerful.

2. Systematic statistics of the form

$$n^{-1}\sum_{i=1}^{n} L\left(\frac{i}{n}\right) W_{i:n}, \tag{4.96}$$

where $L(u) \geq 0$. When $L\left(\frac{i}{n}\right) = 1$, it is the cumulative TTT-statistic in (4.43). Bickel and Doksum (1969) considered selected types of such statistics for exponentiality (p. 72).

3. The one-sided Kolmogorov statistic in (4.28) suggested the distance function

$$D_n^- = \sup_{1\leq i\leq n}\left[W_{i:n} - \frac{i}{n}\right], \tag{4.97}$$

which is the same as the one-sided Kolmogorov statistic based on the K-transformation considered by Seshadri et al. (1969) (p. 56).

Ahmad (1975) presented a non-parametric test for exponentiality against monotone IFR alternatives based on U-statistics. His test statistic was based on a lemma which he used to show that F is IFR if and only if, for $x, y \geq 0$,

$$\bar{F}^2\left[(x+y)/2\right] \geq \bar{F}(x)\bar{F}(y).$$

Ahmad (1975) defined his measure of deviation from exponentiality as

$$\begin{aligned} \Delta_F &= \iint \left\{\bar{F}^2\left[(x+y)/2\right] - \bar{F}(x)\bar{F}(y)\right\} dF(x)dF(y) \\ &= \iint \bar{F}^2\left[(x+y)/2\right] dF(x)dF(y) - \frac{1}{4} \\ &:= \delta_F - \frac{1}{4} \end{aligned} \tag{4.98}$$

and then his test statistic is

$$\delta_{F_n} = \iint \bar{F}_n^2\left[(x+y)/2\right] dF_n(x)dF_n(y) \tag{4.99}$$

An equivalent form of δ_{F_n} is the following U-statistic:

$$U_n = \left[4/\left(n(n-1)(n-2)(n-3)\right)\right]\sum_C g\left[\min(X_{\alpha_1}, X_{\alpha_2}), (X_{\alpha_3}+X_{\alpha_4})/2\right], \tag{4.100}$$

where $\sum_C$ extends over all combinations $1 \leq \alpha_i \leq n$, $i = 1, \ldots, 4$ such that $\alpha_1 \neq \alpha_2, \alpha_1 \neq \alpha_4, \alpha_2 \neq \alpha_3, \alpha_2 \neq \alpha_4, \alpha_1 < \alpha_2$ and $\alpha_3 < \alpha_4$, with $g(x, y) = 1$ if $x > y$ and 0 otherwise.

H_0 is rejected in favour of $H_1^{(1)}$ for large values of U_n.

The U_n-test is consistent and unbiased against all IFR alternatives. Also, U_n is asymptotically normal under H_0. A study of the asymptotic relative efficiency of the U_n-test relative to the tests V_n in (4.86), W_1 in (4.88) and J_n in (4.113) of respectively Proschan and Pyke (1967), Bickel and Doksum (1969), Hollander and Proschan (1972) and the total time on test statistic V in (4.43), showed that the U_n-test performs well.

Barlow and Campo (1975) developed tests for exponentiality against IFRA alternativaes based on the total time on test plot, which is a plot of the points $(i/n, W_i)$, $i = 1, \ldots, n$, where W_i is as in (1.15).

Barlow and Campo (1975) proposed using L_n, the number of crossings between the TTT-plot and the 45°-line, as a test statistic, rejecting the hypothesis of exponentiality in favour of $H_1^{(2)}$ when L_n is small.

Bergman (1977) derived the exact and asymptotic distributions of L_n, but showed that a test based on the TTT-statistic given in (4.43) is uniformly more powerful than the test based on L_n for the class of Weibull alternatives and suspected it to be true in the more general IFRA class.

Lee et al. (1980) considered the class of statistics

$$T_\alpha = \left(n^{-1}\sum_{i=1}^{n} X_i^\alpha\right)/\bar{X}^\alpha, \quad 0 < \alpha < 1 \; or \; \alpha > 1, \tag{4.101}$$

which were first suggested by Kimball (1947) and also considered by Darling (1953), where they have been considered as omnibus tests using two-sided rejection regions. However, Lee et al. (1980) proved that tests based on T_α are inconsistent against certain kinds of non-exponential distributions and should therefore not be used as omnibus tests.

They proved further that, for $\alpha < 1$, lower-tail tests based on T_α are consistent for DFRA alternatives, while upper-tail tests are consistent for IFRA alternatives. On the other hand, for $\alpha > 1$, lower-tail tests are consistent for IFRA alternatives, and upper-tail tests are consistent for DFRA alternatives.

Based on a Monte Carlo power study, Lee et al. (1980) recommended a lower-tail test based on $T_{1/2}$ for DFRA alternatives. If the alternative hypothesis is that the distribution is IFRA, then a lower-tail test based on T_2 was recommended.

These two statistics, $T_{1/2}$ and T_2, outperformed the following test statistics: S_1^* in (4.91) and S_3^* in (4.93) of Bickel and Doksum (1969), T_n in (4.50) of Jackson (1967), χ_z^2 in (4.35) of Wang and Chang (1977), as well as the one-sided Kolmogorov-Smirnov statistics in (4.7) and (4.8).

Deshpande (1983) developed a test for exponentiality against IFRA alternatives based on property (2.1) of IFRA distributions. Recall that F is IFRA if and only if $\bar{F}(\alpha t) \geq \bar{F}^\alpha(t)$ for all $0 < \alpha < 1$ and $t \geq 0$.

He defined the parameter $M(F) = \int_0^\infty \bar{F}(bx)dF(x)$, so that $M(F) = (b+1)^{-1}$ if F belongs to H_0, and $M(F) > (b+1)^{-1}$ for all F belonging to $H_1^{(2)}$. Thus,

$$M(F) - (b+1)^{-1} \tag{4.102}$$

is a measure of deviation of F from H_0.

Deshpande (1983) then defined the function

$$h_b(X_1, X_2) = \begin{cases} 1, & X_1 > bX_2 \\ 0, & \text{otherwise,} \end{cases}$$

where b is a fixed number from $(0,1)$, which was used as kernel to define the U-statistic J_b, i.e.

$$J_b = (n(n-1))^{-1} \textstyle\sum^* h_b(X_i, X_j), \tag{4.103}$$

where $\sum^*$ denotes summation over $1 \le i \le n$, $1 \le j \le n$ such that $i \ne j$.

The value of the statistic ranges from 1/2 to 1, and H_0 is rejected in favour of $H_1^{(2)}$ for large values of J_b.

The statistic J_b has the advantage that it can easily be calculated as follows: Multiply each observation by b. Arrange $X_1, \ldots, X_n$ and $bX_1, \ldots, bX_n$ together in increasing order of magnitude. Let R_i be the rank of X_i in the combined order. Then

$$S = \sum_{i=1}^{n} R_i - \frac{1}{2}n(n+1) - n$$

is the number of pairs of (X_i, bX_j) for $i \ne j$, such that X_i is larger than bX_j, and

$$J_b = \{n(n-1)\}^{-1} S.$$

Deshpande (1983) proved that a test based on J_b is unbiased, and since J_b is a U-statistic, the asymptotic distribution of $n^{1/2}[J_b - M(F)]$ is normal. A test based on J_b is consistent against continuous IFRA distributions, and for $b = k^{-1}$, $k = 2, 3, \ldots$, the J_b-test is also consistent against continuous NBU distributions.

Following calculations regarding the Pitman asymptotic relative efficiency, the test $J_{0.9}$ was recommended when the alternative is the IFRA class, and $J_{0.5}$ was recommended when the alternative is the larger NBU class. The tests based on $J_{0.9}$ and $J_{0.5}$ outperformed the cumulative total time test in (4.91) of Bickel and Doksum (1969), as well as the NBU test in (4.113) of Hollander and Proschan (1972).

Bandyopadyay and Basu (1990) studied the efficiency properties of J_b and recommended $b = 0.44$ as the optimal value for b.

Klefsjö (1983a) defined tests for exponentiality against IFR and IFRA alternatives by using the fact that the TTT-plot (also considered by Barlow and Campo (1975), p. 75) converges to the scaled TTT-transform in (1.12) , together with Theorem 2.1, p. 20 and Theorem 2.3, p. 21.

He proposed two tests against IFR alternatives: A positive (negative) value of

$$A_1 = \sum_{j=1}^{n-1} (2W_j - W_{j-1} - W_{j+1}), \tag{4.104}$$

with W_i defined in (1.15), indicates that F is IFR (DFR) but not exponential. However, a test based on A_1 is not consistent against the whole IFR class.

The second test statistic is

$$A_2 = \sum_{j=0}^{n-2} \sum_{k=2}^{n-j} \sum_{\nu=1}^{k-1} \{k(W_{j+\nu} - W_j) - \nu(W_{j+k} - W_j)\},$$

which is positive (negative) if F is IFR (DFR) but not exponential.

A_2 can be written as

$$A_2 = \sum_{j=1}^{n} \alpha_j D_j / T_n, \tag{4.105}$$

where D_j are the normalized spacings, T_n is the total time on test at $X_{n:n}$ defined in (1.14) and $\alpha_j = \frac{1}{6}\left[(n+1)^3 j - 3(n+1)^2 j^2 + 2(n+1) j^3\right]$.

Klefsjö (1983a) proposed

$$B = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (jW_i - iW_j),$$

for testing exponentiality against IFRA alternatives. A positive (negative) value of B indicates that F is IFRA (DFRA).

Similar to A_2, B can be written as

$$B = \sum_{j=1}^{n} \beta_j D_j / T_n, \tag{4.106}$$

where $\beta_j = \frac{1}{6}\left[2j^3 - 3j^2 + j(1 - 3n - 3n^2) + 2n + 3n^2 + n^3\right]$.

Both A_2 and B were proven to be asymptotically normal, under general assumptions on the distribution F. A_2 is consistent against the class of IFR distributions and B is consistent against the class of IFRA distributions.

Kochar (1985) used the definition of an IFRA distribution (Def. 2.2, p. 20) to define

$$h(s,t) = s \log \bar{F}(t) - t \log \bar{F}(s), \quad s \geq t \geq 0.$$

Under H_0, $h(s,t) = 0$, but under $H_1^{(2)}$, $h(s,t) \geq 0$ for all $s \geq t \geq 0$. Thus, a measure of deviation between H_0 and $H_1^{(2)}$ was defined as

$$\Delta(F) = \iint_{s \geq t} h(s,t) dF(s) dF(t).$$

Using the empirical distribution function, F_n, the following asymptotically equivalent statistic was obtained:

$$\tilde{T}_n = \frac{1}{n} \sum_{i=1}^{n} J\left(\frac{i}{n+1}\right) X_{i:n}, \tag{4.107}$$

where $J(u) = 2(1-u)[1 - \log(1-u)] - 1$.

In order to make the statistic scale invariant, Kochar (1985) proposed $T_n = \tilde{T}_n / \bar{X}_n$. The test rejects H_0 in favour of $H_1^{(2)}$ for large values of T_n.

According to calculations on the Pitman asymptotic relative efficiency, the test based on T_n compares well to the B-test (p. 77) of Klefsjö (1983a), the tests $J_{0.9}$ and $J_{0.5}$ in (4.103) of Deshpande (1983) and the TTT-test in (4.43).

Comparing the powers of these tests for small sample sizes ($n < 7$) against Weibull alternatives, Kochar (1985) found that the T_n-test outperformed the B-test and the $J_{0.5}$-test. However, for large sample sizes the T_n-test had slightly smaller power than the B-test. When testing against linear IFR distributions, the new T_n-test was clearly superior to its competitors.

Epps and Pulley (1986) presented a test for exponentiality against IFR distributions, which they derived as a weighted integral of the difference between the empirical characteristic function $\phi_n(t)$ given in (4.76) and the exponential characteristic function $\phi_0(t) = \int e^{itx} dF_0(x)$. They defined

$$C(\hat{\theta}) = \int_{-\infty}^{\infty} D\left\{\phi_n(t), \phi_0(t)\right\} dW(t),$$

as a class of statistics for testing exponentiality. $D\{\cdot\}$ is an appropriate distance function and $W(t)$ is a complex-valued weighting function which smoothes the oscillations in $\phi_n(t)$.

Epps and Pulley (1986) used $D\{\phi_n, \phi_0\} = \phi_n - \phi_0$ and $dW(t) = (2\pi)^{-1}\phi_0(-t)dt$ to obtain the following scale-invariant test statistic:

$$EP_n = (48n)^{1/2} \left(n^{-1} \sum_{j=1}^{n} e^{-X_j/\bar{X}} - 1/2 \right), \tag{4.108}$$

which is asymptotically standard normally distributed under H_0.

H_0 is rejected for large values of $|EP_n|$, where large positive values indicate an IFR alternative, while large negative values indicate a DFR alternative. The test based on EP_n is consistent against all alternative distributions with monotone hazard rate, provided that F is absolutely continuous, $F(0) = 0$ and $0 < \mu < \infty$.

Monte Carlo power estimates showed that the test based on EP_n performes similar to the Gini-test in (4.65) of Gail and Gastwirth (1978a), and usually better than the S_3^*-test given in (4.93) of Bickel and Doksum (1969) as well as the NBU test in (4.113) of Hollander and Proschan (1972). Comparing the convenience of the test based on EP_n and the Gini-test, Epps and Pulley (1986) found that, although their test has a computational advantage, the Gini-test rapidly converges to normality and therefore has a smaller need for special tables.

Link (1989) developed a test for exponentiality against monotone failure rate alternatives that is related to the test J_b defined in (4.103). Following Deshpande (1983) (p. 76), he noted that $E[J_b]$ is equal to, less than, or greater than $(b+1)^{-1}$, for F exponential, F DFRA but not exponential, and F IFRA but not exponential, respectively. Thus large (small) values of J_b indicate that the average failure rate is increasing (decreasing).

Link (1989) then rewrote the statistic J_b in the form

$$J_b = \frac{1}{2} + (n(n-1))^{-1} \sum_{i \neq j} I\left(\frac{\min(X_i, X_j)}{\max(X_i, X_j)} > b\right),$$

and proposed the statistic

$$\Gamma = (n(n-1))^{-1} \sum_{i \neq j} \frac{\min(X_i, X_j)}{\max(X_i, X_j)}, \tag{4.109}$$

which is scale invariant and can be expressed in the simple form

$$\Gamma = \frac{2}{n(n-1)} \sum_{i<j} \frac{X_{i:n}}{X_{j:n}}.$$

The statistic Γ is a U-statistic estimate of $2\mu(F) - 1$, where

$$\mu(F) = \int_0^1 \int_0^\infty \bar{F}(tx) dF(x) dt.$$

The parameter $\mu(F)$ can be considered as a measure of departure from exponentiality in the class of monotone failure rate average distributions, since $\mu(F) = \log 2$ for F exponential, and is larger (smaller) for F IFRA (DFRA) but not exponential.

A Monte Carlo study of the small sample power of the new test Γ, the tests $J_{0.5}$ and $J_{0.9}$ defined in (4.103) of Deshpande (1983), and the T_n-test in (4.107) of Kochar (1985), against Weibull alternatives with $\nu = 1/2$ (DFRA) and $\nu = 2$ (IFRA) showed that Γ clearly outperformed $J_{0.5}$

in both cases, and that Γ is superior to T_n for $\nu = 1/2$. For the case $\nu = 2$ there was little difference in the power of Γ and T_n.

Similar studies against chi-square alternatives showed that Γ outperformed T_n for IFRA and DFRA alternatives. The statistic Γ also has the advantage of being much simpler to compute than its competitors.

Ahmad (2001) introduced a test statistic for testing exponentiality against IFR alternatives based on sample moments. Using Corollary 3.2, he defined

$$\delta_{r+2}^{(1)} = E[X_{1:2}^{(2r+2)}] - \binom{2r+2}{r+1}\left(\frac{1}{2}\right)^{2r+2}(E[X^{(r+1)}])^2,$$

as a measure of deviation from exponentiality in favour of IFR distributions.

To make the test scale invariant, Ahmad (2001) used $\Delta_{r+2}^{(1)} = \delta_{r+2}^{(1)}/\mu^{2r+2}$, which was estimated by

$$\hat{\Delta}_{r+2}^{(1)} = \hat{\delta}_{r+2}^{(1)}/\bar{X}^{2r+2}, \tag{4.110}$$

where

$$\hat{\delta}_{r+2}^{(1)} = \frac{2}{n(n-1)}\sum\sum_{i<j}\left\{\min\left(X_i^{2r+2}, X_j^{2r+2}\right) - \binom{2r+2}{r+1}\left(\frac{1}{2}\right)^{2r+2} X_i^{r+1}X_j^{r+1}\right\}.$$

H_0 is rejected in favour of $H_1^{(1)}$ for significantly large values. The test performs basically as well as the W_1-test in (4.88) of Bickel and Doksum (1969).

The value of r can be chosen small enough to keep calculations simple, or in cases when the alternative is known, r can be chosen to maximize the asymptotic efficiency. For the Weibull case, with $r = 4$, the new test has greater efficiency than the W_1-test in (4.88).

Ahmad (2004) developed a test for exponentiality against IFR alternatives based on the measure of deviation from H_0

$$\delta_F^{(1)} = \int_0^\infty\int_0^\infty \bar{F}^2\left(\frac{x+y}{2}\right)dxdy - \int_0^\infty\int_0^\infty \bar{F}(x)\bar{F}(y)dxdy,$$

which can be rewritten as

$$\delta_F^{(1)} = \int_0^\infty\int_0^\infty \bar{F}^2\left(\frac{x+y}{2}\right) - \mu^2.$$

Note that $\delta_F^{(1)}$ is similar to (4.98), except that the integration is with respect to $dxdy$ and not $dF(x)dF(y)$.

Under H_0, $\delta_F^{(1)} = 0$. If X_1 and X_2 are independent copies of the variable X with distribution F, then

$$\delta_F^{(1)} = 2E[X_{1:2})]^2 - \mu^2,$$

with unbiased estimate

$$\hat{\delta}^{(1)}_{F_n} = \frac{1}{n(n-1)} \sum\sum_{i \neq j} \left\{ 2[\min(X_i, X_j)]^2 - X_i X_j \right\}. \tag{4.111}$$

To make the test scale invariant, Ahmad (2004) considered $\hat{\Delta}^{(1)}_{F_n} = \hat{\delta}^{(1)}_{F_n} / \bar{X}^2$. The statistic is asymptotically normal, and H_0 is rejected for large values.

4.4 Goodness-of-fit tests for exponentiality against NBU alternatives

One-sided tests for exponentiality against NBU alternatives have been developed by Hollander and Proschan (1972), Koul (1977), Deshpande (1983), Kumazawa (1983) and Ahmad (2001). The hypothesis to be tested is:

$$H_0: \quad F(x) = 1 - e^{-\theta x}, \quad x > 0,\ \theta > 0$$

against

$$H_1^{(3)}: \quad F \text{ is NBU and not exponential.}$$

Hollander and Proschan (1972) used the definition of the NBU class (Def. 2.4, p. 24) to define the following measure of deviation between H_0 and $H_1^{(3)}$:

$$\begin{aligned} \gamma(F) &:= \iint \left\{ \bar{F}(t)\bar{F}(s) - \bar{F}(t+s) \right\} dF(t) dF(s) \\ &= \frac{1}{4} - \iint \bar{F}(t+s) d(F) dF(s) \\ &=: \frac{1}{4} - \Delta(F). \end{aligned} \tag{4.112}$$

By replacing F with the empirical distribution function F_n, they proposed to reject H_0 in favour of $H_1^{(3)}$ for small values of $\iint \bar{F}_n(t+s) dF_n(s) dF_n(t)$, with asymptotically equivalent U-statistic

$$J_n = 2[n(n-1)(n-2)]^{-1} \sum{}' g\left(X_{\alpha_1}, X_{\alpha_2} + X_{\alpha_2}\right), \tag{4.113}$$

where

$$g(a,b) = \begin{cases} 1 & if\ a > b \\ 0 & if\ a \leq b \end{cases}$$

and the summation is over all $n(n-1)(n-2)/2$ triples of $(\alpha_1, \alpha_2, \alpha_3)$ of three integers such that $1 \leq \alpha_i \leq n$, $\alpha_1 \neq \alpha_2$, $\alpha_1 \neq \alpha_3$ and $\alpha_2 < \alpha_3$.

The test that rejects H_0 for small values of J_n is unbiased and consistent against NBU alternatives. The asymptotic normality of J_n follows directly from the U-statistic theory of Hoeffding (1948).

The test based on J_n was compared to the following tests for IFR alternatives, since no other tests for NBU existed at that time: V_n in (4.86) of Proschan and Pyke (1967), the total time on test statistic in (4.43) and W_1 in (4.88) of Bickel and Doksum (1969).

When the underlying distribution was IFR, the IFR tests in general performed better, as was to be expected. However, the NBU test performed distinctly better than the IFR tests when the underlying distribution was NBU but not IFR.

Koul (1977) developed a test against NBU alternatives by defining

$$D(F) = \inf_{s,t\geq 0} \left[\bar{F}(s+t) - \bar{F}(s)\bar{F}(t)\right],$$

so that under H_0, $D(F) = 0$, and under $H_1^{(3)}$, $D(F) < 0$. The test statistic was defined as

$$D_n = D(F_n) = \inf_{s,t\geq 0} \left[\bar{F}_n(s+t) - \bar{F}_n(s)\bar{F}_n(t)\right].$$

A statistic which is easier to compute is

$$T_n := n^2 D_n = \min_{1\leq i\leq j\leq n} \left[nS_{ij} - (n-i)(n-j)\right], \tag{4.114}$$

where $S_{ij} = \sum_{k=1}^{n} I[X_{k:n} > X_{i:n} + X_{j:n}] = n\bar{F}_n(X_{i:n} + X_{j:n})$. H_0 is rejected in favour of $H_1^{(3)}$ for small values of D_n.

The test was shown to be unbiased and consistent, however the asymptotic null distribution is not normal.

Deshpande (1983) developed a test against NBU alternatives which turned out to be a linear combination of J_n in (4.113) from Hollander and Proschan (1972) and U in (4.100) from Ahmad (1975). By substituting t and x in (2.2) with $(x+y)/2$, he defined $D(F)$ as measure of deviation between H_0 and $H_1^{(3)}$:

$$\begin{aligned} D(F) &= \int_0^\infty \bar{F}^2\left(\frac{x+y}{2}\right) dF(x)dF(y) - \int_0^\infty \bar{F}\left(x+y\right) dF(x)dF(y) \\ &= P\left[\min(X_1, X_2) > \frac{X_3+X_4}{2}\right] - P\left[X_1 > X_2 + X_3\right], \end{aligned}$$

where X_1, X_2, X_3, X_4 are i.i.d. random variables with distribution function F.

The U-statistic with expectation $D(F)$, is $S = U - J_n$, which is asymptotically normal. The test rejects H_0 in favour of $H_1^{(3)}$ if S is significantly large.

The Pitman asymptotic relative efficiency of the new S-test was calculated relative to V_n in (4.86) of Proschan and Pyke (1967), as well as to the two statistics of which it is a linear combination. The new test was often a considerable improvement.

Kumazawa (1983) derived his test statistic by noting that the NBU property (2.2) implies that $\bar{F}(mx) \leq \bar{F}(x)^m$ for all $x \geq 0$ and every integer $m \geq 1$. He defined the measure of deviation

$$\begin{aligned}\Gamma_m(F,\xi) &= \int_0^\infty \left[\xi\left(\bar{F}(x)^m\right) - \xi\left(\bar{F}(mx)\right)\right] dF(x) \\ &= \int_0^1 \xi(x^m)dx - \int_0^\infty \xi\left(\bar{F}(mx)\right) dF(x),\end{aligned} \tag{4.115}$$

where ξ is a non-decreasing, right continuous function such that $\xi : [0,1) \rightarrow [0,\infty)$, $\xi(0) = 0$ and $\int_0^\infty \xi(x^2)dx < \infty$. Note that $\int_0^1 \xi(x^m)dx \leq \int_0^1 \xi(x^2)dx < \infty$ for $m \geq 2$.

Kumazawa (1983) proposed rejecting H_0 in favour of $H_1^{(3)}$ for small values of

$$L_n(\xi,m) = \int_0^\infty \xi\left(\bar{F}_n(mx)\right) dF_n(x),$$

which can be written as

$$L_n(\xi,m) = n^{-1}\sum_{i=1}^{n} \xi\left(n^{-1}S_i^{(m)}\right), \tag{4.116}$$

where $S_i^{(m)} = \sum_{j=1}^{n} I[X_{j:n} > mX_{i:n}]$ for $1 \leq i \leq n$.

A test based on $L_n(\xi,m)$ is consistent and unbiased for $m \geq 2$. With $\xi(u) = u$, $L_n(\xi,m)$ reduces to a U-statistic, and is thus asymptotically normal.

In calculating the Pitman asymptotic relative efficiency, Kumazawa (1983) considered $\xi(u) = u^\alpha$, $\alpha \geq 1/2$, and found that the $L_n(\xi,m)$-test with $\alpha = 1/2$ and $m = 3$ is asymptotically more efficient than the tests J_n in (4.113) and T_n in (4.114) of Hollander and Proschan (1972) and Koul (1978) respectively, except for gamma alternatives. $L_n(\xi,m)$ with $\alpha = 1$ and $m = 6$ has higher efficiency for gamma alternatives.

Ahmad (1994) introduced a test which can be used to test exponentiality against either IFRA or NBU alternatives. He proved and used the following extended definitions of the IFRA and NBU classes:

Definition 4.1 *F is IFRA if and only if for all non-negative real numbers $x_1, \ldots, x_k$, $k \geq 1$ and all $0 < \alpha < 1$,*

$$\bar{F}(x_1 + \cdots + x_k) \leq \prod_{i=1}^{k} \left[\bar{F}(\alpha x_i)\right]^{1/\alpha}.$$

Definition 4.2 *F is NBU if and only if for all integers $k \geq 2$,*

$$\bar{F}(x_1 + \cdots + x_k) \leq \prod_{i=1}^{k} \left[\bar{F}(x_i)\right].$$

From these definitions, Ahmad (1994) derived

$$\delta_k(F) = \left(\frac{\alpha}{\alpha+1}\right)^k - \int_0^\infty \cdots \int_0^\infty \bar{F}(x_1 + \cdots + x_k) dF(\alpha x_1) \ldots dF(\alpha x_k),$$

and

$$\eta_k(F) = \left(\frac{1}{4}\right)^k - \int_0^\infty \cdots \int_0^\infty \bar{F}(x_1 + \cdots + x_k) dF(x_1) \ldots dF(x_k),$$

which are generalisations of a testing measure considered by Ahmad (in 1974) in an unpublished article and the testing measure defined in (4.112) considered by Hollander and Proschan (1972). The S-test of Deshpande (1983) (p. 82) was also based on a special case of the measure $\delta_k(F)$. These generalisations, $\delta_k(F)$ and $\eta_k(F)$, are both special cases of the following functional:

$$J(F;\alpha,k) = \left(\frac{\alpha}{\alpha+1}\right)^k - \int_0^\infty \cdots \int_0^\infty \bar{F}(x_1 + \cdots + x_k) dF(\alpha x_1) \ldots dF(\alpha x_k),$$

for $0 < \alpha \leq 1$ and $k \geq 1$.

Note that $\delta_k(F) = J(F;\alpha,k)$, $0 < \alpha < 1$, $k \geq 1$ and $\eta_k(F) = J(F;1,k)$, $k \geq 2$.

Under IFRA alternatives, $J(F;\alpha,k) > 0$, $0 < \alpha < 1$, $k \geq 1$ and under NBU alternatives, $J(F;1,k) > 0$, $k \geq 2$. An estimate of $J(F;\alpha,k)$ can thus be used to test against IFRA alternatives for $0 < \alpha < 1$, $k \geq 1$ and to test against NBU alternatives for $\alpha = 1$, $k \geq 2$.

By using the empirical distribution function, the test statistic has the following U-statistic version:

$$\hat{J}(F_n;\alpha,k) = \left(\frac{\alpha}{\alpha+1}\right)^k - \left[n\binom{n-1}{k}\right]^{-1} \textstyle\sum_C I\left(\alpha X_{i_1} > X_{i_2} + \cdots + X_{i_{k+1}}\right), \qquad (4.117)$$

where $\sum_C$ extends over all indices i_j such that $1 \leq i_2 < \cdots < i_{k+1} \leq n$ and $i_1 \neq i_j$, $j = 2, \ldots, k+1$.

Note that $E\left[\hat{J}(F_n;\alpha,k)\right] = J(F;\alpha,k)$, and if F is exponential, then $J(F;\alpha,k) = 0$. Thus, large values of $\hat{J}(F_n;\alpha,k)$ is an indicator of departure from H_0 in favour of $H_1^{(2)}$ or $H_1^{(3)}$, depending on the values of k and α.

The case $0 < \alpha < 1$, $k \geq 1$ is used if the alternative is suspected to be IFRA, and the case $\alpha = 1$, $k \geq 2$ is used if the alternative is suspected to be NBU. Note that if $k = 1$, $\hat{J}(F_n;\alpha,k)$ is a test statistic developed by Ahmad (in 1974), and it is equivalent to the test of Deshpande (1983). When $\alpha = 1$ and $k = 2$, $\hat{J}(F_n;\alpha,k)$ is the J_n-test in (4.113) of Hollander and Proschan (1972).

The test based on $\hat{J}(F_n;\alpha,k)$ was shown to be unbiased and consistent for all continuous IFRA ($0 < \alpha < 1$) or NBU ($\alpha = 1$) alternatives. Also, $\sqrt{n}\left(\hat{J}(F_n;\alpha,k) - J(F_n;\alpha,k)\right)$ is asymptotically normal.

The values of α and k can be chosen to maximise the asymptotic Pitman efficiency of the test against a specified alternative, or to maximise the power function for a specific alternative. Ahmad (1994) recommended $k \leq 4$ and $\alpha = 0.4$ for testing against IFRA and $k = 3$ or $k = 4$ for testing against NBU. The computations also showed that the convergence to normality speeds up as k increases.

Ahmad (2001) used equation (3.8) with $k = r + 2$ to define the measure of deviation

$$\delta_{r+2}^{(2)} = (E[X])^{r+2} - E[X^{r+2}]/(r+2)!. \tag{4.118}$$

For testing exponentiality against NBU alternatives, he then proposed the scale invariant statistic

$$\hat{\Delta}_{r+2}^{(2)} = \hat{\delta}_{r+2}^{(2)}/\bar{X}^{r+2}, \tag{4.119}$$

where

$$\hat{\delta}_{r+2}^{(2)} = \frac{(r+2)!}{n(n-1)\dots(n-r-1)} \sum\nolimits_c \left[\prod_{j=1}^{r+2} X_{i_j} - X_{i_1}^{r+2}/(r+2)! \right],$$

where $\sum_c$ extends over all indices $1 \leq i_1 < \cdots < i_{r+2} = n$.

The test based on (4.119) has higher efficiency than the J_n-test given in (4.113) of Hollander and Proschan (1972). Again, just as with Ahmad's test against IFR alternatives defined in (4.110), the value of r can be chosen to maximize the asymptotic efficiency.

It will be seen in Section 4.7 that $\hat{\Delta}_{r+2}^{(2)}$ can also be used to test against HNBUE alternatives.

Ahmad (2004) developed a test for exponentiality against NBU alternatives based on the following measure of deviation:

$$\delta_F^{(2)} = \int_0^\infty \int_0^\infty \bar{F}(x)\bar{F}(y)dxdy - \int_0^\infty \int_0^\infty \bar{F}(x+y)dxdy, \tag{4.120}$$

which is

$$\delta_F^{(2)} = \mu^2 - E(X_1^2)/2.$$

The test statistic is

$$\hat{\Delta}_{F_n}^{(2)} = \hat{\delta}_{F_n}^{(2)}/\bar{X}^2, \tag{4.121}$$

where

$$\hat{\delta}_{F_n}^{(2)} = \frac{1}{n(n-1)} \sum\sum_{i \neq j} X_i X_j - \frac{1}{2n} \sum_i X_i^2.$$

Note that $\hat{\Delta}_{F_n}^{(2)}$ is equivalent to

$$\hat{\Delta}_{F_n}^{(2)} = (n+1)(\bar{X}^2 - S^2)/2(n-1)\bar{X}^2, \tag{4.122}$$

where $\bar{X}$ and S^2 are the sample mean and variance respectively.

The test statistic in (4.122) is equivalent to the coefficient of variation test of Borges et al. (1984) (p. 92). The measure of deviation (4.120) can also be considered as a measure of deviation for the NBUE and the HNBUE classes, and a test based on (4.122) is therefore also consistent against these classes.

4.5 Goodness-of-fit tests for exponentiality against DMRL alternatives

The literature contains only a few tests for exponentiality against the decreasing mean residual life (DMRL) class of alternatives. The alternative hypothesis in this case is:

$$H_1^{(4)}: \ F \text{ is DMRL and not exponential.} \tag{4.123}$$

Recall from Section 2.3.1 that a distribution F is DMRL if, for all $0 \leq s \leq t$,

$$\varepsilon_F(s) \geq \varepsilon_F(t),$$

where $\varepsilon_F(s)$ defined in (1.8) is the mean residual life at time s.

A one-sided test for DMRL alternatives was developed by Bandyopadyay and Basu (1990), while two-sided tests for DMRL/IMRL alternatives were developed by Hollander and Proschan (1975) and Bergman and Klefsjo (1989).

Hollander and Proschan (1975) considered the following measure of deviation from H_0 to $H_1^{(4)}$:

$$\Delta(F) = \iint_{s<t} \bar{F}(s)\bar{F}(t)\left\{\varepsilon_F(s) - \varepsilon_F(t)\right\} dF(s)dF(t).$$

By using the empirical distribution function F_n to estimate $\Delta(F)$, they obtained a test statistic which they called T. However, they also obtained a statistic V, asymptotically equivalent to T, defined as

$$V = n^{-2}\sum_{i=0}^{n-1}\sum_{j=i+1}^{n} A_{ij}, \tag{4.124}$$

where $A_{ij} = n^{-2}\left[(n-j)(T_n - T_i) - (n-i)(T_n - T_j)\right]$ and T_j in (1.14) denotes the total time on test at $X_{j:n}$.

The statistic in (4.124) can be rewritten as

$$V = n^{-4}\sum_{i=1}^{n} c_{in} X_i, \tag{4.125}$$

where $c_{in} = \frac{4}{3}i^3 - 4ni^2 + 3n^2 i - \frac{1}{2}n^3 + \frac{1}{2}n^2 - \frac{1}{2}i^2 + \frac{1}{6}i$, which is easier to compute.

The scale invariant statistic $V^* = V/\bar{X}_n$ was proposed for testing exponentiality against DMRL alternatives. Large values of V^* indicate that F is DMRL, while small values of V^* indicate that F is IMRL.

Klefsjö (1983a) used the scaled TTT-transform $\varphi_F(t)$ defined in (1.12) to derive the following test statistic for testing exponentiality against DMRL alternatives:

$$M = \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} \{(n-j)(1-W_i) - (n-i)(1-W_j)\}, \tag{4.126}$$

with W_j defined in (1.15). He noted that M is proportional to the test statistic V^* in (4.125) proposed by Hollander and Proschan (1975).

Bergman and Klefsjo (1989) presented and studied a family of statistics for testing exponentiality against DMRL alternatives, which includes the test statistic in (4.125) proposed by Hollander and Proschan (1975). They generalised the idea of Hollander and Proschan by using the weights $\bar{F}^j(s)\bar{F}^k(t)$, with positive integers j and k, to define the distance measure

$$\Delta_{jk}(F) = \iint_{s<t} \bar{F}^j(s)\bar{F}^k(t) \{\varepsilon_F(s) - \varepsilon_F(t)\} \, dF(s)dF(t).$$

By substituting the empirical distribution function they derived the statistic

$$\begin{aligned} \Delta_{jk}(F_n) &= \sum_{\nu=1}^{n} \left\{ a_1 \left(\frac{n-\nu+1}{n}\right) + a_2 \left(\frac{n-\nu+1}{n}\right)^{k+1} \right. \\ &\qquad \left. + a_3 \left(\frac{n-\nu+1}{n}\right)^{j+k+2} \right\} (X_{\nu:n} - X_{\nu-1:n}), \end{aligned} \tag{4.127}$$

where $a_1 = -\frac{1}{k(k+1)(j+k+1)}$, $a_2 = \frac{1}{k(j+1)}$ and $a_3 = -\frac{j+k+2}{(j+1)(k+1)(j+k+1)}$.

The statistic $V_{jk} = \Delta_{jk}(F_n)/\bar{X}_n$ was proposed as a scale-free test for exponentiality. A large positive (negative) value indicates that F is DMRL (IMRL).

For $j = k = 1$, the test statistic V_{11} is equal to the test statistic V^* given in (p. 86) of Hollander and Proschan (1975). The choice of j and k was seen to influence the power of the test in different situations. For linear failure rate and Pareto distributions, the statistic V_{11} performed better than its competitors, while for Weibull and gamma distributions the statistic V_{55} was the best.

The test was also generalised to be used with censored samples.

Bandyopadyay and Basu (1990) developed a test based on the following characterization of the DMRL class of life distributions:

Theorem 4.1 *F is DMRL if and only if* $\varepsilon_F(\kappa x) \geq \varepsilon_F(x)$ *for all* $0 < \kappa < 1$ *and all* $x \geq 0$.

They defined

$$\delta(F;\kappa) = \int_0^\infty D_\kappa(x;F)dF(x), \tag{4.128}$$

as a measure of deviation from H_0 to $H_1^{(4)}$, where

$$D_\kappa(x;F) = \bar{F}(x)\bar{F}(\kappa x)\left[\varepsilon_F(\kappa x) - \varepsilon_F(x)\right]$$

for $0 < \kappa < 1$.

Thus, under H_0, $\delta(F;\kappa) = 0$, and under $H_1^{(4)}$, $\delta(F;\kappa) > 0$.

By substituting the empirical distribution function F_n for F, they obtained the following U-statistic:

$$V^*(\kappa, n) = [n(n-1)(n-2)]^{-1} \sum\nolimits_{p3} \ \phi_\kappa\left(X_{i1}, X_{i2}, X_{i3}\right),$$

where the sum $\sum_{p3}$ is taken over all permutations i_1, i_2, i_3 of 3 distinct integers chosen from $(1, 2, \ldots, n)$, $n \geq 3$ with $\phi_\kappa(x_1, x_2, x_3) = \phi_1(x_1, x_2, x_3; \kappa) - \phi_2(x_1, x_2, x_3; \kappa)$, where

$$\phi_1(x_1, x_2, x_3; \kappa) = (x_1 - \kappa x_3)I(x_1 > \kappa x_3)I(x_2 > x_3),$$

and

$$\phi_2(x_1, x_2, x_3; \kappa) = (x_1 - \kappa x_3)I(x_1 > x_3)I(x_2 > \kappa x_3).$$

In order to make the test scale invariant, Bandyopadyay and Basu (1990) proposed $V_n(\kappa) = V^*(\kappa, n)/\bar{X}_n$. H_0 is rejected in favour of $H_1^{(4)}$ for significantly large values of $V_n(\kappa)$.

By calculating Pitman's asymptotic relative efficiency and conducting Monte Carlo power studies, Bandyopadyay and Basu (1990) concluded that the $V_n(0.000001)$-test performed uniformly better than the family of tests in (4.127) of Bergman and Klefsjo (1989) and the V^*-test in (4.125) of Hollander and Proschan (1975), for all alternatives except linear failure rate distributions for large samples.

For the special case where $\kappa = 0$, $V_n(0)$ turns out to be the U-statistic version of Hollander and Proschan's (1975) test defined in (4.131) for NBUE alternatives.

4.6 Goodness-of-fit tests for exponentiality against NBUE alternatives

Recall from Section 2.3.1 that a distribution F is NBUE if $\mu = \varepsilon_F(0) \geq \varepsilon_F(t)$ for all $t \geq 0$, where $\varepsilon_F(s)$ defined in (1.8) is the mean residual life at time s. Thus, F is NBUE if, for all $t \geq 0$, the mean residual life at time t is not greater than the mean residual life at time 0 (i.e. the mean life of a new item).

The hypothesis to be tested in this section is:

$$H_0: \quad F(x) = 1 - e^{-\theta x}, \quad x > 0,\ \theta > 0$$

against

$$H_1^{(5)}: \quad F \textit{ is NBUE and not exponential.} \tag{4.129}$$

One-sided tests for NBUE alternatives were developed by Koul (1978) and Ahmad (2001), while two-sided tests for NBUE/NWUE alternatives were developed by Hollander and Proschan (1975), Borges et al. (1984) and Kanjo (1993).

Hollander and Proschan (1975) defined the following measure of deviation of F from H_0:

$$\gamma(F) = \int_0^\infty \bar{F}(t)\left\{\varepsilon_F(0) - \varepsilon_F(t)\right\} dF(t). \tag{4.130}$$

Replacing F by the empirical distribution function F_n gives

$$\begin{aligned} K &= n^{-1}\sum_{i=1}^{n-1}\left\{\frac{(n-i)}{n}\bar{X} - \frac{1}{n}\sum_{j=i+1}^{n}(X_{j:n} - X_{i:n})\right\} \\ &= n^{-2}\sum_{i=1}^{n}\left(\frac{3n}{2} - 2i + \frac{1}{2}\right)X_{i:n}. \end{aligned}$$

In order to make K scale invariant, Hollander and Proschan (1975) proposed $K^* = K/\bar{X}$, which turns out to be just a linear function of the TTT-statistic in (4.43), of which the asymptotic normality is well-known. Hollander and Proschan (1975) showed that a test based on K^* is consistent against the NBUE class. Significantly large values of K^* suggest NBUE alternatives, while significantly small values suggest NWUE alternatives.

Koul (1978) observed that two classes of tests statistics, given in (4.95) and (4.97), suggested by Barlow and Doksum (1972) for testing exponentiality against IFR alternatives, are also consistent for testing exponentiality against NBUE alternatives.

It is easy to show that

$$F \text{ is } NBUE \Leftrightarrow \int_0^t \bar{F}(u)du \geq \mu F(t) \textit{ for } t \geq 0,$$

which is equivalent to

$$F \text{ is } NBUE \Leftrightarrow \int_t^\infty \bar{F}(u)du \leq \mu \bar{F}(t) \textit{ for all } 0 < t < \infty.$$

Let $H_F^{-1}(t)$ in (1.11) denote the TTT-transform of F. Then, if F is continuous, F is NBUE $\Leftrightarrow$ $H_F^{-1}(t) \geq H_F^{-1}(1)t$, $0 \leq t \leq 1$.

Now, let J be a non-decreasing continuous function on $[0, 1]$ with

$$b(J, F) = \int_0^\infty J\left(\mu^{-1}L(y)\right) dF(y), \tag{4.131}$$

where $L(y) = \int_0^y \bar{F}(x)dx$. Then $F \in$ NBUE implies that

$$\int_0^\infty J(F(y))dF(y) \leq b(J, F) \leq J(1),$$

and equality implies that F is exponential. A class of test statistics follows by substituting F by F_n to obtain

$$b_n(J) = n^{-1} \sum_{i=1}^{n} J[W_{i:n}], \tag{4.132}$$

with $W_{i:n}$ defined as in (4.94).

Koul (1978) considered the following distance measure:

$$D = \sup_{0<y<\infty} \left[\mu^{-1} \int_0^y \bar{F}(x)dx - F(y)\right], \tag{4.133}$$

with $D = 0$ if F is exponential. From (4.133), he derived the statistic

$$D_n = \sup_{1 \leq i \leq n} \left[W_{i:n} - \frac{i}{n}\right]. \tag{4.134}$$

These two test statistics, $b_n(J)$ and D_n, are the same as the statistics given in (4.95) and (4.97), derived by Barlow and Doksum (1972) as tests for exponentiality against IFR alternatives. H_0 is rejected in favour of $H_1^{(5)}$ in (4.129) for large values of $b_n(J)$ and D_n.

If $J(u) = u$, the statistic $W_n = b_n(J)$ is the same as K^* (p. 89) of Hollander and Proschan (1975), which was equivalent to the TTT-statistic in (4.43).

By calculating the Bahadur efficiency, Koul (1978) showed that the W_n-test consistently gives a small level of significance at Weibull alternatives more cheaply than the D_n-test in (4.134), and at the same time the W_n-test has power equal to that of the D_n-test. However, for life distributions which are only NBUE, the D_n-test has much higher Bahadur efficiency than the W_n-test.

Bergman (1979) considered the TTT-plot (p. 75) to derive a test for exponentiality against NBUE alternatives, in the same way he did for the IFR-class (p. 75). Recall from Section 2.3.1, p. 26 that a life distribution F is NBUE if and only if $\varphi_F(t) \geq t$, $0 \leq t \leq 1$, where $\varphi_F(t)$ is the scaled TTT-transform defined in (1.12).

Bergman (1979) also considered two classes he called "New Better than Some Used in Expectation" and "New Better than Old in Expectation", and used the TTT-plot to construct tests for exponentiality in each of these classes.

The TTT-transform and TTT-plot were also used by **Klefsjö (1983a)** to find tests for exponentiality against IFR, IFRA (p. 77), DMRL (p. 87) and NBUE alternatives. He proposed

$$C = \sum_{j=1}^{n} \left(W_j - \frac{j}{n} \right) = \sum_{j=1}^{n-1} W_j - \frac{n-1}{2},$$

as a test statistic, with W_j defined in (1.15).

However, Klefsjö (1983a) noted that $C = V - \frac{n-1}{2} = nK^*$, where V is the TTT-statistic given in (4.43) and K^* (p. 89) was introduced by Hollander and Proschan (1975). Thus, C, V and K^* are equivalent test statistics.

Borges et al. (1984) followed a different approach to that of Hollander and Proschan (1975) and Koul (1978). Instead of replacing F by the empirical distribution function F_n, they first developed a measure of exponentiality based on the moments of F and then replaced the moments by sample moments to obtain a test statistic.

Their measure of deviation was defined as

$$\Delta(F) = \int_0^{\infty} \bar{F}(t)[\mu - \varepsilon_F(t)]dt,$$

where $\varepsilon_F(t)$ is the mean residual life at time t defined in (1.8) and $\mu = \varepsilon_F(0)$.

Borges et al. (1984) observed that if F is NBUE and has positive finite mean μ, then F has finite p-th moment for all $p \geq 1$.

Note that the difference between this measure of deviation and the one in (4.130) defined by Hollander and Proschan (1975), is that dt is integrated and not $dF(t)$.

Borges et al. (1984) then proved that if F is NBUE, a necessary and sufficient condition in order that F is exponential, is that $CV(F) = 1$, where $CV(F)$ denotes the coefficient of variation of F. This suggested using the sample coefficient of variation $S/\bar{X}$ as a test for exponentiality against NBUE alternatives.

Under H_0, $S/\bar{X}$ is asymptotically normal with mean 1 and variance $1/n$.

The test rejects H_0 for large values of $\left|\sqrt{n}\left(S/\bar{X} - 1\right)\right|$. In particular, large negative values of $\sqrt{n}\left(S/\bar{X} - 1\right)$ is evidence towards the NBUE property, while large positive values of $\sqrt{n}\left(S/\bar{X} - 1\right)$ is evidence towards the NWUE property.

On the basis of Pitman asymptotic relative efficiency, the $S/\bar{X}$-statistic was found to perform better than the K^*-statistic in (4.131) of Hollander and Proschan (1975) in some cases, and in other cases the reverse held. When considering Bahadur efficiency, the $S/\bar{X}$-statistic is more efficient than K^* as the distribution gets closer to exponentiality. The $S/\bar{X}$-test has the advantage of being much simpler in structure than its predecessors.

Kanjo (1993) gave an exact test for testing exponentiality against NBUE alternatives based on the characterization in Theorem 3.31, p. 46 regarding the equilibrium distribution. He defined the following measure of deviation:

$$\begin{aligned}
\Delta' &= \int_0^\infty \left[\bar{F}(t) - \bar{T}_F(t)\right] dF(t) \\
&= \int_0^\infty \left[T_F(t) - F(t)\right] dF(t) \\
&= \frac{1}{2} - \int_0^\infty F(t) dT_F(t),
\end{aligned}$$

so that $\Delta' > 0$ favours $H_1^{(5)}$. Equivalently, $0 < \Delta := \int_0^\infty F(t)dT_F(t) < \frac{1}{2}$ favours $H_1^{(5)}$. Also, $\frac{1}{2} < \Delta < \infty$ implies that F is NWUE. Thus, small values of Δ support NBUE alternatives, while large values support NWUE alternatives.

The hypothesis of exponentiality is thus equivalent to H_0: $\Delta = \frac{1}{2}$ against $H_1{''}$: $\Delta \neq \frac{1}{2}$, which is a two-sided test where the left side corresponds to F in NBUE and the right side corresponds to F in NWUE. If F in NWUE is not feasible, then the required test is H_0: $\Delta = \frac{1}{2}$ against $H_1{'''}$: $\Delta < \frac{1}{2}$.

The sample analogue of Δ is taken as the test statistic:

$$V_n = \sum_{j=1}^{n} \frac{j}{n} D_j / T_n, \tag{4.135}$$

where D_j is the normalized spacings defined in (1.10) and $T_n = n\bar{X}_n = \sum_{i=1}^{n} D_i$.

Tables of critical values for testing exponentiality against both the two-sided and one-sided alternative hypotheses, $H_1{''}$ and $H_1{'''}$, were given.

The test is consistent against NBUE alternatives and asymptotically normal, but the convergence to normality seems to be very slow.

For testing exponentiality against NBUE alternatives, **Ahmad (2001)** used Theorem 3.22 to define the measure of deviation

$$\delta_{r+2}^{(3)} = \mu E[X^{(r+1)}] - E[X^{(r+2)}]/(r+2). \tag{4.136}$$

He then proposed the scale invariant test

$$\hat{\Delta}_{r+2}^{(3)} = \hat{\delta}_{r+2}^{(3)}/\bar{X}^{r+2}, \tag{4.137}$$

where

$$\hat{\delta}_{r+2}^{(3)} = \frac{1}{n(n-1)} \sum\sum_{i \neq j} \left\{ X_i^{r+1} X_j - X_i^{r+2}/(r+2) \right\}.$$

When $r = 0$, $\hat{\delta}_{r+2}^{(3)}$ is equal to $\hat{\delta}_2^{(2)}$ in (4.119), Ahmad's test for NBU alternatives. Ahmad (2001) claims that the test based on (4.137) performs better than the test in (4.131) of Hollander and Proschan (1975) on the basis of Pitman asymptotic efficiency.

4.7 Goodness-of-fit tests for exponentiality against HNBUE alternatives

Recall from Section 2.4 that a distribution F is HNBUE if

$$\frac{1}{\frac{1}{x}\int_0^x \frac{1}{\varepsilon_F(t)} dt} \leq \mu \qquad \forall x > 0,$$

where $\varepsilon_F(t)$ is the mean residual life at time t defined in (1.8).

The hypothesis to be tested is:

$$H_0: \quad F(x) = 1 - e^{-\theta x}, \quad x > 0, \ \theta > 0$$

against

$$H_1^{(6)}: \quad F \text{ is HNBUE and not exponential.} \tag{4.138}$$

One-sided tests for HNBUE alternatives were developed by Kochar and Deshpande (1985) and Aly (1992), while two-sided tests for HNBUE/HNWUE alternatives were developed by Klefsjö (1983b), Basu and Ebrahimi (1985), Singh and Kochar (1986), Hendi, Al-Nachawati and Alwasel (1998) and Klar (2000).

Klefsjö (1983b) firstly proved that V, the TTT-statistic in (4.43), is consistent against the larger class of HNBUE alternatives. Hollander and Proschan (1975) have already proven that the TTT-statistic is consistent against NBUE alternatives, and not only against IFR alternatives.

Klefsjö (1983b) then used the following properties of the HNBUE class to derive two tests for exponentiality against HNBUE alternatives:

Corollary 4.1 *If a life distribution F with mean μ is HNBUE (HNWUE), then*

$$\int_0^\infty \bar{F}^\nu(x)dx \geq (\leq)\frac{\mu}{\nu}, \quad for\ \nu = 2, 3, 4, \ldots$$

Corollary 4.2 *If a life distribution F with mean μ is HNBUE (HNWUE), then*

$$\int_0^\infty \left\{1 - \bar{F}^\nu(x)\right\} dx \leq (\geq)\mu \sum_{j=1}^{\nu} \frac{1}{j}, \quad for\ \nu = 2, 3, 4, \ldots$$

With

$$J_1(u) = -1/\nu + \nu(1-u)^{\nu-1},$$

and

$$Q_1 = \sum_{j=1}^{n} J_1(j/n) X_{j:n}/S_n, \tag{4.139}$$

Klefsjö (1983b) used Corollary 4.1 to prove that a test based on Q_1 is, independently of $\nu \geq 2$, a consistent test against continuous HNBUE (HNWUE) life distributions. He proposed $\nu = 3$, thus $J_1(u) = -1/3 + 3(1-u)^2$.

Similarly, based on Corollary 4.2, with

$$J_2(u) = \sum_{j=1}^{\nu} \frac{1}{j} - \nu u^{\nu-1},$$

Klefsjö (1983b) proposed

$$Q_2 = \sum_{j=1}^{n} J_2(j/n) X_{j:n}/S_n, \tag{4.140}$$

as a test statistic against continuous HNBUE (HNWUE) distributions, again with $\nu = 3$.

The exact null distributions of both Q_1 and Q_2 were given.

The test statistic Q_2 has larger asymptotic efficiency than Q_1 for linear failure rate distributions and the Pareto distribution, but smaller efficiency values for Weibull and gamma distributions.

Klefsjö (1983b) proposed that a test statistic Q_3 based on a linear combination of $J_1(u)$ and $J_2(u)$, i.e. $J_3(u) = J_1(u) + aJ_2(u)$ for a suitable $a > 0$, could be a reasonable compromise between Q_1 and Q_2.

Power simulations were done to compare the new tests against the following test statistics: A_1 in (4.104), A_2 in (4.105) and B in (4.106) introduced by Klefsjö (1983a) for IFR and IFRA alternatives, the cumulative TTT-statistic in (4.43), the V-test in (4.124) of Hollander and Proschan (1975) for DMRL alternatives, and the L-test in (4.61) of Gail and Gastwirth (1978b). Klefsjö (1983b) recommended Q_1 as a good statistic.

Basu and Ebrahimi (1985) also proposed two tests for testing exponentiality against HNBUE (HNWUE) alternatives, both constructed by using the definitions of the TTT-transform in (1.11) and the scaled TTT-transform in (1.12). These definitions were also used by Klefsjö (1983a) for testing against IFR or IFRA alternatives (Section 4.3, p. 77).

The first statistic of Basu and Ebrahimi (1985) was based on the fact that F is HNBUE (HNWUE) if and only if

$$\int_0^t \left\{ \frac{1 - F(x)}{1 - \varphi_F(F(x))} - 1 \right\} dx \geq (\leq) 0, \qquad \forall\, t \geq 0.$$

They estimated the left hand side of this inequality for the specified times $X_{1:n}, \ldots, X_{n:n}$ and the subsequent test statistic was given by

$$\begin{aligned} A_n &= \sum_{r=1}^{n} \sum_{j=1}^{r} \left[\frac{(n-j+1)/n}{1 - W_{j-1}} \right] \left[X_{j:n} - X_{(j-1):n} \right] \\ &= \left[\frac{1}{n} \sum_{j=1}^{n} (n-j+1) \frac{D_j}{1 - W_{j-1}} \right] - T_n, \end{aligned} \tag{4.141}$$

where D_j is the normalized spacings defined in (1.10), T_j is the total time on test in (1.14) at $X_{j:n}$ and $W_j = T_j / T_n$ in (1.15).

The second test statistic of Basu and Ebrahimi (1985) was based on the fact that F is HNBUE (HNWUE) if and only if

$$\ln\left(1 - \varphi_F(F(t))\right) \leq (\geq) -\frac{t}{\mu},$$

for all $t \geq 0$. The left hand side of this inequality was again estimated for the specified times $X_{1:n}, \ldots, X_{n:n}$ and the subsequent test statistic was given as

$$B_n = \left[\sum_{j=1}^{n} \ln(1 - Y_{j-1}) \right] + n. \tag{4.142}$$

The distribution of A_n under H_0 could be found only for $n = 2$, and it was shown that the complexity of the distribution of A_n increases very rapidly as the number of observations increases. However, the distribution of B_n under H_0 was derived, and a test based on B_n is unbiased and consistent for testing exponentiality against HNBUE alternatives.

For both A_n and B_n, large positive (negative) values indicate that F is HNBUE (HNWUE).

On the basis of Pitman asymptotic relative efficiency, a test based on B_n was found to be more efficient than the $L_n(m, \xi)$-test given in (4.116) with $\alpha = 1/2$ and $m = 3$ proposed by Kumazawa (1983).

Kochar and Deshpande (1985) proved that the exponential scores statistics

$$T_n = -n^{-1} \sum_{i=1}^{n} \log(1 - W_{i:n}),$$

suggested by Barlow and Doksum (1972) (p. 73) for testing exponentiality against IFR alternatives, is also consistent against the HNBUE class of distributions. H_0 is rejected in favour of $H_1^{(6)}$ for small values of T_n.

Singh and Kochar (1986) used the following measure of deviation:

$$\Delta(F) = \int_0^\infty \left[\mu e^{-t\mu} - \int_t^\infty \bar{F}(x)dx \right] dG(t).$$

Then a necessary and sufficient condition that F is exponential is that $\Delta(F) = 0$.

They proved that a test based on the statistic

$$T_n = n^{-1} \sum_{i=1}^{n} \exp(-X_{i:n}/\bar{X}) \tag{4.143}$$

is consistent against HNBUE (HNWUE) alternatives. Large positive (negative) values indicate that F is HNBUE (HNWUE).

T_n was proven to be asymptotically normal under H_0.

Doksum and Yandell (1984) have proven that a test based on (4.143) is asymptotically most powerful in the class of all similar tests for testing exponentiality against the Makeham distribution.

Kochar and Gupta (1988) performed a Monte Carlo study to compare the powers of the following tests for exponentiality:

- A test based on the sample coefficient of variation, $S/\bar{X}$, which has been shown to be consistent against IFR and IFRA distributions (Doksum and Yandell 1984), as well as against NBUE distributions (Borges et al. 1984) (p. 92). Kochar and Gupta (1988) claimed it to be consistent against HNBUE distributions as well.

- A test based on the scaled total time on test up to the i-th failure,

$$n^{-1}\sum_{i=1}^{n}\left[-\log(1-W_{i:n})\right],$$

which was suggested by Barlow and Doksum (1972) (p. 73) for testing against IFR alternatives and proven by Kochar (1985) to be consistent against HNBUE alternatives (p. 96).

- The tests S_2 in (4.92) and W_2 in (4.89) from Bickel and Doksum (1969).

Kochar and Gupta (1988) found that the test based on the sample coefficient of variation performed the best, followed by S_2.

Aly (1992) introduced a non-parametric graphical technique based on a HNBUE-plot, as an exploratory data analysis tool for testing exponentiality against HNBUE alternatives. He considered both the full sample and the randomly right censored sample case. The test is based on Theorem 2.6, p. 29 which is equivalent to

$$\Delta(t)\geq 0, \quad 0\leq t\leq 1,$$

where $\Delta(t)=\varphi(t)+e^{-F^{-1}(t)/\mu}-1$, $0\leq t\leq 1$, is the so-called HNBUE-plot function, with corresponding sample HNBUE-plot function defined by

$$\Delta_n(t)=\varphi_n(t)+e^{-F_n^{-1}(t)/\bar{X}}-1, \quad 0\leq t\leq 1.$$

If H_0 is true, the plot of $\Delta_n(t)$ should closely evolve around the horizontal axis.

Further, based on the measure of deviation

$$\gamma(F)=\mu\int_0^1\Delta(F,t)dt=\int_0^\infty\bar{F}^2(x)dx-\int_0^\infty\bar{F}(x)e^{-x/\mu}dx,$$

Aly (1992) proposed the test statistic

$$t_n=\gamma(F_n)/\bar{X}_n=\sum_{i=1}^{n}\left(1-\frac{i-1}{n}\right)^2\left(X_{i:n}-X_{i-1:n}\right)/\bar{X}_n+\frac{1}{n}\sum_{i=1}^{n}e^{-X_{i:n}/\bar{X}_n}-1,$$

for testing exponentiality against $H_1^{(6)}$. H_0 is rejected for large values of t_n.

This test was shown to be equally efficient and equally powerful to the K^*-test in (4.131) of Hollander and Proschan (1975).

Hendi et al. (1998) used the definition of the HNBUE class in (2.6) to note that if $F <_C M_\mu$, then

$$\int_0^\infty xF(x)dF(x)\leq\frac{3\mu}{4},$$

which suggested the measure of deviation

$$\Delta'_F = \frac{3}{4} - \Delta_F,$$

where $\Delta_F = \frac{1}{\mu}\int_0^\infty xF(x)dF(x)$.

This compares to the measure of Hollander and Proschan (1975) defined as

$$K = \frac{1}{\mu}\int_0^\infty xJ\left(F(x)\right)dF(x),$$

with $J(u) = 3/2 - 2u + (1/2n)$ for testing against NBUE (p. 89).

Note that $\Delta'_F > 0$ favours $H_1^{(6)}$, or equivalently, $0 \leq \Delta_F < \frac{3}{4}$ favours $H_1^{(6)}$. Also, $\frac{3}{4} < \Delta_F < \infty$ favours $H_1^{(6)'}$: F is HNWUE. Thus, small values of Δ_F support HNBUE alternatives, while large values support HNWUE alternatives.

Similar to the case of Kanjo (1993) (p. 92), the hypothesis of exponentiality is thus equivalent to H_0: $\Delta_F = \frac{3}{4}$ against ${H_1}''$: $\Delta_F \neq \frac{3}{4}$, which is a two-sided test where the left side corresponds to F in HNBUE and the right side corresponds to F in HNWUE. If F in HNWUE is not feasible, then the required test is H_0: $\Delta_F = \frac{3}{4}$ against ${H_1}'''$: $\Delta_F < \frac{3}{4}$.

The sample analogue of Δ_F gives the scale invariant and consistent test statistic

$$V^* = \frac{1}{\overline{X}}\sum_{j=1}^{n} X_{j:n}\left(\frac{j}{n}\right)/n,$$

which can also be written in terms of the normalized spacings:

$$V^* = \sum_{j=1}^{n} e_{jn}D_j / \sum_{j=1}^{n} D_j,$$

where $e_{jn} = (1/2)\left((j/n) + 1\right)$.

Tables of critical values for testing exponentiality against both the two-sided and one-sided alternative hypotheses, ${H_1}''$ and ${H_1}'''$, were given. The asymptotic distribution of V^* converges very slowly to normality.

The test was also extended to handle the case of right censored data.

Klar (2000) reformulated the definition of the HNBUE class in (2.6) by using the integrated distribution function (idf) $\xi(t)$ defined in (4.13). Note that $\xi(t)$ is related to the TTT-transform in (1.11) by

$$H_F^{-1}(t) + \xi_F\left(F^{-1}(t)\right) = \mu, \quad 0 < t < 1.$$

Then F is HNBUE if

$$\xi_F(t) \leq \xi(t, 1/\mu), \quad \forall t \geq 0,$$

where $\xi(t,\theta) = \exp(-\theta t)/\theta$ is the integrated distribution function of the exponential distribution with mean $1/\theta$.

Let $\xi_n(t)$ be the empirical idf, then Klar (2000) proposed the weighted integral

$$T_{n,a} = \hat{\theta}_n^2 \int_0^\infty \left(\xi_n(t) - \xi(t,\hat{\theta}_n)\right) e^{-a\hat{\theta}_n t} dt,$$

where $\hat{\theta}_n = 1/\bar{X}_n$ and $a \geq 0$ is a non-negative constant, as a test statistic for testing exponentiality against HNBUE alternatives.

H_0 is rejected in favour of $H_1^{(6)}$ for large negative values of $T_{n,a}$, while a test for HNWUE alternatives uses an upper rejection region.

It was shown that the choice of a has a definite influence on the power of the test. The case $a = 1$ gives

$$T_{n,1} = n^{-1} \sum_{j=1}^{n} \left(e^{-Y_j} - \frac{1}{2}\right),$$

where $Y_j = X_j/\bar{X}_n$. Doksum and Yandell (1984) have shown that the test based on $T_{n,1}$ is asymptotically most powerful for testing H_0 against the Makeham distribution.

Singh and Kochar (1986) established consistency of $T_{n,1}$ against continuous HNBUE alternatives and $T_{n,1}$ was also considered by Jammalamadaka and Lee in 1998.

Also note that, up to one-to-one transformations, $T_{n,0}$ coincides with the statistic of Greenwood (1946) and with the sample coefficient of variation, $S/\bar{X}$. Doksum and Yandell (1984) proved that a test based on $\sqrt{n}T_{n,0}$ is asymptotically most powerful for testing exponentiality against linear failure rate distributions, and Kochar and Gupta (1988) remarked that a test based on $S/\bar{X}$ is consistent against continuous HNBUE alternatives.

$T_{n,a}$ is scale-invariant, asymptotically normal, and consistent against each HNBUE alternative. It is important to note that the consistency result holds without the usual assumption of an absolutely continuous alternative distribution as in Aly (1992) (p. 97), Klefsjö (1983b) (p. 94), Kochar (1985) (p. 96) and Singh and Kochar (1986) (p. 96).

Simulation studies were done to compare $T_{n,a}$ to the following tests:

- The linear combination $T_n = (2 + 2\rho(0.5, 3))^{-1/2} (T_{n,0.5} + T_{n,3})$, where $\rho(a,b) = \frac{\sqrt{(2a+1)(2b+1)}}{a+b+1}$;
- $Q_{1,3}$ in (4.139) and $Q_{2,3}$ in (4.140) recommended by Klefsjö (1983b);
- The total time on test statistic V in (4.43).

$T_{n,1}$ performed the best amongst the $T_{n,a}$-tests against Weibull alternatives, while the linear combination T_n outperformed all the tests in this case. Against linear failure rate distributions, $T_{n,0}$ performed the best, while $T_{n,3}$ and $T_{n,5}$ were the most powerful against gamma alternatives. The test T_n was recommended when nothing was known about the HNBUE alternative, since it distributed its power more evenly over the range of alternatives.

Ahmad (2001) proposed the test $\hat{\Delta}^{(2)}_{r+2}$ in (4.119) for NBU alternatives, but noted that it can also be used to test against HNBUE alternatives. This test has higher efficiency than the tests defined in (4.139) and (4.140) of Klefsjö (1983b) and those defined in (4.141) and (4.142) of Basu and Ebrahimi (1985).

Chapter 5

New characterizations and goodness-of-fit tests

5.1 Introduction

Throughout the discussion below the exponential distribution function (df) will be denoted by F_0, i.e.,

$$F_0(x) = 1 - e^{-\lambda x}, \quad \text{for } x > 0, \tag{5.1}$$
$$= 0, \quad \text{elsewhere,}$$

for some unknown constant λ, $0 < \lambda < \infty$.

Let $\mathcal{H}$ denote the class consisting of all almost everywhere (with respect to Lebesgue measure) continuous df's F such that $F(0) = 0$, $F(x) < 1$ for all $x \in (0, \infty)$, $0 < \mu := \int_0^\infty x dF(x) < \infty$ and $F \in$ NBUE. Recall that $F \in$ NBUE if and only if $\varepsilon_F(0) \geq \varepsilon_F(t)$ for all $t \geq 0$, where $\varepsilon_F(t)$ is the mean residual life at time t, defined in (1.8).

Also recall that if $F \in$ NBUE and has positive finite mean μ, then F has finite p-th moment for all $p \geq 1$ (p. 91).

Suppose X_1 and X_2 are two independent copies of a nonnegative random variable X with some df $F \in \mathcal{H}$.

In Section 5.2 two new characterizations of the exponential distribution are presented. In Section 5.3 two new goodness-of-fit tests based on these characterizations are developed and discussed.

5.2 New characterizations

The two new characterizations of the exponential distribution are presented and proved in two theorems:

Theorem 5.1 The following two statements are equivalent:

1. $F = F_0$

2. $\sigma^2 = \mu E(X_{2:2} - X_{1:2})$ and $F \in \mathcal{H}$,

where $\mu := E(X)$ and $\sigma^2 := Var(X)$.

Proof. It is well known that σ^2 can be written as

$$\begin{aligned} \sigma^2 &= \int_0^\infty \int_0^\infty \{F(\min(x,y)) - F(x)F(y)\}\, dydx \qquad (5.2) \\ &= 2\int_0^\infty F(x) \int_x^\infty (1 - F(y))\, dydx \\ &= 2\int_0^\infty F(x)(1 - F(x))\, \varepsilon_F(x) dx. \end{aligned}$$

Also, it follows from David (1981) that

$$E(X_{2:2} - X_{1:2}) = 2\int_0^\infty F(y)(1 - F(y))\, dy. \qquad (5.3)$$

Hence, from (5.2) and (5.3) and the fact that $\mu = \varepsilon_F(0)$ we obtain

$$\begin{aligned} \Delta_1(F) &:= \mu E(X_{2:2} - X_{1:2}) - \sigma^2 \qquad (5.4) \\ &= 2\int_0^\infty F(x)(1 - F(x)) \{\varepsilon_F(0) - \varepsilon_F(x)\}\, dx, \end{aligned}$$

so that $\Delta_1(F) \geq 0$ for all $F \in \mathcal{H}$.

Obviously, if $F = F_0$ then $\Delta_1(F_0) = 0$ and $F_0 \in \mathcal{H}$. Conversely, if $\Delta_1(F) = 0$ and $F \in \mathcal{H}$, the continuity of F and the fact that $F \in$ NBUE immediately imply that

$$\varepsilon_F(x) = \varepsilon_F(0) \ a.e.(dF). \qquad (5.5)$$

A well-known characterization theorem of Dallas (1973) states that the exponential distribution F_0 is the only distribution for which (5.5) holds. Hence, $F = F_0$ which completes the proof of the theorem.

Remark. It is clear from the proof of Theorem 5.1 that, if $F \in \mathcal{H}$ and $F \neq F_0$, then the continuity of F implies (see (5.4)) that $\Delta_1(F) > 0$. Furthermore, since $E(X_{1:2}) = 2\mu - E(X_{2:2})$, it easily follows that $\Delta_1(F) = 0$ is equivalent to

$$\frac{2}{\mu} E(X_{2:2}) - \frac{1}{\mu^2} E(X^2) - 1 = 0. \qquad (5.6)$$

Theorem 5.2 The following two statements are equivalent:

1. $F = F_0$

2. $F \in \mathcal{H}$ and

$$\frac{a}{\mu^2} E\left(X_{1:2}^2\right) + \frac{(a+2b)}{\mu} E(X_{1:2}) - \frac{c}{\mu^2} E(X_1^2) - (a+b-2c) = 0,$$

for any constants $a \geq 0$, $b \geq 0$, $c \geq 0$, such that $a+b+c > 0$.

Proof. Consider the following measure of the deviation of F from F_0,

$$\begin{aligned}
\Delta_2(F) &:= 2\int_0^\infty \{(ay+b\mu)f(y)+c\}(1-F(y))\{\varepsilon_F(0)-\varepsilon_F(y)\}\,dy \\
&= 2\mu\int_0^\infty \{(ay+b\mu)f(y)+c\}(1-F(y))\,dy \\
&\quad -2\int_0^\infty \{(ay+b\mu)f(y)+c\}\int_y^\infty (1-F(t))\,dtdy. \qquad (5.7)
\end{aligned}$$

Note that $\Delta_2(F) \geq 0$ for all $F \in \mathcal{H}$, with $\Delta_2(F_0) = 0$.
From David (1981) it follows that

$$E(X_1) = \int_0^\infty (1-F(y))\,dy, \qquad (5.8)$$

$$E(X_1^2) = 2\int_0^\infty y\,(1-F(y))\,dy, \qquad (5.9)$$

$$E(X_{1:2}) = 2\int_0^\infty yf(y)\,(1-F(y))\,dy, \qquad (5.10)$$

$$E(X_{1:2}^2) = 2\int_0^\infty y\,(1-F(y))^2\,dy. \qquad (5.11)$$

Interchanging the order of integration and using expressions (5.8) and (5.10), it easily follows that

$$\begin{aligned}
\Delta_2(F) &= a\mu E(X_{1:2}) + b\mu^2 + 2c\mu^2 - 2a\int_0^\infty (1-F(t))\int_0^t ydF(y)dt \\
&\quad -2b\mu\int_0^\infty (1-F(t))\int_0^t dF(y)dt - 2c\int_0^\infty (1-F(t))\int_0^t dydt. \qquad (5.12)
\end{aligned}$$

Note that by applying partial integration and the expression given in (5.11) we obtain

$$\begin{aligned}
& 2\int_0^\infty (1-F(t))\int_0^t ydF(y)dt \\
= \; & -2\int_0^\infty (1-F(t))\int_0^t yd\,(1-F(y))\,dt \\
= \; & -2\int_0^\infty t\,(1-F(t))^2\,dt + 2\int_0^\infty (1-F(t))\int_0^t (1-F(y))\,dydt \\
= \; & -E(X_{1:2}^2) + 2\int_0^\infty (1-F(t))\int_0^t (1-F(y))\,dydt \\
=: \; & -E(X_{1:2}^2) + \mathcal{I}. \qquad (5.13)
\end{aligned}$$

Furthermore,

$$
\begin{aligned}
\mathcal{I} &= 2\int_0^\infty (1-F(t))\left(\mu - \int_t^\infty (1-F(y))\,dy\right)dt \\
&= 2\mu^2 - 2\int_0^\infty (1-F(t))\int_t^\infty (1-F(y))\,dydt \\
&= 2\mu^2 - 2\int_0^\infty (1-F(y))\int_0^y (1-F(t))\,dtdy \\
&= 2\mu^2 - \mathcal{I},
\end{aligned}
$$

so that

$$\mathcal{I} = \mu^2. \tag{5.14}$$

Also, from (5.3) we deduce that

$$
\begin{aligned}
&2\int_0^\infty (1-F(t))\int_0^t dF(y)dt \\
=\ & 2\int_0^\infty F(t)\,(1-F(t))\,dt \\
=\ & E(X_{2:2} - X_{1:2}),
\end{aligned} \tag{5.15}
$$

and from (5.9) it follows that

$$
\begin{aligned}
&2\int_0^\infty (1-F(t))\int_0^t dydt \\
=\ & E(X_1^2).
\end{aligned} \tag{5.16}
$$

Hence, substituting (5.13)-(5.16) into (5.12) we find that

$$
\begin{aligned}
\Delta_2(F) &= a\mu E(X_{1:2}) + b\mu^2 + 2c\mu^2 + aE(X_{1:2}^2) - a\mu^2 - b\mu E(X_{2:2} - X_{1:2}) - cE(X_1^2) \\
&= aE(X_{1:2}^2) + (a+2b)\mu E(X_{1:2}) - cE(X_1^2) - (a+b-2c)\mu^2,
\end{aligned} \tag{5.17}
$$

by using the fact that $E(X_{2:2}) = 2\mu - E(X_{1:2})$.

Obviously, if $F = F_0$ then $\Delta_2(F_0) = 0$ and $F_0 \in \mathcal{H}$. Conversely, if $\Delta_2(F) = 0$ and $F \in \mathcal{H}$ for some constants $a \geq 0$, $b \geq 0$ and $c \geq 0$, such that $a+b+c > 0$, the continuity of F and the fact that $F \in$ NBUE immediately imply that

$$\varepsilon_F(x) = \varepsilon_F(0) \quad \text{a.e. (w.r.t. Lebesgue measure)}. \tag{5.18}$$

Again using the characterization theorem of Dallas (1973) we conclude that $F = F_0$ and the proof is completed.

Remarks.

1. It is clear from the proof of Theorem 5.2 that, if $F \in \mathcal{H}$ and $F \neq F_0$, then the continuity of F implies (see first line of (5.7)) that $\Delta_2(F) > 0$.

2. Some interesting comments can be made regarding the equality in Theorem 5.2 (which is satisfied by only F_0 in $\mathcal{H}$) for certain choices of the constants a, b and c. A few special cases are discussed here:

   (a) For $a = 1$, $b = 0$ and $c = 0$, the equality becomes

   $$\frac{1}{\mu^2} E\left(X_{1:2}^2\right) + \frac{1}{\mu} E(X_{1:2}) - 1 = 0. \tag{5.19}$$

   (b) For $a = 0$, $b = 0$ and $c = 1$, the equality becomes

   $$\begin{aligned} \frac{1}{\mu^2} E(X_1^2) - 2 &= 0 \\ \Leftrightarrow E(X_1^2) &= 2\mu^2, \end{aligned} \tag{5.20}$$

   which is equivalent to the characterization on which the coefficient of variation test (p. 92) proposed by Borges et al. (1984) is based.

   (c) Letting $a = 0$, $b > 0$ and $c = 0$, we obtain

   $$\begin{aligned} \frac{2b}{\mu} E(X_{1:2}) - b &= 0 \\ \Leftrightarrow E(X_{1:2}) &= \frac{\mu}{2}, \end{aligned} \tag{5.21}$$

   which is equivalent to the characterization on which the well-known Gini test statistic (p. 64) is based.

   (d) For $a = 0$, $b = 2c$ and $c \neq 0$, the equation simplifies to

   $$E(X_1^2) = 4\mu E(X_{1:2}). \tag{5.22}$$

   (e) By choosing $c = \frac{a+b}{2}$, we obtain

   $$\frac{a}{\mu^2} E\left(X_{1:2}^2\right) + \frac{(a+2b)}{\mu} E(X_{1:2}) - \frac{(a+b)}{2\mu^2} E(X_1^2) = 0.$$

   In particular, for $a = 1$ and $b = \frac{1}{2}$, this reduces to

   $$4E(X_{1:2}^2) + 8\mu E(X_{1:2}) - 3E(X_1^2) = 0. \tag{5.23}$$

   (f) For the case $a = b = c \neq 0$, the equality becomes

   $$E(X_{1:2}^2) + 3\mu E(X_{1:2}) - E(X_1^2) = 0.$$

5.3 The proposed new goodness-of-fit test statistics

Consider a nonnegative continuous random variable X, with df F, representing a lifetime of some component or system. Suppose a random sample $X_1, \ldots, X_n$ of lifetimes from F is observed and we wish to test H_0: $F = F_0$ against H_1: $F \in$ NBUE and not exponential.

Suppose for the moment that μ is **known**.

From (5.6) define

$$\gamma_1(F) := \frac{2}{\mu} E(X_{2:2}) - \frac{1}{\mu^2} E\left(X^2\right) - 1. \tag{5.24}$$

Note that $\gamma_1(F) = 0$ under H_0 and $\gamma_1(F) > 0$ for all $F \in$ NBUE but $F \neq F_0$. An unbiased estimator of $\gamma_1(F)$ is the U-statistic

$$T_n := \frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le n} h_1(X_i, X_j), \tag{5.25}$$

with

$$h_1(x_1, x_2) = \frac{2}{\mu} \max(x_1, x_2) - \frac{1}{2\mu^2}(x_1^2 + x_2^2) - 1. \tag{5.26}$$

H_0 will be rejected for large values of T_n.

Similarly, from (5.17), define

$$\gamma_2(F) := \frac{\Delta_2(F)}{\mu^2} = \frac{a}{\mu^2} E\left(X_{1:2}^2\right) + \frac{(a+2b)}{\mu} E(X_{1:2}) - \frac{c}{\mu^2} E(X_1^2) - (a + b - 2c). \tag{5.27}$$

Note that $\gamma_2(F) = 0$ under H_0 and $\gamma_2(F) > 0$ for all $F \in$ NBUE but $F \neq F_0$. An unbiased estimator of $\gamma_2(F)$ is the U-statistic

$$S_n := \frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le n} h_2(X_i, X_j), \tag{5.28}$$

with

$$h_2(x_1, x_2) = \frac{a}{\mu^2} \left\{\min(x_1, x_2)\right\}^2 + \frac{(a+2b)}{\mu} \min(x_1, x_2) - \frac{c}{2\mu^2}(x_1^2 + x_2^2) - (a + b - 2c). \tag{5.29}$$

H_0 will be rejected for large values of S_n.

5.3.1 Consistency and Limiting Distributions

Note that

$$\gamma_1(F) = E_F\left\{h_1(X_1, X_2)\right\},$$

with h_1 defined in (5.26). It easily follows that $\gamma_1(F)$ can be written as

$$\gamma_1(F) = \frac{2}{\mu} \int_0^\infty \left(1 - F^2(t)\right) dt - \frac{2}{\mu^2} \int_0^\infty t\left(1 - F(t)\right) dt - 1. \tag{5.30}$$

(See Serfling (1980, pp. 46-47)).

Consider

$$g_1(x) := \int_0^\infty h_1(x,y)dF(y) - \gamma_1(F). \tag{5.31}$$

Using some algebra, it easily follows that

$$g_1(x) = \frac{2x}{\mu} + \frac{2}{\mu}\int_x^\infty (1 - F(y))\,dy - \frac{1}{2\mu^2}\left(x^2 + E(X^2)\right) - 1 - \gamma_1(F).$$

If $F = F_0$, then $g_1(x)$ becomes

$$g_{1,0}(x) = 2x\lambda + 2e^{-\lambda x} - \frac{1}{2}\lambda^2 x^2 - 2.$$

Also, define

$$\sigma_{g_1}^2 := Var\left(g_1(X)\right).$$

Since $E\left(g_1(X)\right) = 0$, it follows that for g_1 defined in (5.31) we have that

$$\begin{aligned}\sigma_{g_1}^2 &= \int_0^\infty g_1^2(x)dF(x) \qquad (5.32)\\ &= \int_0^\infty \left\{\frac{2x}{\mu} + \frac{2}{\mu}\int_x^\infty (1 - F(y))\,dy - \frac{1}{2\mu^2}\left(x^2 + E(X^2)\right) - 1 - \gamma_1(F)\right\}^2 dF(x).\end{aligned}$$

We are now able to establish the asymptotic distribution of T_n.

Theorem 5.3 If $E(X^4) < \infty$ and $\sigma_{g_1}^2 > 0$, then as $n \to \infty$,

$$\sqrt{n}\left(T_n - \gamma_1(F)\right) \xrightarrow{d} N(0, 4\sigma_{g_1}^2). \tag{5.33}$$

Proof. Note that $E(X^4) < \infty$ implies that $E_F\left(h_1^2(X_1, X_2)\right) < \infty$. Since T_n is a U-statistic, the proof of the theorem follows directly from Theorem A, p. 192 of Serfling (1980).

Remarks.

1. Under H_0, we have that $\gamma_1(F_0) = 0$ and $\sigma_{g_1}^2$ becomes $\sigma_{g_{1,0}}^2 = \frac{5}{6}$. We conclude that under H_0,
$$T_n \sim N\left(0, \frac{10}{3n}\right) \text{ as } n \to \infty.$$
2. The condition $E(X^4) < \infty$ is automatically satisfied if $F \in$ NBUE.
3. The test which rejects H_0 for large values of T_n is consistent against NBUE alternatives. This follows easily from Theorem 5.3 and the fact that $\gamma_1(F) > 0$ for $F \in$ NBUE and not exponential.

Similarly, note that

$$\gamma_2(F) = E_F\left\{h_2(X_1, X_2)\right\},$$

with h_2 defined in (5.29). It then follows that $\gamma_2(F)$ can be written as

$$\gamma_2(F) = \frac{2a}{\mu^2}\int_0^\infty t\,(1-F(t))^2\,dt + \frac{(a+2b)}{\mu}\int_0^\infty (1-F(t))^2\,dt - \frac{2c}{\mu^2}\int_0^\infty t\,(1-F(t))\,dt - (a+b-2c). \tag{5.34}$$

Define

$$g_2(x) := \int_0^\infty h_2(x,y)dF(y) - \gamma_2(F). \tag{5.35}$$

After tedious algebra, it follows that

$$\begin{aligned} g_2(x) &= \frac{x}{\mu}\left(\frac{x}{\mu}(a-\frac{c}{2}) + (a+2b)\right) - \frac{2a}{\mu^2}\int_0^x yF(y)dy - \frac{(a+2b)}{\mu}\int_0^x F(y)dy \\ &\quad -\frac{c}{2\mu^2}\int_0^\infty y^2 dF(y) - (a+b-2c) - \gamma_2(F). \end{aligned} \tag{5.36}$$

If $F = F_0$, then $g_2(x)$ becomes

$$g_{2,0}(x) = -2a\lambda x e^{-\lambda x} - (3a+2b)e^{-\lambda x} - \frac{c}{2}\lambda^2 x^2 + (2a+b+c).$$

Also, define

$$\sigma_{g_2}^2 := Var\left(g_2(X)\right).$$

Since $E\left(g_2(X)\right) = 0$, it follows that for g_2 defined in (5.35) we have that

$$\begin{aligned} \sigma_{g_2}^2 &= \int_0^\infty g_2^2(x)dF(x) \qquad (5.37) \\ &= \int_0^\infty \left\{\frac{x}{\mu}\left(\frac{x}{\mu}(a-\frac{c}{2}) + (a+2b)\right) - \frac{2a}{\mu^2}\int_0^x yF(y)dy - \frac{(a+2b)}{\mu}\int_0^x F(y)dy \right. \\ &\quad \left. -\frac{c}{2\mu^2}\int_0^\infty y^2 dF(y) - (a+b-2c) - \gamma_2(F)\right\}^2 dF(x). \end{aligned}$$

Again following standard U-statistics theory, we obtain the asymptotic distribution of S_n:

Theorem 5.4 If $E(X^4) < \infty$ and $\sigma_{g_2}^2 > 0$, then as $n \to \infty$,

$$\sqrt{n}\left(S_n - \gamma_2(F)\right) \xrightarrow{d} N(0, 4\sigma_{g_2}^2). \tag{5.38}$$

Proof. The proof is similar to that of Theorem 5.3.

Remarks.

1. Under H_0, we have that $\gamma_2(F_0) = 0$ and $\sigma^2_{g_2}$ becomes

$$\sigma^2_{g_{2,0}} = \frac{17}{27}a^2 + \frac{1}{3}b^2 + 5c^2 + \frac{8}{9}ab - \frac{5}{2}ac - \frac{3}{2}bc. \tag{5.39}$$

Hence, under H_0,

$$S_n \sim N\left(0, \frac{4}{n}\left(\frac{17}{27}a^2 + \frac{1}{3}b^2 + 5c^2 + \frac{8}{9}ab - \frac{5}{2}ac - \frac{3}{2}bc\right)\right) \text{ as } n \to \infty.$$

2. The test which rejects H_0 for large values of S_n is consistent against NBUE alternatives. This follows easily from Theorem 5.4 and the fact that $\gamma_2(F) > 0$ for $F \in$ NBUE and not exponential.

3. For the case where the parameter μ is **unknown**, μ can be estimated by $\hat{\mu} = \bar{X}$, the sample mean. Let T_n^* and S_n^* be defined as T_n in (5.25) and S_n in (5.28), respectively, with μ replaced by $\bar{X}$ in (5.26) and (5.29). Hence, T_n^* and S_n^* are scale-free statistics. By applying a result of Randles (1982) regarding U-statistics with estimated parameters, as was done by Janssen, Swanepoel and Veraverbeke (2005), it can easily be proved, under conditions similar to those stated in Theorems 1 and 2 of the latter authors, that $\sqrt{n}\,(T_n^* - \gamma_1(F)) \xrightarrow{d} N(0, \tau_1^2)$ and $\sqrt{n}\,(S_n^* - \gamma_2(F)) \xrightarrow{d} N(0, \tau_2^2)$, where

$$\tau_1^2 = 4\sigma^2_{g_1} + E\left[(X_1 - \mu)^2\left(\frac{d}{d\mu}\gamma_1(F)\right)^2\right] + 4E\left[g_1(X_1)(X_1 - \mu)\frac{d}{d\mu}\gamma_1(F)\right],$$

and

$$\tau_2^2 = 4\sigma^2_{g_2} + E\left[(X_1 - \mu)^2\left(\frac{d}{d\mu}\gamma_2(F)\right)^2\right] + 4E\left[g_2(X_1)(X_1 - \mu)\frac{d}{d\mu}\gamma_2(F)\right].$$

Under H_0, we have that $\gamma_1(F_0) = 0$, and after some algebra, τ_1^2 becomes

$$\tau^2_{1,0} = 4\sigma^2_{g_{1,0}} - 1 = \frac{7}{3}. \tag{5.40}$$

Thus, under H_0, $\sqrt{n}T_n^* \xrightarrow{d} N(0, \frac{7}{3})$.

Similarly, under H_0, we have that $\gamma_2(F_0) = 0$, and after tedious calculations, τ_2^2 becomes

$$\tau^2_{2,0} = 4\sigma^2_{g_{2,0}} - \frac{1}{4}(3a + 2b - 8c)^2 \tag{5.41}$$

Thus, under H_0,

$$\sqrt{n}S_n^* \xrightarrow{d} N\left(0, 4\left(\frac{17}{27}a^2 + \frac{1}{3}b^2 + 5c^2 + \frac{8}{9}ab - \frac{5}{2}ac - \frac{3}{2}bc\right) - \frac{1}{4}(3a + 2b - 8c)^2\right).$$

Hence, we suggest the following procedures for testing exponentiality against NBUE alternatives: Let

$$\tilde{T}_n := \frac{\sqrt{n}T_n^*}{\tau_{1,0}} \tag{5.42}$$

and

$$\tilde{S}_n := \frac{\sqrt{n}S_n^*}{\tau_{2,0}} \tag{5.43}$$

be the standardized test statistics. Then H_0 is rejected if $\tilde{T}_n > C_T(\alpha)$. Similarly, a test based on $\tilde{S}_n$ rejects H_0 if $\tilde{S}_n > C_S(\alpha)$. $C_T(\alpha)$ and $C_S(\alpha)$ are the critical values for $\tilde{T}_n$ and $\tilde{S}_n$ respectively, with significance level α.

4. Table 5.1 contains possible choices of a, b and c, together with the corresponding values of $\tau_{2,0}^2$.

   - The test statistic $\tilde{S}_n^{(1)}$ was obtained by setting $a = c = 0$, and since $a + b + c > 0$, we have that $b > 0$. Without loss of generality we can use $b = 1$. Similarly, for $\tilde{S}_n^{(2)}$ we choose $a = b = 0$ and $c = 1$, and for $\tilde{S}_n^{(3)}$ we choose $b = c = 0$ and $a = 1$. Note from remark 2(c) on p. 105 that $\tilde{S}_n^{(1)}$ is equivalent to the Gini test statistic of Gail and Gastwirth (1978a) and from remark 2(b) on p. 105 that $\tilde{S}_n^{(2)}$ is equivalent to the coefficient of variation test statistic of Borges et al. (1984).
   - The test statistics $\tilde{S}_n^{(4)}$ and $\tilde{S}_n^{(14)}$ were obtained by setting $a = c$ and partially differentiating the first term of $\tau_{2,0}^2$ to b and c respectively. Setting the partial derivatives equal to zero and solving for c in terms of b, we obtain $a = c = \frac{33b}{338}$, $b > 0$ for $\tilde{S}_n^{(4)}$ and $a = c = \frac{12b}{11}$, $b > 0$ for $\tilde{S}_n^{(14)}$. Without loss of generality we can again take $b = 1$. Similarly, for $\tilde{S}_n^{(5)}$ and $\tilde{S}_n^{(15)}$ we set $b = c$ to obtain $b = c = \frac{29a}{138}$, $a > 0$ and $b = c = \frac{612a}{783}$, $a > 0$, respectively. We take $a = 1$. For $\tilde{S}_n^{(16)}$ and $\tilde{S}_n^{(17)}$ we set $a = b$ to obtain $a = b = \frac{27c}{25}$, $c > 0$ and $a = b = \frac{5c}{2}$, $c > 0$, respectively. We use $c = 1$.
   - The test statistics $\tilde{S}_n^{(6)}$ and $\tilde{S}_n^{(7)}$ were obtained by setting $a = 0$, partially differentiating the first term of $\tau_{2,0}^2$ to b and c respectively, and solving for c in terms of b. For $\tilde{S}_n^{(6)}$ we have that $c = \frac{3b}{20}$, $b > 0$, and for $\tilde{S}_n^{(7)}$ we have that $c = \frac{4b}{9}$, $b > 0$. We use $b = 1$.

     Similarly, $\tilde{S}_n^{(8)}$ and $\tilde{S}_n^{(9)}$ were obtained by setting $b = 0$, giving $c = \frac{a}{4}$, $a > 0$ and $c = \frac{68a}{135}$, $a > 0$, respectively. We take $a = 1$.
   - The values of a, b and c for $\tilde{S}_n^{(10)}$ to $\tilde{S}_n^{(13)}$ were obtained by setting $c = 0$ and choosing a and b such that $a + b = 1$.

- Finally, the statistics $\tilde{S}_n^{(18)}$ to $\tilde{S}_n^{(20)}$ were obtained from remarks 2(d) to 2(f) on p. 105. We use $c = 1$ for $\tilde{S}_n^{(18)}$ and $a = b = c = 1$ for $\tilde{S}_n^{(20)}$.

In further discussions below we will consider $\tilde{S}_n^{(1)}$ to $\tilde{S}_n^{(13)}$, and disregard $\tilde{S}_n^{(14)}$ to $\tilde{S}_n^{(20)}$ due to their large variances under H_0.

Table 5.1: Value of $\tau_{2,0}^2$ for different choices of a, b and c.

Statistic	a	b	c	$\tau_{2,0}^2$
$\tilde{S}_n^{(1)}$	0	1	0	0.333
$\tilde{S}_n^{(2)}$	0	0	1	4.000
$\tilde{S}_n^{(3)}$	1	0	0	0.269
$\tilde{S}_n^{(4)}$	$\frac{33}{338}$	1	$\frac{33}{338}$	0.643
$\tilde{S}_n^{(5)}$	1	$\frac{29}{138}$	$\frac{29}{138}$	1.085
$\tilde{S}_n^{(6)}$	0	1	$\frac{3}{20}$	0.723
$\tilde{S}_n^{(7)}$	0	1	$\frac{4}{9}$	2.012
$\tilde{S}_n^{(8)}$	1	0	$\frac{1}{4}$	1.019
$\tilde{S}_n^{(9)}$	1	0	$\frac{68}{135}$	2.291
$\tilde{S}_n^{(10)}$	$\frac{1}{4}$	$\frac{3}{4}$	0	0.308
$\tilde{S}_n^{(11)}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0.289
$\tilde{S}_n^{(12)}$	$\frac{2}{3}$	$\frac{1}{3}$	0	0.280
$\tilde{S}_n^{(13)}$	$\frac{3}{4}$	$\frac{1}{4}$	0	0.276
$\tilde{S}_n^{(14)}$	$\frac{12}{11}$	1	$\frac{12}{11}$	10.581
$\tilde{S}_n^{(15)}$	1	$\frac{612}{783}$	$\frac{612}{783}$	6.135
$\tilde{S}_n^{(16)}$	$\frac{27}{25}$	$\frac{27}{25}$	1	9.670
$\tilde{S}_n^{(17)}$	$\frac{5}{2}$	$\frac{5}{2}$	1	21.234
$\tilde{S}_n^{(18)}$	0	2	1	9.333
$\tilde{S}_n^{(19)}$	1	$\frac{1}{2}$	$\frac{3}{4}$	5.130
$\tilde{S}_n^{(20)}$	1	1	1	9.157

5.3.2 Pitman and approximate Bahadur asymptotic efficiency

In this section we consider both the Pitman and approximate Bahadur asymptotic efficiencies (see, e.g., Serfling (1980)) of our test statistics, compared to that of the following tests for exponentiality against NBUE alternatives: the coefficient of variation test $S/\bar{X}$ of Borges et al. (1984), as discussed on p. 92, the test based on K^* defined by Hollander and Proschan (1975) and discussed on p. 89, and the test based on $\hat{\Delta}^{(3)}_{r+2}$ of Ahmad (2001) (see p. 93), with $r = 0$.

Consider a sequence of alternative distribution functions indexed by the parameter θ_n, where $\theta_n = \theta_0 + cn^{-\frac{1}{2}}$, c is an arbitrary positive constant, and θ_0 corresponds to the exponential distribution F_0. The Pitman asymptotic efficiency of a test statistic V_n is defined for a given alternative df F_θ as

$$e(V_n) = \frac{(\eta'(F_{\theta_0}))^2}{\sigma_0^2},$$

where $\eta(F_\theta)$ is the asymptotic expected value of V_n, $\eta'(F_{\theta_0}) = \frac{d}{d\theta}\eta(F_\theta)|_{\theta\to\theta_0}$, and $\sigma_0^2 := \lim_{n\to\infty} Var_{\theta_0}(\sqrt{n}V_n)$.

We know that $\tilde{T}_n$ is an asymptotically unbiased estimator of $\gamma_1(F)$ in (5.24) and that $\tau^2_{1,0} = \frac{7}{3}$. Therefore, $e(\tilde{T}_n) = \frac{(\gamma_1'(F_{\theta_0}))^2}{\frac{7}{3}}$.

After tedious algebra, using $\bar{F}_{\theta_0}(x) = \bar{F}_0 = e^{-x}$ (choose $\lambda = 1$ in (5.1) without loss of generality, since all the test statistics are scale-free) and the fact that $\mu'(F_{\theta_0}) = \int_0^\infty \bar{F}'_{\theta_0}(x)dx$, where $\bar{F}'_{\theta_0}(x) = \frac{d}{d\theta}\bar{F}_\theta(x)|_{\theta\to\theta_0}$, we obtain

$$\begin{aligned} \gamma_1'(F_{\theta_0}) &= \frac{d}{d\theta}\gamma_1(F_\theta)|_{\theta\to\theta_0} \\ &= \int_0^\infty \bar{F}'_{\theta_0}(x)\left[5 - 4e^{-x} - 2x\right] dx. \end{aligned} \tag{5.44}$$

Similarly, $\tilde{S}_n$ is an asymptotically unbiased estimator of $\gamma_2(F)$ given in (5.34) and

$$\tau^2_{2,0} = 4\left(\frac{17}{27}a^2 + \frac{1}{3}b^2 + 5c^2 + \frac{8}{9}ab - \frac{5}{2}ac - \frac{3}{2}bc\right) - \frac{1}{4}(3a + 2b - 8c)^2.$$

Therefore, $e(\tilde{S}_n) = \frac{(\gamma_2'(F_{\theta_0}))^2}{\tau^2_{2,0}}$, with

$$\begin{aligned} \gamma_2'(F_{\theta_0}) &= \frac{d}{d\theta}\gamma_2(F_\theta)|_{\theta\to\theta_0} \\ &= \int_0^\infty \bar{F}'_{\theta_0}(x)\left[4axe^{-x} + 2(a+2b)e^{-x} - 2cx - \left(\frac{3a}{2} + b - 4c\right)\right] dx. \end{aligned} \tag{5.45}$$

Table 5.2 contains the Pitman asymptotic efficiencies of the test statistics for the following alternative distributions:

- The Weibull distribution: $F_1(x) = 1 - e^{-x^{\theta}}$, $x \geq 0$, $\theta \geq 1$,

- The linear failure rate (LFR) distribution: $F_2(x) = 1 - e^{-x-\frac{\theta}{2}x^2}$, $x \geq 0$, $\theta \geq 0$,

- The Makeham distribution: $F_3(x) = 1 - e^{-x-\theta(x+e^{-x}-1)}$, $x \geq 0$, $\theta \geq 0$.

All three these distributions are IFR and hence NBUE.

For the Weibull and Makeham distributions, the highest Pitman efficiencies are associated with the test statistics $\tilde{S}_n^{(1)}$, $\tilde{S}_n^{(4)}$, $\tilde{S}_n^{(6)}$, $\tilde{S}_n^{(10)}$ and K^*, all of which are of similar magnitude. For linear failure rate alternatives, $\tilde{S}_n^{(5)}$, $\tilde{S}_n^{(7)}$, $\tilde{S}_n^{(8)}$ and $\tilde{S}_n^{(9)}$ have efficiencies close to 1, which is the efficiency of $\tilde{S}_n^{(2)}$ and $S/\bar{X}$.

Table 5.2: Pitman efficiencies.

Test statistic	**Weibull**	**Linear Failure Rate**	**Makeham**
$\tilde{T}_n$	0.31	0.41	0.02
$\tilde{S}_n^{(1)}$	1.44	0.75	0.08
$\tilde{S}_n^{(2)}$	1.00	1.00	0.06
$\tilde{S}_n^{(3)}$	1.08	0.93	0.07
$\tilde{S}_n^{(4)}$	1.38	0.86	0.08
$\tilde{S}_n^{(5)}$	1.13	0.97	0.07
$\tilde{S}_n^{(6)}$	1.36	0.88	0.08
$\tilde{S}_n^{(7)}$	1.24	0.96	0.08
$\tilde{S}_n^{(8)}$	1.06	0.98	0.07
$\tilde{S}_n^{(9)}$	1.04	0.99	0.07
$\tilde{S}_n^{(10)}$	1.39	0.81	0.08
$\tilde{S}_n^{(11)}$	1.31	0.86	0.08
$\tilde{S}_n^{(12)}$	1.25	0.89	0.08
$\tilde{S}_n^{(13)}$	1.21	0.91	0.08
$S/\bar{X}$	1.00	1.00	0.06
K^*	1.44	0.75	0.08
$\hat{\Delta}_2^{(3)}$	1.00	1.00	0.06

Definition 5.1 *For a sequence $\{V_n\}$ of real-valued statistics based on a random sample of size n from the distribution P_θ, $\theta \in \Omega$, and such that*

- $P_\theta(V_n \leq x) \to G(x)$ *if* $n \to \infty$ *for all* $\theta \in \Omega_0 \subset \Omega$ *and all* $x \in \mathcal{R}$, *where* G *is a proper absolutely continuous distribution;*
- $\log(1 - G(x)) = -\left(\frac{kx^2}{2}\right)(1 + o(1))$ *if* $x \to \infty$;
- $\frac{V_n}{\sqrt{n}} \to m(\theta)$ *in probability for all* $\theta \in \Omega - \Omega_0$, *where* $m(\theta)$ *is a real-valued function with* $0 < m(\theta) < \infty$;

the approximate Bahadur slope of V_n *is defined as* $h(\theta) = k.m^2(\theta)$ *(Borges et al. 1984).*

Now, under H_0, $\tilde{T}_n \to N(0,1)$. Therefore, G in Definition 5.1 is the standard normal df $\Phi(x)$, and it is known that as $x \to \infty$,

$$\log(1 - \Phi(x)) = \frac{-x^2}{2}(1 + o(1)),$$

so that $k = 1$. Further, for $\theta \neq \theta_0$,

$$\frac{\tilde{T}_n}{\sqrt{n}} \xrightarrow{P_\theta} \frac{\gamma_1(F_\theta)}{\tau_{1,0}}.$$

Therefore, the approximate Bahadur slope of $\tilde{T}_n$ is given by (see (5.24) and (5.40)),

$$h_{\tilde{T}_n}(\theta) = \frac{3}{7}\left(\frac{2}{\mu}E(X_{2:2}) - \frac{1}{\mu^2}E(X^2) - 1\right)^2. \tag{5.46}$$

Similarly, for $\tilde{S}_n$, we find that

$$h_{\tilde{S}_n}(\theta) = \frac{\left(\frac{a}{\mu^2}E\left(X_{1:2}^2\right) + \frac{(a+2b)}{\mu}E(X_{1:2}) - \frac{c}{\mu^2}E(X_1^2) - (a + b - 2c)\right)^2}{4\left(\frac{17}{27}a^2 + \frac{1}{3}b^2 + 5c^2 + \frac{8}{9}ab - \frac{5}{2}ac - \frac{3}{2}bc\right) - \frac{1}{4}(3a + 2b - 8c)^2}. \tag{5.47}$$

In Table 5.3 we present the approximate Bahadur slopes for the different test statistics. The same distributions as in Table 5.2 are considered, together with the gamma distribution,

$$F_4(x) = \frac{1}{\Gamma(\theta)}\int_0^x t^{\theta-1}e^{-t}dt, \ x \geq 0, \quad \theta \geq 1.$$

For Weibull alternatives, K^* is outperformed by $\tilde{S}_n^{(1)}$ and $\tilde{S}_n^{(10)}$ to $\tilde{S}_n^{(13)}$. However, for gamma alternatives, K^* has the largest approximate Bahadur slope. For linear failure rate alternatives, all the tests based on $\tilde{S}_n$ have a much larger approximate Bahadur slope than K^*. For Makeham alternatives, the test statistics $\tilde{T}_n$, $\tilde{S}_n^{(2)}$, $S/\bar{X}$, K^* and $\hat{\Delta}_2^{(3)}$ are outperformed by the remaining test statistics, which have almost identical Bahadur slopes.

Table 5.3: Approximate Bahadur slopes.

Test statistic	**Weibull** ($\theta = 1.5$)	**Weibull** ($\theta = 2$)	**Weibull** ($\theta = 2.5$)	**Weibull** ($\theta = 3$)
$\tilde{T}_n$	0.03	0.04	0.04	0.03
$\tilde{S}_n^{(1)}$	0.20	0.52	0.80	1.06
$\tilde{S}_n^{(2)}$	0.07	0.13	0.17	0.19
$\tilde{S}_n^{(3)}$	0.16	0.44	0.71	1.00
$\tilde{S}_n^{(4)}$	0.17	0.42	0.63	0.83
$\tilde{S}_n^{(5)}$	0.13	0.32	0.47	0.63
$\tilde{S}_n^{(6)}$	0.16	0.38	0.56	0.73
$\tilde{S}_n^{(7)}$	0.12	0.27	0.38	0.48
$\tilde{S}_n^{(8)}$	0.12	0.27	0.40	0.53
$\tilde{S}_n^{(9)}$	0.10	0.22	0.32	0.40
$\tilde{S}_n^{(10)}$	0.20	0.51	0.80	1.08
$\tilde{S}_n^{(11)}$	0.19	0.50	0.79	1.07
$\tilde{S}_n^{(12)}$	0.18	0.49	0.77	1.06
$\tilde{S}_n^{(13)}$	0.18	0.48	0.76	1.04
$S/\bar{X}$	0.10	0.23	0.33	0.40
K^*	0.17	0.41	0.63	0.84
$\hat{\Delta}_2^{(3)}$	0.05	0.08	0.10	0.12

Test statistic	**Gamma** ($\theta = 1.5$)	**Gamma** ($\theta = 2$)	**Gamma** ($\theta = 2.5$)	**Gamma** ($\theta = 3$)
$\tilde{T}_n$	0.01	0.03	0.03	0.04
$\tilde{S}_n^{(1)}$	0.07	0.19	0.30	0.42
$\tilde{S}_n^{(2)}$	0.03	0.06	0.09	0.11
$\tilde{S}_n^{(3)}$	0.04	0.13	0.22	0.32
$\tilde{S}_n^{(4)}$	0.06	0.15	0.25	0.34
$\tilde{S}_n^{(5)}$	0.04	0.11	0.18	0.24
$\tilde{S}_n^{(6)}$	0.06	0.15	0.23	0.31
$\tilde{S}_n^{(7)}$	0.04	0.11	0.17	0.22
$\tilde{S}_n^{(8)}$	0.04	0.10	0.15	0.21
$\tilde{S}_n^{(9)}$	0.03	0.08	0.13	0.17
$\tilde{S}_n^{(10)}$	0.06	0.18	0.29	0.40
$\tilde{S}_n^{(11)}$	0.06	0.16	0.27	0.38
$\tilde{S}_n^{(12)}$	0.05	0.15	0.26	0.36
$\tilde{S}_n^{(13)}$	0.05	0.15	0.25	0.35
$S/\bar{X}$	0.03	0.08	0.14	0.18
K^*	0.15	0.74	1.90	3.76
$\hat{\Delta}_2^{(3)}$	0.14	0.98	3.58	9.06

Continued on next page

Table 5.3: continued

Test statistic	**LFR** ($\theta = 0.5$)	**LFR** ($\theta = 1$)	**LFR** ($\theta = 1.5$)	**LFR** ($\theta = 2$)
$\tilde{T}_n$	0.02	0.03	0.03	0.04
$\tilde{S}_n^{(1)}$	0.04	0.07	0.10	0.12
$\tilde{S}_n^{(2)}$	0.02	0.04	0.05	0.07
$\tilde{S}_n^{(3)}$	0.04	0.07	0.09	0.10
$\tilde{S}_n^{(4)}$	0.04	0.07	0.09	0.11
$\tilde{S}_n^{(5)}$	0.04	0.06	0.08	0.09
$\tilde{S}_n^{(6)}$	0.04	0.06	0.08	0.11
$\tilde{S}_n^{(7)}$	0.03	0.05	0.07	0.09
$\tilde{S}_n^{(8)}$	0.03	0.06	0.07	0.08
$\tilde{S}_n^{(9)}$	0.03	0.05	0.07	0.08
$\tilde{S}_n^{(10)}$	0.04	0.07	0.10	0.12
$\tilde{S}_n^{(11)}$	0.04	0.08	0.10	0.11
$\tilde{S}_n^{(12)}$	0.04	0.08	0.10	0.11
$\tilde{S}_n^{(13)}$	0.04	0.07	0.10	0.11
$S/\bar{X}$	0.03	0.05	0.07	0.09
K^*	0.02	0.03	0.03	0.04
$\hat{\Delta}_2^{(3)}$	0.01	0.01	0.01	0.01

Test statistic	**Makeham** ($\theta = 0.5$)	**Makeham** ($\theta = 1$)	**Makeham** ($\theta = 1.5$)	**Makeham** ($\theta = 2$)
$\tilde{T}_n$	0.01	0.01	0.02	0.02
$\tilde{S}_n^{(1)}$	0.01	0.04	0.05	0.05
$\tilde{S}_n^{(2)}$	0.01	0.02	0.03	0.04
$\tilde{S}_n^{(3)}$	0.01	0.03	0.05	0.06
$\tilde{S}_n^{(4)}$	0.01	0.03	0.05	0.05
$\tilde{S}_n^{(5)}$	0.01	0.03	0.04	0.05
$\tilde{S}_n^{(6)}$	0.01	0.03	0.05	0.05
$\tilde{S}_n^{(7)}$	0.01	0.02	0.04	0.05
$\tilde{S}_n^{(8)}$	0.01	0.02	0.04	0.05
$\tilde{S}_n^{(9)}$	0.01	0.02	0.03	0.04
$\tilde{S}_n^{(10)}$	0.01	0.04	0.06	0.06
$\tilde{S}_n^{(11)}$	0.01	0.04	0.05	0.06
$\tilde{S}_n^{(12)}$	0.01	0.04	0.05	0.06
$\tilde{S}_n^{(13)}$	0.01	0.04	0.05	0.06
$S/\bar{X}$	0.01	0.02	0.03	0.05
K^*	0.01	0.02	0.02	0.02
$\hat{\Delta}_2^{(3)}$	0.00	0.00	0.00	0.00

Remark. Note that the test based on $\tilde{S}_n^{(2)}$ which rejects H_0 for large positive values is equivalent to the coefficient of variation test of Borges et al. (1984) which rejects H_0 for large negative values of $\sqrt{n}\left(\frac{S}{\bar{X}}-1\right)$, as well as the test based on $\hat{\Delta}_{r+2}^{(3)}$ of Ahmad (2001) with $r=0$, which also rejects H_0 for large positive values.

Surely the power of these tests will be the same and from Table 5.2 the Pitman efficiencies are also the same. However, the approximate Bahadur efficiencies for these three tests differ slightly, which can be seen as follows:

From Borges et al. (1984) it follows that the approximate Bahadur efficiency of the coefficient of variation test is given by $\left(\frac{\sigma}{\mu}-1\right)^2$. By setting $a=b=0$ and $c=1$ in equation (5.47), we obtain that the approximate Bahadur efficiency of $\tilde{S}_n^{(2)}$ is equal to

$$
\begin{aligned}
h_{\tilde{S}_n^{(2)}}(\theta) &= \frac{\left(-\frac{E(X_1^2)}{\mu^2}+2\right)^2}{4} \\
&= \frac{\left(\frac{\sigma^2}{\mu^2}-1\right)^2}{4} \\
&= \frac{\left(\frac{\sigma}{\mu}-1\right)^2\left(\frac{\sigma}{\mu}+1\right)^2}{4} \\
&\leq \left(\frac{\sigma}{\mu}-1\right)^2, \qquad (5.48)
\end{aligned}
$$

since in the NBUE-class of alternative distributions $\frac{\sigma}{\mu}\leq 1$ (see p. 42), with equality if and only $F=F_0$.

This is an example that shows that the approximate Bahadur efficiency is not a convincing measure of efficiency, except for alternatives close to the null distribution (for the example above this is distributions for which $\frac{\sigma}{\mu}$ is close to 1). This agrees with a result obtained by Verbeke (1989, pp. 54-56).

Similarly, the approximate Bahadur slope of $\hat{\Delta}_2^{(3)}$ is found to be

$$
\begin{aligned}
h_{\hat{\Delta}_2^{(3)}}(\theta) &= \frac{\left(\mu^2-\frac{E(X_1^2)}{2}\right)^2}{1} \\
&= \frac{\mu^4}{4}\left(\frac{\sigma}{\mu}-1\right)^2\left(\frac{\sigma}{\mu}+1\right)^2 \\
&\leq \mu^4\left(\frac{\sigma}{\mu}-1\right)^2. \qquad (5.49)
\end{aligned}
$$

If $\mu\leq 1$, then $h_{\hat{\Delta}_2^{(3)}}(\theta)\leq\left(\frac{\sigma}{\mu}-1\right)^2$. This is also clear from Table 5.3, since for the Weibull, linear failure rate and Makeham distributions for which $\mu<1$, the Bahadur slope of $\hat{\Delta}_2^{(3)}$ is much smaller than that of $S/\bar{X}$.

5.3.3 Power comparisons

In this section we present the results of Monte Carlo studies conducted to determine the power of tests based on $\tilde{T}_n$ and $\tilde{S}_n$, the K^*-statistic of Hollander and Proschan (1975) (p. 89) and the V_n-test of Kanjo (1993) (p. 92). All calculations were done using double precision arithmetic in FORTRAN and routines from the IMSL library.

Recall that H_0 is rejected if $\tilde{T}_n > C_T(\alpha)$. Similarly, a test based on $\tilde{S}_n$ rejects H_0 if $\tilde{S}_n > C_S(\alpha)$. From Hollander and Proschan (1975) we know that, under H_0, $\sqrt{n}K^* \xrightarrow{d} N\left(0, \frac{1}{12}\right)$ and H_0 is rejected if $\sqrt{12n}K^* > C_{K^*}(\alpha)$. For V_n of Kanjo (1993) we have that, under H_0, $\sqrt{n}\left(V_n - \frac{1}{2}\right) \xrightarrow{d} N\left(0, \frac{1}{3}\right)$. However, Kanjo (1993) remarked that the asymptotic test performs very poor compared to the exact test due to slow convergence to normality. We therefore use the exact test which rejects H_0 if $V_n < C_{V_n}(\alpha)$.

The critical values used for the simulations are given in Table 5.4. For the right-sided tests, the critical values were obtained as the 95-th percentiles of 200,000 simulated test statistics from the exponential distribution. For the left-sided test of Kanjo (1993), the 5-th percentile was obtained. All standard errors of the estimated critical values were found to be negligibly small and are therefore not reported in Table 5.4. The estimated critical values of $\tilde{S}_n^{(1)}$, $\tilde{S}_n^{(4)}$, $\tilde{S}_n^{(10)}$ and $\tilde{S}_n^{(11)}$ are quite close to the asymptotic critical value of 1.645.

Power estimates were calculated as the proportion of 20,000 Monte Carlo samples that resulted in rejection of H_0 at significance level $\alpha = 0.05$ for the four alternative distributions considered in the previous section.

We also consider the following two distributions, which are NBUE:

$$\begin{aligned} F_5(x) &= 0, & 0 \le x < \theta \\ &= 1 - \frac{1}{\beta}e^{-(x-\theta)}, & x \ge \theta \end{aligned}$$

with $0 < \theta < 1$, $1 \le \beta \le \frac{1}{1-\theta}$ and $\frac{1}{\beta} < e^{-\theta}$, and

$$\begin{aligned} F_6(x) &= \frac{x}{\theta + 1}, & 0 \le x < \theta \\ &= 1 - \frac{1}{\theta+1}e^{-(x-\theta)}, & x \ge \theta \end{aligned}$$

with $\theta \ge 0$.

In Tables 5.5 - 5.10 we present power comparisons for sample sizes $n = 20, 30, \ldots, 100$ for different parameter choices of each of the alternative distributions. The standard errors of the estimated probabilities in these tables are no greater than $\sqrt{0.25/20,000} = 0.0035$.

Table 5.4: Estimated critical values based on 200,000 trials.

n	$C_{\tilde{T}_n}(\alpha)$	$C_{\tilde{S}_n^{(1)}}(\alpha)$	$C_{\tilde{S}_n^{(2)}}(\alpha)$	$C_{\tilde{S}_n^{(3)}}(\alpha)$	$C_{\tilde{S}_n^{(4)}}(\alpha)$	$C_{\tilde{S}_n^{(5)}}(\alpha)$	$C_{\tilde{S}_n^{(6)}}(\alpha)$	$C_{\tilde{S}_n^{(7)}}(\alpha)$
20	0.989	1.681	1.209	1.551	1.602	1.452	1.563	1.437
30	1.088	1.673	1.281	1.548	1.610	1.471	1.571	1.475
40	1.152	1.680	1.326	1.558	1.609	1.489	1.584	1.496
50	1.193	1.658	1.350	1.551	1.618	1.500	1.593	1.511
60	1.230	1.660	1.378	1.559	1.614	1.508	1.591	1.508
70	1.257	1.658	1.393	1.567	1.611	1.516	1.598	1.527
80	1.282	1.642	1.409	1.558	1.611	1.529	1.602	1.532
90	1.301	1.651	1.422	1.574	1.613	1.529	1.607	1.542
100	1.315	1.651	1.435	1.574	1.620	1.536	1.598	1.547

n	$C_{\tilde{S}_n^{(8)}}(\alpha)$	$C_{\tilde{S}_n^{(9)}}(\alpha)$	$C_{\tilde{S}_n^{(10)}}(\alpha)$	$C_{\tilde{S}_n^{(11)}}(\alpha)$	$C_{\tilde{S}_n^{(12)}}(\alpha)$	$C_{\tilde{S}_n^{(13)}}(\alpha)$	$C_{V_n}(\alpha)$	$C_{K^*}(\alpha)$
20	1.387	1.336	1.656	1.617	1.596	1.583	0.421	1.608
30	1.424	1.383	1.652	1.620	1.593	1.588	0.431	1.620
40	1.447	1.413	1.643	1.614	1.593	1.581	0.438	1.631
50	1.466	1.432	1.636	1.611	1.597	1.585	0.443	1.620
60	1.477	1.452	1.636	1.609	1.601	1.585	0.447	1.630
70	1.491	1.455	1.636	1.614	1.591	1.584	0.450	1.638
80	1.498	1.471	1.634	1.607	1.604	1.587	0.453	1.635
90	1.505	1.475	1.635	1.614	1.610	1.596	0.455	1.629
100	1.516	1.490	1.635	1.625	1.609	1.595	0.457	1.638

From these tables we can conclude the following:

- For the Weibull distribution, all the tests perform rather well, except $\tilde{T}_n$.
- The best tests against Weibull and gamma alternatives are $\tilde{S}_n^{(1)}$, $\tilde{S}_n^{(4)}$ and $\tilde{S}_n^{(6)}$, followed by $\tilde{S}_n^{(10)}$ and $\tilde{S}_n^{(7)}$.
- Against linear failure rate alternatives all the tests yield approximately the same power.
- Against Makeham alternatives, all the tests except $\tilde{T}_n$ have almost the same power, the latter performing slightly worse.
- For $F_5(x)$ the test based on $\tilde{S}_n^{(1)}$ outperforms all the other tests based on $\tilde{S}_n$.
- Against $F_6(x)$ alternatives it is again $\tilde{S}_n^{(1)}$, $\tilde{S}_n^{(4)}$ and $\tilde{S}_n^{(6)}$ (with almost identical power), followed by $\tilde{S}_n^{(10)}$ and $\tilde{S}_n^{(7)}$, which perform the best.
- For all the alternatives, the best tests based on $\tilde{S}_n$ perform similar to the tests based on V_n and K^*.

Table 5.5: Estimated power functions for Weibull alternatives based on 20,000 trials.

Weibull ($\theta = 1.5$):

n	$C_{T_n^*}(\alpha)$	$C_{\tilde{S}_n^{(1)}}(\alpha)$	$C_{\tilde{S}_n^{(2)}}(\alpha)$	$C_{\tilde{S}_n^{(3)}}(\alpha)$	$C_{\tilde{S}_n^{(4)}}(\alpha)$	$C_{\tilde{S}_n^{(5)}}(\alpha)$	$C_{\tilde{S}_n^{(6)}}(\alpha)$	$C_{\tilde{S}_n^{(7)}}(\alpha)$
20	0.35	0.64	0.62	0.55	0.63	0.58	0.64	0.63
30	0.59	0.82	0.79	0.74	0.82	0.77	0.82	0.81
40	0.73	0.91	0.88	0.85	0.91	0.88	0.91	0.91
50	0.82	0.96	0.94	0.92	0.96	0.94	0.96	0.96
60	0.88	0.98	0.97	0.96	0.98	0.97	0.98	0.98
70	0.93	0.99	0.98	0.98	0.99	0.98	0.99	0.99
80	0.95	0.99	0.99	0.99	0.99	0.99	0.99	0.99
90	0.97	0.99	0.99	0.99	0.99	0.99	0.99	0.99
100	0.98	0.99	0.99	0.99	0.99	0.99	0.99	0.99
n	$C_{\tilde{S}_n^{(8)}}(\alpha)$	$C_{\tilde{S}_n^{(9)}}(\alpha)$	$C_{\tilde{S}_n^{(10)}}(\alpha)$	$C_{\tilde{S}_n^{(11)}}(\alpha)$	$C_{\tilde{S}_n^{(12)}}(\alpha)$	$C_{\tilde{S}_n^{(13)}}(\alpha)$	$C_{V_n}(\alpha)$	$C_{K^*}(\alpha)$
20	0.57	0.58	0.63	0.61	0.59	0.58	0.63	0.63
30	0.76	0.76	0.81	0.80	0.78	0.77	0.82	0.82
40	0.87	0.87	0.91	0.90	0.89	0.88	0.92	0.91
50	0.93	0.93	0.96	0.95	0.94	0.94	0.96	0.96
60	0.96	0.96	0.98	0.98	0.97	0.97	0.98	0.98
70	0.98	0.98	0.99	0.99	0.98	0.98	0.99	0.99
80	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99
90	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99
100	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99

Weibull ($\theta = 2$):

n	$C_{T_n^*}(\alpha)$	$C_{\tilde{S}_n^{(1)}}(\alpha)$	$C_{\tilde{S}_n^{(2)}}(\alpha)$	$C_{\tilde{S}_n^{(3)}}(\alpha)$	$C_{\tilde{S}_n^{(4)}}(\alpha)$	$C_{\tilde{S}_n^{(5)}}(\alpha)$	$C_{\tilde{S}_n^{(6)}}(\alpha)$	$C_{\tilde{S}_n^{(7)}}(\alpha)$
20	0.37	0.97	0.97	0.95	0.97	0.96	0.97	0.97
30	0.86	0.99	0.99	0.99	0.99	0.99	0.99	0.99
40	0.98	0.99	0.99	0.99	0.99	0.99	0.99	0.99
50	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00
60	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00
70	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00
80	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
90	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
100	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
n	$C_{\tilde{S}_n^{(8)}}(\alpha)$	$C_{\tilde{S}_n^{(9)}}(\alpha)$	$C_{\tilde{S}_n^{(10)}}(\alpha)$	$C_{\tilde{S}_n^{(11)}}(\alpha)$	$C_{\tilde{S}_n^{(12)}}(\alpha)$	$C_{\tilde{S}_n^{(13)}}(\alpha)$	$C_{V_n}(\alpha)$	$C_{K^*}(\alpha)$
20	0.96	0.96	0.97	0.97	0.96	0.96	0.97	0.97
30	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99
40	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99
50	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
60	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
70	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
80	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
90	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
100	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

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Table 5.5: continued

Weibull ($\theta = 2.5$):								
n	$C_{T_n^*}(\alpha)$	$C_{\tilde{S}_n^{(1)}}(\alpha)$	$C_{\tilde{S}_n^{(2)}}(\alpha)$	$C_{\tilde{S}_n^{(3)}}(\alpha)$	$C_{\tilde{S}_n^{(4)}}(\alpha)$	$C_{\tilde{S}_n^{(5)}}(\alpha)$	$C_{\tilde{S}_n^{(6)}}(\alpha)$	$C_{\tilde{S}_n^{(7)}}(\alpha)$
20	0.13	0.99	0.99	0.99	0.99	0.99	0.99	0.99
30	0.61	1.00	1.00	1.00	1.00	1.00	1.00	1.00
40	0.94	1.00	1.00	1.00	1.00	1.00	1.00	1.00
50	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00
60	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
70	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
80	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
90	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
100	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
n	$C_{\tilde{S}_n^{(8)}}(\alpha)$	$C_{\tilde{S}_n^{(9)}}(\alpha)$	$C_{\tilde{S}_n^{(10)}}(\alpha)$	$C_{\tilde{S}_n^{(11)}}(\alpha)$	$C_{\tilde{S}_n^{(12)}}(\alpha)$	$C_{\tilde{S}_n^{(13)}}(\alpha)$	$C_{V_n}(\alpha)$	$C_{K^*}(\alpha)$
20	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99
30	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
40	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
50	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
60	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
70	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
80	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
90	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
100	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 5.6: Estimated power functions for gamma alternatives based on 20,000 trials.

Gamma ($\theta = 1.5$):								
n	$C_{T_n^*}(\alpha)$	$C_{\tilde{S}_n^{(1)}}(\alpha)$	$C_{\tilde{S}_n^{(2)}}(\alpha)$	$C_{\tilde{S}_n^{(3)}}(\alpha)$	$C_{\tilde{S}_n^{(4)}}(\alpha)$	$C_{\tilde{S}_n^{(5)}}(\alpha)$	$C_{\tilde{S}_n^{(6)}}(\alpha)$	$C_{\tilde{S}_n^{(7)}}(\alpha)$
20	0.16	0.29	0.26	0.22	0.28	0.24	0.28	0.28
30	0.23	0.40	0.35	0.30	0.38	0.33	0.39	0.38
40	0.28	0.49	0.41	0.37	0.48	0.40	0.48	0.46
50	0.34	0.58	0.48	0.44	0.55	0.47	0.56	0.54
60	0.38	0.65	0.54	0.50	0.63	0.54	0.63	0.61
70	0.42	0.72	0.60	0.56	0.70	0.60	0.70	0.67
80	0.46	0.77	0.63	0.61	0.75	0.64	0.74	0.72
90	0.50	0.82	0.68	0.65	0.79	0.70	0.79	0.76
100	0.53	0.85	0.72	0.69	0.83	0.73	0.83	0.80
n	$C_{\tilde{S}_n^{(8)}}(\alpha)$	$C_{\tilde{S}_n^{(9)}}(\alpha)$	$C_{\tilde{S}_n^{(10)}}(\alpha)$	$C_{\tilde{S}_n^{(11)}}(\alpha)$	$C_{\tilde{S}_n^{(12)}}(\alpha)$	$C_{\tilde{S}_n^{(13)}}(\alpha)$	$C_{V_n}(\alpha)$	$C_{K^*}(\alpha)$
20	0.24	0.24	0.28	0.26	0.25	0.24	0.28	0.28
30	0.32	0.32	0.38	0.36	0.34	0.33	0.40	0.40
40	0.38	0.39	0.47	0.44	0.42	0.41	0.49	0.49
50	0.45	0.46	0.55	0.52	0.50	0.48	0.58	0.58
60	0.51	0.52	0.63	0.59	0.56	0.55	0.65	0.65
70	0.57	0.58	0.69	0.66	0.63	0.62	0.72	0.72
80	0.62	0.62	0.74	0.71	0.68	0.66	0.77	0.77
90	0.67	0.67	0.79	0.76	0.73	0.71	0.82	0.82
100	0.71	0.71	0.83	0.79	0.77	0.75	0.85	0.85

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Table 5.6: continued

Gamma ($\theta = 2$):								
n	$C_{T_n^*}(\alpha)$	$C_{\tilde{S}_n^{(1)}}(\alpha)$	$C_{\tilde{S}_n^{(2)}}(\alpha)$	$C_{\tilde{S}_n^{(3)}}(\alpha)$	$C_{\tilde{S}_n^{(4)}}(\alpha)$	$C_{\tilde{S}_n^{(5)}}(\alpha)$	$C_{\tilde{S}_n^{(6)}}(\alpha)$	$C_{\tilde{S}_n^{(7)}}(\alpha)$
20	0.26	0.62	0.55	0.48	0.60	0.51	0.61	0.59
30	0.44	0.79	0.70	0.64	0.77	0.68	0.77	0.75
40	0.57	0.89	0.80	0.76	0.88	0.80	0.88	0.86
50	0.65	0.95	0.87	0.85	0.94	0.87	0.94	0.92
60	0.72	0.97	0.91	0.90	0.97	0.92	0.97	0.96
70	0.78	0.98	0.94	0.94	0.98	0.95	0.98	0.97
80	0.83	0.99	0.96	0.96	0.99	0.97	0.99	0.99
90	0.86	0.99	0.97	0.98	0.99	0.98	0.99	0.99
100	0.89	0.99	0.98	0.98	0.99	0.99	0.99	0.99
n	$C_{\tilde{S}_n^{(8)}}(\alpha)$	$C_{\tilde{S}_n^{(9)}}(\alpha)$	$C_{\tilde{S}_n^{(10)}}(\alpha)$	$C_{\tilde{S}_n^{(11)}}(\alpha)$	$C_{\tilde{S}_n^{(12)}}(\alpha)$	$C_{\tilde{S}_n^{(13)}}(\alpha)$	$C_{V_n}(\alpha)$	$C_{K^*}(\alpha)$
20	0.50	0.51	0.59	0.56	0.54	0.52	0.61	0.61
30	0.66	0.67	0.76	0.73	0.71	0.69	0.79	0.78
40	0.78	0.78	0.87	0.85	0.82	0.81	0.89	0.89
50	0.86	0.86	0.94	0.92	0.90	0.89	0.95	0.95
60	0.91	0.91	0.97	0.95	0.94	0.93	0.97	0.97
70	0.94	0.94	0.98	0.97	0.97	0.96	0.98	0.98
80	0.96	0.96	0.99	0.99	0.98	0.98	0.99	0.99
90	0.98	0.98	0.99	0.99	0.99	0.99	0.99	0.99
100	0.98	0.98	0.99	0.99	0.99	0.99	0.99	0.99

Gamma ($\theta = 2.5$):								
n	$C_{T_n^*}(\alpha)$	$C_{\tilde{S}_n^{(1)}}(\alpha)$	$C_{\tilde{S}_n^{(2)}}(\alpha)$	$C_{\tilde{S}_n^{(3)}}(\alpha)$	$C_{\tilde{S}_n^{(4)}}(\alpha)$	$C_{\tilde{S}_n^{(5)}}(\alpha)$	$C_{\tilde{S}_n^{(6)}}(\alpha)$	$C_{\tilde{S}_n^{(7)}}(\alpha)$
20	0.29	0.84	0.78	0.71	0.83	0.75	0.83	0.82
30	0.60	0.95	0.90	0.87	0.94	0.89	0.95	0.94
40	0.75	0.99	0.96	0.94	0.98	0.96	0.98	0.98
50	0.83	0.99	0.98	0.98	0.99	0.98	0.99	0.99
60	0.89	0.99	0.99	0.99	0.99	0.99	0.99	0.99
70	0.93	0.99	0.99	0.99	0.99	0.99	0.99	0.99
80	0.95	0.99	0.99	0.99	0.99	0.99	0.99	0.99
90	0.97	1.00	0.99	0.99	1.00	0.99	1.00	1.00
100	0.98	1.00	0.99	0.99	1.00	0.99	1.00	1.00
n	$C_{\tilde{S}_n^{(8)}}(\alpha)$	$C_{\tilde{S}_n^{(9)}}(\alpha)$	$C_{\tilde{S}_n^{(10)}}(\alpha)$	$C_{\tilde{S}_n^{(11)}}(\alpha)$	$C_{\tilde{S}_n^{(12)}}(\alpha)$	$C_{\tilde{S}_n^{(13)}}(\alpha)$	$C_{V_n}(\alpha)$	$C_{K^*}(\alpha)$
20	0.73	0.74	0.82	0.8	0.77	0.76	0.84	0.84
30	0.88	0.88	0.94	0.93	0.91	0.90	0.95	0.95
40	0.95	0.95	0.98	0.98	0.97	0.97	0.99	0.99
50	0.98	0.98	0.99	0.99	0.99	0.98	0.99	0.99
60	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99
70	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99
80	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99
90	0.99	0.99	1.00	0.99	0.99	0.99	1.00	1.00
100	0.99	0.99	1.00	1.00	1.00	0.99	1.00	1.00

Table 5.7: Estimated power functions for LFR alternatives based on 20,000 trials.

Linear failure rate ($\theta = 0.5$):

n	$C_{T_n^*}(\alpha)$	$C_{\tilde{S}_n^{(1)}}(\alpha)$	$C_{\tilde{S}_n^{(2)}}(\alpha)$	$C_{\tilde{S}_n^{(3)}}(\alpha)$	$C_{\tilde{S}_n^{(4)}}(\alpha)$	$C_{\tilde{S}_n^{(5)}}(\alpha)$	$C_{\tilde{S}_n^{(6)}}(\alpha)$	$C_{\tilde{S}_n^{(7)}}(\alpha)$
20	0.16	0.18	0.19	0.18	0.18	0.18	0.18	0.18
30	0.23	0.24	0.24	0.24	0.24	0.24	0.24	0.24
40	0.30	0.30	0.32	0.31	0.31	0.32	0.32	0.32
50	0.35	0.36	0.38	0.37	0.36	0.37	0.36	0.37
60	0.41	0.42	0.44	0.44	0.43	0.44	0.43	0.45
70	0.47	0.48	0.51	0.50	0.49	0.50	0.49	0.50
80	0.52	0.53	0.55	0.55	0.54	0.55	0.53	0.55
90	0.57	0.57	0.60	0.59	0.58	0.60	0.58	0.59
100	0.62	0.61	0.65	0.64	0.63	0.65	0.64	0.65

n	$C_{\tilde{S}_n^{(8)}}(\alpha)$	$C_{\tilde{S}_n^{(9)}}(\alpha)$	$C_{\tilde{S}_n^{(10)}}(\alpha)$	$C_{\tilde{S}_n^{(11)}}(\alpha)$	$C_{\tilde{S}_n^{(12)}}(\alpha)$	$C_{\tilde{S}_n^{(13)}}(\alpha)$	$C_{V_n}(\alpha)$	$C_{K^*}(\alpha)$
20	0.18	0.18	0.18	0.18	0.18	0.18	0.18	0.18
30	0.24	0.24	0.24	0.24	0.24	0.24	0.24	0.23
40	0.32	0.32	0.31	0.32	0.32	0.32	0.31	0.31
50	0.37	0.37	0.37	0.37	0.37	0.37	0.36	0.36
60	0.44	0.44	0.43	0.44	0.44	0.44	0.42	0.42
70	0.50	0.51	0.49	0.50	0.50	0.50	0.48	0.47
80	0.55	0.55	0.53	0.55	0.54	0.55	0.52	0.52
90	0.60	0.60	0.58	0.59	0.59	0.59	0.57	0.57
100	0.65	0.65	0.63	0.64	0.64	0.65	0.61	0.61

Linear failure rate ($\theta = 1$):

n	$C_{T_n^*}(\alpha)$	$C_{\tilde{S}_n^{(1)}}(\alpha)$	$C_{\tilde{S}_n^{(2)}}(\alpha)$	$C_{\tilde{S}_n^{(3)}}(\alpha)$	$C_{\tilde{S}_n^{(4)}}(\alpha)$	$C_{\tilde{S}_n^{(5)}}(\alpha)$	$C_{\tilde{S}_n^{(6)}}(\alpha)$	$C_{\tilde{S}_n^{(7)}}(\alpha)$
20	0.24	0.29	0.30	0.28	0.29	0.28	0.29	0.29
30	0.37	0.40	0.41	0.40	0.40	0.41	0.40	0.40
40	0.48	0.50	0.52	0.50	0.51	0.52	0.51	0.51
50	0.57	0.59	0.62	0.60	0.60	0.61	0.60	0.61
60	0.64	0.67	0.70	0.68	0.68	0.69	0.68	0.70
70	0.72	0.73	0.76	0.75	0.75	0.76	0.75	0.76
80	0.78	0.80	0.82	0.81	0.81	0.81	0.81	0.82
90	0.83	0.83	0.86	0.85	0.85	0.86	0.84	0.86
100	0.86	0.87	0.89	0.89	0.88	0.89	0.88	0.89

n	$C_{\tilde{S}_n^{(8)}}(\alpha)$	$C_{\tilde{S}_n^{(9)}}(\alpha)$	$C_{\tilde{S}_n^{(10)}}(\alpha)$	$C_{\tilde{S}_n^{(11)}}(\alpha)$	$C_{\tilde{S}_n^{(12)}}(\alpha)$	$C_{\tilde{S}_n^{(13)}}(\alpha)$	$C_{V_n}(\alpha)$	$C_{K^*}(\alpha)$
20	0.29	0.29	0.29	0.29	0.29	0.29	0.28	0.28
30	0.40	0.40	0.40	0.41	0.41	0.40	0.40	0.39
40	0.52	0.51	0.51	0.51	0.52	0.52	0.50	0.50
50	0.61	0.61	0.60	0.61	0.61	0.61	0.59	0.59
60	0.69	0.69	0.68	0.69	0.69	0.69	0.67	0.67
70	0.76	0.76	0.74	0.75	0.76	0.76	0.73	0.73
80	0.81	0.81	0.81	0.81	0.81	0.81	0.79	0.79
90	0.86	0.86	0.84	0.85	0.85	0.85	0.83	0.83
100	0.89	0.89	0.88	0.89	0.89	0.89	0.87	0.87

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Table 5.7: continued

Linear failure rate ($\theta = 1.5$):

n	$C_{T_n^*}(\alpha)$	$C_{\tilde{S}_n^{(1)}}(\alpha)$	$C_{\tilde{S}_n^{(2)}}(\alpha)$	$C_{\tilde{S}_n^{(3)}}(\alpha)$	$C_{\tilde{S}_n^{(4)}}(\alpha)$	$C_{\tilde{S}_n^{(5)}}(\alpha)$	$C_{\tilde{S}_n^{(6)}}(\alpha)$	$C_{\tilde{S}_n^{(7)}}(\alpha)$
20	0.30	0.36	0.38	0.36	0.36	0.36	0.37	0.37
30	0.46	0.51	0.52	0.51	0.51	0.52	0.52	0.52
40	0.59	0.62	0.64	0.63	0.64	0.64	0.64	0.64
50	0.69	0.72	0.75	0.74	0.73	0.75	0.73	0.74
60	0.77	0.80	0.82	0.81	0.81	0.82	0.82	0.83
70	0.83	0.86	0.88	0.87	0.87	0.88	0.87	0.88
80	0.89	0.91	0.92	0.92	0.91	0.92	0.91	0.92
90	0.92	0.93	0.94	0.94	0.94	0.94	0.94	0.94
100	0.94	0.95	0.96	0.96	0.96	0.96	0.96	0.96

n	$C_{\tilde{S}_n^{(8)}}(\alpha)$	$C_{\tilde{S}_n^{(9)}}(\alpha)$	$C_{\tilde{S}_n^{(10)}}(\alpha)$	$C_{\tilde{S}_n^{(11)}}(\alpha)$	$C_{\tilde{S}_n^{(12)}}(\alpha)$	$C_{\tilde{S}_n^{(13)}}(\alpha)$	$C_{V_n}(\alpha)$	$C_{K^*}(\alpha)$
20	0.37	0.36	0.37	0.37	0.37	0.37	0.36	0.36
30	0.51	0.51	0.51	0.52	0.52	0.51	0.51	0.51
40	0.64	0.64	0.64	0.64	0.64	0.64	0.63	0.62
50	0.74	0.74	0.74	0.75	0.74	0.74	0.73	0.73
60	0.82	0.82	0.81	0.82	0.82	0.82	0.81	0.80
70	0.87	0.88	0.87	0.87	0.88	0.87	0.86	0.86
80	0.92	0.92	0.91	0.92	0.92	0.92	0.90	0.90
90	0.94	0.94	0.94	0.94	0.94	0.94	0.93	0.93
100	0.96	0.96	0.96	0.96	0.96	0.96	0.95	0.95

Table 5.8: Estimated power functions for Makeham alternatives based on 20,000 trials.

Makeham ($\theta = 0.5$):

n	$C_{T_n^*}(\alpha)$	$C_{\tilde{S}_n^{(1)}}(\alpha)$	$C_{\tilde{S}_n^{(2)}}(\alpha)$	$C_{\tilde{S}_n^{(3)}}(\alpha)$	$C_{\tilde{S}_n^{(4)}}(\alpha)$	$C_{\tilde{S}_n^{(5)}}(\alpha)$	$C_{\tilde{S}_n^{(6)}}(\alpha)$	$C_{\tilde{S}_n^{(7)}}(\alpha)$
20	0.09	0.11	0.11	0.10	0.11	0.10	0.11	0.11
30	0.12	0.13	0.13	0.13	0.13	0.13	0.13	0.13
40	0.13	0.14	0.14	0.14	0.15	0.14	0.15	0.15
50	0.15	0.18	0.17	0.16	0.17	0.17	0.17	0.17
60	0.17	0.20	0.19	0.19	0.20	0.19	0.20	0.20
70	0.18	0.22	0.21	0.20	0.22	0.21	0.22	0.21
80	0.19	0.24	0.22	0.22	0.23	0.22	0.23	0.23
90	0.21	0.26	0.24	0.23	0.26	0.25	0.26	0.25
100	0.22	0.28	0.25	0.25	0.28	0.26	0.28	0.27

n	$C_{\tilde{S}_n^{(8)}}(\alpha)$	$C_{\tilde{S}_n^{(9)}}(\alpha)$	$C_{\tilde{S}_n^{(10)}}(\alpha)$	$C_{\tilde{S}_n^{(11)}}(\alpha)$	$C_{\tilde{S}_n^{(12)}}(\alpha)$	$C_{\tilde{S}_n^{(13)}}(\alpha)$	$C_{V_n}(\alpha)$	$C_{K^*}(\alpha)$
20	0.10	0.10	0.11	0.11	0.10	0.10	0.10	0.10
30	0.13	0.12	0.13	0.13	0.13	0.13	0.13	0.13
40	0.14	0.14	0.15	0.15	0.14	0.14	0.15	0.15
50	0.17	0.17	0.18	0.17	0.17	0.17	0.18	0.18
60	0.19	0.19	0.20	0.20	0.19	0.19	0.20	0.20
70	0.20	0.21	0.22	0.21	0.21	0.21	0.22	0.22
80	0.22	0.22	0.23	0.23	0.22	0.23	0.23	0.23
90	0.24	0.24	0.26	0.25	0.24	0.24	0.26	0.26
100	0.25	0.25	0.28	0.27	0.27	0.26	0.28	0.28

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Table 5.8: continued

Makeham ($\theta = 1$):								
n	$C_{T_n^*}(\alpha)$	$C_{\tilde{S}_n^{(1)}}(\alpha)$	$C_{\tilde{S}_n^{(2)}}(\alpha)$	$C_{\tilde{S}_n^{(3)}}(\alpha)$	$C_{\tilde{S}_n^{(4)}}(\alpha)$	$C_{\tilde{S}_n^{(5)}}(\alpha)$	$C_{\tilde{S}_n^{(6)}}(\alpha)$	$C_{\tilde{S}_n^{(7)}}(\alpha)$
20	0.14	0.17	0.17	0.16	0.17	0.16	0.17	0.17
30	0.18	0.22	0.21	0.20	0.21	0.21	0.22	0.21
40	0.23	0.28	0.26	0.25	0.28	0.26	0.28	0.28
50	0.26	0.33	0.31	0.30	0.32	0.31	0.32	0.32
60	0.30	0.37	0.35	0.34	0.37	0.36	0.38	0.38
70	0.33	0.42	0.39	0.38	0.42	0.39	0.42	0.41
80	0.37	0.47	0.44	0.43	0.47	0.44	0.47	0.46
90	0.40	0.50	0.47	0.46	0.50	0.48	0.50	0.50
100	0.43	0.55	0.51	0.50	0.55	0.52	0.55	0.54
n	$C_{\tilde{S}_n^{(8)}}(\alpha)$	$C_{\tilde{S}_n^{(9)}}(\alpha)$	$C_{\tilde{S}_n^{(10)}}(\alpha)$	$C_{\tilde{S}_n^{(11)}}(\alpha)$	$C_{\tilde{S}_n^{(12)}}(\alpha)$	$C_{\tilde{S}_n^{(13)}}(\alpha)$	$C_{V_n}(\alpha)$	$C_{K^*}(\alpha)$
20	0.16	0.16	0.17	0.17	0.17	0.16	0.17	0.17
30	0.21	0.21	0.22	0.21	0.21	0.21	0.22	0.21
40	0.26	0.26	0.28	0.27	0.27	0.27	0.28	0.28
50	0.30	0.30	0.32	0.32	0.31	0.31	0.33	0.33
60	0.35	0.35	0.37	0.37	0.36	0.36	0.37	0.37
70	0.39	0.39	0.42	0.41	0.40	0.40	0.42	0.42
80	0.44	0.44	0.47	0.46	0.45	0.45	0.47	0.47
90	0.47	0.47	0.50	0.50	0.49	0.48	0.50	0.51
100	0.51	0.51	0.55	0.54	0.53	0.53	0.55	0.55

Makeham ($\theta = 1.5$):								
n	$C_{T_n^*}(\alpha)$	$C_{\tilde{S}_n^{(1)}}(\alpha)$	$C_{\tilde{S}_n^{(2)}}(\alpha)$	$C_{\tilde{S}_n^{(3)}}(\alpha)$	$C_{\tilde{S}_n^{(4)}}(\alpha)$	$C_{\tilde{S}_n^{(5)}}(\alpha)$	$C_{\tilde{S}_n^{(6)}}(\alpha)$	$C_{\tilde{S}_n^{(7)}}(\alpha)$
20	0.18	0.24	0.23	0.21	0.23	0.22	0.24	0.23
30	0.25	0.30	0.29	0.28	0.30	0.29	0.31	0.30
40	0.32	0.39	0.37	0.36	0.39	0.37	0.39	0.39
50	0.37	0.47	0.44	0.43	0.46	0.44	0.46	0.46
60	0.42	0.53	0.50	0.48	0.52	0.50	0.53	0.53
70	0.47	0.59	0.56	0.54	0.59	0.56	0.59	0.58
80	0.52	0.64	0.61	0.60	0.64	0.61	0.64	0.64
90	0.56	0.69	0.65	0.65	0.69	0.67	0.69	0.68
100	0.60	0.73	0.69	0.69	0.73	0.70	0.73	0.72
n	$C_{\tilde{S}_n^{(8)}}(\alpha)$	$C_{\tilde{S}_n^{(9)}}(\alpha)$	$C_{\tilde{S}_n^{(10)}}(\alpha)$	$C_{\tilde{S}_n^{(11)}}(\alpha)$	$C_{\tilde{S}_n^{(12)}}(\alpha)$	$C_{\tilde{S}_n^{(13)}}(\alpha)$	$C_{V_n}(\alpha)$	$C_{K^*}(\alpha)$
20	0.22	0.22	0.23	0.23	0.23	0.22	0.23	0.23
30	0.28	0.28	0.30	0.30	0.29	0.29	0.30	0.30
40	0.37	0.37	0.39	0.39	0.38	0.38	0.39	0.39
50	0.43	0.43	0.46	0.46	0.45	0.45	0.47	0.47
60	0.49	0.49	0.53	0.52	0.51	0.51	0.53	0.53
70	0.55	0.56	0.59	0.58	0.57	0.57	0.59	0.59
80	0.60	0.60	0.64	0.63	0.62	0.62	0.64	0.64
90	0.65	0.66	0.69	0.68	0.67	0.67	0.69	0.69
100	0.69	0.69	0.73	0.72	0.71	0.71	0.73	0.73

Table 5.9: Estimated power functions for $F_5(x)$ alternatives based on 20,000 trials.

$F_5(x)$ $(\beta = 2,\ \theta = 0.5)$:

n	$C_{T_n^*}(\alpha)$	$C_{\tilde{S}_n^{(1)}}(\alpha)$	$C_{\tilde{S}_n^{(2)}}(\alpha)$	$C_{\tilde{S}_n^{(3)}}(\alpha)$	$C_{\tilde{S}_n^{(4)}}(\alpha)$	$C_{\tilde{S}_n^{(5)}}(\alpha)$	$C_{\tilde{S}_n^{(6)}}(\alpha)$	$C_{\tilde{S}_n^{(7)}}(\alpha)$
20	0.04	0.40	0.25	0.25	0.36	0.27	0.37	0.33
30	0.09	0.57	0.33	0.36	0.52	0.38	0.52	0.46
40	0.13	0.69	0.40	0.44	0.64	0.47	0.63	0.56
50	0.17	0.80	0.48	0.53	0.74	0.56	0.73	0.65
60	0.20	0.88	0.54	0.62	0.82	0.64	0.82	0.74
70	0.22	0.92	0.59	0.67	0.87	0.69	0.86	0.79
80	0.25	0.95	0.63	0.73	0.91	0.74	0.90	0.83
90	0.28	0.97	0.68	0.77	0.94	0.79	0.93	0.86
100	0.31	0.98	0.71	0.82	0.96	0.83	0.95	0.90

n	$C_{\tilde{S}_n^{(8)}}(\alpha)$	$C_{\tilde{S}_n^{(9)}}(\alpha)$	$C_{\tilde{S}_n^{(10)}}(\alpha)$	$C_{\tilde{S}_n^{(11)}}(\alpha)$	$C_{\tilde{S}_n^{(12)}}(\alpha)$	$C_{\tilde{S}_n^{(13)}}(\alpha)$	$C_{V_n}(\alpha)$	$C_{K^*}(\alpha)$
20	0.26	0.25	0.37	0.34	0.31	0.30	0.39	0.39
30	0.35	0.35	0.52	0.47	0.44	0.41	0.57	0.57
40	0.43	0.42	0.65	0.58	0.54	0.52	0.70	0.70
50	0.51	0.50	0.75	0.69	0.64	0.62	0.80	0.81
60	0.59	0.58	0.84	0.78	0.73	0.71	0.88	0.88
70	0.64	0.63	0.88	0.83	0.79	0.77	0.92	0.92
80	0.69	0.68	0.92	0.88	0.83	0.81	0.95	0.95
90	0.74	0.72	0.95	0.91	0.87	0.85	0.97	0.97
100	0.78	0.76	0.97	0.94	0.91	0.89	0.98	0.98

$F_5(x)$ $(\beta = 1.25,\ \theta = 0.2)$:

n	$C_{T_n^*}(\alpha)$	$C_{\tilde{S}_n^{(1)}}(\alpha)$	$C_{\tilde{S}_n^{(2)}}(\alpha)$	$C_{\tilde{S}_n^{(3)}}(\alpha)$	$C_{\tilde{S}_n^{(4)}}(\alpha)$	$C_{\tilde{S}_n^{(5)}}(\alpha)$	$C_{\tilde{S}_n^{(6)}}(\alpha)$	$C_{\tilde{S}_n^{(7)}}(\alpha)$
20	0.11	0.23	0.20	0.17	0.22	0.18	0.22	0.22
30	0.14	0.30	0.24	0.21	0.29	0.23	0.30	0.28
40	0.17	0.37	0.28	0.25	0.35	0.28	0.36	0.33
50	0.20	0.45	0.33	0.29	0.41	0.32	0.42	0.39
60	0.22	0.51	0.36	0.33	0.48	0.37	0.48	0.45
70	0.24	0.56	0.40	0.37	0.53	0.41	0.53	0.49
80	0.26	0.62	0.43	0.40	0.58	0.44	0.57	0.53
90	0.28	0.66	0.46	0.43	0.62	0.47	0.61	0.56
100	0.30	0.71	0.49	0.47	0.66	0.51	0.66	0.60

n	$C_{\tilde{S}_n^{(8)}}(\alpha)$	$C_{\tilde{S}_n^{(9)}}(\alpha)$	$C_{\tilde{S}_n^{(10)}}(\alpha)$	$C_{\tilde{S}_n^{(11)}}(\alpha)$	$C_{\tilde{S}_n^{(12)}}(\alpha)$	$C_{\tilde{S}_n^{(13)}}(\alpha)$	$C_{V_n}(\alpha)$	$C_{K^*}(\alpha)$
20	0.18	0.18	0.22	0.20	0.19	0.19	0.23	0.23
30	0.22	0.22	0.28	0.26	0.24	0.23	0.30	0.30
40	0.26	0.27	0.35	0.32	0.30	0.29	0.38	0.38
50	0.30	0.31	0.42	0.38	0.35	0.33	0.45	0.45
60	0.34	0.35	0.47	0.43	0.40	0.38	0.51	0.51
70	0.38	0.39	0.53	0.48	0.45	0.43	0.56	0.56
80	0.41	0.42	0.57	0.53	0.49	0.47	0.62	0.62
90	0.44	0.45	0.62	0.57	0.52	0.50	0.66	0.66
100	0.48	0.48	0.66	0.60	0.56	0.54	0.71	0.71

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Table 5.9: continued

$F_5(x)$ ($\beta = 3,\ \theta = 0.8$):								
n	$C_{T_n^*}(\alpha)$	$C_{\tilde{S}_n^{(1)}}(\alpha)$	$C_{\tilde{S}_n^{(2)}}(\alpha)$	$C_{\tilde{S}_n^{(3)}}(\alpha)$	$C_{\tilde{S}_n^{(4)}}(\alpha)$	$C_{\tilde{S}_n^{(5)}}(\alpha)$	$C_{\tilde{S}_n^{(6)}}(\alpha)$	$C_{\tilde{S}_n^{(7)}}(\alpha)$
20	0.00	0.84	0.42	0.57	0.78	0.59	0.78	0.69
30	0.01	0.95	0.53	0.74	0.91	0.75	0.91	0.83
40	0.02	0.99	0.64	0.85	0.97	0.85	0.96	0.91
50	0.03	0.99	0.73	0.92	0.99	0.91	0.98	0.95
60	0.04	0.99	0.79	0.96	0.99	0.95	0.99	0.98
70	0.05	0.99	0.84	0.98	0.99	0.97	0.99	0.99
80	0.07	1.00	0.87	0.99	0.99	0.98	0.99	0.99
90	0.08	1.00	0.90	0.99	0.99	0.99	0.99	0.99
100	0.09	1.00	0.92	0.99	1.00	0.99	0.99	0.99
n	$C_{\tilde{S}_n^{(8)}}(\alpha)$	$C_{\tilde{S}_n^{(9)}}(\alpha)$	$C_{\tilde{S}_n^{(10)}}(\alpha)$	$C_{\tilde{S}_n^{(11)}}(\alpha)$	$C_{\tilde{S}_n^{(12)}}(\alpha)$	$C_{\tilde{S}_n^{(13)}}(\alpha)$	$C_{V_n}(\alpha)$	$C_{K^*}(\alpha)$
20	0.54	0.51	0.79	0.73	0.69	0.66	0.83	0.83
30	0.69	0.65	0.92	0.88	0.85	0.82	0.95	0.95
40	0.80	0.76	0.98	0.95	0.93	0.92	0.99	0.99
50	0.87	0.84	0.99	0.98	0.97	0.96	0.99	0.99
60	0.92	0.89	0.99	0.99	0.99	0.98	0.99	0.99
70	0.95	0.92	0.99	0.99	0.99	0.99	0.99	0.99
80	0.96	0.95	0.99	0.99	0.99	0.99	1.00	1.00
90	0.98	0.96	0.99	0.99	0.99	0.99	1.00	1.00
100	0.98	0.97	1.00	0.99	0.99	0.99	1.00	1.00

Table 5.10: Estimated power functions for $F_6(x)$ alternatives based on 20,000 trials.

$F_6(x)$ ($\theta = 0.5$):								
n	$C_{T_n^*}(\alpha)$	$C_{\tilde{S}_n^{(1)}}(\alpha)$	$C_{\tilde{S}_n^{(2)}}(\alpha)$	$C_{\tilde{S}_n^{(3)}}(\alpha)$	$C_{\tilde{S}_n^{(4)}}(\alpha)$	$C_{\tilde{S}_n^{(5)}}(\alpha)$	$C_{\tilde{S}_n^{(6)}}(\alpha)$	$C_{\tilde{S}_n^{(7)}}(\alpha)$
20	0.08	0.11	0.11	0.09	0.11	0.10	0.11	0.11
30	0.09	0.13	0.12	0.11	0.13	0.11	0.13	0.13
40	0.10	0.15	0.12	0.11	0.14	0.12	0.15	0.14
50	0.11	0.18	0.14	0.13	0.16	0.14	0.17	0.16
60	0.11	0.19	0.15	0.14	0.18	0.15	0.18	0.17
70	0.12	0.21	0.16	0.14	0.20	0.16	0.20	0.19
80	0.13	0.23	0.18	0.16	0.22	0.17	0.21	0.20
90	0.14	0.25	0.19	0.17	0.23	0.19	0.23	0.22
100	0.14	0.26	0.19	0.18	0.25	0.20	0.25	0.23
n	$C_{\tilde{S}_n^{(8)}}(\alpha)$	$C_{\tilde{S}_n^{(9)}}(\alpha)$	$C_{\tilde{S}_n^{(10)}}(\alpha)$	$C_{\tilde{S}_n^{(11)}}(\alpha)$	$C_{\tilde{S}_n^{(12)}}(\alpha)$	$C_{\tilde{S}_n^{(13)}}(\alpha)$	$C_{V_n}(\alpha)$	$C_{K^*}(\alpha)$
20	0.10	0.10	0.11	0.10	0.10	0.10	0.11	0.11
30	0.11	0.11	0.13	0.12	0.12	0.11	0.13	0.13
40	0.12	0.12	0.14	0.13	0.13	0.12	0.15	0.15
50	0.13	0.14	0.17	0.16	0.15	0.14	0.18	0.18
60	0.14	0.14	0.18	0.17	0.16	0.15	0.19	0.19
70	0.15	0.16	0.19	0.18	0.17	0.17	0.21	0.20
80	0.17	0.17	0.21	0.20	0.18	0.18	0.23	0.23
90	0.18	0.18	0.23	0.21	0.20	0.19	0.25	0.25
100	0.18	0.19	0.25	0.22	0.21	0.20	0.26	0.26

Continued on next page

Table 5.10: continued

$F_6(x)$ $(\theta = 1)$:

n	$C_{T_n^*}(\alpha)$	$C_{\tilde{S}_n^{(1)}}(\alpha)$	$C_{\tilde{S}_n^{(2)}}(\alpha)$	$C_{\tilde{S}_n^{(3)}}(\alpha)$	$C_{\tilde{S}_n^{(4)}}(\alpha)$	$C_{\tilde{S}_n^{(5)}}(\alpha)$	$C_{\tilde{S}_n^{(6)}}(\alpha)$	$C_{\tilde{S}_n^{(7)}}(\alpha)$
20	0.15	0.25	0.23	0.21	0.25	0.22	0.25	0.24
30	0.20	0.33	0.29	0.27	0.32	0.29	0.33	0.31
40	0.25	0.41	0.34	0.33	0.40	0.35	0.40	0.39
50	0.29	0.48	0.40	0.39	0.46	0.41	0.47	0.45
60	0.32	0.55	0.45	0.44	0.53	0.47	0.53	0.51
70	0.35	0.60	0.49	0.49	0.59	0.51	0.58	0.56
80	0.38	0.67	0.53	0.54	0.64	0.56	0.64	0.61
90	0.41	0.71	0.57	0.57	0.68	0.60	0.68	0.65
100	0.44	0.74	0.60	0.61	0.72	0.63	0.72	0.68

n	$C_{\tilde{S}_n^{(8)}}(\alpha)$	$C_{\tilde{S}_n^{(9)}}(\alpha)$	$C_{\tilde{S}_n^{(10)}}(\alpha)$	$C_{\tilde{S}_n^{(11)}}(\alpha)$	$C_{\tilde{S}_n^{(12)}}(\alpha)$	$C_{\tilde{S}_n^{(13)}}(\alpha)$	$C_{V_n}(\alpha)$	$C_{K^*}(\alpha)$
20	0.22	0.22	0.24	0.23	0.23	0.22	0.25	0.25
30	0.28	0.28	0.32	0.31	0.30	0.29	0.33	0.33
40	0.34	0.34	0.40	0.38	0.37	0.36	0.41	0.41
50	0.39	0.39	0.47	0.45	0.43	0.42	0.49	0.49
60	0.45	0.44	0.54	0.51	0.49	0.48	0.55	0.55
70	0.49	0.49	0.59	0.56	0.54	0.53	0.60	0.60
80	0.54	0.53	0.64	0.62	0.59	0.58	0.66	0.66
90	0.57	0.57	0.69	0.66	0.63	0.62	0.71	0.71
100	0.61	0.61	0.72	0.69	0.67	0.66	0.74	0.74

$F_6(x)$ $(\theta = 1.5)$:

n	$C_{T_n^*}(\alpha)$	$C_{\tilde{S}_n^{(1)}}(\alpha)$	$C_{\tilde{S}_n^{(2)}}(\alpha)$	$C_{\tilde{S}_n^{(3)}}(\alpha)$	$C_{\tilde{S}_n^{(4)}}(\alpha)$	$C_{\tilde{S}_n^{(5)}}(\alpha)$	$C_{\tilde{S}_n^{(6)}}(\alpha)$	$C_{\tilde{S}_n^{(7)}}(\alpha)$
20	0.24	0.40	0.38	0.35	0.39	0.36	0.39	0.39
30	0.37	0.53	0.49	0.48	0.52	0.50	0.53	0.52
40	0.45	0.64	0.58	0.58	0.64	0.59	0.64	0.62
50	0.51	0.73	0.66	0.67	0.72	0.68	0.72	0.71
60	0.57	0.80	0.72	0.74	0.79	0.75	0.79	0.78
70	0.62	0.86	0.78	0.79	0.85	0.81	0.84	0.83
80	0.67	0.90	0.82	0.84	0.89	0.84	0.88	0.87
90	0.71	0.92	0.85	0.87	0.91	0.88	0.91	0.90
100	0.74	0.94	0.88	0.91	0.94	0.91	0.94	0.92

n	$C_{\tilde{S}_n^{(8)}}(\alpha)$	$C_{\tilde{S}_n^{(9)}}(\alpha)$	$C_{\tilde{S}_n^{(10)}}(\alpha)$	$C_{\tilde{S}_n^{(11)}}(\alpha)$	$C_{\tilde{S}_n^{(12)}}(\alpha)$	$C_{\tilde{S}_n^{(13)}}(\alpha)$	$C_{V_n}(\alpha)$	$C_{K^*}(\alpha)$
20	0.36	0.36	0.39	0.38	0.38	0.37	0.39	0.39
30	0.49	0.49	0.53	0.52	0.51	0.50	0.53	0.53
40	0.58	0.58	0.64	0.63	0.61	0.61	0.64	0.64
50	0.67	0.66	0.73	0.71	0.70	0.69	0.73	0.74
60	0.74	0.73	0.80	0.79	0.77	0.77	0.80	0.80
70	0.79	0.79	0.85	0.84	0.83	0.82	0.86	0.86
80	0.83	0.83	0.89	0.88	0.87	0.86	0.89	0.89
90	0.87	0.86	0.92	0.91	0.90	0.89	0.92	0.92
100	0.90	0.89	0.94	0.93	0.93	0.92	0.94	0.94

5.3.4 Conclusions and recommendations

In this section we summarize our main conclusions:

- $\tilde{S}_n^{(1)}$, $\tilde{S}_n^{(4)}$, $\tilde{S}_n^{(6)}$, $\tilde{S}_n^{(10)}$ and K^* have the highest Pitman efficiencies against Weibull and Makeham distributions. However, $\tilde{S}_n^{(4)}$ and $\tilde{S}_n^{(6)}$ have much larger Pitman efficiencies than $\tilde{S}_n^{(1)}$ and K^* against linear failure rate alternatives.

- $\tilde{S}_n^{(2)}$, $\tilde{S}_n^{(5)}$, $\tilde{S}_n^{(7)}$, $\tilde{S}_n^{(8)}$ and $\tilde{S}_n^{(9)}$ have the largest Pitman efficiencies against linear failure rate alternatives.

- Based on the approximate Bahadur slopes against Weibull alternatives, $\tilde{S}_n^{(1)}$ and $\tilde{S}_n^{(10)}$ to $\tilde{S}_n^{(13)}$ outperform all the other tests. Except for $\tilde{T}_n$, the coefficient of variation test performs the worst. The following test statistics all perform similar or better than the test based on K^*: $\tilde{S}_n^{(1)}$, $\tilde{S}_n^{(3)}$, $\tilde{S}_n^{(4)}$ and $\tilde{S}_n^{(10)}$ to $\tilde{S}_n^{(13)}$.

- Against gamma alternatives, K^* clearly outperforms the other test statistics based on approximate Bahadur efficiency. For the test statistics based on $\tilde{S}_n$, $\tilde{S}_n^{(1)}$ and $\tilde{S}_n^{(10)}$ perform the best.

- For linear failure rate alternatives all the tests based on $\tilde{S}_n$ have a much higher approximate Bahadur slope than K^*.

- For Makeham alternatives, the test statistics $\tilde{T}_n$, $\tilde{S}_n^{(2)}$, $S/\bar{X}$, K^* and $\Delta_2^{(3)}$ are outperformed by the remaining test statistics, which have almost identical Bahadur slopes.

- Based on the power comparisons against Weibull, gamma and $F_6(x)$ alternatives, the best tests are those based on $\tilde{S}_n^{(1)}$, $\tilde{S}_n^{(4)}$ and $\tilde{S}_n^{(6)}$. Against linear failure rate and Makeham alternatives, all the tests have similar power.

- From the power comparisons we conclude that the tests based on $\tilde{S}_n^{(1)}$, $\tilde{S}_n^{(4)}$ and $\tilde{S}_n^{(6)}$ perform similar to the tests based on V_n and K^*. However, as was seen earlier, the test based on K^* is often outperformed in terms of Pitman and approximate Bahadur efficiency.

Recommendation: *Based on its overall good performance, we propose the test statistics $\tilde{S}_n^{(1)}$, $\tilde{S}_n^{(4)}$ and $\tilde{S}_n^{(6)}$ as effective procedures for testing exponentiality against NBUE alternatives.*

5.4 Applications to real data

In this section we apply our best test statistics to three examples from the literature.

Example 1. Hollander and Proschan (1975) and Kanjo (1993) used data from Bryson and Siddiqui (1969) which are survival times, in days from diagnosis, of patients suffering from chronic granulocytic leukemia. The data set contains 43 observations:

$$7, 47, 58, 74, 177, 232, 273, 285, 317, 429, 440, 445, 455, 468, 495, 497, 532,$$

$$571, 579, 581, 650, 702, 715, 779, 881, 900, 930, 968, 1077, 1109, 1314, 1334,$$

$$1367, 1534, 1712, 1784, 1877, 1886, 2045, 2056, 2260, 2429, 2509.$$

We calculated the values of $\tilde{S}_n^{(1)}$, $\tilde{S}_n^{(4)}$ and $\tilde{S}_n^{(6)}$ as 1.681, 1.661 and 1.645, respectively, with critical values 1.672, 1.609 and 1.584. All three tests thus reject the null hypothesis in favour of an NBUE distribution at a 5% significance level. This corresponds with the conclusions of both Hollander and Proschan (1975) and Kanjo (1993).

Example 2. Lee et al. (1980) used an example from Lieblein and Zelen (1956) regarding fatigue life of deep groove ball bearings where the data set consists of the number of revolutions (in millions) before failure of 23 bearings:

$$17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.48, 51.84,$$

$$51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12,$$

$$93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.$$

They rejected exponentiality at a 1% significance level, in favour of an IFRA distribution. The value of $\tilde{S}_n^{(1)}$ for this data set is 3.511 and the values of $\tilde{S}_n^{(4)}$ and $\tilde{S}_n^{(6)}$ are 3.144 and 3.011, respectively. Comparing these values to the critical values 1.675, 1.601 and 1.568 at a 5% significance level, the null hypothesis is rejected in favour of an NBUE distribution. The critical values at a 1% significance level are 2.367, 2.203 and 2.125 respectively, resulting in H_0 being rejected in all three cases.

Example 3. Kochar (1985) considered an experiment from Florida State University in which the effect of methylmercury poisoning on the life lengths of fish were studied. At one dosage level, the ordered times to death (in days) of goldfish subjected to methylmercury were

$$42, 43, 51, 61, 66, 69, 71, 81, 82, 82.$$

He rejected exponentiality in favour of the IFRA property, at a 1% significance level.

The value of $\tilde{S}_n^{(1)}$ is 3.945, which is larger than the critical value of 2.389, the value of $\tilde{S}_n^{(4)}$ is 3.452, which is larger than the critical value of 2.159, and the value of $\tilde{S}_n^{(6)}$ is 3.207, which is larger than the critical value of 2.075. Thus in all three cases, H_0 is rejected in favour of an NBUE alternative at a 1% significance level.

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