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Conserved Vectors, Analytic Solutions and Numerical Simulation of Soliton Collisions of the Modified Gardner Equation

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Abstract: This paper aims to study the modified Gardner (mG) equation that was proposed in the literature a short while ago. We first construct conserved vectors of the mG equation by invoking three different techniques; namely the method of multiplier, Noether's theorem, and the conservation theorem owing to Ibragimov. Thereafter, we present exact solutions to the mG equation by invoking a complete discrimination system for the fifth degree polynomial. Finally, we simulate collisions of solitons for the mG equation.

Keywords: Modified Gardner equation; Lie symmetries; conserved vectors; Noether's theorem; multiplier method; numerical simulation

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1. Introduction

Nonlinear evolution equations (NLEEs) are partial differential equations (PDEs) which contain time t as one of their independent variables. Many real world phenomena in physics, plasma waves, fluid mechanics, and material science are governed by NLEEs. Thus, such equations arise in many scientific disciplines of study. For instance, Euler, porous media, and Navier–Stokes equations are encountered in fluid mechanics, the nonlinear Klein–Gordon equation appears in quantum mechanics, and the Cahn–Hilliard equation comes up in material science which, among others, are examples of NLEEs. Due to their vast applications, NLEEs have being studied by many scientists and mathematicians who work in nonlinear sciences (see, for instance, Refs. [1–9]).

One such nonlinear evolution partial differential equation is

$$u_t + uu_x + \frac{1}{6}\varepsilon^2 u^2 u_x + u_{xxx} = 0,\ \varepsilon \in \mathbb{R}, \tag{1}$$

which is usually referred to as the Gardner equation. It can also be seen as the combination of the Korteweg–de Vries (KdV) and modified KdV equations. The nonlinear wave steepening is described by the terms uu_x and u^2u_x, whereas u_{xxx} represents the dispersive wave effects. Gardner equation is a completely integrable NLEE and its solitary wave solutions are solitons. It was introduced by the mathematician C. S. Gardner in the late sixties of 20th century (see [10] and the references therein). Gardner equation can be constructed by the Galilean transformation of the modified Korteweg–de Vries equation, and its solutions can be compared to the solution to KdV equation. The Gardner equation arises in various branches of physics. This includes solid-state physics, quantum field theory, fluid dynamics, and plasma physics. Several authors have studied the Gardner equation and constructed its exact solutions (see, for example, Ref. [11] and its references).

Recently, the modified Gardner (mG) equation,

$$u_t + auu_x + bu^2u_x + cu^3u_x + \frac{1}{2}u_{xxx} = 0, \tag{2}$$

which has an additional term cu^3u_x, was derived in [12], where the mG equation was shown to have failed the Painlevé test, thus suggesting that the equation is not integrable. When the constant $c = 0$ in (2), the equation reduces to the standard Gardner equation.

In this work, we investigate the mG Equation (2). Firstly, we compute conserved vectors of mG equation by invoking three distinct procedures: the method of multiplier, Noether's theorem, and the theorem owing to Ibragimov. Thereafter, we call on a complete discrimination system related to the fifth degree polynomial, and construct the exact explicit solutions to (2). The integrability of the mG equation is then investigated numerically through the simulation of soliton collisions. Conclusions are delivered in the end.

2. Conserved Vectors of the mG Equation

We establish the conserved vectors for mG Equation (2) by invoking three different methods, namely method of multipliers, Noether's technique, and conservation theorem owing to Ibragimov.

2.1. Conserved Vectors Utilizing Multiplier Approach

Firstly, we invoke multiplier method to determine the conserved vectors of the mG Equation (2). Firstly, we seek the first-order multipliers, and later look for the second-order multipliers, and use them to construct the conserved vectors. The conserved vectors are the physical properties (i.e., measurable quantities) that remain unchanged in the course of time inside a physical system. For instance, the momentum, energy, angular momentum, and electric charge of a physical system [13–22].

2.1.1. First-Order Multipliers

We determine first-order multipliers $Q = Q(t, x, u, u_x)$ by utilizing the determining equation,

$$\frac{\delta}{\delta u}\Big\{Q(t, x, u, u_x)\Big(u_t + auu_x + bu^2u_x + cu^3u_x + \frac{1}{2}u_{xxx}\Big)\Big\} = 0, \tag{3}$$

where $\delta/\delta u$ is the Euler operator, furnished by

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_x\frac{\partial}{\partial u_x} - D_t\frac{\partial}{\partial u_t} - D_x^3\frac{\partial}{\partial u_{xxx}} \tag{4}$$

and D_t and D_x are the total derivatives [13]. Working out (3), and separating different derivatives of u, yields

$$Q_{u_x} = 0,\ Q_{uu} = 0,\ Q_{xu} = 0,\ Q_t + auQ_x + bu^2Q_x + cu^3Q_x + \frac{1}{2}Q_{xxx} = 0,$$

which, on solving, gives

$$Q = C_1u + C_2$$

with constants C_1 and C_2. Therefore, we obtain two first-order multipliers for the mG Equation (2), namely $Q_1 = u$ and $Q_2 = 1$. To generate conserved vectors (T^t, T^x) associated with these multipliers, we use

$$Q\{u_t + auu_x + bu^2u_x + cu^3u_x + \frac{1}{2}u_{xxx}\} = D_tT^t + D_xT^x, \tag{5}$$

where T^t is the density and T^x is the flux. After a bit of computation, we acquire the conserved vector of (2) associated with $Q_1 = u$,

$$\begin{aligned}T_1^t &= \frac{1}{2}u^2,\\ T_1^x &= \frac{1}{2}uu_{xx} - \frac{1}{4}u_x^2 + \frac{1}{3}au^3 + \frac{1}{4}bu^4 + \frac{1}{5}cu^5\end{aligned}$$

and the conserved vector of (2) corresponding to $Q_2 = 1$,

$$\begin{aligned}T_2^t &= u,\\ T_2^x &= \frac{1}{2}u_{xx} + \frac{1}{2}au^2 + \frac{1}{3}bu^3 + \frac{1}{4}cu^4.\end{aligned}$$

Remark: the multiplier $Q_2 = 1$ reveals that the mG Equation (2) is, itself, a conservation law.

2.1.2. Second-Order Multipliers

We now seek second-order multipliers $Q = Q(t, x, u, u_x, u_{xx})$. Following the course of action of the foregoing subsection, we acquire three second-order multipliers, namely

$$Q_1 = 1, \;\; Q_2 = u, \;\; Q_3 = u_{xx} + au^2 + \frac{2}{3}bu^3 + \frac{1}{2}cu^4 \tag{6}$$

and, subsequently, the associated conserved vectors (T^t, T^x) are

$$\begin{aligned}T_1^t &= u,\\ T_1^x &= \frac{1}{2}u_{xx} + \frac{1}{2}au^2 + \frac{1}{3}bu^3 + \frac{1}{4}cu^4;\end{aligned}$$

$$\begin{aligned}T_2^t &= \frac{1}{2}u^2,\\ T_2^x &= \frac{1}{2}uu_{xx} - \frac{1}{4}u_x^2 + \frac{1}{3}au^3 + \frac{1}{4}bu^4 + \frac{1}{5}cu^5;\end{aligned}$$

$$\begin{aligned}T_3^t &= \frac{1}{2}uu_{xx} + \frac{1}{3}au^3 + \frac{1}{6}bu^4 + \frac{1}{10}cu^5,\\ T_3^x &= \frac{1}{2}u_tu_x - \frac{1}{2}uu_{tx} + \frac{1}{4}u_{xx}^2 + \frac{1}{2}au^2u_{xx} + \frac{1}{4}a^2u^4 + \frac{1}{3}bu^3u_{xx}\\ &\quad + \frac{1}{3}abu^5 + \frac{1}{4}cu^4u_{xx} + \frac{1}{9}b^2u^6 + \frac{1}{4}acu^6 + \frac{1}{6}bcu^7 + \frac{1}{16}c^2u^8.\end{aligned}$$

Remark. We note that the use of second-order multiplier for (2) produced one extra conserved vector (T_3^t, T_3^x) compared to when we used the first-order multiplier. We also observe that the first two conserved vectors, (T_1^t, T_1^x) and (T_2^x, T_2^x), are identical in both cases.

2.2. Conserved Vectors Utilizing Noether's Theorem

We calculate the conserved vectors for the mG Equation (2) utilizing Noether's theorem [15–17]. Equation (2) has no Lagrangian. We let $u = V_x$, and this transforms Equation (2) into the fourth-order equation,

$$V_{tx} + aV_xV_{xx} + bV_x^2V_{xx} + cV_x^3V_{xx} + \frac{1}{2}V_{xxxx} = 0, \tag{7}$$

which has a Lagrangian and, consequently, we can invoke Noether's theorem. A second-order Lagrangian of (7) is

$$\mathcal{L} = -\frac{1}{2}V_xV_t - \frac{1}{6}aV_x^3 - \frac{1}{12}bV_x^4 - \frac{1}{20}cV_x^5 + \frac{1}{4}V_{xx}^2 \tag{8}$$

since $\delta\mathcal{L}/\delta V = 0$ on (7), where $\delta/\delta V$ is the Euler operator,

$$\frac{\delta\mathcal{L}}{\delta V} = \frac{\partial}{\partial V} - D_t\frac{\partial}{\partial V_t} - D_x\frac{\partial}{\partial V_x} + D_x^2\frac{\partial}{\partial V_{xx}} - D_x^3\frac{\partial}{\partial V_{xxx}} + \cdots, \tag{9}$$

with D_t and D_x being the total derivatives.

The vector field

$$N = \tau(t,x,V)\frac{\partial}{\partial t} + \xi(t,x,V)\frac{\partial}{\partial x} + \eta(t,x,V)\frac{\partial}{\partial V}, \tag{10}$$

is a Noether symmetry of (7) provided that N satisfies

$$N^{[2]}(\mathcal{L}) + \mathcal{L}\{D_x(\xi) + D_t(\tau)\} = D_t(B^t) + D_x(B^x), \tag{11}$$

with $B^t(t,x,V)$ and $B^x(t,x,V)$ being the gauge functions, and $N^{[2]}$ denotes second extension of N, described as

$$N^{[2]} = N + \zeta_1\frac{\partial}{\partial V_t} + \zeta_2\frac{\partial}{\partial V_x} + \zeta_{22}\frac{\partial}{\partial V_{xx}} \tag{12}$$

with ζ_1, ζ_2, and ζ_{22} given in [23]. Working out (11), and splitting on different derivatives of V, we acquire

$$\begin{aligned}
&\tau_v = 0,\ \tau_x = 0,\ \xi_v = 0,\ \eta_{vv} = 0,\ \eta_{xx} = 0,\ \xi_x - \eta_v = 0,\ \eta_v - 2\xi_x = 0,\\
&2\eta_{xv} - \xi_{xx} = 0,\ a\xi_x - a\eta_v - \frac{2}{3}b\eta_x = 0,\ \xi_t - a\eta_x = 0,\ \tau_t + \xi_t - 2\eta_v = 0,\\
&b\xi_x - b\eta_v - \frac{3}{4}c\eta_x = 0,\ \eta_t + 2B_v^x = 0,\ \eta_x + 2B_v^t = 0,\ B_t^t + B_x^x = 0.
\end{aligned}$$

The solution to the above PDEs, thus, give Noether symmetries, and the corresponding gauge terms as

$$\begin{aligned}
N_1 &= \frac{\partial}{\partial t}, \quad B^t = 0,\ B^x = 0,\\
N_2 &= \frac{\partial}{\partial x}, \quad B^t = 0,\ B^x = 0,\\
N_h &= h(t)\frac{\partial}{\partial V}, \quad B^x = -\frac{1}{2}h'(t)V,\ B^t = 0.
\end{aligned}$$

We now use these Noether symmetries to determine the conserved vectors of (7). Invoking the formulae in [16] for the conserved vector (T^t, T^x), we acquire the subsequent three conserved vectors connected to the Noether symmetries N_1, N_2, and N_h as

$$\begin{aligned}
T_1^t &= -\frac{1}{6}aV_x^3 - \frac{1}{12}bV_x^4 - \frac{1}{20}cV_x^5 + \frac{1}{4}V_{xx}^2, \\
T_1^x &= \frac{1}{2}V_t^2 + \frac{1}{2}aV_tV_x^2 + \frac{1}{3}bV_tV_x^3 + \frac{1}{4}cV_tV_x^4 + \frac{1}{2}V_tV_{xxx} - \frac{1}{2}V_{tx}V_{tt}; \\
T_2^t &= \frac{1}{2}V_x^2, \\
T_2^x &= \frac{1}{3}aV_x^3 + \frac{1}{4}bV_x^4 + \frac{1}{5}cV_x^5 - \frac{1}{4}V_{xx}^2 + \frac{1}{2}V_xV_{xxx}; \\
T_h^t &= -\frac{1}{2}h(t)V_x^2, \\
T_h^x &= -\frac{1}{2}h(t)V_t - \frac{1}{2}ah(t)V_x^2 - \frac{1}{3}bh(t)V_x^3 - \frac{1}{4}ch(t)V_x^4 - \frac{1}{2}h(t)V_{xxx} + \frac{1}{2}h'(t)V,
\end{aligned}$$

respectively. We now return to the variables t, x, and acquire two non-local and one local conserved vectors of (2) as

$$\begin{aligned}
T_1^t &= -\frac{1}{6}au^3 - \frac{1}{12}bu^4 - \frac{1}{20}cu^5 + \frac{1}{4}u_x^2, \\
T_1^x &= \frac{1}{2}\left(\int u_t dx\right)^2 + \frac{1}{2}au^2\int u_t dx + \frac{1}{3}bu^3\int u_t dx + \frac{1}{4}cu^4\int u_t dx \\
&\quad + \frac{1}{2}u_{xx}\int u_t dx - \frac{1}{2}u_t u_x; \\
T_2^t &= \frac{1}{2}u^2, \\
T_2^x &= \frac{1}{3}au^3 + \frac{1}{4}bu^4 + \frac{1}{5}cu^5 - \frac{1}{4}u_x^2 + \frac{1}{2}uu_{xx}; \\
T_h^t &= -\frac{1}{2}h(t)u, \\
T_h^x &= -\frac{1}{2}h(t)\int u_t dx - \frac{1}{2}ah(t)u^2 - \frac{1}{3}bh(t)u^3 - \frac{1}{4}ch(t)u^4 - \frac{1}{2}h(t)u_{xx} + \frac{1}{2}h'(t)ux.
\end{aligned}$$

Remark. Because of the appearance of function $h(t)$, we acquire endless nonlocal conserved vectors.

2.3. Conserved Vectors Utilizing Ibragimov's Theorem

Lastly, we determine conserved vectors for mG Equation (2) utilizing Ibragimov's theorem [22].

First, we write down the adjoint equation of (2), which is

$$F^* \equiv \frac{\delta}{\delta u}\left\{v\left(u_t + auu_x + bu^2u_x + cu^3u_x + \frac{1}{2}u_{xxx}\right)\right\} = 0, \tag{13}$$

where $\delta/\delta u$ is the Euler operator,

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} - D_x^3\frac{\partial}{\partial u_{xxx}} + \cdots$$

and $v(x)$ is a new variable.

Equation (13) yields the adjoint equation,

$$F^* \equiv av_xu + bv_xu^2 + cv_xu^3 + v_t + \frac{1}{2}v_{xxx} = 0. \tag{14}$$

The Lagrangian $\mathcal{L}$ for (2) and (14) is

$$\mathcal{L} = 3vu_t - vu_x + 3vu_x^2 - v_x u_{xx}.$$

The mG Equation (2) has only two Lie point symmetries, and these are the translation symmetries $X_1 = \partial/\partial t$ and $X_2 = \partial/\partial x$. Now, using the following formula [22]:

$$C^i = \xi^i \text{Ł} + W^\alpha \left(\frac{\partial \text{Ł}}{\partial u_i^\alpha} - D_k \frac{\partial \text{Ł}}{\partial u_{ik}^\alpha} \right) + D_k(W^\alpha) \frac{\partial \text{Ł}}{\partial u_{ik}^\alpha}, \tag{15}$$

we compute the conserved vectors related to these two symmetries. Here, W^α is the Lie characteristic function, $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$.

Consequently, the conserved vectors related to the two symmetries are

$$\begin{aligned} T_1^t &= au_x uv + bu_x u^2 v + cu_x u^3 v + \frac{1}{2} u_{xxx} v, \\ T_1^x &= -au_t uv - bu_t u^2 v - cu_t u^3 v - \frac{1}{2} vu_{txx} - \frac{1}{2} u_t v_{xx} + \frac{1}{2} v_x u_{tx}, \end{aligned}$$

and

$$\begin{aligned} T_2^t &= au_x uv + bu_x u^2 v + cu_x u^3 v + \frac{1}{2} u_{xxx} v, \\ T_2^x &= -au_t uv - bu_t u^2 v - cu_t u^3 v - \frac{1}{2} vu_{txx} - \frac{1}{2} u_t v_{xx} + \frac{1}{2} v_x u_{tx}. \end{aligned}$$

3. Exact Solutions to mG Equation (2)

We construct traveling wave solutions for mG Equation (2). Thus, by letting

$$u = \Phi(\xi),\ \xi = x - \nu t,$$

the mG Equation (2) reduces to the non-linear ordinary differential equation

$$-\nu\Phi' + a\Phi\Phi' + b\Phi^2\Phi' + c\Phi^3\Phi' + \frac{1}{2}\Phi''' = 0. \tag{16}$$

Integrating Equation (16) twice gives

$$\Phi'^2 = -\frac{c}{5}\Phi^5 - \frac{b}{3}\Phi^4 - \frac{2a}{3}\Phi^3 + 2\nu\Phi^2 + 4K_1\Phi + 4K_2 \tag{17}$$

with K_1 and K_2 being constants of integration. We let

$$\mathcal{Q} = \left(-\frac{c}{5}\right)^{1/5}\Phi + \left(-\frac{b}{15}\right)\left(-\frac{c}{5}\right)^{-4/5},\ \ z = \left(-\frac{c}{5}\right)^{1/5}\xi, \tag{18}$$

Equation (17) transforms to

$$\mathcal{Q}'^2 \equiv f(\mathcal{Q}) = \mathcal{Q}^5 + p\mathcal{Q}^3 + q\mathcal{Q}^2 + r\mathcal{Q} + s, \tag{19}$$

where

$$p = \frac{2(3ac - b^2)5^{3/5}}{9(-c)^{8/5}}, \tag{20}$$

$$q = \frac{2\nu}{(-c/5)^{2/5}} - \frac{2ab}{15(-c/5)^{7/5}} - \frac{4b^3}{675(-c/5)^{12/5}}, \tag{21}$$

$$r = \frac{4C_1}{(-c/5)^{1/5}} + \frac{4\nu b}{15(-c/5)^{6/5}} - \frac{2ab^2}{225(-c/5)^{11/5}} - \frac{b^4}{3375(-c/5)^{16/5}}, \tag{22}$$

$$s = -\frac{4C_1 b}{3c} + \frac{2\nu b^2}{9c^2} + \frac{2ab^3}{81c^3} - \frac{4b^5}{1215c^4} + 4C_2. \tag{23}$$

We now employ complete discrimination system (CDS) on $f(\mathcal{Q})$ [24,25], where discr(f) represents Bezout's matrix [26] of $f(\mathcal{Q})$ and $f'(\mathcal{Q})$. Thus, we obtain

$$
\begin{aligned}
D_2 &= -p, && (24)\\
D_3 &= 40rp - 12p^3 - 45q^2, && (25)\\
D_4 &= 12p^4r + 117prq^2 - 4p^3q^2 - 88r^2p^2 + 125ps^2 - 40qp^2s - 27q^4 \\
&\quad + 160r^3 - 300qrs, && (26)\\
D_5 &= -1600qsr^3 + 2000ps^2r^2 - 3750ps^3q + 16p^3q^3s - 4p^3q^2r^2 \\
&\quad - 900rs^2p^3 + 144pq^2r^3 + 825q^2p^2s^2 + 16p^4r^3 + 2250q^2rs^2 \\
&\quad + 108p^5s^2 - 27q^4r^2 - 128r^4p^2 + 256r^5 + 108q^5s - 72p^4rsq + 3125s^4 \\
&\quad - 630prq^3s + 560r^2p^2sq, && (27)\\
F_2 &= 3q^2 - 8rp, && (28)\\
E_2 &= 160r^2p^3 - 48rp^5 + 900q^2r^2 + 1500rpsq + 60q^2p^2r + 16q^2p^4 \\
&\quad - 1100qp^3s - 3375q^3s + 625s^2p^2. && (29)
\end{aligned}
$$

In accordance with CDS (24)–(29), we have the following twelve cases for which the corresponding integrals can be acquired [27,28].

Case 1. When $D_3 > 0, D_4 = D_5 = 0, E_2 \neq 0$, we have

$$f(\mathcal{Q}) = (\mathcal{Q} - \beta_1)^2(\mathcal{Q} - \pi)^2(\mathcal{Q} - \pi_3) \tag{30}$$

with π_1, π_2, π_3 being real numbers, and $\pi_1 \neq \pi_2 \neq \pi_3$. For $\mathcal{Q} > \pi_3$, we obtain

$$
\begin{aligned}
\pm\frac{\pi_1 - \pi_2}{2}(\xi - \xi_0) &= \sqrt{\pi_3 - \pi_1}\tan^{-1}\left(\frac{\sqrt{\mathcal{Q} - \pi_3}}{\sqrt{\pi_3 - \pi_1}}\right) \\
&\quad - \sqrt{\pi_3 - \pi_2}\tan^{-1}\left(\frac{\sqrt{\mathcal{Q} - \pi_3}}{\sqrt{\pi_3 - \pi_2}}\right), \\
&\quad \pi_3 > \pi_1, \pi_3 > \pi_2,
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
\pm\frac{\pi_1 - \pi_2}{2}(\xi - \xi_0) &= \sqrt{\pi_3 - \pi_1}\tan^{-1}\left(\frac{\sqrt{\mathcal{Q} - \pi_3}}{\sqrt{\pi_3 - \pi_1}}\right) \\
&\quad - \frac{1}{\sqrt{\pi_2 - \pi_3}}\ln\left|\frac{\sqrt{\mathcal{Q} - \pi_3} - \sqrt{\pi_2 - \pi_3}}{\sqrt{\mathcal{Q} - \pi_3} + \sqrt{\pi_2 - \pi_3}}\right|, \\
&\quad \pi_3 > \pi_1, \pi_3 < \pi_2,
\end{aligned} \tag{32}
$$

$$\pm\frac{\pi_1-\pi_2}{2}(\xi-\xi_0)=\frac{1}{2\sqrt{\pi_1-\pi_3}}\ln\left|\frac{\sqrt{\mathcal{Q}-\pi_3}-\sqrt{\pi_1-\pi_3}}{\sqrt{\mathcal{Q}-\pi_3}+\sqrt{\pi_1-\pi_3}}\right| - \sqrt{\pi_3-\pi_2}\tan^{-1}\left(\frac{\sqrt{\mathcal{Q}-\pi_3}}{\sqrt{\pi_3-\pi_2}}\right), \quad \pi_3<\pi_1, \pi_3>\pi_2, \tag{33}$$

$$\pm\frac{\pi_1-\pi_2}{2}(\xi-\xi_0)=\frac{1}{2\sqrt{\pi_1-\pi_3}}\ln\left|\frac{\sqrt{\mathcal{Q}-\pi_3}-\sqrt{\pi_1-\pi_3}}{\sqrt{\mathcal{Q}-\pi_3}+\sqrt{\pi_1-\pi_3}}\right| - \frac{1}{2\sqrt{\pi_2-\pi_3}}\ln\left|\frac{\sqrt{\mathcal{Q}-\pi_3}-\sqrt{\pi_2-\pi_3}}{\sqrt{\mathcal{Q}-\pi_3}+\sqrt{\pi_2-\pi_3}}\right|, \quad \pi_3<\pi_1, \pi_3<\pi_2. \tag{34}$$

Case 2. When $D_2\neq 0, D_3=D_4=D_5=0, F_2\neq 0$, we obtain

$$f(\mathcal{Q})=(\mathcal{Q}-\pi_1)^3(\mathcal{Q}-\pi_2)^2 \tag{35}$$

with π_1, π_2 being real, and $\pi_1\neq\pi_2$. For $\mathcal{Q}>\pi_1$, we have

$$\pm\frac{\pi_1-\pi_2}{2}(\xi-\xi_0)=-\frac{1}{\sqrt{\mathcal{Q}-\pi_1}}-\sqrt{\pi_1-\pi_2}\tan^{-1}\frac{\sqrt{\mathcal{Q}-\pi_1}}{\sqrt{\pi_1-\pi_2}}, \quad \pi_1>\pi_2, \tag{36}$$

or

$$\pm\frac{2}{\pi_1-\pi_2}(\xi-\xi_0)=-\frac{1}{\sqrt{\mathcal{Q}-\pi_1}}-\frac{1}{2\sqrt{\pi_2-\pi_1}}\ln\left|\frac{\sqrt{\mathcal{Q}-\pi_1}-\sqrt{\pi_2-\pi_1}}{\sqrt{\mathcal{Q}-\pi_1}+\sqrt{\pi_2-\pi_1}}\right|, \quad \pi_1<\pi_2. \tag{37}$$

Case 3. For $D_2\neq 0, D_3=D_4=D_5=0, F_2=0$, we obtain

$$f(\mathcal{Q})=(\mathcal{Q}-\pi_1)^4(\mathcal{Q}-\pi_2) \tag{38}$$

with π_1, π_2 being real, and $\pi_1\neq\pi_2$. When $\mathcal{Q}>\pi_1$, we have

$$\pm\frac{\pi_1-\pi_2}{2}(\xi-\xi_0)=-\frac{\sqrt{\mathcal{Q}-\pi_2}}{2(\mathcal{Q}-\pi_1)}-\frac{1}{2\sqrt{\pi_1-\pi_2}}\tan^{-1}\left(\frac{\sqrt{\mathcal{Q}-\pi_2}}{\sqrt{\pi_2-\pi_1}}\right), \quad \pi_1<\pi_2, \tag{39}$$

or

$$\pm\frac{\pi_1-\pi_2}{2}(\xi-\xi_0)=-\frac{\sqrt{\mathcal{Q}-\pi_2}}{2(\mathcal{Q}-\pi_1)}-\frac{1}{4\sqrt{\pi_1-\pi_2}}\ln\left|\frac{\sqrt{\mathcal{Q}-\pi_2}-\sqrt{\pi_1-\pi_2}}{\sqrt{\mathcal{Q}-\pi_2}+\sqrt{\pi_1-\pi_2}}\right|, \quad \pi_1>\pi_2. \tag{40}$$

Case 4. When $D_2=D_3=D_4=D_5=0$, we obtain

$$f(\mathcal{Q})=(\mathcal{Q}-\pi_1)^5 \tag{41}$$

with π_1 being real. For $\mathcal{Q}>\pi_1$, we obtain

$$\pm(\xi-\xi_0)=-\frac{2}{3}(\mathcal{Q}-\pi_1)^{-\frac{2}{3}}. \tag{42}$$

Case 5. For $D_3 < 0, D_4 = D_5 = 0, E_2 \neq 0$, we obtain

$$f(\mathcal{Q}) = (\mathcal{Q} - \pi_1)(\mathcal{Q}^2 + r\mathcal{Q} + s)^2 \tag{43}$$

with π_1 being real, and $r^2 - 4s < 0$. For $\mathcal{Q} > \pi_1$, we obtain

$$\begin{aligned} \pm(\xi - \xi_0) = -\frac{2}{\rho\sqrt{4s - r^2}} \Bigg[& \cos\vartheta \tan^{-1}\left(\frac{2\rho\sin\vartheta\sqrt{\mathcal{Q} - \pi_1}}{\mathcal{Q} - \pi_1 - \rho^2}\right) + \frac{\sin\vartheta}{2} \\ & \ln\left|\frac{\mathcal{Q} - \pi_1 + \rho^2 - 2\rho\cos\vartheta\sqrt{\mathcal{Q} - \pi_1}}{\mathcal{Q} - \pi_1 + \rho^2 + 2\rho\cos\vartheta\sqrt{\mathcal{Q} - \pi_1}}\right|\Bigg], \end{aligned} \tag{44}$$

where

$$\begin{aligned} \vartheta &= \frac{1}{2}\tan^{-1}\left(\frac{\sqrt{4s - r^2}}{-2\pi_1 - r}\right), \\ \rho &= (\pi_1^2 + r\pi_1 + s)^{\frac{1}{4}}. \end{aligned} \tag{45}$$

Case 6. When $D_4 > 0, D_5 = 0$, we obtain

$$f(\mathcal{Q}) = (\mathcal{Q} - \pi_1)^2(\mathcal{Q} - \alpha_1)(\mathcal{Q} - \alpha_2)(\mathcal{Q} - \alpha_3) \tag{46}$$

with $\pi_1, \alpha_1, \alpha_2, \alpha_3$ being real, and $\alpha_1 \neq \alpha_2 \neq \alpha_3$. We have

$$\pm(\xi - \xi_0) = -\frac{2}{(\pi_1 - \alpha_2)\sqrt{\alpha_2 - \alpha_3}}\left\{F(\vartheta, k) - \frac{\alpha_1 - \alpha_2}{\alpha_1 - \pi_1}\Pi\left(\vartheta, \frac{\alpha_1 - \alpha_2}{\alpha_1 - \pi_1}, k\right)\right\}, \tag{47}$$

where $\pi_1 \neq \alpha_1, \pi_1 \neq \alpha_2, \pi_1 \neq \alpha_3$ and

$$F(\vartheta, k) = \int_0^{\vartheta} \frac{d\vartheta}{\sqrt{1 - k^2\sin^2\vartheta}}, \tag{48}$$

$$\Pi(\vartheta, n, k) = \int_0^{\vartheta} \frac{d\vartheta}{(1 + n\sin^2\vartheta)\sqrt{1 - k^2\sin^2\vartheta}}. \tag{49}$$

Case 7. For $D_5 = 0, D_4 = 0, D_3 < 0$, and $E_2 = 0$, we obtain

$$f(\mathcal{Q}) = (\mathcal{Q} - \pi_1)^3((\mathcal{Q} - l_1)^2 + s_1^2), \tag{50}$$

where π_1, l_1 and s_1 are real numbers. When $\mathcal{Q} > \pi_1$, for $\pi_1 \neq l_1 + s_1$, we have

$$\begin{aligned} \pm(\xi - \xi_0) = & \frac{\tan\theta + \cot\theta}{2(s_1\tan\theta - l - \pi_1)\sqrt{\frac{s_1}{\sin^3 2\theta}}}F(\vartheta, k) \\ & - \frac{s_1\tan\theta + s_1\cot\theta}{s_1\cot\theta + l_1 + \pi_1}\left\{\frac{\tan\theta + l_1 + \pi_1}{(s_1\cot\theta + l_1 - \pi_1)\sin\vartheta}\sqrt{1 - k^2\sin^2\vartheta} + F(\vartheta, k) - E(\vartheta, k)\right\}. \end{aligned} \tag{51}$$

If $\pi_1 = l_1 + s_1$, we obtain

$$\pm(\xi - \xi_0) = \sqrt{\frac{\sin^3 2\theta}{4s^3}}\left(\frac{1}{k}\arcsin(k\sin\vartheta) - F(\vartheta, k)\right), \tag{52}$$

where

$$\tan 2\theta = \frac{s_1}{\pi_1 - l_1},\ k = \sin\theta,\ 0 < \theta < \frac{\pi}{2},\ E(\vartheta, k) = \int_0^{\vartheta} \sqrt{1 - k^2 \sin^2 \psi}\, d\psi. \tag{53}$$

Case 8. For $D_5 = 0$ and $D_4 < 0$, we obtain

$$f(\mathcal{Q}) = (\mathcal{Q} - \pi_1)^2 (\mathcal{Q} - \pi_2)((\mathcal{Q} - l_1)^2 + s_1^2), \tag{54}$$

with π_1, l_1 and s_1 being real numbers. Whenever $\mathcal{Q} > \pi_2$, if $\pi_1 \neq l_1 - s_1 \tan\theta, \pi_1 \neq l_1 + s_1 \cot\theta$, we obtain

$$\begin{aligned} \pm(\xi - \xi_0) &= \frac{\tan\theta + \cot\theta}{2(s_1 \tan\theta - l - \pi_1)\sqrt{\frac{s_1}{\sin^3 2\theta}}} F(\vartheta, k) \\ &- \frac{s_1 \tan\theta + s_1 \cot\theta}{s_1 \cot\theta + l_1 + \pi_1} \left\{ \frac{\tan\theta + l_1 + \pi_1}{(s_1 \cot\theta + l_1 - \pi_1)\sin\vartheta} \sqrt{1 - k^2 \sin^2 \vartheta} + F(\vartheta, k) - E(\vartheta, k) \right\}. \end{aligned} \tag{55}$$

If $\pi_1 = l_1 - s_1 \tan\theta$, then we obtain

$$\pm(\xi - \xi_0) = \sqrt{\frac{\sin^3 2\theta}{4s^3}} \left(\frac{1}{k} \arcsin(k \sin\vartheta) - F(\vartheta, k) \right). \tag{56}$$

If $\pi_1 = l_1 + s_1 \cot\theta$, then we obtain

$$\pm(\xi - \xi_0) = \sqrt{\frac{\sin^3 2\theta}{4s^3}} \left(F(\vartheta, k) - \frac{1}{\sqrt{1 - k^2}} \ln \frac{\sqrt{1 - k^2 \sin^2 \vartheta} + \sqrt{1 - k^2} \sin\vartheta}{\cos\vartheta} \right), \tag{57}$$

where

$$\tan 2\theta = \frac{s_1}{\pi - l_1}, \quad k = \sin\theta, \quad 0 < \theta < \frac{\pi}{2}. \tag{58}$$

Case 9. For $D_5 = 0, D_4 = 0, D_3 > 0$, and $E_2 = 0$, we obtain

$$f(\mathcal{Q}) = (\mathcal{Q} - \pi_1)^3 (\mathcal{Q} - \pi_2)(\mathcal{Q} - \pi_3), \tag{59}$$

where π_1, π_2 and π_3 are real numbers. When $\mathcal{Q} > \pi_1 > \pi_2 > \pi_3$, we obtain

$$\begin{aligned} \pm\frac{\pi_1 - \pi_2}{2}(\xi - \xi_0) = \frac{1}{\pi_1 - \pi_3} E\left(\arcsin\sqrt{\frac{\pi_1 - \pi_3}{\mathcal{Q} - \pi_3}}, \sqrt{\frac{\pi_2 - \pi_3}{\pi_1 - \pi_3}} \right) \\ - \sqrt{\frac{\mathcal{Q} - \pi_2}{(\mathcal{Q} - \pi_3)(\mathcal{Q} - \pi_1)}}. \end{aligned} \tag{60}$$

Likewise, for other cases, such as $\mathcal{Q} > \pi_2 > \pi_1 > \pi_3$, one can write their associated solutions, which we exclude here.

Case 10. For $D_5 > 0, D_4 > 0, D_3 > 0$, and $D_2 > 0$, we obtain

$$f(\mathcal{Q}) = (\mathcal{Q} - \alpha_1)(\mathcal{Q} - \alpha_2)(\mathcal{Q} - \alpha_3)(\mathcal{Q} - \alpha_4)(\mathcal{Q} - \alpha_5) \tag{61}$$

and the solutions may be written in the form of hyper-elliptic functions (integrals), such as

$$\pm(\xi - \xi_0) = \int \frac{d\mathcal{Q}}{\sqrt{(\mathcal{Q} - \alpha_1)(\mathcal{Q} - \alpha_2)(\mathcal{Q} - \alpha_3)(\mathcal{Q} - \alpha_4)(\mathcal{Q} - \alpha_5)}}. \tag{62}$$

Case 11. For $D_5 < 0$, we obtain

$$f(\mathcal{Q}) = (\mathcal{Q} - \alpha_1)(\mathcal{Q} - \alpha_2)(\mathcal{Q} - \alpha_3)((\mathcal{Q} - l_1)^2 + s_1^2) \tag{63}$$

and the solutions may be written in the form of hyper-elliptic functions (integrals), such as

$$\pm(\xi - \xi_0) = \int \frac{d\mathcal{Q}}{\sqrt{(\mathcal{Q} - \alpha_1)(\mathcal{Q} - \alpha_2)(\mathcal{Q} - \alpha_3)((\mathcal{Q} - l_1)^2 + s_1^2)}}. \tag{64}$$

Case 12. Finally, when $D_5 > 0$ and ($D_4 \leqslant 0$ or $D_3 \leqslant 0$ or $D_2 \leqslant 0$), we obtain

$$f(\mathcal{Q}) = (\mathcal{Q} - \pi_1)((\mathcal{Q} - l_1)^2 + s_1^2)((\mathcal{Q} - l_2)^2 + s_2^2) \tag{65}$$

and the associated solutions may be written in terms of hyper-elliptic functions (integrals) as

$$\pm(\xi - \xi_0) = \int \frac{d\mathcal{Q}}{\sqrt{(\mathcal{Q} - \pi_1)((\mathcal{Q} - l_1)^2 + s_1^2)((\mathcal{Q} - l_2)^2 + s_2^2)}}. \tag{66}$$

Thus, returning to original variables t, x, and u, we obtain traveling wave solutions to the mG Equation (2).

4. Numerical Simulation of Soliton Collisions for the mG Equation

Roughly speaking, integrable systems are systems of DEs with sufficiently many conserved quantities (first integrals). With regard to PDEs, integrability is occasionally linked to having infinitely many conservation laws, the presence of a Lax pair, and elastic collisions among solitons. In an elastic collision, the objects detach after the impact, and no net kinetic energy is lost.

In this section, we numerically investigate the integrability of the mG Equation (2). For this purpose, we simulate collisions between solitons in the mG equation. Elastic collisions are anticipated from these simulations for integrable equations. Although integrability is not established decisively, this fact furnishes a strong proposition that the equation could be integrable.

With regard to our numerical study of the mG Equation (2), we limit the scope to the equation of the form

$$u_t + 2u^3 u_x + \frac{1}{2} u_{xxx} = 0, \tag{67}$$

i.e., we take $a = b = 0$ and $c = 2$ in (2). We introduce a truncated spatial domain $x \in [-L/2, L/2]$ with $L = 200$ being the length of interval, and discretize the domain through the introduction of a grid with $N = 2000$ points. Moreover, we utilize periodic boundary conditions. The spatial derivatives are approximated by means of the standard central difference approximations. The resulting system of ODE's are integrated numerically by means of the fourth-order Runge–Kutta formula with a timestep of $\Delta t = 10^{-4}$, which is small enough to ensure numerical stability.

To construct an initial condition that is appropriate for the simulation of a soliton collision, we use the one-soliton solution that is determined easily by changing to a moving frame coordinate,

$$\xi = x - Wt,$$

and integrating Equation (67) twice [29], which yields the solution

$$u(x,t) = \sqrt[3]{5W} \operatorname{sech}^{\frac{2}{3}} \left(3\sqrt{\frac{W}{2}} \xi \right), \tag{68}$$

where ξ represents the moving frame variable, and W affects the amplitude, width, and the speed of the soliton. The initial condition is

$$u(x,0) = \sqrt[3]{5W}\operatorname{sech}^{\frac{2}{3}}\left(3\sqrt{\frac{W}{2}}x\right). \tag{69}$$

As time increases, $u(x,t)$ will shift to the right. In order to validate the numerical scheme, we simulated the single soliton solution by using the initial condition (69) with $W = 1$. The solution shows strong agreement with the analytical solution at $t = 100$.

For the initial condition, let the first soliton be $f(x) = u(x,0)$ with speed $W = 0.5$, and the second soliton be $g(x) = u(x,0)$ with speed $W = 1$. We then shift the faster soliton 30 units behind the first soliton so that the new initial condition becomes $h(x) = f(x) + g(x+30)$.

We now consider the numerical results for the simulation of two solitons for the mG equation in the case where one soliton is faster than the other. Figure 1 shows the results before and after the collision. In Figure 1a, we show the solitary waves at the initial state. As they start to move with time, they approach each other as shown in Figure 1b, where $t = 30$. After this, the collision takes place, as shown in Figure 1c,d, where the solution is shown for $t = 50$ and $t = 60$, respectively. During this period the two solitons become almost indistinguishable. After the collision, Figure 1e shows a perturbation around the interval $0 < x < 20$, where $t = 80$, a good indication that the collision is not elastic. In Figure 1f, the two solitary waves re-emerges at $t = 100$.

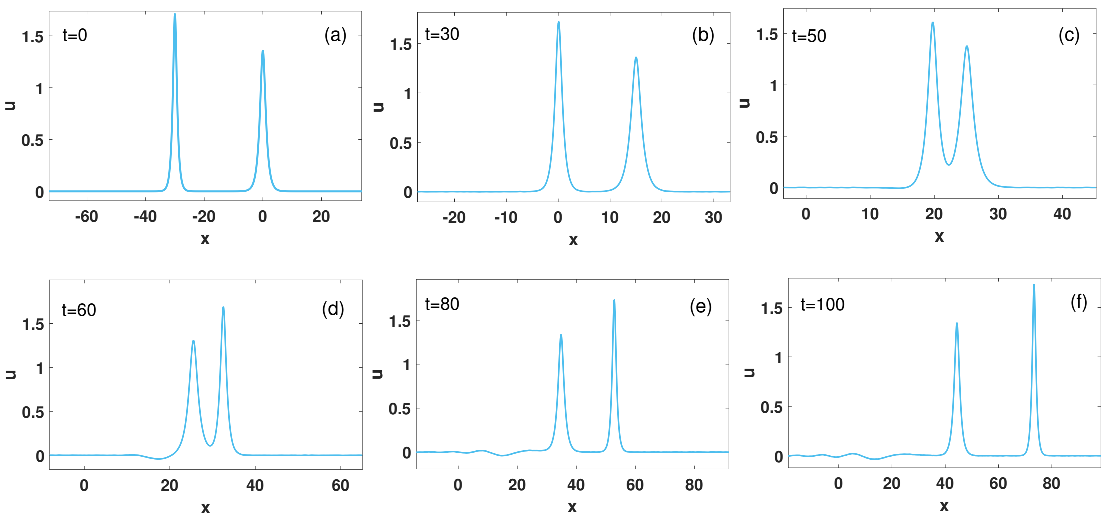

Figure 1. Plots of two solitons moving at (**a**) $t = 0$, (**b**) $t = 30$, (**c**) $t = 50$, (**d**) $t = 60$, (**e**) $t = 80$ and (**f**) $t = 100$.

To further investigate the effects of the collision, we plot these solutions in a moving frame in order to track the speed of the soliton after the collision. For an elastic collision, we expect the curve after the collision to remain vertical. If it diverts from the vertical shape, it corresponds to a change in speed. In particular, if the wave curves to the right, it corresponds to an increase in speed, while if it curves to the left, it corresponds to a reduction in speed.

Figure 2a shows the moving frame for the slow soliton, where it is moving at the speed of 0.5. The slow soliton loses speed after the collision, as is clearly seen from the leftward straying curve that arises after the collision, indicated by the arrows.

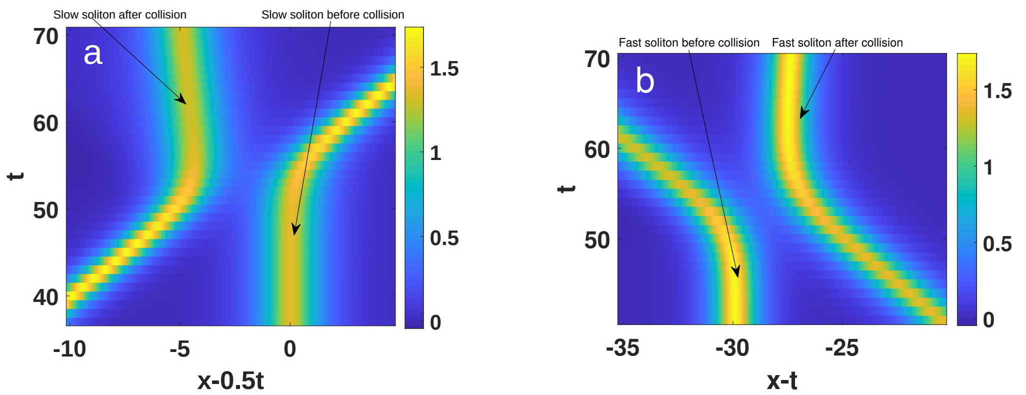

Figure 2. (**a**) Solution plotted in a moving frame with speed W = 0.5. (**b**) solution plotted in a moving frame with speed W = 1.

Figure 2b shows the solution plotted in a moving frame corresponding to the fast soliton. When the soliton moves with a speed of 1, the resulting curve formed by the fast soliton is stationary in this frame, as is seen for $40 < t < 45$. However, after the collision, the fast soliton speeds up. This is clear from the yellow curve that strays rightward after the collision, as indicated by the arrows. Besides the change in speed, the shapes of the solitons are also slightly altered following the collision. In particular, the amplitude of the fast soliton is larger, while the amplitude of the slow soliton is slightly smaller. Finally, a clear indication of the inelastic collision is the formation of a wave train in the wake of the slow soliton. This is clearly seen in panels (e) and (f) of Figure 1.

Our results suggest that this form of the modified Gardner equation is not integrable. It should be noted that this is consistent with [29], where it was shown that this form of the mG equation fails the Painlevé test. In addition, collisions between a supersoliton and a soliton were simulated in [12] for the modified Gardner equation with $a = 1$, $b = -\sqrt{3.6}$ and $c = 1$. The results also showed inelastic collisions. All of these results strongly suggest that the modified Gardner equations are not integrable for $c \neq 0$.

5. Conclusions

In this paper, the modified Gardner Equation (2) was investigated. Firstly, we determined conserved vectors of (2) using three approaches: the multiplier method, Noether's theorem, and Ibragimov's theorem. Thereafter, we utilized the traveling wave variable to transform (2) into a nonlinear ordinary differential equation. Moreover, using a CDS for polynomials and the elementary integral method, we considered twelve and computed traveling wave solutions to (2).

Lastly, we explored the integrability of mG Equation (2) numerically. In this regard, we simulated collisions of solitons of (2). In order to prove the elastic or inelastic character of collisions, we utilized two criteria that were resolved by the visual displacement of the results. Firstly, we examined whether there was any background perturbation that arose during the collision. Secondly, we determined the speeds of solitons after the collision. The results demonstrated that the soliton collision of mG Equation (2) produced a background perturbation during the collision. In addition, the slow soliton lost speed, while the fast soliton gained speed after the collision. This suggested that the equation is not integrable. Besides the lack of elastic soliton collisions, the lack of integrability implies that there may only be a finite number of conservation laws. In addition, this indicates that there may not exist a Lax pair, so that an inverse scattering transform may not exist for the mG equation.

In our future work, we shall use the double reduction theory of a PDE when a conservation law is associated with Noether symmetry of Euler–Lagrange equation and find more exact solutions to the underlying equation.

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