Constructing feasible portfolios under tracking error, beta, alpha, utility and asset weight constraints

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Preface

This dissertation has been assembled and completed under the article format.

The theoretical work described in this dissertation was carried out whilst in the employment of Bloomberg L.P. (Cape Town, South Africa) and later at Bloomberg L.P. (London, United Kingdom). Some theoretical and practical work was carried out in collaboration with the Department of Risk Management, School of Economics, North-West University (South Africa) under the supervision of Prof Gary van Vuuren and co-supervision of Prof Chris van Heerden.

These studies represent the original work of the author and have not been submitted in any form to another university. Where use was made of the work of others, this has been duly acknowledged in the text.

Unless otherwise stated, all data were obtained from Bloomberg™, non-proprietary internet sources, and non-proprietary financial databases of Bloomberg L.P., Cape Town, South Africa and Bloomberg L.P., London, United Kingdom. The results associated with the work presented in Chapter 3 (Feasible portfolios under tracking error, β, α, and utility constraints) have been published in *Investment Management and Financial Innovations* (February 2018). Although no copyright is involved for this article (since it was published under an open access agreement) a permission agreement has been signed and is provided in Appendix B.

The work described in Chapter 4 (Portfolio performance under tracking error and asset weight constraints) has been submitted for publication in the *Macroeconomics and Finance in Emerging Market Economies* (August 2019).

The results obtained from these articles and the contributions they make to the existing body of knowledge are summarised in Chapter 5 which also assesses future research opportunities.

MICHAEL DALY
01 October 2019

Acknowledgements

What began as a simple project to develop a portfolio optimisation model replicating the work theorised by Jorion (2003), has matured in something far greater than we could possibly have anticipated. Without a clear direction or foreknown conclusion, the blind, and optimistic pursuit of adding value to the field of portfolio optimisation kick-started a journey that has brought fulfilment, success and far too many sleepless nights. From tirelessly reviewing current methodologies to questioning seemingly basic concepts, the endless gratitude I have to the following people for their encouragement, insight and expertise has been incalculable. I would like to mention:

Gary van Vuuren, my supervisor, mentor and close friend. You have brought an unrivalled level of optimism and ambition that has been incredibly inspiring and motivational. I truly am grateful to having met you. Your attention to detail, ambition and positive attitude in the way you approach life is testament to all you have achieved, and I can honestly say that I am where I am today as a result of your influence.

Michael Maxwell, my co-author, we have been close friends since high school. Having taken separate paths in engineering only to meet up again in our honour's degree to tackle this beast has been incredibly rewarding. Your grit, determination and dynamic approach has helped tremendously towards the success of this journey and you couldn't be more deserving of your career.

My parents, for having raised the man I am today. I cannot thank you enough for your endless love and support and for providing me with the resources and educational platform to launch my career.

Rowan Daly, my brother, for your unconditional encouragement. You have been my role model and best friend since as long I can remember. I admire your approach to life. You've helped me understand true value when I've been too short-sighted to see it. I cannot be prouder of who you are and know that success will follow wherever your heart takes you.

Lastly, I'd like to thank all other friends and influences that have contributed in their own personal way.

Abstract

Active portfolio managers are constrained by mandates which prevent them from taking on unnecessary absolute portfolio risk when pursuing returns in excess of the benchmark. By deviating away from index weightings, an element of active risk is introduced called the tracking error (TE) – defined as the standard deviation of the difference between the returns of an investment and the prescribed benchmark. The returns and associated risk of a TE constrained portfolio form a tilted ellipse in mean/variance space which has interesting properties that may be exploited for investment purposes.

Recently (2018), a new constraint has been proposed which isolates the portfolio with the highest risk-adjusted return, i.e. through maximisation of the Sharpe ratio for a given TE. Traditionally the TE constraint has been used in conjunction with other performance indicators based on the investment policy of the portfolio. This dissertation explores the severity of the restrictive practice of constructing efficient TE-constrained portfolios, while simultaneously imposing other constraints, such as α, β and utility. Additionally, the effects of long-only portfolio selection and asset weight allocation constraints are also investigated. The imposition of such limitations on TE-constrained portfolios has not been done before. This dissertation contributes by establishing the impossibility of satisfying more than two constraints simultaneously and explores the behaviour of these constraints on the maximum risk-adjusted return portfolio (defined arbitrarily here as the optimal portfolio). In doing so, this dissertation answers the first of two research questions.

Active fund managers are responsible for driving capital gains while observing other restrictions (over and above the TE), most commonly, allocated asset weights. These boundaries are defined by upper or lower limits, acceptable ranges or – for example – long only limitations, depending on the active managers mandate. The locus of acceptable risk/return coordinates for active funds subject to these restrictions is also derived for the first time, thereby answering the second of two research questions.

Key words: Tracking error, alpha, beta, utility, active management, asset allocation.

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Chapter 1

Introduction

1.1 Background

Active managers are often given a portfolio of assets with the task of outperforming a benchmark, most frequently this is a market index. Naively, this problem is often assumed to be one of *portfolio return maximisation* i.e. without consideration for induced absolute portfolio risk.

The establishment of the Markowitz efficient frontier determines the boundary of all feasible portfolios in mean-variance space, characterised by maximal absolute returns for associated risk levels. This formulation led to the development of the tracking error (TE) frontier (and thence the constant TE frontier), which allows investors to isolate portfolios that maximise expected returns for a set of TEs (defined as the annualised standard deviation of the difference between the fund and benchmark returns (whether *ex post* or *ex ante*)).

Jorion (2003) developed the framework of the constant TE frontier, a boundary in risk/return space, where all feasible portfolios are constrained within the limits of a pre-determined TE. By constraining a portfolio to the same volatility as that of the benchmark, Jorion (2003) found that fund managers were able to achieve superior returns without taking on any additional absolute risk. Jorion (2003) suggested investment in portfolios with equal risk to the benchmark, but positioned on the constant TE frontier, where returns are maximised for a given TE constraint. Extending Jorion's (2003) work, Maxwell, Daly, Thomson & van Vuuren (2018) proposed and implemented a maximally efficient risk-adjusted portfolio return – selected through the maximisation of the Sharpe ratio – located on the TE frontier.

In portfolio management, a set of strict constraints regulates portfolio risk aversion, preventing managers from wagering in unrealistic return margins. Restricting a portfolio to a TE as a relative performance consistency measure is often used in hedge funds, mutual funds and exchange traded funds (ETFs), where profits are less predictable and investments more volatile. Optimising portfolio performance over a benchmark, while constraining TE, is non-trivial. Other constraints include portfolio α (Alexander & Baptista, 2010) and β (Roll, 1992, and Bertrand, 2009, 2010) – from the capital asset pricing model (CAPM), Value at Risk (VaR) from a risk point of view (Palomba & Riccetti, 2013, and Rodposhti & Sharareh, 2015), and utility, which combines risk and return (Stowe, 2014). The problem is that active managers sometimes combine multiple restrictions that are incompatible. Increasing portfolio β, for example, decreases risk-adjusted returns and assembling portfolios which obey strict VaR requirements as well as TE constraints are often impossible. Despite these mutually exclusive objectives, active fund managers must comply with mandates bearing these impositions, where indeed, their performance (and subsequent remuneration) is based on strict compliance with mandates (Riccetti, 2010 and Rodposhti & Sharareh, 2015) and their ability to outperform their prescribed benchmarks - a non-trivial enterprise. Fund managers must simultaneously maximise excess returns (over the benchmark), limit risk, and observe mandatory constraints such as those affecting TEs, β and sometimes α. Active fund managers must generate capital gains while observing additional mandated restrictions, such as those imposed on asset weights. These boundaries set upper or lower limits, acceptable ranges or – for example – long only limitations, depending on the active manager's mandate. The behaviour of active portfolios subject to these multiple constraints is complex and opaque, but it is considerably important for fund managers and their agents.

1.2 Problem Statement

Investors naively assume that portfolio managers can construct portfolios while constrained by maximum TEs, αs, βs, utility and restrictions on asset weights. It is impossible to construct portfolios that impose all these restrictions simultaneously.

1.3 Research questions

i. Is it possible to generate a TE-constrained portfolio that simultaneously optimises α, β and a utility constraint? What compromises must be made to reach these objectives as closely as possible?

ii. How does the imposition of constraints on portfolio weights influence the investable universe of TE-constrained portfolios?

1.4 Dissertation structure

The dissertation proceeds as follows:

Chapter 2 presents the history governing portfolio construction and optimisation using effective asset allocation and risk reduction techniques. The framework around modern portfolio theory, with respect to the efficient frontier on the mean-variance plane, and the development of the CAPM is investigated. Active portfolio managers are finding it more and more difficult to outperform a benchmark as a result of a progressively evident imposition of investment restrictions. As markets become increasingly efficient and saturated, tracking real-time portfolio performance has become a necessity for companies that wish to maintain an 'edge' or survive in our current economy. Chapter 2 identifies current market approaches and explores various theoretical optimisation techniques that can be used to select an optimal portfolio tailored to investor needs. The mathematical methodologies surrounding the vast arsenal of possible investor constraints (such as TE, β and portfolio weight constraints) on portfolio selection is examined and the effects of these restrictions on the investable universe is analysed in detail, where various optimisation techniques (such as maximum Sharpe, β and α frontier construction and utility performance) are explored, that aim to navigate a portfolio manager through the labyrinth of effective portfolio selection and management.

Chapter 3 traces the development of various frontiers and boundaries in risk/return (and sometimes in mean/variance) space, which define the limitations of the active fund manager's investable universe. These limits characterise efficient portfolios, in the sense that they establish maximal returns for given levels of risk, i.e. β, α, VaR or other parameters or combinations of these. Inevitably, the regions bordered by these limits shrink as constraints are added. Depending on the severity of the constraints, the potential universe of permitted investments is sometimes undefined. Navigating this narrow arena of possibilities – and optimising the returns generated from it – is a complex task. The contribution of this chapter is to assemble these frontiers and then populate the return/risk space with them, using the same, small but stylised, asset universe to demonstrate the consequences of their limitations. This chapter answers the first research question.

Chapter 4 investigates the effect of constraining individual, or groups, of assets to a long only position subject to a given TE. In addition, the consequence of imposing weight allocation ceilings on portfolio assets are also explored, whereby new constrained frontiers are established governing a new and diminished investable universe in mean variance space. This chapter addresses the second research question.

Chapter 5 concludes the dissertation and sets out future possible research.

1.5 Research objectives

The research objectives are divided into general and specific objectives outlined below:

1.5.1 General objectives

The general objective of this research is to construct and establish the α, β and utility frontier in mean/variance and risk/return space which will be used to investigate the range of possible portfolios, given various mandated constraints. In addition, the effect on adding a weight allocation restriction on long-only TE-constrained portfolios will be explored.

1.5.2 Specific objectives

The specific objectives of this research are:

i. Establish in mean/standard deviation space the range of possible investments when constrained by a TE;

ii. Evaluate the behaviour of α, β and utility constraints on the constant TE frontier in mean/variance as well as return/risk space;

iii. Verify that the simultaneous imposition of more than two constraints leads to inefficient portfolios (i.e. those that do not exhibit maximum return or minimum risk); and

iv. Evaluate the imposing of a weight allocation limitation on long-only TE-constrained portfolios and determine how the investable universe of feasible portfolios is affected.

1.6 Research design

1.6.1 Literature review

The literature review focuses on the origin, development, history and applications of the issues identified through problem statements and research questions surrounding optimal portfolio selection subject to various constraints. The literature study explains and clarifies the problem of portfolio optimality and validates how previous studies have addressed the problem. Alternative approaches that illustrate the limitations of constrained portfolio performance are presented for the first time.

1.6.2 Data

Data requirements, frequency and source are shown in Table 1.1 below.

Table 1.1: Data requirements, frequency and source.

#	Topic	Data required	Frequency	Sources
1	Feasible portfolios under tracking error, β, α, and utility constraints	Historical asset returns and covariance matrices (including individual asset volatilities and correlations) Portfolio weights, β, α, and utility calculations	Monthly	Bloomberg, IRESS (formerly INET BFA), Opendata, and other non-proprietary internet databases
2	Portfolio performance under combined tracking error and investment constraints			

1.6.3 Research output

Table 1.2: Research output.

#	Topic	Mathematics	Research methodology
1	Daly, M., Maxwell, M. & van Vuuren, G. 2018. Feasible portfolios under tracking error, β, α, and utility constraints. *Investment Management and Financial Innovations,* 50(54): 5846–5858	Proprietary, bespoke Microsoft Excel™ models using: Calculus (differentiation techniques to identify optimal solutions) Linear algebra Lagrangian dynamics	Portfolio optimisation approaches. Asset selection under prescribed portfolio constraints
2	Daly, M. & van Vuuren, G. 2019. Portfolio performance under combined tracking error and asset weight constraints. Submitted for publication in *Macroeconomics and Finance in Emerging Market Economies*.		

1.7 Conclusion

The conclusion presents a summary of the key findings of both topics and provides detailed recommendations for possible future research. The next chapter presents a literature survey governing the background information relevant to the dissertation.

Chapter 2

Literature study

Inherent in the success of a portfolio manager is the skill of portfolio optimisation, forming part of Modern Portfolio Theory, or Portfolio Selection Theory (PST). The principal objective of PST (in addition to maximising returns) is to find the optimal allocation of investments between different assets, given the investor's risk profile/preference, i.e. to diversify away as much risk (volatility of returns) as possible. This 'optimal' allocation results in a portfolio of selected assets whose risk matches the investors risk preference, thus maximising their utility (Ghosh & Mahanti, 2014). This suggests that the trade-off between risk and return (mean/variance trade-off) is different for each investor, but Markowitz (1952) indicated that, although this may be the case, the preferences of all investors lie on a curve, namely the efficient frontier. The efficient frontier comprises efficient (diversified) portfolios which have the lowest risk for a given level of return or, equivalently, the highest return for a given level of risk, i.e. the set of the best risk/return combinations forms this frontier.

Portfolio optimisation falls in as one of the phases of what is known as the investment management process. This process encapsulates the general procedure followed by portfolio managers when selecting an 'optimal' portfolio for the investor. The other phases involve capturing the risk preference/profile of an investor, recorded in their investment policy along with; the investment objectives, constraints, permitted investment allocations in asset classes and/or sectors, the investment strategy (passive/active, value/growth), and the performance measures and evaluators (Fabozzi & Markowitz, 2011). These form part of portfolio planning and optimisation models used to solve the portfolio optimisation (or selection) problem.

The portfolio optimisation problem, formulated by Markowitz (1952), consists of two criteria, namely expected return (mean) and risk (standard deviation), measuring the volatility/variability of returns. Markowitz (1952) formulated this problem into a single investment period model, in which the investor allocates the capital amongst several assets. Over the course of the investment period, a random rate of return is generated by the portfolio resulting in greater or lower capital value at the end of the period (relative to the principal amount). Subsequent research has extended this model to multiple periods and it remains the foundation upon which Modern Portfolio theory is based (Mansini, et al., 2014).

Markowitz (1952) stated that the portfolio selection process can be divided into two main phases. The first phase couples experience and observation in order to forecast the future performance of the assets of interest, and the second uses these forecasts in choosing the most suitable portfolio. The first stage depends largely on the ability of the fund manager, the models used for forecasting and the way estimation error is dealt with. There has been extensive research done on this and it is beyond the scope of this dissertation. Markowitz (1952) concludes that investors should focus on return and risk in conjunction when selecting desired portfolios. In doing so, they will most likely choose a portfolio aligned with their preferences, thereby maximising their utility.

Utility maximisation is ultimately the desired outcome for the investor, and forms part of an extensively researched field of economics called Utility Theory. The utility of an investor is the total satisfaction received from consumption or investment of capital. It is described by a utility function, which assigns numeric values to all possible choices faced by the investor where the higher the numeric value of a choice, the greater the satisfaction derived from it (Fabozzi & Markowitz, 2011). As such, PST sets out to find the optimal choice (portfolio) resulting in the maximum possible utility, given a set of investor constraints. This is achieved using indifference curves. An indifference curve represents a set of choices (in this case portfolios with different risk/return combinations) for which the investor derives the same level of utility from each and is therefore indifferent to which is chosen. These indifference curves can be mapped out in the same space (mean/variance) as the efficient frontier, enabling the portfolio manager to select the optimal portfolio from the point where the maximum indifference curve is tangential to the efficient frontier (Fabozzi & Markowitz, 2011). This results in the selection of an efficient portfolio which is optimal for the investor, i.e. satisfies their risk profile/preferences (Larsen, & Resnick, 2001).

Markowitz (1952, 1959) formulated mean variance optimisation into a quadratic programming model, providing a quantitative tool to use when making the investment allocation decision by considering the trade-off between risk and return (mean and variance) of a portfolio of assets (Ghosh & Mahanti, 2014). Markowitz (1959) extended his 1952 work and transformed it into the Markowitz model (aforementioned), in which this optimal allocation of holdings/investments is determined through the solution of this quadratic programming model (Ghosh & Mahanti, 2014). This mean variance model has been altered in various ways since its inception, namely; the single index/market model which ignores the covariance between asset returns, the CAPM (Capital Asset Pricing Model) as an extension of the single index model (considering the returns of securities to depend on the market index and not the covariance between asset pairs), and the multiple period Mean Variance model.

Ghosh and Mahanti (2014) restated that an important implication of Modern Portfolio Theory (based on the work of Markowitz (1952, 1959)) is that when selecting an asset for a portfolio, the risk and return of an asset should not be considered in isolation but rather in conjunction with the correlation of that asset with the other constituents. This co-movement with other assets, if negligible or in the opposite direction, can reduce the risk (volatility of returns) of the portfolio significantly, whilst maintaining the same overall portfolio return. The process of adding additional uncorrelated or negatively correlated assets to reduce the overall risk of portfolio is known as diversification (Clarke, de Silva, & Thorley, 2002).

Once the optimal portfolio is selected, its performance (and hence the manager's) must be measured and evaluated, a fundamental issue in portfolio management. Various performance measures and attribution models (performance evaluation) have been proposed, two of which are used in this paper to ‘reverse engineer’ the optimal portfolio. The most noteworthy performance attribution model is the Fama Decomposition of Total Return (Fabozzi & Markowitz, 2011), which identifies the sources of the portfolio’s return, indicating how much of the return can be attributed to the manager and how and why he/she earned that return. Notable portfolio measures include: the Treynor ratio (“ratio of excess returns, above risk-free rate, to Beta (systematic risk)”) indicating the manager’s skill in market timing; the Jensen index, as an absolute measure, indicating the ability of the manager in forecasting returns and portfolio diversification against risk (Ghosh & Mahanti, 2014); the Information ratio (“ratio of excess return, above the benchmark, to the TE of the portfolio”), not only highlighting the manager’s ability to generate excess returns but also the consistency of those returns. Lastly, the Sharpe ratio (“ratio of excess return, above the risk-free rate, to total portfolio risk”), indicating the manger’s aptitude in security selection; and the TE, measuring the volatility of the excess returns above the benchmark. These are used in the final phase of the Investment management process and give the investor an overall picture of the ability of the manager, the performance of the portfolio, and the level of satisfaction the investor has derived (whether performance is aligned with their preferences).

The principal objective of active portfolio managers is to select and manage a portfolio that achieves returns in excess of the benchmark. Successful portfolio management lies in the long-term mix of assets that exhibit relatively low correlations with each other. Diversification is a theoretical risk mitigation technique used to smoothen the effects of unsystematic risk associated with inversely performing investments. The rationale behind this logic is that the combination of dissimilar portfolios will, on average, yield higher returns at lower level of risk than that of individual securities (Menchero & Hu, 2006). Effective asset allocation is non-trivial and is considered the most important decision in portfolio construction, more so than individual security selection. Ghosh and Mahanti (2014) suggested however, that inter-asset correlations should always be considered in individual asset selection as the co-movement between assets can reduce portfolio variance while maintaining overall portfolio return.

Retaining the optimal portfolio mix involves the constant rebalancing of portfolios, whereby over-valued securities are repeatedly reweighted/substituted for undervalued securities. Fund managers are evaluated (and ultimately remunerated) on their ability to exceed benchmark returns tantamount to a positive expected TE (Riccetti, 2010). In contrast, reducing TE is equivalent to reducing relative portfolio risk. Modern portfolio theory assumes that investors are risk averse, meaning that given two portfolios of equal expected returns, an investor will always favour the less risky portfolio. The trade-off associated with the risk/return portfolio can be described by a hyperbolic curve known as the efficient frontier, which categorises the highest expected return possible for any given level of risk. A TE, defined as the standard deviation between the portfolio return and the market index, is used to quantify the consistency and performance of a portfolio relative to a benchmark over time (Plaxco & Arnott, 2002).

Traditional portfolio management follows a two-dimensional performance approach. Investors are evaluated on their ability to exceed benchmark returns synonymous with a positive expected TE. In contrast, reducing TE is equivalent to reducing relative portfolio variance. Roll (1992) identified that investors targeting the highest possible excess expected return above the benchmark while concurrently minimising TE were naively disregarding absolute portfolio risk. The construction of the TE frontier (Figure 2.1) illustrates the maximum expected return away from a benchmark subject to a TE constraint. Analytically, it can be shown that the TE curve is derived independent of benchmark returns and unless the index lies directly on the efficient frontier, these portfolios are inefficient (Jansen & van Dijk, 2002). Furthermore, if active portfolio managers were incentivised exclusively on maximising excess return then a portfolio that sits on the upper half on the efficient frontier would be far more favourable. In practice, expected returns are noisy and unpredictable. Instead fund managers are constrained by a TE that prevents excessive amounts of absolute risk being taken on in search for superior relative returns. Failing to adhere to this mandate would result in severe retributive action (El-Hassan & Kofman, 2003).

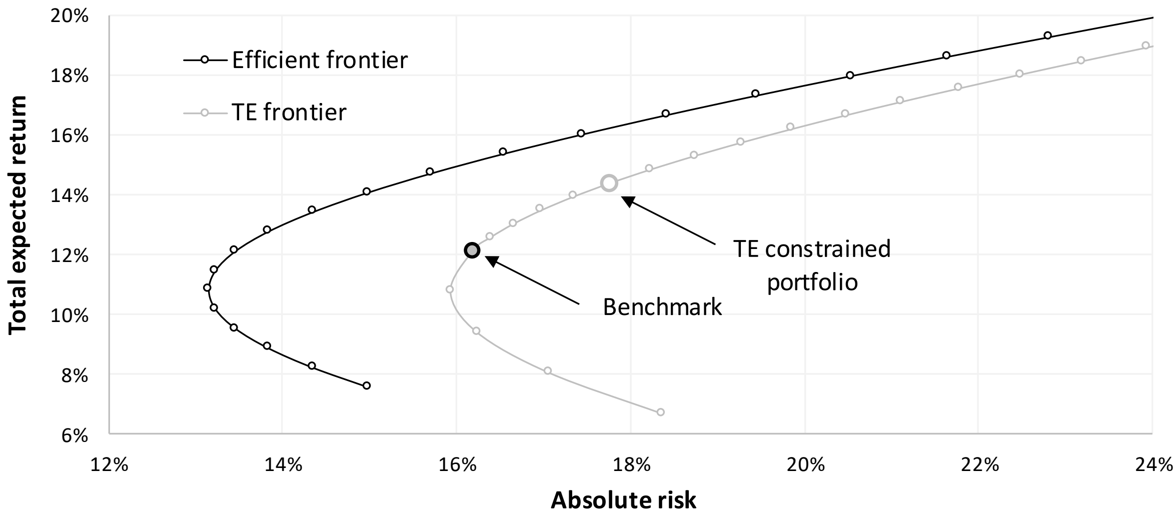

Figure 2.1: TE frontier and TE-constrained portfolio. In this example, $TE = 5\%$ and the TE constrained portfolio position shows the maximal return allowable for that level of TE.

Source: Roll (1992) and own calculations.

Each point on the TE frontier represents the maximum total expected excess return possible for a given TE. Markers indicate intervals of 1% TE deviation away from the benchmark i.e. the enlarged TE-constrained portfolio marker shown above describes the maximum excess return possible above a benchmark for a $TE = 5\%$.

Jorion (2003) investigated whether the naïve characteristic of active portfolios taking on systematically higher risk than that of the benchmark could be solved while maintaining a TE constraint. He proposed an alternative investment decision based on selecting portfolios with the same benchmark risk, but situated on the constant TE frontier as shown in Figure 2.2. Jorion (2003) showed that because of the 'flatness' of the ellipse, the addition of a total portfolio volatility constraint significantly improved portfolio performance, especially with less efficient benchmarks and lower TEs. Jorion (2003) illustrated this by constraining the portfolio volatility to that of the benchmark, whereby portfolios of higher relative returns (but lower absolute returns) could be targeted.

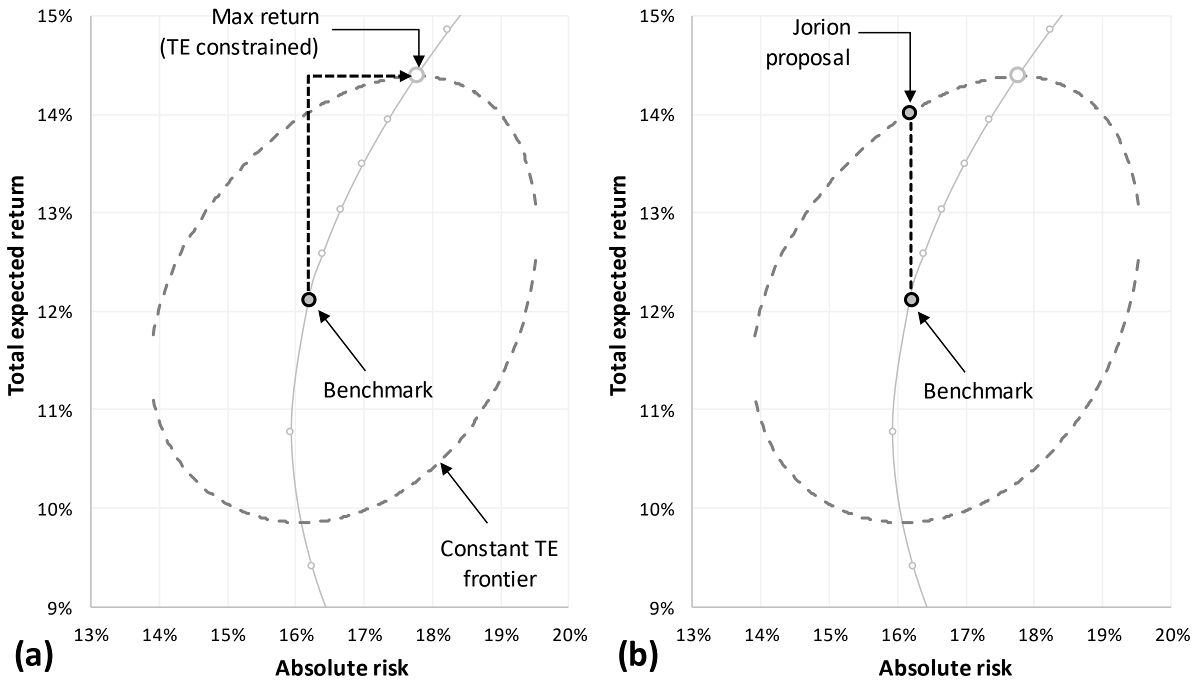

Figure 2.2: TE frontier, TE-constrained portfolio and constant TE frontier (with TE $= 5\%$). (a) Shows the naïve portfolio: excess return is maximised for a given TE constraint. (b) shows Jorion's (2003) suggestion: observe constraints from (a) but restrict portfolio risk to that of the benchmark.

Source: Roll (1992), Jorion (2003) and own calculations.

By mapping out the boundary of all possible portfolios constrained by a TE, Jorion (2003) formulated the constant TE frontier which is described as an ellipse in the traditional mean/variance space (Figure 2.2). This can also be represented in risk/return space as a tilted ellipse in which the eccentric symmetry of the ellipse is abolished.

Figure 2.3 illustrates the constant TE frontier (in risk/return space) for various levels of TE. The grey shadow shown in the $TE = 0\%$ plane represents the realm of feasible portfolios where the outer boundaries trace the efficient frontier. For $TE = 0\%$, the constant TE frontier exists as a single point where the absolute return and risk profile equals that of the benchmark portfolio. For $TE > 0\%$, the constant TE frontier ellipse initially expands outwards until reaching the efficient frontier, as portfolios are able to take on more and more relative portfolio risk. Further increases of TE (in this example, for $TE > 7\%$) pushes the constant TE frontier *away* from the efficient frontier, showing that taking on unnecessary absolute risk results in less efficient portfolios.

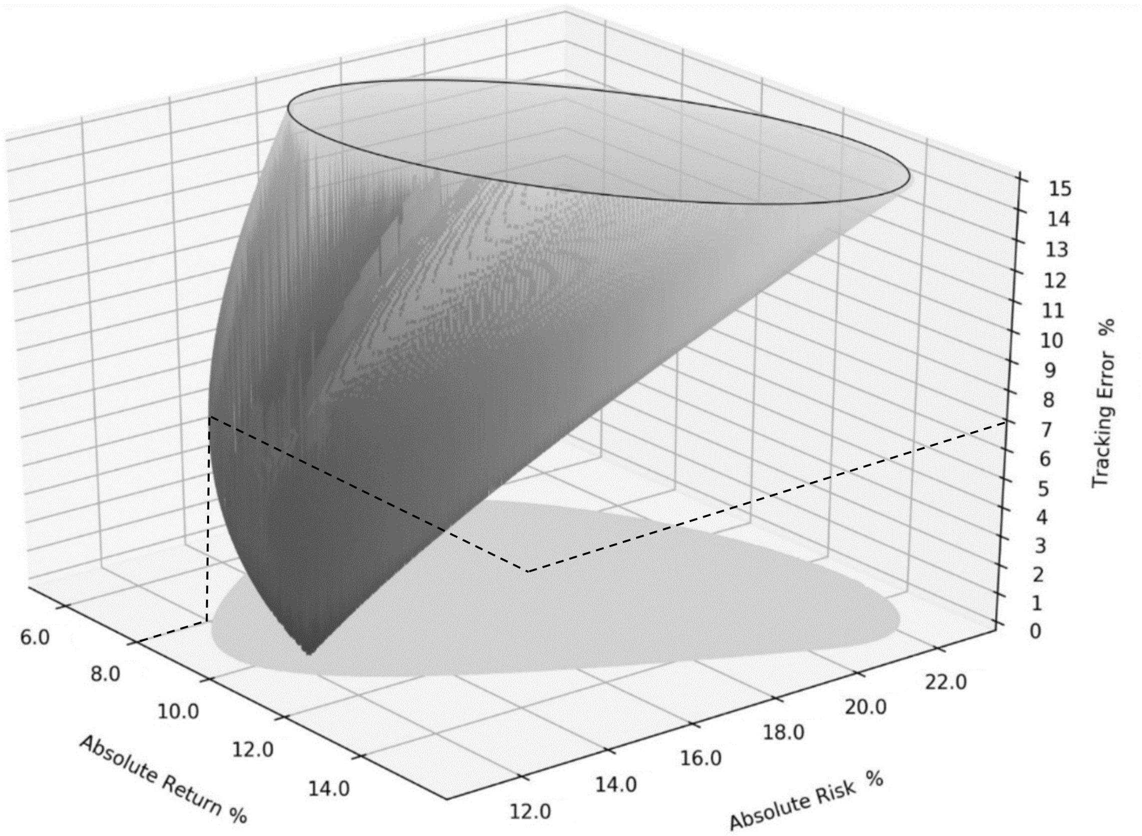

Figure 2.3: Constant TE-constrained frontier $0\% \leq TE \leq 15\%$.

Source: Compiled by authors.

Maxwell, Daly, Thomson & van Vuuren (2018) took this one step further by calculating the position of the portfolio with the highest risk-adjusted return through the addition of another constraint – maximisation of the Sharpe ratio for a given TE. These portfolios sacrifice marginal portfolio return (but hold significantly less absolute risk) than the maximum return portfolio (Figure 2.4). Depending on the risk-free rate, the maximum Sharpe portfolio could have a positive, or negative, volatility relative to the index. In countries such as the United Kingdom, where the interest rates are low (0.75% in May 2019), the maximum Sharpe-constrained portfolio has a higher expected return and lower risk than the market return (where $\beta < 1$).

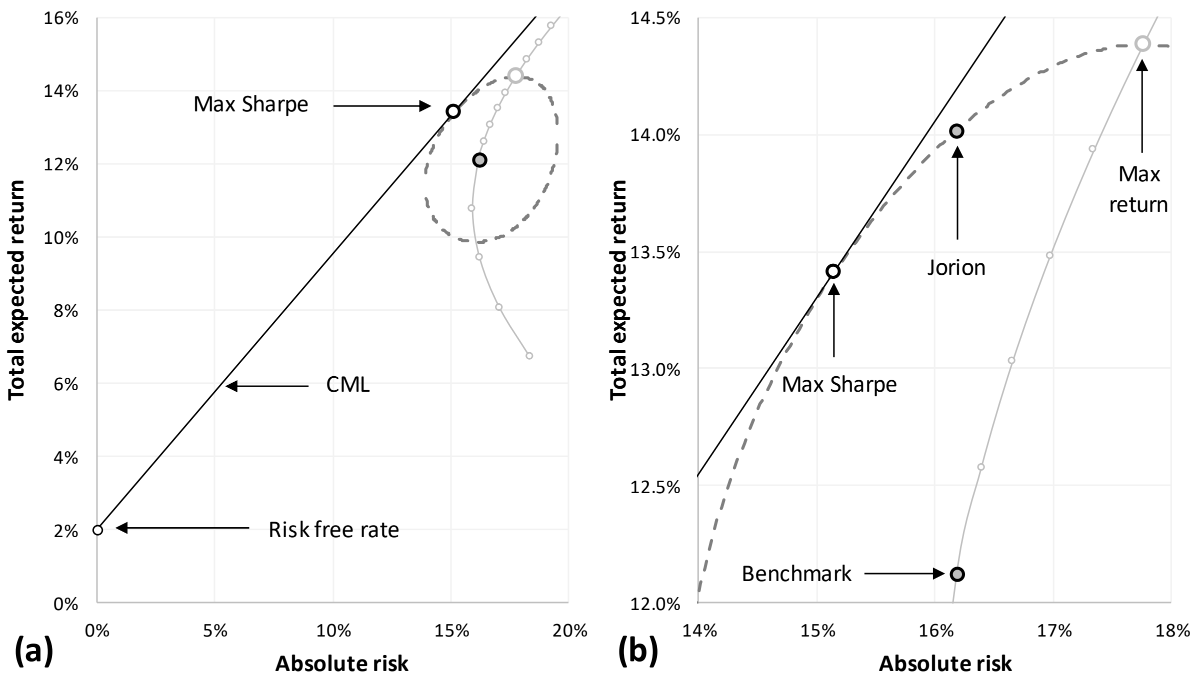

Figure 2.4: (a) TE-constrained portfolio, constant TE frontier and CML with optimal portfolio and (b) enlarged view showing all three portfolios. TE $= 5\%$ and $r_f = 2\%$.

Source: Jorion (2003) and own calculations.

Portfolio managers are often asked to build β-constrained portfolios as higher βs are theoretically indicative of higher returns, however, Roll (1992) proved this to be incorrect. Portfolios that have a negative volatility (and a higher expected performance) relative to the index have $\beta < 1$ on the benchmark. Subsequently, portfolios with a positive performance of $\beta > 1$ are symptomatically inefficient and lie further to the right of the efficient set. For $\beta = 1$ (that is not the benchmark) would sit on the TE frontier but will always be less efficient to that of the benchmark. The addition of the β constraint allowed Roll (1992) to prove the impossibility of producing β constrained portfolios that simultaneously minimises TE and outperforms the market return. The position of Jorion's (2003) proposed portfolio, and the maximum Sharpe portfolio, agrees with Roll's (1992) findings that higher performing portfolios exhibit portfolios with $\beta < 1$.

In Chapter 3 (Paper 1) Daly, Maxwell & van Vuuren (2018) stylised the mathematics governing the β, α-TE and investor utility frontier for portfolios bound by a TE and determined that it is impossible to simultaneously satisfy more than two constraints onto the constant TE frontier (Figure 2.5). The α-TE frontier differs from other constraints in that it shows the minimum TE for various levels of *ex-ante* α. This means that for every TE constrained portfolio, a maximum

α would exist at the boundary of the constant TE. A β frontier that coincides with the maximum α-TE constrained portfolio would be a maximum as defined by Roll (1992) ($\beta < 1$), although this is not useful in practice.

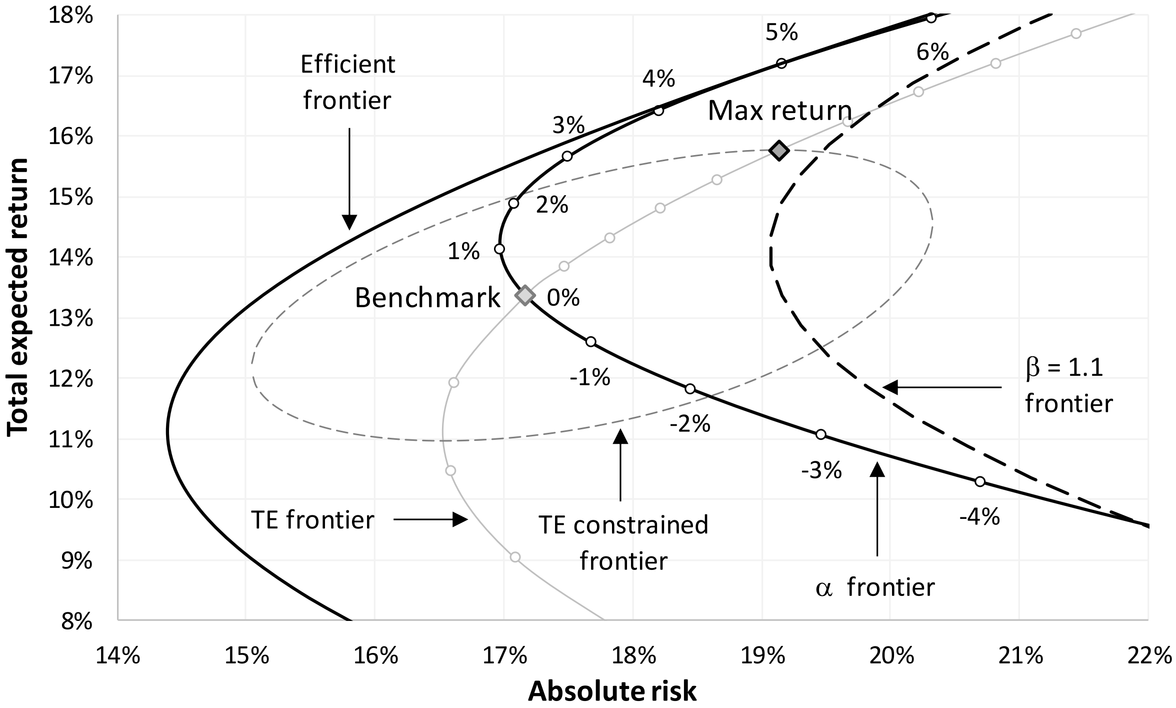

Figure 2.5: The α-TE frontier for various levels of α. Other frontiers are shown for comparison. Levels of α are indicated on the graph. TE $= 5\%, r_f = 2\%$.

Source: Compiled by authors.

Total utility is the quantitative measure of investor satisfaction. Portfolio selection that maximises utility does not necessarily mean choosing a fund that maximises returns, minimises risk or maximises risk-adjusted returns. This implies that utility optimisation is a subjective constraint specific to each investor. Indifference curves show the resulting satisfaction gained based on investment decisions, where each point along the indifference curve would represents the same level of satisfaction for various risk/return combinations (Fabozzi & Markowitz, 2011). When modelled on the mean variance plane, an optimal portfolio can be selected where the maximum indifference curve is tangential to the efficient frontier. Here the risk/return profile (as well as investor utility) is maximised. In Chapter 3 (Paper 1) Daly, Maxwell & van Vuuren (2018) investigated the utility function of TE-constrained portfolios at the maximum Sharpe portfolio and found that risk aversion augmented with increasing TE, up until a point.

Maxwell & van Vuuren (2019) stylised the behaviour of alternative portfolio assemblies on the constant TE frontier (Figure 2.6). These portfolios where characterised as maximally diversified, exhibit risk parity, have minimal intra-correlation, and minimum risk for varying levels of TE. Every point along this frontier was investigated and it was discovered that such portfolios behaved adversely to mean variance efficient (unconstrained) portfolios.

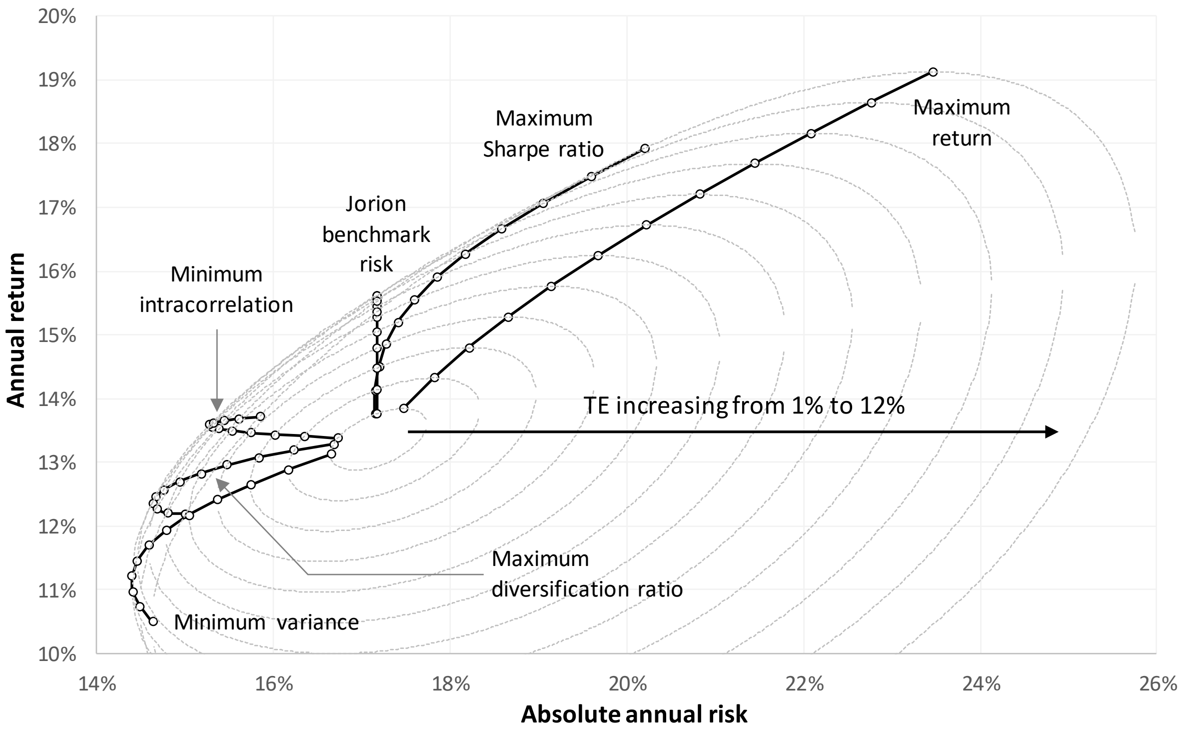

Figure 2.6: Loci of relevant portfolios in mean/risk space for $1\% \leq TE \leq 12\%$.

Evans & van Vuuren (2019) analysed six active TE constrained performance strategies, using numerous performance measures to assess the relative performance on the investment on the ellipse. It was found that the performance ratios reach plateaus for high TEs because of the roughly linear nature of the efficient frontier in risk/return space. Because the constant TE ellipse remains in contact with the efficient frontier for high TEs, the maximum Sharpe ratio for the former will always be approximately the same as the latter.

Bertrand (2010) investigated the effect of fixing the investors level of risk aversion and allowed the TE to float between 0 (the benchmark) and infinity (the minimum variance portfolio). This resulted in the generation of what was called 'iso-aversion frontiers' where all optimal portfolios had the same information ratio, allowing for fund managers to make a portfolio selection based on a desired volatility. The problem with this research is that actively managed funds are generally constrained by a TE mandate, rendering Bertrand's (2010) findings obsolete.

Although modern portfolio theory dictates that specific risk can be removed through portfolio diversification, systematic risk cannot be eliminated e.g. interest rate, transactional fees, economic recession etc. The capital assets pricing model (CAPM) is a theoretical tool that can be used to determine an expected rate of return based on the calculated market risk of the investment. Stowe (2014) identified that the application of β and TE constraints on portfolio selection, assured favourable improvements to investor utility, and if implemented correctly, prudent managers would be able to produce more efficient portfolios.

Alexander and Baptista (2010) proposed an innovative methodology of reducing the sub optimality associated with portfolios that do not lie on the efficient set. The formulation of the α-TE frontier is helpful to practitioners who evaluate the performance of a fund manager based on ex-post α of the investment. In addition, the α-TE frontier allows managers to identify less risky, utilitarian portfolios that are not typically selected by most active managers (Wu & Jakshoj, 2011).

Constructing mean-variance efficient portfolios often involve taking extreme long and short positions, hence the need for active portfolio managers to impose an asset weights constraint. The imposition of a weights constraint is common is active portfolio management, as funds are committed by their own prospectus to a constrained portfolio concentration. Ammann and Zimmermann (2001) examined the relationship between TE and restricted asset weight deviations away from the benchmark and found that imposing large tactical asset allocation ranges implies tracking errors much smaller than expected. Tactical asset allocation restrictions were also found to not only restrict the tactical ranges of the individual asset classes, but also the tracking of the individual asset classes. Bajeux-Besnainou, Belhaj, Maillard & Portait (2011) determined an optimal asset allocation of such agency mandated portfolios and analysed the implications of weight restrictions on managerial performance. Bajeux-Besnainou, Belhaj, Maillard & Portait (2011) investigated the limitations associated with weight constraints on TE-constrained portfolios and found that these restrictions were mutually binding. Also, because of the weight constraint, the information ratio decreases when the fund manager deviates further from the benchmark.

Although various TE-constrained portfolio optimisation techniques have been suggested above, literature regarding constraining long-only asset allocation on TE-constrained portfolios has been limited. Chapter 3 (Paper 1) contributes by developing the mathematical framework surrounding β , α and utility frontiers on TE constrained portfolios and Chapter 4 (Paper 2) evaluates the consequential effect of long-only weight restricted portfolios on the constant TE frontier.

Chapter 3

Feasible portfolios under tracking error, $\boldsymbol{\beta}$, $\boldsymbol{\alpha}$, and utility constraints

Michael Daly,[1] Michael Maxwell[2] and Gary van Vuuren[3]

Abstract

The investment nous of active managers is judged on their ability to outperform specified benchmarks while complying with strict constraints on, for example, tracking errors, β and Value at Risk. Tracking error (TE) constraints give rise to a TE *frontier* – an ellipse in risk/return space which encloses theoretically possible (but not necessarily efficient) portfolios. The β frontier is a parabola in risk/return space and defines the threshold of portfolios subject to a specified β requirement. An α-TE frontier is similarly shaped: portfolios on this frontier have a specified α for a maximum TE. Utility and associated risk aversion have also been explored for constrained portfolios. This paper contributes by establishing the impossibility of satisfying more than two constraints simultaneously and explores the behaviour of these constraints on the maximum risk-adjusted return portfolio (defined arbitrarily here as the optimal portfolio).

Keywords	Tracking error frontier, β, α, utility, optimal portfolios
JEL classification	C52, G11

3.1 Introduction

Active portfolio managers aim to outperform their benchmarks while adhering to constraints imposed by principals. One of the most commonly-used of these constraints is the TE,[4] the annualised standard deviation of the difference between the fund and benchmark returns (whether *ex post* or *ex ante*). Optimising portfolio performance over a benchmark, while constrained to a TE, is non-trivial. Problems begin with the definition of *optimal*. These portfolios have been variously defined as those – constrained by a TE – which outperform the benchmark by the greatest amount with no regard to portfolio volatility (Roll, 1992), which have the same volatility as the benchmark and highest excess return over the benchmark (Jorion,

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[4] Researchers also refer to this quantity as the TE *volatility*, but the phrase has largely fallen out of use amongst practitioners and become simply *TE*.

2003), the highest risk-adjusted return (Maxwell, Daly, Thomson & van Vuuren, 2018) and others.

Contemporary active managers are not only constrained by TE: others include portfolio α (Alexander & Baptista, 2010) and β (Roll, 1992, and Bertrand, 2009, 2010) – from the capital asset pricing model (CAPM), Value at Risk (VaR) from a risk point of view (Palomba & Riccetti, 2013, and Rodposhti & Sharareh, 2015), or utility, which combines risk and return (Stowe, 2014). These multitudinous restrictions are often incompatible. Increasing portfolio β, for example, decreases risk-adjusted returns. Assembling portfolios which obey strict VaR requirements as well as TE constraints is often impossible. Despite these mutually exclusive objectives, active fund managers must comply with mandates bearing these impositions, indeed, their performance (and subsequent remuneration) is based on strict compliance with these mandates.

This paper traces the development of various frontiers and boundaries in risk/return (and sometimes in mean/variance) space, which define the limits of the active fund manager's investable universe. These limits characterise *efficient* portfolios, in the sense that they establish maximal returns for given levels of risk, or β, α, VaR or other parameters, or combinations of these. Inevitably, the regions bordered by these limits shrink as constraints are added. Depending on the severity of the constraints, the potential universe of permitted investments is sometimes undefined. Navigating this narrow arena of possibilities – and optimising the returns generated from it – is a complex task. The contribution of this paper is to assemble these frontiers and then populate the return/risk space within them, using the same, small but stylised asset universe to demonstrate the consequences of the limitations.

This paper proceeds as follows. Section 3.2 discusses the relevant literature governing some of the constraints imposed on active managers and traces the mathematical development which describes the plausible (and ever-diminishing) investment universe given the array of mandated constraints. The relevant mathematics is introduced, defined and contextualised in Section 3.3. The data used are also described in this section. Section 3.4 presents the results and discusses ramifications of active portfolio optimisation constraints. Section 3.5 concludes.

3.2 Literature survey

The Markowitz framework, in a fund management context, establishes the relationship between expected portfolio returns and the variance of those returns given a universe of investable assets. This relationship gives rise to the well-known efficient frontier; the parabolic delineation in mean return/variance space. Active portfolio managers are, however, constrained by restrictions specified by the fund sponsors: poorly-assembled benchmarks (which are seldom Markowitz efficient), a maximum TE (defined as the standard deviation of differences between the benchmark and the active portfolio's returns), a minimum outperformance of the benchmark, an active fund β, a VaR, etc.

Active fund managers are commonly rewarded for generating expected returns (by outperforming mandated benchmarks) while simultaneously minimising specified TE. Roll (1992) called this the TEV (TE volatility) criterion and established conclusively that in attempting to satisfy it, fund managers *intentionally* do not produce mean/variance efficient Makowitz portfolios under all but the rarest of circumstances. Portfolios selected by active fund managers would *always* be dominated by other portfolios with higher average returns and lower volatilities.[5]

Roll (1992) formalised the problem of TE-constrained portfolios and established an elegant solution for the "TE frontier" (Figure 3.1), i.e. portfolios having a maximum total expected return possible for a given TE. Markers are placed at intervals of 1% in Figure 3.1, so the TE-constrained portfolio indicated represents the maximum excess return possible for a fund relative to its benchmark with a TE constraint of 4%, the point above and to the right of it, TE $= 5\%$, and so on.

Jorion (2003) augmented Roll's (1992) solutions by establishing the shape of constant TE portfolios, i.e. the locus of active portfolios with the same TE, being equidistant from the benchmark. Jorion (2003) established that this locus is an ellipse in mean/variance space, but not in the efficient frontier (μ/σ) plane, where μ represents the portfolio expected return and σ the active portfolio volatility (Figure 3.2). The shape of the constant TE frontier in μ/σ space is a *distorted* ellipse in which the bi-axial symmetry associated with ellipse is lost. "Ellipse" will be used here when referring to the shape in either space.

[5] Although not lower *TEs*.

In Figure 3.2(a), the active manager's dilemma is evident: the portfolio subject to a TE constraint which also generates the maximum outperformance of the benchmark has *higher* risk than the benchmark. Because of the flat shape (referring to the generally shallow angle of the ellipse's long axis to the volatility axis) of the ellipse, Jorion (2003) suggested active managers invest in the portfolio indicated in Figure 3.2(b): portfolios with the same risk as the benchmark (on the constant TE frontier). The decrease in expected return (from the maximum expected return) is minimal – again because of the ellipse's flat shape, the portfolio outperforms the benchmark and has the same risk as the benchmark. Jorion (2003) also found that this constraint improved managed portfolio performance, particularly those with lower TEs and less efficient benchmarks. For these portfolios, the information ratio (IR), given by:

$$IR = \frac{r_p - r_b}{TE} \tag{1}$$

(where r_p are the portfolio returns and r_b the benchmark returns), is not maximised.

Maxwell, Daly, Thomson, et al (2018) further explored portfolio optimisation under TE constraints and set forth arguments in favour of maximising the risk-adjusted expected returns (i.e. the maximum Sharpe ratio) on the constant TE frontier. Depending on the risk-free rate or return, this portfolio can lie to the left or right of Jorion's (2003) suggestion. In the current (2017) low interest rate environment, Maxwell et al's (2018) active portfolios lie to the left of Jorion's (2003) so these portfolios have a higher expected return and lower risk than the benchmark, the highest risk-adjusted rate of expected return and they satisfy the TE constraint. These portfolios have lower expected returns than Jorion's (2003) (whose returns are, in turn, lower than the maximum expected return), but again the flat shape of the ellipse means that these portfolios' other credentials more than compensate for this decrease.

Roll (1992) found that all actively-managed portfolios (under the TE constraint) with positive expected performance have $\beta > 1$, while portfolios that have higher expected returns and lower total volatility have $\beta < 1$. Roll (1992) generated TE frontiers with a β constraint and proved that it is impossible to produce a portfolio that is simultaneously constrained by a TE, a given expected performance and a specified β.

Bertrand (2010) allowed the TE to vary, but fixed the investor's level of risk aversion, thereby generating what he called *iso-aversion frontiers*. Bertrand's (2010) $\beta = 1$ iso-aversion frontier coincided with Roll's (1992) $\beta = 1$ frontier and found that, to take advantage of an expected rise in the market (i.e. have a $\beta > 1$) the constructed portfolio must be assembled in the context of iso-aversion frontiers, not constant TE frontiers. While these conclusions are compelling, the clear majority (if not all) actively managed funds have a mandated TE, not a mandated iso-aversion level, so Bertrand's (2010) work is moot.

Stowe (2014) noted that the conventional practices of β constraints, studied in Roll (1992), and TE volatility constraints, studied in Jorion (2003), assure utility improvements for the investor. If these constraints are sensibly implemented, the fund manager will be forced to manage a portfolio which is more efficient than the benchmark. Stowe's (2014) principal contribution was to establish the conditions under which fund managers could increase portfolio utility and found that the β constraint always has the potential to increase utility while the TE constraint (which may increase utility) always lies *below* the constrained β frontier.

Alexander and Baptista (2010) devised a solution for determining the α-TE frontier – i.e. that frontier which exhibits the minimum TE for various levels of ex-ante α. The authors showed that sensible choices of ex-ante α lead to the selection of less-risky portfolios than active fund managers may otherwise select.

3.3 Data and methodology

3.3.1 Data

The data comprised simulated realistic weights, returns, volatilities and correlations for a small benchmark comprising three assets. Portfolio constituents were derived only from the benchmark universe (including short-selling of benchmark components). We followed the example of Stowe (2014) who chose a simple simulated portfolio comprising four assets, and the descriptive statistics for which were chosen somewhat arbitrarily but mainly for ease of exposition. Like Stowe (2014), we believe these examples are representative of a realistic scenario. The relevant inputs are provided in the Appendix of this paper.

Note that the "assets" which constitute the portfolio could be asset *classes* (such as equity, bonds or cash) or specific industry *sectors* within an asset class (e.g. an industrial equity index, a banking and finance index, etc.) or individual *assets* such as single name stocks or bonds.

3.3.2 Methodology

To establish the methodologies required for the various frontiers, some definitions are necessary. These are recreated below in line with the notation developed by Roll (1992) and perpetuated by Jorion (2003).

Fund managers, tasked with outperforming benchmarks, must take positions in assets which may or may not be components of the benchmark (depending on the fund's mandate). The following definitions will be used throughout this paper.

$\boldsymbol{q_b}$:	vector of benchmark weights for a sample of N assets
$\boldsymbol{x}$:	vector of deviations from the benchmark
$\boldsymbol{q_p}$ $(= \boldsymbol{q_b} + \boldsymbol{x})$:	vector of portfolio weights
$\boldsymbol{E}$:	vector of expected returns, and
$\boldsymbol{V}$:	covariance matrix of asset returns.

Net short sales *are* allowed, so the total active weight $\boldsymbol{q}_i + \boldsymbol{x}_i$ may be negative for any individual asset, i. The universe of assets can generally exceed the components of the benchmark, but for Roll's (1992) methodology, assets in the benchmark *must* be included.

Expected returns and variances are expressed in matrix notation as:

$\mu_b = \boldsymbol{q}_{\boldsymbol{b}}'\boldsymbol{E}$:	expected benchmark return
$\sigma_b^2 = \boldsymbol{q}_{\boldsymbol{b}}'\boldsymbol{V}\boldsymbol{q}_{\boldsymbol{b}}$:	variance of benchmark return
$\mu_\varepsilon = \boldsymbol{x}'\boldsymbol{E}$:	expected excess return
$G = \boldsymbol{r_p} - \boldsymbol{r_b}$:	the *gain*, the expected performance relative to the benchmark
$\sigma_\varepsilon^2 = \boldsymbol{x}'\boldsymbol{V}\boldsymbol{x}$:	TE variance (defined as TE^2) and
$\beta = \boldsymbol{q}_{\boldsymbol{p}}'\boldsymbol{V}\boldsymbol{q}_{\boldsymbol{b}}/\boldsymbol{\sigma}_{\boldsymbol{b}}^{\boldsymbol{2}}$:	the sponsor-specified level of market risk (relative to the benchmark).

The active portfolio's expected return and variance are given by:

$$\mu_p = (\boldsymbol{q_b} + \boldsymbol{x})'\boldsymbol{E} = \mu_b + \mu_\varepsilon \qquad (2)$$

$$\begin{aligned}\sigma_p^2 &= (\boldsymbol{q_b} + \boldsymbol{x})'\boldsymbol{V}(\boldsymbol{q_b} + \boldsymbol{x}) = \sigma_b^2 + 2\boldsymbol{q}_{\boldsymbol{b}}'\boldsymbol{V}\boldsymbol{x} + \boldsymbol{x}'\boldsymbol{V}\boldsymbol{x} \\ &= \sigma_b^2 + 2\boldsymbol{q}_{\boldsymbol{b}}'\boldsymbol{V}\boldsymbol{x} + \sigma_\varepsilon^2\end{aligned} \qquad (3)$$

The portfolio must be fully invested, so:

$$(\mathbf{q_b} + \mathbf{x})'\mathbf{1} = 1 \tag{4}$$

where $\mathbf{1}$ represents an N-dimensional vector of 1s.

Using Merton's (1972) terminology, the following parameters are also defined:

$$a = \boldsymbol{E}'\boldsymbol{V}^{-\mathbf{1}}\boldsymbol{E},\ b = \boldsymbol{E}'\boldsymbol{V}^{-\mathbf{1}}\mathbf{1},\ c = \mathbf{1}'\boldsymbol{V}^{-\mathbf{1}}\mathbf{1}, \text{ and } d = a - b^2/c,$$

$$\Delta_1 = \mu_B - \frac{b}{c}$$

where $b/c = \mu_{MV}$ and

$$\Delta_2 = \sigma_B^2 - \frac{1}{c}$$

where $1/c = \sigma_{MV}^2$.

Roll (1992) showed that the three parameters, a, b and c are related to the means and variances of two important portfolios on the efficient frontier. The first, portfolio P_0, is the global minimum variance portfolio and the second, portfolio P_1, is located where a line drawn from the origin passes through the global minimum variance portfolio and intersects the efficient frontier. Both are shown in Figure 3.1. These portfolios have properties indicated in Table 3.1.

Table 3.1: Properties of portfolios 0 and 1 in terms of a, b and c.

Portfolio	Mean	Variance	Weights
P_0	$E_0 = b/c$	$\sigma_0^2 = 1/c$	$\boldsymbol{q_0} = \boldsymbol{V}^{-\mathbf{1}}\mathbf{1}/c$
P_1	$E_1 = a/b$	$\sigma_1^2 = a/b^2$	$\boldsymbol{q_1} = \boldsymbol{V}^{-\mathbf{1}}\boldsymbol{E}/b$

3.3.2.1 TE frontier

The TE frontier is generated by maximising $\boldsymbol{x}'\boldsymbol{E}$ subject to $\boldsymbol{x}'\mathbf{1} = 0$ and $\boldsymbol{x}'\boldsymbol{V}\boldsymbol{x} = \sigma_\varepsilon^2$. The solution for the vector of deviations from the benchmark, $\boldsymbol{x}$, is:

$$\boldsymbol{x} = \pm\sqrt{\frac{\sigma_\varepsilon^2}{d}}\boldsymbol{V}^{-1}\left(\boldsymbol{E} - \frac{b}{c}\mathbf{1}\right)$$

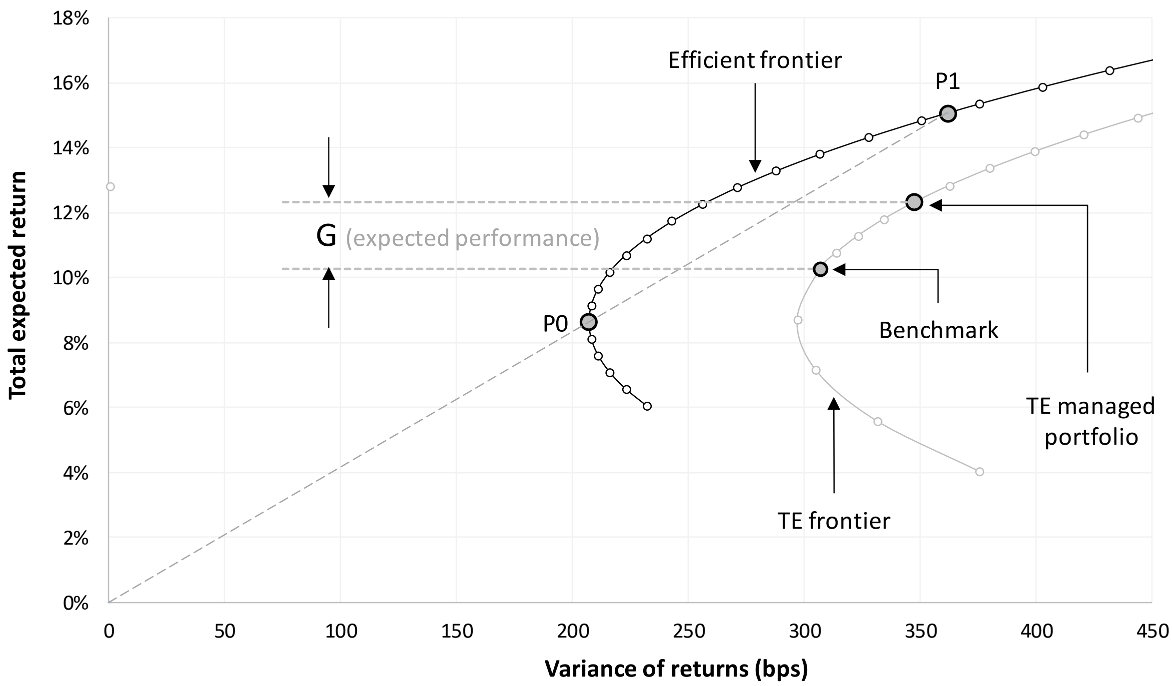

Figure 3.1: Positions of portfolios P_0 and P_1 on the efficient frontier and the gain $G = r_p - r_b$, the fund manager's outperformance target.

Source: authors.

3.3.2.2 Constant TE frontier

To generate the constant TE frontier, maximise $\boldsymbol{x}'\boldsymbol{E}$ subject to $\boldsymbol{x}'\boldsymbol{1} = 0$, $\boldsymbol{x}'\boldsymbol{V}\boldsymbol{x} = \sigma_\varepsilon^2$ and $(\boldsymbol{q_b} + \boldsymbol{x})'\boldsymbol{V}(\boldsymbol{q_b} + \boldsymbol{x}) = \sigma_p^2$. The solution for the vector of deviations from the benchmark, $\boldsymbol{x}$, is:

$$\boldsymbol{x} = -\frac{1}{\lambda_2 + \lambda_3}\boldsymbol{V}^{-1}(\boldsymbol{E} + \lambda_1 + \lambda_3\boldsymbol{V}\boldsymbol{q_b}) \tag{5}$$

where

$$\lambda_1 = -\frac{\lambda_3 + b}{c} \tag{6}$$

$$\lambda_2 = \pm(-2)\sqrt{\frac{d\Delta_2 - \Delta_1^2}{4\sigma_\varepsilon^2\Delta_2 - y^2}} - \lambda_3 \tag{7}$$

$$\lambda_3 = -\frac{\Delta_1}{\Delta_2} \pm \frac{y}{\Delta_2}\sqrt{\frac{d\Delta_2 - \Delta_1^2}{4\sigma_\varepsilon^2\Delta_2 - y^2}} \tag{8}$$

Jorion (2003) defined

$$z = \mu_p - \mu_b \tag{9}$$

and

$$y = \sigma_p^2 - \sigma_b^2 - \sigma_\varepsilon^2 \tag{10}$$

and established that the relationship between y and z is:

$$dy^2 + 4\Delta_2 z^2 - 4\Delta_1 yz - 4\sigma_\varepsilon^2(d\Delta_2 - \Delta_1^2) = 0$$

which is a quadratic equation in both y and z. Solving for z gives:

$$\mathrm{z} = \frac{\Delta_1 \mathrm{y} \pm \sqrt{(\Delta_1^2 - \mathrm{d}\Delta_2) \cdot (\mathrm{y}^2 - 4\Delta_2\sigma_\varepsilon^2)}}{2\Delta_2} \tag{11}$$

which describes an ellipse – a *constant* TE frontier – in return/variance space (and a distorted ellipse in risk/return space – Figure 3.2).

In Figure 3.2, each point on and inside the ellipse represents a portfolio with $TE = 5\%$. The point on the ellipse corresponding to the largest outperformance of the portfolio over the benchmark is common to both the TE frontier and the *constant* TE frontier. Managers attempting to maximise excess return need to move up and to the right of the benchmark – so the portfolio will always exhibit higher risk than that of the benchmark. This led Jorion (2003) to propose a constraint on total risk. Jorion (2003) suggested that the portfolio risk could be constrained to equal that of the benchmark (i.e. that $\sigma_p = \sigma_b$), which implies that $2\boldsymbol{q}'\boldsymbol{V}\boldsymbol{x} = -\sigma_\varepsilon^2$.

3.3.2.3 Constant $\boldsymbol{\beta}$ frontier

Assume a fund manager is mandated to assemble a portfolio p which minimises the TE, generates an expected outperformance (or gain) G and maintains a specified β against the benchmark portfolio b. This optimisation problem can be expressed as:

Minimise $\boldsymbol{x}'\boldsymbol{V}\boldsymbol{x}$ subject to $\boldsymbol{x}'\mathbf{1} = 0$, $\boldsymbol{x}'\boldsymbol{E} = G$ and $\boldsymbol{q_p'}\boldsymbol{V}\boldsymbol{q_b} = \beta\boldsymbol{\sigma_b^2}$.

The final constraint may be rearranged and written as:

$$(\boldsymbol{q_b} - \boldsymbol{x})\boldsymbol{V}\boldsymbol{q_b} = \beta\boldsymbol{\sigma_b^2}$$
$$\boldsymbol{x}'\boldsymbol{V}\boldsymbol{q_b} = \boldsymbol{\sigma_b^2}(\beta - 1)$$

Using Lagrange multipliers, the solution for the weights, $\boldsymbol{x}$, of the relevant portfolio which satisfies the above constraints is $\boldsymbol{x} = \gamma_1\boldsymbol{q_1} + \gamma_0\boldsymbol{q_0} + \gamma_b\boldsymbol{q_b}$ where:

$$\gamma_1 = \frac{G(\sigma_b^2 - \sigma_0^2) + \sigma_b^2(\beta - 1)(E_0 - \mu_b)}{(E_1 - E_0)(\sigma_b^2 - \sigma_{b*}^2)}$$

$$\gamma_0 = \frac{G\left(\frac{\mu_b}{b - \sigma_b^2}\right) + \sigma_b^2(\beta - 1)(\mu_b - E_1)}{(E_1 - E_0)(\sigma_b^2 - \sigma_{b*}^2)}$$

$$\gamma_b = \frac{G\left(\frac{\sigma_0^2 - \mu_b}{b}\right) + \sigma_b^2(\beta - 1)(E_1 - E_0)}{(E_1 - E_0)(\sigma_b^2 - \sigma_{b*}^2)}$$

Roll (1992) also established that $G = \gamma_1 \boldsymbol{E_1} + \gamma_0 \boldsymbol{E_0} + \gamma_b \mu_b$ and $\boldsymbol{x'Vq_b} = \sigma_b^2(\beta - 1)$ as required.

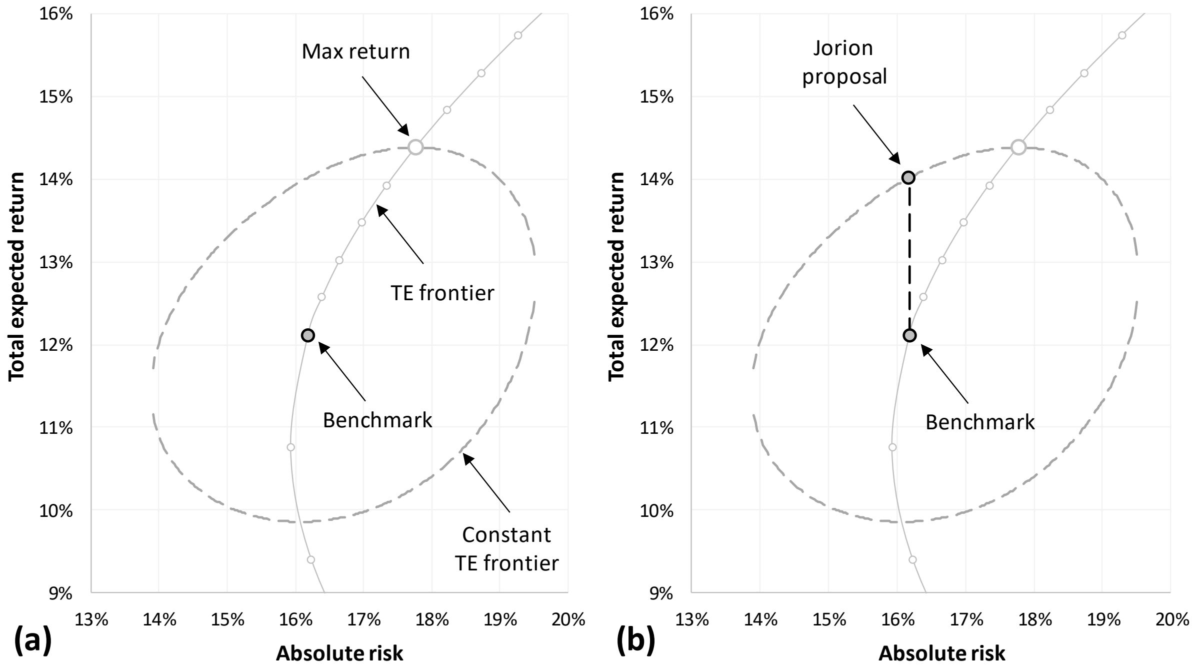

Figure 3.2: TE frontier, TE-constrained portfolio and constant TE frontier (with $TE = 5\%$). (a) shows the naïve portfolio: excess return is maximised for a given TE constraint. (b) shows Jorion's (2003) suggestion: observe constraints from (a) but restrict portfolio risk to that of the benchmark.

Source: authors.

3.3.2.4 The α-TE frontier

A portfolio is deemed to be on the α-TE frontier if there is no portfolio with the same α and a smaller TE. The methodology to generate this frontier involves first calculating three useful parameters:

$$k_1 = \frac{b - \left(\frac{\mu_b - r_f}{\sigma_b^2}\right)}{c}$$

$$k_2 = a + \left(\frac{r_f^2 - \mu_b}{\sigma_b^2}\right) - \frac{\left[b - \left(\frac{\mu_b - r_f}{\sigma_b^2}\right)\right]^2}{c}$$

$$k_3 = \frac{d}{c} + \left(\frac{b}{c} - \mu_b\right) \cdot \left(\frac{\mu_b - r_f}{\sigma_b^2}\right)$$

where r_f is the risk-free rate and the other symbols have been defined previously.

Define

$$\gamma_{0,\alpha} = \frac{-\alpha c k_1}{k_2}$$

$$\gamma_{1,\alpha} = \frac{\alpha b}{k_2}$$

then the vector of portfolio weights on the α-TE frontier, q_α, are generated from:

$$\boldsymbol{q_\alpha} = \gamma_{0,\alpha}\boldsymbol{q_0} + \gamma_{1,\alpha}\boldsymbol{q_1} + (1 - \gamma_{0,\alpha} - \gamma_{1,\alpha})\boldsymbol{q_b}$$

where q_0 and q_1 retain the definitions established in Table 3.1.

3.3.2.5 Fund utility

Stowe (2014, 2017) used the popular "quadratic-style" utility function to find the portfolio, constrained by a TE, which maximised the investor's utility which increases with expected return, $\boldsymbol{q_p'E}$, and decreases with risk, $\sqrt{\boldsymbol{q_p'Vq_p}}$. The relationship is:

$$\mathrm{U} = \mu_\mathrm{p} - \theta\sigma_\mathrm{p}^2 \tag{12}$$

where U is the utility function, μ_p the fund's expected return, σ_p^2 the variance of the portfolio's return and θ the coefficient of risk aversion (Stowe, 2014 & 2017). The maximisation setup is:

$$\max_{\mathrm{w}} \ \boldsymbol{q_p'E} - \theta\boldsymbol{q_p'Vq_p}$$

for which Stowe (2014, 2017) provides elegant derivations for the optimal portfolio. We use his results, but do not find the optimal utility portfolio on the constant TE frontier. Instead, we investigate the utility around the portfolio which lies on the constant TE frontier and has maximal risk-adjusted return, i.e. the maximum Sharpe ratio portfolio constrained to the constant TE frontier. This augments and extends our work on this portfolio under TE constraints (Maxwell et al., 2018).

Differentiating (12) gives:

$$\frac{\mathrm{dU}}{\mathrm{d}\sigma_\mathrm{p}} = 2\sigma_\mathrm{p}\theta \tag{13}$$

and where this slope takes the value of the maximum Sharpe ratio, the investor's utility for the maximum Sharpe ratio portfolio is determined. Maxwell, et al (2018) showed that solving for μ_p in (14) gives the coordinates of the maximum Sharpe ratio portfolio in mean/standard deviation space (from whence it is trivial to determine the maximum Sharpe ratio, knowing r_f:

$$\frac{(\mathrm{r_f} - \mu_\mathrm{b})}{\sigma_\mathrm{p}^2} + \frac{\dfrac{(\Delta_1^2 - \mathrm{d}\Delta_2)\cdot\left(\sigma_\mathrm{p}^2 - \sigma_\mathrm{b}^2 - \sigma_\varepsilon^2\right)}{\sqrt{(\Delta_1^2 - \mathrm{d}\Delta_2)\left[\left(\sigma_\mathrm{p}^2 - \sigma_\mathrm{b}^2 - \sigma_\varepsilon^2\right)^2 - 4\Delta_2\sigma_\varepsilon^2\right]}} + \Delta_1}{\Delta_2} - \frac{\sqrt{(\Delta_1^2 - \mathrm{d}\Delta_2)\left[\left(\sigma_\mathrm{p}^2 - \sigma_\mathrm{b}^2 - \sigma_\varepsilon^2\right)^2 - 4\Delta_2\sigma_\varepsilon^2\right]} + \Delta_1\cdot\left(\sigma_\mathrm{p}^2 - \sigma_\mathrm{b}^2 - \sigma_\varepsilon^2\right)}{2\Delta_2\sigma_\mathrm{p}^2} = 0 \tag{14}$$

with $\Delta_1 = \mu_B - \frac{b}{c}$ $(b/c = \mu_{MV})$ and $\Delta_2 = \sigma_B^2 - \frac{1}{c}(1/c = \sigma_{MV}^2)$ as defined by Jorion (2003).

Equating the maximum Sharpe ratio and (13) gives:

$$\frac{dU}{d\sigma_p} = 2\sigma_p^2\theta_S = \text{Maximum Sharpe ratio}$$

so:

$$\theta_S = \frac{\text{Maximum Sharpe ratio}}{2\sigma_p}. \tag{15}$$

where θ_s is the risk aversion coefficient tangent to the constant TE frontier at the maximum Sharpe ratio portfolio.

3.4 Results and discussion

Armed with the mathematics and methodologies detailed in the previous section, the various frontiers were constructed and a small portfolio comprising three stylised assets used to investigate the range of possible portfolios, given various mandated constraints.

3.4.1 The constant TE frontier

Roll (1992) developed the TE frontier (Figure 3.1). Jorion (2003) first described the *constant* TE frontier (Figure 3.2), the boundary whose upper edge defines the maximum return possible, at various risk levels, for a given, fixed TE. This region, an ellipse in mean/variance space, led to the proposition that a portfolio with the same risk as the benchmark on this frontier would perform better than the naïve 'maximum return' portfolio (whose risk is greater than that of the benchmark).

3.4.2 The β frontier

Adding a β constraint generates a β frontier on which all portfolios have the same CAPM β. Portfolios not geared to the market (in this case, it is assumed that the 'market' is the benchmark) have $\beta = 1$: this frontier necessarily passes through the benchmark (Figure 3.3a). For $\beta < 1$ the frontier moves to the left, so portfolios with lower risk yet higher returns than the benchmark, in bear markets, are possible. For $\beta > 1$, the frontier moves to the right in risk/return space, so only portfolios with higher volatilities than the benchmark are feasible. This result was also found by Roll (1992).

For all values of β, only one sensible intersection with the constant TE frontier exists (i.e. on the upper half of the ellipse). This point may not be the maximum return possible (given the TE constraint) nor the maximum risk-adjusted return – the crux is that this intersection point may not be 'optimal' in any sense. Active fund managers with portfolios subject to this combination of constraints (TE and β) and wishing to maximise excess returns are confined to a single point in risk/return space: a highly restrictive arrangement.

Figure 3.3b shows the range of possible β values for various levels of TE. This range is effectively the values of β at which the β frontier tangentially intersects the constant TE frontier at either end of the ellipse. Either end represents, respectively, the active fund's minimum and maximum volatility, so it is doubtful active fund managers would be interested in these extreme portfolios in the first place. Thus, the range of possible βs is lower than the stylised values in Figure 3.3b.

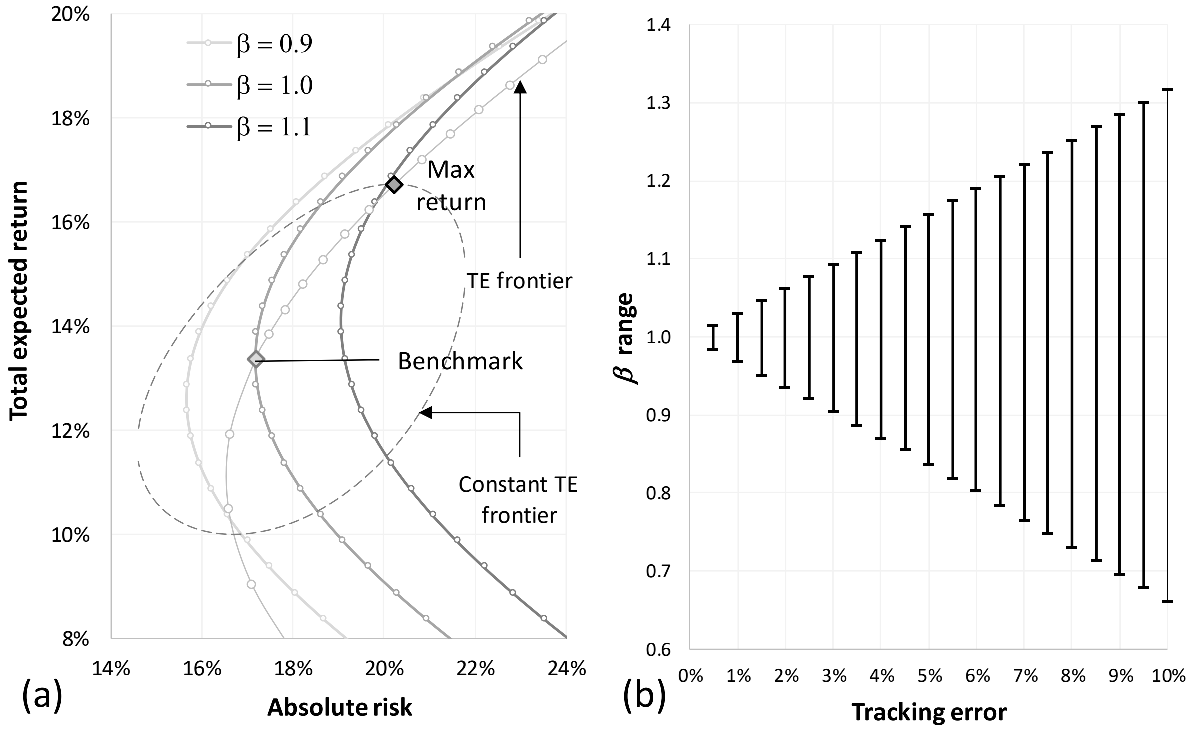

Figure 3.3: (a) Position of β frontier for $\beta = 0.9$, 1.0 and 1.1 and (b) maximum and minimum β values for changing TEs.

Source: authors.

3.4.3 The α-TE frontier

Figure 3.4 shows the α-TE frontier, that frontier which exhibits the minimum TE for various levels of *ex-ante* CAPM α. These αs are indicated on the frontier. The elements of Figure 3.4 were generated using stylised asset parameters, and it is clear that feasible active portfolios that simultaneously satisfy α and prescribed TE constraints as well as maximising excess returns, are impossible (except in the case of the vanishingly small probability that the α constraint coincides with the intersection on the TE frontier). Combining α, β and TE constraints is impossible as shown in Figure 3.4. Note, however, that using the α-frontier leads to the selection of less-risky portfolios than managers might otherwise select. In Figure 3.4, for example, the (naïve) maximum return portfolio is considerably riskier than the intersection point of the α-frontier and the constrained TE frontier. Note that the $\alpha = 0\%$ portfolio is necessarily coincident with the benchmark.

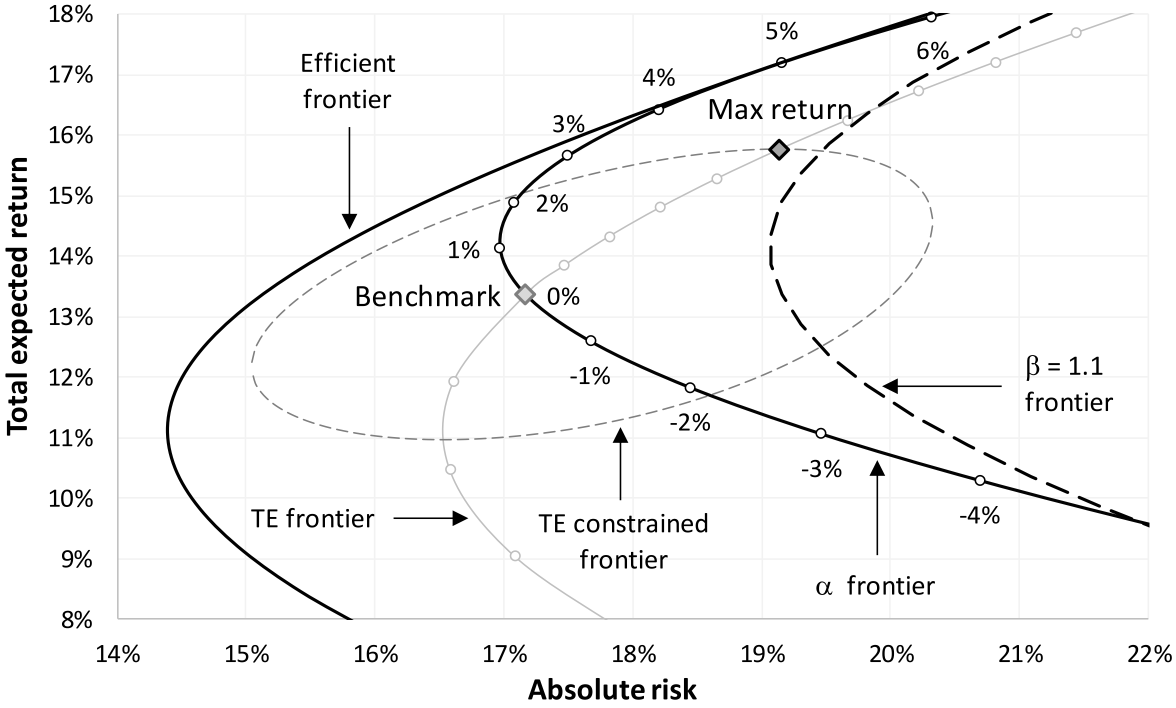

Figure 3.4: The α-TE frontier for various levels of α. Other frontiers are shown for comparison. Levels of α are indicated on the graph. $TE = 5\%, r_f = 2\%$.

Source: authors.

3.4.4 Utility constraints

While variance minimisation is an approach to analysing portfolio selection, maximising investor utility is another. The parameter θ, the coefficient of risk aversion, measures the sensitivity or the trade-off. Stowe (2014) found that the weights for the portfolio which maximised utility under TE constraints were given by:

$$\boldsymbol{w} = \frac{1}{2\theta}\boldsymbol{V}^{-1}\left(\boldsymbol{E} - \frac{\boldsymbol{E}\boldsymbol{V}^{-1}\boldsymbol{1}}{\boldsymbol{1}'\boldsymbol{V}^{-1}\boldsymbol{1}}\boldsymbol{1}\right) + \boldsymbol{q_b}$$

We did not use Stowe's (2014) maximum utility portfolio. Instead, we determined the utility function which is tangent to the constant TE frontier at the maximum Sharpe ratio portfolio (Figure 3.5a) using (15) and backed out the appropriate risk aversion coefficient.

Figure 3.5b shows the θ surface as a function of r_f and TE. While θ decreases for increasing r_f (understandable because, all else equal, the slope of the maximum Sharpe ratio decreases with increasing r_f), a clear maximum θ exists at a certain TE, for all r_f. It is not obvious why this should be.

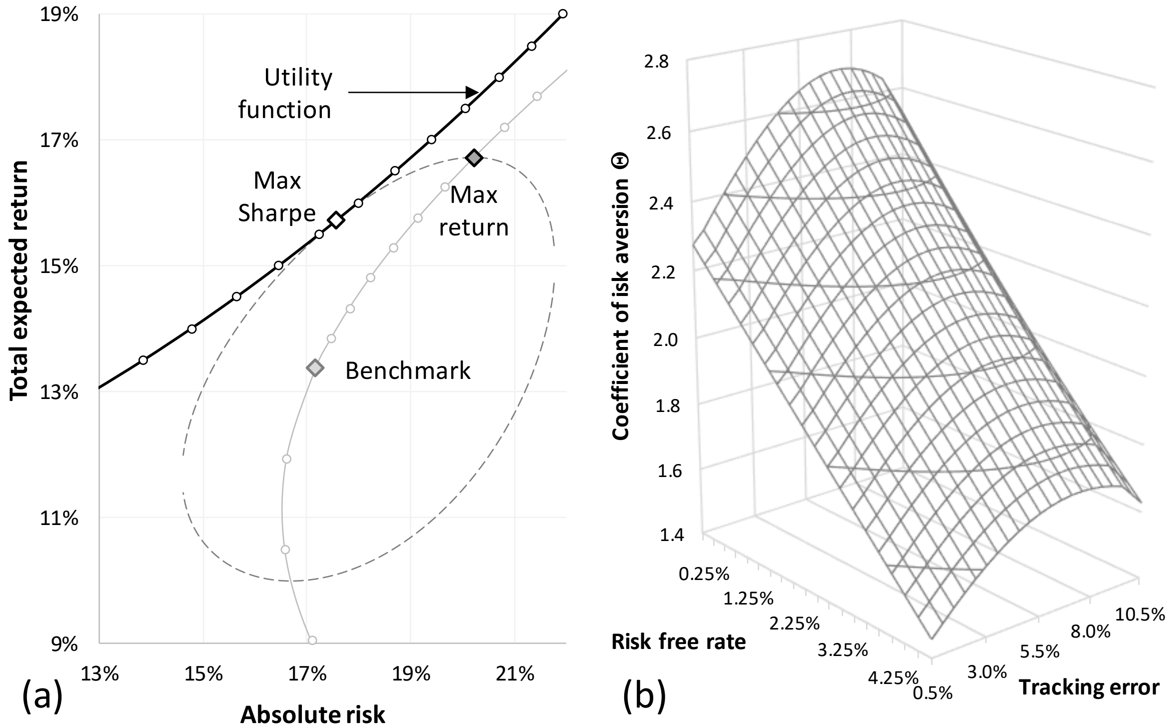

Figure 3.5: (a) Utility function tangential to the maximum Sharpe ratio portfolio on the constant TE frontier and (b) θ as a function of TE and risk-free rate.

Source: authors.

Figure 3.6a further explores θ as a function of TE. A maximum θ occurs, for this stylised example, at about TE $= 7\%$. Maxwell, et al., (2018) established that the maximum Sharpe ratio also increases with TE before flattening off and decreasing slightly for large TEs ($> 12\%$). Thus, a portfolio exists for which the Sharpe ratio itself is maximised. The portfolios characterised by a maximum θ and a maximum Sharpe ratio are not the same – thus a maximum Sharpe ratio does not describe the reason for the maximum θ.

The locus of the maximum Sharpe ratio portfolio, as TE increases, moves up (increased return) and to the left (decreased risk) of the benchmark. At higher levels of TE, the portfolio continues to move up, but then moves to the right, i.e. absolute risk increases. At some value of TE, the maximum Sharpe ratio portfolio will be coincident upon Jorion's (2003) proposal (i.e. where $\sigma_b = \sigma_p$) and for higher TE values, the portfolio will continue to move up and right eventually coincident with the maximum return portfolio (coincident with the TE frontier and ellipse) in the clockwise direction (see Maxwell et al., 2018). Investor risk aversion increases dramatically with increasing TE (labels indicated on the graph) and increasing σ_p – the portfolio risk of the Sharpe ratio portfolio at that level of TE (Figure 3.6b).

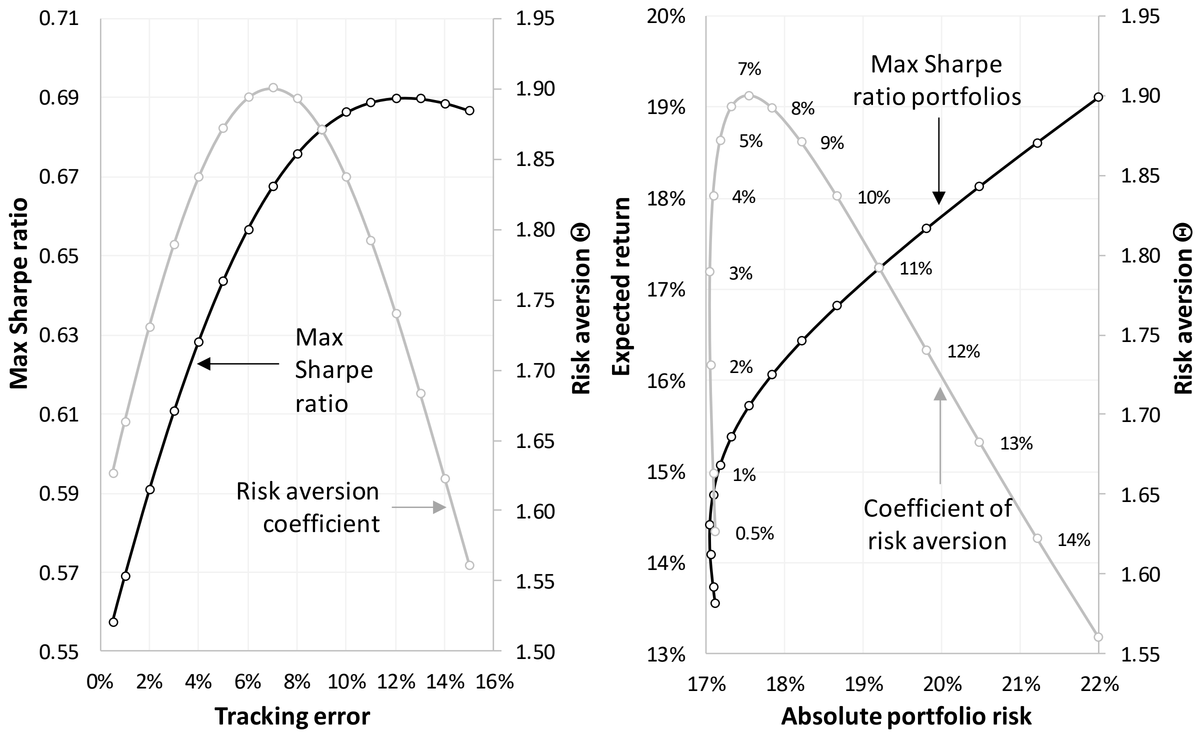

Figure 3.6: (a) Maximum Sharpe ratio as a function of TE and risk-free rate and (b) utility function (risk aversion) as a function of TE and portfolio risk.

Source: authors.

3.5 Conclusions and suggestions

Portfolios subject to TE constraints are always sub-optimal to those that are not. Furthermore, as the TE constraint is in excess return space and relative to an investor- or a somewhat arbitrarily-defined benchmark (in general), there is potential for greater inefficiency should fund managers naïvely pursue maximum excess returns as a sole investment objective. As TE is mandatory in active portfolio management, optimisation under it is of great interest to practitioners.

Before addressing the selection of an optimal portfolio under TE, α and β constraints and what that means in terms of maximising excess and risk-adjusted returns, and the subsequent impact upon investor utility, it is important to note what is meant by an optimal portfolio. An optimal portfolio is one which maximises investor's utility, not necessarily that which minimises risk, maximises returns, nor that which maximises risk-adjusted returns. Therefore, an optimal portfolio for one investor may not be the same as the optimal portfolio for another investor. Unfortunately, to assign a number to utility, one looks at risk aversion (or risk tolerance), but *this* does not encompass the full utility an investor derives from an investment (whether that be avoiding large losses or realising large gains). There is a subjective component of utility, or investor satisfaction, which cannot be captured numerically. Furthermore, investors are not always aware what provides the most utility/satisfaction from an investment. Although the peak of both the utility curve and the maximum Sharpe curve do not coincide, and the maximum risk-adjusted portfolio is not identified as the optimal portfolio (in utility terms), it could be argued that it *should* be. Understanding of optimality, risk and utility (and their interaction) requires re-adjustment. As an investor's utility cannot be fully captured, it cannot be concluded that the maximum Sharpe portfolio is not the optimal portfolio as it is the maximum *risk-adjusted* return portfolio and will provide the investor with far more utility than a more risk-averse portfolio. Therefore, as they are generally uninformed, investors should attempt to maximise risk-adjusted returns, rather than avoid risk.

Another important discussion is that between risk and volatility. A high volatility portfolio is not necessarily riskier than a lower volatility portfolio, as the volatility estimate employs both positive and negative returns. A high volatility portfolio could comprise of predominantly positive ("good") returns and a lower volatility portfolio could comprise of predominantly negative ("bad") returns. Portfolio risk, on the other hand, is the risk of losing capital. In the above example, the lower volatility portfolio contributes more risk than the higher volatility portfolio. Reducing portfolio risk would thus reduce the potential capital losses, and metrics such as VaR, conditional VaR and downside deviation should therefore be optimised. One could adjust the Sharpe ratio and attempt to maximise the *Sortino* ratio instead; this may provide substantially more utility to the risk-averse investor.

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Appendix

Correlation matrix:

1	+0.09	+0.16
	1	+0.12
		1

Volatility vector:	28%	25%	18%
Benchmark weights:	50%	22%	28%
Annualised r:	15%	19%	6%

Chapter 4

Portfolio performance under tracking error and asset weight constraints

Michael Daly[6] and Gary van Vuuren[7]

Abstract

Active managers are assessed on their ability to outperform benchmarks. Attaining outperformance is non-trivial: fund managers must simultaneously maximise excess returns (over the benchmark), limit risk, and observe other mandatory constraints. Active fund managers are responsible for driving capital gains while observing other restrictions, most commonly, allocated asset weight parameters. These boundaries can be defined by upper or lower limits, acceptable ranges or – for example – long only limitations. The behaviour of active portfolios subject to these multiple constraints is complex and opaque. This article investigates the behaviour of such portfolios and establishes the locus of acceptable risk/return coordinates for active funds subject to these restrictions. Considerable reduction in possible investable portfolios is observed, even for limited asset weight restrictions. This effect is amplified if multiple restrictions are simultaneously imposed. The reduction is driven by both a reduced area of possible portfolios in risk/return space enclosed by the constant tracking error frontier and a change in the long-axis slope of the frontier from positive to negative, which compresses the efficient set.

Keywords	Tracking error frontier, optimal portfolios, investment constraints, asset weights
JEL classification	C52, G11

4.1 Introduction

Passive investment attempts to match the performance of certain market indices rather than attempting to outperform these through specific stock assessment and selection. Active investment managers invest in funds whose constituents' worth are independently assessed, while their passive counterparts construct portfolios of market indices in the proportion they are held in the specific index and rebalance these proportions as the market changes.

The goal of active management is to beat the market or outperform agency-determined benchmarks. These benchmarks are often inefficient comprising assets with arbitrarily defined upper and lower limits as a result of partial securitization and free float restrictions.

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Active investing is generally costlier than passive investing (research analysts and portfolio managers require compensation and frequent trading also incurs costs) and many active managers do not beat the index after expenses are accounted for. Active managers must also comply with strict tracking error (TE) (the variance of the difference between portfolio and benchmark returns) ceilings, where punitive penalties for non-compliance can be severe. For these reasons, passive investing often outperforms active investments.

The competition between advocates of active and passive management has, however, recently (2018) intensified. Historically, passive investing has taken preference globally, however, recent evidence identifies a change in this trend (Torr, 2018). Today's (2018) narrow bull market has a fundamental weakness of being top heavy with the "big five" (Facebook, Apple, Amazon, Microsoft and Google) S&P 500 companies driving the large US index-tracking market. These high concentration levels reduce diversification and the emphasis on tracking the S&P 500 has led to overcrowding of the ETF market (Brenchley, 2018). A reversal of this tide would provide an opportunity for real dominance by active managers (Gilreath, 2017).

The success of active portfolio managers is measured relative to a benchmark. The components of these benchmarks are frequently defined by principals who may know little about asset allocation and whose principal interest is often to limit risk through the imposition of strict weight allocation ranges. Asset allocation weights can depend on tax considerations, geographical restrictions (such as foreign exchange limits) or simply agent preferences. Active portfolio components can be under- or over-weighted relative to the benchmark, but overall weights can also be negative, i.e. short positions. These constraints coupled with strict TE limits generally inhibits fund manager performance.

This paper contributes by investigating the effect of imposing long-only portfolio component weights on active portfolios subject to TE constraints. In addition, the effect of asset weight ranges on such portfolios is also explored and new constrained frontiers are established as a result. Properties of such frontiers are of considerable interest to active fund managers and investors.

The remainder of this article proceeds as follows: Section 2 explores the relevant literature governing *efficient frontiers* subject to no constraints, *TE frontiers* which describe portfolios achieving maximal returns while subject to a TE constraint, *constant TE frontiers* which embrace all portfolios subject to various TEs and finally *constrained constant TE frontiers* subject to a specific TE limit and various asset weight range constraints. Section 3 describes the stylised data used for the study and defines the mathematics relevant to the exposition. The results are displayed and discussed in Section 4 and Section 5 concludes.

4.2 Literature survey

For the fund to outperform the benchmark, the generation of a positive expected TE is implied (Roll, 1992:17). An important criterion for evaluating active manager performance is the minimising of the TE (the volatility of the difference between managed fund and benchmark returns). Roll (1992) explored this criterion and asserted that portfolios with the smallest total return volatility for a given expected total return meant that TE-constrained funds would pursue portfolios with a minimum TE for a given expected performance measured relative to a benchmark. This approach neglects absolute portfolio risk which means these portfolios are not optimal (in mean-variance space) and they are riskier than the benchmark (Ma & Jagannathan, 2002).

Roll (1992:19) also set out the fundamentals for establishing the TE frontier: a frontier comprising maximum excess returns (relative to a benchmark) and associated risk coordinates for different levels of TE. Jorion (1992 & 2003) expanded on Roll's (1992:19) work and set out the relevant mathematics to construct a *constant* TE frontier, i.e. a locus of points comprising *all* returns and associated risk for different levels of TE. This locus was found to be an ellipse in mean/variance space and a distorted ellipse in mean/risk space. Jorion (2003) also proposed that an optimal measure of TE-constrained portfolio performance should be the return on the constant TE frontier at a risk equal to that of the benchmark. The tilt of the constant TE frontier meant that the difference between the maximum return at a given TE level and the return associated with Jorion's (2003) suggestion was relatively minor.

Maxwell, Daly, Thomson & van Vuuren (2018:5848) extended Jorion's (2003) work and established the analytical solution for the optimal Sharpe ratio portfolio on the constant TE frontier. This portfolio is the TE-constrained analogue of the optimal (efficient) portfolio on the standard efficient frontier. Daly, Maxwell & van Vuuren (2018) explored β, α and utility on constant TE frontier. The β frontier is a parabola in risk/return space which defines feasible portfolios subject to a specified β. An α-TE frontier is assembled from portfolios which have a specified α for a maximum TE. Utility and associated risk aversion were also explored for TE-constrained portfolios.

Maxwell & van Vuuren (2019) investigated the behaviour of various portfolio strategies (maximally diversified, exhibit risk parity, have minimal intra-correlation, and minimum risk) on the constant TE frontier. Such portfolios were found to behave differently to those which form part of the efficient set. Evans & van Vuuren (2019) used several performance indicators to evaluate the benchmark outperformance of six active portfolio strategies – subject to a TE constraint – on the constant TE frontier.

This catalogue of work involving TE-constrained portfolios – while comprehensive – ignores a fundamental reality of active portfolios: mandated constraints on constituent asset weights. Work which covers this important aspect of active portfolios is limited.

Ammann & Zimmermann (2001:40) investigated the relation between statistical TE measures and asset allocation restrictions expressed as admissible weight ranges by simulating investment strategies subject to such constraints. Using market data, Ammann & Zimmermann (2001) found that imposing large tactical asset allocation ranges implied surprisingly small TEs. More recently, Bajeux-Besnainou, Belhaj, Maillard, & Portait, (2011) considered portfolio performance under simultaneous TE and weight constraints compliance. Both equality and inequality weights constraints were considered and the analytical and geometrical solutions for both cases were derived.

In this article, we use a stylised example of realistic market data like that employed by Bajeux-Besnainou et al (2011) and explore the shape of the constant TE frontier under various weight constraints for different TEs. We also examine the Sharpe ratio and information ratio profiles of portfolios subject to these constraints for the first time.

4.3 Data and methodology

4.3.1 Data

The data comprised simulated realistic weights, returns, volatilities and correlations for a small, standardised benchmark comprising equal weights in seven assets (five domestic: d_1 to d_5, and two foreign: f_1 and f_2) with the stylised description as given in Table 1. This numerical example is like that used by Besnainou et al., (2011) for continuity and comparison.

We also tested portfolios comprising assets with higher correlations, with correlations < 0 and using some negative expected returns. The results were broadly the same, even though some of the calculated weights for the optimisation procedure become negative as a consequence of these changes.

Table 4.1: Stylised input data.

	Domestic					Foreign	
Assets	d_1	d_2	d_3	d_4	d_5	f_1	f_2
Benchmark weights	20%	20%	20%	20%	20%	0%	0%
Annual return	16.7%	16.0%	14.9%	15.8%	14.5%	12.0%	15.3%
Annual volatility	19%	27%	22%	26%	21%	27%	26%
Correlation matrix	**1**	0.3	0.3	0.3	0.3	0.2	0.2
	0.3	**1**	0.3	0.3	0.3	0.2	0.2
	0.3	0.3	**1**	0.3	0.3	0.2	0.2
	0.3	0.3	0.3	**1**	0.3	0.2	0.2
	0.3	0.3	0.3	0.3	**1**	0.2	0.2
	0.2	0.2	0.2	0.2	0.2	**1**	0.3
	0.2	0.2	0.2	0.2	0.2	0.3	**1**

Portfolio constituents were derived from the universe of investable assets (i.e. domestic and foreign components). Note that the assets which constitute the portfolio in the following examples could be asset *classes* (such as equity, bonds and cash), specific industry *sectors* within an asset class (e.g. an industrial equity index, a banking and finance index, etc.) or individual *assets* such as single name stocks or bonds.

Like Besnainou et al., (2011) we consider a portfolio manager charged with managing two funds with the same benchmark but constrained by different TEs.

4.3.2 Methodology

To establish the methodologies required for the various frontiers, some definitions are first required. This section proceeds by introducing and describing the relevant variables and algebraic components. The mathematics governing the generation of the efficient frontier is then set out, followed by the algebra which defines the TE frontier and then the *constant* TE frontier.

Active fund managers are tasked with outperforming specified benchmarks and the active asset positions they take may or may not be benchmark components (depending on the mandate governing the fund). The algebra required to derive the relevant investment strategy weights uses the same underlying variables, matrices and matrix notation, defined below.

$\boldsymbol{q}$: vector of benchmark weights for a sample of N assets

$\boldsymbol{x}$: vector of deviations from the benchmark

$\boldsymbol{q_P}$ $(= \boldsymbol{q} + \boldsymbol{x})$: vector of portfolio weights

$\boldsymbol{E}$: vector of expected returns,

$\boldsymbol{\sigma}$: vector of benchmark component volatilities

ρ: benchmark correlation matrix

$\boldsymbol{V}$: covariance matrix of asset returns and

r_f: the risk-free rate.

Net short sales *are* allowed in this formulation, so the total active weight $\boldsymbol{q}_i + \boldsymbol{x}_i$ may be negative for any individual asset, i. The universe of assets can generally exceed the components of the benchmark, but for Roll's (1992) methodology, assets in the benchmark *must* be included.

Expected returns and variances are expressed in matrix notation as:

$\mu_B = \boldsymbol{q}'\boldsymbol{E}$: expected benchmark return

$\sigma_B^2 = \boldsymbol{q}'\boldsymbol{V}\boldsymbol{q}$: variance of benchmark return

$\mu_\varepsilon = \boldsymbol{x}'\boldsymbol{E}$: expected excess return; and

$\sigma_\varepsilon^2 = \boldsymbol{x}'\boldsymbol{V}\boldsymbol{x}$: TE variance (i.e. TE^2).

The active portfolio expected return and variance is given by $\mu_P = (\boldsymbol{q} + \boldsymbol{x})'\boldsymbol{E} = \mu_B + \mu_\varepsilon$ and $\sigma_P^2 = (\boldsymbol{q} + \boldsymbol{x})'\boldsymbol{V}(\boldsymbol{q} + \boldsymbol{x})$ respectively. The portfolio must be fully invested, so $(\boldsymbol{q} + \boldsymbol{x})'\boldsymbol{1} = 1$ where $\boldsymbol{1}$ represents an N-dimensional vector of 1s.

Merton (1972) defined $a = \boldsymbol{E}'\boldsymbol{V}^{-1}\boldsymbol{E},\ b = \boldsymbol{E}'\boldsymbol{V}^{-1}\boldsymbol{1}, c = \boldsymbol{1}'\boldsymbol{V}^{-1}\boldsymbol{1}, d = a - \frac{b^2}{c}$ and $\Delta_1 = \mu_B - \frac{b}{c}$ where $b/c = \mu_{MV}$ and $\Delta_2 = \sigma_B^2 - \frac{1}{c}$ with $1/c = \sigma_{MV}^2$ where MV is the minimum variance portfolio.

Note that, deviations from the benchmark are represented by $\boldsymbol{x}$, total portfolio component weights are $\boldsymbol{q}+\boldsymbol{x}\ (=\boldsymbol{q_P})$, and where the investment strategy is unaffected by TE constraints, the relevant portfolio weights, $\boldsymbol{w}$, are used.

Also note that when $\Delta_1 = \mu_B - \mu_{MV} > 0$ the long axis of the ellipse has a positive slope and when $\Delta_1 = \mu_B - \mu_{MV} < 0$ the ellipse's long axis has a negative slope. This is important as the slope changes sign as restrictions increase (Section 4.4).

Mean variance frontier in risk/return space

Minimise $\boldsymbol{q_P'Vq_P}$ subject to $\boldsymbol{q_P'1} = 1$ and $\boldsymbol{q_P'E} = G$ where G is the target return. The vector of portfolio weights is $\boldsymbol{q_P} = \left(\frac{a-bG}{d}\right)\boldsymbol{q_{MV}} + \left(\frac{bG-\frac{b^2}{c}}{d}\right)\boldsymbol{q_{TG}}$ where q_{MV} is the vector of asset weights for the minimum variance portfolio given by $\boldsymbol{q_{MV}} = \boldsymbol{V}^{-1}\frac{\boldsymbol{1}}{c}$ and $\boldsymbol{q_{TG}}$ is the vector of asset weights for the tangent (optimal) portfolio, i.e. $\boldsymbol{q_{TG}} = \boldsymbol{V}^{-1}\frac{\boldsymbol{E}}{b}$.

The locus of points in return/risk space is the efficient frontier, not subject to sales constraints, i.e. short-selling of assets is permitted (as shown in Figure 4.1). Imposing a long-only constraint on efficient portfolios requires a recursive algorithm to solve for constituent weights, there is no closed-form solution to this problem.

Unconstrained TE frontier in risk/return space

Maximise $\boldsymbol{x'E}$ subject to $\boldsymbol{x'1} = 0$ and $\boldsymbol{x'Vx} = \sigma_\varepsilon^2$. The solution for the vector of deviations from the benchmark is $\boldsymbol{x} = \pm\sqrt{\frac{\sigma_\varepsilon^2}{d}}\boldsymbol{V}^{-1}\left(\boldsymbol{E} - \frac{b}{c}\boldsymbol{1}\right)$. The solution to this optimisation problem generates the TE frontier, a portfolio's maximal return at a given risk level and subject to a TE constraint (as shown in Figure 4.1).

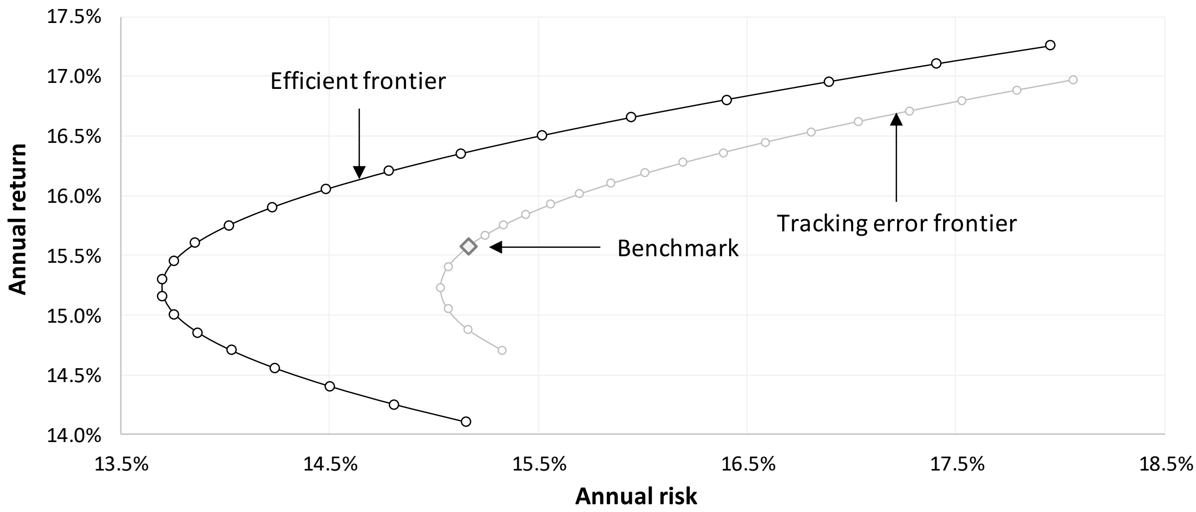

Figure 4.1: Efficient frontier, TE frontier and constant TE frontier in mean/standard deviation space. The square marker indicates the maximum Sharpe ratio on the global efficient frontier with no constraints imposed. $TE = 7\%$ and $r_f = 5\%$.

Source: authors.

The benchmark – also indicated in Figure 4.1 – may be efficient, in which case it would lie on the efficient frontier. In reality, the benchmark is somewhat arbitrarily selected (a mix of stock and bonds or an inefficient market index) so it frequently is not a member of the efficient portfolio set. This TE frontier passes through the benchmark position where $TE = 0$.

Unconstrained constant TE frontier

Maximise $\boldsymbol{x}'\boldsymbol{E}$ subject to $\boldsymbol{x}'\boldsymbol{1} = 0$, $\boldsymbol{x}'\boldsymbol{V}\boldsymbol{x} = \sigma_\varepsilon^2$ and $(\boldsymbol{q}+\boldsymbol{x})'\boldsymbol{V}(\boldsymbol{q}+\boldsymbol{x}) = \sigma_P^2$.

The vector of deviations from the benchmark is $\boldsymbol{x} = -\frac{1}{\lambda_2+\lambda_3}\boldsymbol{V}^{-1}(\boldsymbol{E}+\lambda_1+\lambda_3\boldsymbol{V}\boldsymbol{q})$ where $\lambda_1 = -\frac{\lambda_3+b}{c}$, $\lambda_2 = \pm(-2)\sqrt{\frac{d\Delta_2-\Delta_1^2}{4\sigma_\varepsilon^2\Delta_2-y^2}} - \lambda_3$ and $\lambda_3 = -\frac{\Delta_1}{\Delta_2} \pm \frac{y}{\Delta_2}\sqrt{\frac{d\Delta_2-\Delta_1^2}{4\sigma_\varepsilon^2\Delta_2-y^2}}$.

Jorion (2003:81) established that the solution for this optimisation describes an ellipse – a *constant* TE frontier – in return/risk space: the unconstrained[8] constant TE frontier (Figure 4.1). This frontier establishes the boundary of possible risk/return combinations of (i.e. satisfy mandatory TE constraints), active portfolios. The upper segment of this frontier, shown in Figure 4.2, is bounded on the left by the minimum variance portfolio and above by the maximum return portfolio. The arc between these portfolios on the unconstrained constant TE

[8] Unconstrained because there are no restrictions on the values of the relative asset weights, x.

frontier represents the efficient set of portfolios subject to a specific TE (in this example 5%). The locus of the constant TE ellipse in return/risk space under increasing TEs is informative.

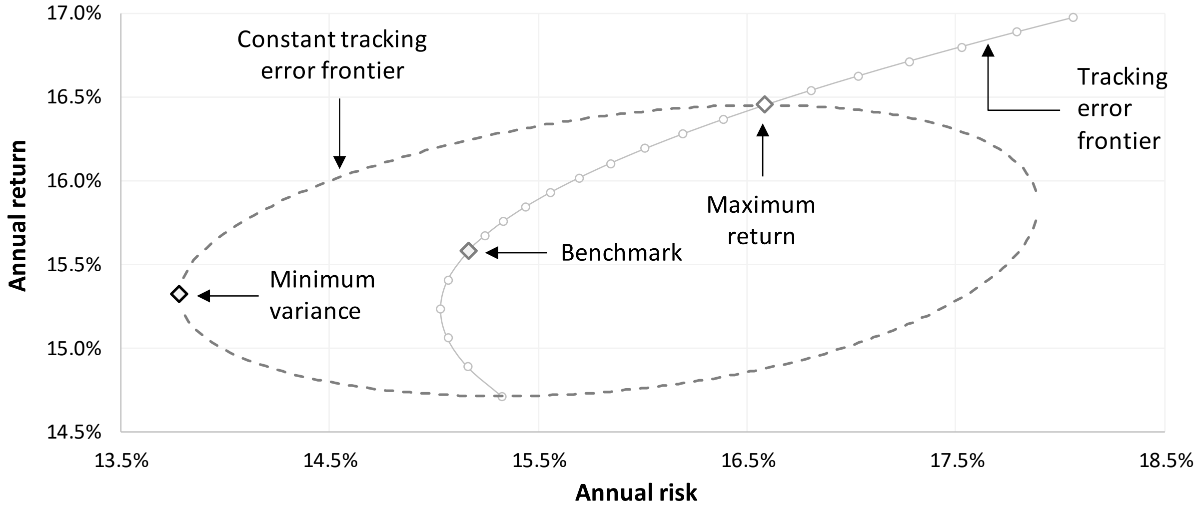

Figure 4.2: Constant (unconstrained) TE frontier for $TE = 5\%$ and TE frontier for $0\% \leq TE \leq 8\%$.[9]

Source: authors.

We display the entire TE ellipse to demonstrate the domain of all possible TE-constrained portfolios, even inefficient ones. In our experience, although portfolio managers do not consider portfolios with high risk given the same return, sometimes the vagaries of market conditions render their portfolios inefficient. Note that each risk/return combination within a TE ellipse (even the benchmark risk/return) has the same TE. The benchmark's risk/return combination may be achieved via many different constituent weight combinations – not just the unique benchmark configuration. If the benchmark's risk/return coordinate is within the TE ellipse, a combination of constituent weights exist which lead to the same risk/return benchmark coordinate. The asset weights will nevertheless be sufficiently different from the benchmark weights to warrant a TE $\neq 0$.

At first localised on the benchmark portfolio position, the ellipse expands (long and short axes increase) as TE increases until the left end coincides with the efficient frontier's turning point region. Increasing TE still further drags the ellipse back to the right and increasing TE further shifts the frontier so far to the right that the benchmark is eventually excluded, i.e. it lies

[9] The TE frontier is the locus of risk/maximum return points for many different TEs. The constant TE frontier is the locus of points which enclose all possible risk/return combinations for a single TE. The maximum return obtainable for a given TE on the constant TE frontier is, by definition, also a point on the TE frontier – hence their intersection at the maximum return.

outside the ellipse. In such cases, the TE is sufficiently loose to permit a high enough level of risk between the portfolio and the benchmark as to exclude benchmark constituents entirely.

Long-only, inequality weights-constrained constant TE frontier

Here, we use the results of Besnainou et al (2011). The weights are assumed to be positive (no short positions) and consider a subset ℓ of the N traded assets. These are the limited, or restricted assets because portfolios will be constrained to a limited weight, or limited range of weights in these ℓ assets. then $\mathbf{1}_{\boldsymbol{\ell}}$ is the N-dimensional vector such that

$$\mathbf{1}_{\boldsymbol{\ell}} = \begin{cases} 1 \text{ if asset}_i \in \ell \\ 0 \text{ if asset}_i \notin \ell \end{cases}$$

Consider an inequality constraint $\mathbf{1}_{\boldsymbol{\ell}}'\boldsymbol{q_P} \leq \omega$ where ω is a weight constraint such that $0 \leq \omega \leq 1$, that is, the weights are positive, so the inequality constraints are all positive. An example from Table 4.1 might be that the sum of domestic asset weights must not constitute more than half the portfolio, or

$$\sum_{i=1}^{N} d_i \leq 50\%.$$

Besnainou et al (2011) define a risk aversion parameter, γ, and a risk tolerance parameter, θ, which are related by:

$$\theta = \frac{\mathbf{1}'\boldsymbol{V}^{-\mathbf{1}}\boldsymbol{E}}{\gamma}.$$

The objective is to maximise $\boldsymbol{E}'\boldsymbol{x} - \frac{\gamma}{2}\boldsymbol{x}'\boldsymbol{V}\boldsymbol{x}$ (recall that $\boldsymbol{x} = \boldsymbol{q_P} - \boldsymbol{q}$ i.e. deviations from the benchmark) subject to $\mathbf{1}'\boldsymbol{x} = 0$ and $\mathbf{1}_{\boldsymbol{\ell}}'\boldsymbol{q_P} \leq \omega$. Besnainou, et al (2011) provide derivations and solutions for this, and other formulations.

The loci of the inequality weights-constrained constant TE frontier – as constraints change – is shown in Figure 4.3, and the consequences of the changing shape discussed in Section 4.

4.4 Results and discussion

Using the stylised example set out in Section 3, comprising domestic and foreign assets (Table 4.1), various constraints were placed on portfolio constituents to explore their effects on the shape of the constant TE frontier.

A common restriction on component weights is that long-only positions are permitted, i.e. $q_P > 0$. Individual assets may still be under-weight the benchmark, in which case $x < 0$, but $q + x > 0$. Imposing this restriction shrinks the ellipse of feasible portfolios, where similar returns for low levels of risk can be achieved (i.e. the south-west end of the ellipse). However, as more and more risk is taken on, a greater deviation from the unconstrained constant TE frontier can be observed (the eastern end). The east and north ends of the ellipse contracts (i.e. both the long – south-west to north-east and north-west to south-east – axes shorten) and the slope of the long axis decreases.

Some results are presented in Figure 4.3 – all for a constant $TE = 5\%$.

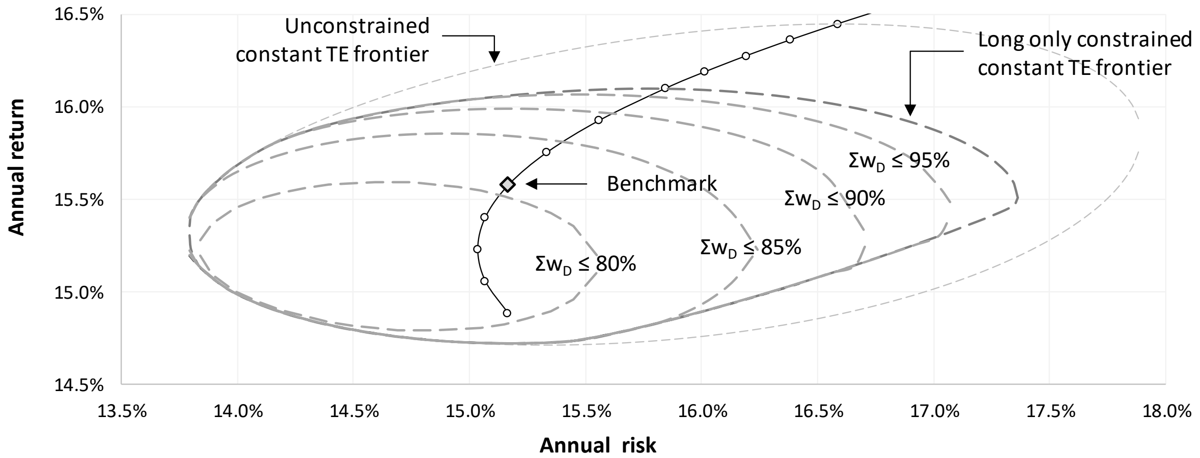

Figure 4.3: Long-only constrained constant TE frontier for $TE = 5\%$ and the sum of domestic weights $\leq 95\%, 90\%, 85\%$ and 80%.

Source: authors.

The grey dashed line shows the unconstrained constant TE frontier. This is the locus of points in return/risk space defined by the maximum and minimum obtainable annual returns at each level of risk subject to a portfolio TE constraint only (i.e. weights can be positive or negative).

Imposing the inequality $\sum_{i=1}^{N} d_i \leq \omega$ (for decreasing ωs) continues to shrink both long and short axes with the result that the original ellipse 'deflates' while remaining hinged to the south-west end. As $\sum_{i=1}^{N} d_i$ recedes from 100%, the portfolio's maximum returns – even for low σ_Ps – decrease. At $\sum_{i=1}^{N} d_i \leq 80\%$ the benchmark already falls outside the frontier – meaning that for $TE = 5\%$ and an inequality restriction that the sum of the domestic weights must be less than 80%, the benchmark falls outside the ellipse (in this stylised example) meaning that the benchmark's unique combination of risk and return is not possible for a portfolio subject to these constraints. The important point here is that – even for relatively mild limits on acceptable asset weights – the range of portfolios which satisfy the relevant constraints becomes severely diminished, and even the benchmark portfolio is rendered unattainable.

The slope of the ellipse's long axis also decreases at the asset weight restrictions increase. For the unconstrained constant TE, the slope is positive (and ≈ 0.13) for $TE = 5\%$ as shown in Figure 4.4 for the unconstrained constant TE frontier (showing a positive long axis slope) and the constant TE frontier where $\sum_{i=1}^{N} d_i \leq 80\%$ (showing a negative long axis slope).

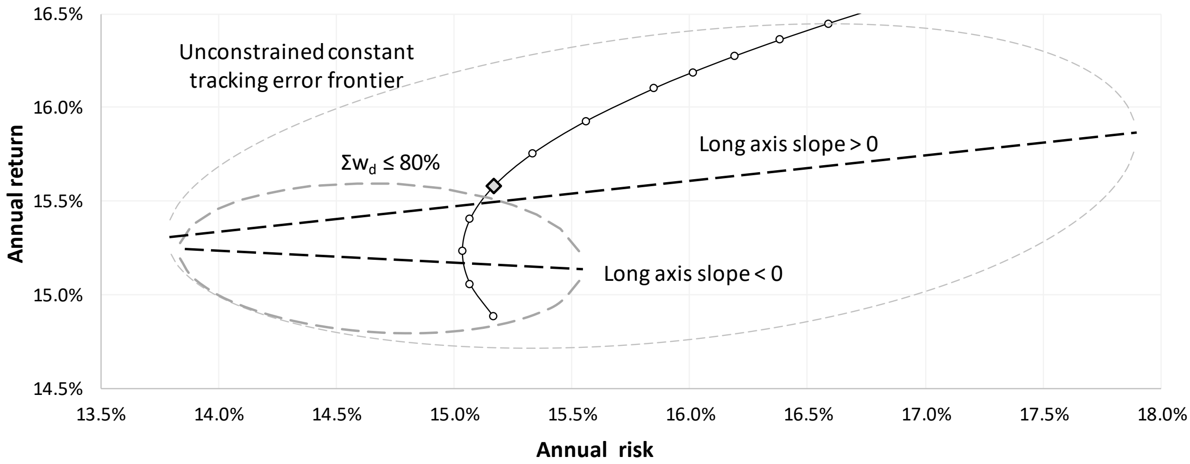

Figure 4.4 Sign of slopes of long axes for unconstrained constant $TE = 5\%$ frontier and $TE = 5\%$ frontier with domestic weights constrained to sum to $\leq 80\%$.

Source: authors.

For increasing constraints on $\sum_{i=1}^{N} d_i$, the slope diminishes but less rapidly as the constraint increases. At the relatively benign constraint of $\sum_{i=1}^{N} d_i \leq 88\%$, the long axis slope becomes negative, as shown in Figure 4.5. This reversal of the constant TEs long-axis slope occurs where $\Delta_1 = \mu_B - \mu_{MV} < 0$ has important consequences for asset allocation and manager investment styles. Even a mild (in the sense that this restriction is common in active portfolio management and often more so than this) constraint (that the weight of the domestic asset components $\leq 88\%$) results in a portfolio which has $\mu_B < \mu_{MV}$ – i.e. clearly inefficient.

Figure 4.5: Slope of long axis for $TE = 5\%$ and $\sum_d w_d \leq 100\%, 95\%, 90\%, 85\%$ and 80%. The long axis slope becomes negative ($\Delta_1 = \mu_B - \mu_{MV} < 0$) at the relatively mild constraint of $\sum_d w_d \leq 88\%$.

Source: authors.

The inclusion of a risk-free asset in the portfolio was also considered. Inverting a VCV matrix with one (or more) component $= 0\%$, however, results in divisions by 0 and associated intractable mathematical problems, so a small – but non-zero risk – "risk free" security was included.[10] This addition did not alter the results – the position of the ellipse in return/risk space was altered, but the orientation of the ellipse's axes and the reduction in area of possible portfolios was negligible unless the weight of the risk-free security were unrealistically high.

Allowing for short selling did not change the results either. When the weights restrictions were imposed, the allocation calculation of portfolio weights was always $> 0\%$ for the configuration employed in this work (i.e. volatilities, expected returns, correlations and benchmark weights). For correlations < 0 or for negative expected returns, weights $< 0\%$ become relevant as the benefits from diversification are invoked to maximise portfolio returns. Again, however, this did not change the orientation of the ellipse's axes nor the rate of reduction in investable portfolio area as shown in Figure 4.6.

[10] Such a security is not unrealistic.

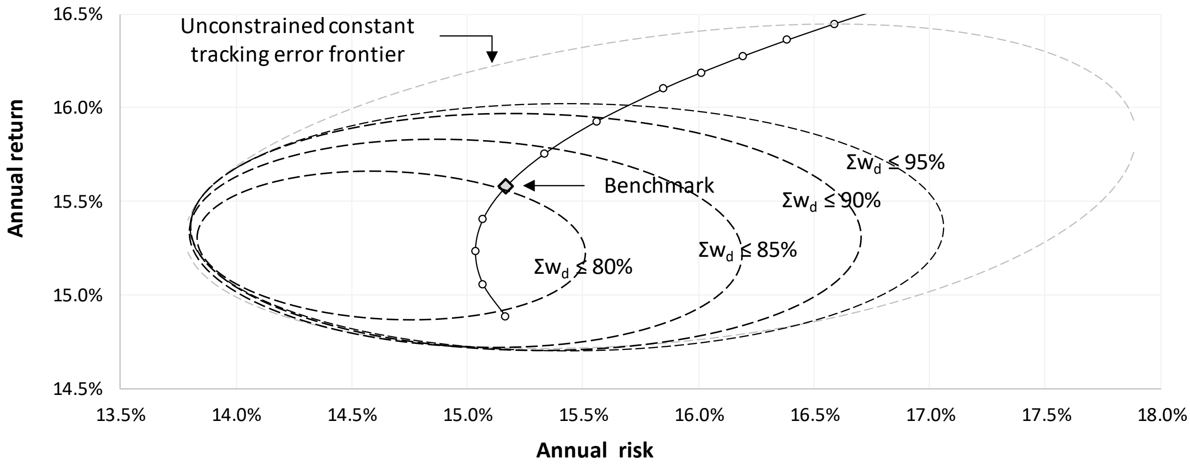

Figure 4.6: Unconstrained (long and short positions permissible) constant TE frontier for $TE = 5\%$ and the sum of domestic weights $\sum_i w_{d_i} \leq 95\%, 90\%, 85\%$ and 80%.

Source: authors.

We also considered multiple constraints, for example, the simultaneous imposition of weights restrictions on both domestic and foreign securities simultaneously. The universe of possible portfolios shrinks faster – with increasing restrictions – than that observed for restrictions on only domestic assets. Portfolio constraints in the form of weight restrictions reduces the portfolio choice considerably, to the point of vanishing relevance as these constraints become more limiting as shown in Figure 4.7. Note again the change in sign of long axis slope as constraints increase.

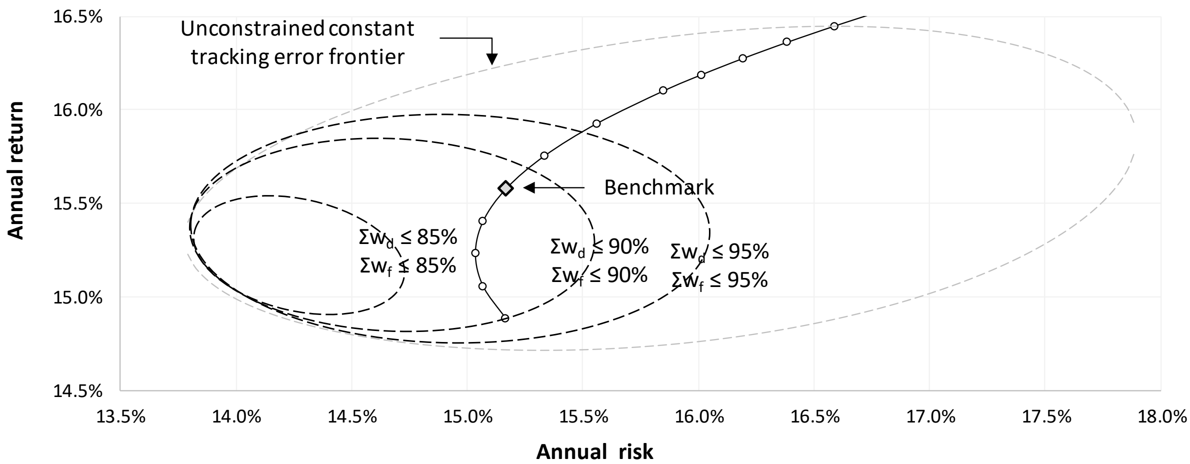

Figure 4.7: Unconstrained (long and short positions permissible) constant TE frontier for $TE = 5\%$, sum of domestic weights $\sum_i w_{d_i} \leq 95\%$ and 90% and sum of foreign weights $\sum_i w_{f_i} \leq 95\%, 90\%$. Note that, to satisfy these constraints, investment in a 'risk-free' asset becomes necessary.

Source: authors.

Imposing weights constraints reduces the number of investable portfolios due to an overall reduction in TE ellipse region as well as a telescoped efficient arc brought about by a negative long-axis slope as shown in Figure 4.8. Maximum Sharpe ratio portfolios occur where the "capital market line" (originating on the return axis at the risk-free rate) is tangent to the constant TE frontiers. The Jorion (2003) portfolio has the same risk as the benchmark, but a return on the efficient portfolio set for a given TE. For the larger constant TE ellipse (unconstrained) the long axis slope is positive, and the range of portfolio efficiency spans a wide range of portfolio risk. For the smaller constant TE ellipse, i.e. one for which a weights constraint has been imposed (in this case $\sum w_f \leq 85\%$ and $\sum w_d \leq 85\%$), not only has the TE ellipse shrunk considerably, but the change in the long-axis slope from positive to negative, thereby reducing the efficient arc length. The combined effect of these TE ellipse changes reduces the potential investment possibilities substantially.

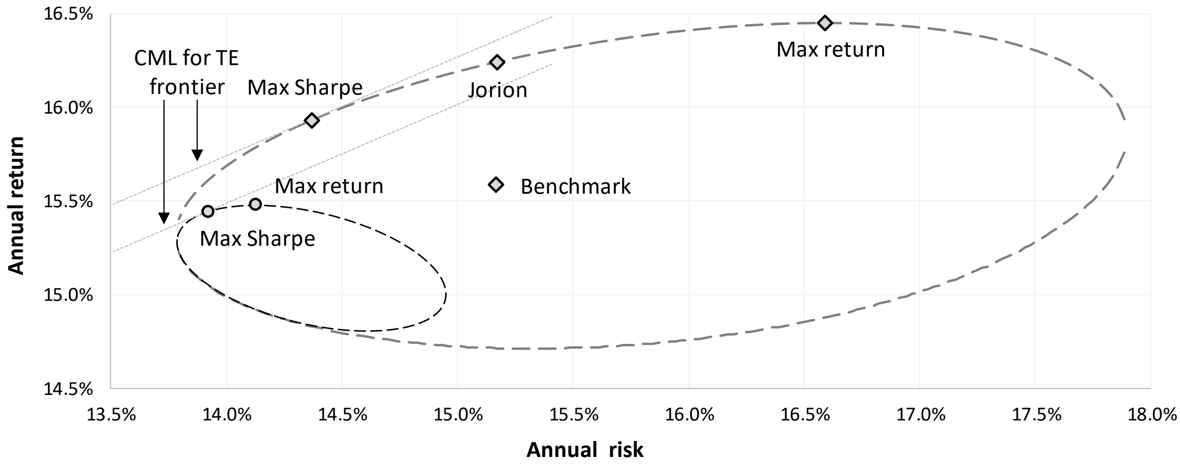

Figure 4.8: Impact of weights constraints on investable portfolios. Note the truncated axes. Source: authors.

The Sharpe ratio was calculated for increasing domestic asset weight restrictions between 80% $\leq w_D \leq$ 100% on the constrained constant TE frontier, given a constant TE of 5%. The constraints precipitate an imploding investible universe, where annual returns of a portfolio, and subsequently the return of the maximum Sharpe ratio portfolio, exponentially decreases, for each feasible risk exposure as shown in Figure 4.9.

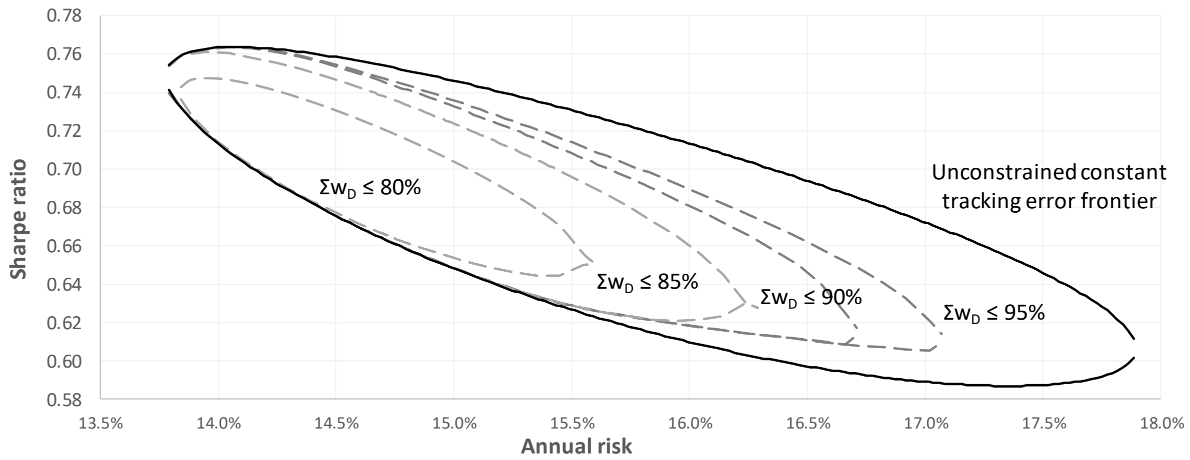

Figure 4.6: Sharpe ratios versus annual portfolio risk for $TE = 5\%$ and $\sum_d w_d \leq$ $95\%, 90\%, 85\%$ and 80%.

Source: authors.

It is interesting to note that, although the *range* of attainable Sharpe ratios diminish with increasing domestic asset weight constraints, the maximum Sharpe ratio for each constrained portfolio does not change much. This is because the north west corners of these ellipses are hinged – which coincide with the position of the maximum Sharpe ratio because of the strong negative long axis slope.

The information ratio (IR) is

$$IR = \frac{\text{Excess return}}{TE}$$

where "excess return" is the return in excess of the benchmark return. Figure 4.10 plots the IR and the Sharp ratio for changing domestic asset weights. Bajeux-Besnainou et al., (2011) introduced the adjusted IR (AIR), i.e. the IR of a portfolio subject to a weight constraint. Such a portfolio may be understood by invoking an adjusted benchmark, defined as the closest portfolio to the benchmark which satisfies the TE and other relevant constraints. The adjusted benchmark is, however, not observable, and must be assembled using empirical historical returns. For this reason, we avoided the use of the AIR.

Both IR and Sharpe ratios diminish with increasing restrictions. Unconstrained portfolios occupy the largest area in risk/return space and this space diminishes as constraints are added. The more restrictive the constraint, the smaller the potential investment area and the lower the performance ratio attainable. The IR diminishes to zero (and can become negative for more severe constraints on asset weights) because at these high weight constraints, the excess return over the benchmark approaches 0% (see Figure 4.3).

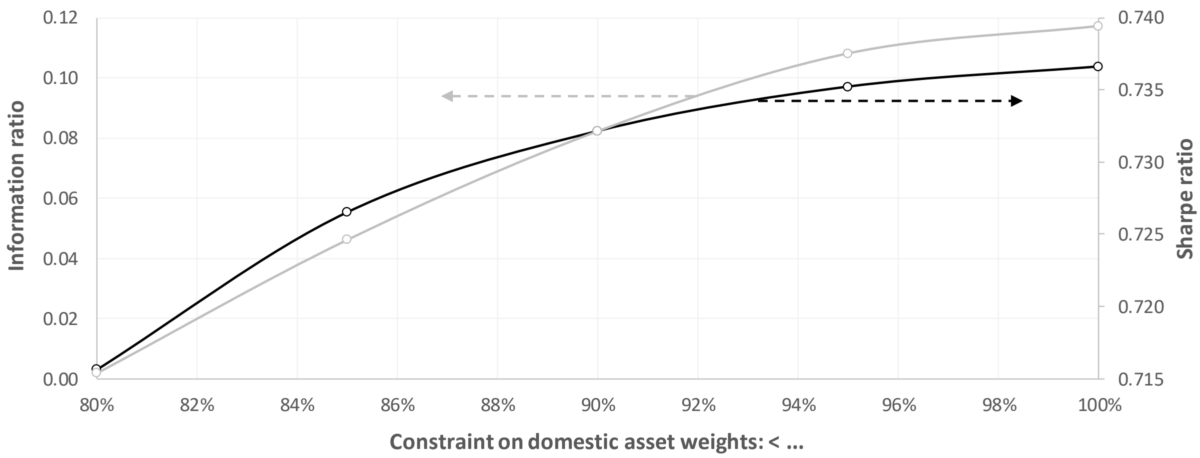

Figure 4.10 IR and Sharpe ratios for $TE = 5\%$ and unconstrained domestic weights and then $\sum_d w_d \leq 95\%, 90\%, 85\%$ and 80%.

Source: authors.

4.5 Conclusion

Optimising portfolios' risk and return originated with Markowitz (1956). The literature has since seen a proliferation of research which has evolved from the original mean/variance space to relative mean/variance space (i.e. active funds whose performance is measured relative to a benchmark). Research then followed an interrogation of TE-constrained portfolios, but to date, not much work has been conducted on (e.g.) TE-constrained and asset weight constrained portfolio simultaneously. Portfolio optimisation in excess return space subject to these multiple constraints is also an area which lacks fundamental research.

This work – using stylised market data for simulations – developed the constant TE frontier subject to asset-weight constraints and established the region in mean/variance space of possible risk/return coordinates for increasingly restrictive boundaries. Unconstrained (i.e. long and short absolute positions permitted) portfolios subject to a TE occupied the largest possible area in risk/return space. Each subsequent restriction diminished this area by shortening both the long and short axis of the constant TE ellipse. This shortening was asymmetrical: the left end of the long axis and the bottom end of the short axis were fixed for increasing restrictions on asset components – thus these constraints reduce the maximum returns attainable while reducing the risk. The range of possible investable portfolios (i.e. the region in risk/return space enclosed by the constant TE frontier) shrinks rapidly and considerably with increasing severity of restrictions, even for relatively small constraints. Combining multiple constraints, such as restrictions on both domestic and foreign weights amplifies this reduction. The change in slope of the constant TE frontier's long axis reduces the range of investable portfolios by shrinking the range of the efficient portfolio set.

At sufficiently high constraints, the benchmark lies outside the realm of possible portfolios. This means that – unsurprisingly – for suitably restrictive constraints, the region of possible risk/return combinations do not embrace a sufficiently large area to include the benchmark: benchmark and portfolio are considerably different and acceptable risk/return coordinates for the former do not apply to the latter.

Information and Sharpe ratios diminish with increasing restrictions, as expected. Unconstrained portfolios occupy the largest area in risk/return space: this space diminishes as constraints are added. The more restrictive the constraint, the smaller the potential investment area. This could have important consequences for fund managers and the way in which they are compensated.

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Chapter 5

Conclusions and suggestions for future research

5.1. Summary and conclusions

Extensive research has shown that active portfolio managers who are driven to outperform market returns while adhering to their mandate, do so without consideration for absolute portfolio risk (e.g. Jansen, & van Dijk, 2002, Wu & Jakshoj, 2011 and Bertrand, 2010). The imposition of a TE constraint restricts performance deviations away from the benchmark, ensuring more risk-adverse portfolio construction. Imposing of further constraints, such as α and β, utility and restrictions on asset weights further compounds the difficulty of attaining these goals simultaneously and limits the potential investable universe. For the first time, the severity of these restrictions on portfolio performance was investigated and reported.

5.1.1. Paper 1: Feasible portfolios under tracking error, β, α, and utility constraints

Portfolios subject to TE constraints are considered sub-optimal in that they do not maximise investor utility - rather than maximise returns, minimise risk nor maximise risk-adjusted returns. 'Optimal' portfolio selection is a subjective measure that cannot be defined solely using risk aversion. Considerations involving individual investor performance targets, investment policies and variances in asset type selection distinguish satisfaction levels between fund managers. As utility is a conceptual depiction of portfolio optimisation and cannot be measured, it can be concluded that maximising *risk-adjusted* return portfolios (*maximum Sharpe portfolio*) are 'optimal', as they provide investors with greater utility than more risk-averse portfolios i.e. investors should aim to maximise risk-adjusted returns, rather than avoid risk.

In practice, investors are not only constrained by TE, other metrics include α, VaR and utility constraints. Adding a CAPM β constraint on TE constrained portfolios generates a β frontier, where $\beta = 1$ would pass through the benchmark at TE = 0. This depiction is useful in showing how portfolios with lower risk, yet higher returns than the index, can only be constructed in bearish markets when $\beta < 1$. The inverse is also true in that all portfolios constrained to a $\beta > 1$ frontier will have a higher volatility than the benchmark. Investors bound by both a TE and β constraint yields a single intersection point that provides a positive return, where volatility is at a maximum – an unfavourable result. Lastly, imposing a β constraint, where the intersection (of the β frontier with the constant TE frontier) holds higher risk (sits further right) than maximum return portfolio is non-sensical.

The α-TE frontier, an additional constrained from CAPM theory, depicts the minimum TE for various levels of *ex-ante* α. Here the impossibility of simultaneously imposing an α - TE constraint as well as maximising excess returns is shown (except in the increasingly rare case where the α constraint coincides with the intersection on the TE frontier). The α-frontier adds value in promoting the selection of less-risky portfolios than managers may want to consider. This paper addressed the first three specific objectives, namely, it established, in mean/standard deviation space, the range of possible investments when constrained by a TE; it evaluated the behaviour of α, β and utility constraints on the constant TE frontier in mean/variance as well as return/risk space and it confirmed that the simultaneous imposition of more than two constraints leads to inefficient portfolios (i.e. those that do not exhibit maximum return or minimum risk).

5.1.2. **Paper 2:** Portfolio performance under combined tracking error and asset weight constraints

Fund managers are responsible for driving capital gains. Effective portfolio selection and asset weight allocation is paramount to their success and compensation. Using stylised market data, the constant TE frontier subject to asset allocation constraints was formulated. One of the most common restrictions on component weights is the long-only constraint, where individual asset *absolute* weights must be positive (but can still be under- or over- weight the benchmark). Here is has been shown that its imposition asymmetrically shrinks the TE-constrained ellipse of feasible portfolios. Characteristically, the resulting *possible* long-only portfolios are far more risk adverse, where possible maximum annual returns (and associated maximum annual risks) are significantly diminished, and the long axis of the long-only TE-constrained ellipse is at a less steep angle to that of TE-constrained portfolio. This tilt has a noteworthy impact on an investors ability to outperforming the benchmark, as excess returns become less lucrative and less achievable.

Imposing an asset allocation ceiling on long-only TE-constrained portfolios is shown to further implode the realm of feasible portfolios, where maximal returns (and risks) continue to exponentially decrease, and the long axis of the ellipse continues to tilt clockwise. An important consideration for fund managers is that at a sufficiently high asset allocation restriction, the benchmark falls outside the area of feasible portfolios and is thus rendered unattainable. As the gradient of the ellipses' long axis tends to 'flatten', the maximum Sharpe portfolio slides towards the left (to lower levels of annual risk) – and severely lower levels of annual return, where the benefits associated to the rate of change of maximising risk-adjusted returns described by Maxwell et al. (2018) are amplified. This paper addressed the final specific objective, namely it evaluated the effect of imposing weight allocation limitations on long-only TE-constrained portfolios and determined how the investable universe of feasible portfolios was affected.

5.1.3. Suggestions for future research

This work has demonstrated that overly constrained portfolio mandates do not accomplish their stated goals and only serve to limit the portfolio manager's ability to outperform the benchmark in any meaningful ways. The combination of only TE and β constraints limits the available investable universe to a single portfolio. Combing these restrictions with an α constraint leads to an impossible scenario – there is no portfolio that can simultaneously satisfy all three constraints: unless the portfolio is inefficient. Nevertheless, now that this work has demonstrated the inefficacy of these restraints, future work can set about optimising available assets weights subject to these restrictions. The resulting portfolios cannot (by construction) be efficient, but perhaps maximal efficiency (or minimal inefficiency) portfolios can be still obtained given the constraints.

Future strategies could also attempt to account for tax considerations, regulatory constraints and rebalancing costs, once real data (rather than a stylised example) have been implemented and the results thoroughly back-tested. These should be simple to implement in practice: due diligence should be observed in selecting assets and forecasting expected returns and adjustments for dividends and stock splits.

Future work could involve studies which interrogate the impact of changing these factors on portfolio performance. While the variation of optimal TE-constrained portfolio performance, dependent on the level of TE, β and α constraints, has been demonstrated, results which investigate other aspects of portfolio construction would be of great benefit to active portfolio managers seeking optimal performance. It is also well-known that β is time-varying, while the CAPM model assumes it is constant. This, too, should be addressed in future work.

It has been shown that Information and Sharpe ratios diminish with increasing restrictions. Unconstrained portfolios occupy the largest area in risk/return space and this region constricts as constraints are added. The more restrictive the constraint, the smaller the potential investment area – a feature that could have important consequences for fund managers and the way in which they are compensated.

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Appendix A

Collaboration affidavit

This serves to advise that I, as co-author of the article below, concur that Michael Daly undertook the bulk of the work. I was involved with only a small part of the data analysis and I provided some minimal assistance with the interpretation of results.

I hereby grant Mr Daly permission to include this article in his dissertation.

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Michael Maxwell

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Appendix B

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